On the Capacity Gap of Gaussian Multi-Way Relay Channels

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***Abstract*—Multi-way relaying is a promising approach to enhance the spectral efficiency in multi-user communication systems. Several relaying strategies have been proposed recently to be used in multi-user communication systems. In this paper, we analyze the gap between the achievable rate of some of these relaying techniques and the capacity of the Gaussian multi-way relay channels (GMWRCs). To this end, for a symmetric GMWRC with K users, we prove that lattice-based relaying guarantees a gap less than $\frac{1}{2(K-1)}$ bit from the capacity upper bound. Also, we show that decode-and-forward and amplify–and-forward relaying may have a larger capacity gap than $\frac{1}{2(K-1)}$ bit depending on the relay and users' SNR. Then, we find the SNR regions where these two techniques also ensure a $\frac{1}{2(K-1)}$-bit gap.**

***Index Terms*—Gaussian multi-way relay channels, capacity gap, achievable rate.**

I. Introduction

Combination of multi-way channels along with relaying initially appeared in the form of two-way relay channels (TWRCs) [1]–[3]. Later, this idea was extended to a system with more than two users in [4], where multi-way relay channels (MWRCs) have been proposed. In an MWRC, several users try to exchange their data usually without having direct links between themselves. To enable data communication between users, one or more relays are used.

The performance of MWRCs highly depends on their relaying approach. In addition to more common amplify-and-forward (AF) and decode-and-forward (DF) relaying techniques, compress-and-forward (CF) [5] is also well-applicable to MWRCs. In fact, it is shown that for MWRCs, CF is very efficient and for a symmetric MWRC, it can achieve to within $\frac{1}{2(K-1)}$ bit of the capacity where K is the number of users [4]. Further, relaying based on lattice codes is proposed in [6] for TWRCs and later for MWRCs in [7]. This technique is also proven to achieve to within $\frac{1}{2}$ bit of the capacity region of each user [8] for Gaussian TWRCs. Ong *et al.* show that this approach is indeed optimal for binary MWRCs [7]. Then, authors suggest a lattice-based relaying strategy for Gaussian MWRCs (GMWRCs), called functional-decode-forward (FDF) in [9]. Authors show that under some conditions, FDF achieves the capacity of GMWRCs. Furthermore, they briefly discuss the capacity gap of FDF when all users and the relay have equal power.

In this paper, we study the capacity gap of FDF, DF and AF in a symmetric GMWRC with $K \geq 2$ users. More specifically, we prove that similar to CF, FDF has a gap less than $\frac{1}{2(K-1)}$ bit between its achievable rate and the capacity region of each user. This is basically the generalization of the results provided in [6] for Gaussian TWRCs to GMWRCs. In addition, it will be shown that AF and DF do not guarantee a gap within $\frac{1}{2(K-1)}$ bit of the capacity bound and may have a larger capacity gap. However, for DF and AF, we find the SNR regions where they guarantee a capacity gap less than $\frac{1}{2(K-1)}$ bit.

The paper is organized as follows: Section II provides the system model and some definitions. The capacity gap analysis is discussed in Section III and Section IV concludes the paper. Further, all proofs are provided in Appendix.

II. Preliminaries

In a GMWRC, $K \geq 2$ users want to communicate their data without having direct user-to-user links. We name users by $u_1, u_2, \ldots, u_K$ and their data by $X_1, X_2, \ldots, X_K$. Each user has a limited power P, thus, for all i, $E[X_i^2] \leq P$. To enable the data communication between users, a relay $\mathcal{R}$ is employed. Here, each user aims to decode all other users data as well as to transmit its data to all other users. In this GMWRC, data communication consists of uplink and downlink phases. In the uplink phase, users transmit their data over a multiple access channel. Assuming a zero-mean Gaussian noise with unit variance for relay, N_R, the received signal at $\mathcal{R}$ is

$$Y_R = \sum_{i=1}^{K} X_i + N_R. \tag{1}$$

Depending on the relaying strategy, $\mathcal{R}$ forms its message, X_R, based on Y_R. Then, X_R is broadcast to all users during the downlink phase. The received signal by u_i in the downlink phase is

$$Y_i = X_R + N_i \tag{2}$$

where N_i is u_i's receiver Gaussian noise with zero mean and unit variance. Due to the power limit at the relay, we assume that $E[X_R^2] < P_r$. Figure 1(a) and 1(b) depicts the uplink and downlink phases in an MWRC. Please notice that in this work, we assume symmetric GMWRCs, i.e. all users have similar channel conditions.

In this paper, we focus on the common rate capacity of GMWRCs. The common rate capacity is the data rate that all users can reliably transmit and receive data with this rate. According to this definition, if we denote the uplink rate for

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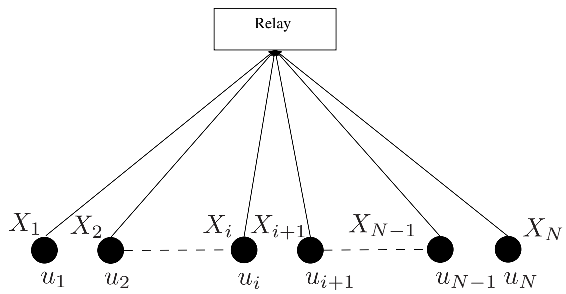
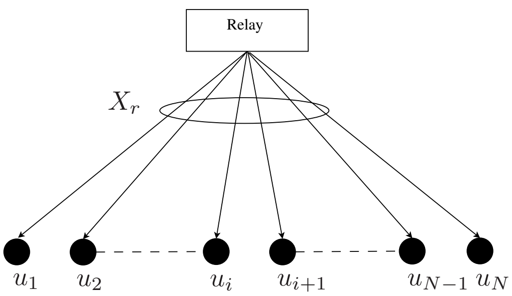


(a) Uplink phase



(b) Downlink phase

Fig. 1. Demonstration of the uplink and downlink phases

u_i by R_i^u and the downlink data rate by R_i^d, then, the common rate capacity, R^c, is

$$R^c = \sup\{\min\{R_1^u, R_2^u, \ldots, R_K^u, R_1^d, R_2^d, \ldots, R_K^d\}\}. \quad (3)$$

For more details on common rate definition and its applications in MWRCs, the reader is encouraged to see [4] and [9]. Please notice that for a GMWRC, R^c is yet to be unknown. In the following, we use the capacity upper bound for our capacity gap analysis.

III. Capacity Gap Analysis

In this section, we first discuss the capacity upper bound for GMWRCs. This bound is then used to study the capacity gap for different relaying schemes. To start the discussion, we state the following lemma on the upper bound of R^c. Please notice that in this paper, $\log(\cdot)$ represents the logarithm to the base 2.

Lemma 1: The upper bound on the common data rate of a GMWRC is

$$R_{\mathrm{UB}}^c = \min\left\{\frac{\log(1+(K-1)P)}{2(K-1)}, \frac{\log(1+P_r)}{2(K-1)}\right\} \quad (4)$$

Proof: Please see [4].

A. Capacity Gap of FDF

For a GMWRC with FDF, the uplink and downlink phases are divided into $K-1$ multiple-access (MAC) and broadcast (BC) slots respectively [9]. In a MAC slot, a pair of users transmits data to the relay and $\mathcal{R}$ directly decodes the sum of the users' data using nested lattice coding [10]. Then, relay broadcasts the decoded message for all users in one BC slot. This continues for $K-1$ MAC and BC slots. Now, each user has received enough independent linear combinations of the other users' data, hence, it can decode the data of all of them. The achievable rate of lattice-based relaying was first studied in [6] for TWRC. Later, the following lemma was proposed [9] for the achievable rate of FDF over GMWRCs.

Lemma 2: The maximum achievable common rate of FDF over a GMWRC is

$$R_{\mathrm{FDF}}^c = \min\left\{\frac{\log\left(\frac{1}{2}+\frac{KP}{2}\right)}{2(K-1)}, \frac{\log(1+P_r)}{2(K-1)}\right\}. \quad (5)$$

Proof: Please see [9].

The following theorem states the performance of FDF in comparison with the capacity upper bound.

Theorem 1: The gap between the achievable rate of FDF and the capacity of a K-user GMWRC is less than $\frac{1}{2(K-1)}$ bit.

Proof: See Appendix.

As numerical illustrations, Figure 2, 3 and 4 depicts the achievable rate of FDF and the capacity upper bound for several cases. In Figure 2, the effect of the users SNR on the capacity gap is studied while the effects of the relay SNR and K are presented in Figure 3 and Figure 4 respectively. As it can be seen, the capacity gap of FDF is always less than $\frac{1}{2(K-1)}$ bit. Further, when downlink limits the rate, FDF achieves the capacity.

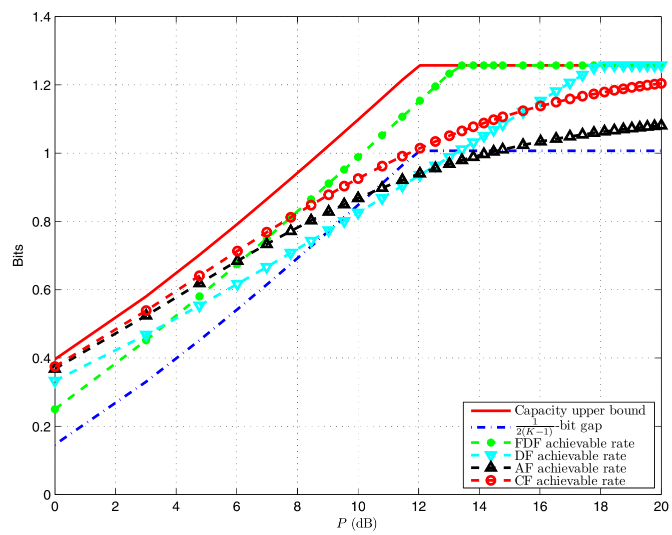


Fig. 2. Achievable rates of relaying schemes when $K = 3$ and $P_r = 15$ dB.

B. Capacity Gap of DF

In a DF GMWRC, all users simultaneously transmit their data in the uplink phase. Then, $\mathcal{R}$ decodes the data of all users and sends them back to all users in the downlink phase. The achievable rate of DF GMWRCs has been studied in [4].

Lemma 3: The maximum achievable common rate for a GMWRC with DF relaying is

$$R_{\mathrm{DF}}^c = \min\left\{\frac{\log(1+KP)}{2K}, \frac{\log(1+P_r)}{2(K-1)}\right\}. \quad (6)$$

Proof: See [4].

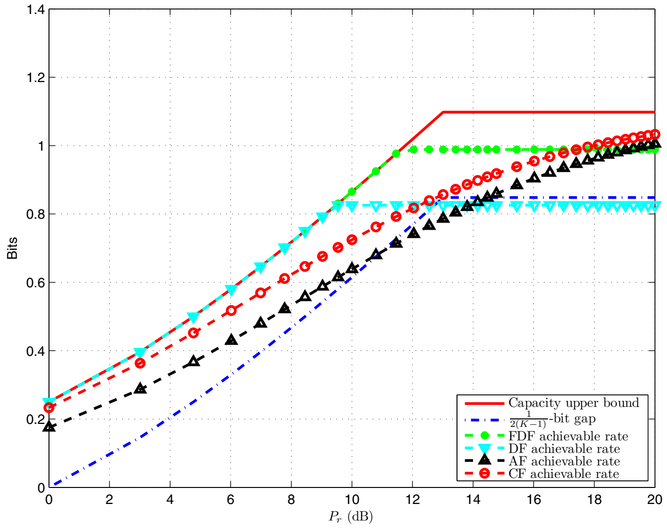


Fig. 3. Achievable rates of relaying schemes when $K = 3$ and $P = 10$ dB.

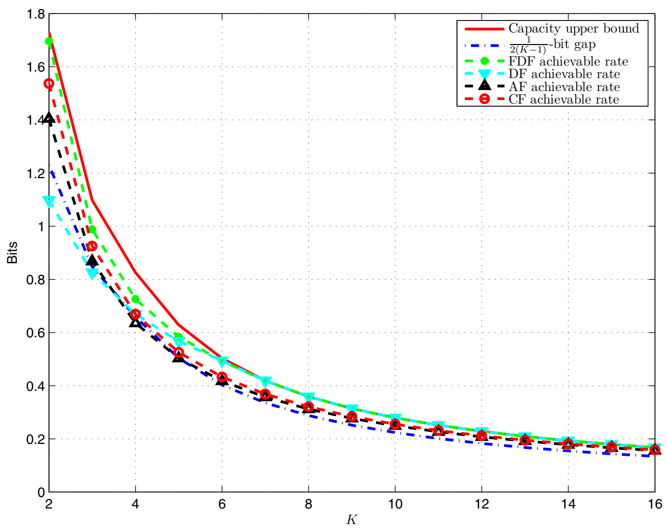


Fig. 4. Achievable rates of relaying schemes when $P = 10$ dB and $P_r = 15$ dB.

Our analysis reveals that DF does not necessarily guarantee a $\frac{1}{2(K-1)}$-bit gap. The result is summarized in the following theorem.

Theorem 2: The gap between R^c_{DF} and R^c_{UB} is less than $\frac{1}{2(K-1)}$ bit if either $P_r < \min\{2(1+KP)^{\frac{K-1}{K}} - 1, (K-1)P\}$ or $(K-1)P < P_r$ and $(K-1)P < 2(1+KP)^{\frac{K-1}{K}} - 1$.

Proof: Please see Appendix.

As it can be seen in Figure 2, 3 and 4, in some SNR regions and depending on the number of users, the capacity gap might be larger than $\frac{1}{2(K-1)}$ bit for DF.

C. Capacity Gap of AF

In a GMWRC with AF, all users simultaneously transmit their data over the channel in the uplink phase. After receiving Y_R, relay simply amplifies the received signal and broadcasts it back to all users without any further processing. Since AF GMWRC is seen as a MAC channel form the users side, it is easy to prove the following lemma on the achievable rate of AF.

Lemma 4: In a K-user GMWRC,

$$R^c_{\text{AF}} = \frac{1}{2(K-1)} \log\left(1 + \frac{(K-1)PP_r}{1+KP+P_r}\right) \qquad (7)$$

is the maximum common rate that AF can achieve.

Now, we state the following theorem on the capacity gap of AF.

Theorem 3: The gap between R^c_{UB} and R^c_{AF} is less than $\frac{1}{2(K-1)}$ if $P_r \le (K-1)P$ and $P_r^2 - (K-2)PP_r < KP$ or $(K-1)P < P_r$ and $K(K-1)P^2 - P - 1 < P_r + (K-1)PP_r$. Otherwise, the gap is larger.

Proof: Please see Appendix.

For the presented numerical cases, depending on the SNR and number of users, the achievable rate of AF falls under the $\frac{1}{2(K-1)}$-bit gap from the capacity upper bound.

IV. Conclusion

In this paper, we studied the capacity gap for different relaying techniques in a symmetric GMWRC with $K \ge 2$ users. Our study revealed that FDF always have a capacity gap less than $\frac{1}{2(K-1)}$ bit. In addition, we showed that depending on the users and relay SNR, AF and DF may also have a capacity gap larger than $\frac{1}{2(K-1)}$.

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Appendix

First, based on Lemma 1, 2 and 3, one can simply conclude the following propositions.

Proposition 1: If $P_r \le (K-1)P$, i.e. downlink is the rate bottleneck, we have

$$R^c_{\text{UB}} = \frac{\log(1+P_r)}{2(K-1)} \qquad (8)$$

and

$$R_{\mathrm{UB}}^{c}=\frac{\log\left(1+(K-1)P\right)}{2(K-1)} \tag{9}$$

if uplink limits the common rate. ■

Proposition 2: In a GMWRC, downlink is the bottleneck if $P_r \le \frac{K}{2}P-\frac{1}{2}$ resulting in

$$R_{\mathrm{FDF}}^{c}=\frac{\log\left(1+P_r\right)}{2(K-1)} \tag{10}$$

and

$$R_{\mathrm{FDF}}^{c}=\frac{\log\left(\frac{1}{2}+\frac{KP}{2}\right)}{2(K-1)} \tag{11}$$

otherwise. ■

Proposition 3: When $P_r \le (1+KP)^{\frac{K-1}{K}}-1$, downlink constrains the common data rate and

$$R_{\mathrm{DF}}^{c}=\frac{\log\left(1+P_r\right)}{2(K-1)}. \tag{12}$$

Further, when $(1+KP)^{\frac{K-1}{K}}-1<P_r$,

$$R_{\mathrm{DF}}^{c}=\frac{\log\left(1+KP\right)}{2K}. \tag{13}$$

and uplink is the rate bottleneck.

A. Proof of Theorem 1

Using Proposition 1 and 2, we partition the range of P_r and P into different regions and then compare R_{UB}^{c} and R_{FDF}^{c} in these regions. These regions specify which constraints in (4) and (5) are active. Since $K\ge 2$, we have $\frac{K}{2}P-\frac{1}{2}<(K-1)P$. To this end, the regions of interest are $P_r\le\frac{K}{2}P-\frac{1}{2}$, $\frac{K}{2}P-\frac{1}{2}<P_r\le(K-1)P$, and $(K-1)P<P_r$. We denote these regions by A_1^{FDF}, A_2^{FDF} and A_3^{FDF} respectively.

Capacity Gap on A_1^{FDF}: In this region, the achievable rate of FDF as well as the upper bound is determined by downlink. Using Proposition 1 and 2, it is seen that $R_{\mathrm{FDF}}^{c}=R_{\mathrm{UB}}^{c}$. In other words, FDF achieves the capacity upper bound and the gap between R_{UB}^{c} and R_{FDF}^{c}, $G_U=R_{\mathrm{UB}}^{c}-R_{\mathrm{FDF}}^{c}$, is 0.

Capacity Gap on A_2^{FDF}: Here, the achievable rate of FDF is bounded by uplink while the rate upper bound is forced by downlink. Thus, the gap in this region is

$$\begin{aligned}G_U&=\frac{1}{2(K-1)}\left[\log(1+P_r)-\log(\frac{1}{2}+\frac{K}{2}P)\right] \qquad (14)\\&=\frac{1}{2(K-1)}\log\left(\frac{1+P_r}{\frac{1}{2}+\frac{K}{2}P}\right).\end{aligned}$$

Since $\log(\cdot)$ is an increasing function, the maximum of G_U happens when P_r has its maximum value on A_2. Since $P_r<(K-1)P$, it is easy to show that

$$\frac{1+P_r}{\frac{1}{2}+\frac{K}{2}P}<2. \tag{15}$$

As a consequence, $G_U<\frac{1}{2(K-1)}$.

Capacity Gap on A_3^{FDF}: On this region, both R_{FDF}^{c} and R_{UB}^{c} are limited by the uplink. Hence, using Proposition 1 and 2

$$G_U=\frac{1}{2(K-1)}\log\left(\frac{1+(K-1)P}{\frac{1}{2}+\frac{K}{2}P}\right). \tag{16}$$

Now, (16) results in $G<\frac{1}{2(K-1)}$. ■

B. Proof of Theorem 2

Similar to FDF, we study the capacity gap for DF over different regions determining whether uplink limits the rate or downlink. Notice that $(1+KP)^{\frac{K-1}{K}}<(K-1)P$. Thus, we define three SNR regions namely A_1^{DF}, A_2^{DF}, and A_3^{DF}. Here, these regions denote $P_r\le(1+KP)^{\frac{K-1}{K}}-1$, $(1+KP)^{\frac{K-1}{K}}-1<P_r\le(K-1)P$, and $(K-1)P<P_r$.

Capacity gap on A_1^{DF}: In this region, both R_{UB}^{c} and R_{DF}^{c} are limited by downlink. Using Proposition 1 and 3, it is seen that $G_U=R_{\mathrm{UB}}^{c}-R_{\mathrm{DF}}^{c}=0$.

Capacity gap on A_2^{DF}: The gap between the capacity upper bound and the achievable rate of DF in this region is

$$\begin{aligned}G_U&=\frac{\log(1+P_r)}{2(K-1)}-\frac{\log(1+KP)}{2K} \qquad (17)\\&=\frac{1}{2(K-1)}\log\left(\frac{1+P_r}{(1+KP)^{\frac{K-1}{K}}}\right).\end{aligned}$$

Now, the capacity gap is less than $\frac{1}{2(K-1)}$ bit if

$$P_r<2(1+KP)^{\frac{K-1}{K}}-1. \tag{18}$$

Considering that $(1+KP)^{\frac{K-1}{K}}\le P_r<(K-1)P$, it is easy to show that (18) does not necessarily hold for all values of P_r and P in this region.

Capacity gap on A_3^{DF}: Here, both R_{UB}^{c} and R_{DF}^{c} are limited by uplink. Thus,

$$G_U=\frac{1}{2(K-1)}\log\left(\frac{1+(K-1)P}{(1+KP)^{\frac{K-1}{K}}}\right) \tag{19}$$

and $G_U<\frac{1}{2(K-1)}$ if

$$(K-1)P<2(1+KP)^{\frac{K-1}{K}}-1 \tag{20}$$

which does not necessarily hold for all P and P_r values within A_3^{DF}. ■

C. Proof of Theorem 3

Based on Proposition 1, we define two regions A_1^{AF} and A_2^{AF}. The first region is where $P_r\le(K-1)P$ and the second region includes $(K-1)P<P_r$.

Capacity Gap on A_1^{AF}: In this region, we have

$$\begin{aligned}G_U^{\mathrm{AF}}&=R_{\mathrm{UB}}^{c}-R_{\mathrm{AF}}^{c} \qquad (21)\\&=\frac{1}{2(K-1)}\log\left(\frac{(1+P_r)(1+KP+P_r)}{1+KP+P_r+(K-1)PP_r}\right)\end{aligned}$$

Now, from (21), it is easy to show that $G_U^{\mathrm{AF}}<\frac{1}{2(K-1)}$ if

$$P_r^2-(K-2)PP_r<KP. \tag{22}$$

Capacity Gap on A_2^{AF}: In this case, we have

$$G_U^{\mathrm{AF}}=\frac{1}{2(K-1)}\log\left(\frac{(1+(K-1)P)(1+KP+P_r)}{1+KP+P_r+(K-1)PP_r}\right) \tag{23}$$

Using (23), it is easy to conclude that if

$$K(K-1)P^2-(K-1)PP_r<1+P_r+P \tag{24}$$

then AF has a capacity gap smaller than $\frac{1}{2(K-1)}$. ■