Reed-Solomon Virtual Codes Based Novel Algorithm for Sparse Channel Estimation in OFDM Systems

Fatma Abdelkefi
University of Carthage
$COSIM$ Laboratory
High School of Communications of Tunis (SUPCOM), Tunisia
Email: fatma.abdelkefi@supcom.rnu.tn

Jaouhar Ayadi
ECLEXYS - Communication Systems,
CH-6826 Riva San Vitale,
Switzerland
Email: jaouhar.ayadi@eclexys.com

***Abstract*—In this paper, we present a novel efficient algorithm for the estimation of the Channel Impulse Response (CIR) when this CIR is sparse (meaning a big number of the CIR coefficients are equal to zero) for multicarrier systems using Orthogonal Frequency-Division Multiplexing (OFDM) transmission. The derivation of this CIR estimation algorithm investigates first the sparse structure of the channel through the modeling of the sparse CIR as a Bernoulli-Gaussian process. This established modeling will allow us to exploit the relationship between the Reed-Solomon (RS) codes and the OFDM modulator to efficiently estimate the sparse CIR. To do so, we consider the pilot tones that are usually scattered among the information sequence for the synchronization or equalization purposes, as syndromes in order to estimate the sparse channel coefficients, and we prove that using our proposed algorithm, the obtained estimates are unbiased and that the estimation error is quasi-optimum. Furthermore, our proposed technique keeps valid even in the case where the pilots tones are assumed to be not uniformly placed in the transmitted sequence provided that their positions satisfy a repartition condition. Simulation results are presented to illustrate the performance of our proposed algorithm and to support our claims.**

***Index Terms*—Sparse channel estimation, OFDM transmission, Reed-Solomon codes, Peterson-Zierler-Gorenstein, pilot tones.**

I. Introduction

Channel estimation generally for the purpose of equalization is a typical problem in communications and signal processing. In this work, we focus on channel estimation, when the channel has a specific structure. Indeed, we consider the estimation of sparse channels, that is, channels whose time domain response contains a large number of zero taps. We focus also, for this kind of channel estimation problem, on multicarrier systems using Orthogonal Frequency-Division Multiplexing (OFDM) transmission.

The problem of sparse channel estimation in multicarrier OFDM systems, attracted recently a big research attention. Sparse channel estimation has been significantly studied for frequency-selective radio channels based on, i.e., subspace fitting [5], generalized Akaike information criterion [6], zero-tap detection [7], or Monte Carlo Markov Chain methods [8]. Recent solutions proposed in the literature include: the matching pursuit technique [3] and the basis pursuit one as described in [4] and where the authors investigated the use of the compressive sensing paradigm as a mean to solve this estimation problem.

Compared to what was proposed in the literature, our technique presents the advantage to estimate in real-time the channel parameters. In addition, contrarily to the most existing solutions where an averaging on some parameters is requested, which would increase the computation complexity of the considered method, our proposed algorithm does not require this averaging constraint.

This paper is organized as follows, Section II assesses the system model. Section III introduces the proposed technique in order to estimate the channel coefficients based on a coding theory approach. Section IV describes the simulation results that illustrate the performance of the proposed channel estimation algorithm and the last section is dedicate to enumerate the conclusions of our investigations.

II. Notations

The superscripts $.^H$, $.^*$ and $.^T$ denote the conjugate transpose, element wise conjugation and transpose, respectively. By $\|\mathbf{x}\|$, we refer to the Euclidean distance and $\mathcal{E}$ is the expectation operator..

III. System Model

Let $\mathbf{x}$ be a discrete-time complex baseband equivalent transmitted signal that have a finite duration M. Furthermore, let us model the discrete-time complex baseband channel that is assumed to be sparse with finite delay-spread Channel Impulse Radio (CIR) represented by the vector $\mathbf{h} = [h_0, \ldots, h_L]^T$, where L denotes its finite length L and we assume that $L < M$. The CIR coefficients h_k are here modeled as zero mean normal complex random variables with equal real and imaginary parts variances σ_h^2, and

$$h_k = \sum_{l=0}^{L-1} h_l \delta(l - k\, d_l),$$

978-1-4673-1881-5/12/$31.00 ©2012 IEEE

where $0 \leq d_0 \leq \ldots \leq d_{L-1} < M$ are considered as unknown parameters.

The output of the transmission system can be expressed as:

$$y_m = \sum_{l=0}^{L-1} h_l \, x_{m-l} + w_m, \, n = 0, \ldots, M-1,$$

where w_m is a zero mean complex Additive White Gaussian Noise (AWGN) with equal real and imaginary parts variance σ^2.

Let $\Omega = \{p_1, p_2, \ldots, p_\alpha\}$ be the set of pilot tones in the OFDM symbol and α is the number of pilot tones such that $\alpha \in \{0, 1, \ldots, M-1\}$.

In our paper, we consider a discrete OFDM system, that is easily obtained by computing M samples of the signal to be sent through the sparse channel during one OFDM symbol of duration T_s and in the case of a rectangular pulse prototype filter of duration $M \, T_s$. Therefore, the spacing between the carries is equal to $\frac{1}{N \, T_s}$. The samples of the input of the OFDM modulator, $\mathbf{c}$, samples read as:

$$c_k = \sum_{m=0}^{M-1} x_m \, e^{j2\pi \frac{km}{M}}, \, k = 0, \ldots, M-1,$$

which is exactly the Inverse Discrete Fourier Transform (IDFT) of the transmitted sequence $\mathbf{x}$. When we assume that some components of the transmitted signal $\mathbf{x}$ are considered as pilot tones, these symbols are recuperated at the OFDM receiver side. This means after demodulating the signal via a DFT and when one or more impulsive errors occur in the channel the received pilot tone symbols will change. However, we proved in [1] that the OFDM modulator can be seen as a complex Reed-Solomn (RS) coder and we showed how the exploitation of this equivalence between the OFDM modulator and the RS coder in the case of the presence of impulsive noise, can efficiently lead to estimate the impulsive errors. We propose in the sequel to approach the our system model such that this exploitation can be re-used. Consequently, in order to estimate the channel parameters, that are d_k and h_k, we propose to follow the same reasoning and to exploit the relationship that exists between the RS coder and the OFDM modulator.

IV. Equivalence between RS and OFDM modulator

It has been shown in [2], that the ideas of spectral coding theory can be translated in the frequency-domain, over a field $\mathbb{F}$ such as the field of complex numbers $\mathbb{C}$, the field of real numbers $\mathbb{R}$...

If $\mathbb{F} = \mathbb{C}$, then the Bose, Chaudhuri and Hocquenghem (BCH) codes are analogous to the RS ones. The RS codes can be defined as follows [2]:

Definition 1: Let $\mathbb{F}$ contains an element of order M. The $(M, M-2t)$ RS block length M with symbols in $\mathbb{F}$ is the set of all vectors $\mathbf{c}$ whose spectrum in $\mathbb{F}$ satisfies: $c_k = 0$, $\forall k \in \mathcal{A}$ where $\mathcal{A} = \{k_0+1, k_0+2, \ldots, k_0+2t\}$. This is described briefly as an $(M, M-2t)$ RS code over $\mathbb{F}$.

The spectrum of a RS code word lives in the same field as information symbols. Then, to form a RS code, a block of $2t$ consecutive spectral components are chosen as parity frequencies (to be set to zero) and the remaining ones are information symbols. Taking the OFDM scheme as an illustration example, it was proved in [1] that the OFDM modulator can be seen as complex RS code: the key remark that we use is that if a a discrete sequence of complex numbers containing $2t$ pilot tones is transmitted over the OFDM system, therefore, the output of the OFDM modulator can be considered as a RS code word. After transmission over a channel, the DFT of the received discrete time sequence can no longer be equal to the transmitted $2t$ pilot tones values, and this is due only to the channel. Hence, the following BCH bound gives the correction capacity of a complex RS code:

Theorem 1: **BCH bound** If ($2t$ consecutive frequencies belong to $\mathcal{A}$) then (the minimal distance is at least $2t-1$), where $\mathcal{A}$ is the set of the $2t$ zeros.

The BCH bound proves that t errors in any codeword of a RS code can always be corrected, because every pair of codewords differs in at least $2t+1$ places. So the correction capacity is upper bounded by $\lceil \frac{2t+1}{2} \rceil$.

However, strictly speaking, in practice there are more than $\lceil \frac{2t+1}{2} \rceil$ errors: all samples are polluted by noise. Therefore, we concentrate on the removal of the sole impulsive noise, considering the Gaussian component as background noise. The classical decoding techniques have to be modified in order to take into account the presence of this background noise. Generally, considering consecutive zeros, i.e., in OFDM system, has no meaning in practice because these zeros do not correspond to a part of the spectrum which is actually available and only a small part of these zeros can be used. In many cases, however, pilot tones are transmitted for synchronization or channel estimation purposes. Such pilot tones consist in known symbols that are scattered among the information ones. Hence, we can use them for performing impulse noise correction, if we can obtain some flexibility in the possible locations of the pilot tones. The same reasoning can be considered for the case where the pilot tones are not uniformity distributed among the transmitted sequence once their position verifies the necessary condition derived in [9].

As the RS code is a cyclic one, then it can be decoded with any decoding algorithms of the cyclic codes such as the Peterson-Zierler-Gorenstein (PGZ) or Berlekamp-Massy or Euclid decoding algorithms. In the sequel, we propose to use the PGZ algorithm as a decoding scheme and we assume that there is only burst or isolated errors.

V. Derivation of the sparse CIR estimation algorithm

Assume a cyclic prefix added to the input of the OFDM modulator. At the receiver side a DFT is performed and the received signal is converted into its frequency-domain representation $\mathbf{y}$, which is given by:

$$\mathbf{y} = \mathbf{X}\mathbf{H} + \mathbf{W}, \tag{1}$$

where $\mathbf{X}$ is a diagonal matrix that contains in its diagonal the Fourier transform of the transmitted sequence $\mathbf{x}$, $\mathbf{H} = [H_0, \dots, H_M]^T$ where H_k corresponds to the Fourier transform of the sparse CIR channel h and $\mathbf{W}$ is the Fourier transform of the noise sequence $\mathbf{w}$.

Now, we come to the step where we investigate the relationship between the RS codes and the OFDM modulator. Without loss of generality, the sparse CIR can be modeled as a Bernoulli Gaussian sequence. Assume a memoryless transmission CIR channel, corrupted by a background noise $\mathbf{w}$. That means that each sample h_k can be modeled as $h_k = l_k g_k$ where l_k stands for a Bernoulli process, an i.i.d. sequence of zeros and ones with $prob(l_k = 1) = p$ and $prob(l_k = 0) = 1 - p$, and g_k is a complex Gaussian noise with zero mean and variance σ_h^2. We assume that the h_k components are scattered among the sequence along time.

Based on the OFDM system, we have shown in [1] that the transmitted sequence $\mathbf{x}$ can be seen as a RS code. Therefore, the output of the OFDM modulator can be assimilated to a complex-valued RS codeword where the correction capacity is upper bounded by $\lceil \frac{\alpha-1}{2} \rceil$, where α is the cardinal of the set of the pilot tones positions in the transmitted sequence $\mathbf{x}$. This means that, when we dispose of α pilot tones, we can estimate a sparse CIR with at most $\lceil \frac{\alpha-1}{2} \rceil$ non-zero components in the presence of a background condition. This implies that the CIR length, L, must verify $L < \lceil \frac{\alpha-1}{2} \rceil$. In terms of coding, we can make the following equivalences:

- the information sequence is $\mathbf{c}$ and the codeword is $\mathbf{x}$ (i.e., $\mathbf{c} = \mathbf{F}_M^H \mathbf{x}$, where $\mathbf{F}_M$ denotes the Fourier transform of order M),
- the RS generator matrix is the Fourier transform of order M.

Given the received sequence $\mathbf{y}$, the receiver has the task of decoding $\mathbf{x}$ from $\mathbf{y}$ using the syndrome vector that will be defined in the sequel. For further illustration starting from the received signal expression given by (1), assume a given constellation at the OFDM modulator, then the demodulated signal is:

$$\mathbf{z} = \mathbf{H} + \mathbf{X}^{-1}\mathbf{W} = \mathbf{H} + \mathbf{B}^{'},$$

where $\mathbf{z} = \mathbf{X}^{-1}\mathbf{y}$ and $\mathbf{B}^{'} = \mathbf{X}^{-1}\mathbf{W}$. As $\mathbf{X}$ is a diagonal matrix such that $\mathbf{X} = \mathbf{F}_M \mathbf{K} \mathbf{F}_M^H$ where $\mathbf{K}$ is a Toeplitz matrix. Consequently,

$$\mathbf{z} = \mathbf{F}_M(\mathbf{h} + \mathbf{K}^{-1}\mathbf{w})\mathbf{F}_M^{-1} = \mathbf{H} + \mathbf{B}^{'}. \qquad (2)$$

As $\mathbf{w}$ is assumed to be an AWGN, and each component of this vector has zero mean and variance σ^2, then $\mathbf{b} = \mathbf{K}^{-1}\mathbf{w}$ is also a Gaussian noise with zero mean and covariance matrix $\mathbf{R}_{bb}$ such that:

$$\begin{aligned}
\mathbf{R}_{bb} &= \mathcal{E}(\mathbf{b}\mathbf{b}^H) \\
&= \mathcal{E}(\mathbf{K}^{-1}\mathbf{w}\mathbf{w}^H(\mathbf{K}^{-1})^H) \\
&= \mathbf{K}^{-1}\mathcal{E}(\mathbf{w}\mathbf{w}^H)(\mathbf{K}^{-1})^H \\
&= \sigma^2\mathbf{K}^{-1}(\mathbf{K}^{-1})^H.
\end{aligned}$$

The vector $\mathbf{B}^{'}$ is the equivalent resulted background noise at the receiver. It corresponds to $\mathbf{X}^{-1}\mathbf{W}$, where $\mathbf{W} = \mathbf{F}_M\mathbf{w}$. As $\mathbf{w}$ is assumed to be an AWGN vector with zero mean and $\sigma^2\mathbf{I}_M$ as covariance matrix, therefore $\mathbf{W}$ is also a Gaussian noise with zero mean and $\sigma^2\mathbf{I}_M$ as a covariance matrix. Hence, the resulted background noise vector at the receiver, $\mathbf{B}^{'}$ is also a Gaussian vector with zero mean and a covariance matrix $\mathbf{R}_{B'B'}$ that is equal to:

$$\begin{aligned}
\mathbf{R}_{B'B'} &= \mathcal{E}(\mathbf{B}^{'}(\mathbf{B}^{'})^H) \\
&= \mathcal{E}(\mathbf{X}^{-1}\mathbf{W}\mathbf{W}^H(\mathbf{X}^{-1})^H) \\
&= \mathbf{X}^{-1}\mathcal{E}(\mathbf{W}\mathbf{W}^H)(\mathbf{X}^{-1})^H \\
&= \sigma^2\mathbf{X}^{-1}(\mathbf{X}^{-1})^H,
\end{aligned}$$

which is a diagonal one. Since the syndrome vector is defined at the receiver side, taking into account its main role and definition, we conceive the syndrome vector as follows:

$$\mathbf{S}_k = \mathbf{z}_{p_k} \qquad (3)$$

$$= \frac{\mathbf{y}_{p_k}}{x_{p_k}}, \; k = 1, \dots \alpha \qquad (4)$$

where $\{x_{p_k}\}_{k=1,\dots,\alpha}$ are the pilot tones.

To estimate the sparse CIR parameters and using the results obtained in [1], we propose to use a modified version of the classical RS decoding algorithm, known as Peterson-Gorenstein-Zieler (PGZ) algorithm that is matched to the presence of background noise. We consider a modified PGZ algorithm to locate the non-null CIR channel components, based only on a syndrome vector evaluation (eq.4).

The proposed version of this PGZ algorithm will contain the following steps:

Modified PGZ Algorithm for the CIR channel estimation

Step 1: *estimate the number of the non-null CIR channel components,*
Step 2: *seek the parameters d_k, that are mapped to the placements of the non-null CIR channel components, by the mean of a generator polynomial whose roots correspond to the impulse locations in the data sequence,*
Step 3: *estimate the amplitude of* $\mathbf{h}$*,*
Step 4: *perform an a posteriori control step that is dedicated to detect the malfunction of the CIR channel estimation: estimate first the syndrome vector $\tilde{\mathbf{s}}$, then compare $\|\mathbf{S} - \tilde{\mathbf{S}}\|$ to a chosen threshold, $Thresh$: if $\|\mathbf{S} - \tilde{\mathbf{S}}\| < Thresh$, then the channel impulses are well estimated, else re-estimate the CIR channel by performing a truncated enumeration of all possible impulse localizations.*

Note that this truncated enumeration of **Step 4** is necessary due to the presence of the background noise which can introduce some fuzziness in the performed computation. More importantly, contrarily to other related existing works, this proposed algorithm allows the possibility to estimate the number of non sparse channel components if this information is not accessible.

VI. Simulation results

In the performed simulations, we assume that the pilots are consecutively distributed in the transmitted sequence. First, we consider the case where no background noise where is present and then the case there is a presence of the background noise .

In Fig.1, for different values of the received signal length M, we vary the number of tones to transmit (the length of the syndrome sequence), meaning that we give different values to the parameter α. We also vary the probability of occurrence, p, of non null channel components, as a function (percentage) of the parameter of α.

We consider a channel with a CIR delay-spread $L > 1$ (we omit the case of channels with all coefficients equal to zero). For each value of p, we perform 1000 Monte-Carlo run of the noise-free system. The taps of the channel are generated as i.i.d. complex Gaussian random variables with zero mean and variance $\sigma_h^2 = 1$.

In Fig.1, we plot the curves illustrating the Normalized-Mean-Square-Error (NMSE) of the sparse CIR estimation, versus the probability of occurrence p and for different number of pilot tones α. The obtained results do illustrate the efficiency of our proposed algorithm as we can conclude that that when $p < 0.04$, the NMSE is closed to zero, which means that the estimation error is quasi-optimum and that the obtained CIR coefficient estimates are unbiased and close to their real values. We note that when p increases, the NMSE represents sometimes spikes that are due essentially to the fact that for some runs of the simulations, an overflow of the correction occurs. This overflow becomes rare when α increases.

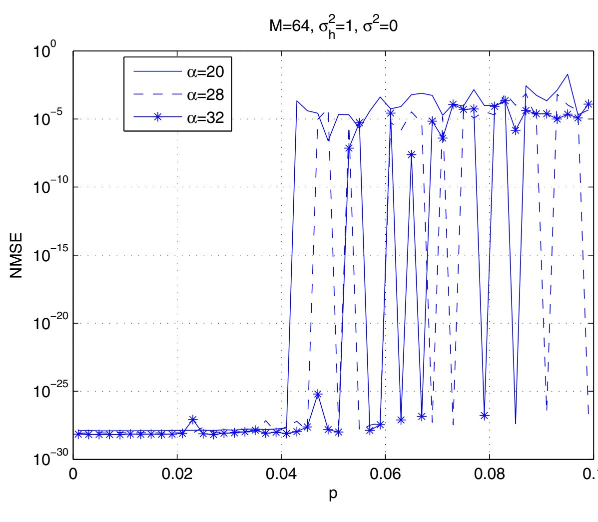

Fig. 1. NMSE of h in a noise-free system versus the probability of occurrence of a non null sparse channel component for difference number of pilot tones in the transmitted sequence

For both Fig.2 and Fig.3, we plot the NMSE of the sparse CIR estimation versus the SNR in dB, for the noisy system case. The background noise $\mathbf{w}$ is a zero mean AWGN complex vector, with equal real and imaginary parts variance σ^2 and we define our SNR $= \frac{E_s}{\sigma^2}$ and E_s is the energy per symbol of the 4QAM transmitted sequence.

In Fig.2, we vary the number of CIR channel impulses (i.e., $L \in \{1, 2, 3\,4\}$), when the number of pilot tones is put to 20. As the PGZ algorithm is known to be sensitive to the background noise, therefore this has an impact essentially on its localization step. This aspect is illustrated by the curve corresponding to $L = 1$ where only one position is located. When the SNR increases the NMSE performance curves corresponding to the different values of L become close.

In Fig.3, the probability of occurrence of one CIR channel impulse is taken as $p = 0.005$ and the number of pilot tones is also put to 20. We vary in this simulation the length of the transmitted sequence where three situations are considered: $M = 40$, $M = 64$ and $M = 100$. We note that for $M = 64$ and $M = 40$ the corresponding NMSE performance curves are close. But when M increases, the occurrence of an impulse becomes possible, which means that the mean number of impulses increases also and as the PGZ algorithm is sensitive to the background noise for small SNR levels, a confusion between a channel impulse and the background noise, could occur.

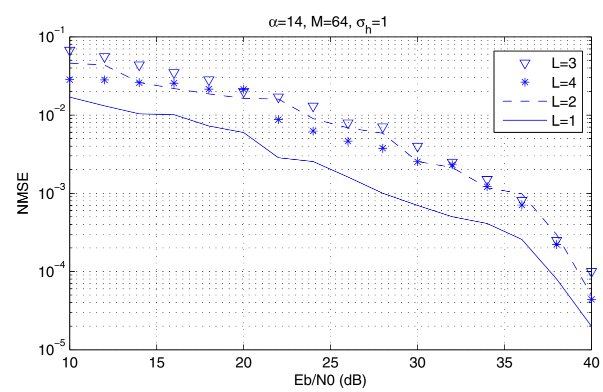

Fig. 2. NMSE of h versus SNR for different values of the CIR channel impulses L

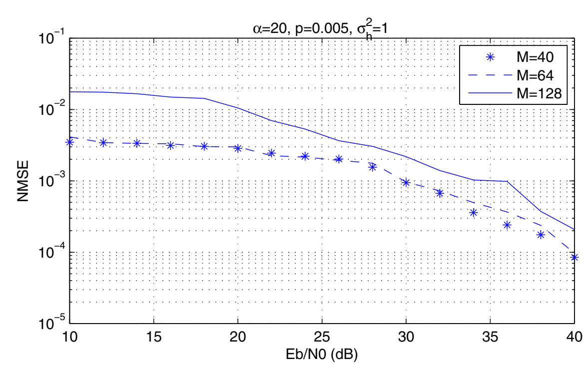

Fig. 3. NMSE of h versus SNR for different values of the length of the transmitted sequence M

VII. Conclusions

Using a multi-carrier transmission scheme, pilot tones can be seen as virtual RS coder. Since the CIR channel is a sparse channel, we proposed to model it in this paper as a Bernoulli Gaussian channel: Consequently, the channel recovery can be performed by a modified version of the classical RS PGZ decoding algorithm. Using a relationship between the multi-carrier transmission scheme and the RS coder, we have shown that the number of pilot tones needed to estimate sufficiently a sparse channel is equal to $\lceil \frac{2\alpha+1}{2} \rceil$, which is necessary to estimate α CIR channel impulses.

For a noise-free system, we can perfectly estimate the CIR. Otherwise, the efficient decoding estimation algorithm contains three main steps: 1) estimate the number of non sparse CIR channel coefficients, 2) estimate the position of the non sparse channel components, 3) estimate the amplitude of the channel and 4) an *a posteriori* control step that is dedicated to detect the malfunction of the CIR estimation.

The obtained simulation results showed the efficiency of our proposed CIR estimation algorithm for the sparse channels case.

References

[1] F. Abdelkefi, P. Duhamel and F. Alberge, *"Impulsive Noise Cancellation in Multi-carrier"*, IEEE Transactions on Communications, vol.5, no.1, 2005.

[2] R. E. Blahut, *"Theory and practice of error control codes"*, Addison-Wesley, 1983.

[3] S. G. Mallat and Z. Zhang, *"Matching pursuits with time-frequency dictionaries"*, IEEE Transactions on Signal Processing, vol. 41, pp.33973415, Dec. 1993.

[4] C. R. Berger, S. Zhou, J. Preisig and P. Willett, *"Sparse Channel Estimation for Multicarrier Underwater Acoustic Communication: From Subspace Methods to Compressed Sensing"*, IEEE Transonction on Signal Processing, Vol. 58, No. 3, pp. 1708-1721, March 2010.

[5] W. Chun-Jung and D. W. Lin, *"Sparse channel estimation for OFDM transmission based on representative subspace fitting"*, Vehicular Technology Conference (VTC-Spring), June 2005.

[6] M. R. Raghavendra and K. Srikrishna Bhashyam Giridhar, *"Improving channel estimation in OFDM systems sparse multipath channels"*, IEEE 5th Workshop on Signal Processing Advances in Wireless Communications, 2004.

[7] C. Carbonelli, U. Vedantam and S. Mitra, *"Sparse Channel Estimation with Zero Tap Detection "*, IEEE Transactions on Wireless Communications, vol.6, pp.1743–1763, May 2007.

[8] H. Wan, R-R. Chen, J. W.n Choi, A. Singer and J. Preisig, B. Farhang-Boroujeny, *"Markov Chain Monte Carlo detection for underwater acoustic channels"*, Information Theory and Applications Workshop (ITA), 2010.

[9] F. Abdelkefi, P. Duhamel and F. Alberge, *"A necessary condition on the location of pilot tones to impulse noise cancellation in OFDM system and is application in Hiperlan2"*, IEEE Globecom Conference, 2003.