Convex Optimization-based Beamforming in Cognitive Radio Multicast Transmission

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***Abstract*— A novel algorithm for transmit beamforming to single cochannel multicast group is presented in this paper. We consider the max-min fairness (MMF) based beamforming problem where the maximization of the smallest receiver signal-to-noise ratio (SNR) over the secondary users subject to constraints on the transmit power and interference caused to the primary users. It is shown that this problem, which is nonconvex NP-hard, can be approximated by a convex second-order cone programming (SOCP) problem. Then, an iterative algorithm which successively improves the SOCP approximation is presented. Simulation results show the superior performance of the proposed approach, together with a reduced computational complexity, as compared to the state-of-the-art approach[1].**

I. Introduction

Radio spectrum is a precious resource for wireless communications. Yet, according to the FCC most spectrum is underutilized [1]. In [2], cognitive radio (CR), a new paradigm for exploiting the spectrum resources in a dynamic way, was proposed. In CR, the primary (licensed) users (PUs) have the priority rights to access the spectrum, while the secondary (cognitive) users (SUs) can occupy the spectrum only if they do not interrupt the communication of PUs. Spectrum holes are the most obvious opportunities to be exploited by CR [3].

The utilization efficiency of the radio spectrum can be enhanced if the network of SUs operates simultaneously with network of the PUs. However, secondary spectrum usage is possible only if the SUs cause an acceptably small performance degradation to the PUs. Hence, the challenge is to construct spectrum sharing algorithms that protect PUs from excessive interference and, at the same time, ensure a meaningful quality-of-service (QoS) to the SUs. Transmit beamforming, that has been studied in the context of multiple-input multiple-output (MIMO) systems, can improve the performance of cognitive transmission of the secondary systems at the frequency band of the primary system. Thus, the interference to the primary network can be managed through the use of MIMO antenna system and beamforming. Recently, a number of beamforming techniques have been proposed for CR networks to control interference [4], [5], [6], [7], [8].

In this work, we consider the max-min fairness (MMF) based beamforming problem where the maximization of the smallest receiver signal-to-noise ratio (SNR) over the SUs subject to constraints on the transmit power and interference caused to the PUs. The original problem is approximated by a convex second-order cone programming (SOCP) problem whose solution, if it exists, is always feasible for the original problem. Then, an iterative procedure in which the convex approximation is successively improved is developed.

The remainder of this paper is structured as follows. The system model is introduced in Section II and the optimal beamforming problem is formulated. In Section III a procedure is proposed to address the beamforming problem. Section IV provides simulation results of the proposed procedure. Finally, Section V summarizes the main conclusions.

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II. Problem Formulation

Let us consider a wireless communication system where a secondary transmitter has N antennas transmiting the same information-bearing signal s to M secondary receivers. Each receiver is equipped with single antenna. A primary network of L users operates concurrently with the secondary transceivers. Let $\boldsymbol{l}_j = [l_{j1}, l_{j2}, ..., l_{jN}]^T$ be the channel gain between the cognitive transmitter and the primary receiver j, while $\boldsymbol{h}_i = [h_{i1}, h_{i2}, ..., h_{iN}]^T$ is the channel gain between the secondary transmitter and the secondary receiver i. The noise n_i at the cognitive receiver i and the noise $\overline{n}_j$ at the primary receiver j are zero-mean and white with variances σ_i^2 and $\overline{\sigma}_j^2$, respectively. It is further assumed that the channel values are known to the transmitter. The transmitted signal s is also zero-mean and white with unit variance. The symbol $\boldsymbol{x} \in \mathbb{C}^n$ is denoted by the beamforming weight vector applied to the N transmit antennas. Therefore, the received signal at the user i is

$$d_i = \boldsymbol{x}^H \boldsymbol{h}_i s + n_i, \tag{1}$$

and the interference signal on a PU j is:

$$\overline{d}_j = \boldsymbol{x}^H \boldsymbol{l}_j s + \overline{n}_j, \tag{2}$$

The SNR at the receiver i is given by

$$\text{SNR}_i = \frac{\left|\boldsymbol{x}^H \boldsymbol{h}_i\right|^2}{{\sigma_i}^2}, \tag{3}$$

and the interference caused by the secondary transmission on PU j is defined by

$$\text{INT}_j = \frac{\left|\boldsymbol{x}^H \boldsymbol{l}_j\right|^2}{\overline{\sigma}_j^2}. \tag{4}$$

The MMF based beamforming problem where the maximization of the smallest receiver SNR over the SUs subject to constraints on the transmit power and interference caused to the PUs is mathematically is formulated as follows:

$$\underset{\boldsymbol{x} \in \mathbb{C}^N}{\text{maximize}} \quad \min_{i=1,2,...,M} \text{SNR}_i \tag{5}$$

subject to

$$||\boldsymbol{x}||^2 \leq P, \tag{6}$$

$$\text{INT}_j \leq \beta_j, \; j = 1, ..., L. \tag{7}$$

For the sake of notational simplicity, the following rescaling is performed

$$\boldsymbol{h}_i \rightarrow \frac{1}{\sigma_i}\boldsymbol{h}_i, \; \boldsymbol{l}_j \rightarrow \frac{\sqrt{\beta}}{\overline{\sigma}_i\sqrt{\beta_j}}\boldsymbol{l}_j, \tag{8}$$

where

$$\beta = \max_{j=1,2,...,L} \beta_j. \tag{9}$$

Then, the optimization problem (5)–(7) is equivalently simplified

$$\underset{\boldsymbol{x} \in \mathbb{C}^N}{\text{maximize}} \quad \min_{i=1,2,...,M} \{\left|\boldsymbol{x}^H \boldsymbol{h}_i\right|^2\} \tag{10}$$

subject to

$$||\boldsymbol{x}||^2 \leq P, \tag{11}$$

$$\left|\boldsymbol{x}^H \boldsymbol{l}_j\right|^2 \leq \beta, \; j = 1, ..., L. \tag{12}$$

Introducing a new variable t, the problem (10)–(12) can be equivalently rewritten as the following optimization problem:

$$\underset{t,\, \boldsymbol{x} \in \mathbb{C}^N}{\text{maximize}} \quad t \tag{13}$$

subject to

$$\left|\boldsymbol{x}^H \boldsymbol{h}_i\right|^2 \geq t, \; i = 1, ..., M, \tag{14}$$

$$||\boldsymbol{x}||^2 \leq P, \; t \geq 0, \tag{15}$$

$$\left|\boldsymbol{x}^H \boldsymbol{l}_j\right|^2 \leq \beta, \; j = 1, ..., L. \tag{16}$$

The problem defined in (13)–(16) is a non-convex optimization problem since the constraints in (14) are nonconvex. More precisely, it belongs to the class of nonconvex quadratically constrained quadratic programming (QCQP) problems. The nonconvex QCQP is NP-hard [9]. Consequently, a QCQP is generally difficult to be solved.

The standard way to address the problem (13)–(16) is using semidefinite programming (SDP) [7], [9]. By the variable change $\boldsymbol{X} = \boldsymbol{x}\boldsymbol{x}^H$, the problem defined in (13)–(16) is in fact the following non-convex rank-1 constrained optimization problem

$$\underset{t, \boldsymbol{x} \in \mathbb{C}^N, \boldsymbol{X} \in \mathbb{C}^{N \times N}}{\text{maximize}} \quad t \tag{17}$$

subject to

$$\text{tr}\left(\boldsymbol{X}\boldsymbol{h}_i\boldsymbol{h}_i^H\right) \geq t, \; i = 1, ..., M, \tag{18}$$

$$\text{tr}\left(\boldsymbol{X}\right) \leq P, \;\; t \geq 0, \tag{19}$$

$$\text{tr}\left(\boldsymbol{X}\boldsymbol{l}_j\boldsymbol{l}_j^H\right) \leq \beta, \; j = 1, ..., L, \tag{20}$$

$$\boldsymbol{X} = \boldsymbol{x}\boldsymbol{x}^H. \tag{21}$$

The problem (17)–(21) is a difficult optimization problem because of the nonconvex rank-1 constraint (21).

By dropping the constraint (21), the problem (17)–(21) is relaxed to the following problem

$$\underset{t,\boldsymbol{X}\in\mathbb{C}^{N\times N}}{\text{maximize}} \quad t \tag{22}$$

subject to

$$\operatorname{tr}\left(\boldsymbol{X}\boldsymbol{h}_i\boldsymbol{h}_i^H\right)\geq t,\ i=1,...,M, \tag{23}$$

$$\operatorname{tr}\left(\boldsymbol{X}\right)\leq P,\ \ t\geq 0, \tag{24}$$

$$\operatorname{tr}\left(\boldsymbol{X}\boldsymbol{l}_j\boldsymbol{l}_j^H\right)\leq \beta,\ j=1,...,L, \tag{25}$$

Clearly, the problem (22)–(25) falls into the category of convex programming. More precisely, it is an SDP. To solve it, the package CVX for specifying and solving convex programs is used [11]. When the solution $\widehat{\boldsymbol{X}}$ of (22)–(25) is of rank one, then the optimal values of the objective functions in (13), (17) and (22) are equal. More precisely, when $\widehat{\boldsymbol{X}}=\widehat{\boldsymbol{x}}\widehat{\boldsymbol{x}}^H$, then $\widehat{\boldsymbol{x}}$ is the optimal solution of (13)–(16). However, if rank $\left(\widehat{\boldsymbol{X}}\right)>1$, the optimal value of (22)–(25) is just a lower bound to that of (13)–(16). After solving the convex optimization problem in (22)–(25), a randomization technique is implemented according to the procedure described in [7, subsection VI.D] to generate feasible solutions of (13)–(16) from the optimal solution $\widehat{\boldsymbol{X}}$.

In the next Section, a simple iterative method for beamforming in the scenario of single group multicast will be presented. As will be shown, this proposed method demonstrates better performance at a reduced computational complexity as compared with the SDP-based approach (22)–(25).

III. PROPOSED METHOD

The newly proposed method iterates a sequence of quadratic programming problems under linear inequality constraints and convex quadratic inequality constraints. Initializing from a feasible solution $\boldsymbol{x}_{(0)}$ of (13)–(16), a new feasible solution is found from the following program:

$$\underset{t,\,\boldsymbol{x}\in\mathbb{C}^{N}}{\text{maximize}} \quad t \tag{26}$$

subject to

$$\left|\boldsymbol{x}_{(0)}^H\boldsymbol{h}_i\right|^2+2\Re\left(\boldsymbol{x}_{(0)}^H\boldsymbol{h}_i\boldsymbol{h}_i^H\Delta\boldsymbol{x}_{(0)}\right)\geq t,\ i=1,...,M, \tag{27}$$

$$\left|\left|\boldsymbol{x}\right|\right|^2\leq P,\ t\geq 0, \tag{28}$$

$$\left|\boldsymbol{x}^H\boldsymbol{l}_j\right|^2\leq \beta,\ j=1,...,L, \tag{29}$$

where $\Delta\boldsymbol{x}_{(0)}=\boldsymbol{x}-\boldsymbol{x}_{(0)}$. The convex problem in (26)–(29) is formed by linearizing all the nonconvex constraints in (14) around the original feasible point $\boldsymbol{x}_{(0)}$. It is an SOCP problem and can be solved by, e.g. CVX [11], to produce a new feasible point $\boldsymbol{x}_{(1)}$ with a lower objective value. It is important to note that for each i, the left hand side of the constraint (27) is an affine lower bound on the function $\left|\boldsymbol{x}^H\boldsymbol{h}_i\right|^2$. This means that the feasible set of the problem (26)– (29) is a convex subset of the original nonconvex feasible set. By linearizing the original nonconvex constraints, a set of convex constraints that are tighter than the original ones is obtained. If one linearizes again the problem around $\boldsymbol{x}_{(1)}$ and repeat the procedure, a sequence of feasible points with decreasing objective values is generated. To prove this claim, let $\boldsymbol{x}_{(1)}$ be the solution of (26)– (29). Combining the obvious inequality $\left(\boldsymbol{x}_{(1)}-\boldsymbol{x}_{(0)}\right)^H\boldsymbol{h}_i\boldsymbol{h}_i^H\left(\boldsymbol{x}_{(1)}-\boldsymbol{x}_{(0)}\right)\geq 0$ with (27), when $\boldsymbol{x}=\boldsymbol{x}_{(1)}$, it is straightforward to see that $\left|\boldsymbol{x}_{(1)}^H\boldsymbol{h}_i\right|^2\geq t$, i.e., $\boldsymbol{x}_{(1)}$ is feasible for the original problem (13)–(16). Next, remark that $\boldsymbol{x}_{(0)}$ is feasible for (26)– (29). This implies that $\left|\left|\boldsymbol{x}_{(1)}\right|\right|^2\leq\left|\left|\boldsymbol{x}_{(0)}\right|\right|^2$. Note, however, that there is no guarantee that the generated sequence will have strictly decreasing objective values. The algorithm stops when $||\boldsymbol{x}_{(k)}-\boldsymbol{x}_{(k+1)}||<0.01$ for some k. As a final remark, note that the N-dimensional zero vector represents a feasible solution of (13)–(16); hence the problem of finding a starting feasible point is easily resolved.

This reformulation/linearization-based method (26)–(29) is also known as the convex-concave procedure (CCP) [9]. To the best of the author's knowledge, the MMF based beamforming problem (13)–(16) has not been previously addressed using the CCP. The simulation results in Section IV will assess its effectiveness.

In the multicast beamforming problem (13)–(16), the beamformer was designed under the assumption that full CSI is available at the secondary transmitter. In practice, only partial CSI is available at the design center, due to errors in the estimation of the channels vectors. Hence, similar to the approach in [7], the perfect CSI assumption is relaxed and robust design for the MMF based beamforming problem is presented. The perturbation $\boldsymbol{\delta}$ is modeled as a deterministic one with bounded norm, leading to a worst cast optimization. In the sequel, it will be shown that the CCP approach can also be applied to the MMF based beamforming with imperfect CSI.

The robust counterpart of the MMF based beamforming problem (13)–(16) can be written as

$$\underset{t,\,\boldsymbol{x}\in\mathbb{C}^{N}}{\text{maximize}} \quad t \tag{30}$$

subject to

$$\boldsymbol{x}^H \boldsymbol{H}_i \boldsymbol{x} \geq t,\ i = 1, ..., M, \tag{31}$$

$$||\boldsymbol{x}||^2 \leq P, t \geq 0 \tag{32}$$

$$\boldsymbol{x}^H \boldsymbol{G}_j \boldsymbol{x} \leq \beta,\ j = 1, ..., L, \tag{33}$$

$\boldsymbol{H}_i = \boldsymbol{h}_i \boldsymbol{h}_i^H + \epsilon^2 \boldsymbol{I}_N - 2\epsilon \sqrt{\boldsymbol{h}_i^h \boldsymbol{h}_i} \boldsymbol{I}_N,\ i = 1, ..., M,$, $\boldsymbol{G}_j = \boldsymbol{l}_j \boldsymbol{l}_j^H + \epsilon^2 \boldsymbol{I}_N + 2\epsilon \sqrt{\boldsymbol{l}_j^h \boldsymbol{l}_j} \boldsymbol{I}_N,\ j = 1, ..., L,$ and ϵ denotes the upper-bound of channel error vector norms. See [7] for more details.

The CCP involves breaking up each constraint function in (31) into a sum of one concave and one convex function. Then each iteration of the CCP procedure approximates the concave part by its tangent and solves the resulting convex optimization problem. Let $\boldsymbol{x}_{(0)}$ be a feasible starting point. Then, $\boldsymbol{x}_{(1)}$ is obtained as the solution of the following convex optimization problem

$$\underset{t, \boldsymbol{x} \in \mathbb{C}^N}{\text{maximize}} \quad t \tag{34}$$

subject to

$$\boldsymbol{x}^H \boldsymbol{H}_i^+ \boldsymbol{x} + t \leq \boldsymbol{x}_{(0)}^H \boldsymbol{H}_i^- \boldsymbol{x}_{(0)} + 2\Re\left(\boldsymbol{x}_{(0)}^H \boldsymbol{H}_i^- \Delta \boldsymbol{x}_{(0)}\right), \forall i, \tag{35}$$

$$||\boldsymbol{x}||^2 \leq P, t \geq 0 \tag{36}$$

$$\boldsymbol{x}^H \boldsymbol{G}_j \boldsymbol{x} \leq \beta,\ \forall j, \tag{37}$$

where $\boldsymbol{H}_i^+ = 2\epsilon \sqrt{\boldsymbol{h}_i^H \boldsymbol{h}_i} \boldsymbol{I}_N$ and $\boldsymbol{H}_i^- = \boldsymbol{h}_i \boldsymbol{h}_i^H + \epsilon^2 \boldsymbol{I}_N$. The problem (34)–(37) is a convex minimization problem and can be solved using classical and efficient convex algorithms, such as CVX [11]. (Note that the matrices $\boldsymbol{G}_j$, $j = 1, ..., L,$ are positive semidefinite). As a final remark, it should be noted that for $\epsilon = 0$ the problem (34)–(37) is identical to the one in (13)–(16).

IV. Performance Results

Extensive simulations have been performed to compare the performance of the proposed method in Section III with the existing beamforming algorithm. In all simulations presented here, the number of Monte Carlo runs is $M_c = 1000$, the number of transmit antennas is $N = 8$, the number of primary users is $L = 2$ and the number of secondary users is $M = 16$. The entries of the channel vectors $\boldsymbol{h}_i$, $i = 1, ..., M$, and $\boldsymbol{l}_j$, $j = 1, ..., L$, are independent identically distributed circularly symmetric Gaussian random variables with zero mean and unit variance. Both the perfect and imperfect CSI cases are investigated. In the latter case, as in [7], the error vector $\boldsymbol{\delta}$ is randomly and uniformly generated in a sphere centered at the origin with the radius $\epsilon = 0.01$.

The performance of the newly presented method in Section III will be compared with the performance of the SDP-based approach (22)–(25), denoted here by "SDP". The solid-circle and the solid-plus line represent the performance of the proposed method for the case of perfect and imperfect CSI, respectively, and $T = 1$, where T is the number of trials performed to obtain the most suitable beamforming vector. The dashed-circle and the dashed-plus line represent the performance of the SDP-based approach for the case of perfect and imperfect CSI, respectively, whereas the solid line represents the SDP upper bound for the perfect CSI case. Note that the solution of the SDP relaxation provides an upper bound on possible performance. The number of the samples generated in the procedure of randomization is $K = 2000$.

A. MMF Based Beamforming

The scenario which corresponds to the case of fixed transmit power, P, and varying interference threshold, β, is considered. Figures 1 and 2 display the SNR of the worst user versus the interference threshold for $P = 15$ and $P = 5$, respectively. Figures reveal that the performance of the worst user increases provided that the interference threshold also increases. This holds true for both the new and SDP method. This conclusion is intuitively appealing since when the acceptable interference level is higher, the feasible set over which a solution of the corresponding optimization problem is searched for is larger. More importantly, it is evident that the new approach outperforms significantly the SDP-based approach. To illustrate this fact, consider $\beta =$ 0 dB. From figures 1 and 2, one can see that the new method demonstrates a gain of 1.5 dB and 2 dB, respectively, when compared with the SDP method. It should also be noted that, as expected, when the exact CSI is unavailable, the performance of the beamforming techniques degrades significantly.

B. Complexity analysis

Apart from performance, the comparison of complexity between the new method and the existing approach is also important. Without loss of generality, the perfect CSI case is analyzed, as all the conclusions presented below also hold for the case of imperfect CSI. The complexity of solving the SOCP problem (13)–(16) is $O\left((M + L + 2)^{1/2}\left((N + 2)^2 + 9L + 4M + 1\right)\right)$. See [12] for more details. This complexity scales linearly with the number of iterations I. The simulations have shown that a small number of iterations (e.g. $I = 10$) is sufficient to ensure the convergence of the algorithm.

The complexity of solving the SDP (22)–(25) is equal to

$$O\left(n_{sdp}^{1/2}\left(m_{sdp}n_{sdp}^{3}+m_{sdp}^{2}n_{sdp}^{2}+m_{sdp}^{3}\right)\right),$$

where $n_{sdp} = N + M + L + 2$ and $m_{sdp} = M + L + 1$. See [12]–[14] for more details. The overall complexity of the SDP-based approach is that of a single SDP problem along with the complexity of the randomization step. This is clearly higher than the complexity of the new method. Consequently, one can conclude that the computational complexity of the new method is smaller compared to that of the state-of-the-art approach.

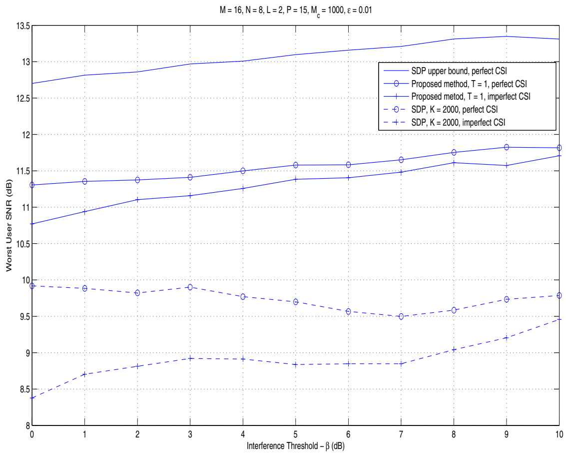

Fig. 1. Max-min fairness based beamforming: worst user SNR versus interference threshold for $M = 16$, $N = 8$, $L = 2$, $P = 15$, $M_c = 1000$ and $\epsilon = 0.01$.

V. Conclusions

The MMF based beamforming problem was revisited in this work. It is common to use the semi-definite relaxation technique to approximate the problem by an SDP problem. By doing this, the non-convex feasible set of the problem is approximated by a convex set which contains the original set as a subset. To generate a feasible solution from the solution of the SDP, the Gaussian randomization technique [7] is applied.

The feasible set was restricted to a subset which is a convex approximation of the original set. This approximation was then iteratively refined around the current solution. The simulation results have confirmed that the newly proposed method outperforms significantly the state-of-the-art method.

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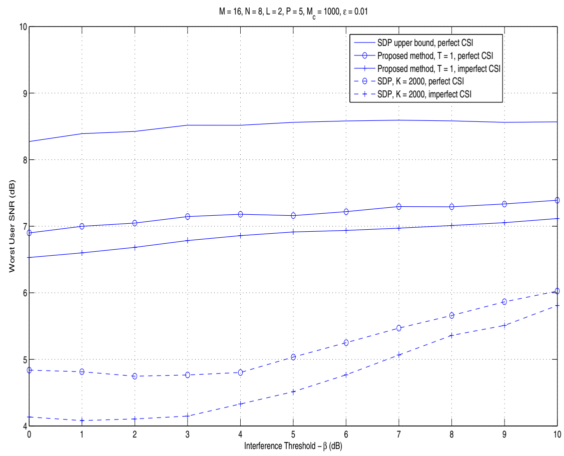

Fig. 2. MMF based beamforming: worst user SNR versus interference threshold for $M = 16$, $N = 8$, $L = 2$, $P = 5$, $M_c = 1000$ and $\epsilon = 0.01$.

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