Analytical Performance Evaluation of an Efficient Reduced-Complexity Time Synchronization Approach for OFDM Systems

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Abstract— In this paper, we analytically study the performance of a recently proposed efficient reduced complexity time synchronization approach for orthogonal frequency division multiplexing systems. This method uses a preamble of two identical parts and proceeds in two stages. In the first stage, the repetitive structure of the preamble is exploited to provide the coarse time estimate respecting the algorithm of Cox and Schmidl. In the second stage, a fine metric, based on differential correlation, is carried over a reduced time window centered on the coarse estimate. We here study the performance of the fine stage, assuming a successful coarse stage whereby the fine search window is centered on the correct frame start. We approximate the fine metric by a Gaussian distribution to derive a closed form expression of the frame start correct detection probability. To this end, a statistical characterization of the fine metric is achieved by its mean and variance computation. Simulations are used to validate the results of the analysis. Indeed, the evaluated rate of correct detection perfectly concords with the theoretical probability in both additive white Gaussian noise and multipath channels.

Keywords— OFDM, frame detection, preamble-based synchronization, Gaussian process

I. INTRODUCTION

Orthogonal Frequency Division Multiplexing (OFDM) technology is nowadays extensively used for its good performance in high data rate systems and its simplified equalization in severe frequency selective channels. Even if the insertion of a guard interval, generally a Cyclic Prefix (CP), larger than the channel delay spread, protects against Inter Symbol Interference (ISI), deriving an accurate Time Offset (TO) estimation enables to tackle more dispersive channels.

Several approaches for time synchronization are proposed in literature. Namely CP induced cyclostationarity based blind synchronization methods [1]-[2] and data-aided preamble based synchronization methods [3]-[11] exist. We are here concerned with the latter family of synchronization methods, where a specifically repeated pattern is designed for the preamble. In the receiver, a metric adequately exploiting the so designed preamble is computed. The methods proposed in [3]-[7] compute sliding correlation based metrics with low complexity. However, the frame start detection performance is poor. Other methods, proposed in [8]-[9], rather compute differential correlation based metrics, characterized by a higher complexity. Nevertheless, such methods lead to a greatly enhanced detection performance. A good tradeoff between performance and complexity is realized through a two stage Reduced Complexity (RC) scheme, proposed in [10] and [11], respectively for Additive White Gaussian Noise (AWGN) and multipath channels. In the coarse stage, a Cox and Schmidl like metric, based on a sliding correlation, is carried exploiting the repetitive structure of the preamble. In the fine stage, a differential correlation based metric is computed over a reduced time window centered on the coarse estimate. This approach was shown to greatly enhance the detection performance, compared to former methods, through experimental studies.

The contribution of this work is to analytically assess the fine stage performance assuming successful coarse synchronization. Using the Central Limit Theorem (CLT), the fine metric is approximated by a Gaussian distribution that is characterized by its mean and variance values for different time indexes. Based on the statistical characterization, the frame start Probability of Correct Detection (PCD) is derived in its closed form expression. In this work, the performance is averaged over random preamble choice where frequency domain data is drawn from QPSK. Numerical results show a good match between the analytical PCD and the simulated rate of correct frame start detection, thus validating the analytical performance study for both AWGN and multipath channels.

This paper is organized as follows. Section II presents the adopted system model. The analytical performance study is provided in section III. In Section IV, the numerical and simulation results are given. Finally, the paper concludes in section V.

II. SYSTEM MODEL

We consider an OFDM system of $N_u = 2^m$ sub-carriers, where a CP of length Ng samples is prepended to each OFDM symbol. The modulation used here is QPSK with modulated

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Symbols $s_k = \pm\sqrt{E_s/2} \pm j\sqrt{E_s/2}$, $k \in \mathbb{Z}$ with symbol energy E_s. We consider a multipath finite impulse response channel of length L_H, assumed to be shorter than N_g, and coefficients denoted by h_l, $\{l = 0 \ldots L_H - 1\}$. The received signal is affected by a frequency offset, which's normalized version, with respect to sub-carriers spacing, is denoted by υ. The channel also introduces a zero-mean AWGN with variance E_ω. The k^{th} received sample is then expressed as

$$r_k = e^{j2\pi\upsilon/N_u} \sum_{l=0}^{L_h-1} h_l s_{k-l} + \omega_k \tag{1}$$

At the receiver, timing reference is shifted by τ which corresponds to the shortest path delay of propagation with respect to the timing reference at the transmission. The aim of the synchronization process is to precisely detect it.

Depending on the transmission mode, the preamble is sent either periodically for continuous transmission or at the start of a data stream transmission in the bursty packet mode. The preamble of the studied approach has two identical parts (in time domain) of length $L_u = N_u/2$ each, with samples denoted by $\left[p_0, p_1, \ldots, p_{L_u-1}, p_{L_u}, \ldots, p_{N_u-1}\right]$. The preamble halves are made identical by transmitting a data on the even frequencies, while zeros are transmitted on the odd frequencies. The preamble is also extended by a CP of N_g samples. Thanks to CP insertion and channel causality, the two preamble halves remain identical after passing through the channel, up to a phase difference caused by the frequency offset. The repetitive structure of the preamble is exploited only in the coarse synchronization.

During the first stage of the studied approach, the coarse time estimate $\hat{\tau}_c$ of τ is provided with low complexity as in [3]. $\hat{\tau}_c$ may fall within the ISI free part but the accuracy offered is unsatisfactory, mainly at low SNR. To fine tune it, the second stage computes a differential correlation based metric over a time window of length $2\Delta\tau + 1$, centered on $\hat{\tau}_c$. The fine stage is carried over $\left[\hat{\tau}_c - \Delta\tau, \hat{\tau}_c + \Delta\tau\right]$ to find the fine estimate $\hat{\tau}_f$ that maximizes the metric, which will be detailed in the next section. We note that the differential correlation performed during the fine stage is specified by the sequence α generated from the preamble as

$$\alpha_k = p^*_k p_{k+q}, \; k \in [0, L_u - 1], \tag{2}$$

where q stands for the correlation shift which must be different from 0 and L_u integer multiples and for $k + q \geq L_u$, $p_{k+q} \equiv p_{k'}$, where $k + q \equiv k' [L_u]$.

III. Performance Analysis

In this section, we assume that the coarse time estimation is successful and focus on the analysis of the fine stage. The fine timing metric of the studied scheme is first detailed. Second, adopting the CLT, the fine metric $M(k)$ is approximated by a Gaussian distribution with mean μ_k and variance σ_k^2 for each time index k [12]. Then, supposing the independence of the metric values for different time indexes, the Probability of Correct Detection (PCD) of the frame start is derived in its closed form.

A. Fine timing metric

The studied fine timing metric is calculated over the uncertainty interval $2\Delta\tau + 1$ around τ as

$$M(k) = \sum_{l=0}^{Lu-1} \alpha_l^* Y_{k+l}, k \in [\tau - \Delta\tau, \tau + \Delta\tau], \tag{3}$$

where Y is a differentially-demodulated version of the received signal generated as

$$Y_k = r_k^* r_{k+q}, k \in [\tau - \Delta\tau, \tau + \Delta\tau + L_u]. \tag{4}$$

The fine time estimate $\hat{\tau}_f$ is selected as the argument that maximizes the metric in (3).

The fine timing metric and the coarse one are presented in figure (1), in the monopath channel. As shown in the figure, the coarse metric exhibits a plateau, which leads to poor detection, where the fine metric has a high sharp peak, which greatly enhances the detection performance. The fine synchronization is then necessary to get accurate frame start detection.

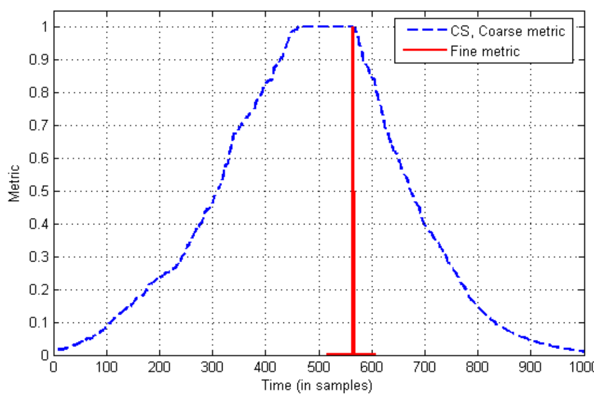

Figure 1. Coarse and fine timing metrics in the AWGN channel under noiseless conditions

We suppose that the preamble is the first sent symbol, so $p_l = s_l$ for all values of l in $\left[0, L_u - 1\right]$. Substituting (2) and (4) in (3), the fine metric yields

$$M(k) = \sum_{l=0}^{L_u-1} r^*_{k+1} r_{k+l+q} s_l s_{l+q}. \tag{5}$$

In the following, we do not take in account the frequency offset υ, which adds a phase shift of $e^{j2\pi\upsilon/N_u}$ to the metric. Inserting (1) in (5), the fine timing metric turns into

$$\begin{aligned} M(k) &= \sum_{l=0}^{L_u-1} \left[\begin{array}{l} \left(\sum_{i=0}^{L_H-1} h_i s_{k+l-i} + \omega_{k+1} \right)^* \left(\sum_{j=0}^{L_H-1} h_j s_{k+l+q} + \omega_{k+l+q} \right) \\ s_l s^*_{l+q} \end{array} \right] \\ &= \sum_{l=0}^{L_u-1} \left(\begin{array}{l} \sum_{i=0}^{L_H-1} \sum_{j=0}^{L_H-1} h^*_i h_j s^*_{k+l-i} s_{k+l+q-j} s_l s^*_{l+q} + \sum_{i=0}^{L_H-1} h^*_i s^*_{k+l-i} \omega_{k+1+q} s_l s^*_{l+q} + \\ \sum_{j=0}^{L_H-1} h_j s_{k+l+q-j} \omega^*_{k+1+q} s_l s^*_{l+q} + \omega^*_{k+l} \omega_{k+1+q} s_l s^*_{l+q} \end{array} \right) \end{aligned} \tag{6}$$

B. Statistical characterization of the fine metric

The fine timing metric expressed in (3) sums up L_u random variables. Assuming the independence between the summed terms in $M(k)$ and for L_u sufficiently large, the CLT enables to approximate $M(k)$ by a Gaussian distribution [12]. Consequently, $M(k)$ is completely characterized by its mean μ_k and variance σ_k^2, which are hereafter derived. Since s_k and ω_k are mutually uncorrelated and ω_k is centered, the second and third terms of (6) have zero mean. As ω_k is white and $q \neq 0$, the last term in (6) is also zero mean. Then, the mean value of $M(k)$, given by its expectation, reduces to

$$\begin{aligned} E(M(k)) &= \sum_{l=0}^{L_u-1} E\left(\sum_{i,j=0}^{L_H-1} h^*_i h_j s^*_{k+l-i} s_{k+l+q-j} s_l s^*_{l+q} \right) \\ &= \sum_{l=0}^{L_u-1} h^*_i h_j E\left(\sum_{i,j=0}^{L_H-1} s^*_{k+l-i} s_{k+l+q-j} s_l s^*_{l+q} \right) \end{aligned} \tag{7}$$

The expectation in (7) is non zero only for time index $k = i = j$, where k coincides with an effective path. Then, the last expression becomes

$$E(M(k)) = \sum_{l=0}^{L_u-1} |h_k|^2 E\left(|s_l|^2 |s_{l+q}|^2\right) = L_u |h_k|^2 E_s^2 \tag{8}$$

For all the other values of k, the expectation is zero because the samples s^*_{k+l-i}, $s^*_{k+l+q-j}$, s_l and s^*_{l+q} are mutually uncorrelated.

To determine $E(|M(k)|^2)$, we use once again the fact that the samples s_k and ω_k are uncorrelated, which leads to a zero valued expectation of the cross terms of $E(|M(k)|^2)$. This latter is then expressed as

$$\begin{aligned} E\left(|M(k)|^2\right) &= \sum_{l,l'=0}^{L-1} \Bigg(\sum_{i,j,i',j'=0}^{L_H-1} h^*_i h_j h^*_{i'} h_{j'} E \left[\begin{array}{l} s^*_{k+l-i} s_{k+l+q-j} s_l s^*_{l+q} s_{k+l'-i'} \\ s^*_{k+l'+q-j'} s^*_{l'} s^*_{l'+q} \end{array} \right] \\ &+ \sum_{i,i'=0}^{L_u-1} h^*_i h_{i'} E\left[s^*_{k+l-i} \omega_{k+l+q} s_l s^*_{l+q} s_{k+l'-i'} \omega^*_{k+l'+q} s^*_{l'} s_{l'+q} \right] \\ &+ \sum_{i,i'=0}^{L_H-1} h_j h^*_{j'} E\left[s_{k+l+q-j} \omega^*_{k+l+q} s_l s^*_{l+q} s^*_{k+l'+q-j'} \omega_{k+l'+q} s^*_{l'} s_{l'+q} \right] \\ &+ E\left[\omega^*_{k+l} \omega_{k+l+q} s_l s^*_{l+q} \omega_{k+l'} \omega^*_{k+l'+q} s^*_{l'} s_{l'+q} \right]. \end{aligned} \tag{9}$$

In the case where k coincides with an effective path, the last expression becomes

$$\begin{aligned} E\left(|M(k)|^2\right) &= L_u^2 |h_k|^4 E_s^4 + L_u \sum_{i,j=0}^{L_H-1} |h_i|^2 |h_j|^2 E_s^4 - L_u |h_k|^4 E_s^4 \\ &+ 2L_u \sum_{i=0}^{L_H-1} |h_i|^2 E_s^3 E_\omega + L_u E_s^2 E_\omega^2. \end{aligned} \tag{10}$$

In the other case, where k falls out of the set of paths, the expectation becomes

$$\begin{aligned} E\left(|M(k)|^2\right) &= L_u \sum_{i,j=0}^{L_H-1} |h_i|^2 |h_j|^2 E_s^4 + 2L_u \sum_{i=0}^{L_H-1} |h_i|^2 E_s^2 E_\omega + \\ & L_u E_s^2 E_\omega^2. \end{aligned} \tag{11}$$

As mentioned above, for these values of k, $\mu_k = 0$. So the variance $\sigma_k^2 = E(|M(k)|^2)$. Otherwise, for k corresponding to an effective path, the variance of the metric yields

$$\sigma_k^2 = L_u \sum_{i,j=0}^{L_H-1} |h_i|^2 |h_j|^2 E_s^4 - L_u |h_k|^4 E_s^4 + 2L_u \sum_{i=0}^{L_H-1} |h_i|^2 E_s^3 E_\omega + L_u E_s^2 E_\omega^2. \quad (12)$$

Note that the monopath AWGN channel, reduces to a channel with a single path ($L_H = 1$) whose gain is $h_0 = 1$.

C. Probqbility of correct detection

The probability of correct detection is the probability that, for all values of the time index k within the uncertainty interval, the amplitude of the timing metric $|M(k)|$ is lower than that of $|M(k_c)|$, were k_c stands for the correct frame start. We denote $|M(k_c)|$ by ε. Therefore, assuming the independence between the different values of $M(k)$ and $M(k_c)$ for $k_c \neq k$, this probability can be expressed as

$$PCD = \int_0^{+\infty} \prod_{k \neq k_c} F_k(\varepsilon) P_{k_c}(\varepsilon) d(\varepsilon), \quad (13)$$

where $F_k(\varepsilon)$ presents the Cumulative Distribution Function (CDF) of the fine timing metric at the time index k and P_k stands for the Probability Density Function (PDF) of $M(k)$.

From the statistical characterization of the fine timing metric over the interval $[\tau_c - \Delta\tau, \tau + \Delta\tau]$, we note that each of the samples whose index k coincides with an effective path, has his own Gaussian distribution depending on its mean and variance. However, for the other values of k out of the set of effective paths, the Gaussian distribution is uniform as they have all the same mean and variance.

For notational convenience, we introduce the random discrete variable X, corresponding to $|M(k)|$. We recall that X is non-uniformly Gaussian distributed. The CDF is calculated for positive values of ε because we consider the metric amplitude. The CDF F_k in the probability expression (13) is determined as follows

$$F_{|X|}(\varepsilon) = P(|X| \leq \varepsilon) = P(X \leq \varepsilon) + P(-X \leq \varepsilon)$$
$$= \int_0^{\varepsilon} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx + \int_{-\varepsilon}^{0} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx. \quad (14)$$

Applying variable change of x by $x' = \dfrac{x-\mu}{\sigma}$, we get

$$P(|X| \leq \varepsilon) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\frac{-\varepsilon-\mu}{\sigma}}^{\frac{\varepsilon-\mu}{\sigma}} e^{-\frac{x'^2}{2}} \sigma dx' = \frac{1}{\sqrt{2\pi}} \int_{\frac{-\varepsilon-\mu}{\sigma}}^{+\infty} e^{-\frac{x'^2}{2}} dx'$$
$$- \frac{1}{\sqrt{2\pi}} \int_{\frac{\varepsilon-\mu}{\sigma}}^{+\infty} e^{-\frac{x'^2}{2}} dx' = Q\left(\frac{-\varepsilon-\mu}{\sigma}\right) - Q\left(\frac{\varepsilon-\mu}{\sigma}\right), \quad (15)$$

where $Q(x)$ stands for the Marcum function.

Now, we determine the PDF at the correct frame start k_c. To this end, we can derive the CDF expressed in (15) that gives

$$P_{k_c}(\varepsilon) = \frac{1}{\sqrt{2\pi\sigma_{k_c}^2}} \left(e^{\frac{-(\varepsilon-\mu_{k_c})^2}{2\sigma_{k_c}^2}} + e^{\frac{-(-\varepsilon-\mu_{k_c})^2}{2\sigma_{k_c}^2}} \right). \quad (16)$$

IV. Numerical and Simulation Results

The performance of the studied synchronization scheme is considered in the AWGN and multipath channels over a practical SNR range. The uncertainty interval, over which the fine metric is carried, is set to N_g centered on $\tau_c = \tau$.

We recall that we here compare the theoretical and the experimental frame start probability of correct detection of the fine stage. This means that we neglect the error caused by the coarse stage, supposing that the coarse time estimate is well positioned. Under the assumption that the samples are modeled as independent Gaussian random variables, the PCD can be expressed as

$$PCD = \begin{cases} \int_0^{+\infty} \left(F_k(\varepsilon)\right)^{N_g-1} P_{k_c}(\varepsilon) d\varepsilon, & \text{AWGN channel} \\ \int_0^{+\infty} \left(F_{k \notin I_H}(\varepsilon)\right)^{N_g - I_H} \prod_{k \neq k_c} \left(F_{k \in I_H \setminus k_c}(\varepsilon)\right) P_{k_c}(\varepsilon) d\varepsilon, & \text{Multipath channel,} \end{cases} \quad (17)$$

where I_H corresponds to the effective paths (with non-zero gain) positions. In (17), for k out of the set of channel paths, F_k can be chosen arbitrarily from any of the time indexes, as they are identically distributed ($k \in [\tau_c - N_g/2, \tau_c - N_g/2] \setminus I$).

The simulation is performed for 10^4 Monte Carlo trials. The performance is averaged over random preambles taken from QPSK modulation where $s_k = \pm\sqrt{1/2} \pm j\sqrt{1/2}$ (having $E_s = 1$). We evaluate the approximated probabilities of correct detection using (17) for AWGN and multipath channels. The considered multipath channel has 7 paths uniformly separated by 6 samples and an exponential power delay profile. Consequently, the channel impulse response lengths is 42 samples. The ratio of the first path to the last path is set to 12 dB with regular adjacent paths gain ratio. The parameter m is set to 10 leading to a system of useful part length $N_u = 2^{10} = 1024$ samples and $N_g = 102$ samples. The correlation shift q is set to 1 .

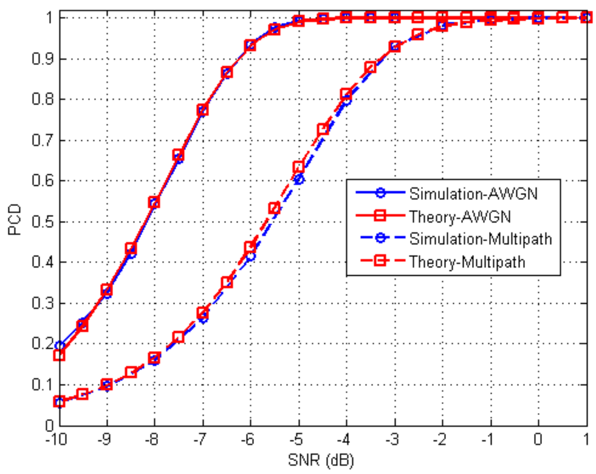

Figure 2. Probability of correct detection in the case of AWGN and exponential power profile multipath channels: simulation versus analysis.

Figure 2 illustrates the probability of correct detection in the considered channels. The manipulated signals are here complex and the channel coefficient gains are also complex, with phase randomly chosen from a uniform distribution over $[0, 2\pi]$. As shown in the figure, in the AWGN channel, the analytical probability concords perfectly with the simulated one. In the multipath channel, the experimental curve matches the numerical curve very well, for an SNR lower than -6 dB and higher than -3 dB. In between, the theoretically approximated values of the PCD are very close to the experimental values. A slight gap of about 0.1 dB emerges. This gap may be related to the $M(k)$ independence assumption limit for different k values. We also note that the frame start detection is perfect (PCD $= 1$) from an SNR value of -5 dB and -1 dB respectively in the AWGN and multipath channels.

The obtained results prove that the assumptions made in the analysis are valid and the simulation results validate the theoretical analysis. In fact, the analytical frame start PCD concords perfectly with the simulated one in the case of AWGN channel, where a slight difference is observed in the case of multipath channel. This difference may be related to the validity limits of the independence assumption between the timing metric values. This assumption is corrupted in the case of multipath channel due to the channel memory.

V. CONCLUSION

In this work, we provided a theoretical performance analysis of a two-stage reduced complexity time synchronization approach applied for OFDM systems. The performance was evaluated, for the fine stage, in terms of correct detection probability of the frame start in both AWGN and multipath channels. The fine timing metric, carried over the uncertainty interval, was first analyzed and approximated by a Gaussian distribution characterized by its mean and variance. Based on the obtained mean and variance, the expression of the probability of frame start correct detection was derived in its closed form. Simulations were performed to validate the theoretical analysis. It was shown that the experimental results perfectly agree with the theoretical ones in the case of AWGN channel and realize a good match with the theoretical results in the case of multipath channel.

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