A SIMPLE PROOF OF THE GENERALIZED OPTIMUM CONTINUOUS RUNNING-APPROXIMATION BASED ON A CLASS OF MULTI-LEGGED-TYPE SIGNALS

Yuichi Kida and Takuro Kida

The School of Pharmaceutical Sciences, Ohu University, Koriyama-shi, Japan
‡Professor Emeritus, Tokyo Institute of Technology, Tokyo Japan

ABSTRACT

In this paper, with respect to band-limited signals $f(t)$ contained in a given set of signals Ξ, we present very general optimum continuous running approximations that minimize various continuous worst-case measures of running approximation error simultaneously. Firstly, as a means of proof, we consider a multi-legged-type signal $m(t)$ that is a combined-signal of a given infinite number of one-dimensional band-limited signals $h_n(t)$ $(n = \dots, -2, -1, 0, 1, 2, \dots)$ in Ξ. A series of given finite segments $\sigma_n(t)$ of $h_n(t)$ are arranged sequentially and make backbone of $m(t)$. Sets of two other signals of $h_n(t)$ make feet of $m(t)$. With respect to these $h_n(t)$, we consider a series of the Kida's optimum approximations, $g_n(t)$, each of which uses a given finite number of generalized sample values of $h_n(t)$. Also, we define a similar multi-legged-type approximation $y(t)$ of $m(t)$ using these $g_n(t)$. Secondly, under a slight modification of Ξ, when the backbone of $m(t)$ itself is a band-limited signal $f(t)$ in Ξ, we prove that backbone of $y(t)$ becomes the corresponding optimum continuous running approximation of $f(t)$. Although this paper treats pure theoretical topics, we believe that it is important for multi-paths communication and signal processing systems, such as MIMO systems, to present a fundamental method of constructing the optimum running approximation including various extended multi-paths transmission systems without using difficult high-level mathematical concept.

1. INTRODUCTION

With respect to approximate restoration of signals by a finite number of sample values, theory of interpolation approximation of signals contained in a given set of signals has been considered as a powerful technique in the digital signal processing [2],[3],[4], [5], [6],[7]. However, the minimization of error associated with a running approximation by a FIR filter bank is one of the most important problems of the signal processing.

In this paper, we introduce new concepts of multi-legged-type signal $m(t)$, its multi-legged-type approximation $y(t)$ and the corresponding multi-legged-type error $\epsilon(t) = m(t) - y(t)$, respectively. These multi-legged-type signals are used as a means of proof in this paper and are not the final object-signals of the proposed running approximation.

With respect to the multi-legged-type error $\epsilon(t) = m(t) - y(t)$, we define measures of error that are identical with the prescribed measures of error, in a position of backbone of $\epsilon(t)$. Further, we define that these measures of error become zero, in positions of feet of $\epsilon(t)$. Then, if backbone of $m(t)$ itself is a given band-limited signal $f(t)$, we notice that the corresponding backbone of $y(t)$ becomes a running approximation $g(t)$ of $f(t)$. Under the condition that Kida's optimum approximation using the corresponding finite number of sample values is used in this approximation, we prove that this running approximation becomes the optimum approximation that minimizes various continuous worst-case measures of running approximation error at the same time.

Throughout these procedures, we prove that the running approximation that is identical with side-by-side sequential connection of Kida's approximation is the final optimum running approximation.

2. OUTLINE OF THE OPTIMUM APPROXIMATION

In many signal processing of communication systems, sensor systems or remote-sensing systems, it is necessary to estimate an unknown object signal $f(t)$ by using an approximated signal $g(t)$ that is equal to a certain linear combination of discrete sampled-data of $f(t)$ [2], [3], [9].

Before entering the main part, in this section, we begin with a short summary of known optimum approximations of one-dimensional signals $f(t)$ which are given by linear combinations of general inverse transformations of components in extended spectrum-vectors $\mathbf{F}(\omega)$. For example, these components of $\mathbf{F}(\omega)$ are corresponding to Fourier transforms of a finite number of music apparatuses and $f(t)$ is a total music itself. Although the signals in this paper are one-dimensional signals, if multi-dimensional signals are treated, the components of the spectrum-vectors are corresponding to Fourier transforms of a $3D$ image captured by a finite number of $2D$ cameras and the object signal of the approximation is the above $3D$ image itself. Therefore, both the signal and the corresponding approximation treated in this paper are one-

978-1-4673-0990-5/12/$31.00 ©2012 IEEE

dimensional signals and not vectors. The idea of this paper is given by the authors, for example, [8]. However, because of mistakes of definitions and notations and logical errors and confusion contained in these conference papers and technical papers presented by the authors by 2011 with respect to the topics in this paper, as the reviewers pointed out, these results should be better organized in order to improve readability. Therefore, in the following, we will present the necessary analysis of this approximation again.

As is mentioned above, we assume that $f(t)$ is given by a generalized linear combination of inverse transformation of components of an extended spectrum-vector $\mathbf{F}(\omega)$. With respect to this spectrum-vector $\mathbf{F}(\omega)$, we consider a finite number of analysis-filter-matrix operators that correspond to a finite number of observation systems. Further, we consider discrete kernel vector-operators that give generalized sample values of linear combinations of output signals of the above analysis filter-matrix-operator systems. The approximation $g(t)$ of $f(t)$ is defined as a linear combination of these sample values which has time-dependent coefficients called interpolation functions.

The approximation shown in [8] minimizes the upper limit of all the measures of error defined as arbitrary operators of error. This property of the approximation plays an essential part of the main proof of this paper. The following is the outline of this approximation.

Let R and Z be the set of all the real numbers and the set of all the integer numbers, respectively. We denote by Λ a finite subset of Z and we denote by Δ a subset of Z, respectively. Further, we denote by R^n the set of all the real $n-$dimensional vectors, where n is a positive integer. Also, we denote by Z^n the set of all the $n-$dimensional integer vectors.

Suppose Θ is a Hilbert space with respect to vectors $\mathbf{F}(\omega) = (F_1(\omega), F_2(\omega), \ldots, F_\nu(\omega))$, where ν is a given positive integer and q in $F_q(\omega)$ $(q = 1, 2, \ldots, \nu)$ are index numbers [1].

An inner product between $\mathbf{F}(\omega)$ and $\mathbf{G}(\omega)$ in Θ is expressed by $< \mathbf{F}(\omega), \mathbf{G}(\omega) >$. Further, the notation $||\mathbf{F}(\omega)||$ indicates a norm of $\mathbf{F}(\omega)$ in Θ.

Suppose that $V^{1/2}$ is a positive operator matrix on Θ. We assume that the range of $V^{1/2}$ is Θ. Further, we denote by $V^{-1/2}$ the inverse operator matrix of $V^{1/2}$. We assume that its domain and range are Θ also. Let $V = V^{1/2}V^{1/2}$ and $V^{-1} = V^{-1/2}V^{-1/2}$. Sometimes, we consider that $V^{1/2} = \mathrm{diag}\big(\sqrt{W_1(\omega)}, \ldots, \sqrt{W_\nu(\omega)}\big)$.

Fig. 1 is an approximation system treated in the following discussion. In this figure, the notations H_k $(k \in \Lambda)$ are operator matrices on Θ having domain and range in Θ. We call these H_k $(k \in \Lambda)$ analysis-filter-matrix operators.

Now, we consider a given kernel-function vector $\mathbf{s}(\omega, t) = (s_1(\omega, t), s_2(\omega, t), \ldots, s_\nu(\omega, t)) \in \Theta$ that has a variable ω and a parameter t. Further, suppose that $\mathbf{s}_{k,p}(\omega) = (s^1_{k,p}(\omega), s^2_{k,p}(\omega), \ldots, s^\nu_{k,p}(\omega)) \in \Theta$ is a given function vector of ω, where $k \in \Lambda$ and $p \in \Delta$.

Now, let us assume that B_0 is a subset of Θ composed of $\mathbf{H}(\omega)$ that satisfies $||V^{-1/2}\mathbf{H}(\omega)||^2 \le A$, where A is the prescribed positive number. Further, assume that $\mathbf{H}(\omega)$ is added to the linear matrix system $V^{1/2}$. We denote by $\mathbf{F}(\omega)$ the corresponding output vector. Then, we obtain $\mathbf{F}(\omega) = V^{1/2}\mathbf{H}(\omega)$, that is, $\mathbf{H}(\omega) = V^{-1/2}\mathbf{F}(\omega)$. Hence, we can prove that $||V^{-1/2}\mathbf{H}(\omega)||^2 = ||V^{-1}\mathbf{F}(\omega)||^2 \le A$ holds.

In the following, we denote by B a given subset of Θ that is composed of $\mathbf{F}(\omega)$ satisfying $||V^{-1}\mathbf{F}(\omega)||^2 \le A$.

Now, the signal $f(t)$ in this discussion is defined by

$$f(t) =< V^{-1}\mathbf{F}(\omega), \mathbf{s}(\omega, t) > \quad (\mathbf{F}(\omega) \in B) \tag{1}$$

As shown in the later example, this expression is corresponding to a generalized inverse Fourier transform from $\mathbf{F}(\omega)$ to $f(t)$. The set of these signals $f(t)$ is denoted by Ξ.

The approximation system in the following discussion is shown by Fig. 1. Let $\mathbf{H}_k$ $(k \in \Lambda)$ be operator matrices on Θ having domain and range on Θ. These $\mathbf{H}_k$ $(k \in \Lambda)$ are called analysis-filter-matrix operators. We assume that the operation $\mathbf{H}_k\mathbf{F}(\omega)$ in the functional theory means the product $\mathbf{H}_k\mathbf{F}(\omega)'$ in the ordinary matrix manipulation, where the symbol $\mathbf{F}(\omega)'$ means the transpose of vectors $\mathbf{F}(\omega)$.

Let $\mathbf{G}_k = \mathbf{G}_k(\omega)$ be the adjoint operator matrix of $\mathbf{V}^{-1}\mathbf{H}_k\mathbf{V}$, [1]. Then, the (generalized) sample values of $f(t)$ are defined by

$$\begin{aligned} f_{k,p} &=< \mathbf{V}^{-1}\mathbf{H}_k\mathbf{F}(\omega), \mathbf{s}_{k,p}(\omega) >= \\ &< \mathbf{V}^{-1}\mathbf{F}(\omega), \mathbf{G}_k\mathbf{s}_{k,p}(\omega) > \ (k \in \Lambda, p \in \Delta) \end{aligned} \tag{2}$$

Then, the approximation vector $g(t)$ of $f(t)$ is defined by

$$g(t) = \sum_k \sum_p f_{k,p}\, \psi_{k,p}(t) \tag{3}$$

For convenience sake, we call $\psi_{k,p}(t)$ $(k \in \Lambda, p \in \Delta)$ (generalized) interpolation functions.

For example, let $\nu = 2$, $\mathbf{F}(\omega) = (F_1(\omega), F_2(\omega))$ and $\mathbf{H}_k\mathbf{F}(\omega) = (H_k(\omega)F_1(\omega), F_2(\omega))$. Further,

$$V\mathbf{F}(\omega) = (W_1(\omega)F_1(\omega), W_2(\omega)F_2(\omega)) \tag{4}$$

$$V^{-1}\mathbf{F}(\omega) = (F_1(\omega)/W_1(\omega), F_2(\omega)/W_2(\omega)) \tag{5}$$

$$(\mathbf{F}(\omega), \mathbf{G}(\omega)) = \frac{1}{2\pi}\sum_{q=1}^{2}\int_{-\infty}^{\infty} W_q(\omega)F_q(\omega)\overline{G_q(\omega)}d\omega$$

$$\mathbf{s}(\omega, t) = (\exp(-j\omega t),\ 0\ )$$

$$\mathbf{s}_{k,p}(\omega) = (\exp(-j\omega t_{k,p}), a(t_{k,p})\exp(-j\omega t_{k,p})) \tag{6}$$

where $t_{k,p}$ are given sample points and $a(t)$ is a given multiplicative noise signal. Then,

$$\begin{aligned} f(t) &=< V^{-1}\mathbf{F}(\omega), \mathbf{s}(\omega, t) > \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} W_1(\omega)F_1(\omega)/W_1(\omega)\exp(j\omega t)d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\omega)\exp(j\omega t)d\omega = f_1(t) \end{aligned} \tag{7}$$

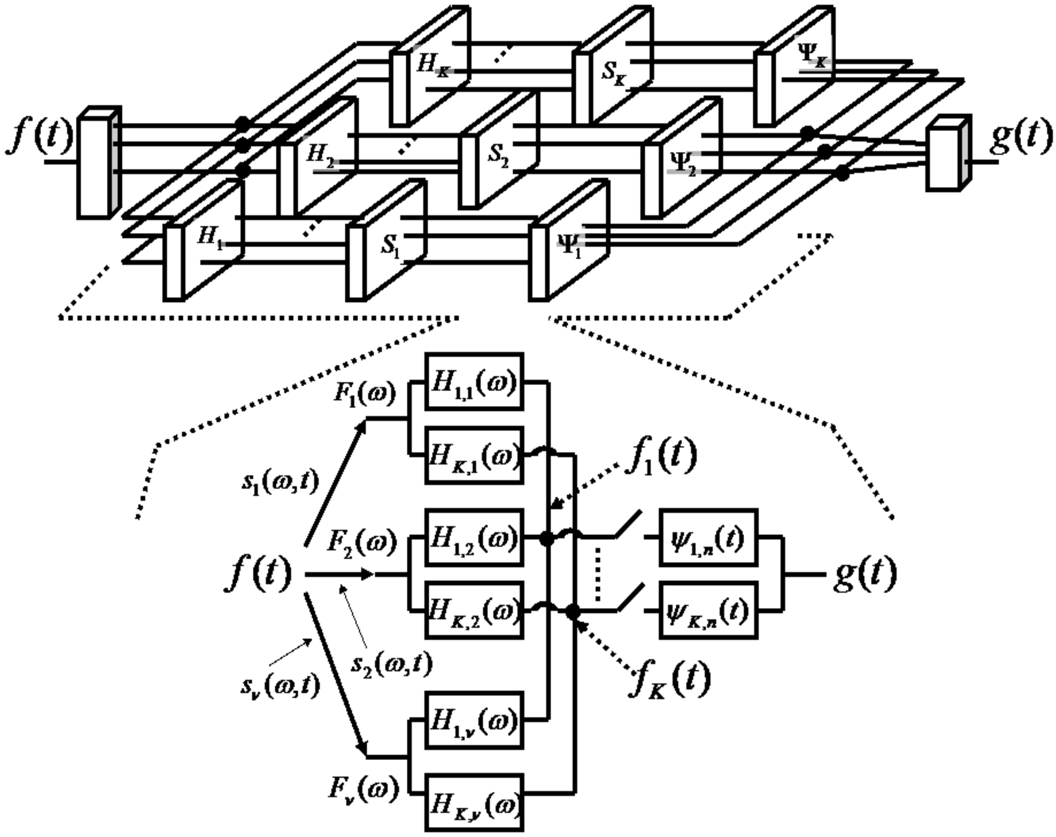

Fig. 1. Filter bank systems using multi-observation systems

$$
\begin{aligned}
f_{k,p} &=< V^{-1}\mathbf{H}_k\mathbf{F}(\omega), \mathbf{s}_{k,p}(\omega) > \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} W_1(\omega)H_k^1(\omega)F_1(\omega)/W_1(\omega) \\
&\quad \exp(j\omega t_{k,p})d\omega \\
&+\frac{1}{2\pi}\int_{-\infty}^{\infty} W_2(\omega)F_2(\omega)/W_2(\omega) \\
&\quad a(t_{k,p})\exp(j\omega t_{k,p})d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} H_k^1(\omega)F_1(\omega)\exp(j\omega t_{k,p})d\omega \\
&+\frac{1}{2\pi}\int_{-\infty}^{\infty} H_k^2(\omega)F_2(\omega)a(t_{k,p})\exp(j\omega t_{k,p})d\omega \\
&= f_k^1(t_{k,p}) + a(t_{k,p})f_2(t_{k,p}) \qquad (8)
\end{aligned}
$$

Hence, this example is corresponding to the approximation of $f_1(t)$ based on the sample values $f_k^1(t_{k,p})+a(t_{k,p})f_k^2(t_{k,p})$ with modulated noise $a(t_{k,p})f_k^2(t_{k,p})$. Therefore, the presented approximation minimizes the error of the approximation having the above noise. [The example is finished].

The approximation error is given by

$$e(t) = f(t) - g(t) \qquad (9)$$

Let $\mathbf{S}(\omega,t) = \mathbf{s}(\omega,t) - \sum_k\sum_p \overline{\psi_{k,p}(t)}\mathbf{G}_k\mathbf{s}_{k,p}(\omega)$. Then, $e(t) = f(t) - g(t)$ can be expressed by

$$e(t) =< V^{-1}\mathbf{F}(\omega), \mathbf{S}(\omega,t) > \qquad (10)$$

In order to derive the interpolation functions, we consider the measure of error $E_{max}(t)$ defined by

$$E_{max}(t) = \sup_{f(t)\in\Xi}\{| e(t) |\} \qquad (11)$$

that is the upper limit of $e(t)$ for all the $f(t)$ in Ξ.

In [8], it is proved that, by differentiating $E_{max}(t)^2$ with respect to $\psi_{m,q}(t)$ $(m \in \Lambda, q \in \Delta)$ and putting the resultant formulas into zero, we can obtain a set of following linear equations with a constant coefficient-matrix for the real optimum interpolation functions $\psi_{k,p}(t)$ $(k \in \Lambda, p \in \Delta)$ minimizing $E_{max}(t)$.

$$
\begin{aligned}
&\Sigma_k\Sigma_p\psi_{k,p}(t) < \mathbf{G}_m\mathbf{s}_{m,q}(\omega), \mathbf{G}_k\mathbf{s}_{k,p}(\omega) > \\
&=< \mathbf{G}_m\mathbf{s}_{m,q}(\omega), \mathbf{s}(\omega,t) > \qquad (m \in \Lambda, q \in \Delta)(12)
\end{aligned}
$$

We assume that the coefficient matrix of Eqs.(12) has sufficiently large rank.

Further, in [8], it is proved that, if the above interpolation functions $\psi_{k,p}(t)$ satisfying Eq.(12) are used in $g(t)$, the corresponding errors $e(t)$ satisfy the following two conditions.

CONDITION 1: $e(t)|_{f(t)=e(t)} = e(t)$.

CONDITION 2: $\Xi_e \subseteq \Xi_0$, where Ξ_e is set of $e(t)$.

Now, let $\hat{g}(t) = v[\{f_{k,p}\};t]$ be a linear or nonlinear approximation for $f(t)$ in Ξ using the same sample values $f_{k,p} =< V^{-1}\mathbf{H}_k\mathbf{F}, \mathbf{s}_{k,p}(\omega) >$ $(k \in \Lambda, p \in \Delta)$. We assume that $\hat{g}(t) = v[\{f_{k,p}\};t]$ is zero when all the $f_{k,p}$ $(k \in \Lambda, p \in \Delta)$

are zero. Since the error $\hat{e}(t) = f(t) - \hat{g}(t)$ by this approximation depends on the signal $f(t)$, we express the error as $\hat{e}(t) = \widehat{\xi}[f(t)]$.

Let $d(t) = \gamma[\hat{e}(t); t]$ be an arbitrary kind of linear/nonlinear operator/functional/function of the approximation error $\hat{e}(t)$ between $f(t)$ in Ξ and its approximation $\hat{g}(t)$. Further, we define new upper-limit measure of error by

$$E_{sup}(t) = \sup_{f(t)\in\Xi} \{\gamma[\hat{e}(t); t]\} \tag{13}$$

Then, in the literature [8], it is proved that, as the direct consequence of the above two conditions, the presented approximation $g(t)$ minimizes any measures of error Eq.(13 for any operator/functional/function of error $\gamma[\hat{e}(t); t]$.

Now, we enter the main part of this paper.

3. MULTI-LEGGED-TYPE SIGNALS IN A HYPER DOMAIN

We introduce the following two assumptions.

Assumption 1: All the components of the analysis-filter-matrix operators $\mathbf{H}_k$ $(k \in \Lambda)$ are given FIR filters having the frequency characteristics that are equal to linear combination of a finite number of functions $e^{jn\tau\omega}$ with given integers n and bounded real or complex coefficients, where τ is a given positive constant.

Assumption 2: All the components of the sampling-kernel vectors $\mathbf{s}_{k,p}(\omega)$ in Eq.(2) are given linear combination of a finite number of functions $e^{jnM\tau\omega}$ with given integers n and bounded real or complex coefficients, where M is a given positive integer.

In running approximation using FIR synthesis filters, the time axis is divided into a series of small segments I^q $q \in Z$, where, in each I^q, the same set of a finite number of sample values is used in the running approximation. Changing the integer-index q sequentially, the running approximation moves on the time axis [5].

Now, we consider an infinite series of band-limited signals $\{\tilde{f}_q(t)\}$ $(q \in Z)$ in the set of signals Ξ. With respect to each $\tilde{f}_q(t)$, let $\tilde{g}_q(t)$ be the above optimum approximation of $\tilde{f}_q(t)$ using sample values $f_{k,p}$ $((k,p) \in D_{k,p})$, where $D_{k,p}$ is a given pair of k in Λ and p which is contained in a given subsets of Z. These $D_{k,p}$ are defined according to the running approximation. Let $\tilde{e}_q(t) = \tilde{f}_q(t) - \tilde{g}_q(t)$ $(q \in Z)$.

Further, we assume that both the support of the signal $\tilde{f}_q(t)$ and the support of the approximation $\tilde{g}_q(t)$ are folded into form of character C of right angle like a needle of a stapler as shown in Fig. 2. In this operation, we consider that the function-values of $\tilde{f}_q$ and $\tilde{g}_q$ in the both side-wings of this folded supports become $\tilde{f}_q(\tau)$ and $\tilde{g}_q(\tau)$ in the new hyper domain with a variable τ, respectively.

Besides, we put this signal, that is bent like a needle of a stapler, such that the part of the middle of the needle of the stapler is just on each I^q in the time axis as shown in Fig. 2. These parts of the middle of $\tilde{f}_q(t)$ or $\tilde{g}_q(t)$ located on I^q are called backbone of $\tilde{f}_q(t)$ or backbone of $\tilde{g}_q(t)$. We denote these backbones by $[\tilde{f}_q(t)]_B$ and $[\tilde{g}_q(t)]_B$, respectively. Besides, we call the parts of the both sides of $[\tilde{f}_q(t)]_B$ and $[\tilde{g}_q(t)]_B$ the wing $\tilde{f}_q(\xi)$ and the wing $\tilde{g}_q(\xi)$ in the hyper domain with the variable τ. We denote by $[\tilde{f}_q(\tau)]_b$ and $[\tilde{g}_q(\tau)]_b$ those wings, respectively.

For a given q, a merger of $[\tilde{g}_q(t)]_B$ and $[\tilde{g}_q(\tau)]_b$ together is the optimum approximation of a merger of $[\tilde{f}_q(t)]_B$ and $[\tilde{f}_q(\tau)]_b$, because we only change the position of the corresponding graphs of $\tilde{f}_q(t)$ and $\tilde{g}_q(t)$.

Such operations are repeated along the time axis t by changing q. As is shown in Fig. 2, the wing $\tilde{f}_q(\xi)$ and the wing $\tilde{g}_q(\xi)$ extend to the same direction for all $q \in Z$. But, we assume that the wings corresponding to q are lengthen in the opposite direction to the wings corresponding to $q + 1$, respectively. Further, as is shown in Fig. 2, we assume that $q \in Z$, $[\tilde{f}_q(t)]_B$ and $[\tilde{g}_q(t)]_B$ are adjacent to $[\tilde{f}_{q+1}(t)]_B$ and $[\tilde{g}_{q+1}(t)]_B$, but $[\tilde{f}_q(\xi)]_b$ and $[\tilde{g}_q(\xi)]_b$ are never piled up $[\tilde{f}_{q+1}(\xi)]_b$ and $[\tilde{g}_{q+1}(\xi)]_b$. We call these signals multi-legged-type signals in a hyper domain X and we denote a multi-legged-type signal by $f_{multi}(t, \xi)$. Suppose that $\Gamma[\Xi]$ is the set of all the multi-legged-type signals $f_{multi}(t, \xi)$.

In particular, we denote a merger of the backbones $[\tilde{f}_q(t)]_B$ $(q \in Z)$ in the multi-legged-type signals in $\Gamma[\Xi]$ in the hyper domain X by $f_A(t)$. Also, we denote a merger of the backbones $[\tilde{g}_q(t)]_B$ $(q \in Z)$ in the multi-legged-type approximation in the hyper domain X by $g_A(t)$.

As special cases of the multi-legged-type signals in the

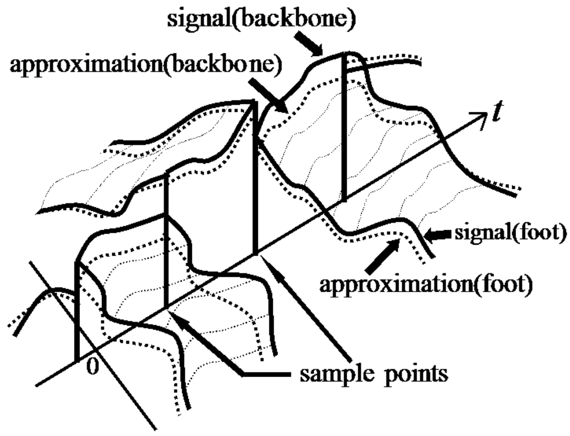

Fig.2 Multi-legged-type signals

hyper domain, we define the following sequence of band-limited signals $\{\tilde{f}_q(t)\}$ $(q \in Z)$ satisfying next (a) and (b).

(a) Every $\tilde{f}_q(t)$ is equal to a $f(t)$ in Ξ.

(b) Each $\tilde{f}_q(t)$ $(q \in Z)$ is bent such that the merger of the corresponding backbones makes $f(t)$ itself. That is, the corresponding $f_A(t)$ becomes $f(t)$ itself.

Using these special set of $\tilde{f}_q(t)$ $(q \in Z)$, we make a set of all the multi-legged-type signals. These multi-legged-type signals in the hyper domain X are called the target multi-legged-type signals in the hyper domain. Also, we call the other multi-legged-type signals in the hyper domain X the non-target multi-legged-type signals.

Then, if a multi-legged-type signal in the hyper domain is identical to a target multi-legged-type signal in the hyper domain associated with the given signal $f(t)$ shown in the above (a), we can recognize easily that the corresponding merger of the backbones $[\tilde{g}_q(t)]_B$ $(q \in Z)$ is the running approximation of the signal $f(t)$ in (a). We denote by $\Gamma_{tar}[\Xi]$ and $\Gamma_{n-tar}[\Xi]$ the set of the target multi-legged-type signals and the set of the non-multi-legged-type signals in the hyper domain X, respectively.

If $\tilde{g}_q(t)$ $(q \in Z)$ are the above optimum approximations of $\tilde{f}_q(t)$ $(q \in Z)$ using the sample values $f_{k,p}$ $((k,p) \in D^q)$ mentioned so far, we denote by $g_{multi}(t,\xi)$ and $e_{multi}(t,\xi)$ the merger of $\tilde{g}_q(t)$ $(q \in Z)$ and $\tilde{e}_q(t) = \tilde{f}_q(t) - \tilde{g}_q(t)$ $(q \in Z)$, respectively.

Now, let $\gamma\{e_{multi}(t,\xi)\}$ be an arbitrary operator/functional/function of $e_{multi}(t,\xi)$ satisfying $\gamma\{\hat{e}_{multi}(t,\xi)\} = 0$ if $\hat{e}_{multi}(t,\xi)$ is a merger of functions that are zero identically. Then, we can obtain the next lemma.

Lemma 1: The above $g_{multi}(t,\xi)$ is the optimum approximation of $f_{multi}(t,\xi)$ that minimizes various worst-case measures of error

$$E_{sup}(t,\xi) = \sup_{f_{multi}(t,\xi) \in \Gamma\ [\Xi]} \gamma\{e_{multi}(t,\xi)\} \quad (14)$$

(Lemma 1 finishes).

Further, as is shown above, $\gamma\{e_{multi}(t,\xi)\}$ is any linear /nonlinear operator/functional/function of $e_{multi}(t,\xi)$. Hence, if we consider $\gamma\{e_{multi}(t,\xi)\}$ that becomes the proposed measures of error for $e_A(t) = f_A(t) - g_A(t)$ in the position of the backbone made by the corresponding running approximation $g_A(t)$ and becomes zero about the other errors, we can easily prove that the presented extended optimum approximation minimizes various continuous worst-case measures of the running approximation error at the same time.

However, let us assume that a multi-legged-type signal having continuous backbone $\tilde{f}(t)$ has an error shown in the point $\tilde{e}(t)$ in Fig.3 in the proposed approximation. Further, another a multi-legged-type signal $\hat{f}(t)$ having non-continuous backbone in the approximation to be compared has the the same error $\tilde{e}(t)$. Besides, if the value of the treated measure of error shown in the point $\tilde{e}(t)$ in Fig.3 is very large compared to other measures of error in Fig. 3. Then, you can easily recognize that we have to adopt just this large measure of error in our approximation because we consider minimization of the maximum value of the measure of error in the present approximation. On the other hand, if we restrict our set of multi-legged-type signals in the domain of the multi-legged-type signals having continuous backbone, the above large measure of error is not adopted in the approximation to be compared. Therefore, in the process of this operation, perhaps, the maximum value of the measure of error changes and it will happen that the presented approximation is not the optimum one. Hence, we introduce a very small peak signal to all the multi-legged-type signals having non-continuous backbones at the position of $t = \tau/2$. Because of Assumption 1 and Assumption 2 shown above, this peak signal does not have any effect on the corresponding approximation signals because sampling procedure is done on the integer time.

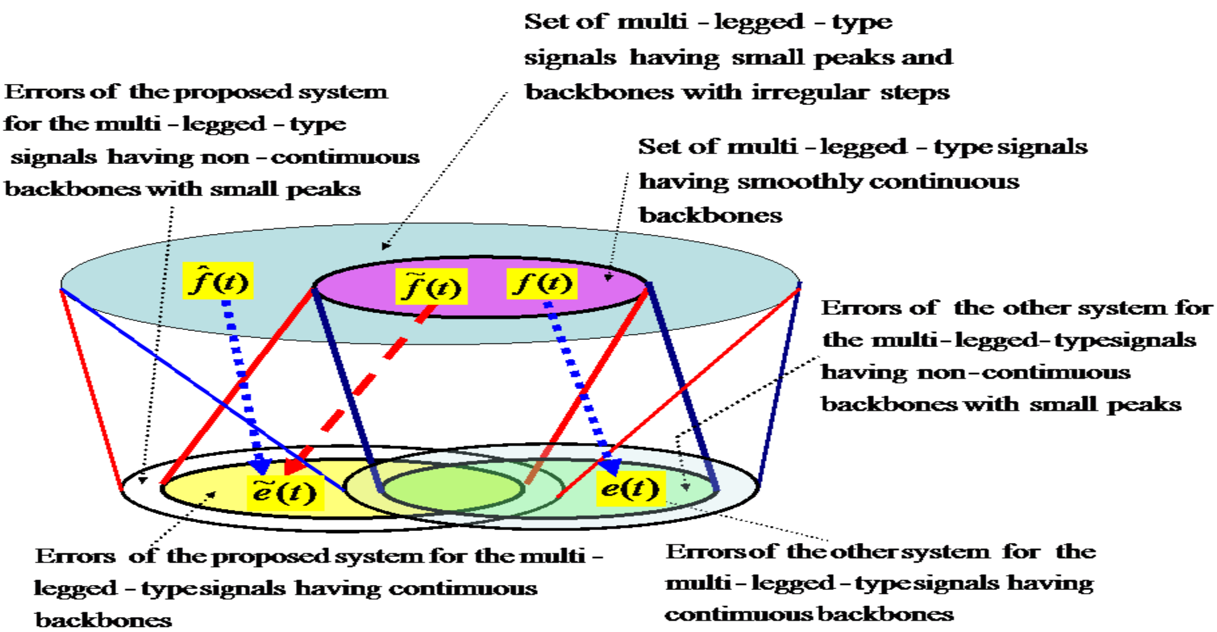

Fig. 3 Multi-legged-type signals

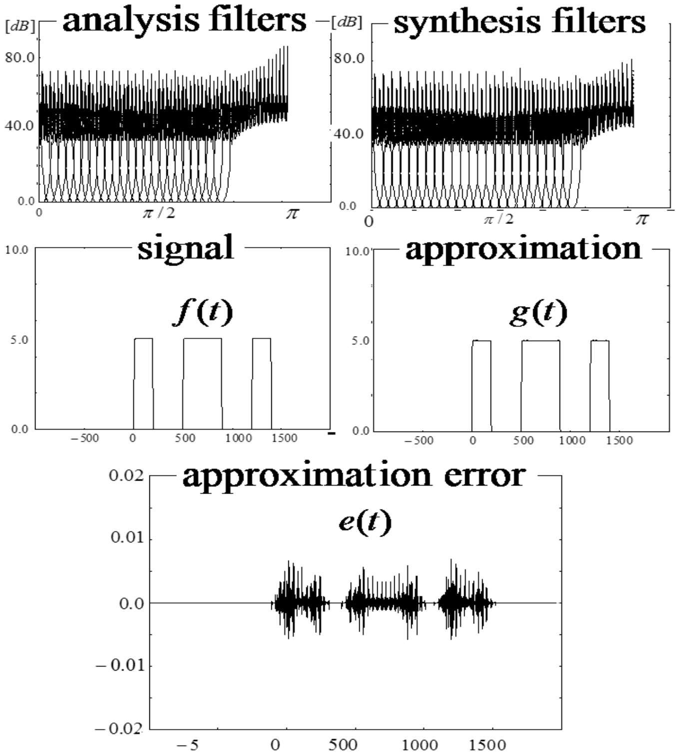

Fig. 4 almost square pulses and approximations

Therefore, the corresponding errors contain this peak signal because the signal has this peak signal but the corresponding approximation does not have the peak signal.

Hence, under the condition that the proposed measures of error is continuous and Assumption 1 and Assumption 2, we can assume that the above set function identifies the multi-legged signals having this small peak signal or not and has the value of zero if the error contains the peak signal. On the other hand, because the proposed measure of error is assumed to be continuous, even if an error contains the peak signal remains after the restriction of the set of multi-legged-type signals, the corresponding change of value of the measure of error is quite small. That is, under these assumptions, we can prove that the presented extended optimum running approximation minimizes various continuous worst-case measures of the continuous error at the same time.

We show a concrete example of an under-sampling filter bank as is suggested by the reviewers. We use initially 4×24 analysis-filter matrix composed by 4 parallel-interleaved cosine conversion type filters operating each time interval of 4 times of the unit interval. We use 24 path 128-tap FIR synthesis filter bank made by the presented optimum interpolation functions. The sampling intervals of 24 samplers in the middle of this system are 32. We repeat the exchange of analysis filters and synthesis filters 40 times and obtain the final result. The presented approximation is powerful in the approximation of fine detail of signals. But, in this example, $f(t)$ is simple positive square signal. Hence, the running average among each adjacent three sampled data is adopted after the presented approximation for smoothing. The error is under 1 percent. Pay attention that this system is an under-sampling system and it is impossible to make so-called perfect reconstruction system.

4. CONCLUSION

New concept of multi-legged-type signal is introduced that is a combined-signal of many one-dimensional band-limited signals. We propose an approximation method of the multi-legged-type signals and we prove that this approximation is the optimum. Then, we define measures of error that become the proposed measures of error in the position of the backbone made by the corresponding running approximation and become small about the other errors. For these measures of error, we prove that the presented extended optimum approximation minimizes various worst-case measures of the running approximation error at the same time.

5. REFERENCES

[1] Yoshida, K.: *"Functional Analysis"*, Springer, no.5, 1978.

[2] Vaidyanathan, P.: *" Multirate Systems and Filter Banks"*, Prentice-Hall, Englewood Cliffs, New Jersey, 1993.

[3] Mitra, S.K.: *"Digital Signal Processing - A Computer Based Approach"*, MacGraw-Hill Series in Electrical and Computer Engineering, International Edition, 2002.

[4] Kida, T.: *"On Restoration and Approximation of Multi-Dimensional Signals Using Sample Values of Transformed Signals"*, IEICE Trans., vol. E77-A, no.7, 1994.

[5] Kida,T., "The Optimum Approximation of Multidimensional Signals Based on the Quantized Sample Values of Transformed Signals," IEICE Trans., E78-A, No.2, pp.208-234, 1995.

[6] Kida, Y. and Kida, T.: *"The Optimum Discrete Approximation of Band-Limited Signals with an Application to Signal Processing on Internet"*, IEICE Trans. Fundamentals, E82-A, No.8, pp.1592-1607, 1999.

[7] Ogawa, H.: *"Sampling Theory and Isao Someya: A Historical Note"*, Sampling Theory in Signal and Image Processing, vol.5, no.3, pp.247-256, 2006.

[8] Kida, Y. and Kida, T.: *"The optimum approximate reconstruction of a signal from the discrete sample values of the prescribed multiple waves"*, Proceedings of ISITA-ISSSTA 2010, Au-Tu-2:Signal Processing, pp. 621-626, 2010.

[9] http://www.ntt.co.jp/mirai/organization/organization0304 .html