New Results on the Connectivity in Wireless Ad Hoc Networks

Min-Kuan Chang, Che-Ann Shen and Min-Han Chuang

Graduate Institute of Communication Engineering and Department of Electrical Engineering,
National Chung Hsing University, Taichung, Taiwan
E-mail: minkuanc@dragon.nchu.edu.tw

***Abstract*—We first investigate when it is possible for two nodes in a wireless network to communicate with each other. Based on the result from bond percolation in a two-dimensional lattice, as long as the probability that a sub-square is close is less than 0.5 and each sub-square contains at least four nodes, percolation occurs. Following that, we establish the conditions for full connectivity in a network graph. How two adjacent sub-squares are connected differentiates this work from others. Two adjacent sub-squares are connected if there exists a communicating path between them instead of a direct communication link. The full connectivity occurs almost surely if each sub-square contains at least one node and the probability of having an open sub-edge is no less than 0.3822. Simulations are conducted to validate the proposed conditions for percolation and full connectivity. We also apply the results to SINR model [1]. We find each node can tolerate more interference than that stated in [1]. Last, we extend the results to the case of unreliable transmission. Under this situation, increasing nodes per sub-square or maintaining the probability of successful transmission above a certain threshold seems to be two possible approaches to achieve percolation and full connectivity.**

***Index Terms*—Wireless ad hoc network, Poisson point process, percolation, full connectivity, unreliable transmission**

I. Introduction

Wireless ad hoc networks have gained much attention and attracted many research activities in recent year. Since a wireless ad hoc network is self-organized without the need of infrastructure, nodes in such a network can communicate with one another only through multi-hop relaying. Due to this characteristic of multi-hop relaying, one question will be naturally raised: Is it possible for two arbitrary nodes in a wireless ad hoc network to communicate with each other? If yes, will this be always possible without fail? The first question is related to the problem of when the percolation occurs and the second one is when the full connectivity in a network graph will take place in almost sure sense.

Under the preferable conditions in either [1] or [2], all four sub-edges of a sub-square are open regardless the locations of nodes in sub-squares. Based on this fact, full connectivity and percolation ensue in [1] and [2], respectively. Furthermore, this fact also implies two individual nodes located in two adjacent sub-squares are connected if and only if there exists a direct communication link between them. However, from the viewpoint of communication, two nodes in two adjacent sub-squares can be deemed as being connected as long as there exists a communicating path between these two nodes even if the sub-edge shared by these two adjacent sub-squares is closed. Thus, the existence of direct communication links between any pairs of nodes in two adjacent sub-squares can be relaxed to the existence of a communicating path between any pair of nodes located in two different sub-squares in almost sure sense. In this work, we will base on this relaxation to derive new results on percolation and full connectivity.

We first make an analogy with the conditions for bond percolation in two-dimensional square lattice [3] and propose a theorem to identify the conditions of when percolation would occur in a two-dimensional network, where nodes are randomly deployed according to the Poisson point process. We find that as long as each sub-square contains at least four nodes and the probability that a sub-square is closed is less than 0.5, the network graph will be percolated. In the second part of this work, we address the conditions of when the network graph is fully connected in almost sure sense and any two arbitrary nodes can always communicate with each other. We will first map the network graph on a lattice and give the condition of when a sub-edge is open based on the derived bounds on the probabilities of isolated components and then, claim full connectivity on the original network graph. Based on that, we derive the necessary probability of having an open sub-edge in a sub-square and we find that as long as the probability that a sub-edge is open is greater than 0.3822, the network graph will be almost surely (abbreviated as a.s.)[1] fully connected.

Section II will describe a probabilistic lattice model. In Section III and IV, the conditions for percolation and full connectivity will be given. In Section V, simulation results will be given to validate the proposed theorems in this work, which is followed by the concluding remark in Section VI.

II. Model

In this section, we will give several definitions where some of them are directly adopted from [1] to help develop our own conditions for percolation and full connectivity in the following sections. We first construct a square lattice over an infinite plane denoted by $\pounds$, with edge length d. Let $\pounds'$ be the dual lattice of $\pounds$, obtained by simply shifting $\pounds$ by $d/2$ horizontally and vertically, as described in Fig. 1. Based on this construction, an one-to-one correspondence between the edges of $\pounds'$ and the edges of $\pounds$ can be established. In addition, without loss of generality, the origin O of the plane is set at a vertex of $\pounds'$. Next, the locations of nodes in this infinite planar network follow the Poisson point process in $\mathbb{R}^2$ with density λ. Therefore, on average, there are λd^2 nodes in each

[1]Hereafter, "almost surely" and "a.s." will be used interchangeably.

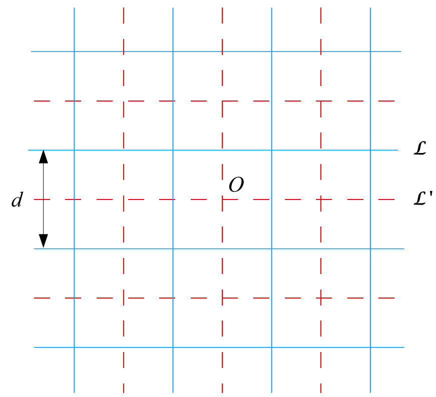

Fig. 1: Lattice $\pounds$ (blue plain) and its dual lattice, $\pounds'$, (red dashed)

square of $\pounds$. Each square of lattice $\pounds$ is further divided into k^2 sub-squares of size $(d/k) \times (d/k)$, where $k \in \mathbb{N}$. Several definitions are given below to help us establish the conditions for percolation and full connectivity in a network graph.

Definition 1: A square is said to be populated if all its sub-squares contain at least one node.

Definition 2: The sub-edge is said to be open if the following conditions are fulfilled:

- The corresponding adjacent sub-square contains at least one node.
- The transmission coverage covers the corresponding adjacent sub-square completely.

Definition 3: An sub-edge (or an edge) a' of $\pounds'$ is said to be open if and only if the crossed sub-edge (or an edge) of $\pounds$ is open.

Definition 4: A path (in $\pounds$ or $\pounds'$) is said to be open (resp. closed) if all sub-edges (or edges) forming this path are open. [2]

These definitions are chosen such that an open sub-edge guarantees connectivity in the continuous model [1]. An open edge in this work is less obvious than that in [1]. Later on, the definition of an open edge shared by two adjacent squares will be derived and given formally.

In the following section, we will first establish the conditions for percolation. When percolating, the communication between two arbitrary nodes is possible. However, the full connectivity is not guaranteed. Then we will investigate under what conditions the network graph is a.s. fully connected. We will derive the lower bound on the probability of having an open sub-edge such that there a.s. exists a communicating path between two nodes located at different sub-squares. Following this, the conditions for the existence of an infinite fully-connected component of the original network graph in almost sure sense will be shown and proved.

[2] A path and a communicating path are used interchangeably in this work.

III. Conditions for Percolation

In this part, we will develop the conditions for percolation by making an analog with the conditions for the bond percolation in a two-dimensional square lattice [3], where percolation occurs when the probability of having an open edge between two adjacent vertexes is greater than 0.5. In addition, there are at most four independent open edges emanating from an arbitrary vertex, where these open edges connect this vertex with the respective adjacent vertexes, when being in percolation. These two observations motivate us to propose the following theorem.

Theorem 1: The network graph is percolated if the following conditions are fulfilled:

1) The probability that a sub-square is closed, called $P^c_{sub,sq}$, is less than 0.5.
2) There are at least four nodes in a sub-square.

The proof of this theorem is omitted due to the length constraint. The simulation results will confirm the correctness of this theorem.

IV. Conditions for Full Connectivity

We have known under what conditions the network graph is percolated and the communication between two nodes is possible. However, when two arbitrary nodes can always communicate with each remains unclear to us. Next, we will see when the network graph is a.s. fully connected.

The main difference between [1] and this work lies in how two adjacent sub-squares are connected. In [1], as long as each of two adjacent sub-squares contains at least one node, they are connected with probability one. Therefore, the location of a node in a sub-square will not affect the connectivity. In this work, whether a node in a sub-square can communicate with nodes in the adjacent sub-squares depends on its location in a sub-square. Thus, nodes in two adjacent sub-squares may not be able to directly communicate with each other. In this case, instead, we seek for a communicating path between these two nodes. If such a communicating path exists, these two nodes are able to communicate with each other and the corresponding two adjacent sub-squares are declared to be connected. With this mindset, we would like to investigate when the full connectivity occurs in a network graph. In the followings, we first propose the following theorem to guarantee the full connectivity in a network graph.

Theorem 2: Let $P_{sub,e}$ be the probability that a sub-edge of a sub-square is open. The network graph is a.s. fully connected if $P_{sub,e} \geq 0.3822$.

Proof: The proof of this theorem consists of two parts. In the first part, we will show that when $P_{sub,e}$ of each edge of all sub-squares is greater than or equal to 0.3822, there a.s. exists a communicating path between any two arbitrary sub-squares in a $d \times d$ square. That is, we will prove the full connectivity in a square. In the second part, we will show that if there a.s. exists a communicating path between any two arbitrary sub-squares in a $d \times d$ square, an infinite fully-connected component on the continuous plane in $\mathbb{R}^2$ exists almost surely.

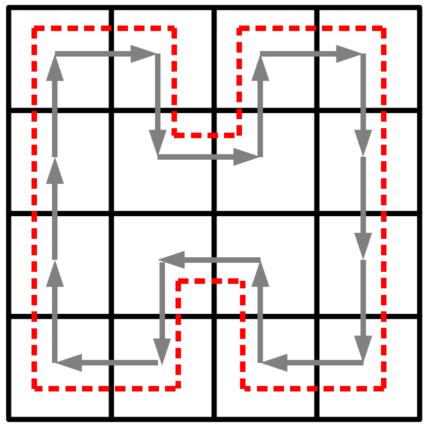

Fig. 2: An example of a communicating path.

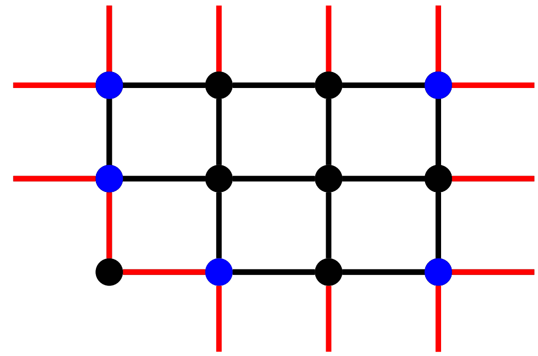

Fig. 3: An example of the isolated island. (Bonds in red are of zero weight and vertexes in blue are those vertexes in the corner.)

A. Full connectivity in a square

Note that a communication link between two adjacent sub-squares exists if and only if the corresponding sub-edge shared by these two sub-squares is open and in this case, we say these two sub-squares are connected. A communicating path consists of a sequence of such open sub-edges. An example of a communicating path (the dashed line) is shown in Fig. 2. In Fig. 2, the arrows indicate which sub-edge is open. But when will there a.s. exist a communicating path between any two sub-squares in a square?

To answer this question, let us see what happens when $P^c_{sub,sq} = 0$ and assume each of the four regions in a grey area contains a node. According to Theorem 1, the network percolates and furthermore, all sub-squares in a square are connected since there surely exists an open sub-edge between any two adjacent sub-squares. If we view each sub-square as a vertex, there is an open bond between any two vertexes. In addition, each bond contains two un-directional links due to the fact the each sub-square has four un-directional links connecting to its four adjacent sub-squares and we say the weight of the bond is 2.

Now we know that when $P^c_{sub,sq} = 0$ and each sub-square in a square has four un-directional links, the network graph in a square, called $\mathcal{G}_s$, is surely fully connected. Next, we will investigate the impact of randomly removal of one or more un-directional links of a sub-square in $\mathcal{G}_s$ on the connectivity in a square. This random removal of links would result in the nonexistence of direct links between two adjacent sub-squares. Will all sub-squares still be connected? Here we will discuss three scenarios, which can help us to establish the conditions under which there a.s. exists a communicating path between any two sub-squares.

1) Scenario I: One of four un-directional links of a sub-square in $\mathcal{G}_s$ is randomly removed: This random removal of a node in each sub-square in $\mathcal{G}_s$ causes the number of un-directional links of a sub-square to be 3. Thus, the weight of each bond is either 0, 1, or 2. Two adjacent vertexes are said to be disconnected if the weight of the bond connecting them is 0. In this scenario, only when either the horizontal or vertical strip of bonds of zero weight occurs, the network graph in a square will not be fully connected. In a square with k^2 sub-squares, the probability of having such a horizontal strip is $(1/4 \times 1/4)^k = (1/16)^k$. When k is large enough, that probability goes to zero. Thus, the network graph in a square stay a.s. fully connected.

2) Scenario II: Two of four un-directional links of a sub-square in $\mathcal{G}_s$ are randomly removed: Now the number of un-directional links of a sub-square to be 2 and the weight of each bond is still either 0, 1, or 2. Under this situation, the horizontal and vertical strips of bonds of zero weight could occurs but their chances of occurrence are close to zero as well. In addition, in this scenario, the isolated islands could occur. An isolated island is formed by the vertexes surrounded by bonds of zero weight. An example of the isolated islands is shown in Fig. 3. The probability of the occurrence of an isolated island is upper-bounded by

$$\left(\frac{1}{6}\right)^{m_1}\left(\frac{1}{4}\right)^{m},$$

where m is the number of vertexes in the perimeter of the isolated island, which are adjacent to vertexes right outside this island, m_1 is the number of vertexes in the corner among these m vertexes, and $m \geq m_1 \geq 4$. Even for $m = m_1 = 4$, which corresponds to the case of the smallest isolated island, the probability to have this isolated island is bounded by $(1/6)^4(1/4)^4 \approx 3 \times 10^{-6}$. For $m = m_1 = 5$, the corresponding isolated island occurs with probability less than 1.26×10^{-7}. An isolated island with larger m and m_1 would be even more unlikely to happen. Furthermore, the probability to have more than two isolated islands at the same time is less than 9×10^{-12}. Consequently, the network graph in a square almost surely remains fully connected.

3) Scenario III: Two or three of four un-directional links of a sub-square in $\mathcal{G}_s$ are randomly removed: In this scenario, each sub-square has either 1 or 2 un-directional links. The chance to have a strip of bonds of zero weight is still negligible as in the previous scenario. The probability of the occurrence of a isolation island is upper-bounded by

$$\left(\frac{1}{2}\right)^{2m+2m_1}\left(\frac{3}{4}\right)^{2m}.$$

When $m = m_1 = 4$, the corresponding probability of having such an island is less than 1.5×10^{-6}. As for $m = m_1 = 5$, the probability of having the corresponding island is less than 5.4×10^{-8}. If there is one isolated island with $m = m_1 = 4$ in a square, the percentage of connected sub-squares is $(k^2-4)/k^2$. As k approaches infinity, these four isolated sub-squares are negligible. Thus, in this case, the full connectivity in a square is still ensured almost surely. Since an isolated island with larger size or multiple isolated islands are almost impossible to form, the full connectivity in a square can be a.s. achieved.

There is another type of isolation in this scenario. We call it the isolated line. The probability of having an isolated line with length m, where m is the number of vertexes in this line, is bounded by

$$\left(\frac{1}{2}\right)^{4m+6}\left(\frac{3}{4}\right)^{2m+2} = \left(\frac{1}{2}\right)^{2m+2}\left(\frac{3}{8}\right)^{2m+2},$$

where $m \geq 2$. When $m = 2$, this corresponding to the case of the shortest isolated line and the probability of having such a line is less than 4.3×10^{-5}. When $m = 3$, the corresponding probability becomes 1.5×10^{-6}. To sum up, due to the fact that multiple isolated components, namely isolated islands, isolated lines or the hybrids of them, occur with probability close to zero, all sub-squares in a square can be a.s. fully connected if each sub-square has at least one un-directional link connecting to one of its adjacent sub-squares. That is, when each sub-square has at least one open sub-edge, the full connectivity in a square can be a.s. ensured. Thus, according to the above discussion, we propose the following lemma.

Lemma 1: If each sub-square in a square has at least one open sub-edge, all sub-squares are a.s. connected.

When each sub-square has at least one open sub-edge, any two adjacent sub-squares may not have a direct link to connect each other. However, there a.s. exists a communicating path between any pair of sub-squares. In the following lemma, we will specify that given a sub-square, under what circumstance the existence of having at least one open sub-edge is possible.

Lemma 2: If a square is populated, when the transmission radius is larger than or equal to $\sqrt{10}d/2k$, the probability that at least one open sub-edge exists is equal to 1.

From Lemma 2, we have already known when at least one sub-edge is open. Next, question of interest is the probability that a sub-edge is open, $P_{sub,e}$. Let us observe Fig. 4 first. Since the nodes are randomly deployed at a sub-square, given the transmission radius of $\sqrt{10}d/2k$, the probability that the right sub-edge of this sub-square is open is the same as the probability that there is a node located in the shaded area in Fig. 4. It is clear that if the transmission radius is larger than $\sqrt{10}d/2k$, we will have larger shaded area and thus, $P_{sub,e}$ will increase as well. In the following, we will assume that the transmission radius is $\sqrt{10}d/2k$ and derive the lower bound on $P_{sub,e}$ when a square is populated.

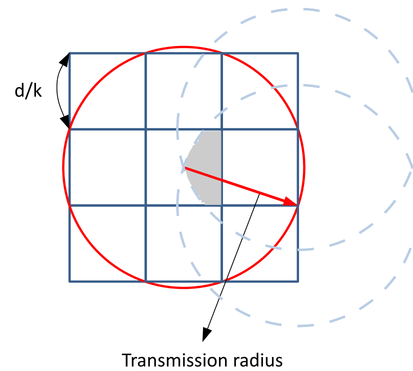

Fig. 4: The red circle, whose radius is $\sqrt{10}d/2k$, is the transmission range. When nodes fall in the shaded area, an adjacent sub-square to the right can be covered by those nodes in the shadow area completely.

Since the locations of nodes follow the Poisson point process, the probability of having nodes in the shaded area shown in Fig. 4 is equivalent to the ratio of the area of the shaded area to that of a sub-square. We first find the area of the shaded area, A_{shaded}. It is noted that as long as there is at least one node in the shaded area, the right adjacent sub-square and the sub-square in the middle are connected. Thus, to find this area, we have to know how this shaded area is formed. The shaded area is the intersection of the sub-square in the middle, the red circle and two blue circles as shown in Fig. 4. The two blue circles are the transmission ranges of the two nodes located at vertices on the right edge of the right adjacent sub-square as shown in Fig. 4. These two nodes are the farthermost nodes that can be reached by those nodes in the sub-square in the middle. As long as the nodes are in the intersection of these two blue circles and the sub-square in the middle, these two farthermost vertices can be reached and hence, the right adjacent sub-square can be fully covered. Based on this construction, A_{shaded} can be found by

$$\begin{aligned} A_{\text{shaded}} &= 2\left(\int_{\frac{1}{2}}^{1}\sqrt{\left(\frac{\sqrt{10}}{2}\right)^2 - x^2 dx} - 1 \times \frac{1}{2}\right)\left(\frac{d}{k}\right)^2 \\ &= 0.3822\left(\frac{d}{k}\right)^2. \end{aligned}$$

As a consequence, the probability that a sub-edge is open is given by

$$P_{sub,e} = \frac{\text{Area of the shaded area}}{\text{Area of a sub-square}} = \frac{A_{\text{shaded}}}{\left(\frac{d}{k}\right)^2} = 0.3822.$$

By Lemma 1 and Lemma 2, there exists a.s. a communicating path from one sub-square to any sub-square.

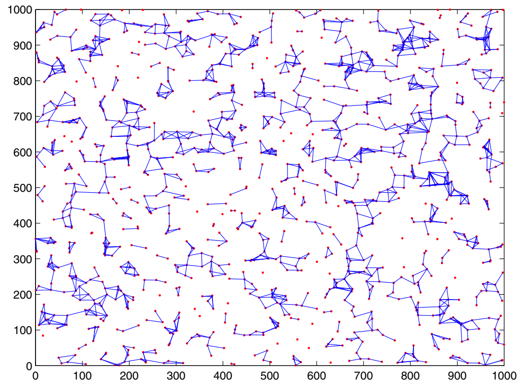

Fig. 5: The network size of 1000×1000 when $P^c_{sub,sq} = 3/7$.

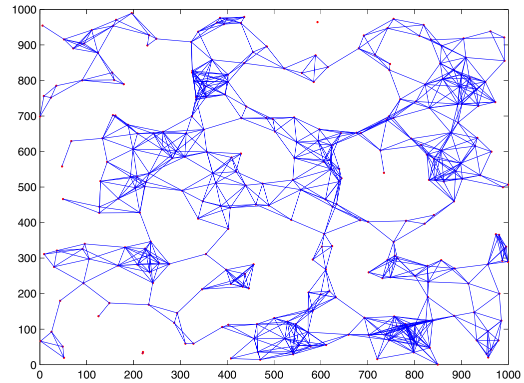

Fig. 6: The network size of 1000×1000 when $P_{sub,e} = 0.3822$.

B. Reverse mapping and full-connectivity on $\Re^2$

Next, we will prove that the existence of a communicating path between two arbitrary sub-squares guarantees these two sub-squares belong to the same cluster a.s. The formal statement of this is written in the following lemma.

Lemma 3: If $P_{sub,e}$'s of all sub-edges of any sub-square in a populated square are at least 0.3822, then all nodes in the square a.s. belong to the same cluster. Furthermore, if two adjacent squares fulfill the same condition, all nodes inside these two squares a.s. belong the same cluster.

The proof of this lemma is neglected due to the length constraint. We now can give the formal definition of an open edge.

Definition 5: An edge e of $\pounds$ is said to be open a.s. if all $P_{sub,e}$'s in these two squares adjacent to edge e are larger than or equal to 0.3822.

Putting Lemma 1, Lemma 2 and Lemma 3 together, when $P_{sub,e} \geq 0.3822$, all nodes in this network a.s. belongs to the same cluster and hence, the network graph a.s. is fully connected. This concludes the proof of Theorem 2. ■

V. Simulation

A. Verification of Theorem 1

As we can see in Fig. 5, there exists an open path from the top to the bottom. This implies an infinite open path exists and the network is percolated. This verifies the correctness of Theorem 1.

B. Verification of Theorem 2

To have each square populated, in theory, we have to let either the density of node approach to infinity or let the area of a sub-square infinite. Both are not possible in real world. Thus, we find the equivalent condition by choosing proper size of a sub-square and the transmission radius under a fixed density of node λ. Assuming that at least one sub-edge will be open in a sub-square, to have $P_{sub,e} = 0.3822$, we have to let the probability that there is no node in a sub-square other than the shaded area as shown in Fig. 4 is equal to $1 - P_{sub,e} = 0.6178$. Based on this and the Poisson point process, the area of a sub-square is given by $(-\ln 0.6178)/\lambda$. In the simulation, we set $\lambda = 3 \times 10^{-4}$ and let $P_{sub,e}$ of each sub-edge is 0.3822. As a result, the edge length of a sub-square is 40.0662 unit length and the transmission radius of a node is 89.5908 unit length. According to Fig. 6, if there is an infinite network and $P_{sub,e}$ is greater than or equal to 0.3822, all nodes in this network a.s belong the same cluster and the infinite network is fully connected.

VI. Conclusion

We had investigated the problem of when a network graph is percolated and when a network graph is fully connected in almost sure sense. First, if $P^c_{sub,sq}$ is less than 0.5 and each sub-square has at least four nodes, the network graph would be percolated. Secondly, we found that when being populated, as long as $P_{sub,e} \geq 0.3822$, the network graph is a.s. fully connected.

VII. Acknowledgment

This work was supported by the National Science Council of the Republic of China under Grant NSC 100-2221-E-005-048.

References

[1] O. Dousse, P. Thiran, and M. Hasler, "Impact of interferences on connectivity in ad hoc networks," *IEEE J. Sel. Areas Commun.*, vol. 13, no. 2, pp. 425 – 436, Apr. 2005.

[2] E. N. Gilbert, "Random plane networks," *Society for Industrial and Applied Mathematics*, vol. 9, no. 4, pp. 533 – 543, Dec. 1961.

[3] G. Grimmett, *Percolation*, 2nd ed. John Wiley and Sons, 1999.