# A Trellis Coded Modulation Scheme for the Fading Relay Channel


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***Abstract*—A decode and forward protocol based Trellis Coded Modulation (TCM) scheme for the half-duplex relay channel, in a Rayleigh fading environment, is presented. The proposed scheme can achieve any spectral efficiency greater than or equal to one bit per channel use (bpcu). A near-ML decoder for the suggested TCM scheme is proposed. It is shown that the high Signal to Noise Ratio (SNR) performance of this near-ML decoder approaches the performance of the optimal ML decoder. Based on the derived Pair-wise Error Probability (PEP) bounds, design criteria to maximize the diversity and coding gains are obtained. Simulation results show a large gain in SNR for the proposed TCM scheme over uncoded communication as well as the direct transmission without the relay.**


## I. Preliminaries and Background

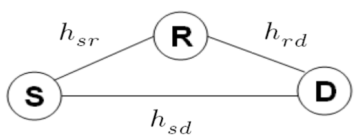



Fig. 1. The Relay Channel

We consider the Rayleigh fading relay channel shown in Fig. 1, consisting of the source node $S$, the relay node $R$ and the destination node $D$. It is assumed that R can operate only in the half-duplex mode and it has perfect knowledge about the instantaneous value of the fade coefficient associated with the S-R link and D has perfect knowledge about the instantaneous values of the fade coefficients associated with the S-R, R-D and S-D links. Throughout, the phase during which the relay is in reception mode is referred to as Phase 1 and the phase during which the relay is in transmission mode is referred to as Phase 2.

In all practical scenarios the S-R and R-D links are stronger than the S-D link. As a result, the use of coding schemes which involve both S and R, can potentially outperform the coding schemes involving S alone. The problem addressed in this paper is the design of a Trellis Coded Modulation (TCM) scheme for the fading relay channel, which achieves a spectral efficiency greater than or equal to one bit per channel use (bpcu) and provides a large gain compared to the TCM scheme for the direct transmission without the relay as well as the uncoded transmission with the relay.

### A. The Proposed Scheme

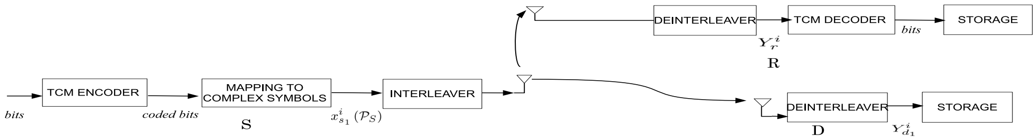
The proposed TCM scheme for the fading relay channel is shown in Fig. 2(a) and Fig. 2(b) (shown at the top of the next page). It is assumed that TCM encoding at S during Phase 1 and Phase 2 and at R during Phase 2 take place using the same encoder, but the labeling schemes used for mapping coded bits onto signal points can be different. By a trellis, we refer to the states of the TCM encoder, the edges connecting these states in two successive stages and the complex numbers which are labeled on these edges. By a path in the trellis, we refer to the sequence of connected edges (the complex numbers labeled on the edges are not included). Let $\mathcal{T}_{S_1}$, $\mathcal{T}_{S_2}$ and $\mathcal{T}_R$ represent the trellises used for encoding by S during Phase 1, by S during Phase 2 and by R during Phase 2 respectively. Since the encoder corresponding to the trellises $\mathcal{T}_{S_1}$, $\mathcal{T}_{S_2}$ and $\mathcal{T}_R$ are the same, they differ only in the complex numbers labeled on the edges. Let $\zeta$ denote the set of edges connecting the states in two successive stages of the trellis. The labeling of the edges in the trellis $\mathcal{T}_{S_1}$ is given by the map $\mathcal{X}_{s_1} : \zeta \longrightarrow \mathcal{S}$, where $\mathcal{S}$ is the signal set used at S and R. Similarly, the maps $\mathcal{X}_{s_2}$ and $\mathcal{X}_r$ are defined for the the trellises $\mathcal{T}_{S_2}$ and $\mathcal{T}_R$ respectively. For a spectral efficiency of $r$ bpcu ($2r$ bits get transmitted in the two phases of relaying), the cardinality of the signal set $\mathcal{S}$ is $2^{2r+1}$ .

During Phase 1, input bits at $S$ are encoded by the TCM encoder, mapped onto complex symbols from the signal set $\mathcal{S}$ and interleaved by the block interleaver as shown in Fig. 2(a). Each transmission made is assumed to be a block of $L$ complex symbols. After each transmission, the encoder state is brought back to the all zeros state by appending zero input bits at the end. Let $\mathcal{P}_S$ denote the path in the trellis corresponding to the output of the TCM encoder at S. Let $x^i_{s_1}(\mathcal{P}_S)$ denote the symbol to be transmitted corresponding to the $i^{th}$ branch of the path $\mathcal{P}_S$, where $1 \leq i \leq L$.

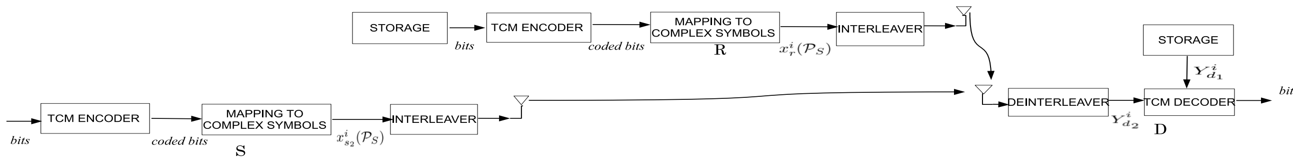
The received complex signal sequence at R and D during Phase 1 are given by,

$$\left.\begin{array}{ll} Y^i_r & = h^i_{sr} x^i_{s_1}(\mathcal{P}_S) + z_r; \\ Y^i_{d_1} & = h^i_{sd_1} x^i_{s_1}(\mathcal{P}_S) + z_{d_1}; \end{array}\right\} 1 \leq i \leq L$$

where $h^i_{sr}$ and $h^i_{sd_1}$ are the independent zero mean circularly symmetric complex Gaussian fading coefficients associated with the S-R and S-D links respectively with the corresponding variances given by $\sigma^2_{sr}$ and $\sigma^2_{sd}$. The additive noises $z_r$ and




(a) Phase 1




(b) Phase 2

Fig. 2. The TCM Scheme

$z_{d_1}$ are assumed to be $CN(0,1)$, where $CN(0,1)$ denotes the standard circularly symmetric complex Gaussian random variable. An interleaver with sufficiently large block length is assumed to make the block fading channel appear as a fading channel with independent fade coefficients for successive channel uses. At R, after deinterleaving, an ML decoder using the Viterbi algorithm is used to decode the bits (Fig. 2(a)). Let $\mathcal{P}_R$ denote the path in the trellis to which R decodes. At D, the received complex numbers are deinterleaved and stored (Fig. 2(a)).

During Phase 2, S encodes the same bits used during Phase 1 and R encodes the bits it decoded during Phase 1, using the TCM encoder. The encoded bits at S and R are mapped onto complex symbols from the signal set $\mathcal{S}$ (possibly with different mappings and also possibly different from the mapping used at S during Phase 1) and interleaved using the block interleavers (same as the interleaver used at S during Phase 1) as shown in Fig. 2(b). Let $x^i_{s_2}(\mathcal{P}_S)$ and $x^i_r(\mathcal{P}_R)$ denote the symbols transmitted by S and R respectively during Phase 2, corresponding to the $i^{th}$ branch of the paths $\mathcal{P}_S$ and $\mathcal{P}_R$, where $1 \leq i \leq L$.

The received complex signal sequence at D during Phase 2 is given by,

$$Y^i_{d_2} = h^i_{sd_2} x^i_{s_2}(\mathcal{P}_S) + h^i_{rd} x^i_r(\mathcal{P}_R) + z_{d_2},$$

where $1 \leq i \leq L$. The random variables $h^i_{sd_2}$ and $h^i_{rd}$ are the independent zero mean circularly symmetric complex Gaussian fading coefficients associated with the S-D and R-D links respectively with the corresponding variances given by $\sigma^2_{sd}$ and $\sigma^2_{rd}$. The additive noise $z_{d_2}$ is $CN(0,1)$. Throughout, it is assumed that $\sigma^2_{sd} \ll \sigma^2_{rd}$, due to the proximity of the relay to the destination than the source to the destination.

During Phase 2, at D the received complex numbers are deinterleaved and along with the complex numbers stored during Phase 1 are fed to the proposed near-ML TCM decoder described in Section II.

### B. Background and Related Work

Achievable rates and upper bounds on the capacity for the discrete memoryless relay channel were obtained in [1]. A turbo-coding scheme for the full duplex and more practical half-duplex Rayleigh fading relay channels were proposed in [2], [3]. These coding schemes achieve a spectral efficiency strictly less than one bpcu, i.e., they pay in bandwidth. In this paper, we propose a Trellis Coded Modulation (TCM) scheme for the half-duplex fading relay channel for any spectral efficiency greater than or equal to one bpcu.

Implementation as well as the performance analysis of the optimal ML decoder for the relay channel operating in the DF mode is complicated [4], [5]. In view of these problems, several sub-optimal decoders for uncoded DF scheme were proposed in [4], [5] and [6]. These problems carry over to coded communication using DF protocol as well. If the decoder at D is designed assuming that R always decodes the bits correctly, the diversity order obtained is dependent on whether the relay decoded correctly or not. To circumvent this problem, relay selection strategies [7], fountain codes [8], Cyclic Redundancy Check (CRC) codes [9] etc. are used. To solve these problems of finding a decoder with implementable complexity and tractable performance analysis, we propose a near-ML decoder, whose performance at high SNR approaches the performance of the optimal ML decoder. The form of this decoder allows a DF scheme in which the relay need not check whether it has decoded correctly before forwarding the message. This eliminates the need for embedding CRC bits as well as the need for feedback from the relay to the source. Furthermore, the diversity order of the proposed scheme does not depend on whether the relay decoded correctly or not.

The proposed TCM scheme is different from the Co-operative Multiple Trellis Coded Modulation (CMTCM) scheme proposed in [9] in the following aspects: (i) The presence of embedded CRC bits and feedback from R to S is assumed in [9], whereas we make no such assumptions. (ii) Interleaving/Deinterleaving is not assumed in [9] whereas we assume interleaving/deinterleaving which makes the quasi-static fading scenario appear as a fast fading scenario.

### C. Contributions and Organization

The main contributions of this paper are as follows:

- A TCM scheme for the fading relay channel is proposed which achieves any spectral efficiency greater than or equal to one bpcu.
- A near-ML decoder is obtained for the proposed TCM scheme (Section II). The relay need not check whether it has decoded correctly before forwarding the message which eliminates the need for embedded CRC bits as well as the feedback from the relay to the source.
- The proposed near-ML decoder enables the formulation of design criteria to maximize the diversity order and the coding gain. (Sections III A, III B and III C).
- The SNR vs BER performance of the near-ML decoder proposed with a non-ideal S-R link, at high SNR, approaches the performance of the optimal ML decoder with an ideal S-R link. This implies that the high SNR performance of the near-ML decoder for the relay channel with a non-ideal S-R link approaches the performance of the optimal ML decoder (Section III D).
- Simulation results show a large gain in the SNR vs BER performance for the proposed TCM scheme over the uncoded transmission scheme as well as the TCM scheme for the direct transmission without the relay (Section V).

***Notations:*** For simplicity, distinction is not made between the random variable and a particular realization of the random variable, in expressions involving probabilities of random variables. In some probability expressions involving conditioning of the fading coefficients, the fact that the probability is conditioned on the values taken by the fading coefficients is not explicitly written, as it can be understood from the context. Throughout, $E_{\mathcal{S}}$ denotes the average energy in dB of the signal set $\mathcal{S}$ used at the source and the relay. Let Q[.] denote the complementary CDF of the standard Gaussian random variable.

*The proofs of Theorems and other claims are omitted due to lack of space but are available in [13] along with more code design examples and more simulation results.*

## II. A NEAR-ML TCM DECODER

The trellis used for decoding at D is constructed as described in the following subsection.

### A. Decoding Trellis at D

Let $\{a^i, a^{i+1}\}$ and $\{b^i, b^{i+1}\}$ respectively denote any two pairs of states in the $i^{th}$ and $i+1^{th}$ stages of the trellises $\mathcal{T}_{S_1}$ and $\mathcal{T}_R$. Let $\mathcal{E}_{a^i,a^{i+1}}$ and $\mathcal{E}_{b^i,b^{i+1}}$ denote the sets of edges from $a^i$ to $a^{i+1}$ and $b^i$ to $b^{i+1}$ respectively.

The trellis used for decoding at D, denoted as $\mathcal{T}_D$, is constructed as follows: The states of $\mathcal{T}_D$ at the $i^{th}$ and $i+1^{th}$ stages are denoted by the two tuples $[a^i, b^i]$ and $[a^{i+1}, b^{i+1}]$ respectively. The set of edges from $[a^i, b^i]$ to $[a^{i+1}, b^{i+1}]$, denoted as $\mathcal{E}_{[a^i,b^i],[a^{i+1},b^{i+1}]}$, contains an edge denoted by the pair $\{e_{a^i,a^{i+1}}, e_{b^i,b^{i+1}}\}$ if $e_{a^i,a^{i+1}} \in \mathcal{E}_{a^i,a^{i+1}}$ and $e_{b^i,b^{i+1}} \in \mathcal{E}_{b^i,b^{i+1}}$. The edge $\{e_{a^i,a^{i+1}}, e_{b^i,b^{i+1}}\}$ is labeled with the four tuple $(\mathcal{X}_{s_1}(e_{a^i,a^{i+1}}), \mathcal{X}_{s_2}(e_{a^i,a^{i+1}}), \mathcal{X}_{s_1}(e_{b^i,b^{i+1}}), \mathcal{X}_r(e_{b^i,b^{i+1}})) \in \mathcal{S}^4$.

For an example of decoding trellis and figures see [13].

The decoding metric for the proposed near-ML decoder is obtained in the following subsection.

### B. Near-ML Decoding Metric

Let $\mathcal{A}_L$ denote the set of all paths of length $L$ in the trellis (since the TCM encoder used at S during Phase 1, at S during Phase 2 and at R during Phase 2 are the same, the set $\mathcal{A}_L$ is the same for the trellises $\mathcal{T}_{S_1}$, $\mathcal{T}_{S_2}$ and $\mathcal{T}_R$). Let $\mathcal{P}_S$ denote the path which corresponds to the symbols transmitted by the source during Phase 1. Let $\mathcal{P}_R$ denote the path to which the Viterbi decoder at $R$ decodes during Phase 1.

The optimal ML decoder at D decides in favor of the path,

$$\hat{\mathcal{P}}_S = \arg \max_{\mathcal{P}_S \in \mathcal{A}_L} Pr\left\{Y_{d_1}^{1,L}, Y_{d_2}^{1,L} | \mathcal{P}_S\right\},$$

where $Y_{d_1}^{1,L}$ and $Y_{d_2}^{1,L}$ denote the sequences $Y_{d_1}^1, Y_{d_1}^2, ...., Y_{d_1}^L$ and $Y_{d_2}^1, Y_{d_2}^2, ...., Y_{d_2}^L$ respectively and $Pr\left\{Y_{d_1}^{1,L}, Y_{d_2}^{1,L} | \mathcal{P}_S\right\}$ is the probability that $Y_{d_1}^{1,L}$ and $Y_{d_2}^{1,L}$ are the received sequences by D during Phase 1 and Phase 2, given that the path corresponding to the complex numbers transmitted by S is $\mathcal{P}_S$. The probability which needs to be maximized is given by,

$$Pr\left\{Y_{d_1}^{1,L}, Y_{d_2}^{1,L} | \mathcal{P}_S\right\} = \sum_{\mathcal{P}_R \in \mathcal{A}_L} Pr\left\{Y_{d_1}^{1,L}, Y_{d_2}^{1,L} | \mathcal{P}_S, \mathcal{P}_R\right\} Pr\left\{\mathcal{P}_R | \mathcal{P}_S\right\}, \tag{1}$$

where $Pr\left\{Y_{d_1}^{1,L}, Y_{d_2}^{1,L} | \mathcal{P}_S, \mathcal{P}_R\right\}$ is the probability that $Y_{d_1}^{1,L}$ and $Y_{d_2}^{1,L}$ are the received sequences by D during Phase 1 and Phase 2, given that the paths corresponding to the complex numbers transmitted by S and R are $\mathcal{P}_S$ and $\mathcal{P}_R$ respectively. The quantity $Pr\left\{\mathcal{P}_R | \mathcal{P}_S\right\}$ is the probability that R decodes to the path $\mathcal{P}_R$ given that $\mathcal{P}_S$ is the path corresponding to the complex symbols transmitted by S.

The probability $Pr\left\{\mathcal{P}_R | \mathcal{P}_S\right\}$ can be upper bounded by the corresponding PEP, i.e.,

$$\begin{aligned} Pr\left\{\mathcal{P}_R | \mathcal{P}_S\right\} &\leq \prod_{i=1}^{L} Q\left[\frac{\left|h_{sr}^i\left(x_{s_1}^i\left(\mathcal{P}_R\right) - x_{s_1}^i\left(\mathcal{P}_S\right)\right)\right|}{\sqrt{2}}\right] \\ &\leq \exp\left\{-\frac{1}{4}\sum_{i=1}^{L}\left|h_{sr}^i\left(x_{s_1}^i\left(\mathcal{P}_S\right) - x_{s_1}^i\left(\mathcal{P}_R\right)\right)\right|^2\right\}, \end{aligned} \tag{2}$$

and we also have,

$$\begin{aligned} &Pr\left\{Y_{d_1}^{1,L}, Y_{d_2}^{1,L} | \mathcal{P}_S, \mathcal{P}_R\right\} = \\ &\frac{1}{\pi^{2L}} \exp\left\{-\sum_{i=1}^{L}\left(\left|Y_{d_1}^i - h_{sd_1}^i x_{s_1}^i\left(\mathcal{P}_S\right)\right|^2\right.\right. \\ &\left.\left. + \left|Y_{d_2}^i - h_{sd_2}^i x_{s_2}^i\left(\mathcal{P}_S\right) - h_{rd}^i x_r^i\left(\mathcal{P}_R\right)\right|^2\right)\right\}. \end{aligned} \tag{3}$$

Substituting (2) and (3) in (1) we get (4), shown at the top of the next page.

If we maximize only the dominant exponential in the upper bound (4), then the decoded path is given by,

$$\hat{\mathcal{P}}_S = \arg \min_{\mathcal{P}_S \in \mathcal{A}_L}\left\{\min_{\mathcal{P}_R \in \mathcal{A}_L} \sum_{i=1}^{L} \phi_i\left(\mathcal{P}_S, \mathcal{P}_R\right)\right\}, \tag{6}$$

where the metric $\phi_i\left(\mathcal{P}_S, \mathcal{P}_R\right)$ is given by (5) (shown at the top of the next page). This near-ML decoder given by (6) involves

$$Pr\left\{Y_{d_1}^{1,L}, Y_{d_2}^{1,L}|\mathcal{P}_S\right\} \leq \frac{1}{\pi^{2L}} \sum_{\mathcal{P}_R \in \mathcal{A}_L} \exp\left\{-\sum_{i=1}^{L}\left(\left|Y_{d_1}^i - h_{sd_1}^i x_{s_1}^i\left(\mathcal{P}_S\right)\right|^2 + \left|Y_{d_2}^i - h_{sd_2}^i x_{s_2}^i\left(\mathcal{P}_S\right) - h_{rd}^i x_r^i\left(\mathcal{P}_R\right)\right|^2 + \frac{1}{4}\left|h_{sr}^i\left(x_{s_1}^i\left(\mathcal{P}_R\right) - x_{s_1}^i\left(\mathcal{P}_S\right)\right)\right|^2\right)\right\} \tag{4}$$

$$\phi_i\left(\mathcal{P}_S, \mathcal{P}_R\right) = \left|Y_{d_1}^i - h_{sd_1}^i x_{s_1}^i\left(\mathcal{P}_S\right)\right|^2 + \left|Y_{d_2}^i - h_{sd_2}^i x_{s_2}^i\left(\mathcal{P}_S\right) - h_{rd}^i x_r^i\left(\mathcal{P}_R\right)\right|^2 + \frac{1}{4}\left|h_{sr}^i\left(x_{s_1}^i\left(\mathcal{P}_S\right) - x_{s_1}^i\left(\mathcal{P}_R\right)\right)\right|^2 \tag{5}$$

minimizing the additive metric over two distinct paths and hence the decoding takes place in the trellis $\mathcal{T}_D$ described in the previous subsection. Throughout, it is assumed that decoding at D takes place using this near-ML decoder.

## III. PEP ANALYSIS OF THE NEAR-ML DECODER AND CODE DESIGN CRITERIA

### A. PEP Analysis of the Near-ML Decoder

For the trellis $\mathcal{T}_{S_1}$, let $\eta_{s_1}(\mathcal{P}_S, \tilde{\mathcal{P}}_S)$ denote the set of values of $i$ for which $x_{s_1}^i(\mathcal{P}_S)$ and $x_{s_1}^i(\tilde{\mathcal{P}}_S)$ are different, for $1 \leq i \leq L$. Similarly, $\eta_{s_2}(\mathcal{P}_S, \tilde{\mathcal{P}}_S)$ and $\eta_r(\mathcal{P}_S, \tilde{\mathcal{P}}_S)$ are defined for the trellises $\mathcal{T}_{S_2}$ and $\mathcal{T}_R$ respectively.

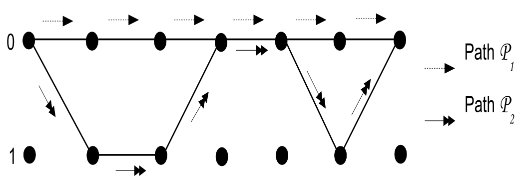



Fig. 3. Example illustrating the notion of unmerged length between two paths

*Definition 1:* The *unmerged length* between the paths $\mathcal{P}_1$ and $\mathcal{P}_2$ in the trellis, denoted as $h(\mathcal{P}_1, \mathcal{P}_2)$ is the number of branches in which the paths $\mathcal{P}_1$ and $\mathcal{P}_2$ differ. For example, for the paths $\mathcal{P}_1$ and $\mathcal{P}_2$ of length 6 shown in Fig. 3, the unmerged length $h(\mathcal{P}_1, \mathcal{P}_2) = 5$. The *unmerged length* of the code is the minimum value of $h(\mathcal{P}_1, \mathcal{P}_2)$, over all possible pairs of paths $(\mathcal{P}_1, \mathcal{P}_2)$ in the trellis.

*Definition 2:* The *effective length* of a pair of paths $(\mathcal{P}_S, \tilde{\mathcal{P}}_S)$ in the trellis $\mathcal{T}_{S_1}$ is defined to be the cardinality of the set $\eta_{s_1}(\mathcal{P}_S, \tilde{\mathcal{P}}_S)$. The *effective length* of the trellis $\mathcal{T}_{S_1}$ is defined to be minimum among the effective lengths of all possible pairs of paths $(\mathcal{P}_S, \tilde{\mathcal{P}}_S)$. In a similar way, the effective lengths of the trellises $\mathcal{T}_{S_2}$ and $\mathcal{T}_R$ can be defined.

*Definition 3:* The *generalized effective length* of a pair of paths $(\mathcal{P}_S, \tilde{\mathcal{P}}_S)$ in the trellis pair $(\mathcal{T}_{S_2}, \mathcal{T}_R)$ is defined to be the cardinality of the set $\left\{\eta_{s_2}(\mathcal{P}_S, \tilde{\mathcal{P}}_S) \cup \eta_r(\mathcal{P}_S, \tilde{\mathcal{P}}_S)\right\}$.

*Theorem 1:* For the proposed TCM scheme, if the trellises $\mathcal{T}_{S_1}$ and $\mathcal{T}_{S_2}$are such that their effective lengths are equal to the unmerged length of the code, at high SNR, the PEP that a transmitted path $\mathcal{P}_S$ in the trellis is decoded as $\tilde{\mathcal{P}}_S$, by the proposed near-ML decoder, is upper bounded by (7) (shown at the top of the next page), where $K$ is a positive constant.

### B. Diversity Criteria

Since the minimum values of both $|\eta_{s_1}(\mathcal{P}_S, \tilde{\mathcal{P}}_S)|$ and $|\eta_{s_2}(\mathcal{P}_S, \tilde{\mathcal{P}}_S) \cup \eta_r(\mathcal{P}_S, \tilde{\mathcal{P}}_S)|$ are upper bounded by the unmerged length of the code, from (7) it can be seen that the diversity order of the proposed TCM scheme is upper bounded by twice the unmerged length of the code. Hence, from (7), it follows that *maximizing the effective lengths of the trellises $\mathcal{T}_{S_1}$ and $\mathcal{T}_{S_2}$* itself is a sufficient condition to guarantee maximum diversity, independent of the labeling $\mathcal{X}_r$. This is not surprising, since during Phase 2 the source encodes the same bits it encoded and transmitted during Phase 1, maximum diversity order is obtained irrespective of whether the relay transmits during Phase 2 or not. But since $\sigma_{sd}^2 \ll \sigma_{rd}^2$, from (7) it can be seen that the coding gain is greatly reduced if the relay does not transmit during Phase 2. It is shown in the following subsection that the effective length of $\mathcal{T}_R$ also should equal the unmerged length of the code to avoid this reduction in coding gain. In the rest of the section it is assumed that the effective lengths of the trellises $\mathcal{T}_{S_1}$, $\mathcal{T}_{S_2}$ and $\mathcal{T}_R$ are made equal to the unmerged length of the code.

From (7), it is clear that an error event corresponding to a pair of paths contributes to the minimum diversity order if and only if the effective length of the pair of paths in $\mathcal{T}_{S_1}$ and the generalized effective length of the pair of paths in $(\mathcal{T}_{S_2}, \mathcal{T}_R)$, are equal to the unmerged length of the code.

### C. Coding Gain Criterion

Let $m_1(\mathcal{P}_S, \tilde{\mathcal{P}}_S)$ denote the product distance of the pair of paths $(\mathcal{P}_S, \tilde{\mathcal{P}}_S)$ in the trellis $\mathcal{T}_{S_1}$, i.e.,

$$m_1(\mathcal{P}_S, \tilde{\mathcal{P}}_S) = \prod_{i \in \eta_{s_1}(\mathcal{P}_S, \tilde{\mathcal{P}}_S)} \left|x_{s_1}^i\left(\mathcal{P}_S\right) - x_{s_1}^i(\tilde{\mathcal{P}}_S)\right|^2.$$

*Definition 4:* The *generalized product distance* of the pair of paths $(\mathcal{P}_S, \tilde{\mathcal{P}}_S)$ corresponding to the trellis pair $(\mathcal{T}_{S_2}, \mathcal{T}_R)$ denoted as $m_2(\mathcal{P}_S, \tilde{\mathcal{P}}_S)$ is defined as,

$$m_2(\mathcal{P}_S, \tilde{\mathcal{P}}_S) = \prod_{i \in \left\{\eta_{s_2}(\mathcal{P}_S, \tilde{\mathcal{P}}_S) \cup \eta_r(\mathcal{P}_S, \tilde{\mathcal{P}}_S)\right\}} \left(\gamma\left|x_{s_2}^i\left(\mathcal{P}_S\right) - x_{s_2}^i(\tilde{\mathcal{P}}_S)\right|^2 + \left|x_r^i\left(\mathcal{P}_S\right) - x_r^i(\tilde{\mathcal{P}}_S)\right|^2\right),$$

where $\gamma = \sigma_{sd}^2/\sigma_{rd}^2$.

*Definition 5:* Consider the trellis triplet $(\mathcal{T}_{S_1}, \mathcal{T}_{S_2}, \mathcal{T}_R)$. The combined product distance of the pair of paths $(\mathcal{P}_S, \tilde{\mathcal{P}}_S)$, denoted by $m(\mathcal{P}_S, \tilde{\mathcal{P}}_S)$, is defined as the product of the product distance of the pair of paths $(\mathcal{P}_S, \tilde{\mathcal{P}}_S)$ in the trellis $\mathcal{T}_{S_1}$ and the generalized product distance of the pair of paths $(\mathcal{P}_S, \tilde{\mathcal{P}}_S)$ corresponding to the trellis pair $(\mathcal{T}_{S_2}, \mathcal{T}_R)$, i.e.,

$$m(\mathcal{P}_S, \tilde{\mathcal{P}}_S) = \left[m_1(\mathcal{P}_S, \tilde{\mathcal{P}}_S) m_2(\mathcal{P}_S, \tilde{\mathcal{P}}_S)\right].$$

*Definition 6:* The combined product distance of the trellis triplet $(\mathcal{T}_{S_1}, \mathcal{T}_{S_2}, \mathcal{T}_R)$, which is also the coding gain metric denoted by $\mathcal{G}$, is defined to be the minimum value of $m(\mathcal{P}_S, \tilde{\mathcal{P}}_S)$, i.e.,

$$\mathcal{G} = \min_{(\mathcal{P}_S, \tilde{\mathcal{P}}_S) \in \mathcal{Z}_{\mathcal{P}}} \left[m(\mathcal{P}_S, \tilde{\mathcal{P}}_S)\right].$$

where $\mathcal{Z}_{\mathcal{P}}$ denotes the set of all pairs of paths $(\mathcal{P}_S, \tilde{\mathcal{P}}_S)$ whose effective length in $\mathcal{T}_{S_1}$ and the generalized effective length in $(\mathcal{T}_{S_2}, \mathcal{T}_R)$ are equal to the unmerged length of the code.

$$Pr\left(\mathcal{P}_S \longrightarrow \tilde{\mathcal{P}}_S\right) \leq K \prod_{i \in \eta_{s_1}(\mathcal{P}_S, \tilde{\mathcal{P}}_S)} \left[\frac{1}{\left|\sigma_{sd}\left(x^i_{s_1}\left(\mathcal{P}_S\right) - x^i_{s_1}(\tilde{\mathcal{P}}_S)\right)\right|^2}\right] \prod_{\substack{i \in \{\eta_{s_2}(\mathcal{P}_S, \tilde{\mathcal{P}}_S) \\ \cup \eta_r(\mathcal{P}_S, \tilde{\mathcal{P}}_S)\}}} \left[\frac{1}{\left|\sigma_{sd}\left(x^i_{s_2}\left(\mathcal{P}_S\right) - x^i_{s_2}(\tilde{\mathcal{P}}_S)\right)\right|^2 + \left|\sigma_{rd}\left(x^i_r\left(\mathcal{P}_S\right) - x^i_r(\tilde{\mathcal{P}}_S)\right)\right|^2}\right] \tag{7}$$

From (7), it can be seen that *the combined product distance of the trellis triplet* $(\mathcal{T}_{S_1}, \mathcal{T}_{S_2}, \mathcal{T}_R)$ *needs to be maximized, to maximize the coding gain*. For a pair of paths $(\mathcal{P}_S, \tilde{\mathcal{P}}_S)$ which contribute to the minimum diversity order, if the effective length of $(\mathcal{P}_S, \tilde{\mathcal{P}}_S)$ in the trellis $\mathcal{T}_R$ is not equal to the unmerged length of the code, then $x^i_r(\mathcal{P}_S) = x^i_r(\tilde{\mathcal{P}}_S)$ for some $i \in \{\eta_r(\mathcal{P}_S, \tilde{\mathcal{P}}_S) \cup \eta_{s_2}(\mathcal{P}_S, \tilde{\mathcal{P}}_S)\}$. In that case $m_2(\mathcal{P}_S, \tilde{\mathcal{P}}_S)$ is greatly reduced since $\gamma \ll 1$. In order to avoid this, the effective length of the trellis $\mathcal{T}_R$ should be made equal to the unmerged length of the code.

### D. Near Optimality of the Proposed Near-ML decoder

The following argument proves the high SNR near optimality of the proposed near-ML decoder for the TCM scheme. Consider the situation where the S-R link is ideal, i.e., the relay decodes all the bits correctly. The optimal ML decoder at D decides in favor of the path given by,

$$\hat{\mathcal{P}}_S = \arg \min_{\mathcal{P}_S \in \mathcal{A}_L} \left( \sum_{i=1}^{L} \left| Y^i_{d_1} - h^i_{sd_1} x^i_{s_1}\left(\mathcal{P}_S\right)\right|^2 + \left| Y^i_{d_2} - h^i_{sd_2} x^i_{s_2}\left(\mathcal{P}_S\right) - h^i_{rd} x^i_r\left(\mathcal{P}_S\right)\right|^2 \right).$$

For this case, the PEP that the path $\mathcal{P}_S$ is decoded as $\tilde{\mathcal{P}}_S$ is upper bounded by (8) (shown at the top of the next page). Taking expectation with respect to the fading coefficients, we get an upper bound which is the same as (7). In other words the high SNR bounds on the PEP for the proposed near ML decoder with a non-ideal S-R link is same as that of the optimal ML decoder with an ideal S-R link, if we consider only the error events giving rise to the minimum diversity order. The simulation results presented in Section VI confirm that the $E_{\mathcal{S}}$ vs BER performance of the proposed near ML decoder with a non-ideal S-R link approaches the performance of optimal ML decoder with an ideal S-R link, at high SNR. The high SNR performance of the optimal ML decoder for the proposed TCM scheme, with a non-ideal S-R link cannot be better than that of optimal ML decoder for the case when the S-R link is ideal. This implies the high SNR performance of the proposed near-ML decoder approaches the performance of the optimal ML decoder.

## IV. TCM CODE DESIGN EXAMPLES

Let the product distance $m_{21}(\mathcal{P}_S, \tilde{\mathcal{P}}_S)$ and the metric $\mathcal{G}_2$ be defined as,

$$m_{21}(\mathcal{P}_S, \tilde{\mathcal{P}}_S) = \prod_{i \in \{\eta_{s_2}(\mathcal{P}_S, \tilde{\mathcal{P}}_S) \cup \eta_r(\mathcal{P}_S, \tilde{\mathcal{P}}_S)\}} \left| x^i_r\left(\mathcal{P}_S\right) - x^i_r(\tilde{\mathcal{P}}_S)\right|^2,$$

$$\mathcal{G}_2 = \min_{(\mathcal{P}_S, \tilde{\mathcal{P}}_S) \in \mathcal{Z}_{\mathcal{P}}} \left[ m_1(\mathcal{P}_S, \tilde{\mathcal{P}}_S) m_{21}(\mathcal{P}_S, \tilde{\mathcal{P}}_S) \right].$$

Since $\gamma \ll 1$, $\mathcal{G} \approx \mathcal{G}_2$ and instead of maximizing $\mathcal{G}$, $\mathcal{G}_2$ which is independent of $\gamma$, can be maximized.

In the examples considered in this subsection, S transmits two information bits to D (spectral efficiency = 1 bpcu) in the two phases of relaying (design examples which achieve higher rates are omitted for lack of space but are available in [13]). In all these examples the values of the variances of the fading links are assumed to be $\sigma^2_{sd} = 0$ dB, $\sigma^2_{sr} = 15$ dB and $\sigma^2_{rd} = 15$ dB, for which $\gamma = 0.0316$. The value of the metrics $\mathcal{G}_2$, $\mathcal{G}$ (for $\gamma = 0.0316$) and the diversity order, for the two, four, eight and sixteen state TCM design examples considered below are given in Table I.

It can be shown that for a 2 state trellis with 8-PSK signal set, Ungerboeck's labeling [10] for $\mathcal{X}_{s_1}$, $\mathcal{X}_{s_2}$ and $\mathcal{X}_r$ guarantees maximum diversity and maximizes the coding gain metric $\mathcal{G}_2$. Consider the case when encoding at S and R take place using a four state trellis (Fig. 4) with symmetric 8 PSK signal set. The points in the 8-PSK signal set are named as $s_1, s_2, ..., s_8$ in the counter-clockwise direction. In the trellis diagram shown, the triple inside [.] shown to the left of each state, when read from left to right, denotes the labellings $(\mathcal{X}_{s_1}, \mathcal{X}_r, \mathcal{X}_{s_2})$ corresponding to the edges emerging from top to bottom (the labellings $\mathcal{X}_{s_1}, \mathcal{X}_r, \mathcal{X}_{s_2}$ are shown in the same trellis diagram instead of three different trellis diagrams). If $\mathcal{X}_{s_1}$, $\mathcal{X}_{s_2}$ and $\mathcal{X}_r$ are chosen based on the design criteria for the point to point fast fading channel (Jamali et al. labeling [12]), the value obtained for the metric $\mathcal{G}_2 = 5.49$. Instead, if the labellings $\mathcal{X}_{s_1}$, $\mathcal{X}_{s_2}$ and $\mathcal{X}_r$ are chosen as shown in Fig. 4, the value of $\mathcal{G}_2 = 16$. The simulation results presented confirm that the labeling shown in Fig. 4 performs better than the case when both S and R use Jamali et al. labeling. For eight and sixteen state trellises with 8-PSK signal set, with Ungerboeck's labeling [10] used for $\mathcal{X}_{s_1}$, $\mathcal{X}_{s_2}$ and $\mathcal{X}_r$, the values of $\mathcal{G}_2$, $\mathcal{G}$ (for $\gamma = 0.0316$) and the diversity order obtained are given in Table I.

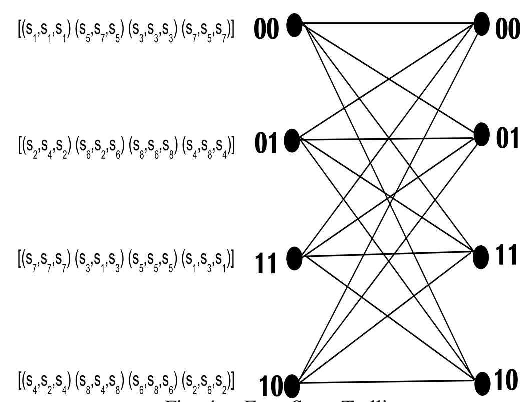


Fig. 4. Four State Trellis

## V. SIMULATION RESULTS

The $E_{\mathcal{S}}$ vs $BER$ plots for the different schemes for the relay channel and the direct transmission, achieving a spectral efficiency of 1 bpcu are shown in Fig. 5. The TCM schemes for the direct transmission were designed based on the design criteria for TCM for fast fading channels [11]. The value of $E_{\mathcal{S}}$ required at a BER of $10^{-4}$ for the different schemes are summarized in Table I. As the number of states of the proposed TCM scheme for the relay channel increases, the gain in $E_{\mathcal{S}}$

$$Pr\left\{\mathcal{P}_S \longrightarrow \tilde{\mathcal{P}}_S\right\} \leq \frac{1}{\pi^{2L}} \exp\left\{\sum_{i=1}^{L}\left(-\frac{1}{2}\left|h_{sd_1}^{i}\left(x_{s_1}^{i}\left(\mathcal{P}_S\right)-x_{s_1}^{i}(\tilde{\mathcal{P}}_S)\right)\right|^2-\frac{1}{2}\left|h_{sd_2}^{i}\left(x_{s_2}^{i}\left(\mathcal{P}_S\right)-x_{s_2}^{i}(\tilde{\mathcal{P}}_S)\right)+h_{rd}^{i}\left(x_{r}^{i}\left(\mathcal{P}_S\right)-x_{r}^{i}(\tilde{\mathcal{P}}_S)\right)\right|^2\right)\right\} \tag{8}$$

TABLE I

COMPARISON OF DIFFERENT SCHEMES FOR A SPECTRAL EFFICIENCY OF 1 BPCU, FOR $\sigma_{sd}^2 = 0\ dB$, $\sigma_{sr}^2 = 15\ dB$ AND $\sigma_{rd}^2 = 15\ dB$.

| Scheme | Diversity Order | $\mathcal{G}_2$ | $\mathcal{G}$, for $\gamma = 0.0316$ | $E_{\mathcal{S}}$ in dB for $BER = 10^{-4}$ |
|---|---|---|---|---|
| Direct Tx. - BPSK | 1 | - | - | 37 dB |
| Direct Tx. - 2 State TCM | 2 | - | - | 18 dB |
| Direct Tx. - 4 State TCM | 3 | - | - | 12 dB |
| Relay Channel - Uncoded 4 PSK | 2 | 4 | 4.13 | 12.5 dB |
| Relay Channel - 2 State 8 PSK TCM | 2 | 16 | 16.51 | 8 dB |
| Relay Channel - 4 State 8 PSK TCM - Jamali et al. Labeling | 4 | 5.49 | 5.67 | 3.6 dB |
| Relay Channel - 4 State 8 PSK TCM - Our Labeling | 4 | 16 | 16.17 | 3 dB |
| Relay Channel - 8 State 8 PSK TCM | 4 | 64 | 66.02 | 1.5 dB |
| Relay Channel - 16 State 8 PSK TCM | 6 | 21.96 | 22.66 | 0 dB |

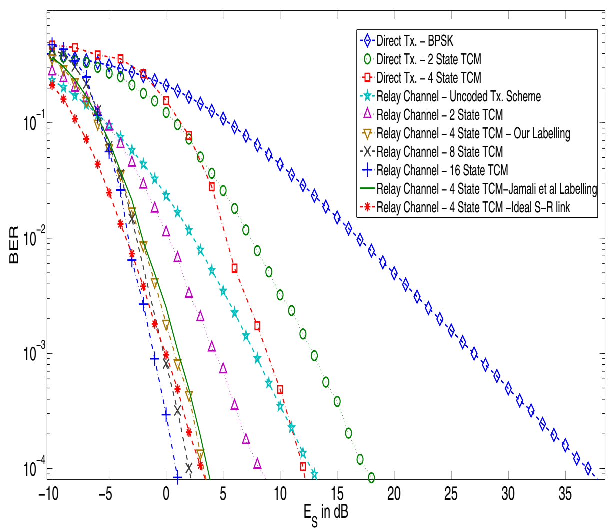



Fig. 5. $E_{\mathcal{S}}$ vs $BER$ for the uncoded transmission scheme using 4 PSK and TCM Schemes for $\sigma_{sd}^2 = 0$ dB, $\sigma_{sr}^2 = 15$ dB and $\sigma_{rd}^2 = 15$ dB

obtained at a BER of $10^{-4}$ over the uncoded transmission using 4 PSK increases. For instance, a large gain of 12.5 dB is obtained for the 16 state 8 PSK TCM scheme over uncoded transmission using 4 PSK. Also, it can be seen from Table I that the 4 state 8 PSK scheme with our labeling provides a gain of 0.6 dB over the case when S and R use Jamali et al. labeling.

Fig. 5 also shows the $E_{\mathcal{S}}$ vs $BER$ plot for the 4 state 8 PSK TCM scheme with our labeling, for the case when the S-R link is ideal. As observed in Section IV D, the high SNR performance of the proposed near-ML decoder with non-ideal S-R link approaches the performance of the optimal ML decoder with an ideal S-R link.

From Table I, it can be seen that the proposed schemes for the relay channel, which exploit the fact that the S-R and R-D links are better than the S-D link, outperform the corresponding schemes for the direct transmission without relay. It can be verified that the value of $E_{\mathcal{S}}$ required for the direct transmission without relay to achieve a capacity of 1 bpcu is 1.5 dB (for details see [13]), which is greater than the value of $E_{\mathcal{S}}$ (0 dB) required for the 16 state TCM scheme for the relay channel, for a BER of $10^{-4}$. This means that under the given assumptions, the proposed 16 state 8 PSK TCM scheme for the relay channel, will outperform even the best possible coding scheme for the direct transmission without relay. The claim made above is valid even for other values of the variances of the fading links, possibly with a higher number of states if not with 16 states.

## VI. DISCUSSION

A TCM scheme for the half duplex fading relay channel was proposed. A near-ML decoder whose high SNR performance approaches the performance of the optimal ML decoder was obtained. Based on the expression for PEP bounds, code design criteria to maximize the diversity order and coding gain were formulated. Simulation results showed a large gain in SNR for the proposed TCM scheme over uncoded communication as well as the direct transmission without relay.

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