Evolutionary Game Theory Notes

James Massey

These notes are intended to be a largely self contained guide to everything you need to know for the evolutionary game theory part of the EC341 module. During lectures we will work through these notes, starting with a quick skim through Chapter 2 (which should be a revision of already known material for most of you) before moving on to Chapters 3 and 4 where we will spend most of our time. One notable thing missing from these notes are diagrams. You will find these very useful for making sense of these notes and should make sure to copy these down together with anything else covered in lectures that may help your understanding. These notes also contain many exercises, which I recommend you attempt. Separately I will also give you some further exercises and a sample exam question to give an idea of the kind of thing I might ask in an exam. Good luck!

Chapter 1

Overview

While non-cooperative game theory assumes hyper-rational agents predicting and best responding to each others' play to lead to equilibria, evolutionary game theory takes a somewhat different approach. Here we dispose with such assumptions in favour of a milder assumption of *evolutionary pressure* towards the increasing representation of *better* strategies and study the results.[1]. Take for example the question of which side of the road to drive a car (absent any laws which solve the problem). If two drivers are heading in opposite directions towards each other, we can identify two mutually advantageous outcomes: where they pass each other with both driving on the left; and where they pass each other with both driving on the right. While non-cooperative game theory identifies both as Nash Equilibrium, it is not obvious how two players should be able to coordinate in this way. The evolutionary game theory way is to look at the environment, see how the various strategies fare and how the environment is therefore likely to change when we take evolutionary pressure into account. If we start at a point in time whereby more people drive the left than the right, then those agents who adopt this strategy are doing better than those who stick to the right. So

[1]Here *better* is defined with respect to the environment, in much the same way as it depends on the strategies of one's opponents in a classical non-cooperative game

evolutionary pressures would increase the representation of agents sticking to the left until everybody favoured this strategy.

In this and many other examples the predictions of evolutionary game theory coincide with those of non-cooperative game theory, while often helping to explain them. In fact as will be seen, the central evolutionary game theoretic concepts of Evolutionarily Stable Strategies and stability in the replicator dynamics give slight refinements to Nash Equilibrium. In an economic context there are three main different ways that I will mention on how one can interpret *evolutionary pressure*: i) the biological interpretation; ii) imitating other more successful agents; iii) observing and myopically best responding to one's environment.

The biological interpretation would be based on natural selection, where we interpret payoffs to strategies as reproductive fitness. Agents with strategies giving higher than average reproductive fitness expand quicker and so the proportion of agents following this strategy grows. To give this credence, consider a setting consisting of several firms who do business by interacting with one another in a market place. A firm adopting a successful strategy will expand and so form a larger part of the market, whereas firms adopting unsuccessful strategies go out of business. The imitative interpretation would allow for slightly more sophisticated firms, who can switch strategies. They can see what strategies others adopt, how well they fare, and can switch strategy to mimic more successful firms than themselves. The third approach is to allow for agents, who are more sophisticated still. Under this form of evolutionary pressure a firm would be able to look at what all other firms are doing and strategically choose a best response to the current distribution of other firms' strategies.

Ever since 1859, with the work of Charles Darwin, biologists have recognised the fact that the genetic make-up of animals and plants evolves over time, with genes more favourable to their environment being *selected* by nature. It is unsurprising therefore, that the field of evolutionary game theory originated in the biological literature with the seminal work of Maynard-Smith and Price in 1973. They developed a mathematical framework leading to the central definition of Evolutionarily Stable Strategy (ESS) in order to explain the level of aggression in animal populations.

The concept of Evolutionarily Stable Strategy attempts to capture the idea of resistance to mutations in a static environment. It asks the following question: if we introduce a small number of mutants into our population, how will they fare? If this is a beneficial mutation, they will have higher reproductive fitness than the overall population and hence proportionately grow. On the other hand, if the mutants have lower reproductive fitness, then with time the proportion of agents with this mutation will decline and tend to zero, and we say this mutation dies out. An ESS is a population state which is resistant to all such small mutations in this way.

Since then, attention has been focused on more dynamic models. In 1978 Taylor and Jonker introduced what is now known as the *replicator dynamic* to explicitly model the process of natural selection. The idea is that agents reproduce asexually according to their reproductive fitness, giving offspring with the same genetically determined behaviour as themselves. The consequence of this is that the proportion of agents with characteristic i, say x_i changes in proportion with x_i and is either increasing or decreasing depending on whether agents with characteristic i do better or worse than the average payoff in the whole population. As well as this biological interpretation the replicator dynamic, there are several other interpretations, including agents imitating other more successful agents. So, while there are other selection dynamics, we will focus on the replicator dynamic due to its widespread appeal.

In the 1990s economists built on this work to create dynamic models allowing for both mutation and selection. The introduction of mutation necessarily means that these are stochastic models, as opposed to the deterministic replicator dynamic. The solution concept used is *stochastic stability* which looks at what state we would expect to observe in the long run, as the probability of mutations tends to zero. While ESS refines the set of Nash Equilibria, stochastic stability refines the set of ESS. It is somewhat akin to risk-dominance and so will often select a unique outcome where the former do not. Unfortunately, due to time constraints, none of this material will be covered, although students interested in this are welcome to ask me for references on this work.

Chapter 2 will introduce the general evolutionary game theory framework and go over some useful results from non-co-operative game theory. Chapter 3 looks at the static notion of ESS and its compatriot, NSS. We show how they refine the set of Nash Equilibria. In Chapter 4 we turn our focus to the replicator dynamic and discuss basins of attraction and stability. We show how stability under the replicator dynamic compares with Nash Equilibrium and ESS.

Chapter 2

Framework and Notation

There are three parts to this chapter: the first lays out the basic notation which will be followed throughout these notes and gives a brief coverage of some non-co-operative game theory that will be useful later. The second and third parts explain the evolutionary game theory model in one and two population settings.

2.1 Notation and basics of non-cooperative game theory

Throughout these notes, the focus will be on games between two players, each with a finite number of strategies. This allows the following representation of a game G as a bimatrix game:

Definition 2.1.1. A **bimatrix game** is $G = (A, B)$ where both A and B are $m \times n$ matrices (where both m and n are integers greater than or equal to 1).

$$G = (A, B) = \begin{pmatrix} a_{11}, b_{11} & a_{12}, b_{12} & \dots & a_{1n}, b_{1n} \\ a_{21}, b_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{m1}, b_{m1} & \dots & \dots & a_{mn}, b_{mn} \end{pmatrix}$$

Player 1 has the pure strategy set $S_1 = \{1, \dots, m\}$ and his strategies correspond to choosing between the m rows. Player 2 has the pure strategy set $S_2 = \{1, \dots, n\}$ and his strategies correspond to choosing between the n columns. If they choose the i^{th} row and j^{th} column, then a_{ij} and b_{ij} are the payoffs to Players 1 and 2 respectively.

These payoffs are interpreted as von-Neumann Morgenstern payoffs to avoid any need for considerations of risk aversion or the such like. This is necessary to allow us to define the payoffs of mixed strategies as the weighted averages of payoffs for pure strategies as we will do so below.

Definition 2.1.2. The set of **mixed strategies for player 1** is

$$\Delta_1 = \left\{ \mathbf{x} = (x_1, \dots, x_m)^T \in \mathbb{R}^m_{\geq 0} : \sum_{i=1}^{m} x_i = 1 \right\}$$

where $\mathbb{R}^m_{\geq 0}$ is the set of m-dimensional vectors with all arguments non-negative and the "T" superscript denotes "transpose". The interpretation of x_i is the probability with which strategy i is played. Similarly, the set of **mixed strategies for player 2** is

$$\Delta_2 = \left\{ \mathbf{y} = (y_1, \dots, y_m)^T \in \mathbb{R}^n_{\geq 0} : \sum_{j=1}^{n} y_j = 1 \right\}$$

A strategy $\mathbf{x} \in \Delta_i$ is called **pure** if $x_j = 1$ for some j, and we denote this strategy $\mathbf{e}^j$. Let $\Delta = \Delta_1 \times \Delta_2$ denote the joint mixed strategy space.

Given a strategy $\mathbf{x}$, define the **support (or carrier)** of $\mathbf{x}$ as $C(\mathbf{x}) = \{j \mid x_j > 0\}$.

We can define the **interior** of Δ_1 as all the strategies not on the boundary[1]:

$$\operatorname{int}(\Delta_1) = \left\{ \mathbf{x} = (x_1, \ldots, x_m)^T \in \mathbb{R}^m_{>0} : \sum_{i=1}^{m} x_i = 1 \right\}$$

We can define $\operatorname{int}(\Delta_2)$ similarly and $\operatorname{int}(\Delta) = \operatorname{int}(\Delta_1) \times \operatorname{int}(\Delta_2)$.

I will use the notation i for a player in $\{1, 2\}$ and $-i$, for the opposing player. Let $u_i(\sigma_i, \sigma_{-i})$ to denote the utility of i playing strategy σ_i when his opponent plays strategy σ_{-i}. With this notation, when player 1 plays $\mathbf{x}$ and player 2 plays $\mathbf{y}$, the utility player 1 receives is

$$u_1(\mathbf{x}, \mathbf{y}) = \mathbf{x} \cdot A\mathbf{y} = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j a_{ij}$$

and the utility player 2 receives is the expected payoff

$$u_2(\mathbf{y}, \mathbf{x}) = \mathbf{y} \cdot B^T \mathbf{x} = \mathbf{x} \cdot B\mathbf{y} = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j b_{ij}$$

Exercise 2.1.1. Consider the general $m = n = 2$ game, $G = \begin{pmatrix} a_{11}, b_{11} & a_{12}, b_{12} \\ a_{21}, b_{21} & a_{22}, b_{22} \end{pmatrix}$ and let player 2's strategy be $\mathbf{y} = (y_1, y_2)$. Find the payoffs to player 1 from employing strategies $\mathbf{e}^1$, $\mathbf{e}^2$ and $\mathbf{x} = (x_1, x_2)$.

With this in place, we can now define the notions of weak and strict dominance.

Definition 2.1.3. Strategy $\mathbf{x} \in \Delta_i$ **strictly dominates** $\mathbf{w} \in \Delta_i$ if $u_i(\mathbf{x}, \mathbf{y}) > u_i(\mathbf{w}, \mathbf{y})\ \forall \mathbf{y} \in \Delta_{-i}$.

Strategy $\mathbf{x} \in \Delta_i$ **weakly dominates** $\mathbf{w} \in \Delta_i$ if $u_i(\mathbf{x}, \mathbf{y}) \geq u_i(\mathbf{w}, \mathbf{y})\ \forall \mathbf{y} \in \Delta_{-i}$. and $\exists \mathbf{y}' \in \Delta_{-i}$ such that $u_i(\mathbf{x}, \mathbf{y}') > u_i(\mathbf{w}, \mathbf{y}')$.

A strategy $\mathbf{x} \in \Delta_i$ is **strictly (weakly) dominant** if it strictly (weakly) dominates all other strategies in Δ_i.

[1] That is with $x_i \neq 0$ for all i. In other words all completely mixed strategies

Note that strict dominance implies weak dominance.

Example 2.1.1. Let $G = \begin{pmatrix} 1,1 & 4,2 \\ 4,0 & 1,1 \\ 2,0 & 2,0 \end{pmatrix}$, then

i) for player 2: $\mathbf{e}^2$ weakly dominates $\mathbf{e}^1$.

ii) for player 1: $(1/2, 1/2, 0)$ strictly dominates $\mathbf{e}^3$.

Given one player's strategy, we can describe the other players best replies as follows:

Definition 2.1.4. Given a strategy $\mathbf{y} \in \Delta_{-i}$ denote the **best response correspondence** of player i by $BR_i(\mathbf{y}) \subseteq \Delta_i$

$$BR_i(\mathbf{y}) = \{\mathbf{x} \in \Delta_i \mid u_i(\mathbf{x}, \mathbf{y}) \geq u_i(\mathbf{w}, \mathbf{y}) \ \forall \mathbf{w} \in \Delta_i\}$$

A **Nash Equilibrium** is a pair of strategies $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \Delta_1 \times \Delta_2 = \Delta$ such that $\bar{\mathbf{x}}$ is a best response to $\bar{\mathbf{y}}$, and $\bar{\mathbf{y}}$ is a best response to $\bar{\mathbf{x}}$.

To be a **strict Nash Equilibrium**, we require $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ to be the unique best responses to each other.

Exercise 2.1.2. Show the following:

i) Only pure strategies can be strictly dominant.

ii) If both players follow strictly dominant strategies then this constitutes a strict Nash Equilibrium, and there are no other Nash Equilibria.

iii) If both players follow weakly dominant strategies then this must constitute a Nash Equilibrium. Must this Nash Equilibrium necessarily be unique?

I state the following well-known result without proof[2]:

Theorem 2.1.1. *Every bi-matrix game has at least one Nash Equilibrium*

[2]Students interested in a proof can find one in Weibull or any standard game theory textbook

Exercise 2.1.3. Find all Nash equilibria of the following games and calculate players' payoffs at these equilibria:

i) $\begin{pmatrix} 4,4 & 0,5 \\ 5,0 & 2,2 \end{pmatrix}$

ii) $\begin{pmatrix} 4,4 & 0,5 \\ 5,0 & -1,-1 \end{pmatrix}$

iii) $\begin{pmatrix} 2,3 & 0,0 \\ 0,0 & 1,1 \end{pmatrix}$

iv) $\begin{pmatrix} 1,0 & 0,1 \\ 0,1 & 1,0 \end{pmatrix}$

Symmetric games

In evolutionary game theory, we are often particularly interested in symmetric games and symmetric equilibria.

Definition 2.1.5. An $m \times n$ **bimatrix game** $G = (A, B)$ **is symmetric** if $m = n$ and $B = A^T$, where A^T is the transpose of A.

A **Nash Equilibrium** $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ **is symmetric** if $\bar{\mathbf{x}} = \bar{\mathbf{y}}$.

If the game is symmetric, then both players have the same strategy space, which I denote by Δ and the payoff function is symmetric. So I drop the player subscript and denote $u(\mathbf{x}, \mathbf{y}) = \mathbf{x} \cdot A\mathbf{y}$ to be the utility from playing strategy $\mathbf{x} \in \Delta$ against $\mathbf{y} \in \Delta$.

In a symmetric game, we denote the set of symmetric Nash Equilibria by $\Delta^{NE} \subseteq \Delta$, and the set of strict symmetric Nash Equilibria by $\Delta^{NE>} \subset \Delta$. Note that any strict Nash Equilibria must be in pure strategies. Generally in a symmetric game there can be equilibria which are not symmetric, but the relevant equilibria for our purposes are the symmetric ones. One useful result (proof not given - see Weibull) is that every symmetric bimatrix game has at least one symmetric Nash Equilibrium:

Proposition 2.1.1. *For every symmetric bi-matrix game* $\Delta^{NE} \neq \emptyset$.

Exercise 2.1.4. For each of the following symmetric games find the set of all Nash Equilibria and Δ^{NE} and $\Delta^{NE>}$:

i) $\begin{pmatrix} 4,4 & 0,5 \\ 5,0 & 2,2 \end{pmatrix}$

ii) $\begin{pmatrix} 4,4 & 0,5 \\ 5,0 & -1,-1 \end{pmatrix}$

iii) $\begin{pmatrix} 1,1 & 0,0 \\ 0,0 & 0,0 \end{pmatrix}$

iv) $\begin{pmatrix} 1,1 & 0,0 \\ 0,0 & 2,2 \end{pmatrix}$

v) $\begin{pmatrix} 0,0 & 1,-1 & -1,1 \\ -1,1 & 0,0 & 1,-1 \\ 1,-1 & -1,1 & 0,0 \end{pmatrix}$

2.2 Evolutionary games of one population

We assume there is a large population of agents, each of whom is genetically hardwired to play a particular strategy and discuss the biological interpretation. Agents meet randomly in pairs and the payoff each agent receives is the expected payoff from being matched against a randomly selected opponent. As we shall argue, in this setting, it is natural to restrict attention to symmetric games and symmetric equilibria

If we had an asymmetric game, when two players meet, one must be drawn in the player 1 role and the other in the player 2 role. But since they are both randomly drawn from the same population, there is no clear way to assign them. One could suppose that in every bilateral meeting each agent occupies each of the two player roles with equal probability, but this has the effect of turning the asymmetric game into a symmetric one.

In this setting we can think of the environment (population state) as describing the prevalence of each of the pure strategies in the population. Hence there is a one-to-one-correspondence between the set of population states and the set of mixed strategies.

Definition 2.2.1. A **population state** $\mathbf{x} \in \Delta$ lists the prevalence of each of the pure strategies in the population.

Remark 2.2.1. There are two possible interpretations for this:

1. Every agent is genetically hardwired to play the mixed strategy $\mathbf{x}$.
2. Every agent is genetically hardwired to play a pure strategy: for each $i \in \{1, \dots, m\}$, x_i is the proportion of agents following the ith pure strategy, $\mathbf{e}^i$.

Given that agents meet randomly in pairs, and the population is sufficiently large, the probability that a specific agent's randomly selected opponent plays the ith pure strategy can be taken to be x_i[3]. Thus an agent following the strategy $\mathbf{y} \in \Delta$ when the population state is $\mathbf{x} \in \Delta$ will receive an expected payoff of $u(\mathbf{y}, \mathbf{x})$, while the average payoff in the population is $u(\mathbf{x}, \mathbf{x})$. Logically an agent following a strategy that does better than average will grow quicker and proportionately expand, shifting the population state. Before we move on to the more formal analysis in Sections 3 and 4, it may be useful to consider how we would expect the population state to change in some simple examples.

Exercise 2.2.1. Considering the interpretation where each agent follows a pure strategy. For each of the following examples, draw a graph with the proportion of agents following the first strategy on the horizontal axis and the utilities to each pure strategy on the vertical axis. How would you expect the dynamics of the population state to change?

i) $\begin{pmatrix} 4,4 & 0,5 \\ 5,0 & 2,2 \end{pmatrix}$

[3] Note the role of the assumption that the population is large. Sincein most applications it makes sense to assume an agent never plays against itself, the actual probability of meeting an agent playing strategy strategy i maybe slightly different from x_i. But as the population size increases, this difference tends to zero.

ii) $\begin{pmatrix} 0,0 & 1,1 \\ 1,1 & 0,0 \end{pmatrix}$

iii) $\begin{pmatrix} 4,4 & 0,5 \\ 5,0 & -1,-1 \end{pmatrix}$

iii) $\begin{pmatrix} 1,1 & 0,0 \\ 0,0 & 2,2 \end{pmatrix}$

What are the rest points of the dynamics?

In these examples one should be able to see similarities between the rest points and the symmetric Nash Equilibria in these games. As demonstrated in Chapters 3 and 4, this is not merely coincidental. Also notice that any asymmetric Nash Equilibria such as those in the second example do not enter our analysis. This is because all agents are randomly drawn to face each other, meaning that there is a chance two $\mathbf{e}^1$ agents are drawn together, and a chance that two $\mathbf{e}^2$ agents are drawn together. If it was the case that $\mathbf{e}^1$ agents are only drawn to play with $\mathbf{e}^2$ agents then something resembling the asymmetric equilibrium can be achieved. However this would be to go down the route of two distinct populations of agents, with only inter-population interaction. This is the topic of the next section.

2.3 Evolutionary games of two populations

We assume two large populations of agents, one of row playing agents representing player 1, the other of column playing agents representing player 2. This setting can be used to analyse asymmetric games, or symmetric games in which players take distinct roles in the game, so we can distinguish whether an agent is in the player 1 or player 2 role. A population state defines the prevalence of each pure strategy in each population. Once again, in each population, there is a one-to-one correspondence between the population state and the set of mixed strategies.

Definition 2.3.1. A **population state** $(\mathbf{x}, \mathbf{y}) \in \Delta_1 \times \Delta_2 = \Delta$ lists the prevalence of each of the pure strategies in each population.

There is a random inter-population matching, so that each population 1 agent is randomly matched with a population 2 agent and vice-versa. So, given population state $(\mathbf{x}, \mathbf{y}) \in \Delta$, a population 1 agent following strategy $\hat{\mathbf{x}}$ receives payoff $u_1(\hat{\mathbf{x}}, \mathbf{y})$ and a population 2 agent following strategy $\hat{\mathbf{y}}$ receives payoff $u_2(\hat{\mathbf{y}}, \mathbf{x})$, while the average payoffs of population 1 and population 2 agents are $u_1(\mathbf{x}, \mathbf{y})$ and $u_2(\mathbf{y}, \mathbf{x})$ respectively. The evolutionary dynamics say that strategies with above average payoff should experience an increase in their proportion of the population. In chapter 4, we will see similarities between the rest points of the population state under such a dynamic and the set of Nash Equilibria (this time including asymmetric equilibria).

Chapter 3

ESS

3.1 What is an ESS?

In this Chapter, we build on Section 2.2. All the analysis is with respect to agents being randomly matched in pairs from a single population. If we are at a Nash Equilibrium then each pure strategy present gets the same payoff and so applying an evolutionary dynamic would not change the population state. In this sense, absent mutations, such a state is a true equilibrium: once entered we won't leave. However as we will see, some equilibria are more stable than others.

An Evolutionarily Stable Strategy (ESS) is a mixed strategy, or population state, that is resistant to small mutations. It asks the following question: if everyone is playing this mixed strategy, say $\mathbf{x} \in \Delta$, is this resistant to the introduction of small mutations? To be more precise, if we introduce a small proportion of mutants playing playing $\mathbf{y} \in \Delta$, will the mutants obtain lower payoff than the rest of the population, and hence this mutation be driven out? If the proportion of mutants is ε then the new population state is $(\varepsilon\mathbf{y} + (1-\varepsilon)\,\mathbf{x})$ and we are asking whether the following equation holds:

$$u\left(\mathbf{x}, \varepsilon\mathbf{y} + (1-\varepsilon)\,\mathbf{x}\right) > u\left(\mathbf{y}, \varepsilon\mathbf{y} + (1-\varepsilon)\,\mathbf{x}\right) \tag{3.1.1}$$

If for every strategy other than $\mathbf{x}$, we can find an invasion barrier such that all mutations of a lower proportion are driven out, then we say that $\mathbf{x}$ is an ESS.

Definition 3.1.1. $\mathbf{x} \in \Delta$ is **ESS** if for every strategy $\mathbf{y} \in \Delta$, $\mathbf{y} \neq \mathbf{x}$, there exists some ε_y such that for all $\varepsilon \in (0, \varepsilon_y)$ equation (3.1.1) holds.

Intuitively, when ε is small, we can see that whether a mutation is driven out will primarily depend on how it fares against the incumbent population. The following proposition shows this to be the case, while how agents fare against mutants is only of secondary importance.

Proposition 3.1.1. *A strategy* $\mathbf{x} \in \Delta$ *is ESS if and only if it meets the following first-order and second-order best reply conditions:*

$$u(\mathbf{y}, \mathbf{x}) \leq u(\mathbf{x}, \mathbf{x}) \;\; \forall \mathbf{y} \tag{3.1.2}$$

$$u(\mathbf{y}, \mathbf{x}) = u(\mathbf{x}, \mathbf{x}) \Rightarrow u(\mathbf{y}, \mathbf{y}) < u(\mathbf{x}, \mathbf{y}) \;\; \forall \mathbf{y} \neq \mathbf{x} \tag{3.1.3}$$

Proof. The key formula for ESS is

$$u(\mathbf{x}, \varepsilon\mathbf{y} + (1-\varepsilon)\mathbf{x}) > u(\mathbf{y}, \varepsilon\mathbf{y} + (1-\varepsilon)\mathbf{x}) \tag{3.1.4}$$

Take $\mathbf{y} \in \Delta$, $\mathbf{y} \neq \mathbf{x}$. There are four possible scenarios:

i) $u(\mathbf{y}, \mathbf{x}) < u(\mathbf{x}, \mathbf{x})$. If $u(\mathbf{y}, \mathbf{y}) \leq u(\mathbf{x}, \mathbf{y})$ then clearly equation (3.1.4) holds $\forall \varepsilon \in (0, 1)$. If $u(\mathbf{y}, \mathbf{y}) > u(\mathbf{x}, \mathbf{y})$ then define

$$\varepsilon_y = \left(\frac{u(\mathbf{y}, \mathbf{y}) - u(\mathbf{x}, \mathbf{y})}{u(\mathbf{x}, \mathbf{x}) - u(\mathbf{y}, \mathbf{x})} + 1 \right)^{-1} \tag{3.1.5}$$

Then $\varepsilon_y \in (0, 1)$ and a few lines of routine algebra show that equation (3.1.4) holds $\forall \varepsilon \in (0, \varepsilon_y)$.

ii) $u(\mathbf{y}, \mathbf{x}) = u(\mathbf{x}, \mathbf{x})$ and $u(\mathbf{y}, \mathbf{y}) < u(\mathbf{x}, \mathbf{y})$. Clearly equation (3.1.4) holds $\forall \varepsilon \in (0, 1)$.

iii) $u(\mathbf{y},\mathbf{x}) = u(\mathbf{x},\mathbf{x})$ and $u(\mathbf{y},\mathbf{y}) < u(\mathbf{x},\mathbf{y})$. Clearly equation (3.1.4) is violated $\forall\varepsilon \in (0,1)$.

iv) $u(\mathbf{y},\mathbf{x}) > u(\mathbf{x},\mathbf{x})$. If $u(\mathbf{y},\mathbf{y}) \geq u(\mathbf{x},\mathbf{y})$ then clearly equation (3.1.4) is violated $\forall\varepsilon \in (0,1)$. If $u(\mathbf{y},\mathbf{y}) < u(\mathbf{x},\mathbf{y})$ then it can be shown that for sufficiently small ε equation (3) is violated - eg $\varepsilon \in (0,\varepsilon_y)$.

Thus we have shown that for each $\mathbf{y} \in \Delta$, $\mathbf{y} \neq \mathbf{x}$, we can find ε_y such that equation (3.1.4) holds $\forall\varepsilon \in (0,\varepsilon_y)$ if and only if $\mathbf{y}$ satisfies equations (3.1.2) and (3.1.3). □

Exercise 3.1.1. Show that if $\mathbf{x} \in \Delta$ is weakly dominated then it can't be an ESS.

Proposition 3.1.1 gives us an alternative, and in many cases easier to apply chacterisation of ESS. We first compare how both the incumbent and mutant populations fare against the incumbent, and only if they do equally well, use the payoffs against mutants as a tie breaker.

To be an ESS, we require that the mutants be driven out. A weaker, but linked condition is that the proportion of mutants does not expand. This is the notion of Neutrally Stable Strategy (NSS)

Definition 3.1.2. $\mathbf{x} \in \Delta$ is **NSS** if for every strategy $\mathbf{y} \in \Delta$, $\mathbf{y} \neq \mathbf{x}$, there exists some ε_y such that for all $\varepsilon \in (0,\varepsilon_y)$,

$$u(\mathbf{x}, \varepsilon\mathbf{y} + (1-\varepsilon)\mathbf{x}) \geq u(\mathbf{y}, \varepsilon\mathbf{y} + (1-\varepsilon)\mathbf{x})$$

Proposition 3.1.2. *A strategy $\mathbf{x} \in \Delta$ is NSS if and only if it meets the following first-order and second-order best reply conditions:*

$$u(\mathbf{y},\mathbf{x}) \leq u(\mathbf{x},\mathbf{x}) \;\; \forall \mathbf{y} \tag{3.1.6}$$

$$u(\mathbf{y},\mathbf{x}) = u(\mathbf{x},\mathbf{x}) \Rightarrow u(\mathbf{y},\mathbf{y}) \leq u(\mathbf{x},\mathbf{y}) \;\; \forall \mathbf{y} \neq \mathbf{x} \tag{3.1.7}$$

Exercise 3.1.2. Prove Proposition 3.1.2

Define Δ^{ESS} and Δ^{NSS} to be the sets of Evolutionarily Stable and Neutrally Stable Strategies respectively.

Exercise 3.1.3. Show that for any symmetric bimatrix game G, the following is true:

$$\Delta^{NE} \supseteq \Delta^{NSS} \supseteq \Delta^{ESS} \supseteq \Delta^{NE>}$$

Further, show why no "$\supseteq$" can be replaced by "$=$" in the above statement.

(Hint: use the game $G = \begin{pmatrix} \alpha, \alpha & 1, 0 \\ 0, 1 & 1, 1 \end{pmatrix}$ and consider 3 cases: i) $\alpha < 0$, ii) $\alpha = 0$, iii) $\alpha > 0$.)

3.2 ESS in some popular symmetric games

First a note on notation: Since all the games considered here are symmetric, for a game $G = (A, B)$, we know that $B = A^T$. So to describe a game, its enough to describe the A_{ij} matrix,

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{m1} & \dots & \dots & a_{mm} \end{pmatrix}$$

Although I will often still add the B_{ij} matrix for clarity.

Prisoner's dilemma

This game is given by $m = 2$, $a_{11} > a_{21}$, $a_{12} > a_{22}$ and $a_{11} < a_{22}$. In words this means the first strategy strictly dominates the second. However, when both players play the first strategy, it gives an outcome which is Pareto dominated by both playing the second. There are many applications of this game, for example two firms deciding whether to co-operate by restricting output and so raise price, or rival nations stockpiling weapons. The one I give below is a biological application.

Example 3.2.1. Large (trait 1) and Small (trait 2) Beetles

Consider a large population of beetles wondering around discovering food sources. When a beetle is seen eating, it attracts a neighbouring beetle who looks to share the food (of size 10). There are two kinds of beetle: small and large. When two beetles of the same size meet, they share the food equally. However when a large beetle meets a small one it is able to use its greater size to intimidate the smaller beetle into giving it most of the food (say 9 units to 1). However, due to their larger bodies, the larger beetles also need more food to maintain their metabolism (a cost of 2 units). This gives the following payoff matrix:

$$G = \begin{pmatrix} 3,3 & 7,1 \\ 1,7 & 5,5 \end{pmatrix}$$

We see that trait 1 (large) strictly dominates trait 2 (small). So for any population state, large beetles have a higher reproductive fitness than small beetles. From this it is easy to see that the unique ESS will consist of the entire population being large, $\mathbf{e}^1$.

Hawk-Dove

This is the classic environment Maynard-Smith and Price developed the tool of ESS to analyse. The scenario is as follows: We have a large population of animals contesting scarce resources (eg food, nesting sites, territory). When two animals meet, some will be prepared to fight for these resources, while others will merely display aggression but if pushed, will back down and concede the resource. The aggressive trait (or strategy) is termed “Hawk”, while the more passive strategy is termed “Dove”. When two Doves meet they each have equal chance of getting the resource of value $V > 0$ (or alternatively we can assume they share the resource); when a Hawk meets a Dove the former intimidates the latter and so takes the resource. If two Hawks meet then they fight for the resource, each winning half the time, but sustaining injury with cost $C > 0$ the other half of the time. Letting the first trait (strategy) be Hawk, this gives the payoff matrix:

$$G = \begin{pmatrix} \frac{V-C}{2}, \frac{V-C}{2} & V, 0 \\ 0, V & \frac{V}{2}, \frac{V}{2} \end{pmatrix} \tag{3.2.1}$$

If $V > C$ then Hawk is a strictly dominant strategy, and so the game becomes a prisoner's dilemma. The norm is to consider $C > V$ which turns it into a game of chicken[1]. In this game there are two asymmetric Nash Equilibria, where one player is a Hawk and the other a Dove. However as argued in Chapter 2.2, they are unachievable in our one population setup. The relevant Nash Equilibrium is the symmetric one in mixed strategies and it turns out that this is an ESS.

Lemma 3.2.1. *The Hawk-Dove Game (equation 3.2.1), with $C > V$, has one ESS. This is $\mathbf{x} = \left(\frac{V}{C}, 1 - \frac{V}{C}\right)$*

Proof. Let $\mathbf{x} = \left(\frac{V}{C}, 1 - \frac{V}{C}\right)$ and note that

i) $u(\mathbf{y}, \mathbf{x}) = u(\mathbf{x}, \mathbf{x})\ \forall \mathbf{y} \in \boldsymbol{\Delta}$

ii) $u(\mathbf{y}, \mathbf{y}) < u(\mathbf{x}, \mathbf{y})\ \forall \mathbf{y} \in \boldsymbol{\Delta},\ \mathbf{y} \neq \mathbf{x}$ (can argue this diagramatically)

Then by Proposition 3.1.1, $\mathbf{x} = \left(\frac{V}{C}, 1 - \frac{V}{C}\right)$ is ESS. Furthermore condition ii) implies there are no other symmetric Nash Equilibria and so this is the unique ESS. □

This game has a nice interpretation: In a population consisting of mainly Doves, a Hawk can come and bully the Doves, most of the time getting the full resource value V without a fight. However if the proportion of Hawks is too high, a Hawkish strategy runs a high chance of a fight and injury and so the Doves do better. So from any starting position, the population state should return to the balance of Hawks and Doves predicted by the ESS.

[1] A symmetric two strategy game is a game of Chicken if $a_{12} > a_{22} > a_{21} > a_{11}$.

Co-ordination games

These are games where the two players want to co-ordinate on the same strategy. In the two strategy case, these are given by $a_{11} > a_{21}$ and $a_{22} > a_{12}$. More generally, in games with any number of strategies, this requires that $a_{ii} > a_{ij}$ for any $j \neq i$. These games have symmetric strict Nash Equilibria where all players play the same strategy. Since these are strict equilibria, they are also ESS. In addition, there are also mixed strategy Nash equilibria. These are not ESS but do have some relevance, as they determine an ESS's resistance to mutations (that is the ε in equation 3.1.1).

Exercise 3.2.1. Consider the game $G = \begin{pmatrix} \alpha, \alpha & 0, 0 \\ 0, 0 & 1, 1 \end{pmatrix}$ where $\alpha > 0$.

i) Find Δ^{NE}.

ii) Verify that $\mathbf{e}^1$ and $\mathbf{e}^2$ are ESS and find the largest proportion of mutations that each can withstand.

The ESS notion is unable to distinguish between strict Nash Equilibria.[2]

Rock-Paper-Scissors

This is a symmetric game, often played by kids, where the first strategy (rock) loses to the second strategy (paper) which loses to the third strategy (scissors) which in turn loses to the first strategy. The general form of this game is given by

$$G = \begin{pmatrix} \alpha, \alpha & 0, 2 & 2, 0 \\ 2, 0 & \alpha, \alpha & 0, 2 \\ 0, 2 & 2, 0 & \alpha, \alpha \end{pmatrix} \tag{3.2.2}$$

where $\alpha \in [0, 2]$, is the payoff from a draw, and is often set equal to 1.

[2] To do this, we would need to consider stochastic stability. When there are only two strategies as in the above example, stochastic stability selects the ESS with the higher invasion barrier. In Ex 3.2.1 this is $\mathbf{e}^1$ when $\alpha > 1$ and $\mathbf{e}^2$ when $\alpha < 1$.

Lemma 3.2.2. *In the Rock-Paper-Scissors game, given by equation 3.2.2, we have the following:*

Case	Δ^{NE}	Δ^{NSS}	Δ^{ESS}
$0 \le \alpha < 1$	$\left\{\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right\}$	$\left\{\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right\}$	$\left\{\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right\}$
$\alpha = 1$	$\left\{\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right\}$	$\left\{\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right\}$	$\varnothing$
$1 < \alpha < 2$	$\left\{\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right\}$	$\varnothing$	$\varnothing$
$\alpha = 2$	$\left\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3, \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right\}$	$\varnothing$	$\varnothing$

Proof. One can show Δ^{NE} column using standard techniques from non-co-operative game theory.

`For` $\alpha = 2$: $\mathbf{e}^1$ is not ESS since it can be invaded by $\mathbf{e}^2$. $\mathbf{e}^2$ is not ESS since it can be invaded by $\mathbf{e}^3$. $\mathbf{e}^3$ is not ESS since it can be invaded by $\mathbf{e}^1$. $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is not ESS since it can be invaded by $\mathbf{e}^1$.

`For` $1 < \alpha < 2$: $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is not ESS since it can be invaded by $\mathbf{e}^1$.

`For` $\alpha = 1$: Define $\mathbf{x} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. Clearly by the zero-sum nature of this game $u(\mathbf{x}, \mathbf{x}) = 1$ and for any $\mathbf{y} \in \Delta$: $u(\mathbf{y}, \mathbf{x})$, $u(\mathbf{x}, \mathbf{y})$, $u(\mathbf{y}, \mathbf{y})$ are all also equal to 1. The result then follows directly from Proposition 3.1.1 and 3.1.2.

`For` $0 \le \alpha < 1$: Define $\mathbf{x} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. Clearly $u(\mathbf{y}, \mathbf{x}) = u(\mathbf{x}, \mathbf{x})$ for all $\mathbf{y} \in \Delta$. Let $\alpha = 1 - \delta$. Then clearly $u(\mathbf{x}, \mathbf{y}) = \frac{\alpha+2}{3} = 1 - \frac{\delta}{3}$ for all $\mathbf{y} \in \Delta$, while

$$\begin{aligned} u(\mathbf{y}, \mathbf{y}) &= \left(y_1^2 + y_2^2 + y_3^2\right)(1 - \delta) + 2y_1y_2 + 2y_1y_3 + 2y_2y_3 \\ &= (y_1 + y_2 + y_3)^2 - \delta\left(y_1^2 + y_2^2 + y_3^2\right) \\ &= 1 - \delta\left(y_1^2 + y_2^2 + y_3^2\right) \end{aligned}$$

Now, since

$$\min_{\mathbf{y} \in \Delta}\left(y_1^2 + y_2^2 + y_3^2\right) = \frac{1}{3}$$

and is only achieved at $\mathbf{y} = \mathbf{x} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ this means that for any $\mathbf{y} \in \Delta \setminus \{\mathbf{x}\}$, $u(\mathbf{y}, \mathbf{y}) < 1 - \frac{\delta}{3}$. So we have shown $u(\mathbf{y}, \mathbf{y}) < u(\mathbf{x}, \mathbf{y})$ for all $\mathbf{y} \in \Delta \setminus \{\mathbf{x}\}$. Hence by Proposition 3.1.1, $\mathbf{x} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is ESS. □

This Lemma shows that, unlike Δ^{NE}, there are finite games where the set of ESS (and even NSS) is empty. On the other hand in games with only two purestrategies, NSS must always exist but ESS might not:

Exercise 3.2.2. Let $m = 2$. Give an example of a symmetric game in which and there are no ESS. What must all such games have in common? Show that every game has at least one NSS. (Hint: Easiest to argue this diagramatically going through all possible cases.)

3.3 Structure of the ESS set

As we have already shown, unlike Nash Equilibria, the set of ESS may sometimes be empty. Another difference is that the set of ESS must be finite.

Example 3.3.1. Infinite Δ^{NE} and Δ^{NSS}

Consider the game

$$G = \begin{pmatrix} 1,1 & 0,1 \\ 1,0 & 0,0 \end{pmatrix}$$

In this game every strategy is both a Nash Equilibrium and NSS. That is, $\Delta^{NE} = \Delta^{NSS} = \Delta$.

Recall that for a strategy $\mathbf{x} \in \Delta$, we define its carrier (or support), $C(\mathbf{x})$ to be the set of pure strategies used with positive probability. We show that the support of an ESS cannot contain the support of another ESS.

Proposition 3.3.1. *If* $\mathbf{x} \in \Delta^{ESS}$ *and* $C(\mathbf{y}) \subseteq C(\mathbf{x})$ *for some strategy* $\mathbf{y} \neq \mathbf{x}$ *then* $\mathbf{y} \notin \Delta^{NE}$ *(and hence* $\mathbf{y} \notin \Delta^{ESS}$*).*

Proof. Suppose $\mathbf{x} \in \Delta^{ESS}$. Then $u(\mathbf{x}, \mathbf{x}) = u(\mathbf{e}^i, \mathbf{x})$ for all $i \in C(\mathbf{x})$ and if $\mathbf{y} \in \Delta$ is such that $C(\mathbf{y}) \subseteq C(\mathbf{x})$ then $u(\mathbf{x}, \mathbf{x}) = u(\mathbf{e}^i, \mathbf{x})$ for all $i \in C(\mathbf{y})$. Thus $u(\mathbf{x}, \mathbf{x}) = u(\mathbf{y}, \mathbf{x})$. By Proposition 3.1.1 and $\mathbf{x} \in \Delta^{ESS}$, we must have $u(\mathbf{y}, \mathbf{y}) < u(\mathbf{x}, \mathbf{y})$ and hence $\mathbf{y} \notin \Delta^{NE}$. $\square$

Corollary 3.3.1. *Finite ESS:*

i) Any symmetric bimatrix game G has a finite number of ESS.

ii) If $\mathbf{x} \in int(\Delta)$ is ESS then $\Delta^{ESS} = \{\mathbf{x}\}$.

iii) Further, if the number of pure strategies is $m \leq 3$ then the game has at most m ESSs.

Proof. i) One conclusion we can draw from Proposition 3.3.1 is that each possible support has at most one ESS. Since the number of possible supports is the number of non-empty subsets of $\{1, 2, \ldots, m\}$ which is $2^m - 1$, the number of ESSs is bounded above by $2^m - 1$. (In fact this isn't a particularly tight bound.)

ii) If $\mathbf{x} \in \text{int}(\Delta)$ then $C(\mathbf{x}) = \{1, 2, \ldots, m\}$ and by Proposition 3.3.1, there are no other symmetric Nash Equilibria.

iii) `For` $m = 2$: If there is an ESS $\mathbf{x}$ with $C(\mathbf{x}) = \{1, 2\}$ then this is the only ESS. Hence the maximum number of ESSs is two, one with support $\{1\}$ and the other support $\{2\}$.

`For` $m = 3$: Seven possible supports: $\{1\}$, $\{2\}$,$\{3\}$,$\{1, 2\}$,$\{1, 3\}$,$\{2, 3\}$,$\{1, 2, 3\}$. The most which are simultaneously compatible with Proposition 3.3.1 is three. □

Exercise 3.3.1. Consider the class of symmetric games with $m = 4$ strategies. For each $n \in \{0, 1, 2, 3, 4\}$ find an example with n ESSs. Further question: Is it possible to have more than four ESSs?

Chapter 4

Replicator Dynamics

In general, an evolutionary process combines two basic elements: a *mutation mechanism* and a *selection mechanism*. While the ESS criterion highlighted the role of mutations, the replicator dynamics focuses on selection. Robustness against mutations is handled indirectly by appealing to dynamic stability criteria.

Recall from Remark 2.2.1 that there are two interpretations of a population state. In Chapter 3, we could work with either, and indeed it was often easier to think in terms of the first interpretation. In this Chapter, we think exclusively in terms of the second: every agent is hardwired to play a pure strategy. The Replicator Dynamics then model how the proportion of agents playing each strategy changes assuming that each agent asexually reproduces offspring who play the same strategy as their parent, where the number of such offspring depends on their reproductive fitness (payoff).

4.1 One Population: The Replicator Dynamic

As in Chapter 3, building on the discussion of Section 2.2 we consider all agents being drawn randomly, and matched in pairs, from a single large population. Thus we focus on symmetric games, $G = (A, B)$, where $B = A^T$

and so are able to define everything in terms of the payoff matrix A.

At any point in time t, let $p(t)$ be the size of the population and $p_i(t)$ be the size of the subpopulation following the ith pure strategy. Note that the proportion of the population following the ith pure strategy is $x_i(t) = \frac{p_i(t)}{p(t)}$. When the population state is $\mathbf{x} \in \Delta$, the expected payoff to an agent following strategy i is $u(\mathbf{e}^i, \mathbf{x})$, while the average payoff is $u(\mathbf{x}, \mathbf{x})$[1].

Payoffs represent the incremental effect from playing the game on an individual's fitness, measured as the number of offspring per time unit. Suppose also that each offspring inherits its single parent's strategy - that is strategies breed true. We suppose reproduction takes place continuously over time, so that the birth rate of a strategy i individual at time t is $\beta + u(\mathbf{e}^i, \mathbf{x}(t))$, where $\beta \geq 0$ is the background fitness of all individuals in the population (independent of the strategies individuals follow in the game in question). Similarly, let the death rate be $\delta \geq 0$ for all individuals. With dots for time derivatives and suppressing the time argument, this gives the following population dynamics:

$$\dot{p}_i = p_i \left[\beta + u\left(\mathbf{e}^i, \mathbf{x}\right) - \delta\right] \tag{4.1.1}$$

Now, taking the time derivative of both sides of the identity $p(t)\, x_i(t) = p_i(t)$ gives[2] $p\dot{x}_i + \dot{p}x_i = \dot{p}_i$. This implies

$$p\dot{x}_i = \dot{p}_i - \dot{p}x_i = p_i \left[\beta + u\left(\mathbf{e}^i, \mathbf{x}\right) - \delta\right] - p\left[\beta + u(\mathbf{x}, \mathbf{x}) - \delta\right] x_i \tag{4.1.2}$$

Simplifying and dividing both sides by p gives

$$\dot{x}_i = x_i \left[u\left(\mathbf{e}^i, \mathbf{x}\right) - u(\mathbf{x}, \mathbf{x})\right] \tag{4.1.3}$$

[1]Note that strictly speaking, in games where one does not interact with oneself, this is an approximation. However as the population size increases, the probability of an agent drawing oneself tends to zero and so these expected payoffs do indeed tend to $u\left(\mathbf{e}^i, \mathbf{x}\right)$ and $u(\mathbf{x}, \mathbf{x})$.

[2]Notation: $\dot{x}_i = \frac{\partial x_i}{\partial t}$,$\dot{p}_i = \frac{\partial p_i}{\partial t}$ and $\dot{p} = \frac{\partial p}{\partial t}$

This equation (4.1.3) defines the **replicator dynamic**. It states that the growth rate $\frac{\dot{x}_i}{x_i}$ of the population share using strategy i equals the difference between the strategy's current payoff and the current average payoff in the population as a whole. Exploiting the linearity of the payoff function, we could write the replicator dynamic slightly more concisely as

$$\dot{x}_i = x_i \left[u \left(\mathbf{e}^i - \mathbf{x}, \mathbf{x} \right) \right] \tag{4.1.4}$$

Properties

Note that the replicator dynamics give the following two properties

$$\sum_{i=1}^{m} \dot{x}_i = 0 \quad \text{and} \quad x_i = 0 \Rightarrow \dot{x}_i = 0$$

This means that starting from any initial state $\mathbf{x}^0 \in \Delta$, the shares of the population following each pure strategy will continue to add up to one and no population share can ever be negative. Thus the replicator dynamics define a dynamic system over the space Δ.

More formally, from any initial population state $\mathbf{x}^0 \in \Delta$ the system of i differential equations given by equation 4.1.3 defines a trajectory $\xi : \mathbb{R} \times \Delta \to \Delta$ where $\xi\left(t, \mathbf{x}^0\right) \in \Delta$ defines the population state at time t. It will be useful to refer to $\xi_i\left(t, \mathbf{x}^0\right) \in [0,1]$ as the proportion of individuals following strategy i at this time t population state.

Note the following two key properties of the replicator dynamic[3]:

1. If a strategy is absent from the population, then it must always have been absent and will always be absent at any time in the future.

2. If a strategy is present in the population then it must always have been present and will always remain present at any time in the future.

[3]See Weibull for proof of these claims

While the second statement says that for any strategy i which is present in the initial population state, $\xi_i\left(t, \mathbf{x}^0\right) > 0$ at any time $t \in \mathbb{R}$, it is possible that some strategies may approach extinction as time tends to infinity: That is, we can have

$$\lim_{t\to\infty} \xi_i\left(t, \mathbf{x}^0\right) = 0$$

Invariance under payoff transformations

Suppose that we replace the payoff function u by $\tilde{u} = \lambda u + \kappa$ (in other words multiply each entry in the payoff matrix by λ and add κ), where $\lambda > 0$ and $\kappa \in \mathbb{R}$. Under $\tilde{u}$, the replicator dynamic becomes

$$\dot{x}_i = x_i\left[\tilde{u}\left(\mathbf{e}^i - \mathbf{x}, \mathbf{x}\right)\right] = x_i\lambda\left[u\left(\mathbf{e}^i - \mathbf{x}, \mathbf{x}\right)\right]$$

Thus we conclude that the replicator dynamic is invariant under positive affine transformations of payoffs, modulo a change of time scale. In other words, all the solution orbits are exactly the same under both dynamics, the only change is the velocity, with progress under $\tilde{u}$ being λ times faster.

Similarly, we can show that local shifts of payoff functions do not affect the replicator dynamics:

Exercise 4.1.1. Show that if we add some constant $c \in \mathbb{R}$ to all entries in some column j of the payoff matrix A then this has no effect on the replicator dynamic. (In fact we can link this to the fact the replicator dynamic is invariant to changes in the birth rate β or death rate δ.)

Stability concepts

Stationarity is a minimal requirement of stability. For $\mathbf{x}$ to be stationary we require that once we reach $\mathbf{x}$, we will never leave (without mutations).

Definition 4.1.1. The set of **stationary** states is given by

$$\Delta^{St} = \{\mathbf{x} \in \Delta : \dot{x}_i = 0 \quad \forall i\}$$

Remark 4.1.1. $\{\mathbf{e}^1, \ldots, \mathbf{e}^m\} \subseteq \Delta^{St} \subseteq \Delta$

Lyapunov stability asks the more demanding question of what happens to the trajectory once we perturb the state slightly away from our state: Do we remain close to it or drift away?

Definition 4.1.2. A state $\mathbf{x} \in \Delta$ is **Lyapunov Stable** if every neighbourhood[4] B of $\mathbf{x}$ contains a neighbourhood B^o of $\mathbf{x}$ such that

$$\mathbf{x}^0 \in B^o \cap \Delta \Rightarrow \xi\left(t, \mathbf{x}^0\right) \in B \quad \forall t \in \mathbb{R}$$

Let $\Delta^{LS} \subseteq \Delta$ denote the set of Lyapunov Stable states.

For those not well-versed, in the mathematics of metric spaces, this definition may seem hard to comprehend, so I give a more amenable one:

Consider the standard Euclidean metric (function measuring distance) over $\mathbb{R}^n$:

$$\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \quad d\left(\mathbf{x}, \mathbf{y}\right) = \sqrt{\left(x_1 - y_1\right)^2 + \left(x_2 - y_2\right)^2 + \ldots + \left(x_n - y_n\right)^2}$$

Then Lyapunov stability requires that for any $\varepsilon > 0$, it is possible to find $\delta > 0$ such that: if we start δ-close to $\mathbf{x}$, then we will forever remain ε-close to $\mathbf{x}$. Note that δ can depend on ε. Often, the smaller ε is, the smaller we need δ to be.

Definition. (Assuming Euclidean metric) A state $\mathbf{x} \in \Delta$ is **Lyapunov Stable** if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$d\left(\mathbf{x}^0, \mathbf{x}\right) < \delta \implies d\left(\xi\left(t, \mathbf{x}^0\right), \mathbf{x}\right) < \varepsilon \quad \forall t \in \mathbb{R}$$

[4] Mathematically speaking, for a topological space X with element $x \in X$, B is a neighbourhood of x if there exists an open set U such that $x \in U \subseteq B \subseteq X$. Although I'm not going to go too deep into the theory of topology here. The necessary skill which is examinable is just to make a coherent argument as to why some things are stable and others are not.

A stricter requirement is Asymptotic stability. This requires that after a small perturbation from $\mathbf{x}$, in addition to remaining close, we will eventually head back towards $\mathbf{x}$.

Definition 4.1.3. A state $\mathbf{x} \in \Delta$ is **Asymptotically Stable** if it is Lyapunov Stable and there exists a neighbourhood B^* of $\mathbf{x}$ such that

$$\mathbf{x}^0 \in B^* \cap \Delta \Rightarrow \lim_{t \to \infty} \xi \left(t, \mathbf{x}^0\right) = \mathbf{x}$$

Let $\Delta^{AS} \subset \Delta$ denote the set of Asymptotically Stable states.

Similarly, we can translate this definition, assuming the Euclidean metric:

Definition. (Assuming Euclidean metric) A state $\mathbf{x} \in \Delta$ is Asymptotically Stable if it is Lyapunov Stable and there exists some $\kappa > 0$ such that

$$d\left(\mathbf{x}^0, \mathbf{x}\right) < \kappa \implies \lim_{t \to \infty} \xi \left(t, \mathbf{x}^0\right) = \mathbf{x}$$

Proposition 4.1.1. *Lyapunov Stability implies stationarity*

Proof. I show the contrapositive: if $\mathbf{x}$ is not stationary it is not Lyapunov Stable. Since $\mathbf{x}$ is not stationary, there exists some finite t such that $\xi\left(t, \mathbf{x}\right) = \mathbf{y}$, where $\mathbf{y} \in \Delta$ and $\mathbf{y} \neq \mathbf{x}$. Since $\mathbf{y}$ and $\mathbf{x}$ are a finite distance away, we can find a neighbourhood B of $\mathbf{x}$ not including $\mathbf{y}$ and hence the system leaves B in finite time if started at $\mathbf{x}$, contradicting Lyapunov Stability. $\square$

Some Examples

Generally, stability under the replicator dynamics gives similar results to the ESS concept of Chapter 3.

As we have already seen, in the prisoner's dilemma, everyone playing the strictly dominant strategy is the unique ESS.

Example 4.1.1. Prisoner's Dilemma

Let $G = \begin{pmatrix} 3,3 & 7,1 \\ 1,7 & 5,5 \end{pmatrix}$

We can derive the replicator dynamics and show that

i) $\Delta^{St} = \{\mathbf{e}^1, \mathbf{e}^2\}$,

ii) $\Delta^{LS} = \Delta^{AS} = \Delta^{ESS} = \{\mathbf{e}^1\}$.

In a Hawk-Dove (or Chicken) game we have shown that the ESS is for a mixture of Hawks

Example 4.1.2. Hawk-Dove ($V = 4$, $C = 8$)

Let $G = \begin{pmatrix} -2,-2 & 4,0 \\ 0,4 & 2,2 \end{pmatrix}$

We can derive the replicator dynamics and show that

i)$\Delta^{St} = \left\{\mathbf{e}^1, \mathbf{e}^2, \left(\frac{1}{2}, \frac{1}{2}\right)\right\}$,

ii) $\Delta^{LS} = \Delta^{AS} = \Delta^{ESS} = \left\{\left(\frac{1}{2}, \frac{1}{2}\right)\right\}$.

In co-ordination games we found all pure strategies are ESS

Example 4.1.3. Co-ordination game

Let $G = \begin{pmatrix} 1,1 & 0,0 \\ 0,0 & 2,2 \end{pmatrix}$

We can derive the replicator dynamics and show that $\Delta^{St} = \left\{\mathbf{e}^1, \mathbf{e}^2, \left(\frac{2}{3}, \frac{1}{3}\right)\right\}$ and

$$\Delta^{LS} = \Delta^{AS} = \Delta^{ESS} = \left\{\mathbf{e}^1, \mathbf{e}^2\right\}$$

As we have shown in Chapter 3, weakly dominated strategies can't be part of an ESS.

Example 4.1.4. Game with NE in weakly dominated strategies

Let $G = \begin{pmatrix} 1,1 & 0,0 \\ 0,0 & 0,0 \end{pmatrix}$

We can derive the replicator dynamics and show that

i)$\Delta^{St} = \{\mathbf{e}^1, \mathbf{e}^2\} = \Delta^{NE}$,

ii) $\Delta^{LS} = \Delta^{AS} = \Delta^{ESS} = \{\mathbf{e}^1\}$.

In Rock-Paper-Scissors we have found that ESS will fail to exist.

Exercise 4.1.2. Let $G = \begin{pmatrix} \alpha,\alpha & 0,2 & 2,0 \\ 2,0 & \alpha,\alpha & 0,2 \\ 0,2 & 2,0 & \alpha,\alpha \end{pmatrix}$

Compute the replicator dynamics and verify the results in the table:

Case	Δ^{St}	Δ^{LS}	Δ^{AS}
$0 \leq \alpha < 1$	$\left\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3, \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right\}$	$\left\{\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right\}$	$\left\{\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right\}$
$\alpha = 1$	$\left\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3, \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right\}$	$\left\{\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right\}$	$\emptyset$
$1 < \alpha < 2$	$\left\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3, \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right\}$	$\emptyset$	$\emptyset$
$\alpha = 2$	$\left\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3, \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right\}$	$\emptyset$	$\emptyset$

The following is the counterpart of exercise 3.2.1

Exercise 4.1.3. Let $G = \begin{pmatrix} \alpha,\alpha & 1,0 \\ 0,1 & 1,1 \end{pmatrix}$ and consider 3 cases: i) $\alpha < 0$, ii) $\alpha = 0$, iii) $\alpha > 0$.

In each case compute the replicator dynamics and find Δ^{St}, Δ^{LS}, Δ^{AS}.

4.2 Stability in the Replicator Dynamic: some general results

Link to Nash Equilibrium

Our first result shows the similarities between Stationarity and Nash Equilibrium. Recall that $\operatorname{int}(\Delta)$ is the interior of Δ, that is population states in which every pure strategy is present.

Proposition 4.2.1. *Stationarity and NE.*

i) $\{\mathbf{e}^1, \ldots \mathbf{e}^m\} \cup \Delta^{NE} \subseteq \Delta^{St}$

ii) $\Delta^{NE} \cap int(\Delta) = \Delta^{St} \cap int(\Delta)$ *which is a convex set.*

Proof. i) Firstly, to argue $\{\mathbf{e}^1, \ldots \mathbf{e}^m\} \subseteq \Delta^{St}$. Let $\mathbf{x} = \mathbf{e}^i$. Then $u(\mathbf{e}^i, \mathbf{x}) = u(\mathbf{x}, \mathbf{x})$ and so $\dot{x}_i = 0$. Also, for all $j \neq i$, $x_j = 0$ and so $\dot{x}_j = 0$. To argue $\Delta^{NE} \subseteq \Delta^{St}$, note that $x \in \Delta^{NE}$ is equivalent to the following two conditions:

$$u\left(\mathbf{e}^i, \mathbf{x}\right) = u(\mathbf{x}, \mathbf{x}) \quad \forall i \in C(\mathbf{x}) \tag{4.2.1}$$

$$u\left(\mathbf{e}^i, \mathbf{x}\right) \leq u(\mathbf{x}, \mathbf{x}) \quad \forall i \notin C(\mathbf{x}) \tag{4.2.2}$$

While just (4.2.1) is sufficient for stationarity.

ii) To argue $\Delta^{NE} \cap \operatorname{int}(\Delta) = \Delta^{St} \cap \operatorname{int}(\Delta)$: Take $\mathbf{x} \in \operatorname{int}(\Delta)$, then $x_i > 0 \; \forall i$ thus $\dot{x}_i = 0 \Leftrightarrow u(\mathbf{e}^i, \mathbf{x}) = u(\mathbf{x}, \mathbf{x})$. Also, with $\mathbf{x} \in \operatorname{int}(\Delta)$ condition (4.2.2) becomes redundant and so condition (4.2.1) is necessary and sufficient for both $\mathbf{x} \in \Delta^{NE}$ and $\mathbf{x} \in \Delta^{St}$.

The convexity argument runs as follows: Take $\mathbf{x}, \mathbf{y} \in \Delta^{St} \cap \operatorname{int}(\Delta)$ and consider $\mathbf{z} = \alpha\mathbf{x} + (1-\alpha)\mathbf{y}$, where $\alpha \in [0, 1]$. Clearly $\mathbf{z} \in \operatorname{int}(\Delta)$ and so all that is left to show is stationarity.

$$\begin{aligned} u\left(\mathbf{e}^i, \mathbf{z}\right) &= \alpha u\left(\mathbf{e}^i, \mathbf{x}\right) + (1-\alpha) u\left(\mathbf{e}^i, \mathbf{y}\right) \\ &= \alpha u(\mathbf{x}, \mathbf{x}) + (1-\alpha) u(\mathbf{y}, \mathbf{y}) \end{aligned}$$

Hence all pure strategies earn the same payoff against $\mathbf{z}$. So for all i, $u(\mathbf{e}^i, \mathbf{z}) = u(\mathbf{z}, \mathbf{z})$ and hence $\mathbf{z}$ is stationary. □

Exercise 4.2.1. Find an example where there is a state $\mathbf{x} \in \Delta^{St}$ which is not in $\{\mathbf{e}^1, \ldots \mathbf{e}^m\} \cup \Delta^{NE}$. (Hint: using the above proposition, it cannot be in the interior.)

The next result says that Lyapunov Stability implies Nash Equilibrium.

Proposition 4.2.2. $\Delta^{LS} \subseteq \Delta^{NE}$

Proof. I show the contrapositive: Suppose $\mathbf{x} \notin \Delta^{NE}$ and show that $\mathbf{x} \notin \Delta^{LS}$. If $\mathbf{x} \notin \Delta^{St}$ then by Proposition 4.1.1, $\mathbf{x} \notin \Delta^{LS}$ and we are done. If $\mathbf{x}$ is stationary but not a Nash Equilibrium then

$$u\left(\mathbf{e}^i, \mathbf{x}\right) = u\left(\mathbf{x}, \mathbf{x}\right) \quad \forall i \in C\left(\mathbf{x}\right)$$

$$\exists j \notin C\left(\mathbf{x}\right) \text{ with } u\left(\mathbf{e}^j, \mathbf{x}\right) > u\left(\mathbf{x}, \mathbf{x}\right)$$

So by continuity of u (since payoff function is linear), there exists $\delta > 0$ and a neighbourhood U of $\mathbf{x}$ bounded away from $\mathbf{e}^j$ such that

$$u\left(\mathbf{e}^j, \mathbf{y}\right) - u\left(\mathbf{y}, \mathbf{y}\right) \geq \delta \quad \forall \mathbf{y} \in U$$

Now for $\varepsilon > 0$, consider a move to $(1-\varepsilon)\,\mathbf{x} + \varepsilon\mathbf{e}^j := \mathbf{x}^o$. If this state is inside U then following the replicator dynamics' trajectory from here we must eventually leave U. This is since the proportion playing strategy j increase will increase at exponential rate since

$$\begin{aligned} \dot{y}_j &\geq y_j\delta \quad \forall \mathbf{y} \in U \\ \Rightarrow \xi_j\left(t, \mathbf{x}^o\right) &\geq \varepsilon e^{\delta t} \end{aligned}$$

for any t until we leave U. Applying the definition of Lyapunov Stable where $B = U$ and noting that for any neighbourhood B^o of $\mathbf{x}$, there is ε small enough such that B^o contains $(1-\varepsilon)\,\mathbf{x} + \varepsilon\mathbf{e}^j := \mathbf{x}^o$, we see that $\mathbf{x}$ cannot be Lyapunov Stable. $\square$

Exercise 4.2.2. Find an example where there is a state $\mathbf{x} \in \Delta^{NE}$ which is not in Δ^{LS}.

The next proposition relates to limit states.

Definition 4.2.1. A state $\mathbf{x} \in \Delta$ is a limit state if there exists $\mathbf{x}^0 \in \operatorname{int}(\Delta)$ such that

$$\lim_{t\to\infty} \xi\left(t, \mathbf{x}^0\right) = \mathbf{x}$$

Proposition 4.2.3. *If* $\mathbf{x} \in \Delta$ *is a limit state then* $\mathbf{x} \in \Delta^{NE}$.

Proof. Suppose that $\mathbf{x}^0 \in \operatorname{int}(\Delta)$ and $\lim_{t\to\infty} \xi(t, \mathbf{x}^0) = \mathbf{x}$ but $\mathbf{x} \notin \Delta^{NE}$ and aim to generate a contradiction. By our supposition:

$$\exists i \text{ with } u\left(\mathbf{e}^i, \mathbf{x}\right) - u(\mathbf{x}, \mathbf{x}) \geq \varepsilon > 0$$

Since $\lim_{t\to\infty} \xi(t, \mathbf{x}^0) = \mathbf{x}$ and u is continuous, $\exists T \in \mathbb{R}$ such that

$$u\left(\mathbf{e}^i, \xi\left(t, \mathbf{x}^0\right)\right) - u\left(\xi\left(t, \mathbf{x}^0\right), \xi\left(t, \mathbf{x}^0\right)\right) \geq \frac{\varepsilon}{2} \quad \forall t \geq T$$

Hence

$$\begin{aligned} \dot{x}_i(t) &> x_i(t)\frac{\varepsilon}{2} \quad \forall t \geq T \\ \Rightarrow \xi_i\left(t, \mathbf{x}^0\right) &> \xi_i\left(T, \mathbf{x}^0\right) e^{(t-T)\varepsilon/2} \\ \Rightarrow \xi_i\left(t, \mathbf{x}^0\right) &\to \infty \end{aligned}$$

and so we have generated a contradiction. Hence $\mathbf{x} \in \Delta^{NE}$. □

Exercise 4.2.3. Argue that any $\mathbf{x} \in \Delta^{NE} \cap \operatorname{int}(\Delta)$ must be a limit state. Find an example where there is a state $\mathbf{x} \in \Delta^{NE}$ which is not a limit state.

The next exercise and example shows that Lyapunov Stability and being a limit state are two distinct concepts.

Exercise 4.2.4. Find an example where there is a state $\mathbf{x} \in \Delta^{LS}$ which is not a limit state.

Example 4.2.1. Lyapunov Stability and Limit states

Let $G = \begin{pmatrix} 0,0 & 1,0 & 0,0 \\ 0,1 & 0,0 & 2,0 \\ 0,0 & 0,2 & 1,1 \end{pmatrix}$

$\mathbf{e}^1$ is a Nash Equilibrium and a limit state. In fact it's the unique limit state since

$$\lim_{t\to\infty} \xi\left(t, \mathbf{x}^0\right) = \mathbf{e}^1 \; \forall \mathbf{x}^0 \in \operatorname{int}(\Delta)$$

However $\mathbf{e}^1$ is not Lyapunov Stable.

Link to ESS

The following proposition says that being an NSS implies Lyapunov stability in the Replicator dynamic. The proof is a bit too advanced for this course and so is omitted but anyone interested can see the Weibull book.

Proposition 4.2.4. $\Delta^{NSS} \subseteq \Delta^{LS}$

An analogous result holds for ESS and being asymptotically stable in the replicator dynamic. Again the proof is too advanced and so omitted, but is in Weibull.

Proposition 4.2.5. $\Delta^{ESS} \subseteq \Delta^{AS}$

These results should make sense on an intuitive level: For $\mathbf{x} \in \Delta$ to be NSS we require that mutants not expand their share of the population, while Lyapunov Stability says that after a mutation causing a small perturbation away from $\mathbf{x}$ the population state should not drift further away. For $\mathbf{x} \in \Delta$ to be ESS we further require that mutants do worse than the incumbent population and so are driven out, thus returning us to our original state as is required by Asymptotic Stability.

Perhaps somewhat surprisingly, the reverse implications do not hold. In fact it is possible for a state to be Asymptotically Stable but not even NSS, as the following example shows:

Example 4.2.2. Asymptotically stable but not NSS

Let $G = \begin{pmatrix} 1,1 & 5,0 & 0,5 \\ 0,5 & 1,1 & 5,0 \\ 5,0 & 0,5 & 4,4 \end{pmatrix}$

$\mathbf{x} = \left(\frac{3}{18}, \frac{8}{18}, \frac{7}{18}\right)$ is Asymptotically Stable but not NSS. It is not NSS due to invasion by $\mathbf{e}^3$ mutants. However if we draw a sample trajectory from a point of the form $(1-\varepsilon)\,\mathbf{x}+\varepsilon\mathbf{e}^3$, we can see that it never deviates too far from

$\mathbf{x}$ and will head back towards $\mathbf{x}$: It's initial movement is a loss of strategy 1 and a gain to strategies 2 and 3 - especially strategy 3. Next strategy 3 starts to decline (since fewer $\mathbf{e}^1$). Next after the growth of $\mathbf{e}^2$ and decline of $\mathbf{e}^3$, the advantage switches to $\mathbf{e}^1$ agents. Then after the growth spurt of $\mathbf{e}^1$, the advantage then shifts to $\mathbf{e}^3$ agents and the cycle repeats. Although it can be shown these spiral take us back toward $\mathbf{x}$ and therefore it is asymptotically stable.

Replicator Dynamics and dominated strategies

The first result tells us that any strictly dominated strategy goes extinct in the long run, provided all strategies are initially present in the population (crucially including the one that strictly dominates it).

Proposition 4.2.6. *If a pure strategy i is strictly dominated then for any $\mathbf{x}^0 \in int(\Delta)$*

$$\lim_{t\to\infty} \xi_i\left(t, \mathbf{x}^0\right) = 0$$

Proof. See Weibull □

In fact, we can go further than this and say that any strategy that is iteratively strictly dominated goes extinct in the long run

Theorem 4.2.1. *If a pure strategy i is iteratively strictly dominated then for any $\mathbf{x}^0 \in int(\Delta)$*

$$\lim_{t\to\infty} \xi_i\left(t, \mathbf{x}^0\right) = 0$$

A formal proof of this Theorem is beyond the scope of this course, but we can see the intuition behind the result: Let K_n be the set of pure strategies eliminated in the nth round of elimination. Then after a sufficiently long time the population state will be, and forever remain, arbitrarily close to the face of Δ where $x_i = 0$ for all $i \in K_1$. Next, all the strategies in K_2 will approach extinction and so on. To make this argument clearer, think of the following example:

Exercise 4.2.5. Consider the following game. Let the initial state be $\left(\frac{8}{10}, \frac{1}{10}, \frac{1}{10}\right)$. Draw a rough sample trajectory.

$$G = \begin{pmatrix} 5,5 & 0,6 & 0,0 \\ 6,0 & 1,1 & 1,2 \\ 0,0 & 2,1 & 2,2 \end{pmatrix}$$

What happens with weak domination is less clear cut. Often weakly dominated strategies will be eliminated, but as the next example shows there is no guarantee.

Example 4.2.3. Survival of weakly dominated strategy

Let $G = \begin{pmatrix} 1,1 & 1,1 & 1,0 \\ 1,1 & 1,1 & 0,0 \\ 0,1 & 0,0 & 0,0 \end{pmatrix}$.

The second pure strategy is weakly dominated by the first. However, starting from any interior point of Δ, both will survive in the long run. The reason is that first, strategy 3 becomes almost extinct, removing the advantage strategy 1 has over strategy 2.

4.3 Replicator Dynamics with two populations

This Section builds on Section 2.3. We have one population of player 1 agents (henceforth known as population 1), and another population of player 2 agents (population 2). At any point in time, the current fitness to a population i agent is the expected fitness from meeting a randomly selected population $j \neq i$ agent. Thus the population state in one's own population has no direct effect on payoff, however it can still have an indirect effect on future payoffs by influencing how the other population evolves.

I denote a typical population state of population 1 by $\mathbf{x} \in \Delta_1$; and for population 2 by $\mathbf{y} \in \Delta_2$. The derivation of the replicator dynamic follows the same principle as in the one population case and gives the pair

$$\dot{x}_i = x_i \left[u_1\left(\mathbf{e}^i, \mathbf{y}\right) - u_1\left(\mathbf{x}, \mathbf{y}\right)\right] \qquad \dot{y}_i = y_i \left[u_2\left(\mathbf{e}^i, \mathbf{x}\right) - u_2\left(\mathbf{y}, \mathbf{x}\right)\right] \tag{4.3.1}$$

Example 4.3.1. We compute and illustrate the replicator dynamics for the game

$$G = \begin{pmatrix} 3,3 & 0,1 \\ 2,0 & 2,1 \end{pmatrix}$$

The definitions of stationarity, Lyapunov and asymptotic stability, and limit states all carry over, with the slight modification that the state space is now $\Delta = \Delta_1 \times \Delta_2$, with typical element $(\mathbf{x}, \mathbf{y})$.

The two population model allows us to study asymmetric as well as symmetric games. Contrary to what one might expect, in symmetric games, we can get very different results depending on whether we have one population or two. A classic example of this is the Hawk-Dove game.

Example 4.3.2. Two Population Hawk-Dove Game

Let $G = \begin{pmatrix} -2,-2 & 4,0 \\ 0,4 & 2,2 \end{pmatrix}$, (from $V = 4$, $C = 8$).

We compute and illustrate the replicator dynamics. We find that the only stable states are $(\mathbf{x}, \mathbf{y}) = (\mathbf{e}^1, \mathbf{e}^2)$ and $(\mathbf{x}, \mathbf{y}) = (\mathbf{e}^2, \mathbf{e}^1)$. That is the two pure strategy asymmetric Nash Equilibria. The interpretation of this in the Hawk-Dove setup is that animals are able to identify themselves as playing one of two roles and everyone in role 1 does one strategy, while everyone in role 2 does the other. For example consider battles over nesting sites. Each animal can identify themselves as being in one of two roles: occupier or intruder. The analysis here shows that there are two stable outcomes: occupiers are Hawks, while intruders are Doves; or occupiers are Doves while intruders are Hawks. The empirical evidence from the biological literature does support this model and shows that in most animal species the first equilibrium outcome prevails where the occupier keeps the nesting site. Although there are one or two animal species in which the second equilibrium outcome prevails and so the nesting site changes hands.

Further Reading

- Primary reference is "Evolutionary Game Theory" by Jorgen Weibull
- Most general Game Theory texts will have chapters on Evolutionary games. Two examples, available online are
  - http://www.cs.cornell.edu/home/kleinber/networks-book/networks-book-ch07.pdf
  - "Game Theory a multi-leveled approach" by Hans Peters, which is available as an e-book from the library.