Optimal and Suboptimal Power Allocations for MIMO based Multi-Hop OFDM Systems†

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***Abstract*— We consider the problem of power allocation for capacity maximization in a decode-and-forward (DF) based multi-Hop MIMO-OFDM systems. We derive the necessary conditions for the optimal solution and propose an optimal power allocation algorithm for a 3-hop system, which can be easily extended to arbitrary number of relays. A suboptimal power allocation algorithm for a 3-hop system is also proposed. The suboptimal solution assumes equal power allocation over frequency domain and uses closed-form expressions for optimal power distribution over all hops for each subcarrier. Simulation results verify that, given the same transmit power, edge users are better served by a multi-hop relay scheme than by direct-link based scheme. Moreover, the proposed suboptimal power allocation yields an excellent sum rate performance which is close to the optimal performance.**

***Keywords*-MIMO-OFDM, Decode-and-Forward, relay, multi-hop, power allocation**

I. Introduction

Orthogonal Frequency Division Multiplexing (OFDM) is a special multi-carrier transmission technique that transforms a wideband signal into an array of narrowband signals with overlapping spectra. It is a popular scheme for wideband communication systems for rendering high spectral efficiency, simple and robust equalization against frequency selective fading and allowing flexible dynamic resource allocation. Under the total transmit power constraint, it is well known that the multicarrier sum rate is maximized by a power allocation method called water-filling [1] if each subcarrier channel gain is known.

Large bandwidth or power is often required for high data rate transmissions. However, by installing multiple antennas on both sides of a link, a Multiple Input Multiple Output (MIMO) based system can offer higher data rate without requiring additional bandwidth or power [2]. The MIMO technique provides extra spatial dimensions for communication, i.e., it yields parallel orthogonal spatial channels when, for instance, the transmit and receive nodes apply the pre-processing and post-processing operations based on the Singular-Value-Decomposition (SVD) of the corresponding MIMO channel matrix.

For terrestrial wideband wireless communications, signals are likely to suffer from significant path loss. Through the aids of Decode-and-Forward (DF) or Amplify-and-forward (AF) based relay stations [3], the path loss can be compensated for. Better Quality-of-Services (QoS) and enhanced link capacity are achieved by properly combining selected waveforms received from both relays and the source using a time division duplex (TDD) mode. This is made possible by judiciously selecting the desired relay(s) and adopting an efficient resource allocation to fully exploit the advantages of relays [4] and [5]. In particular, for a DF-based multiple-hop system in which a relayed link involves more than one relays, each subcarrier's achievable data rate is lower-bounded by the worst hop. Hence, the power distribution over multiple relays plays an pivotal role in optimizing the link capacity.

Systems which combine MIMO-OFDM techniques with multi-hop relays have been considered as a promising architecture for further enhancing wireless capacity and extending the coverage of a base station (BS) [6]. Even with known relay and subcarrier selections, optimal power allocation for a multi-hop MIMO-OFDM system is difficult because one has to deal with the frequency and spatial degrees of freedom available at all hops. In this paper, we propose two power allocation schemes for sum rate maximization in DF based multi-hop MIMO-OFDM systems. As expected, the optimal solution is of the water-filling type. We derive the necessary conditions for the corresponding water levels, present an iterative algorithm that satisfies the necessary conditions and prove the convergence of our algorithm. We also propose a closed-form based suboptimal power allocation.

The rest of this paper is organized as follows. Section II describes the system model and related assumptions for the problem of concern and presents the basic problem formulation. The proposed optimal power allocation solution is given in Section III. In Section IV we propose a low complexity suboptimal solution. Section V gives some simulated performance examples of the proposed schemes and finally, Section VI summarizes the main results of this paper.

II. System Model and Problem Formulation

A. System model and basic assumptions

We consider a general DF based multi-hop (M-hop) MIMO-OFDM link with a source node (BS), a mobile user (MS),

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and $M-1$ relay nodes. The BS, MS and the mth relay node are equipped with N_T, N_R and N_m^r antennas, respectively. The BS transmits a packet to the MS through the M-hop link using N subcarriers where for each subcarrier, a relay node decodes its received packet from previous relay node or the BS and forwards the re-encoded packet to the next node or the destination (MS) in its pre-assigned time slot. We assume a multi-phase (time-slot) transmission protocol with perfect network timing and the availability of all the required channel state information (CSI). Each subcarrier suffers from slow flat Rayleigh fading and the CSI remain during an M-phase period. The received noise at the mth relay is represented by an $N_m^r \times 1$ column vector $\mathbf{N}_{im}$ whose elements are i.i.d zero-mean complex Gaussian random variables with variance σ^2. The channel between the mth and the $m-1$th hops is characterized by the $N_m^r \times N_{m-1}^r$ matrix $\mathbf{H}_{im}$, where i is the subcarrier index and $m = 0, M$ refer to the source and destination nodes such that $N_0^r = N_T$ and $N_M^r = N_R$. Applying the singular value decomposition (SVD), we obtain

$$\mathbf{H}_{im} = \mathbf{U}_{im}\mathbf{\Sigma}_{im}\mathbf{V}_{im}^* \tag{1}$$

where the singular values in $\mathbf{\Sigma}_{im}$ are sorted in a decreasing order along diagonal, and $\mathbf{U}_{im}$ and $\mathbf{V}_{im}^*$ are the associated left and right unitary matrices. For each pair (i, m), $\mathbf{V}_{im}$ and $\mathbf{U}_{im}^*$ are the SVD-based pre-processing and post-processing matrices at the transmit and receive sides of the mth hop over subcarrier i. There are $Q_m = \min(N_{m-1}^r, N_m^r)$ spatial SISO subchannels and the kth subchannel's gain-to-noise ratio (GNR) is denoted by $\alpha_{im}^k = \frac{|\mathbf{\Sigma}_{im}(k,k)|^2 d_m^\gamma}{\sigma^2}$, where d_m is the distance between the $(m-1)$th node and the mth node for the mth hop while γ is the path loss exponent.

The achievable rate of the kth spatial subchannel over the ith subcarrier in the mth hop is

$$r_{im}^k(p_{im}^k) = \log_2(1 + p_{im}^k \alpha_{im}^k) \text{ (bits/sec/Hz)} \tag{2}$$

where p_{im}^k is the power assigned to the kth spatial subchannel over the ith subcarrier in the mth hop. For example, Fig. 1 describes a system block diagram for the system with $M = 3$ and $N_m^r = 2, 0 \le m \le M$.

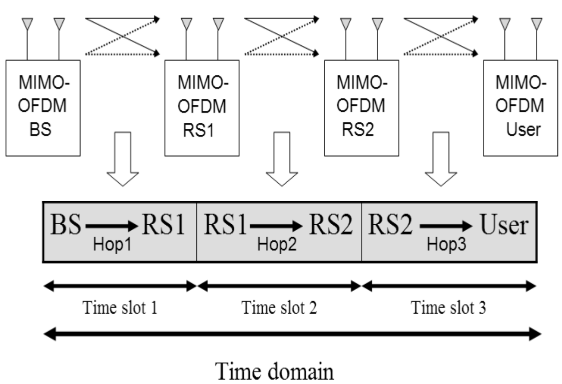

Fig. 1. A MIMO-based three-hop OFDM system.

B. Problem formulation

Based on the above system model, the problem of our concern is the problem of sum rate maximization through M hops transmission under the total power constraint P_T. For subcarrier i, the achievable rate R_i in a DF based multi-hop MIMO-OFDM systems is given by

$$R_i = \min_{1 \le m \le M} \sum_{k=1}^{Q_m} r_{im}^k(p_{im}^k) \tag{3}$$

where Q_m denotes the number of spatial subchannels in the mth hop. Hence to maximum the sum rate over all subcarriers with a total power constraint, we need to solve the optimization problem:

$$\text{(P1):} \quad \max_{\{p_{im}^k\}} \sum_{i=1}^{N} R_i \tag{4}$$

subject to

$$\sum_{i=1}^{N}\sum_{m=1}^{M}\sum_{k=1}^{Q_m} p_{im}^k \le P_T, \; p_{im}^k \ge 0 \tag{5}$$

Two solutions are presented in the next two sections.

III. Optimal Power Allocation

We first obtain the optimal power allocation solution which can be expressed in a water-filling form. The necessary conditions for the water-filling levels of the optimal solution is then derived. In the second subsection, we present an iterative search algorithm that find the water-filling levels which satisfy all the requirements. The algorithm adopts, in principle, a multi-level bisection search approach to locate the total-power-induced level and the levels for each subcarrier associated with each hop.

A. Optimal solution

It is difficult to directly solve the constrained maximization problem (P1) due to many a nonlinear argument (3). However, we observe that the non-constrained problem

$$\max \sum_{i=1}^{N} \min_{1 \le m \le M} a_{im}$$

is equivalent to the constrained optimization problem

$$\max \sum_{i=1}^{N} t_i$$
$$\text{such that } \; \forall\, 1 \le i \le N$$
$$t_i \le a_{im}, \; \forall\, 1 \le m \le M,$$

and use this equivalence to convert (P1) into

$$\text{(P2):} \quad \max \;\; t_1 + t_2 + \dots t_N \tag{6}$$

such that for all $1 \le i \le N$

$$t_i \le \sum_{k=1}^{Q_m} r_{im}^k(p_{im}^k), \forall\, 1 \le m \le M, \text{ and} \tag{7}$$

$$\sum_{i=1}^{N}\sum_{m=1}^{M}\sum_{k=1}^{Q_m} p_{im}^k \le P_T \tag{8}$$

Because (P2) is a convex optimization problem with a convex feasible domain, the Karush-Kuhn-Tucker (KKT) conditions must be the necessary conditions of the optimal solution. We use the method of Lagrange multipliers to form an objective function that incorporates the constraints of (P2):

$$\begin{aligned} L(p_{im}^k, t_i, \lambda_{im}, \lambda_P) = \\ \sum_{i=1}^{N} t_i + \sum_{i=1}^{N}\sum_{m=1}^{M} \lambda_{im}\left(\sum_{k=1}^{Q_m} r_{im}^k(p_{im}^k) - t_i\right) \\ + \lambda_P\left(P_T - \sum_{i=1}^{N}\sum_{m=1}^{M}\sum_{k=1}^{Q_m} p_{im}^k\right) \end{aligned} \tag{9}$$

where $\{\lambda_{im}\}$ and λ_P are nonnegative Lagrange multipliers. With $\{\lambda_{im}\}$ and λ_P known, the KKT condition implies that the optimal power allocation is given by

$$p_{im}^k = \left(\frac{\lambda_{im}}{\ln 2\lambda_P} - \frac{1}{\alpha_{im}^k}\right)^+ \tag{10}$$

where $(t)^+ = \max(0,t)$. The above solution is similar to the conventional water-filling solution with the multiple water-levels determined jointly by the subcarrier index i and the hop index m.

Taking derivative of (9) with respect t_i, we obtain

$$\frac{\partial L(p_{im}^k, t_i, \lambda_{im}, \lambda_P)}{\partial\, t_i} = 0 \Rightarrow \sum_{m=1}^{M} \lambda_{im} = 1 \tag{11}$$

i.e., the summation of all hops' water-levels for all subcarriers should be the same. The water-levels associated with the optimal solution, however, should satisfy another condition given in the following *Lemma*.

Lemma 1: For subcarrier i, $\lambda_{im} > 0, 1 \le m \le M$ of (10) must be such that

$$R_{i1} = R_{i2} = \cdots = R_{iM} \tag{12}$$

where R_{im} represents the rate of the mth hop over the ith subcarrier

$$R_{im} = \sum_{k=1}^{Q_m} r_{im}^k(p_{im}^k) \tag{13}$$

where p_{im}^k is function of λ_P and $\{\lambda_{im}\}$ given by (9).

Proof: Based on the Complementary Slackness condition, we can obtain

$$\lambda_{im}\left(\sum_{k=1}^{Q_m} r_{im}^k(p_{im}^k) - t_i\right) = 0, \forall\ 1 \le m \le M, \tag{14}$$

Since $\lambda_{im} > 0$ for $1 \le m \le M$, the condition is equivalent to $\sum_{k=1}^{Q_m} r_{im}^k(p_{im}^k) = t_i$ for $1 \le m \le M$. It implies that $R_{i1} = R_{i2} = \cdots = R_{iM}$. ■

B. Optimal power allocation algorithm

In this subsection we describe an iterative algorithm to find the optimal power allocation for the special case $M = 3$. Extensions to arbitrary finite M is straightforward. Given $\{\lambda_{im}\}$ and λ_P the optimal power allocation is specified by (10). The remaining problem is to find the corresponding optimal $\{\lambda_{im}\}$ and λ_P with precision ϵ.

Note that each nonnegative λ_{im} is upper-bounded and the total power is also upper-bounded by λ_P. We propose a two-stage iterative process for finding the optimal multipliers. In the first stage, we impose the conditions (11)–(13) to find the local optimal $\{\lambda_{im}\}$ associated with a fixed λ_P using an iterative M sub-stages process. In the second stage, we apply the bi-section method to update λ_P in order to meet the constraint (7). The detailed procedure is summarized in Table I (for $M = 3$).

The following *Lemma* guarantees the convergency of the proposed algorithm.

Lemma 2: For any given $\lambda_P > 0$, each λ_{im} in Step 2 converges to a unique value.

Proof: For a given $\lambda_{i1} > 0$, we define the function $\triangle(\lambda_{i2}) = R_{i2} - R_{i3}$ where $\lambda_{i2} \in [0, 1-\lambda_{i1}]$ based on (10). Because λ_{i2} is continuous, $\triangle(\lambda_{i2})$ is also a continuous function. From (9) and (12), we can easily find the following property of $\triangle(\lambda_{i2})$ as following

$$\triangle(0)\triangle(1-\lambda_{i1}) < 0 \tag{15}$$

When the objective function $f(x)$ is a continuous function defined on an interval $[a,b]$ and $f(a)$ and $f(b)$ have opposite signs, the bisection method is applicable to find the optimal value x^* such that $f(x^*) = 0$. Hence, we can obtain the optimal λ_{i2}^* such that $\triangle(\lambda_{i2}^*) = 0$ implying $R_{i2} = R_{i3}$. Similarly, we can verify that our bisection algorithm results in the optimal λ_{i1}^* and the corresponding λ_{i2}^*, λ_{i3}^* do satisfy (11)-(13). ■

As (11)-(13) are satisfied in each iteration, we need to find λ_P^* that meets the total power constraint (8) by applying the bisection method with a suitable initial interval $[\lambda_P^L, \lambda_P^U]$.

IV. A SUBOPTIMAL POWER ALLOCATION ALGORITHM

The optimal power allocation gives the optimal performance at the cost of high complexity due to the multi-level iterative bisection searches. In order to reduce the complexity, we propose a suboptimal solution which provides closed form expressions for power allocation. We assume that the power assigned to a subcarrier over all hops and all spatial subchannels are the same, i.e.,

$$\sum_{m=1}^{3}\sum_{k=1}^{Q_m} p_{im}^k = \frac{P_T}{N} \tag{16}$$

and each spatial subchannel is given a constant power $p_{im}^k = \bar{p}_{im}$ as well. The power allocation problem now becomes that of solving N identical optimization subproblems of link power allocation for all subcarriers. For subcarrier i, the corresponding optimization subproblem (P3) becomes

$$\text{(P3):} \quad \max_{\{\bar{p}_{im}\}} \min_{1\le m\le 3} \sum_{k=1}^{Q_m} r_{im}^k(\bar{p}_{im}) \tag{17}$$

subject to

$$\sum_{m=1}^{3} \bar{p}_{im} = \sum_{m=1}^{3} \frac{\bar{P}_{im}}{Q_m} = \frac{P_T}{N}. \tag{18}$$

By defining the link power distribution ratio, $\Gamma_{im} = \frac{\bar{p}_{im}}{\bar{p}_{i1}}$, we obtain

$$\bar{p}_{im} = \frac{\Gamma_{im}}{\sum_{m=1}^{3}\Gamma_{im}} \times \frac{P_T}{N} \tag{19}$$

Substituting (19) into (P3) and assuming that there are Q spatial subchannels in all hops, we convert (P3) into

$$\text{(P4):} \quad \max_{\Gamma_{i2},\Gamma_{i3}} \min_{1\le m\le 3} \sum_{k=1}^{Q} r_{im}^k(\bar{p}_{im}) \tag{20}$$

$$\text{subject to } \Gamma_{i2} > 0,\ \Gamma_{i3} > 0 \tag{21}$$

where $\bar{p}_{im} = \frac{\Gamma_{im}}{\sum_{m=1}^{3}\Gamma_{im}} \times \frac{P_T}{QN}$, and $\Gamma_{i1} = 1$. *Lemma* 1 then implies that the optimal power distribution ratio is given by

$$\Gamma_{i2}^* = \left(\prod_{k=1}^{Q} \frac{\alpha_{i1}^k}{\alpha_{i2}^k}\right)^{1/Q}, \ \Gamma_{i3}^* = \left(\prod_{k=1}^{Q} \frac{\alpha_{i1}^k}{\alpha_{i3}^k}\right)^{1/Q} \tag{22}$$

Hence, by restricting the power allocation to the hop domain, we obtain a closed-form suboptimal solution.

V. Numerical Results and Discussions

We consider a 3-hop MIMO-OFDM system with $N_T = N_R = N_1^r = N_2^r = 4$ and $N = 4$ in the simulation results reported in this section. As shown in Fig. 2, we assume that two rely nodes are separated from the BS by the distances 200 m and 400 m, respectively. The distance between the MS and the BS is uniformly distributed within the range $400 \sim 600$m. It is further assumed that each subcarrier suffers from independent Rayleigh fading in any direct or relay link with a path loss d^γ, where d denotes the distance of a hop and

$$\gamma = \begin{cases} -2, & \text{if } d \le 200\text{m} \\ -4, & \text{otherwise} \end{cases}$$

In addition, we define signal-to-noise ratio (SNR) as $\frac{P_T}{N\sigma^2\times 600^{-4}}$, which is the received average SNR measured by the user. For reference purpose, we include in Figs. 3-5, the performance curves of the equal power allocation scheme which distributes power equally over frequency and spatial domains in all hops, i.e. $p_{im}^k = \frac{P_T}{NQM}$, $\forall\ (i, m, k)$.

Fig. 3 indicates that the sum rate performance of the three-hop system is much better than that achieved by direct transmission ($M = 1$) using the same power and transmission time, i.e., the multi-hop relay scheme does provide far superior performance for an user who is far away from the BS.

Furthermore, we compare the sum rate performance of the optimal, suboptimal and equal power allocation schemes in a multi-hop MIMO-OFDM system in Fig. 4 and Fig. 5. In Fig. 4, the optimal power allocation scheme gives the best sum rate. To investigate the performance loss of a non-optimal scheme with respect to the optimal scheme, we define the performance loss factor η_χ of a power allocation scheme χ as

$$\eta_\chi = 1 - \frac{\text{sum rate achieved by } \chi}{\text{sum rate achieved by the optimal scheme}} \tag{23}$$

Fig. 5 depicts the performance loss factor η_χ for the two non-optimal power allocation schemes: the proposed suboptimal power allocation and the equal power allocation schemes. We find that the former scheme suffers from 3.8% to 1% performance loss only while the latter (equal power allocation) has more than 7% performance loss. On the other hand, proposed power distribution over the links of multiple hops in proposed suboptimal power allocation yields 6% performance gain compared with equal power allocation.

Table I: A multi-level bisection based iterative power allocation algorithm for a 3-hop MIMO-OFDM system.

Step 1: (Initialization) Select the upper and lower bounds, λ_P^U, λ_P^L, for λ_P and $\epsilon \ll 1$ and set $\lambda_P \leftarrow (\lambda_P^U + \lambda_P^L)/2$.

Step 2: (Given λ_P find $\{\lambda_{im}\}$ that satisfy (11)-(13))

```
for i=1 to N
 given λ_i1^U ← 1, λ_i1^L ← 0;
 while(λ_i1^U − λ_i1^L > ε)
   λ_i1 ← (λ_i1^U + λ_i1^L)/2.
   λ_i2^U ← 1 − λ_i1 and λ_i2^L ← 0
   while(λ_i2^U − λ_i2^L > ε)
     λ_i2 ← (λ_i2^U + λ_i2^L)/2.
     λ_i3 ← 1 − λ_i1 − λ_i2.
     Compute R_i2 and R_i3 based on (12)
     If R_i2 < R_i3, λ_i2^L ← λ_i2
     else λ_i2^U ← λ_i2 end
   end while
   Compute R_i1 based on (12)
   If R_i1 < R_i2, λ_i1^L ← λ_i1
   else λ_i1^U ← λ_i1 end
 end while
end for
```

Step 3: (Find the optimal λ_P which meets (7))

$\tilde{P}_T = \sum_{i=1}^{N}\sum_{m=1}^{3}\sum_{k=1}^{Q_m} p_{im}^k$

If $(\lambda_P^U - \lambda_P^L > \epsilon)$
  If $\tilde{P}_T < P_T$, $\lambda_P^U \leftarrow \lambda_P$
  else $\lambda_P^L \leftarrow \lambda_P$ **end**
  $\lambda_P \leftarrow (\lambda_P^U + \lambda_P^L)/2$
  goto Step 2;
end

Step 4: (Compute the optimal solution) $\lambda_P^* \leftarrow \lambda_P$ and $\lambda_{im}^* \leftarrow \lambda_{im}$, $\forall\ (i, m)$ and compute $\{p_{im}^k\}$ according to (10).

VI. Conclusion

In this paper, we present two power allocation algorithms for capacity (sum rate) maximization in Decode-and-Forward

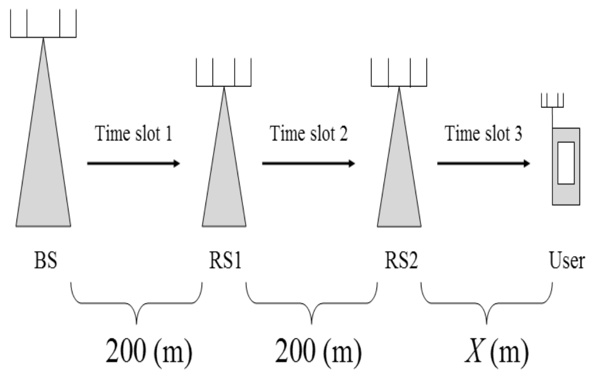

Fig. 2. A three-hop relay system where X is uniformly distributed over [0, 200].

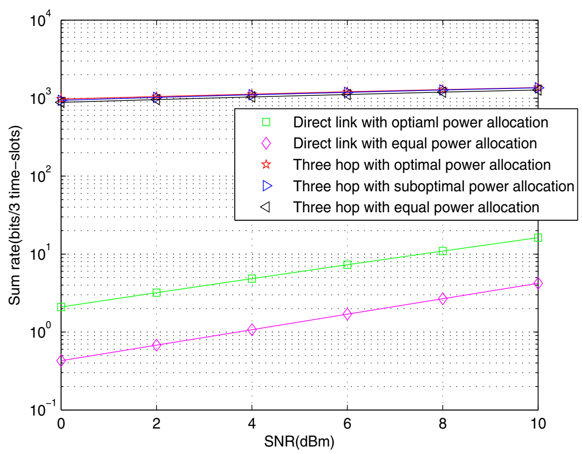

Fig. 3. Sum rate (bits/3 time slots) performance as a function of SNR (dbm); $M = 3, Q = 4$, and $N = 32$.

based multi-hop MIMO-OFDM systems. We derive the necessary conditions for the optimal solution and propose a multi-level bisection based iterative search algorithm to find the optimal power allocation. The convergency of the proposed iterative algorithm is also verified. A practical low complexity suboptimal solution is given as well. Numerical results indicate that our proposed optimal power allocation yields the best performance and proposed closed-form based suboptimal power allocation only suffers minor sum rate performance degradation with respect to the optimal power allocation with iterative computations.

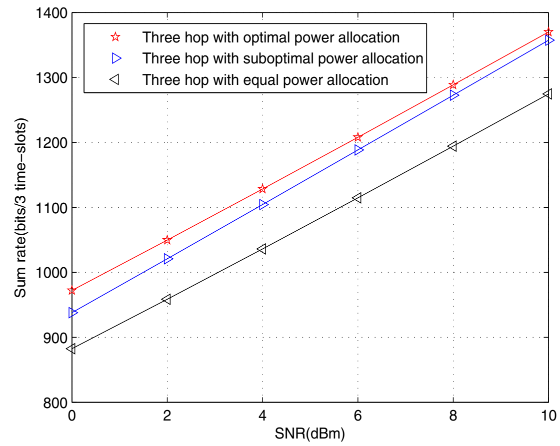

Fig. 4. Sum rate (bits/3 time slots) performance as a function of SNR (dbm). The upper three curves of Fig. 3 are zoomed in so that their performance difference can be better distinguished.

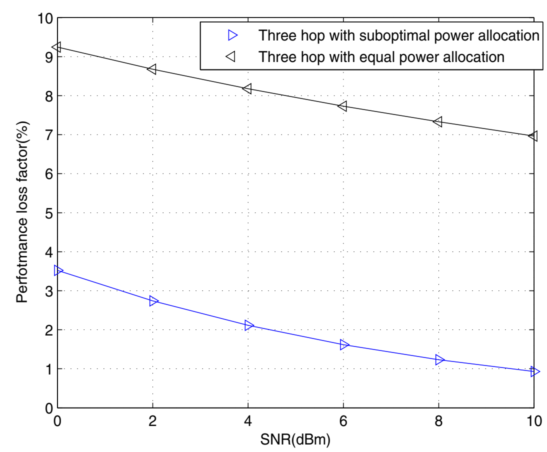

Fig. 5. Performance loss factor η (in %) as a function of SNR (dbm) for the proposed suboptimal power and the equal-power allocation schemes; $M = 3, Q = 4$, and $N = 32$.

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