Novel Robust Adaptive Beamforming

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***Abstract*—Diagonal loading (DL) is one of the widely used techniques against the errors due to steering vector mismatch and finite sample effect. Recently, a variable loading (VL) has shown its advantages over the DL due to using different loading for each eigenvalue of the correlation matrix of the received array data rather than a fixed loading for all of the eigenvalues as the DL. In this paper, a novel robust beamforming method is presented. As compared to the DL and the VL, the weight vector of the proposed method is with a more general form of diagonal loaded correlation matrix. Simulation results show that the proposed method is more robust than the DL and the VL against the errors as mentioned above.**

***Index Terms*—Adaptive beamforming, diagonal loading, variable loading.**

I. Introduction

An adaptive array beamformer is designed for preserving the desired signal and suppressing the interference and noise. For the well-known Capon beamformer or minimum power distortionless response (MPDR) beamformer, the adaptive weights are calculated by minimizing the beamformer's output power subject to the main-beam constraint [1], [2]. Without the errors due to the steering vector mismatch and finite sample effect, the solution to the constrained minimization problem is the optimal one that maximizes the array output signal-to-interference-plus-noise ratio (SINR). However, the adaptive beamformer suffers from performance degradation in the presence of error due to finite sample effect [3], [4], or steering vector mismatch [5], [6].

Diagonal loading (DL) is one of the widely used techniques against the errors due to steering vector mismatch and finite sample effect [3], [7]-[10]. Although the DL is effective, the required loading factor may make the weight vector of the DL deviate from the optimal one [11]. To have a smaller bias, an approach called the variable loading (VL) has been considered in [11], [12], and has shown its advantages over the DL due to using the variable loading, where different loading factor is added to each eigenvalue of the correlation matrix instead of a fixed loading factor for all of the eigenvalues. In this paper, we propose a novel robust adaptive beamforming approach. We formulate an optimization problem of finding an optimal weight vector by imposing a more general form of weight vector norm constraint to the constrained minimization problem. Solving the optimization problem yields a weight vector with a more general form of diagonal loaded correlation matrix. As the DL, the proposed method also needs to determine the loading factor. In the literature, the required loading factor for the DL approaches, namely γ, is chosen according to some optimality criteria [3], [7]-[10]. There seems not a easy task for choosing the loading factor of the proposed method. However, we can follow the concept of [11] or [12] to set the required loading factor of the proposed method as γ^p. Then, it is easy to show that the DL and the VL are two special cases of the proposed method with $p = 1$ and $p = 2$, respectively. Finally, we provide serval simulations to show that the proposed method is more robust to steering vector mismatch and finite sample effect than the aforementioned methods.

II. Problem Formulation

Consider that there are K far-field signal sources including a desired signal and $K-1$ interferers impinging on an M-element antenna array. The received data vector $\mathbf{x}(t)$ can be expressed as

$$\mathbf{x}(t) = \mathbf{a}_0 s_0(t) + \sum_{k=1}^{K-1} \mathbf{a}_k s_k(t) + \mathbf{n}(t) \tag{1}$$

where $\mathbf{x}(t) = [x_1(t)\ x_2(t)\ \cdots\ x_M(t)]^T \in \mathbb{C}^{M\times 1}$, $x_m(t)$, $m = 1, 2, \ldots, M$, is the output of the mth antenna element; $s_k(t)$ denotes the kth signal with zero mean and variance σ_k^2; $\mathbf{a}_k = [a_1(\theta_k)\ a_2(\theta_k)\ \cdots\ a_M(\theta_k)]^T \in \mathbb{C}^{M\times 1}$ represents the $M \times 1$ steering vector from angle θ_k off array broadside; $\mathbf{n}(t) \in \mathbb{C}^{M\times 1}$ is an additive white Gaussian noise vector with zero mean and covariance matrix $\sigma_n^2\mathbf{I}$, where $\mathbf{I}$ is the identity matrix with an appropriate size. For the Capon beamformer or MPDR beamformer, the optimal weight vector is obtained by minimizing the array output power subject to the main-beam constraint [1], [2]

$$\min_{\mathbf{w}} \mathbf{w}^H \mathbf{R}_{xx} \mathbf{w} \text{ subject to } \mathbf{w}^H \mathbf{a}_0 = 1. \tag{2}$$

The solution to (2) is given by

$$\mathbf{w}_{\mathrm{o}} = \frac{\mathbf{R}_{xx}^{-1}\mathbf{a}_0}{\mathbf{a}_0^H \mathbf{R}_{xx}^{-1}\mathbf{a}_0} \tag{3}$$

where $\mathbf{R}_{xx} = E\{\mathbf{x}(t)\mathbf{x}^H(t)\}$ is the ensemble correlation matrix of $\mathbf{x}(t)$. In practice, $\mathbf{R}_{xx}$ is unavailable and the knowledge of $\mathbf{a}_0$ may be inaccurate. A sample matrix inversion

(SMI) approach is commonly used to solve the constrained minimization problem of (2) by using a sample correlation matrix instead of the ensemble one. Under the actual steering vector $\mathbf{a}$, the solution of (2) can be expressed as

$$\mathbf{w}_{\text{smi}} = \frac{\hat{\mathbf{R}}_{xx}^{-1}\mathbf{a}}{\mathbf{a}^H\hat{\mathbf{R}}_{xx}^{-1}\mathbf{a}} \tag{4}$$

where $\mathbf{a} \neq \mathbf{a}_0$ due to steering vector mismatch. The sample correlation matrix $\hat{\mathbf{R}}_{xx}$ is computed from the received data vector $\mathbf{x}(t)$ as follows:

$$\hat{\mathbf{R}}_{xx} = \frac{1}{N}\sum_{n=1}^{N}\mathbf{x}(t_n)\mathbf{x}^H(t_n) \tag{5}$$

where N denotes the number of data snapshots and t_n the nth time instant. Without the steering vector mismatch, $\hat{\mathbf{R}}_{xx}$ converges to $\mathbf{R}_{xx}$ and $\mathbf{w}_{\text{smi}}$ approaches $\mathbf{w}_{\text{o}}$ as N increases. However, the performance degradation of the SMI becomes significantly in the presence of the steering vector mismatch and finite sample effect [3]-[6].

III. Robust Methods

A. Diagonal Loading and Variable Loading

To improve the robustness of the SMI, one of the most well-known techniques is diagonal loading (DL) [3], [7]-[10]. The weight vector of the DL methods can be obtained by adding a scaled identity matrix to $\hat{\mathbf{R}}_{xx}$ of (4) as

$$\mathbf{w}_{\text{dl}} = \frac{(\hat{\mathbf{R}}_{xx}+\gamma\mathbf{I})^{-1}\mathbf{a}}{\mathbf{a}^H(\hat{\mathbf{R}}_{xx}+\gamma\mathbf{I})^{-1}\mathbf{a}} \tag{6}$$

where γ denotes the loading factor which should be appropriately determined. We note that the DL uses $(\hat{\mathbf{R}}_{xx}+\gamma\mathbf{I})^{-1}$ instead of $\hat{\mathbf{R}}_{xx}^{-1}$ to overcome the inherent drawback of the SMI when $N < M$. Under this situation, $\hat{\mathbf{R}}_{xx}$ is not full rank and thus is not invertible. However, there is always a tradeoff between the robustness and interference cancellation and noi0se reduction when using γ. For a small γ, the robustness will diminish. Whereas, a large γ means increased robustness, the capabilities of interference cancellation and noise reduction will decrease. Moreover, the large γ will make the weight vector of the DL deviate from the optimal solution shown by (3) as N approaches infinity. Recently, a method called the variable loading (VL) has been proposed by [11] or [12] in order to have a smaller bias. The VL uses $(\hat{\mathbf{R}}_{xx}^2+\delta\mathbf{I})^{-1}\hat{\mathbf{R}}_{xx}$ instead of $(\hat{\mathbf{R}}_{xx}+\gamma\mathbf{I})^{-1}$ and $\hat{\mathbf{R}}_{xx}^{-1}$ to obtain the weight vector as follows:

$$\mathbf{w}_{\text{vl}} = \frac{(\hat{\mathbf{R}}_{xx}^2+\delta\mathbf{I})^{-1}\hat{\mathbf{R}}_{xx}\mathbf{a}}{\mathbf{a}^H(\hat{\mathbf{R}}_{xx}^2+\delta\mathbf{I})^{-1}\hat{\mathbf{R}}_{xx}\mathbf{a}}. \tag{7}$$

It has been shown that the VL can achieve better performance than the DL due to using a variable loading, where different loading factors are added to each eigenvalue of $\hat{\mathbf{R}}_{xx}$.

B. Proposed Method

The weight vector of the DL shown by (6) can be obtained by imposing an additional constraint on the Euclidean norm of the weight vector in (2), e.g., $\|\mathbf{w}\|^2 \leq T_1$ with T_1 an user parameter. This constraint can be seen as a means of specifying performance under the assumption of white noise [12]. To generalize the assumption, the VL replaces the norm constraint $\|\mathbf{w}\|^2 \leq T_1$ with a quadratic inequality constraint $\mathbf{w}^H\mathbf{R}_{xx}^{-1}\mathbf{w} \leq T_2$. Here we use a more general method to constrain the norm of the weight vector according to $\mathbf{w}^H\mathbf{R}_{xx}^{1-p}\mathbf{w} \leq T_p$, where p is a positive integer. The corresponding optimization problem can be expressed as follows:

$$\begin{aligned}\min_{\mathbf{w}} \mathbf{w}^H\hat{\mathbf{R}}_{xx}\mathbf{w} \text{ subject to } \mathbf{w}^H\mathbf{a} = 1 \\ \text{and } \mathbf{w}^H\hat{\mathbf{R}}_{xx}^{1-p}\mathbf{w} \leq T_p.\end{aligned} \tag{8}$$

According to (8), a cost function is constructed as follows:

$$\begin{aligned}J(\mathbf{w}) = \mathbf{w}^H\hat{\mathbf{R}}_{xx}\mathbf{w} + \lambda(\mathbf{w}^H\mathbf{a}+\mathbf{a}^H\mathbf{w}-2) \\ + \kappa(\mathbf{w}^H\hat{\mathbf{R}}_{xx}^{1-p}\mathbf{w} - T_p)\end{aligned} \tag{9}$$

where we have used an equality constraint for simplicity. λ and κ are the Lagrange multipliers. Differentiating $J(\mathbf{w})$ with respect to $\mathbf{w}^H$ and setting the result to zero yields

$$\begin{aligned}\mathbf{w}_{\text{pr}} &= -\lambda(\hat{\mathbf{R}}_{xx}+\kappa\hat{\mathbf{R}}_{xx}^{1-p})^{-1}\mathbf{a} \\ &= -\lambda(\hat{\mathbf{R}}_{xx}^{p}+\kappa\mathbf{I})^{-1}\hat{\mathbf{R}}_{xx}^{p-1}\mathbf{a}.\end{aligned} \tag{10}$$

Substituting (10) into the first constraint of (8) gives

$$\lambda = -\frac{1}{\mathbf{a}^H(\hat{\mathbf{R}}_{xx}^{p}+\kappa\mathbf{I})^{-1}\hat{\mathbf{R}}_{xx}^{p-1}\mathbf{a}}. \tag{11}$$

Substituting (11) into (10) yields

$$\mathbf{w}_{\text{pr}} = \frac{(\hat{\mathbf{R}}_{xx}^{p}+\kappa\mathbf{I})^{-1}\hat{\mathbf{R}}_{xx}^{p-1}\mathbf{a}}{\mathbf{a}^H(\hat{\mathbf{R}}_{xx}^{p}+\kappa\mathbf{I})^{-1}\hat{\mathbf{R}}_{xx}^{p-1}\mathbf{a}}. \tag{12}$$

We observe that the weight vector of the proposed method $\mathbf{w}_{\text{pr}}$ is with a more general form of diagonal loaded correlation matrix. To see the relations among the aforementioned methods and the proposed method, we rewrite the weight vectors of these methods as follows:

$$\mathbf{w}_{\text{smi}} = \mu_{\text{smi}}\hat{\mathbf{R}}_{xx}^{-1}\mathbf{a} \tag{13}$$

$$\mathbf{w}_{\text{dl}} = \mu_{\text{dl}}(\mathbf{I}+\gamma\hat{\mathbf{R}}_{xx}^{-1})^{-1}\hat{\mathbf{R}}_{xx}^{-1}\mathbf{a} \tag{14}$$

$$\mathbf{w}_{\text{vl}} = \mu_{\text{vl}}(\mathbf{I}+\delta\hat{\mathbf{R}}_{xx}^{-2})^{-1}\hat{\mathbf{R}}_{xx}^{-1}\mathbf{a} \tag{15}$$

$$\mathbf{w}_{\text{pr}} = \mu_{\text{pr}}(\mathbf{I}+\kappa\hat{\mathbf{R}}_{xx}^{-p})^{-1}\hat{\mathbf{R}}_{xx}^{-1}\mathbf{a} \tag{16}$$

where $\mu_{\text{smi}} = (\mathbf{a}^H\hat{\mathbf{R}}_{xx}^{-1}\mathbf{a})^{-1}$, $\mu_{\text{dl}} = [\mathbf{a}^H(\hat{\mathbf{R}}_{xx}+\gamma\mathbf{I})^{-1}\mathbf{a}]^{-1}$, $\mu_{\text{vl}} = [\mathbf{a}^H(\hat{\mathbf{R}}_{xx}^2+\delta\mathbf{I})^{-1}\hat{\mathbf{R}}_{xx}\mathbf{a}]^{-1}$, and $\mu_{\text{pr}} = [\mathbf{a}^H(\hat{\mathbf{R}}_{xx}^{p}+\kappa\mathbf{I})^{-1}\hat{\mathbf{R}}_{xx}^{p-1}\mathbf{a}]^{-1}$. Obviously, we can have

$$\mathbf{w}_{\text{pr}} = \begin{cases} \mu_{\text{smi}}\hat{\mathbf{R}}_{xx}^{-1}\mathbf{a}, & \text{when } p=0 \\ \mu_{\text{dl}}(\mathbf{I}+\gamma\hat{\mathbf{R}}_{xx}^{-1})^{-1}\hat{\mathbf{R}}_{xx}^{-1}\mathbf{a}, & \text{when } p=1 \\ \mu_{\text{vl}}(\mathbf{I}+\delta\hat{\mathbf{R}}_{xx}^{-2})^{-1}\hat{\mathbf{R}}_{xx}^{-1}\mathbf{a}, & \text{when } p=2. \end{cases} \tag{17}$$

Accordingly, the proposed method is the same as aforementioned methods including the SMI when $p=0$, the DL when $p=1$ and κ is replaced by γ, and the VL when $p=2$ and κ

is replaced by δ. Hence, the proposed method can be seen as a generalized approach of those methods. Moreover, let $\hat{\mathbf{R}}_{xx}$ be decomposed as

$$\hat{\mathbf{R}}_{xx} = \sum_{i=1}^{M} \hat{\lambda}_i \hat{\mathbf{e}}_i \hat{\mathbf{e}}_i^H \tag{18}$$

where $\hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \ldots \geq \hat{\lambda}_M$ are the eigenvalues, $\hat{\mathbf{e}}_i$, $i = 1, 2, \ldots, M$, are the corresponding eigenvectors. Based on (12) and (18), $\mathbf{w}_{\mathrm{pr}}$ can be rewritten as

$$\mathbf{w}_{\mathrm{pr}} = \mu_{\mathrm{pr}} \sum_{i=1}^{M} \frac{\hat{\mathbf{e}}_i^H \mathbf{a}}{\hat{\lambda}_i + \kappa/\hat{\lambda}_i^{p-1}} \hat{\mathbf{e}}_i. \tag{19}$$

We can see from (19) that the loaded eigenvalue $(\hat{\lambda}_i+\kappa/\hat{\lambda}_i^{p-1})$ will be significantly large when $\hat{\lambda}_i$ is small and p is large enough. As $\hat{\lambda}_i$ increases, the effect of using κ will be diminished. This indicates that the loaded eigenvalues of the proposed method, unlike the DL, are different according to $\hat{\lambda}_i$, where small loading for the significant eigenvalues and large loading for the least significant ones. On the other hand, from (19), one still needs to determine κ and the optimal choice of κ seems improbable. However, we can follow the concept of [11] or [12] to choose κ. In [11], δ of the VL is chosen as $\delta = \gamma^2$ so that the VL retains the same threshold as the DL. Following this concept, it is reasonable to set $\kappa = \gamma^p$ for the proposed method. Then, the proposed also retains the same threshold. To see this, when $\gamma = \hat{\lambda}_i$, there is 3 dB difference between $1/\hat{\lambda}_i$ and $1/(\hat{\lambda}_i + \gamma)$. Meanwhile, by setting $\kappa = \gamma^p$, there is also 3 dB difference between $1/\hat{\lambda}_i$ and $1/(\hat{\lambda}_i+\kappa/\hat{\lambda}_i^{p-1})$ when $\gamma = \hat{\lambda}_i$. As a result, (12) can be rewritten as

$$\begin{aligned} \mathbf{w}_{\mathrm{pr}} &= \mu_{\mathrm{pr}} (\hat{\mathbf{R}}_{xx}^p + \gamma^p \mathbf{I})^{-1} \hat{\mathbf{R}}_{xx}^{p-1} \mathbf{a} \\ &= \mu_{\mathrm{pr}} \sum_{i=1}^{M} \frac{\hat{\mathbf{e}}_i^H \mathbf{a}}{\hat{\lambda}_i [1 + (\gamma/\hat{\lambda}_i)^p]} \hat{\mathbf{e}}_i. \end{aligned} \tag{20}$$

If we take γ as a threshold to separate the significant eigenvalues and the least significant eigenvalues, we let γ be chosen as follows:

$$\hat{\lambda}_K > \gamma > \hat{\lambda}_{K+1}. \tag{21}$$

Then, from (20), we have

$$\begin{aligned} &\lim_{p\to\infty} (\frac{\gamma}{\hat{\lambda}_i})^p \to 0, \text{ for } i = 1, 2, \ldots, K, \\ &\lim_{p\to\infty} (\frac{\gamma}{\hat{\lambda}_i})^p \to \infty, \text{ for } i = K+1, \ldots, M. \end{aligned} \tag{22}$$

This implies that $\mathbf{w}_{\mathrm{pr}}$ of (20) is proportional to

$$\sum_{i=1}^{K} \frac{\hat{\mathbf{e}}_i^H \mathbf{a}}{\hat{\lambda}_i} \hat{\mathbf{e}}_i \tag{23}$$

when choosing γ as (21) and an infinite p. We note that (23) is proportional to the solution of the eigenspace-based beamformer (ESB) [6], [13]. The ESB has been widely realized as one of the most powerful robust methods against steering vector mismatch [6] and finite sample effect [13].

IV. Simulation Examples

In this section, we present several simulation examples by using the SMI [2], the DL [7], the VL [11], and the proposed method shown by (20) for comparison. For all simulation examples, we use an uniform linear array of $M = 10$ with inter-element spacing equal to half-wavelength. The desired signal and interference are binary phase-shift-keying (BPSK) signals. The desired signal with signal-to-noise ratio (SNR) equal to 10 dB impinges on the array from 0^0 off array broadside. Two interferers with interference-to-noise ratio (INR) equal to 20 dB impinge on the array from -40^0 and 40^0 off array broadside, respectively. The loading-to-noise ratio γ/σ_n^2 is set to 5 dB and the noise variance σ_n^2 is equal to one. The number $N = 500$ of snapshots is used for obtaining the figures 1, 3-5, 7, 8. Moreover, all the simulation results are obtained by averaging 100 independent runs.

Example 1: Here, a scenario with random steering vector mismatch is considered. For this scenario, the actual steering vector $\mathbf{a}$ is defined as

$$\mathbf{a} = \mathbf{a}_0 + \sigma_e \boldsymbol{\Delta} \tag{24}$$

where $\sigma_e \boldsymbol{\Delta}$ is assumed to be the random error vector with zero mean and variance $\sigma_e^2 \mathbf{I}$. For choosing an appropriate p in (20), we present the output SINR versus the value of p when $\sigma_e^2 = 0.2$. We observe from Fig. 1 that the performance of using (20) improves as p increases and achieves a steady state when p is larger than 5. Hence, we use $p = 5$ for the following simulations. Fig. 2 depicts the output SINR versus the number of data snapshots N when $\sigma_e^2 = 0.2$. We observe from the figure that the SMI has the worst performance due to the random steering vector mismatch. The VL achieves better performance than the DL due to using variable loading. However, it dose not perform well enough. In contrast, the proposed method can effectively alleviate the degradation due to the random steering vector mismatch and provides the best performance. Furthermore, the corresponding beampatterns of the aforementioned methods are depicted in Fig. 3. We can see from Fig. 3 that the proposed method is capable of preserving the desired signal and suppressing the interference. In contrast, the other methods work unacceptable. Moreover, the output SINR against σ_e^2 is plotted in Fig. 4. We observe that the performance degradation of each method is more pronounced as σ_e^2 increases. However, the proposed method is more robust against the random steering vector mismatch.

Example 2: In this example, we consider a scenario with nonrandom steering vector mismatch due to look direction mismatch. For this scenario, $\mathbf{a}$ is defined to be

$$\mathbf{a} = \mathbf{a}_0(\theta_0 + \theta_e) \tag{25}$$

where θ_0 is the actual direction of arrival (DOA) of the desired signal and θ_e represents the look direction mismatch. We first observe from Fig. 5 that $p = 5$ is good enough to overcome the look direction mismatch when $\theta_e = -2^0$. Then, $p = 5$ is used for the following simulations. Figures 6 and 7 show the output SINR versus N and the beampatterns of the aforementioned

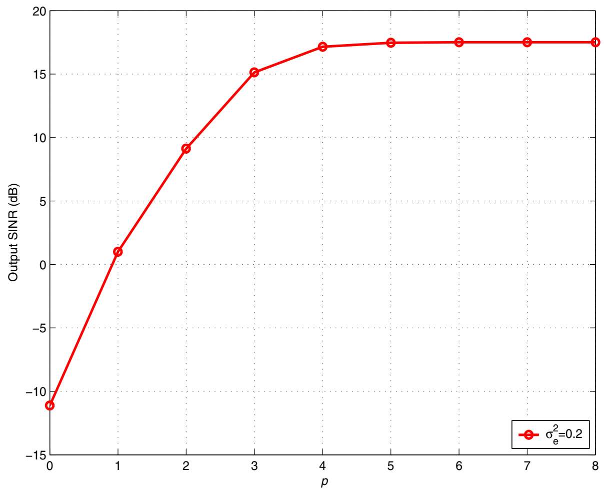

Fig. 1. The output SINR versus p when $\sigma_e^2 = 0.2$ for *Example 1*.

methods when $\theta_e = -2^0$, respectively. We observe from these figures that the proposed method can effectively alleviate the look direction mismatch and outperforms the other methods. Furthermore, we present the output SINR versus θ_e for comparison. From Fig. 8, the proposed method works satisfactorily, whereas the other methods suffer from serious performance degradation as θ_e increases.

Finally, we discuss the parameter p of the proposed method. Based on the simulation results shown in Figs. 1 and 5, $p = 5$ is good enough for the proposed method under the considered scenarios. In fact, the choice of $p = 5$ is determined by experiment. Our experience shows that a small p may be enough for the case with low SNR or large γ. For the case with high SNR or small γ, a large p may be more appropriate.

V. Conclusion

A novel robust adaptive beamforming method has been presented to overcome the performance deterioration due to steering vector mismatch and finite sample effect. The proposed method provides a weight vector with a more general form of diagonal loaded correlation matrix. We have shown that both the diagonal loading (DL) and variable loading (VL) methods are two special cases of the proposed method. Several simulation results have shown the superior performance of the proposed method as compared with the DL and the VL.

Acknowledgements

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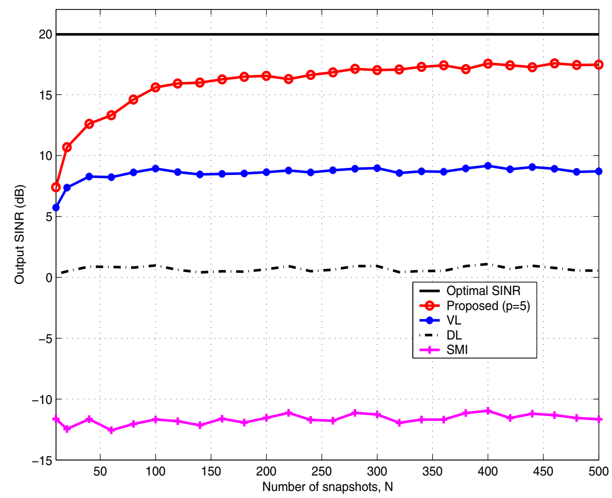

Fig. 2. The output SINR versus N when $\sigma_e^2 = 0.2$ for *Example 1*.

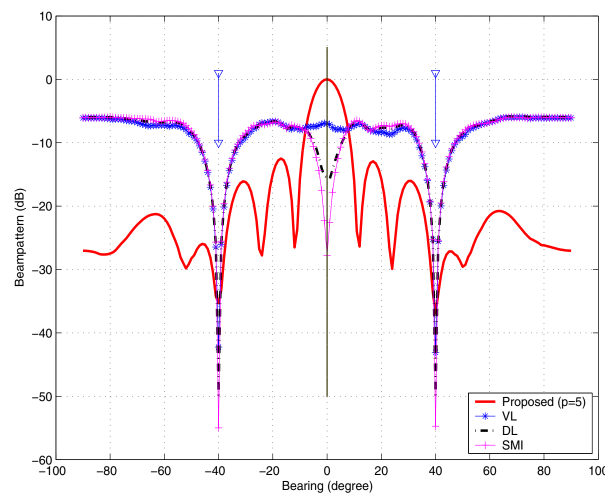

Fig. 3. The beampatterns when $\sigma_e^2 = 0.2$ for *Example 1*.

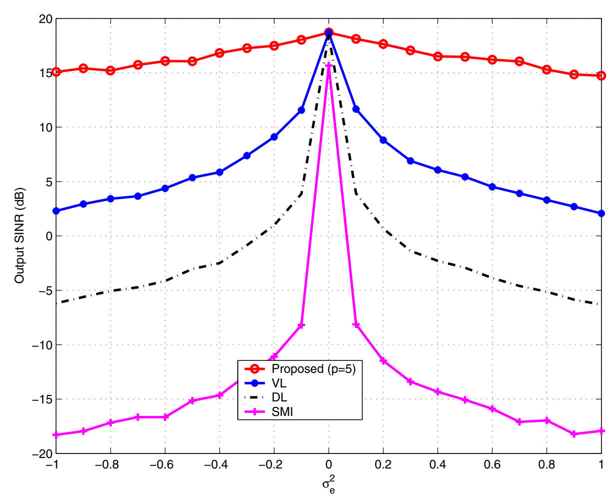

Fig. 4. The output SINR versus σ_e^2 for *Example 1*.

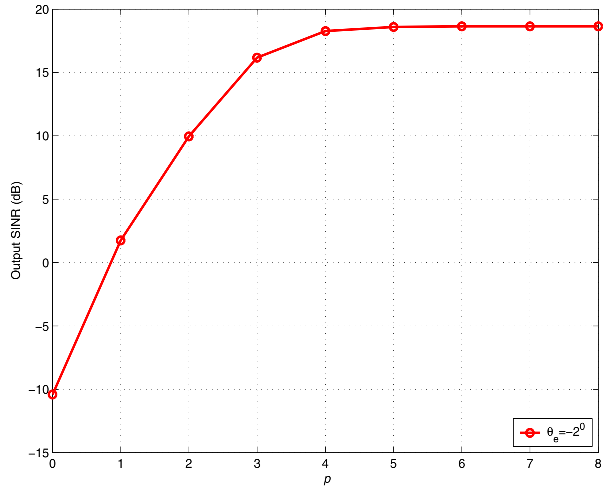

Fig. 5. The output SINR versus p when $\theta_e = -2^0$ for *Example 2*.

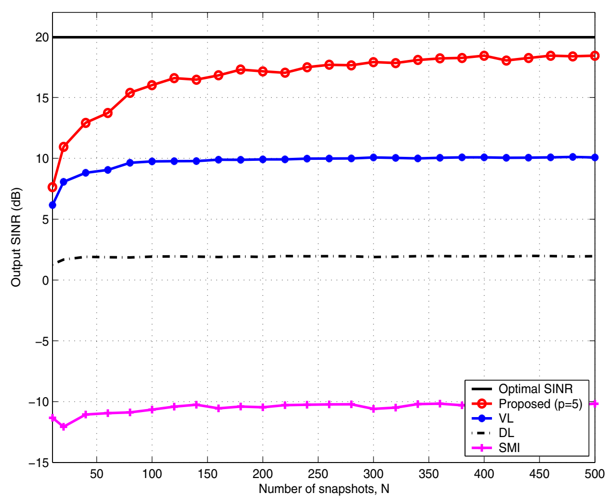

Fig. 6. The output SINR versus N when $\theta_e = -2^0$ for *Example 2*.

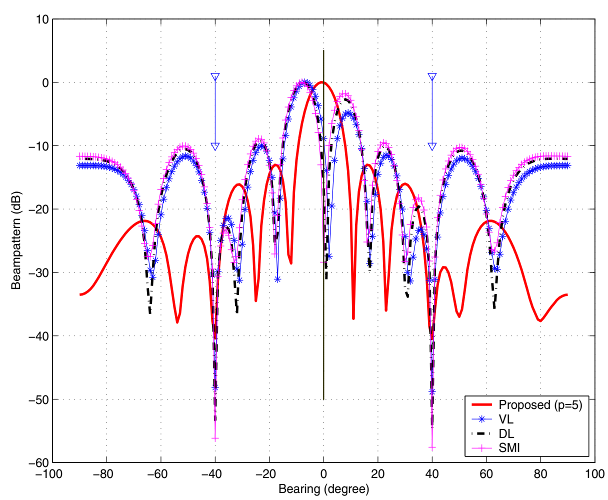

Fig. 7. The beampatterns when $\theta_e = -2^0$ for *Example 2*.

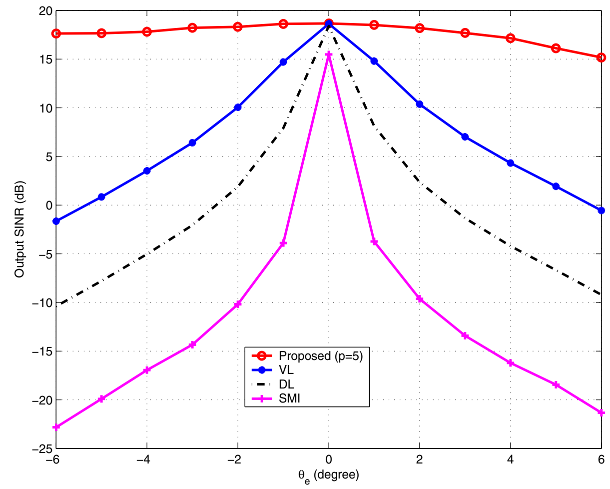

Fig. 8. The output SINR versus θ_e for *Example 2*.

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