Study On The Uplink Sum Capacity of Single Cell Cellular Systems With Minimum SINR Constraint

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***Abstract*—This paper studies the uplink information-theoretical sum capacity of single cell, multi-user cellular systems with minimum SINR constraint. The uplink of single cell multiuser systems is interference limited. If one user increases its transmission power, it will bring more interference to other users. Therefore, power control should be used to balance all users' transmission power. To achieve the maximum system capacity, the optimal power allocation schedules users according to their link gains, which will lead to unfair problem. In order to avoid this problem, a QoS constraint in the form of minimum SINR constraint could be introduced. In previous works, some properties of the optimal allocation under this constraint have been found. However, it is still difficult to get the optimal power allocation. In this paper, we have found some new mathematical properties of the optimal power allocation, and based on these properties, the algorithm for getting the optimal power allocation is proposed. Numerical results are also presented to show the effects of minimum SINR constraint on system capacity.**

I. Introduction

RECENT years, with rapid growth of mobile data traffic, a great deal of attentions have been attracted on the information theoretic capacity of multi-user cellular systems in the industry. The uplink capacity of modern cellular systems is mainly limited by interferences among users. The signal power transmitted by user is not only a necessity for its own data transmission, but also the interferer to other users' communications. To maximize the overall throughput, one should carefully control the power transmitted by each user from causing serious interference.

There are already many works on the capacity of the reverse link of multi-user systems. For the uplink of single cell multi-user systems, the information-theoretical bound has been discussed in [1]. The information-theoretic sum capacity of single cell multi-user systems have been investigated in [2]-[7]. In [2], Inaltekin and Hanly have shown that the optimal solution of the maximum capacity for the uplink of a single cell multi-user system with peak power constraint is binary power control, i.e. each user either transmitting with peak power or be turned off. For two-cell two-user systems [8], binary power control is also optimal. In [9] and [10], it has shown that for a multi-cell system, the optimal power allocation have at most one user transmitting with power between 0 and peak power, no matter whether successive interference cancelation is considered.

As shown in [2] and [3], the optimal power allocation always schedules users successively according to their link gains. It will lead to an unfair resources allocation. To avoid this situation, in practical systems, a QoS constraint is usually imposed [4] [5]. The QoS constraint can be considered as each user has a minimum throughput, which is necessary for low throughput low latency services, such as VoIP. The minimum throughput requirement can be further interpreted as a minimum SINR requirement for each user. In this case, binary power control is not the optimal solution. It has been indicated in [4] that the throughput maximization results is a greedy power allocation whereas the log-throughput maximization results in a proportional fair allocation. In [5], it has shown that the optimal power allocation can have either maximum power or minimum power that can maintain the minimum SINR requirement, with at most one exceptional component. However, it is difficult to find out the exceptional component and calculate the transmitting power of each user.

In this paper, we find some new mathematical properties of the optimal power allocation under the constraints of peak transmit power and minimum SINR constraint for each user: (1) if there is a user transmits at neither peak power nor minimum power that can ensure the SINR requirement, the user with the worst channel condition must transmit at peak power to satisfy its minimum SINR requirement; (2) if the user with the worst channel condition does not transmit at peak power, the optimal power allocation is to schedule users successively according to their link gains. With these properties, we can exactly find out the user that transmitting with neither peak power nor minimum power that satisfies the SINR requirement. Furthermore, we propose a low complexity algorithm to get the optimal power allocation.

The remainder of this paper is organized as follows. Section II gives the problem statement. Section III studies the properties of the optimal power allocation. Based on these properties, section IV proposes a algorithm for the optimal power allocation. The numerical results are given in section V. Finally, we conclude in section VI.

II. Problem statement

Consider a single cell system with M users, whose reverse link gains are $g_1, g_2, g_3, \cdots, g_M$. Without loss of generality,

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assume that $g_1 > g_2 > g_3 > \cdots > g_M > 0$. Define p_i as the i-th user's transmitting power which is the i-th element of the power allocation vector $\mathbf{p} = (p_1, p_2, ..., p_M)$, and is subjected to a peak power constraint:

$$0 \le p_i \le P_{\max}, i = 1, 2, ..., M. \tag{1}$$

Then the signal to interference-plus-noise ratio(SINR) of the i-th user at the base station(BS) can be given by

$$\gamma_i = \frac{p_i g_i}{I + \sum\limits_{\substack{t=1 \\ t \ne i}}^{M} p_t g_t} = \frac{x_i}{A - x_i}, \tag{2}$$

where $x_i = \frac{p_i g_i}{I}$ is the normalized received power of the i-th user, I is background noise at BS, $A = 1 + \sum\limits_{t=1}^{M} x_t$ is rise over thermal (ROT) at BS. The maximum achievable data rate of the i-th user is $\ln(1 + \gamma_i)$ nats/symbol, and the sum capacity of the system is

$$C(\mathbf{x}) = \sum_{i=1}^{M} \ln\left(\frac{A}{A - x_i}\right), \tag{3}$$

where $\mathbf{x} = (x_1, x_2, ..., x_M)$. For a given channel condition $(g_1, g_2, ..., g_M)$ and background noise I, the optimal power allocation always schedules users successively according to their link gains. It will lead to an unfair resources allocation[2]. To avoid this situation, in practical systems, a QoS constraint is usually imposed. The QoS constraint can be considered as each user has a minimum throughput, which is necessary for low throughput low latency services, such as VoIP. The minimum throughput requirement can be further interpreted as a minimum SINR requirement for each user. Let γ be the minimum SINR such that each user has to meet, then $\gamma_i \ge \gamma$, i.e. $x_i \ge \frac{\gamma A}{1+\gamma}$. The necessary and sufficient condition that all users can meet the minimum SINR requirement is [5]:

$$P_{\max} g_M \ge \frac{\gamma I}{1 - \gamma(M-1)}. \tag{4}$$

Our object is to find the properties of $\mathbf{p}$, or equivalently $\mathbf{x}$, which can maximize (3) under the constraints of peak power and minimum SINR.Then the optimization problem will be

$$\max_{\mathbf{x}} \sum_{i=1}^{M} \ln\left(\frac{A}{A - x_i}\right), \tag{5}$$

$$s.t. \left\{ \begin{array}{c} 0 \le x_i \le \frac{g_i P_{\max}}{I}, \\ x_i \ge \frac{\gamma A}{1+\gamma} \end{array} \right\}. \tag{6}$$

The optimal problem above is hard to solve because of the non-convex nature. Nevertheless, we can find some properties of the optimal power allocation.

III. Properties of the Optimal power allocation

Firstly, we divide all users in the system into three groups:

Group 1: users that transmit with maximum power $P_{\max}$ and meet the minimum SINR requirement with inequality. Denote this group as φ_1, then for any $i \in \varphi_1$, $x_i = \frac{g_i P_{\max}}{I}$ and $x_i > \frac{\gamma A}{1+\gamma}$.

Group 2: users that with transmit power smaller than $P_{\max}$ and meet the minimum SINR requirement with inequality. In the following, we will refer such power as in-between value power. Denote this group as φ_2, then for any $i \in \varphi_2$, $\frac{\gamma A}{1+\gamma} < x_i < \frac{g_i P_{\max}}{I}$.

Group 3: users that meet the minimum SINR requirement with equality. Denote this group as φ_3, then for any $i \in \varphi_3$, $x_i = \frac{\gamma A}{1+\gamma}$.

It has been proven in [5] that for the optimal power allocation, there is at most one user belongs to φ_2. Based on this conclusion, we can have the following lemma:

lemma 1: If $\mathbf{x}^*$ is the optimal solution of (5) under the constraint (6) with $\varphi_2 \ne \phi$, where ϕ is empty set, then $x_M^* = \frac{g_M P_{\max}}{I} = \frac{\gamma A^*}{1+\gamma}$. In other words, if the optimal power allocation has one in-between component, the user with the worst channel condition must transmit with the maximum power, and satisfies the minimum SINR constraint with equality.

Proof: Let $\mathbf{x}^*$ be the optimal solution of (5) under the constraint (6). Let k be the number of elements in φ_3 and n be the index of element in φ_2, $0 \le k \le M-1$, $1 \le n \le M$. We assume that $x_M^* < \frac{g_M P_{\max}}{I}$. Define $\mathbf{x}^{**}$ as

$$\mathbf{x}^{**} = \left\{ \begin{array}{cc} x_i^*, & i \in \varphi_1 \\ x_i^* + \Delta, & i = n \\ x_\gamma^{**}, & i \in \varphi_3 \end{array} \right. , \tag{7}$$

where Δ is chosen as in (8) and

$$x_\gamma^{**} = \frac{\gamma(A_{M-k-1}^* + x_n^* + \Delta)}{1 - \gamma(k-1)}, \tag{9}$$

$$A_{M-k-1}^* = 1 + \sum_{i \in \varphi_1} x_i^*. \tag{10}$$

Then all the elements in $\mathbf{x}^{**}$ can satisfy (6). The sum capacity of the system will be (11).

Denote $C_\Delta^{(1)}$ and $C_\Delta^{(2)}$ as the first and second derivations of (11) with respect to Δ, we can get $C_\Delta^{(1)}$ and $C_\Delta^{(2)}$ as (12) and (13), respectively, where

$$H = A_{M-k-1}^* + x_n^* + \Delta + k x_\gamma^{**}, \tag{14}$$

$$E = \frac{\gamma k}{1 - \gamma(k-1)}. \tag{15}$$

It can be shown that when $C_\Delta^{(1)} = 0$, $C_\Delta^{(2)} > 0$. Therefore, the maximum value of C_Δ is at the boundary but not at $\Delta = 0$. This contradicts the optimality assumption. Hence, if $\varphi_2 \ne \phi$, $x_M^* = \frac{g_M P_{\max}}{I}$.

Suppose $x_M^* = \frac{g_M P_{\max}}{I} > \frac{\gamma A^*}{1+\gamma}$. We will show that $\varphi_3 = \phi$. If $\varphi_3 \ne \phi$, there must exist a $x_t^* = \frac{\gamma A^*}{1+\gamma}, t \in \{1, 2, ..., M-1\}, t \ne n$. Construct $\mathbf{x}^{***}$ the same as $\mathbf{x}^*$ except the t-th and M-th elements are $x_t^{***} = \frac{g_M P_{\max}}{I} < \frac{g_t P_{\max}}{I}$ and $x_M^{***} = \frac{\gamma A^*}{1+\gamma}$. i.e.

$$\mathbf{x}^{***} = (x_1^*, ..., x_{t-1}^*, x_t^{***}, x_{t+1}^*, ..., x_{M-1}^*, x_M^{***}). \tag{16}$$

Then the sum rate given by (5) does not change. However, both x_t^{***} and x_M^{***} belong to φ_2, there must exist a power allocation $\hat{\mathbf{x}}$ makes $C(\hat{\mathbf{x}}) > C(\mathbf{x}^{***})$. This contradicts the optimality assumption. Hence, $x_t^* = \frac{g_t P_{\max}}{I}$, $\varphi_3 = \phi$.

$$|\Delta| \le \min \left\{ \begin{array}{c} [1-\gamma(k-1)]x_i^* - \gamma(A_{M-k-1}^* + x_n^*), \forall i \in \varphi_1 \\ \frac{g_i P_{\max}}{I} - x_i^*, \forall i \in \varphi_3 \\ \frac{g_n P_{\max}}{I} - x_n^* \\ x_n^* - \frac{\gamma A_{M-k-1}^*}{1-\gamma k} \end{array} \right\}, \tag{8}$$

$$C_\Delta = \sum_{i\in\varphi_1} \ln(1 + \frac{x_i^*}{A_{M-k-1}^* + x_n^* + \Delta + kx_\gamma^{**} - x_i^*}) + \ln\left(1 + \frac{x_n^* + \Delta}{A_{M-k-1}^* + kx_\gamma^{**}}\right) + k\ln(1+\gamma) \tag{11}$$

$$C_\Delta^{(1)} = \frac{1}{H}\left[\sum_{i\in\varphi_1} \frac{-x_i^*(1+E)}{H-x_i^*} + \frac{H-x_n^*-\Delta-E(x_n^*+\Delta)}{H-x_n^*-\Delta}\right] \tag{12}$$

$$C_\Delta^{(2)} = -\frac{1+E}{H}C_\Delta^{(1)} + \frac{1}{H}\left[\sum_{i\in\varphi_1} \frac{x_i^*(1+E)^2}{(H-x_i^*)^2} - \frac{(H-x_n^*-\Delta)E - E^2(x_n^*+\Delta)}{(H-x_n^*-\Delta)^2}\right] \tag{13}$$

When $\varphi_3 = \phi$ and $x_M \in \varphi_2$, with the discussions above, there must exist a power allocation $\overset{\vee}{\mathbf{x}}$ makes $C(\overset{\vee}{\mathbf{x}}) > C(\mathbf{x}^{***})$. This contradicts the optimality assumption. Therefore, $x_M^* = \frac{g_M P_{\max}}{I} = \frac{\gamma A^*}{1+\gamma}$. ■

Lemma 1 shows that if the optimal power allocation has an in-between component, the user with the worst channel condition must transmit at peak power to satisfy its minimum SINR requirement. Hence, for the optimal power allocation, if the user with the worst channel condition does not transmit at the peak power, all elements should take value at either $\frac{\gamma A^*}{1+\gamma}$ or $\frac{g_i P_{\max}}{I}$. In this case, we have the following lemma:

lemma 2: If $\mathbf{x}^*$ is the optimal solution of (5) under the constraint (6) with $x_M^* \ne \frac{g_M P_{\max}}{I}$, then if $g_t > g_v$, $x_t^* \ge x_v^*$.

Proof: From lemma 1, if $x_M^* \ne \frac{g_M P_{\max}}{I}$, then $x_i^* \in \left\{\frac{\gamma A^*}{1+\gamma}, \frac{g_i P_{\max}}{I}\right\}$, for $\forall i \in \{1,2,...,M\}$. Since $g_t > g_v$, lemma 2 is clearly true when $x_t^* = \frac{g_t P_{\max}}{I}$ or $x_t^* = x_v^* = \frac{\gamma A^*}{1+\gamma}$. Now suppose that $x_t^* = \frac{\gamma A^*}{1+\gamma}, x_v^* = \frac{g_v P_{\max}}{I}$. Construct $\mathbf{x}^{**}$ the same as $\mathbf{x}^*$ except the t-th and v-th elements as $x_t^{**} = \frac{g_v P_{\max}}{I}, x_v^{**} = \frac{\gamma A^*}{1+\gamma}$. Then $C(\mathbf{x}^{**}) = C(\mathbf{x}^*)$, $A^{**} = A^*$, $\frac{\gamma A^{**}}{1+\gamma} < x_t^{**} < \frac{g_t P_{\max}}{I}$, $x_M^{**} = x_M^* \ne \frac{g_M P_{\max}}{I}$. Thus, from the proof of lemma 1, the sum capacity can be further increased by adding a non zero value on x_t^{**}. This contradicts the optimality assumption. ■

IV. PROPOSED ALGORITHM

With lemma 1 and 2, if $\mathbf{x}^*$ is the optimal solution of (5) under constraint (6) with $x_M^* \ne \frac{g_M P_{\max}}{I}$, its corresponding power allocation will have the following structure:

$$p_k^* = \left\{ \begin{array}{ll} P_{\max}, & 1 \le k \le n \\ \frac{\gamma B_n^*}{g_k[1-\gamma(M-n-1)]}, & n < k \le M \end{array} \right., \tag{17}$$

where $1 \le n \le M$, $B_n^* = I + \sum_{i=1}^{n} g_i P_{\max}$. The maximum sum capacity will be (18), where

$$G = \frac{\gamma(M-n)}{1-\gamma(M-n-1)}. \tag{19}$$

It has been derived in [5] that one of the optimal solutions of (5) under the constraint (6) has the following form:

$$\mathbf{x}^{**} = \left(\frac{g_1 P_{\max}}{I}, ..., \frac{g_{i_T-1} P_{\max}}{I}, x_{i_T}^{**}, x_{i_T+1}^{**}, ..., x_M^{**}\right), \tag{20}$$

where i_T is an integer, $1 \le i_T \le M, \frac{\gamma A^{**}}{1+\gamma} \le x_{i_T}^{**} \le \frac{g_{i_T} P_{\max}}{I}$, and $\frac{x_m^{**}}{A^{**} - x_m^{**}} = \gamma$, for any $m > i_T$.

With lemma 1, when $\varphi_2 \ne \phi$, the M-th user must transmit at $P_{\max}$ to meet the minimum SINR requirement. Then we can get

$$\frac{g_M P_{\max}}{I + \sum_{i=1}^{i_T-1} g_i P_{\max} + p_{i_T}^{**} g_{i_T} + (M-i_T-1) g_M P_{\max}} = \gamma. \tag{21}$$

The optimal power allocation will be

$$p_k^{**} = \left\{ \begin{array}{ll} P_{\max}, & 1 \le k \le i_T - 1 \\ \frac{g_M P_{\max} - F_{i_T}}{\gamma g_{i_T}}, & k = i_T \\ \frac{g_M P_{\max}}{g_k}, & i_T < k \le M \end{array} \right., \tag{22}$$

where

$$F_{i_T} = \gamma\left[I + \sum_{i=1}^{i_T-1} g_i P_{\max} + (M - i_T - 1) g_M P_{\max}\right]. \tag{23}$$

The maximum sum capacity will be (24).

Based on the results above, the algorithm for the optimal power allocation is given as follows:

1. Let C_n be the capacity when n users in φ_1, and $M-n$ users in φ_3. C_n and its corresponding power allocation can be get with (17) and (18). Calculate C_n successively with $n = 1, 2, ...$ until $p_M > P_{\max}$ or $n = M$. If $n < M$, set $T = n$, $C_t = 0, t = n, n+1, ..., M$; else set $T = 0$.

$$C_n = \sum_{i=1}^{n} \ln\left[1 + \frac{g_i P_{\max}}{(1+G)B_n^* - g_i P_{\max}}\right] + (M-n)\ln(1+\gamma), \tag{18}$$

$$C'_{i_T} = \sum_{i=1}^{i_T - 1} \ln\left[1 + \frac{\gamma g_i}{(1+\gamma)g_M - \gamma g_i}\right] + \ln\left(1 + \frac{g_M P_{\max} - F_{i_T}}{F_{i_T} + \gamma g_M P_{\max}}\right) + (M-n)\ln(1+\gamma). \tag{24}$$

2. If $T = 1$, go to step 4; else let $N = \arg\max_{n=1,2,\ldots,M} C_n$.

3. If $T = 0$, go to step 5; else take $i_T = T$ into (24) and get C'_T. If $C'_T > C_N$, go to step 4, else go to step 5.

4. Set $p_1 = p_2 = \ldots = p_{T-1} = P_{\max}$, $p_T = \frac{g_M P_{\max} - F_T}{\gamma g_T}$, $p_i = \frac{g_M P_{\max}}{g_i}$ for $i = T+1, \ldots, M$, where $F_T = \gamma[I + \sum_{i=1}^{T-1} g_i P_{\max} + (M-T-1) g_M P_{\max}]$. The maximum system capacity is C'_T. Stop.

5. Set $p_1 = p_2 = \ldots = p_N = P_{\max}$. If $N < M$, set $p_i = \frac{\gamma B_N^*}{g_i[1-\gamma(M-N-1)]}$ for $i = N+1, \ldots, M$, with $B_N^* = I + \sum_{t=1}^{N} g_i P_{\max}$. The maximum system capacity is C_N.

V. Numerical Results

In this section, numerical results for the system capacity are presented. We assume a single hexagonal cellular system with radius of $r = 500$ meters. Users are uniformly distributed in the cell. The link gain between each mobile and BS is modeled as

$$g_i = g_r \, (d_i/r)^{-n} \, 10^{\xi_i/10} \tag{25}$$

where r is the radius of the cell, d_i is the distance between user i and BS, $n = 3.5$ is the path loss exponent, $1/g_r = 121.1$dB is the path loss at the cell edge and $10^{\xi_i/10}$ is shadowing, which is a log-normal variable with a standard deviation of 8.9dB. The peak power $P_{\max} = 0.2$Watt or 23dBm and the thermal noise power I is -138.1dB.

Define outage rate as percentage of users that can not satisfy the following constraint:

$$P_{\max} g \geq \frac{\gamma I}{1 - \gamma(M-1)}. \tag{26}$$

where g represents the link gain between user and base station. Fig. 1 gives the outage rate versus the number of users when minimum SINR is -10dB. Obviously, with the increase of the number of users, the outage rate increases.

Fig. 2 shows the maximum sum capacity of the system with and without minimum SINR constraint(WO SINR constraint). It can be seen that the capacity gap between with and without minimum SINR constraint increases with the increase of the number of users. Besides, this gap increases with the increase of minimum SINR constraint. When the limitation of minimum SINR constraint goes to zero, the gap disappears. It means that the multiuser diversity gain might not compensate the increase of interference from more user access when the value of minimum SINR constraint is high.

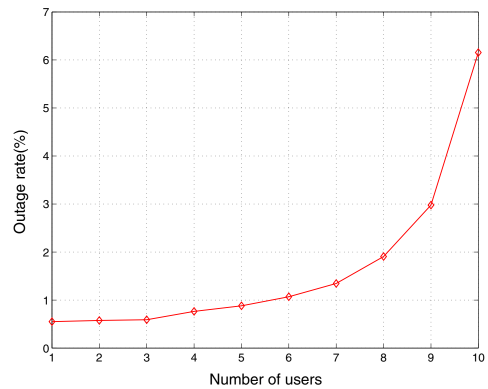

Fig. 1. Outage rate versus number of users when minimum SINR = -10dB.

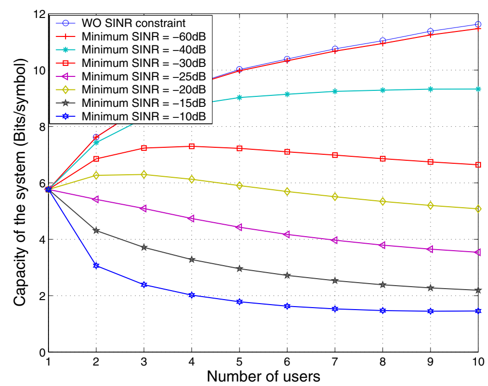

Fig. 2. Sum capacity of system with and without minimum SINR constraint.

VI. Conclusion

In this paper, we have discussed the optimal power allocation for the reverse link of single cell multi-user cellular systems with maximum transmit power and minimum SINR constraint. Some properties of the optimal power allocation are presented. Based on these properties, a simple algorithm for the optimal power allocation is proposed.

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