Simple Formulas for Area Coverage Probability of Cellular Wireless Networks

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Abstract— **Reudink's formula which describes the relationship between cell edge coverage probability and cell area coverage probability is routinely used in link budgets for cellular wireless network dimensioning and designs. A major drawback of this formula is that the effect of handoffs in the cellular network is not taken into account. The formulas derived in this paper overcome this drawback. The new formulas take into account the correlation between two fading handoff links, and allow the shadow fading standard deviations and the RF propagation loss exponents to be different for the two fading handoff paths. In addition, the formulas also allow the correlation between the handoff links to vary as a function of the distance between the mobile and the base stations. These new formulas are easily implemented in a spreadsheet since they are all represented as an integral of an explicit smooth function on the interval [0,1]. The importance and the necessity of the new formulas are demonstrated by applying them to several typical RF design scenarios and show that the widely used link margins in today's RF designs are too optimistic by more than one dB.**

Keywords - RF design; RF signals; shadow fading; slow fading; coverage probability; coverage reliability; fade margin; RF coverage; link budget.

I. INTRODUCTION

In the RF design process of cellular wireless networks, a major design criterion is the RF coverage performance of the network. Due to the presence of random shadow fading, the quality of RF coverage is expressed in terms of coverage probability. In general, a link budget analysis is done to estimate the RF coverage [1]. For example, based on a predefined coverage probability, link budget analysis is done to determine how much margin is required to ensure such coverage probability. However, RF link budget specifies only the maximum allowable path loss between the transmitter and the receiver; therefore the link margin it specifies is to guarantee the quality of performance at the cell edge, such as the cell edge coverage probability. Service providers of the wireless networks, of course, are interested in not only the cell edge coverage probability, but also the overall coverage probability within the intended coverage area. The mapping from the coverage probability at the cell edge to the area coverage probability in the whole cell becomes a key step in the RF design process. A frequently used such mapping is Reudink's formula, which appears in standard RF systems engineering reference books such as [2] [3] and training materials (*e.g.*, [4]). A major drawback of this formula is that it assumes that the cell is isolated; thus the impact of handoffs is not taken into account. In a real cellular network, this is of course not the case since a mobile is able to choose the best signal among the signals from the surrounding cells, and therefore experiences the handoff gain [5] [6] [7]. This handoff gain translates into a smaller required link margin for a given cell edge coverage probability in the link budget analysis. But what is the impact of handoff gain on cell *area* coverage probability? In this paper, we shall answer this question by deriving a unified formula for the area coverage probability, taking into account the impact of soft handoff gain, hard handoff gain, the correlation between the fading handoff links, and allow the shadow fading standard deviations and the RF propagation loss exponents to be different between the two fading handoff paths. Furthermore, we also allow the correlation between the handoff links to vary as a function of the distance between the mobile and the two base stations. Reudink's original no-handoff formula is included as a special case. We consider the case when the mobile is in possible handoff with two RF links. This is the case that should be used in RF link budget design since two-way handoff is likely the most common handoff scenario in a cellular wireless network and is the most conservative among all handoff scenarios.

In addition to being appeared in standard RF systems engineering reference and textbooks and training materials, another reason for the popularity of Reudink's formula among RF systems engineers is its simplicity; its calculation involves only the CDF (Cumulative Distribution Function) and its inverse of the standard normal distribution; thus it can be easily programmed in a spreadsheet. However, when handoffs are taken into account, the problem is represented as an integration of a nontrivial parameterization of the classical two-dimensional probability integral which has been studied extensively by many mathematicians and engineers since the 1940's. To the author's knowledge, reducing a 2-D probability integral to a 1-D probability integral is in general not possible even for independent Gaussian random variables. Fortunately

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by taking advantages of the special structures of this problem we are able to arrive at a general formula that takes a very simple form, which contains only an integration of an explicit smooth function on the interval [0,1]. The general formula can be further simplified for several important special cases. All our formulas can be easily implemented in a spreadsheet. We believe that the RF engineering community will find our formulas useful in RF planning and link budget analyses for cellular wireless networks such as LTE, UMTS, CDMA and GSM.

The paper is organized as follows. Mathematical notations are collected in the next section. The relationship between the edge coverage probability and the corresponding required fade margin is presented in Section III. The new area coverage probability formulas are presented in Section IV. In Section V the formulas obtained in Section IV are used to calculate the coverage probabilities and link margins for some typical RF design scenarios. Comparisons with results using Reudink's formula are made. Conclusions are drawn in Section VI.

II. NOTATION

For easy reference all the notations are listed in this section.

ξ : Gaussian random variables with 0 mean and variance σ^2.

ξ's are used to model shadow fadings.

σ : Standard deviation of the shadow fading.

ρ : Correlation coefficient of two shadow fadings ξ_1 and ξ_2.

z : Standard Gaussian random variable with zero mean and unit variance.

f_z : Probability density function (PDF) of z.

F_z : Cumulative distribution function (CDF) of z.

Γ : Required link margin. Its subscript denotes the corresponding edge coverage probability. *e.g.,* $\Gamma_{0.9}$ is the required link margin for 90% edge coverage probability.

H_h : Hard handoff hysteresis, *i.e.*, the candidate signal has to be H_h dB higher than the host signal for hard handoff to happen.

L_h : $L_h = H_h / \sigma$

P_e : Edge coverage probability.

P_a : Area coverage probability.

r : Distance between the base station and the mobile.

β : Propagation loss exponent.

RF propagation model:

$$\text{Path Loss (in dB)} = A + 10\beta\log(r) + \xi,$$

where A is a morphology dependent constant and β, r, ξ are defined as in the above list.

III. EDGE COVERAGE PROBABILITY

In this section, we shall investigate the relationship between edge coverage probability and the corresponding required fade margin assuming a mobile is capable of taking advantage of soft handoff or hard handoff with handoff hysteresis. A simple unified formula for mapping a fade margin to the edge coverage probability is derived.

Referring to Figure 1, a location between two base stations is considered covered if at least one of the two mobile received signal strengths from the two neighboring base stations is above a given threshold. Equivalently, a location is not covered if both path losses from the two base stations to the mobile are greater than a given threshold. Since the link margin depends only on the random shadow fading distribution and the relative strength of the signals from the different paths, it does not depend on the absolute cell radius R. Thus without loss of generality we can assume $R = 1$. Referring to Figure 1, we have $r_2 = 2 - r_1$.

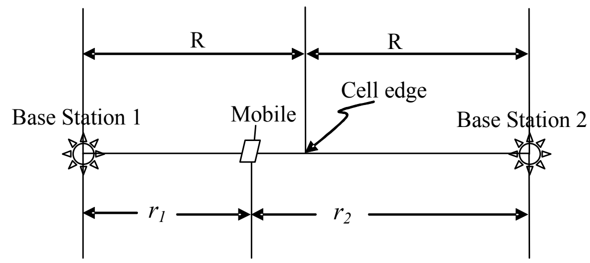

Figure 1. Mobile in handoff between two base stations

The link margin $\Gamma_{x\%}(r_1)$ that is required to achieve $x\%$ of edge coverage probability at distance r_1 satisfies the following equation

$$1 - P\left[\min\begin{Bmatrix}10\beta\log(r_1) + \xi_1, \\ 10\beta\log(2-r_1) + \xi_2\end{Bmatrix} > \Gamma_{x\%}(r_1)\right] = x\%.$$

Adding the hard handoff hysteresis H_h to the path loss of the candidate signal, we have

$$1 - P\left[\min\begin{Bmatrix}10\beta\log(r_1) + \xi_1, \\ 10\beta\log(2-r_1) + \\ \xi_2 + H_h\end{Bmatrix} > \Gamma_{x\%}(r_1)\right] = x\% \tag{1}$$

Note here that the random shadow fading variable ξ_1 and ξ_2 may be correlated. The PDF of the random variable

$$\min\{10\beta\log(r_1) + \xi_1,\ 10\beta\log(2-r_1) + \xi_2 + H_h\}$$

can be expressed as (see [8] and references therein)

$$g_1(y, r_1) + g_2(y, r_1)$$

where

$$g_1(y,r_1) = (1/\sigma_1) f_z\{[y - 10\beta\log(r_1)]/\sigma_1\} \cdot F_z\left\{\rho[y - 10\beta\log(r_1)]/\left(\sigma_1\sqrt{1-\rho^2}\right) - [y - 10\beta\log(2-r_1) - H_h]/\left(\sigma_2\sqrt{1-\rho^2}\right)\right\}$$

and

$$g_2(y,r_1)=(1/\sigma_2)f_z\{[y-10\beta\log(2-r_1)-H_h]/\sigma_2\}\cdot$$
$$F_z\left\{\rho[y-10\beta\log(2-r_1)-H_h]/\left(\sigma_2\sqrt{1-\rho^2}\right)-[y-10\beta\log(r_1)]/\left(\sigma_1\sqrt{1-\rho^2}\right)\right\}$$

Here ρ is the correlation coefficient between ξ_1 and ξ_2.

Letting $r_1=1$, we have the following

Theorem 1. The edge coverage probability is

$$P_e=G_{12}(\Gamma),$$

where $G_{12}(\cdot)$ is the CDF corresponding to the PDF

$$g_{12}(y)=g_1(y,1)+g_2(y,1),$$

Γ is the required fade margin corresponding to the edge coverage probability P_e. ∎

Note that the calculation of edge coverage probability P_e involves integration of 1-D probability integral.

Remark. Since the CDF $G_{12}(\cdot)$ is strictly increasing, there is a one-to-one correspondence between edge coverage P_e and the required fade margin Γ, the notation Γ_{P_e} is well-defined. In fact $\Gamma_{P_e}=G_{12}^{-1}(P_e)$.

IV. Area Coverage Probability as a Function of Edge Coverage Probability

In this section, we shall derive a general formula that calculates the area coverage probability for a given edge coverage probability. Several simpler formulas representing different useful special cases result from the general formula.

From (1) we see that the area coverage probability is given by

$$P_a=1-2\int_0^1 r_1\int_{\Gamma_{x\%}}^{\infty}[g_1(y,r_1)+g_2(y,r_1)]\mathrm{d}y\mathrm{d}r_1$$
$$=2\int_0^1 r_1\int_{-\infty}^{\Gamma_{x\%}}[g_1(y,r_1)+g_2(y,r_1)]\mathrm{d}y\mathrm{d}r_1$$

From the above integral equation, we see that in order to calculate the area coverage probability P_a, we need to evaluate the integrals of two 2-dimensional probability integrals $\int_{-\infty}^{\Gamma_{x\%}}g_1(y,r_1)\mathrm{d}y$ and $\int_{-\infty}^{\Gamma_{x\%}}g_2(y,r_1)\mathrm{d}y$, both parameterized by r_1. Further simplification of this integral equation turns out to be a nontrivial task due to the involvement of 2-D probability integrals.

Evaluation of 2-D probability integrals has been studied extensively since the 1940's due to its importance both in theory and in application. While a general procedure for reducing a 2-D probability integral to a 1-D probability integral is not available even for independent Gaussian random variables, we shall show that calculation of the area coverage probability P_a can be done with just an integration of an explicit smooth function on the interval [0,1]. The integrand involves only 1-D probability integral. This is made possible by the following lemma, which takes advantage of the special structure of this particular problem. The proof of this lemma is omitted here to save space.

Lemma 1. Let

$$g(x,r)=f_z[ax+b(r)]F_z[c(r)x+d(r)],$$

where a is a nonzero constant and the functions $b(r)$, $c(r)$ and $d(r)$ are differentiable. Then

$$\int_{-\infty}^{\Gamma}\frac{\partial}{\partial r}g(x,r)\mathrm{d}x=$$
$$\frac{b'(r)}{a}f_z[a\Gamma+b(r)]F_z[c(r)\Gamma+d(r)]+$$
$$e^{-\frac{1}{2}\frac{(a\cdot d(r)-b(r)\cdot c(r))^2}{a^2+c^2(r)}}\left\{2\pi\left[a^2+c^2(r)\right]\right\}^{-1/2}\cdot$$
$$\left\{\left[d'(r)-\frac{c(r)\cdot b'(r)}{a}\right]F_z(\Lambda)-c'(r)\cdot\left[\left(\frac{\Lambda}{\sqrt{a^2+c^2(r)}}-\Gamma\right)F_z(\Lambda)+\frac{e^{-\Lambda^2/2}}{\sqrt{2\pi}\sqrt{a^2+c^2(r)}}\right]\right\}$$

where $\Lambda=\sqrt{a^2+c^2(r)}\left(\Gamma+\frac{ab(r)+c(r)d(r)}{a^2+c^2(r)}\right)$. ∎

Using integration by parts and Lemma 1 above, we can prove the following theorem.

Theorem 2. Let P_e be the edge coverage probability and Γ_{P_e} be the corresponding fade margin. The area coverage probability

$$P_a=P_e+\int_0^1 J(r)\,\mathrm{d}r,$$

where

$$J(r)=\frac{r10}{\ln 10}\left\{\begin{array}{l}\frac{\beta_1}{\sigma_1}f_z\left[\chi_1(\Gamma_{P_e})\right]F_z\left[\frac{\rho(r)\chi_1(\Gamma_{P_e})-\chi_2(\Gamma_{P_e})}{\sqrt{1-\rho^2(r)}}\right]-\\ \qquad\frac{\beta_2}{\sigma_2}\frac{r}{2-r}f_z\left[\chi_2(\Gamma_{P_e})\right]\cdot\\ \qquad\qquad F_z\left[\frac{\rho(r)\chi_2(\Gamma_{P_e})-\chi_1(\Gamma_{P_e})}{\sqrt{1-\rho^2(r)}}\right]\end{array}\right\}+$$
$$r^2 f_z[\omega(r)]\rho'(r)f_z\left[\eta(\Gamma_{P_e}+\lambda(r))\right]/\sqrt{1-\rho^2(r)}$$

and

$$b_1(r)=-[10\beta_1\log(r)]/\sigma_1$$
$$b_2(r)=-[10\beta_2\log(2-r)+H_h]/\sigma_2$$
$$\chi_1(\Gamma_{P_e})=(\Gamma_{P_e}/\sigma_1)+b_1(r)$$

$$\chi_2\left(\Gamma_{P_e}\right) = \left(\Gamma_{P_e} / \sigma_2\right) + b_2(r)$$

$$\eta = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2} / \left(\sigma_1\sigma_2\sqrt{1-\rho^2}\right)$$

$$\lambda(r) = \sigma_1\sigma_2 \frac{\sigma_2 b_1(r) + \sigma_1 b_2(r) - \rho\left(\sigma_1 b_1(r) + \sigma_2 b_2(r)\right)}{\left(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2\right)}$$

$$\omega(r) = \left[\sigma_1 b_1(r) - \sigma_2 b_2(r)\right] / \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2} \qquad \blacksquare$$

In Theorem 2, the area coverage probability is expressed as the edge coverage probability plus a positive increment. Thus the area coverage probability is always greater than the edge coverage probability. This is obvious in the isolated cell case since the coverage probability increases as the distance between the base station and the mobile decreases. However, it is not immediately obvious in the handoff case since the handoff gain decreases as the mobile moves away from the cell edge toward the base station. Theorem 2 asserts that it is still true in the handoff case. Due to the reduced handoff gain in the interior of the coverage area, for a given edge coverage probability, a smaller area coverage probability should be expected for the handoff case, compared to the isolated cell case.

Remark. In Theorem 2 we allow the fading paths correlation coefficient ρ to be a function of r_1 and $r_2 = 2 - r_1$. For example, it seems reasonable to assume that this correlation coefficient gradually decreases as the mobile gets closer to one base station.

Remark. Since the integrand $J(r)$ is smooth, Theorem 2 can be easily implemented using a simple numerical quadrature such as Simpson's rule, along with spreadsheet's built-in functions NORMSDIST(.), which is the function $F_z(\cdot)$, and NORMSINV(.), which is the function $F_z^{-1}(\cdot)$.

The area coverage probability formula in Theorem 2 simplifies when we assume that the correlation coefficient ρ is a constant. In this case $\rho' = 0$ and we have

Corollary 1. (Constant ρ) Let P_e be the edge coverage probability and Γ_{P_e} be the corresponding fade margin. Assume ρ is a constant. Then the area coverage probability is

$$P_a = P_e + \int_0^1 J_1(r)\,\mathrm{d}r$$

with

$$J_1(r) = \frac{r10}{\ln 10}\left\{\begin{array}{l} \dfrac{\beta_1}{\sigma_1} f_z\left[\chi_1\left(\Gamma_{P_e}\right)\right] F_z\left[\dfrac{\rho\chi_1\left(\Gamma_{P_e}\right) - \chi_2\left(\Gamma_{P_e}\right)}{\sqrt{1-\rho^2}}\right] - \\ \dfrac{\beta_2}{\sigma_2}\dfrac{r}{2-r} f_z\left[\chi_2\left(\Gamma_{P_e}\right)\right] F_z\left[\dfrac{\rho\chi_2\left(\Gamma_{P_e}\right) - \chi_1\left(\Gamma_{P_e}\right)}{\sqrt{1-\rho^2}}\right] \end{array}\right\}$$

where $b_1(r), b_2(r), \chi_1\left(\Gamma_{P_e}\right)$ and $\chi_2\left(\Gamma_{P_e}\right)$ are defined as in Theorem 2. $\blacksquare$

Note that the case $H_h = 0$ corresponds to the soft handoff case. If the hard handoff hysteresis $H_h \to \infty$, *i.e.*, the signal from the second base station is never used, the corresponding case is the same as the isolated cell case, which is treated by Reudink's formula. In the following we see that the well-known Reudink's formula is easily obtained by letting $H_h \to \infty$ in Theorem 2.

Corollary 2. (Reudink's Formula) In the case of isolated cell, the area coverage probability is

$$P_a = P_e + e^{\frac{k\ln(10)}{5}\left[\frac{k\ln(10)}{10} + F_z^{-1}(P_e)\right]}\left[1 - F_z\left(F_z^{-1}\left(P_e\right) + \frac{k\ln(10)}{5}\right)\right]$$

where $k = \sigma / \beta$. $\blacksquare$

Remark. This formula can be found in many standard RF systems engineering references in various forms. Note that we have rewritten this formula in terms of the edge coverage P_e and the parameter $k = \sigma / \beta$. Thus we see that once the edge coverage probability P_e is given, the area coverage probability P_a depends only on the ratio σ / β, not on the individual σ and β.

The following theorem is an extension of Reudink's formula in the case when the handoff cells are in the same morphology, *i.e.*, the shadow fading standard deviations and the path loss exponents are the same for the neighboring cells.

Theorem 3. If $\sigma_1 = \sigma_2 = \sigma$ and $\beta_1 = \beta_2 = \beta$, then area coverage probability P_a depends only on P_e, ρ and the ratios $k = \sigma / \beta$ and $L_h = H_h / \sigma$, and is given by

$$P_a = P_e + \int_0^1 J_2(r)\,\mathrm{d}r \tag{2}$$

with

$$J_2(r) = \frac{r10}{k\ln 10}\left\{\begin{array}{l} f_z\left[\begin{array}{r} G^{-1}\left(P_e\right) + \\ b_{12}(r) \end{array}\right] F_z\left[\begin{array}{c} \dfrac{\rho(r) b_{12}(r) - b_{22}(r)}{\sqrt{1-\rho^2(r)}} - \\ G^{-1}\left(P_e\right)\sqrt{\dfrac{1-\rho(r)}{1+\rho(r)}} \end{array}\right] - \\ \dfrac{r}{2-r} f_z\left[\begin{array}{r} G^{-1}\left(P_e\right) + \\ b_{22}(r) \end{array}\right] F_z\left[\begin{array}{c} \dfrac{\rho(r) b_{22}(r) - b_{12}(r)}{\sqrt{1-\rho^2(r)}} - \\ G^{-1}\left(P_e\right)\sqrt{\dfrac{1-\rho(r)}{1+\rho(r)}} \end{array}\right] \end{array}\right\} + \frac{r^2 f_z\left[\omega_2(r)\right]\rho'(r)}{\sqrt{1-\rho^2(r)}} f_z\left[\sqrt{\frac{2}{1+\rho}}\left(\begin{array}{l} G^{-1}\left(P_e\right) + \\ \dfrac{b_{12}(r) + b_{22}(r)}{2} \end{array}\right)\right]$$

where

$$b_{12}(r) = -\left[10\log(r)\right] / k,$$

$$b_{22}(r) = -[10\log(2-r)]/k - L_h,$$

$$\omega_2(r) = [b_{12}(r) - b_{22}(r)]/\sqrt{2[1-\rho(r)]}$$

and G is the CDF corresponding to the PDF

$$g(x) = f_z(x)\cdot F_z\left(-\sqrt{\frac{1-\rho(r)}{1+\rho(r)}}x + \frac{L_h}{\sqrt{1-\rho^2(r)}}\right) + f_z(x-L_h)\cdot F_z\left(-\sqrt{\frac{1-\rho(r)}{1+\rho(r)}}x - \frac{\rho(r)L_h}{\sqrt{1-\rho^2(r)}}\right). \blacksquare$$

V. Examples

In this section, we apply the formulas derived in the previous section to obtain the area coverage probability. The results are compared with that obtained using Reudink's formula.

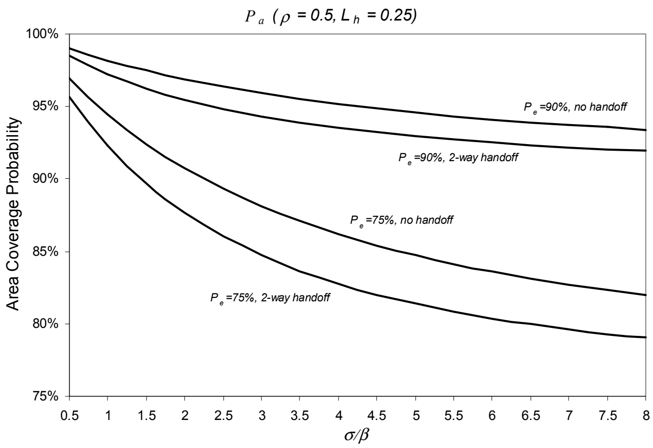

Figure 2. Comparison with Reudink's curves

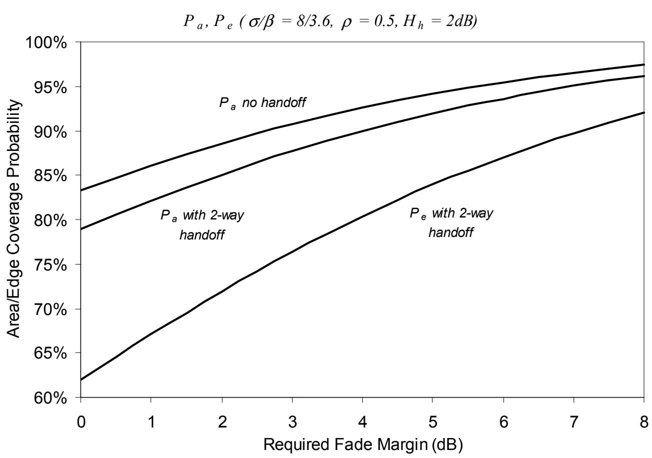

Figure 3. Fade margin vs. corresponding edge/area coverage probabilities

Figure 2 shows the comparison between the Reudink's area coverage probability curves (*i.e.*, the "no-handoff" curves), which is for the case of an isolated cell where there is no handoffs, and the area coverage probability curves using formula (2) (*i.e.*, the "2-way handoff" curves), which is for the case of embedded cells where two-way handoff is assumed. With same edge coverage probability, the area coverage probability is worse for the "2-way handoff" cases is due to the fact that the handoff gain is the greatest along the cell edge, and the gain decreases as the mobile moves closer to the host base station and further away from handoff base station. Thus in the interior of the coverage area, the handoff gain is smaller than that in the cell edge. As depicted in Figure 2, there are large gaps between the "no-handoff" and "2-way handoff" curves, which underlines the necessity and the importance of the new formula (2).

Since the area coverage probability calculated using the classical Reudink's formula is too optimistic, the resulting required fade margin is also too optimistic. Figure 3 shows that the required fade margin is more than one dB too optimistic when Reudink's formula is used. In particular, we see that corresponding to 90% edge coverage probability, the area coverage probability is 95.16% for the "2-way handoff" case while it is 96.63% for the "no-handoff" case; corresponding to 75% edge coverage, the area coverage probability is 86.90% for the "2-way handoff" case while it is 90.07% for the "no handoff" case.

VI. Conclusions

We have presented several general formulas to compute the *area* coverage probability of a cellular wireless network based on its *edge* coverage probability. The new formulas are easily implementable in a spreadsheet. The necessity of the new formulas is highlighted by their application to some typical cellular wireless network RF design scenarios to show that the existing formula underestimates the link margins by well over one dB. The generality and simplicity, plus the necessity of the new formulas should make them indispensable additions to the RF design and link budget analysis toolboxes.

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