Capacity and Power Allocation of Dual-Hop AF Relaying over Rayleigh Fading Channels

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Abstract—**In this paper, we investigate the capacity and optimal power allocation (PA) scheme between the source and relay for a dual-hop amplify-and-forward (AF) system over non-symmetric Rayleigh fading channels with channel information available at the relay. At first, a closed-form expression of the mutual information (MI) between the input and output of the considered channel is obtained. Since only the exponential integral is involved, the derived expression is useful in finding the optimal PA to achieve the capacity. By further considering high and low signal-to-noise ratio (SNR) regimes, we present tight yet simple approximations to this MI, which can be used to show the advantage of knowing channel information at the relay. Then, focusing on the problem of optimal PA, we first derive a closed-form derivative of the MI. A simple bisection method is then proposed to find the optimal PA scheme. While uniform PA is shown to achieve the capacity at any SNR over the symmetric channel, its optimality can only be observed at low SNRs over a non-symmetric channel. In other SNR regimes, numerical results reveal that uniform PA experiences a significant loss. A comparison between the dual-hop and direct transmission scheme is also made, where we show that the dual-hop scheme using the optimal PA can provide impressive rate increases in medium SNR ranges in various network configurations.**

I. Introduction

Relay communications, first proposed in [1], [2], have recently received considerable research attention due to the advantage in increasing range and reliability in wireless networks [3]–[5]. In general, relaying strategies can be categorized as decode-and-forward (DF), compress-and-forward (CF) and amplify-and-forward (AF). Among these protocols, the AF scheme is the simplest to implement since the relay only needs to scale and retransmit the received signal. AF relaying can be further classified according to the availability of channel knowledge. Specifically, when the relay has only channel distribution information (CDI) of the incoming link, the fixed gain amplification coefficient [3] can be used to maintain an average power constraint at the relay. On the other hand, when the relay has instantaneous knowledge of the incoming link, the received signal can be normalized with such knowledge to the desired power level before being forwarded to the destination [5]. The latter method can therefore be referred to as the channel inversion (CI) technique.

Of different AF schemes, the dual-hop AF system without direct link from the source to destination has attracted considerable efforts in the literature. Pioneering work on dual-hop AF relaying by Hasna in [6]–[8] studied the outage probability and error performance. The achievable rate of such systems over fading channels has also gained much attention. For instance, using a uniform power allocation (PA) scheme between the source and relay, an infinite series representation of the mutual information (MI) between the input and output of the dual-hop channel and two closed-form upper bounds were derived in [9] using the CI coefficient. For the dual-hop system with multiple antennas at all nodes and using the CDI coefficient, an expression for the MI was provided in [10] as a finite sum of integrals based on random matrix theory. A single-integral expression of the MI was also given in [11] along with a Taylor approximation for the dual-hop system with CDI. As similar to [9], only uniform PA among the nodes was considered in [11]. Although PA schemes based on outage probability and instantaneous SNR were investigated respectively in [12], [13], to the best of our knowledge, optimal PA strategies to achieve the capacity of the dual-hop AF system have not been addressed in the literature. It is perhaps due to the difficulty in obtaining a closed-form yet insightful expression of the MI.

Inspired by the above observations, this paper studies the capacity and optimal PA between the source and relay for a half-duplex dual-hop AF system over Rayleigh fading channels. Here, we focus on the system with CI. As shall be shown shortly, such a system outperforms the one using CDI in high SNR regions. At first, for a given PA scheme, a closed-form expression of the MI is derived for a general non-symmetric network with arbitrary channel variances. This expression is useful in finding the optimal PA scheme, as it only involves the well-known exponential integral. Tight approximations to the MI at high and low SNRs are also provided, which can be used to demonstrate the advantage of CI over CDI. Then, we turn our attention to the optimal PA scheme that leads to the capacity of the considered system. To this end, we first derive a closed-form derivative of the MI. A simple bisection method is then proposed to optimize the PA. Whilst uniform PA is shown to be the best for the symmetric channel at any SNR, it is only optimal at low SNRs over a non-symmetric one. Finally, simulation results are provided to quantify the gain provided by the proposed PA scheme. Various scenarios in which the dual-hop system outperforms the direct transmission scheme are also analyzed and discussed.

II. Channel Model

The considered dual-hop system consists of three single-antenna nodes: the source S, the relay R, and the destination D. The transmission is carried out in frames composed of two consecutive unit-time phases as follows. In the first phase of a

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given frame i, S sends the signal x_i to R. The received signal at R can be written as

$$r_i = \sqrt{E_s} h_{\text{sr}}^{(i)} x_i + w_i,$$

where E_s is a constant related to the transmitted power that shall be explained shortly; w_i is the zero-mean circularly Gaussian noise with variance N_0, denoted as $\mathcal{CN}(0, N_0)$; and $h_{\text{sr}}^{(i)}$ denotes the S-R complex channel gain at frame i. In the second phase of frame i, R amplifies and forwards the symbol received during the first phase to D. The received signal at D in the second phase is expressed as

$$y_i = h_{\text{rd}}^{(i)} b_i (\sqrt{E_s} h_{\text{sr}}^{(i)} x_i + w_i) + v_i, \tag{1}$$

where $v_i \sim \mathcal{CN}(0, N_0)$, $h_{\text{rd}}^{(i)}$ is the R-D channel gain, and b_i is the amplification coefficient. In this paper, it is assumed that D has full knowledge of the channel gains $\boldsymbol{h}_i = [h_{\text{sr}}^{(i)}, h_{\text{rd}}^{(i)}]$, which change independently from frame to frame. Furthermore, these channel gains are assumed to be independent zero-mean complex circular Gaussian with arbitrary variances as $h_{\text{sr}}^{(i)} \sim \mathcal{CN}(0, \phi_{\text{sr}})$ and $h_{\text{rd}}^{(i)} \sim \mathcal{CN}(0, \phi_{\text{rd}})$, which result in a general non-symmetric relaying channel. The variances ϕ_{sr} and ϕ_{rd} are used to account for different pathloss and shadowing effects over the links.

Let $q_1 = \mathbb{E}[|x_i|^2]$ so that an average power of $q_1 E_s$ is allocated to S in the first phase, and let $q_2 E_s$ be the average power allocated to R in the second phase. When R only knows the second order statistics of the S-R channel, i.e., the relay has CDI, the fixed gain amplification coefficient is given as

$$b_i^{(\text{CDI})} = b^{(\text{CDI})} = \sqrt{\frac{q_2 \rho}{\phi_{\text{sr}} q_1 \rho + 1}}, \tag{2}$$

where $\rho = E_s / N_0$ is the normalized SNR. On the other hand, when the relay has instantaneous knowledge of the S-R link, the CI technique can be used and the variable gain amplification coefficient is given as

$$b_i^{(\text{CI})} = \sqrt{\frac{q_2 \rho}{|h_{\text{sr}}^{(i)}|^2 q_1 \rho + 1}}. \tag{3}$$

In this paper, our main focus will be on the system using the CI coefficient. Relevant comparisons between the CDI and CI systems shall also be discussed throughout the paper.

For either CDI or CI, it can be easily shown from (1) that the MI between the input and output of the dual-hop channel can be maximized when S uses a Gaussian codebook. The conditional MI between x_i and y_i in (1) is then given as

$$I(x_i, y_i | \boldsymbol{h}_i) = \log \left(1 + \frac{\rho q_1 b_i^2 |h_{\text{sr}}^{(i)}|^2 |h_{\text{rd}}^{(i)}|^2}{1 + b_i^2 |h_{\text{rd}}^{(i)}|^2} \right), \tag{4}$$

where $\log(\cdot)$ denotes the logarithm to base two. Given a total power constraint on S and R of $q_t E_s$, i.e., $q_1 + q_2 \le q_t$, and that the transmission of x_i lasts two symbol periods, the power-constrained ergodic capacity (in b/s/Hz) of the dual-hop system can be calculated as

$$C = \frac{1}{2} \max_{\substack{q_1, q_2 \ge 0 \\ q_1 + q_2 \le q_t}} \mathbb{E}_{\boldsymbol{h}}[I(x_i, y_i | \boldsymbol{h}_i)]. \tag{5}$$

III. Mutual Information and Asymptotic Behavior

In this section, the unconditional MI $I(x_i, y_i)$ for a given PA $\boldsymbol{q} = [q_1, q_2]$ shall be evaluated, which is useful for the development of the optimal PA scheme in the next section. The benefit provided by CI over CDI is also discussed.

First, let $g_i = |h_{\text{sr}}^{(i)}|^2$ and $h_i = |h_{\text{rd}}^{(i)}|^2$. The conditional MI of the systems using the CDI and CI-based coefficients in (2) and (3) simplifies respectively to

$$I^{(\text{CDI})}(x_i, y_i | \boldsymbol{h}_i) = \log \left(1 + \frac{\rho^2 q_1 q_2 h_i g_i}{\rho q_2 h_i + \rho q_1 \phi_{\text{sr}} + 1} \right), \tag{6}$$

$$I^{(\text{CI})}(x_i, y_i | \boldsymbol{h}_i) = \log \left(1 + \frac{\rho^2 q_1 q_2 h_i g_i}{\rho q_2 h_i + \rho q_1 g_i + 1} \right). \tag{7}$$

Let $I^{(\text{CI})}(x_i, y_i) = \mathbb{E}_{\boldsymbol{h}}[I^{(\text{CI})}(x_i, y_i | \boldsymbol{h}_i)]$. Factoring (7), the expectation in (5) for the CI system can be expressed as

$$\begin{aligned} I^{(\text{CI})}(x_i, y_i) = &\mathbb{E}_g[\log(1 + \rho q_1 g_i)] + \mathbb{E}_h[\log(1 + \rho q_2 h_i)] \\ &- \mathbb{E}_{g,h}[\log(1 + \rho q_1 g_i + \rho q_2 h_i)]. \end{aligned} \tag{8}$$

To further examine the expectations in (8), we first have the following lemma with regards to the exponential integral.

Lemma 1: Let a_0 be a non-negative constant and $\{\omega_1, \omega_2\}$ be independent exponentially distributed random variables with means ϕ_1 and ϕ_2, respectively. Define

$$\mathcal{J}(x) = \exp(x) E_1(x), \tag{9}$$

where $E_1(\cdot)$ is the exponential integral [14]

$$E_1(x) = \int_x^\infty \frac{e^{-u}}{u} \, du = -\left(\gamma + \ln(x) + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n! n} \right),$$

and γ is the Euler constant. Then for $a_0 > 0$, one has:

$$\begin{aligned} &\mathbb{E}_{\omega_1}[\ln(a_0 + \omega_1)] = \ln(a_0) + \mathcal{J}(a_0/\phi_1), \\ &\mathbb{E}_{\omega_1, \omega_2}[\ln(a_0 + \omega_1 + \omega_2)] = \\ &\begin{cases} 1 + \ln(a_0) + \left(1 - \frac{a_0}{\phi_1}\right) \mathcal{J}\left(\frac{a_0}{\phi_1}\right), & \phi_1 = \phi_2 \\ \ln(a_0) + \frac{\phi_1 \mathcal{J}(a_0/\phi_1) - \phi_2 \mathcal{J}(a_0/\phi_2)}{\phi_1 - \phi_2}, & \phi_1 \neq \phi_2, \end{cases} \end{aligned} \tag{10}$$

and for $a_0 = 0$,

$$\mathbb{E}_{\omega_1}[\ln(\omega_1)] = \ln(\phi_1) - \gamma, \tag{11}$$

$$\mathbb{E}_{\omega_1, \omega_2}[\ln(\omega_1 + \omega_2)] = \begin{cases} 1 - \gamma + \ln(\phi_1), & \phi_1 = \phi_2 \\ \frac{\phi_1 \ln(\phi_1) - \phi_2 \ln(\phi_2)}{\phi_1 - \phi_2} - \gamma, & \phi_1 \neq \phi_2. \end{cases}$$

The proof of this lemma is straightforward and omitted here for brevity of the presentation. Now, from (10) in Lemma 1, the expectation in (8) can be obtained as

$$\begin{aligned} &I^{(\text{CI})}(x_i, y_i) = \\ &\begin{cases} \frac{1}{\ln(2)} \left[\left(1 + \frac{1}{\rho q_1 \phi_{\text{sr}}}\right) \mathcal{J}\left(\frac{1}{\rho q_1 \phi_{\text{sr}}}\right) - 1 \right], & q_1 \phi_{\text{sr}} = q_2 \phi_{\text{rd}} \\ \frac{q_1 \phi_{\text{sr}} \mathcal{J}\left(\frac{1}{\rho q_2 \phi_{\text{rd}}}\right) - q_2 \phi_{\text{rd}} \mathcal{J}\left(\frac{1}{\rho q_1 \phi_{\text{sr}}}\right)}{\ln(2)(q_1 \phi_{\text{sr}} - q_2 \phi_{\text{rd}})}, & q_1 \phi_{\text{sr}} \neq q_2 \phi_{\text{rd}}. \end{cases} \end{aligned} \tag{12}$$

Observe that the MI in (12) is in closed-form and simple to evaluate since it involves only the exponential integral.

To demonstrate the benefit of CI over CDI, we focus on high SNR regions. Ignoring the lower order terms, the conditional MI in (7) can be approximated at high SNRs as

$$I^{(\mathrm{CI})}(x_i, y_i|\boldsymbol{h}_i) \approx \log(\rho q_1 q_2 h_i g_i) - \log(q_2 h_i + q_1 g_i).$$

From (11), the expectation of the above equation becomes

$$I^{(\mathrm{CI})}(x_i, y_i) \approx \tag{13}$$
$$\begin{cases} \log(\rho q_1 \phi_{\mathrm{sr}}) - [\gamma/\ln(2)] - [1/\ln(2)], & q_1\phi_{\mathrm{sr}} = q_2\phi_{\mathrm{rd}} \\ \log(\rho) + \frac{q_1\phi_{\mathrm{sr}}\log(q_2\phi_{\mathrm{rd}}) - q_2\phi_{\mathrm{rd}}\log(q_1\phi_{\mathrm{sr}})}{(q_1\phi_{\mathrm{sr}} - q_2\phi_{\mathrm{rd}})} - \frac{\gamma}{\ln(2)}, & q_1\phi_{\mathrm{sr}} \neq q_2\phi_{\mathrm{rd}} \end{cases}$$

By using a similar analysis, the MI of the system with CDI in (2) can be approximated at high SNRs as

$$I^{(\mathrm{CDI})}(x_i, y_i) \approx \frac{1}{\ln(2)}\left[\ln(\rho q_2 \phi_{\mathrm{rd}}) - 2\gamma - \mathcal{J}\left(\frac{q_1\phi_{\mathrm{sr}}}{q_2\phi_{\mathrm{rd}}}\right)\right] \tag{14}$$

It is then straightforward to show that

$$\begin{aligned} &\mathbb{E}_{\boldsymbol{h}}[I^{(\mathrm{CDI})}(x_i, y_i|\boldsymbol{h}_i) - I^{(\mathrm{CI})}(x_i, y_i|\boldsymbol{h}_i)] \\ &\approx \mathbb{E}_{g,h}[\log(q_1 g_i + q_2 h_i) - \log(q_1\phi_{\mathrm{sr}} + q_2 h_i)] \leq 0. \end{aligned} \tag{15}$$

Therefore, at high SNRs, the capacity of the AF system in (5) grows as $\frac{1}{2}\log(\rho)$ and CI is more beneficial than CDI.

At low SNRs, it is not hard to verify that the systems with CI and CDI provide a similar performance. In particular, the MI for both cases can be approximated as

$$I(x_i, y_i) \approx \mathbb{E}_{g,h}\left[\frac{\rho^2 q_1 q_2 h_i g_i}{\ln(2)}\right] = \frac{\rho^2 q_1 q_2 \phi_{\mathrm{sr}}\phi_{\mathrm{rd}}}{\ln(2)}, \tag{16}$$

where we have used the fact that $\log(1+x) \approx x/\ln(2)$ for small $x > 0$. Note from (16) that the MI of both systems decreases as ρ^2 at low SNR for any $\boldsymbol{q}$.

IV. Optimal Power Allocation

The capacity of the dual-hop AF system in (5) with CI can be found by maximizing the unconditional MI in (12) over the feasible region $q_1 + q_2 \leq q_t$. With a slight abuse of notation, let $I(\boldsymbol{q}|\boldsymbol{h}_i) = I^{(\mathrm{CI})}(x_i, y_i|\boldsymbol{h}_i)$ in (7) and $I(\boldsymbol{q}) = I^{(\mathrm{CI})}(x_i, y_i)$ in (12). By taking the first derivatives, it can be easily shown that $I(\boldsymbol{q}|\boldsymbol{h}_i)$ in (7) is a strictly increasing function of both q_1 and q_2. Hence, $I(\boldsymbol{q})$ in (12) is also strictly increasing with q_1 and q_2, and the power constraint must be tight $q_1 + q_2 = q_t$. Furthermore, the second derivative of (7) over the line segment $q_1 + q_2 = q_t$ ($0 \leq q_1, q_2 \leq q_t$) can be shown to be negative. The maximization of (12) over the line $q_1 + q_2 = q_t$ is therefore a concave optimization problem. The optimal PA $\boldsymbol{q}^\star = [q_1^\star, q_2^\star] = [q_1^\star, q_t - q_1^\star]$ is then unique and can be easily obtained by finding a stationary point in $I(\boldsymbol{q})$. In particular, the derivative of $I([q_1, q_t - q_1])$ can be written when $q_1\phi_{\mathrm{sr}} \neq (q_t - q_1)\phi_{\mathrm{rd}}$ as

$$\frac{dI([q_1, q_t - q_1])}{dq_1} = \frac{-Aq_1^4 - Bq_1^2 + Eq_1 + Hq_1^5 + Fq_1^3 - G}{\rho q_1^2 \phi_{\mathrm{rd}}\phi_{\mathrm{sr}}(q_t - q_1)^2[(q_t - q_1)\phi_{\mathrm{rd}} - q_1\phi_{\mathrm{sr}}]^2} \tag{17}$$

where we have used the fact that $\frac{d}{dx}\mathcal{J}(x) = \mathcal{J}(x) - \frac{1}{x}$ [14]. In (17), $A = \mathcal{J}_2\phi_{\mathrm{rd}}^2(\phi_{\mathrm{rd}} + \phi_{\mathrm{sr}} - \rho q_t \phi_{\mathrm{sr}}^2) + \mathcal{J}_1\phi_{\mathrm{sr}}^2(\rho q_t \phi_{\mathrm{rd}}^2 - \phi_{\mathrm{sr}} - \phi_{\mathrm{rd}}) + \rho q_t \phi_{\mathrm{rd}}\phi_{\mathrm{sr}}(\phi_{\mathrm{sr}}^2 + 4\phi_{\mathrm{rd}}^2 + 5\phi_{\mathrm{rd}}\phi_{\mathrm{sr}})$, $B = q_t^2\phi_{\mathrm{rd}}^2[\rho q_t \phi_{\mathrm{sr}}(\mathcal{J}_1\phi_{\mathrm{sr}} + \phi_{\mathrm{sr}} + 4\phi_{\mathrm{rd}}) + (6\phi_{\mathrm{rd}} + 3\phi_{\mathrm{sr}} - \rho q_t \phi_{\mathrm{sr}}^2)\mathcal{J}_2]$, $H = \rho\phi_{\mathrm{rd}}\phi_{\mathrm{sr}}(\phi_{\mathrm{rd}} + \phi_{\mathrm{sr}})^2$, $E = q_t^3\phi_{\mathrm{rd}}^2(4\mathcal{J}_2\phi_{\mathrm{rd}} + \mathcal{J}_2\phi_{\mathrm{sr}} + \rho q_t \phi_{\mathrm{rd}}\phi_{\mathrm{sr}})$, $F = q_t\phi_{\mathrm{rd}}[\mathcal{J}_1\phi_{\mathrm{sr}}^2(2\rho q_t\phi_{\mathrm{rd}} - 1) + 2\rho q_t \phi_{\mathrm{rd}}\phi_{\mathrm{sr}}(2\phi_{\mathrm{sr}} + 3\phi_{\mathrm{rd}}) + \mathcal{J}_2\phi_{\mathrm{rd}}(4\phi_{\mathrm{rd}} + 3\phi_{\mathrm{sr}} - 2\rho q_t \phi_{\mathrm{sr}}^2)]$, and $G = \mathcal{J}_2 q_t^4 \phi_{\mathrm{rd}}^3$ with $\mathcal{J}_1 = \mathcal{J}(1/[\rho(q_t - q_1)\phi_{\mathrm{rd}}])$ and $\mathcal{J}_2 = \mathcal{J}(1/[\rho q_1 \phi_{\mathrm{sr}}])$. When $q_1\phi_{\mathrm{sr}} = (q_t - q_1)\phi_{\mathrm{rd}}$, $q_1 = (q_t\phi_{\mathrm{rd}})/(\phi_{\mathrm{rd}} + \phi_{\mathrm{sr}})$ and (17) can be simplified to

$$\frac{d}{dq_1} I\left(\left[\frac{q_t\phi_{\mathrm{rd}}}{\phi_{\mathrm{rd}} + \phi_{\mathrm{sr}}}, \frac{q_t\phi_{\mathrm{sr}}}{\phi_{\mathrm{rd}} + \phi_{\mathrm{sr}}}\right]\right) = \frac{(\phi_{\mathrm{rd}}^2 - \phi_{\mathrm{sr}}^2) f_1(\phi_{\mathrm{rd}}, \phi_{\mathrm{sr}}, \rho)}{2q_t^3 \phi_{\mathrm{rd}}^3 \phi_{\mathrm{sr}}^3 \rho^2}, \tag{18}$$

where $f_1(\phi_{\mathrm{rd}}, \phi_{\mathrm{sr}}, \rho) = -\rho q_t \phi_{\mathrm{rd}}\phi_{\mathrm{sr}}(\phi_{\mathrm{sr}} + \phi_{\mathrm{rd}} + \rho q_t \phi_{\mathrm{rd}}\phi_{\mathrm{sr}}) + (\phi_{\mathrm{rd}}^2 + \phi_{\mathrm{sr}}^2 + 2\phi_{\mathrm{rd}}\phi_{\mathrm{sr}} + 2\rho q_t \phi_{\mathrm{rd}}\phi_{\mathrm{sr}}^2 + 2\rho q_t \phi_{\mathrm{rd}}^2\phi_{\mathrm{sr}})\mathcal{J}_3$, and $\mathcal{J}_3 = \mathcal{J}([\phi_{\mathrm{rd}} + \phi_{\mathrm{sr}}]/[\rho q_t \phi_{\mathrm{rd}}\phi_{\mathrm{sr}}])$. The optimal PA can then be obtained by finding the point $0 \leq q_1^\star \leq q_t$ such that the derivative in (17) is equal to zero. Given that (17) is highly non-linear, finding a closed form expression for the stationary point $q_1^\star$ appears difficult for any ϕ_{rd} and ϕ_{sr}. From the concavity of (12) over the line $q_1 + q_2 = q_t$, the optimal PA can be obtained numerically by performing bisection on (17) for $0 \leq q_1 \leq q_t$.

Observe from (18) that $I([q_1, q_t - q_1])$ has a stationary point at $q_1 = (q_t\phi_{\mathrm{rd}})/(\phi_{\mathrm{rd}} + \phi_{\mathrm{sr}})$ when $\phi_{\mathrm{rd}} = \phi_{\mathrm{sr}}$, i.e., when the channel is symmetric. Since $I([q_1, q_t - q_1])$ is concave, $q_1^\star = q_2^\star = q_t/2$ must be the global maximizer for the system with $\phi_{\mathrm{rd}} = \phi_{\mathrm{sr}}$. Hence, uniform PA is optimal for the symmetric network at any SNR. The optimality of uniform PA can also be seen over a general non-symmetric channel $\phi_{\mathrm{rd}} \neq \phi_{\mathrm{sr}}$ but only at a sufficiently low SNR. In particular, from the arithmetic/geometric mean inequality, it can be seen from (16) that the capacity of the system at low SNRs is achieved by uniform PA $q_1^\star = q_2^\star = q_t/2$ regardless of ϕ_{sr} and ϕ_{rd}. For other SNR regimes, uniform PA becomes sub-optimal, as shall be demonstrated shortly via numerical results. Finally, it should be noted that at high enough SNRs, the optimal PA $\boldsymbol{q}^\star$ that achieves the capacity in (5) can be obtained by maximizing the approximation in (13) over the line $q_1 + q_2 = q_t$. It can be seen from (13) that the optimal PA is then asymptotically independent of ρ at high SNRs.

V. Illustrative Results

In this section, simulation results are provided to confirm the analysis carried out in previous sections. For all simulations, we adopt the linear network model. With this model, it is assumed that the relay is in the line between the source and destination. Furthermore, the S-D distance is normalized to 1, while the S-R and R-D distances are d and $(1-d)$, respectively, where $0 \leq d \leq 1$. As a result, $\phi_{\mathrm{sr}} = 1/d^\nu$ and $\phi_{\mathrm{rd}} = 1/(1-d)^\nu$, where ν is the pathloss exponent. For convenience, the results in this section are plotted against SNR ρ, $q_t = 2$, and $\nu = 3$ unless otherwise stated.

To verify the analytical expressions in (12) and the high SNR approximations in (13), Fig. 1 shows the MI of the dual-hop system using the CI amplification coefficient in (3). In Fig. 1, uniform PA $\boldsymbol{q} = [1, 1]$ is assumed for all systems and two different relay locations, $d = 0.1$ and 0.5, are considered. The MI of the system using the CDI coefficient in (2) is also included for comparison along with its high SNR

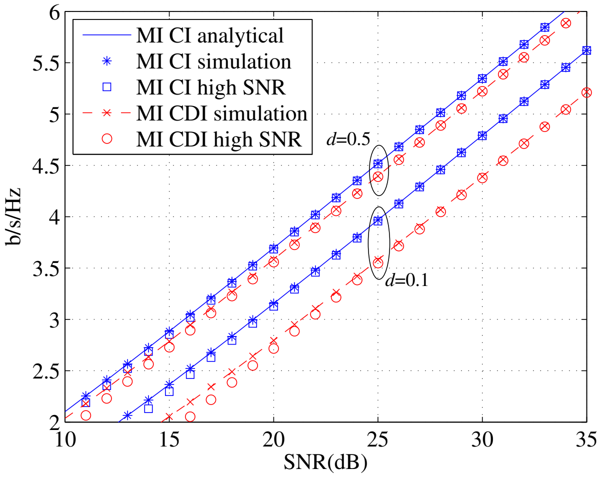

Fig. 1. MI of dual-hop systems using the CI and CDI coefficients along with high SNR approximations for $\boldsymbol{q} = [1, 1]$ and $d = 0.1, 0.5$.

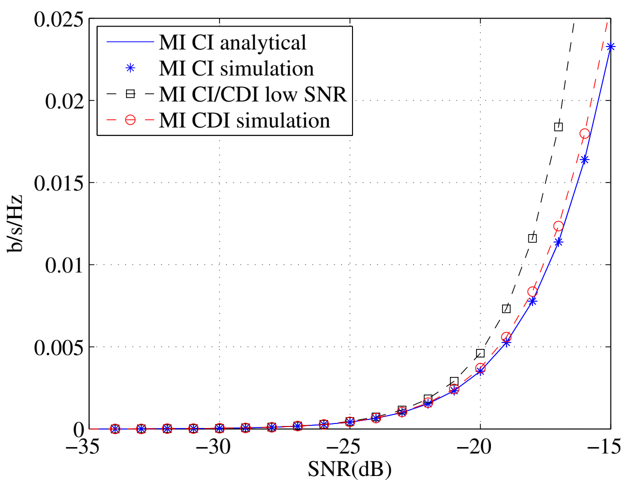

Fig. 2. MI of the dual-hop systems using the CI and CDI coefficients along with the low SNR approximation for $\boldsymbol{q} = [1, 1]$ and $d = 0.5$.

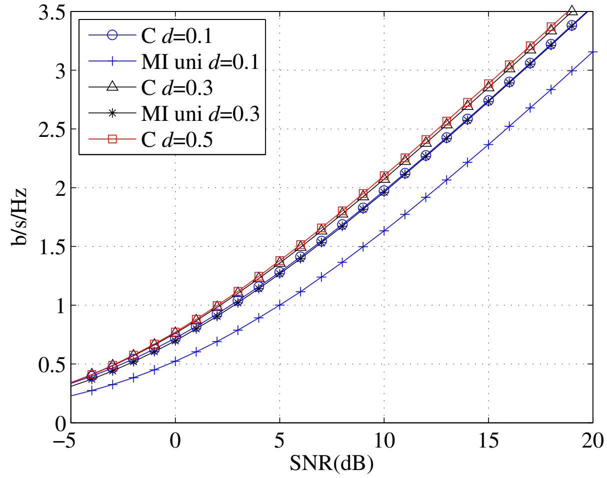

Fig. 3. Capacity of dual-hop systems along with MI of the system with uniform PA for $d = 0.1$, 0.3 and 0.5.

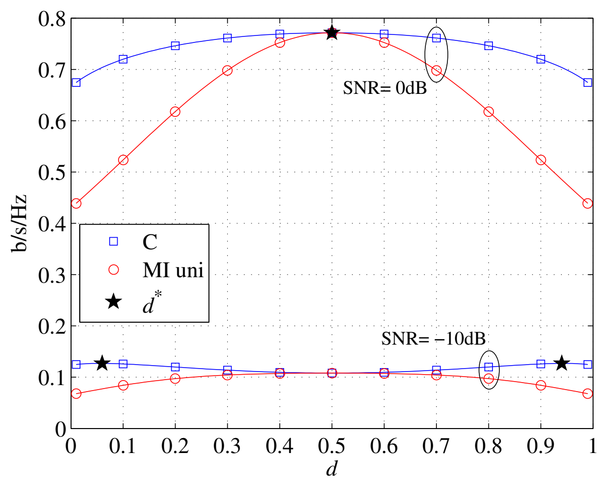

Fig. 4. Capacity of dual-hop systems along with MI of the system with uniform PA at 0dB and -10dB for $0 \le d \le 1$.

approximation in (14). First, note from Fig. 1 that the MI with the CI coefficient obtained through Monte Carlo simulations and the MI derived analytically in (12) coincide for both relay locations. Also observe that the approximations in (13) and (14) are tight at sufficiently high SNRs. As proved in (15), it can be seen from Fig. 1 that the system using the CI coefficient outperforms the one using the CDI in high SNR ranges.

To verify the low SNR approximation in (16), Fig. 2 shows the MI of the systems using the CI and CDI coefficients with uniform PA and $d = 0.5$. Note again that the MI with the CI coefficient obtained through simulations matches perfectly with the analysis in Sec. III. As previously discussed, it can be seen from Fig. 2 that the systems using the CI and CDI coefficients have similar performance at sufficiently low SNRs. The approximation in (16) is tight at low SNRs for both systems, as expected.

To quantify the advantage of the optimal PA, Fig. 3 shows the capacity of the CI systems in (5) along with the MI of the systems using the uniform allocation scheme for $d = 0.1$, 0.3 and 0.5. First, recall from (18) that uniform PA is optimal for the symmetric network with $d = 0.5$. For the other relay locations, it can be seen from Fig. 3 that the optimal PA provides significant gains over the uniform allocation scheme. In particular, asymptotic gains of 0.7dB and 2.4dB can be achieved for $d = 0.3$ and 0.1, respectively.

To examine the effect of relay location, Fig. 4 shows the capacity and MI with uniform PA of the dual-hop CI systems operating at 0dB and -10dB for $0 \le d \le 1$. First, note that due to symmetric structure of the conditional MI in (7), the rates in Fig. 4 are symmetric with respect to the $d = 0.5$ line. As demonstrated earlier, the optimal PA provides rate increases for all relay locations except $d = 0.5$, where uniform PA is optimal. It is important to note from Fig. 4 that the optimal relay location $d^\star$ of the system using the optimal PA, obtained numerically, is SNR dependent and not necessarily unique.

Finally, to compare the relaying system with the direct

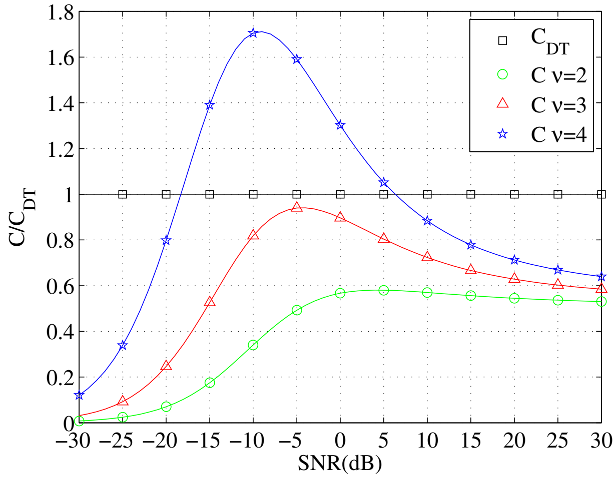

Fig. 5. Capacity of the DT scheme along with the capacity of the dual-hop systems for $d = 0.5$ and $\nu = 2$, 3, 4.

transmission (DT) scheme, Fig. 5 shows the capacity of the dual-hop schemes for $d = 0.5$ and $\nu = 2$, 3, 4. The capacities in Fig. 5 are normalized by the capacity of the DT scheme $C_{\rm DT} = \frac{1}{\ln(2)}\mathcal{J}\left(\frac{1}{\rho\phi_{\rm sd}}\right)$, where $\phi_{\rm sd} = 1$ for the considered network configuration. Note that for $q_t = 2$, both the DT and dual-hop systems have the same total average power constraint. Observe from Fig. 5 that the dual-hop system suffers a great rate penalty at high SNRs due to the half-duplex constraint of the relay. Specifically, since $C_{\rm DT}$ above grows as $\log(\rho)$ at high SNRs, the normalized capacities of the dual-hop system in Fig. 5 approach $1/2$. The dual-hop system also experiences a rate loss at low SNRs since its capacity decreases quadratically rather than linearly with ρ as $C_{\rm DT}$. However, it can be seen from Fig. 5 that for $\nu = 4$, the dual-hop system is able to outperform the DT scheme in some SNR ranges, providing an impressive rate increase of up to 1.7 times $C_{\rm DT}$. This is because when $\phi_{\rm sr} = \phi_{\rm rd} = \phi$ is large, the capacity of the dual-hop scheme can be approximated from (5) and (12) as $C \approx \frac{1}{2\ln(2)}\mathcal{J}\left(\frac{1}{\rho\phi}\right)$. Since $\lim_{x\to 0^+}\mathcal{J}(x) \to +\infty$, the dual-hop system can recover from the half-duplex constraint in medium SNR ranges as long as ϕ is large. Therefore, even if the source has enough power to communicate directly to the destination, the dual-hop protocol is still advantageous under these scenarios.

VI. Conclusions

In this paper, the capacity and optimal PA have been investigated for a dual-hop AF system over Rayleigh fading channels. A closed-form expression of the MI in terms of the exponential integral was first derived, along with its tight approximations at high and low SNRs. A bisection method on the derivative of the mutual information was then proposed to obtain the optimal PA for the general network configuration. Uniform power sharing was then shown to be optimal for the symmetric network at any SNR and for the non-symmetric one at low SNRs. In other SNR regions, the proposed PA provides significant gains over uniform allocation. Although the dual-hop system was shown to be inferior to the DT scheme at high and low SNRs, it can provide significant rate increases in medium SNRs when the intermediate channels are significantly stronger than the direct link.

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