Performance Characterization of AOA Geolocation Systems using the von Mises Distribution

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***Abstract*—Circular (angular) random variables, such as the angle of arrival (AOA) in direction finding systems and the phase error in digital phase-locked loops and digital phase interferometers, are best modeled by the von Mises distribution, not the ubiquitous Gaussian distribution. However, the Gaussian distribution has been commonly preferred in the performance analysis of AOA geolocation systems, mostly due to its mathematical tractability. This paper demonstrates the use of the von Mises distribution to model AOA measurement errors in direction finding systems and derives simple expressions for the Fisher information matrix (FIM) and the Cramer-Rao lower bound for the average miss distance. These results show that the use of the von Mises model in the performance analysis of AOA geolocation systems is entirely practicable.**

***Index Terms*—Angle of arrival (AOA), direction finding, emitter geolocation, von Mises distribution, Fisher information matrix, Cramer-Rao lower bound (CRLB)**

I. The Von Mises Distribution for Angle of Arrival (AOA) Measurements

Although the von Mises distribution is the best statistical model for circular parameters such as phase error and angle of arrival (AOA) [1]-[6], surprisingly, it has never been widely used to characterize the performance of AOA geolocation systems. Previously reported investigations [7] commonly modeled angular errors using the mathematically tractable, but less accurate, Gaussian distribution. The main goal of this paper is to fill this gap and use the von Mises distribution to establish an alternative statistical framework for the performance analysis of AOA geolocation systems.

Let $\kappa > 0$ and $\mu \in [-\pi, \pi)$. The von Mises distribution with concentration parameter κ and mean angle μ is defined by the following probability density function:

$$p(\theta|\mu,\kappa) = \frac{e^{\kappa\cos(\theta-\mu)}}{2\pi I_0(\kappa)}, \quad \theta \in [-\pi,\pi) \tag{1}$$

where, I_n, with $n = 0$, is the modified Bessel function of the first kind with order n computed by:

$$I_n(x) = \frac{1}{\pi}\int_0^\pi e^{x cos(t)}\cos(nt)dt, \quad x \in (-\infty,\infty) \tag{2}$$

By identifying the angle $\theta \in [-\pi,\pi)$ with the phasor $e^{j\theta}$, the von Mises distribution is best interpreted as a distribution on the unit circle consisting of all the unit phasors. As a consequence, the mean and variance of the von Mises distribution should be defined appropriately as follows. Let α be a circular random variable obeying the von Mises distribution (1), its mean value, denoted by $E(\alpha)$, is then defined and computed by:

$$\begin{aligned} E(\alpha) &= \angle \int_{-\pi}^{\pi} e^{j\theta} p(\theta|\mu,\kappa)d\theta \\ &= \angle\left[\frac{1}{2\pi I_0(\kappa)}\int_{-\pi}^{\pi} e^{j\theta}\ e^{\kappa cos(\theta-\mu)}d\theta\right] \\ &= \angle\left[e^{j\mu}\int_{-\pi}^{\pi} e^{j\theta}\ e^{\kappa cos\theta}d\theta\right] \\ &= \angle\left[2e^{j\mu}\int_0^{\pi} e^{\kappa\cos\theta}\cos\theta d\theta\right] \\ &= \angle\left[2\pi e^{j\mu} I_1(\kappa)\right] = \mu \end{aligned} \tag{3}$$

i.e., μ is the mean value of the von Mises distribution, or, $e^{j\mu}$ is the 'average value' of the random phasor $e^{j\alpha}$.

To define the variance of α, first note that the circular (angular) distance between the phasors $e^{j\theta}$ and $e^{j\mu}$, denoted by $d(e^{j\theta}, e^{j\mu})$, can be verified to be given by:

$$d(e^{j\theta}, e^{j\mu}) = |(\theta-\mu)\text{mod}(2\pi)| \tag{4}$$

(c.f. Fig. 1). The variance of α, denoted by $\text{Var}(\alpha)$, is then

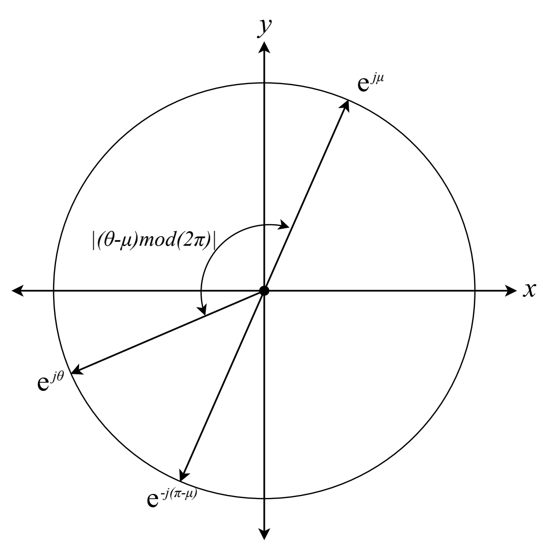

Fig. 1. Circular distance between $e^{j\theta}$ and $e^{j\mu}$.

defined and computed by:

$$\begin{aligned}
\text{Var}(\alpha) &= E\left(\left(d(e^{j\theta}, e^{j\mu})\right)^2\right) \\
&= \int_{-\pi}^{\pi} \left(d(e^{j\theta}, e^{j\mu})\right)^2 p(\theta|\mu,\kappa)d\theta \\
&= \frac{1}{2\pi I_0(\kappa)} \int_{-\pi}^{\pi} |(\theta-\mu)\text{mod}(2\pi)|^2 e^{\kappa\cos(\theta-\mu)}d\theta \\
&= \frac{1}{2\pi I_0(\kappa)} \int_{-\pi}^{\pi} \theta^2 e^{\kappa\cos\theta}d\theta = \frac{g(\kappa)}{I_0(\kappa)} \qquad (5)
\end{aligned}$$

where

$$g(\kappa) = \frac{1}{\pi}\int_0^{\pi} \theta^2 e^{\kappa\cos\theta}d\theta \qquad (6)$$

i.e., $\text{Var}(\alpha)$ is the average of the squared circular distance between the random phasor $e^{j\alpha}$ and the mean phasor $e^{j\mu}$. As will be shown later in this paper, the larger the value of κ, the more concentrated the von Mises distribution is towards the mean angle μ, with $\text{Var}(\alpha) \approx \frac{1}{\kappa}$, for $\kappa \geq 5$. On the other hand, if $\kappa \to 0$, the von Mises distribution approaches the uniform distribution on the unit circle with $\text{Var}(\alpha) \to \frac{\pi^2}{3}$.

II. Maximum Likelihood AOA Emitter Location Estimator Derived from the von Mises Distribution

In the literature, the maximum likelihood AOA emitter location estimator has always been based on a Gaussian model for the AOA measurements [7]-[9]. Here we formulate the maximum likelihood AOA emitter location estimator based on the von Mises model (1).

Assume that there is a single emitter at (x, y) and there are n direction finding (DF) systems located at the spatially dispersed positions $A_l = (u_l, v_l)$, $l = 1, 2, \ldots, n$, and the AOA measurement at A_l, denoted by ϕ_l, is the measured value of the angle, θ_l, between the vector pointing from A_l to the emitter at (x, y) and the positive direction of the x-axis. If the rotation from the positive direction of the x-axis to the vector pointing from A_l to the emitter at (x, y) is counterclockwise, $\theta_l \in [0, \pi)$; otherwise, $\theta_l \in [-\pi, 0)$. The AOA measurement ϕ_l is now modeled by the von Mises distribution with its probability density, denoted by V_l, computed by:

$$V_l(\phi_l) = p(\phi_l|\theta_l, \kappa_l) = \frac{e^{\kappa_l\cos(\phi_l-\theta_l)}}{2\pi I_0(\kappa_l)}, \quad 1 \leq l \leq n \qquad (7)$$

where $\kappa_l > 0$ is the concentration parameter of ϕ_l and θ_l is computed by the four-quadrant arctangent function atan2 in Matlab:

$$\theta_l = \text{atan2}(y - v_l, x - u_l) \qquad (8)$$

Assuming statistical independence of ϕ_l, $1 \leq l \leq n$, the joint probability density function of ϕ_l, $1 \leq l \leq n$, denoted by $F = F(\phi_1, \ldots, \phi_n|x, y)$, is computed by:

$$\begin{aligned}
F &= F(\phi_1, \ldots, \phi_n|x,y) = V_1(\phi_1)V_2(\phi_2)\cdots V_n(\phi_n) \\
&= \frac{e^{\sum_{l=1}^n \kappa_l\cos(\phi_l-\theta_l)}}{\prod_{l=1}^n (2\pi I_0(\kappa_l))} = ce^{-2\sum_{l=1}^n \kappa_l\left(\sin\left(\frac{\phi_l-\theta_l}{2}\right)\right)^2} \qquad (9)
\end{aligned}$$

where $c > 0$ is a constant independent of (x, y). Given the measurements ϕ_l, $1 \leq l \leq n$, the maximum likelihood AOA emitter location estimator, denoted by $(\hat{x}_{ML}, \hat{y}_{ML})$, is then obtained by maximizing the likelihood function $F = F(\phi_1, \ldots, \phi_n|x, y)$, or equivalently, by minimizing the objective function Q defined by:

$$Q(x,y) = \sum_{l=1}^n \kappa_l\left(\sin\left(\frac{\phi_l-\theta_l}{2}\right)\right)^2 \qquad (10)$$

i.e.,

$$(\hat{x}_{ML}, \hat{y}_{ML}) = \text{argmin } Q(x,y) \qquad (11)$$

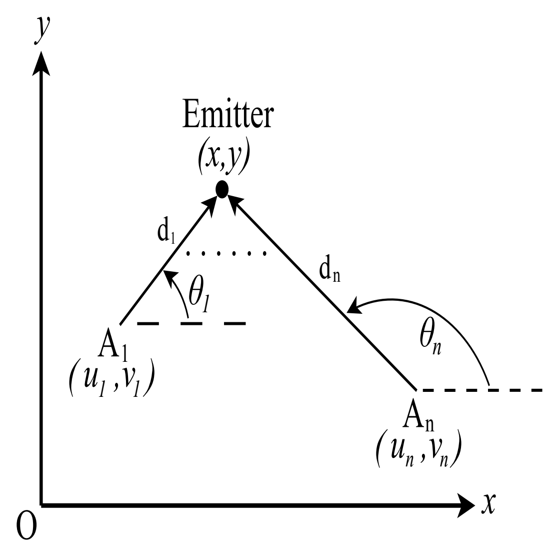

Fig. 2. Positions of n direction finding systems.

The estimator (11) can be implemented using the same approaches for implementing the maximum likelihood AOA emitter location estimator derived from the Gaussian model for ϕ_l, $1 \leq l \leq n$, such as performing a grid search or carrying out a Gauss-Newton type iteration.

III. Fisher Information Matrix and Cramer-Rao Lower Bound Derived from the von Mises Distribution

In this section, the Fisher information matrix and the Cramer-Rao lower bound (CRLB) for the average miss distance for unbiased estimators of (x, y) are derived, assuming the von Mises model (7) for ϕ_l, $1 \leq l \leq n$.

In fact, from (9) and (8) it follows that

$$\begin{aligned}
\frac{\partial\log(F)}{\partial x} &= \sum_{l=1}^n \kappa_l\left[\sin\left(\phi_l-\theta_l\right)\right]\frac{\partial\theta_l}{\partial x} \\
&= \sum_{l=1}^n \kappa_l\left[\sin\left(\phi_l-\theta_l\right)\right]\frac{-(y-v_l)}{(x-u_l)^2+(y-v_l)^2} \\
&= \sum_{l=1}^n \kappa_l\left[\sin\left(\phi_l-\theta_l\right)\right]\frac{(-\sin\theta_l)}{d_l} \qquad (12)
\end{aligned}$$

and

$$\frac{\partial \log(F)}{\partial y} = \sum_{l=1}^{n} \kappa_l \left[\sin\left(\phi_l - \theta_l\right)\right] \frac{\cos\theta_l}{d_l} \tag{13}$$

Hence

$$E\left[\left[\frac{\partial \log(F)}{\partial x}\right]^2\right] = \sum_{l=1}^{n} \kappa_l^2 \frac{\sin^2\theta_l}{d_l^2} E\Big[\left[\sin\left(\phi_l - \theta_l\right)\right]^2\Big] \tag{14}$$

where

$$\begin{aligned}
&E\left[\left[\sin\left(\phi_l - \theta_l\right)\right]^2\right] \\
&= \frac{1}{2\pi I_0(\kappa_l)} \int_{-\pi}^{\pi} \left[\sin\left(\phi_l - \theta_l\right)\right]^2 \exp\left\{\kappa_l \cos\left(\phi_l - \theta_l\right)\right\} d\phi_l \\
&= \frac{1}{2\pi I_0(\kappa_l)} \int_{-\pi}^{\pi} e^{\kappa_l \cos t} \sin^2 t dt \\
&= \frac{1}{\pi I_0(\kappa_l)} \int_{0}^{\pi} e^{\kappa_l \cos t} \sin^2 t \ dt = \frac{f(\kappa_l)}{I_0(\kappa_l)}
\end{aligned} \tag{15}$$

where

$$f(\kappa) = \frac{1}{\pi} \int_0^{\pi} e^{\kappa \cos\theta} \sin^2\theta \ d\theta \tag{16}$$

Here we have used the fact that, for $k \neq l$,

$$\begin{aligned}
&E[(\sin(\phi_k - \theta_k)) \sin(\phi_l - \theta_l)] \\
&= \left(E[\sin(\phi_k - \theta_k)]\right)\left(E[\sin(\phi_l - \theta_l)]\right) = 0
\end{aligned} \tag{17}$$

Substituting (15) into (14) yields:

$$\left[\left[\frac{\partial \log(F)}{\partial x}\right]^2\right] = \sum_{l=1}^{n} \frac{\kappa_l^2 f(\kappa_l)}{I_0(\kappa_l)} \frac{\sin^2\theta_l}{d_l^2} \tag{18}$$

Similarly

$$\left[\left[\frac{\partial \log(F)}{\partial y}\right]^2\right] = \sum_{l=1}^{n} \frac{\kappa_l^2 f(\kappa_l)}{I_0(\kappa_l)} \frac{\cos^2\theta_l}{d_l^2} \tag{19}$$

and

$$E\left[\frac{\partial \log(F)}{\partial x} \frac{\partial \log(F)}{\partial y}\right] = -\sum_{l=1}^{n} \frac{\kappa_l^2 f(\kappa_l)}{I_0(\kappa_l)} \frac{(\cos\theta_l)\sin\theta_l}{d_l^2} \tag{20}$$

Let

$$h(\kappa) = \frac{\kappa^2 f(\kappa)}{I_0(\kappa)} \tag{21}$$

and set

$$\rho_l = h(\kappa_l), \quad 1 \leq l \leq n \tag{22}$$

The Fisher information matrix, $\mathbf{J}$, is then given by (c.f. [8]):

$$\begin{aligned}
\mathbf{J} &= \begin{bmatrix} E\left[\left[\frac{\partial \log(F)}{\partial x}\right]^2\right] & E\left[\frac{\partial \log(F)}{\partial x}\frac{\partial \log(F)}{\partial y}\right] \\ E\left[\frac{\partial \log(F)}{\partial x}\frac{\partial \log(F)}{\partial y}\right] & E\left[\left[\frac{\partial \log(F)}{\partial y}\right]^2\right] \end{bmatrix} \\
&= \begin{bmatrix} \sum_{l=1}^{n} \frac{\rho_l \sin^2\theta_l}{d_l^2} & -\sum_{l=1}^{n} \frac{\rho_l (\cos\theta_l)\sin\theta_l}{d_l^2} \\ -\sum_{l=1}^{n} \frac{\rho_l (\cos\theta_l)\sin\theta_l}{d_l^2} & \sum_{l=1}^{n} \frac{\rho_l \cos^2\theta_l}{d_l^2} \end{bmatrix}
\end{aligned} \tag{23}$$

The average miss distance, δ, of any unbiased emitter location estimator, $(\hat{x}, \hat{y})$, is defined by:

$$\delta^2 = E\left[(x - \hat{x})^2 + (y - \hat{y})^2\right] \tag{24}$$

δ is bounded below by the Cramer-Rao lower bound (c.f. [8]), i.e., $\delta \geq \sqrt{\mathrm{tr}\left(\mathbf{J}^{-1}\right)}$, where tr denotes the trace of a square matrix, with

$$\mathrm{tr}\left(\mathbf{J}^{-1}\right) = \frac{\sum_{l=1}^{n} \frac{\rho_l}{d_l^2}}{\left[\sum_{l=1}^{n} \frac{\rho_l \cos^2\theta_l}{d_l^2}\right]\left[\sum_{l=1}^{n} \frac{\rho_l \sin^2\theta_l}{d_l^2}\right] - \left[\sum_{l=1}^{n} \frac{\rho_l (\cos\theta_l)\sin\theta_l}{d_l^2}\right]^2} \tag{25}$$

IV. Fisher Information Matrix and Cramer-Rao Lower Bound Derived from the Gaussian Distribution

If the AOA measurements, ϕ_l, $1 \leq l \leq n$, are modeled by a Gaussian distribution, the associated Fisher information matrix and Cramer-Rao lower bound (CRLB) for the average miss distance have been derived and widely used in the literature [7]. For comparison, these expressions are reproduced here without proofs.

Assume that ϕ_l is modeled by the Gaussian distribution:

$$G_l(\phi_l) = \frac{1}{\sqrt{2\pi}\sigma_l} e^{-\frac{(\phi_l - \theta_l)^2}{2\sigma_l^2}}, \quad 1 \leq l \leq n \tag{26}$$

where σ_l is the standard deviation of ϕ_l. Based on the Gaussian model (26), the Fisher information matrix, $\mathbf{J}$, is then computed by [7]:

$$\mathbf{J} = \begin{bmatrix} \sum_{l=1}^{n} \frac{\tau_l \sin^2\theta_l}{d_l^2} & -\sum_{l=1}^{n} \frac{\tau_l (\cos\theta_l)\sin\theta_l}{d_l^2} \\ -\sum_{l=1}^{n} \frac{\tau_l (\cos\theta_l)\sin\theta_l}{d_l^2} & \sum_{l=1}^{n} \frac{\tau_l \cos^2\theta_l}{d_l^2} \end{bmatrix} \tag{27}$$

where

$$\tau_l = \frac{1}{\sigma_l^2} = \frac{1}{v_{\phi_l}}, \quad 1 \leq l \leq n \tag{28}$$

with v_{ϕ_l} being the variance of ϕ_l defined using the Gaussian model (26):

$$v_{\phi_l} = \int_{-\infty}^{\infty} (\phi_l - \theta_l)^2 G_l(\phi_l) d\phi_l \tag{29}$$

The Cramer-Rao lower bound for the average miss distance is computed by $\sqrt{\mathrm{tr}\left(\mathbf{J}^{-1}\right)}$, with

$$\mathrm{tr}\left(\mathbf{J}^{-1}\right) = \frac{\sum_{l=1}^{n} \frac{\tau_l}{d_l^2}}{\left[\sum_{l=1}^{n} \frac{\tau_l \cos^2\theta_l}{d_l^2}\right]\left[\sum_{l=1}^{n} \frac{\tau_l \sin^2\theta_l}{d_l^2}\right] - \left[\sum_{l=1}^{n} \frac{\tau_l (\cos\theta_l)\sin\theta_l}{d_l^2}\right]^2} \tag{30}$$

V. Comparisons and Discussions

It can be observed that, in the identities (23) and (25), ρ_l plays the same role that τ_l does in the identities (27) and (30). In fact, if ρ_l and τ_l are equated, the corresponding expressions become completely identical. Therefore, comparisons of the von Mises and Gaussian distributions boil down to comparisons of the expressions for ρ_l and τ_l, $1 \leq l \leq n$.

The parameter, τ_l, is easily computed as the inverse of the variance of ϕ_l evaluated using (29). It is not immediately clear if the parameter, ρ_l, defined as $h(\kappa_l)$, admits a similar interpretation. We next show that ρ_l in fact can also be expressed as the inverse of the variance of ϕ_l, now computed using (5).

For relatively large values of κ, a careful examination of the definite integrals (6) and (16) reveals that the functions $f(\kappa)$ and $g(\kappa)$ should be close to each other, since

$$\sin\theta \approx \theta, \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad e^{\kappa\cos\theta} \approx 0, \quad \frac{\pi}{2} \leq \theta \leq \pi \tag{31}$$

More specifically, if $\kappa > 0$ is relatively large,

$$\begin{aligned} g(\kappa) &= \frac{1}{\pi}\int_0^\pi \theta^2 e^{\kappa\cos\theta} d\theta \\ &\approx \frac{1}{\pi}\int_0^\pi (\sin^2\theta) e^{\kappa\cos\theta} d\theta = f(\kappa) \\ &\approx \frac{1}{\pi}\int_0^\pi \theta(\sin\theta) e^{\kappa\cos\theta} d\theta \\ &= \frac{1}{\pi}\int_0^\pi \theta\left\{\frac{d}{d\theta}\left[-\frac{1}{\kappa}e^{\kappa\cos\theta}\right]\right\} d\theta \\ &= -\frac{e^{-\kappa}}{\kappa} + \frac{I_0(\kappa)}{\kappa} \approx \frac{I_0(\kappa)}{\kappa} \end{aligned} \tag{32}$$

This implies that

$$\frac{f(\kappa)}{I_0(\kappa)} \approx \frac{g(\kappa)}{I_0(\kappa)} \approx \frac{1}{\kappa} \tag{33}$$

Numerical evidence shows that the approximations in (33) are accurate for $\kappa \geq 5$ (c.f. Fig. 3). From (33) it follows that

$$h(\kappa) = \frac{\kappa^2 f(\kappa)}{I_0(\kappa)} = \kappa^2 \frac{f(\kappa)}{I_0(\kappa)} \approx \kappa^2 \frac{1}{\kappa} = \kappa \tag{34}$$

and this approximation is accurate for $\kappa \geq 1$ (c.f. Fig. 4). Since the variance of the von Mises distribution (1) is computed by (5), it also follows from (33) that the variance of ϕ_l, also denoted by v_{ϕ_l} here, is closely approximated by $\frac{1}{\kappa_l}$, provided that the standard deviation of ϕ_l is less than 1 radian, or $\frac{180}{\pi} \approx$ 57 degrees, which can be assumed to be satisfied for virtually all useful DF systems. Thus, for all practical purposes, we can safely write

$$\rho_l = h(\kappa_l) = \frac{1}{v_{\phi_l}}, \quad 1 \leq l \leq n \tag{35}$$

The preceding discussions imply that the two formulas (23) and (27) for the Fisher information matrix can be combined to yield the single unified formula (36) by replacing ρ_l and τ_l by $w_l = \frac{1}{v_{\phi_l}}$, $1 \leq l \leq n$, where v_{ϕ_l} is the variance of ϕ_l, computed using either (5) or (29), depending on whether the von Mises distribution (7) or the Gaussian distribution (26) is used to model the AOA measurements ϕ_l, $1 \leq l \leq n$:

$$\mathbf{J} = \begin{bmatrix} \sum_{l=1}^n \frac{w_l \sin^2\theta_l}{d_l^2} & -\sum_{l=1}^n \frac{w_l(\cos\theta_l)\sin\theta_l}{d_l^2} \\ -\sum_{l=1}^n \frac{w_l(\cos\theta_l)\sin\theta_l}{d_l^2} & \sum_{l=1}^n \frac{w_l\cos^2\theta_l}{d_l^2} \end{bmatrix} \tag{36}$$

Similarly, the formulas (25) and (30) for the Cramer-Rao lower bound are combined to yield the single formula:

$$\operatorname{tr}\left(\mathbf{J}^{-1}\right) = \frac{\sum_{l=1}^n \frac{w_l}{d_l^2}}{\left[\sum_{l=1}^n \frac{w_l\cos^2\theta_l}{d_l^2}\right]\left[\sum_{l=1}^n \frac{w_l\sin^2\theta_l}{d_l^2}\right] - \left[\sum_{l=1}^n \frac{w_l(\cos\theta_l)\sin\theta_l}{d_l^2}\right]^2} \tag{37}$$

Before closing this section, it is pointed out here that the von Mises distribution (7) has a distinct advantage over the Gaussian distribution (26) in that it is definitively more appropriate for modeling very noisy AOA measurements. As can be seen from (5), when $\kappa \to 0$, the standard deviation of the von Mises distribution (1) approaches $\frac{\pi}{\sqrt{3}}$, which is the standard deviation of the uniform distribution on the unit circle. The Gaussian distribution (26) does not have the capability to capture this characteristic of AOA measurements.

VI. Conclusions

This paper develops an alternative theoretical framework for characterizing the performance of an AOA geolocation system by utilizing the von Mises distribution to model angle of arrival (AOA) measurements. It is found that the mathematical expressions for the Fisher information matrix and the Cramer-Rao lower bound for the average miss distance for unbiased emitter location estimators assume identical forms for the von Mises and Gaussian distributions, provided that the variances of AOA measurements are appropriately defined. The results of this paper show that both the von Mises and the Gaussian distributions are useful for modeling angle of arrival (AOA) measurement errors in direction finding systems. However, the von Mises distribution is preferable, for the simple but fundamental reason that AOA measurements are inherently circular in nature.

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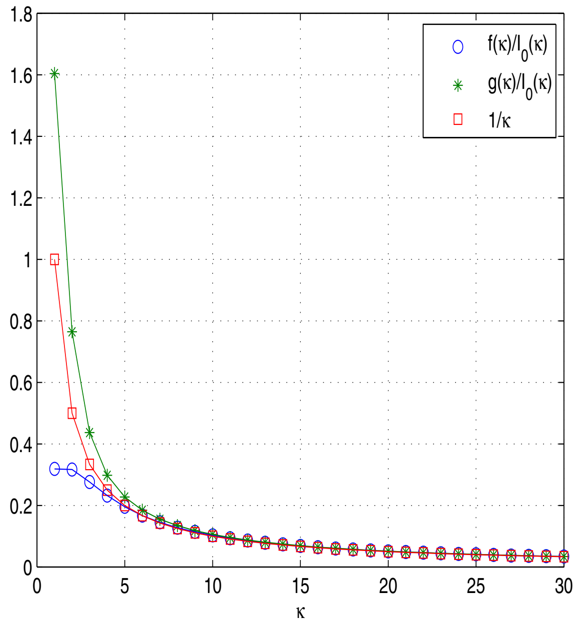

Fig. 3. Comparison of $\frac{f(\kappa)}{I_0(\kappa)}$, $\frac{g(\kappa)}{I_0(\kappa)}$ and $\frac{1}{\kappa}$ on the interval $[1, 30]$.

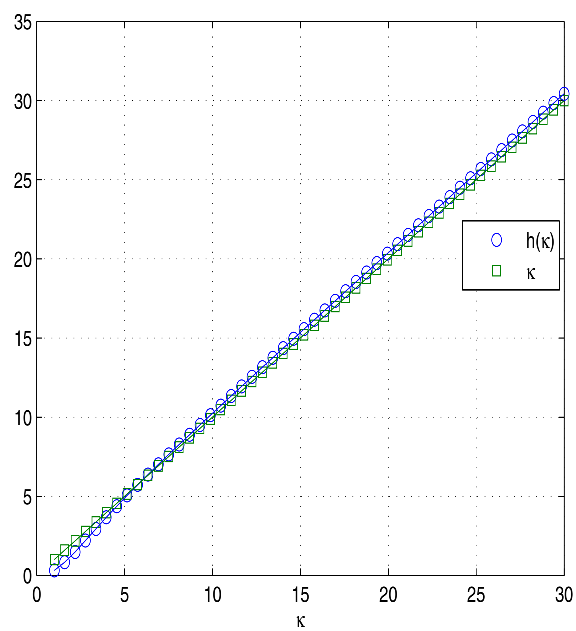

Fig. 4. Comparison of $h(\kappa)$ and κ on the interval $[1, 30]$.