Stochastic NDA CRLB for DOA Estimation over SIMO Systems

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Abstract— **This paper derives for the first time the stochastic Cramér-Rao lower bounds (CRLBs) for direction of arrival (DOA) estimates of any linearly-modulated signals, in presence of additive white circular complex Gaussian noise (AWCCGN). The transmitted symbols are assumed to be completely unknown at the receiver side. The channel is slowly time-varying and assumed to introduce a constant distortion phase during the observation interval. Simulation results show that the CRLBs hold almost the same for all the modulation schemes. We also show that, contrarily to uniform lineair array (ULA) systems, the knowledge of the channel distortion phase does not bring any additional information to the achievable performance in uniform circular array (UCA) systems.**

Index Terms— **QAM signals, stochastic Cramér-Rao lower bound (CRLB), DOA estimation, ULA, UCA.**

I. Introduction

Numerous applications including radar, sonar and wireless cellular communications, raise the problem of signals direction of arrival (DOA) acquisition. Indeed, DOA estimation has attracted a lot of interest [1, 2] and intensive research works have been conducted on this topic. Roughly speaking, there are two major categories of DOA estimators. In the data-aided (DA) estimation mode, the transmitted symbols are supposed to be perfectly known at the array receiver. In the case of non-data-aided (NDA) estimation, DOA estimators are not aided by any *a priori* knowledge about the symbols and base the estimation process on the received samples only [3-7].

In both cases, an overall benchmark against the performance of any DOA estimator is of great importance. In this context, the Cramér-Rao lower bound (CRLB) sets the minimum achievable variance of any unbiased estimator of a given parameter [8]. The evaluation of the stochastic CRLB can always be performed either numerically or empirically, since deriving closed-form expressions for the CRLB of a general modulated signal is a tedious task, if not impossible. This is due to the complex structure of the likelihood function, which cannot be easily simplified. Yet the deterministic CRLB expression (with deterministic signals) was succesfully derived in [9]. But it should be kept in mind that the deterministic CRLBs are not generally attainable, because their likelihood functions are not sufficiently regular, and they remain hence very loose bounds, especially in the low-SNR regime. On the other hand, the stochastic CRLB, is achieved by the stochastic maximum likelihood (ML) estimator, and therefore gained a lot of interest. Multiple works dealing with the computation of the stochastic CRLB have been reported in the literature. In fact, in [10, 11], it was computed for circular Gaussian distributed signals, and then was extended to the case of circular complex Gaussian distributions [12]. Later, an explicit expression for the stochastic CRLB of DOA estimates for non-circular Gaussian sources in the general case of arbitrary unknown Gaussian noise was derived in [13]. Recently, a closed-form expression for the stochastic CRLB of NDA DOA estimators has been established, in the special case of square-QAM-modulated signals [14], generalizing thereby the particular expressions for BPSK and QPSK-modulated signals derived by Delmas and Abeida in [15]. But the evaluation of the stochastic CRLB for DOA estimation in the general case of linearly-modulated signals (PSK, APSK, QAM...) has not been derived yet.

Motivated by these facts, we propose in this contribution a simple but very accurate technique for the evaluation of the stochastic DOA CRLB of any linearly-modulated signal over SIMO systems.

The rest of this paper is organized as follows. In section II, we introduce the system model to be used throughout this article. Section III is dedicated to the derivation of the Fisher information matrix (FIM) and the evaluation of the stochastic CRLB for NDA DOA estimation. In section IV, graphical representations of the obtained results are presented. Finally, we draw out some concluding remarks in section V.

II. System Model

We consider a linearly-modulated signal impinging from a single source on an array of M antennas. The transmitted symbols are supposed to be independent and identically distributed and drawn from an L-ary constellation. Assuming ideal timing and frequency synchronization, the n^{th} received sample on the $\{i^{th}\}_{i=1}^{M}$ antenna element, at the output of the matched filter, is modeled as:

$$y_i(n) = Se^{j\phi}a_i x(n) + w_i(n), \quad i = 1, 2, .., M \tag{1}$$

where a_i stands for the i^{th} element of the steering vector $\boldsymbol{a} = [1, e^{2\pi j f_0(\theta)}, .., , e^{2\pi j f_{M-1}(\theta)}]^T$ which verifies the identity $\|\boldsymbol{a}\|^2 = M$, and j is the complex number verifying $j^2 = -1$, $f_i(\theta)$ is a function of the DOA that depends on the

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antenna array geometry. Moreover, at time index n, $x(n)$ is the transmitted symbol, and $w_i(n)$ is the noise component on the i^{th} antenna branch that is modeled by a zero-mean circular complex Gaussian random variable with independent real and imaginary parts, each of variance $\frac{\sigma^2}{2}$. The parameter S stands for the channel gain that is assumed constant over the observation interval. Moreover, θ is the unknown DOA of the wave impinging from the far-field source. We assume hereafter that the noise components $\boldsymbol{w}(n) = [w_1(n),\ w_2(n), \cdots,\ w_{N_a}(n)]^T$ are spatially uncorrelated, i.e, $E\{\boldsymbol{w}(n)^H\boldsymbol{w}(n)\} = \sigma^2\boldsymbol{I}_M$, where $\boldsymbol{I}_M$ is the $M \times M$ identity matrix. The noise is also assumed to be white over each antenna. In order to provide standard CRLBs that do not depend on the emission power, we also assume the energy of the transmitted signals to be normalized to one, i.e., $E\{|x(n)|^2\} = 1$. Then we define the SNR of the system as follows:

$$\rho = \frac{\mathrm{E}\{|S|^2|x(n)|^2\}}{\sigma^2} = \frac{|S|^2}{\sigma^2}. \tag{2}$$

III. Derivation of the NDA Stochastic CRLB for Linearly-Modulated Signals

In this section, we will develop the stochastic Fisher information matrix (FIM) associated with the two considered parameters (θ, ϕ). We will then evaluate the stochastic CRLB for NDA DOA estimates. The entries of the FIM are defined as :

$$[\boldsymbol{I}(\theta,\phi)]_{i,j} = \mathrm{E}\left\{\frac{\partial L_{\boldsymbol{\alpha}}(n)}{\partial \boldsymbol{\alpha}_i}\frac{\partial L_{\boldsymbol{\alpha}}(n))}{\partial \boldsymbol{\alpha}_j}\right\}, \quad i,j = 1,2. \tag{3}$$

The expectation $\mathrm{E}\{.\}$ is taken with respect to the received signal $\boldsymbol{y}(n)$ and $\{\boldsymbol{\alpha}_i\}_{i=1,2}$ are the elements of the unknown parameter vector $\boldsymbol{\alpha} = [\theta, \phi]^T$. The CRLB for NDA DOA estimation is given by:

$$\mathrm{CRLB}(\theta) = \left[\boldsymbol{I}(\theta,\phi)^{-1}\right]_{1,1}. \tag{4}$$

In (3), $L_{\boldsymbol{\alpha}}(n)$ is the log-likelihood function of the n^{th} received sample, defined as:

$$L_{\boldsymbol{\alpha}}(n) = \ln(p(\boldsymbol{y}(n);\boldsymbol{\alpha})), \tag{5}$$

where $p(\boldsymbol{y}(n);\boldsymbol{\alpha})$ is the probability density function (pdf) of the received vector $\boldsymbol{y}(n)$ parameterized by $\boldsymbol{\alpha}$. In order to obtain the pdf of the received signal $\boldsymbol{y}(n)$, we average its pdf, conditioned on the transmitted symbols, with respect to all the points of the constellation alphabet as follows:

$$p(\boldsymbol{y}(n);\boldsymbol{\alpha}) = \frac{1}{L\pi^M\sigma^{2M}}\sum_{l=1}^{L}\exp\left\{\frac{\|\boldsymbol{y}(n) - Se^{j\phi}x_l\boldsymbol{a}\|^2}{\sigma^2}\right\}, \tag{6}$$

where $\{x_l\}_{l=1}^{L}$ are the L points of the constellation alphabet, and the operator $\|.\|$ returns the Euclidian norm of any vector. A more compact expression of the likelihood function is:

$$p(\boldsymbol{y}(n);\boldsymbol{\alpha}) = \frac{1}{L\pi^M\sigma^{2M}}\exp\left\{\frac{-\|\boldsymbol{y}(n)\|^2}{\sigma^2}\right\}D_{[\theta,\phi]}(n), \tag{7}$$

where all the terms depending on θ and ϕ are gathered in $D_{[\theta,\phi]}(n) = D_{\boldsymbol{\alpha}}(n)$, defined as:

$$\begin{aligned} D_{\boldsymbol{\alpha}}(n) \;=\; & \sum_{l=1}^{L}\Bigg(\exp\left\{\frac{-S^2|x_l^2|M^2}{\sigma^2}\right\}\times \\ & \exp\left\{\frac{2S\Re\{e^{j\phi}x_l\boldsymbol{y}^H(n)\boldsymbol{a}\}}{\sigma^2}\right\}\Bigg), \end{aligned} \tag{8}$$

where $\Re\{.\}$ returns the real part of any complex number. Then, the log-likelihood function, $L_{\boldsymbol{\alpha}}(n)$, is obtained by:

$$L_{\boldsymbol{\alpha}}(n) = -\ln(L\pi^M\sigma^{2M}) - \frac{\|\boldsymbol{y}(n)\|^2}{\sigma^2} + \ln(D_{\boldsymbol{\alpha}}(n)). \tag{9}$$

In the sequel, we detail the derivation of $\mathrm{E}\left\{\left(\frac{\partial L_{\boldsymbol{\alpha}}(n)}{\partial\theta}\right)^2\right\}$ and the other terms of the FIM matrix involved in (3) can be obtained following the same derivation. In fact, in order to differentiate the log-likelihood function with respect to θ, we can proceed as follows:

$$\frac{\partial L_{\boldsymbol{\alpha}}(n)}{\partial\theta} = \frac{\partial L_{\boldsymbol{\alpha}}(n)}{\partial q(n,\theta)}\frac{\partial q(n,\theta)}{\partial\theta}, \tag{10}$$

where

$$q(n,\theta) = \boldsymbol{y}^H(n)\boldsymbol{a}. \tag{11}$$

Since $q(n,\theta)$ is complex-valued, we use the following property:

$$\begin{aligned} \frac{\partial L_{\boldsymbol{\alpha}}(n)}{\partial\theta} \;=\; & \frac{\partial L_{\boldsymbol{\alpha}}(n)}{\partial\Re(q(n,\theta))}\frac{\partial\Re(q(n,\theta))}{\partial\theta} \\ +\; & \frac{\partial L_{\boldsymbol{\alpha}}(n)}{\partial\Im(q(n,\theta))}\frac{\partial\Im(q(n,\theta))}{\partial\theta}, \end{aligned} \tag{12}$$

where $\Im\{.\}$ returns the imaginary part of any complex number. Now using the following identity:

$$\frac{\partial}{\partial(\alpha + j\beta)} = \frac{1}{2}\left(\frac{\partial}{\partial\alpha} - j\frac{\partial}{\partial\beta}\right), \tag{13}$$

equation (11) reduces to:

$$\frac{\partial L_{\boldsymbol{\alpha}}(n)}{\partial\theta} = 2\Re\left\{\frac{\partial L_{\boldsymbol{\alpha}}(n)}{\partial q(n,\theta)}\frac{\partial q(n,\theta)}{\partial\theta}\right\}. \tag{14}$$

Substituting (14) in the expression of $[\boldsymbol{I}(\theta,\phi)]_{1,1}$, we obtain :

$$\mathrm{E}\left\{\left(\frac{\partial L_{\boldsymbol{\alpha}}(n)}{\partial\theta}\right)^2\right\} \;=\; 4\,\mathrm{E}\left\{\left(\Re\left\{\frac{\partial L_{\boldsymbol{\alpha}}(n)}{\partial q(n,\theta)}\frac{\partial q(n,\theta)}{\partial\theta}\right\}\right)^2\right\}. \tag{15}$$

Using the identities $\frac{\partial}{\partial z}e^{\Re(rz)} = \frac{r}{2}e^{\Re(rz)}$ and $\frac{\partial}{\partial z}e^{\alpha z} = \alpha e^{z}$ that are valid for any complex numbers r, α and z, we obtain :

$$\frac{\partial L_{\boldsymbol{\alpha}}(n)}{\partial q(n,\theta)} = \frac{S}{\sigma^2}e^{j\phi}C_{\boldsymbol{\alpha}}(n), \tag{16}$$

with

$$C_{\boldsymbol{\alpha}}(n) \;=\; \frac{\sum_{l=1}^{L}x_l\exp\{\frac{-S^2|x_l^2|M^2}{\sigma^2}\}\exp\{\frac{2S\Re\{e^{j\phi}x_l\boldsymbol{y}^H(n)\boldsymbol{a}\}}{\sigma^2}\}}{\sum_{l=1}^{L}\exp\{\frac{-S^2|x_l^2|M^2}{\sigma^2}\}\exp\{\frac{2S\Re\{e^{j\phi}x_l\boldsymbol{y}^H(n)\boldsymbol{a}\}}{\sigma^2}\}}, \tag{17}$$

In addition, other easy algebraic manipulations on $q(n,\theta)$ lead to :

$$q(n,\theta) = Sx^*(n)e^{-j\phi}M + \sum_{i=1}^{M} w_i^*(n)a_i, \tag{18}$$

where $(.)^*$ is the conjugate operator. Therefore, the derivative of $q(n,\theta)$ with respect to θ is given by :

$$\frac{\partial q(n,\theta)}{\partial\theta} = \frac{\partial}{\partial\theta}\sum_{i=1}^{M} w_i^*(n)a_i = \sum_{i=1}^{M} w_i^*(n)\frac{\partial a_i}{\partial\theta}. \tag{19}$$

Finally we obtain the expression of $[\boldsymbol{I}(\theta,\phi)]_{1,1}$ as follows :

$$[\boldsymbol{I}(\theta,\phi)]_{1,1} = \frac{4S^2}{\sigma^4}\,\mathrm{E}\left\{\left(\Re\left\{e^{j\phi}C_{\boldsymbol{\alpha}}(n)\sum_{i=1}^{M} w_i^*(n)\frac{\partial a_i}{\partial\theta}\right\}\right)^2\right\}. \tag{20}$$

Similar derivation steps lead to the following expressions for the remaining terms of the FIM :

$$[\boldsymbol{I}(\theta,\phi)]_{2,2} = \frac{4S^2}{\sigma^4}\,\mathrm{E}\left\{\left(\Re\left\{q(n,\theta)C_{\boldsymbol{\alpha}}(n)je^{j\phi}\right\}\right)^2\right\}, \tag{21}$$

$$[\boldsymbol{I}(\theta,\phi)]_{1,2} = \frac{4S^2}{\sigma^4}\,\mathrm{E}\left\{\Re\left\{je^{2j\phi}C_{\boldsymbol{\alpha}}^2(n)q(n,\theta)\frac{\partial q(n,\theta)}{\partial\theta}\right\}\right\}, \tag{22}$$

$$[\boldsymbol{I}(\theta,\phi)]_{2,1} = [\boldsymbol{I}(\theta,\phi)]_{1,2}\,, \tag{23}$$

which are then used to evaluate the DOA CRLB as follows:

$$\mathrm{CRLB}(\theta) = \frac{[\boldsymbol{I}(\theta,\phi)]_{2,2}}{\det(\boldsymbol{I}(\theta,\phi))}, \tag{24}$$

where $\det\{.\}$ returns the determinant of any square matrix. Next, we introduce the empirical technique we used in order to evaluate the FIM elements and thereby the CRLB for DOA estimates. We mention here that even though a closed-form expression has been recently established for NDA CRLB for DOA estimators for the square-QAM modulations, the case of general QAM or PSK-modulated signals remains unresolved. The major challenge stems from the complexity (severe non linearity) of the likelihood function for a general likelihood function which renders the expectation operation over the received samples analytically unfeasible.

Two approaches are followed in the open literature in order to evaluate this kind of expectations. Numerical integration is usually performed at low SNRs, where the likelihood function does not exhibit any spike, as it is a smooth function. Consequently, numerical integration works perfectly, even if the sampling step value, for numerical evaluation of the expected value as an integral over the possible noise value weighted by their pdfs, is relatively large (tolerance in sampling step precision). However, when the SNR gets high, the likelihood function becomes more and more spiky and the corresponding spikes get even much closer to each other. At very high SNR values, the spikes become very narrow and the numerical integration method fails completely since the likelihood function can no longer be sampled. Therefore, we opt here for a more efficient empirical technique to evaluate the expectations involved in the different FIM elements. This empirical technique is simply based on Monte Carlo methods. It consists in generating a sufficiently large number K of noise samples then writing down the statistical expectations as a simple sample summation over all the generated noise realizations according to the following formula :

$$\mathrm{E}\{f(X)\} = \frac{1}{K}\sum_{k=1}^{K} f(x_k), \tag{25}$$

where $\{x_k\}_{k=1}^{K}$ are K realizations of the random variable X. This method ensures a very accurate approximation of the expectation, for all the possible SNR values, especially in high SNR regime. In our case, we generate $K = 10000$ random realizations for each noise component on the M receiving antenna branches, i.e., $\{w_i^k(n)\}_{k=1}^{K}$. Then we inject these realizations in (20), (21) and (22) and substitute the statistical expectation by a simple sample sum.

IV. Graphical Representations

In this section, we present the major simulation results for the evaluated stochastic CRLBs of NDA DOA estimates. We plot these CRLBs as a function of the SNR for different modulation types (QAM, PSK, and APSK) and different modulation orders, at different values of the number of receiving antenna elements M, both for ULA and UCA configurations. It should be noted here that steering vectors for these two popular configurations are, respectively, given by : $[1, e^{j\pi sin(\theta)}, e^{2j\pi sin(\theta)}, ..., e^{j(M-1)\pi sin(\theta)}]$ and $[e^{\frac{j\pi cos(\theta)}{2sin(\frac{\pi}{M})}}, e^{\frac{j\pi cos(\theta-2\frac{\pi}{M})}{2sin(\frac{\pi}{M})}}, ..., e^{\frac{j\pi cos(\theta-2(M-1)\frac{\pi}{M})}{2sin(\frac{\pi}{M})}}]$. We verify that our approach of empirically evaluating the CRLBs for general linearly-modulated signals provides the same results as those obtained in the special case of square-QAM signals in [14].

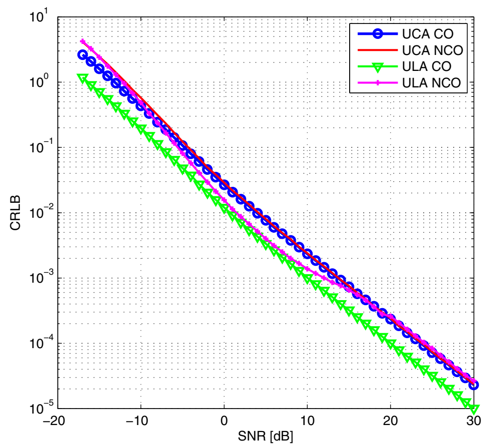

Fig. 1. CRLB for DOA estimates as a function of the SNR, for 8-PSK-modulated signals: , $\theta = 0$, $N = 1$, $M = 3$.

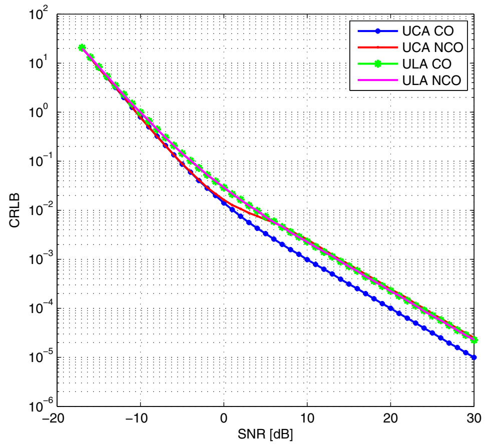

Fig. 2. CRLB for DOA estimates as a function of the SNR, for 16-APSK-modulated signals: , $\theta = 0$, $N = 1$, $M = 3$.

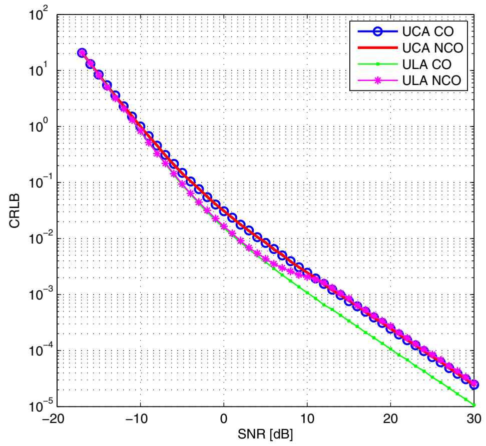

Fig. 3. CRLB for DOA estimates as a function of the SNR, for 32-QAM-modulated signals $\theta = 0$, $N = 1$, $M = 3$.

We see from Figs. 1, 2 and 3 that for a fixed modulation type (i.e. PSK, QAM, or APSK), the CRLB holds almost irrespectively from the modulation order. In addition, for relatively high SNR values, the different CRLBs ultimately coincide for all modulation types meaning that the DOA can be estimated from either of these modulation schemes with the same accuracy.

We see also from the figures that the CRLB for coherent estimation (CO) is always lower than the CRLB for noncoherent estimation (NCO), reflecting thereby better estimation capabilities in the coherent case, especially at high SNR. This difference is less obvious at lower SNR levels. In fact, in the presence of considerably noise-corrupted samples, it is too difficult to accurately estimate the phase distortion. In this case, the estimated phase will not bring too much information that can improve the DOA estimation. Therefore, the advantage of coherent estimation is well motivated especially at high SNR. Indeed, when the SNR is sufficiently high, the phase can be easily estimated then exploited to enhance the estimation of the unknown DOA. Neglecting the phase contribution when the SNR is high is simply a loss of information. Yet, this is no longer the case when the antennas are placed in a circular configuration. It is clearly seen from Figs 1,2 and 3 that, contrarily to the ULA configuration, the a priori knowledge about the phase offset with a UCA configuration does not bring any additional CRLB performance gain. Again, this is due to the circular symmetry of the UCA configuration.

In Fig. 4, it is seen that increasing the number of receiving antennas improves, although slightly, the DOA estimation performance. This result is intuitively expected. In fact, the more receiving antenna branches we use, the more samples we have and thus the more information retrieved about the unknown DOA.

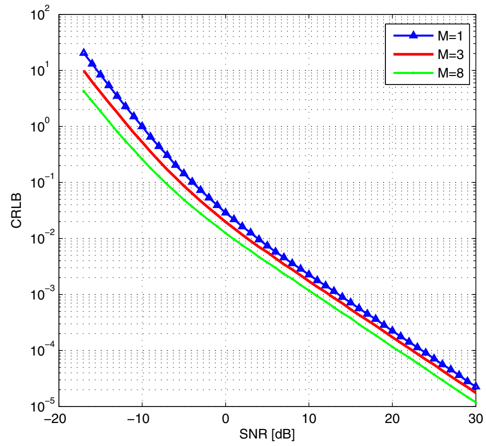

Fig. 4. CRLB for DOA estimates as function of the SNR, for different antenna elements , $\theta = 0$, $N = 1$, 32-QAM, UCA configuration.

V. Conclusion

In this article, we proposed a simple but very accurate technique to evaluate the stochastic CRLBs of the NDA DOA estimates from general linearly-modulated signals over SIMO channels. The received samples are assumed to be corrupted by additive white circular complex Gaussian noise. We established that the achievable performance on the NDA DOA estimates in the presence of an unknown phase offset holds almost the same irrespectively of the modulation order and that it depends on the geometrical configuration of the antenna array. However, the CRLBs obtained in the absence of the phase offset vary from one modulation order to another, especially in the high SNR region. The CRLBs depend in general on the DOA and the antenna array geometry. With a UCA configuration, we have seen that the CRLBs no longer depend on the true DOA due to its circular symmetry.

VI. Appendix A : Derivation of $[\boldsymbol{I}(\theta, \phi)]_{2,2}$

The differentiation of the likelihood function with respect to ϕ gives :

$$\frac{\partial L_{\boldsymbol{\alpha}}(n)}{\partial \phi} = 2\Re\left\{\frac{\partial L_{\boldsymbol{\alpha}}(n)}{\partial h(\phi)}\frac{\partial h(\phi)}{\partial \phi}\right\}, \tag{26}$$

where

$$h(\phi) = e^{j\phi}. \tag{27}$$

Substituting (26) in the expression of $[\boldsymbol{I}(\theta,\phi)]_{2,2}$, we obtain :

$$\mathrm{E}\left\{\left(\frac{\partial L_{\boldsymbol{\alpha}}(n)}{\partial\phi}\right)^2\right\} = 4\,\mathrm{E}\left\{\left(\Re\left\{\frac{\partial L_{\boldsymbol{\alpha}}(n)}{\partial h(\phi)}\frac{\partial h(\phi)}{\partial\phi}\right\}\right)^2\right\}. \tag{28}$$

Having :

$$\frac{\partial L_{\alpha}(n)}{\partial h(\phi)} = \frac{S}{\sigma^2}\boldsymbol{y}^H(n)\boldsymbol{a}C_{\boldsymbol{\alpha}}(n) = \frac{S}{\sigma^2}q(n,\theta)C_{\boldsymbol{\alpha}}(n), \tag{29}$$

and

$$\frac{\partial h(\phi)}{\partial\phi} = je^{j\phi}, \tag{30}$$

and substituting (29 and (30) in (28), we obtain the expression of $[\boldsymbol{I}(\theta,\phi)]_{2,2}$ given by (22).

APPENDIX B
CONSTELLATION POINTS

Constellation	Symbol Values
8PSK	$\exp(\frac{j\pi m}{8})$ where $m = 1,3,5,..,15$
16APSK	$\{R_1\exp(\frac{j\pi m}{4}), R_2\exp(\frac{j\pi n}{12})\}$ $m = 1,3,5,7.n = 1,3,..,23$ $R_1 = 0.4109, R_2 = 1.1301$
32QAM	$\frac{1}{\sqrt{20}}\{\pm1\pm j, \pm3\pm j, \pm1\pm3j, \pm3\pm3j\}$ $\frac{1}{\sqrt{20}}\{\pm1\pm5j, \pm5\pm j, \pm3\pm5j, \pm5\pm3j\}$

REFERENCES

[1] T. S. Rappaport, *Smart antennas: adaptive arrays, algorithms, and wireless position location*, *IEEE Press*, 1998.

[2] S. D. Blostein and H. Leib, "Multiple antenna systems: role and impact in future wireless access," *IEEE Commun. Mag.*, vol. 41, no. 7, pp. 94-101, Jul. 2003.

[3] R. O. Schmidt, "Multiple emitter location and signal parameter estimation," *Proc. RADC, Spectral Estimation Workshop*, Rome, NY, 1979, pp. 243-258.

[4] R. Roy, A. Paulraj and T. Kailath, "ESPRIT - A subspace rotation approach to estimation of parameters of cisoids in noise," *IEEE Trans. Acoust., Speech, Sign. Process.*, vol. ASSP- 34, pp. 1340-1342, Oct. 1986.

[5] P. Stoica and K. C. Sharman, "Maximum likelihood methods for direction-of-arrival estimation," *IEEE Trans. Acoust., Speech, Sign. Process.*, vol. 38, pp. 1132-1143, July 1990.

[6] J. Li, B. Halder, P. Stoica and M. Viberg, "Computationally effcient angle estimation for signals with known waveforms," *IEEE Trans. Sign. Process.*, vol. 43, pp. 2154-2163, Sept. 1995.

[7] M. Agrawal and S. Prasad, "A Modified likelihood function approach to DOA estimation in the presence of unknown spatially correlated gaussian noise using a uniform linear array," *IEEE Trans. Sign. Process.*, vol. 48, no. 10, pp. 2743-2749, Oct. 2000.

[8] P. Stoica, A. G. Larsson and A. B. Gershman, "The stochastic CRB for array processing: A textbook derivation," *IEEE Sign. Process. Lett.*, vol. 8, pp. 148-150, May. 2001

[9] P. Stoica and A. Nehorai, "Performance study of conditional and unconditional direction of arrival estimation," *IEEE Trans. Acoust., Speech, Sign. Process.*, vol. 38, pp. 1783-1795, Oct. 1990.

[10] D. Slepian, "Estimation of signal parameters in the presence of noise," *Trans.* IRE Prof. Group Inform. Theory PG IT-3, pp. 68-89, 1954.

[11] W. J. Bangs, *Array processing with generalized beamformers*, Ph.D. Dissertation, Yale Univ., New Haven, CT, 1971.

[12] P. Stoica and R. Moses, *Introduction to spectral analysis*, Upper Saddle River, NJ: Prentice-Hall, 1997.

[13] J. P. Delmas and H. Abeida, "Gaussian Cramer-Rao bound for direction estimation of noncircular signals in unknown noise fields," *IEEE Trans. Sign. Process.*, vol. 53, no. 12, pp. 4610-4618, Dec. 2005.

[14] F. Bellili, S. Ben Hassen, S. Affes and A. Stéphenne, "Cramér-Rao bound for NDA DOA estimates of square QAM-modulated signals," *IEEE Trans. on Commun*, vol. 59, no. 6, June 2011

[15] J. P. Delmas and H. Abeida, "Cramér-Rao bounds of DOA estimates for BPSK and QPSK modulated signals," *IEEE Trans. Sign. Process.*, vol. 54, no. 1, pp. 117-126, Jan. 2006.