Sample-Autocorrelation-Function-Based Frequency Estimation of a Single Sinusoid in AWGN

Hua Fu and Pooi-Yuen Kam
Department of Electrical and Computer Engineering,
National University of Singapore, Singapore 117583
E-mail: {elefh,elekampy}@nus.edu.sg Tel: (65)6516-8892 Fax: (65)6779-1103

Abstract— **The problem of estimating the frequency of a single sinusoid observed in additive, white, Gaussian noise is addressed. An explicit, sample-autocorrelation-function-based, approximate maximum likelihood (ML) frequency estimator which does not require numerical search is derived. The structure of this estimator reveals that both the magnitude and the angle of the autocorrelation function should be utilized in estimation processing. Simulation results show that the estimator derived attains the Cramer-Rao lower bound at high signal-to-noise ratio, and has better performance than the planar filtered estimator developed in [22, 23] and the time-domain, approximate ML, received-signal-based estimator in [20, 21]. A new phase unwrapping algorithm is presented to faciliate an efficient, recursive implementation of the estimator. To have a better understanding on the sample-autocorrelation-function-based estimator, a geometric interpretation on the autocorrelation function is introduced. This interpretation allows us to propose a further simplified frequency estimator that resorts to the autocorrelation function angle increments to avoid phase unwrapping. By using the autocorrelation function expression, an alternative form to the frequency estimator of [20, 21] is derived.**

Index Terms— **single sinusoid, frequency estimation, sample-autocorrelation-function-based estimator, phase unwrapping**

I. Introduction

Estimating the frequency of a single sinusoid observed in additive, white, Gaussian noise (AWGN) is an important and classic problem in communications and signal processing [1]−[3]. The discrete-time observation case that we are concerned with in this paper was first formulated in [4] and [5]. The most common approach so far is to model the frequency as the unknown parameter, and apply the theory of maximum likelihood (ML) estimation [6]. This has led to the well-known, computationally intensive solution that requires locating the peak of a periodogram in the frequency domain to obtain the ML estimate [5]. It is implemented by making use of Fast-Fourier-Transform (FFT) which first divides the parameter range into small subintervals and performs a preliminary parallel processing for each subinterval, and then conducts a local maximization to get the final estimate. To reduce the estimation errors due to frequency quantization, the data can be zero padded so that a large number of frequency samples are computed. Alternatively, a coarse FFT followed by an exhaustive local fine numerical search (i.e., Newton-Raphson search) can be implemented [1]. The major drawback of this FFT-based periodogram-search solution is that the computational load is very heavy and intensive. In order to get a better trade-off between the computational complexity and the estimation performance, numerous refinements and improvements have been proposed[1] [7]−[14].

In order to reduce the computational complexity, simplified ML estimators are desirable, and this has led to the phase-based, time-domain estimation approach which provides an explicit structure on the frequency estimator that does not require numerical search. The frequency estimator proposed in [15] uses the received signal phase that can be expressed as a sum of the transmitted signal phase and an additive observation phase noise (AOPN) as the only observation data sample to be fed into the estimator. The estimation is done through linear regression of the received signal phase, which is obtained through phase unwrapping. Moreover, an approximate, linearized observation model (referred to hereafter as Tretter's model) for the instantaneous signal phase was introduced in [15] that is valid only for high signal-to-noise ratio (SNR). The work done in [15] is insightful and has inspired many further research [16]−[21]. For instance, in order to avoid phase unwrapping, using the same approximate AOPN model as [15], the weighted phase averager (WPA) estimator is proposed in [16], where the frequency estimate is obtained as a weighted sum of the differenced received signal phases through differential demodulation operation. Recently, an explicit, approximate, iterative, time-domain ML (ITDML) estimator was derived in [20] and [21]. The result shows that both the magnitude and the phase of each received signal sample are used in the estimation. Along with this result, a new AOPN model is proposed through a geometric approach (referred to hereafter as geometric model), which also involves the high SNR assumption, but can lead to a better estimation performance than Tretter's model in [15].

Further works on simplified ML frequency estimation have been done in [22]−[24]. These three approximate ML estimators are derived by making use of the autocorrelation function of the received signal. The estimator obtained in [22] and [23] is called the planar filtered estimator (PFE), and the estimator in [24] is obtained by interchanging the operations of taking the angle and summation of the PFE. The contribution of this paper is four-fold. First, in analogy to the ITDML estimator of [20] and [21], an explicit, approximate, sample-autocorrelation-function-based frequency estimator is derived which shows that both the magnitude and the angle of the sample autocorrelation function should be used. The estimator obtained attains the Cramer-Rao lower bound (CRLB) at high SNR, and has better performance than the PFE and the ITDML estimator. Second, a new phase unwrapping algorithm is developed to faciliate the efficient, recursive implementation of the estimator proposed. Third, a geometric interpretation of the sample-autocorrelation-function-based estimation is presented. This interpretation engenders a new frequency estimator that uses the autocorrelation function angle increments to avoid phase unwrapping. Fourth, by making use of autocorrelation function expression, an alternative form to the ITDML estimator of [20] and [21] is derived.

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[1]Due to page limit, it is difficult to give an extensive literature survey and discussion on this FFT-based, periodogram-search approach, the readers may wish to refer to the references given in [7]−[14] for more details.

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The paper is organized as follows. The estimation problem is formulated in Section II. The sample-autocorrelation-function-based frequency estimators are derived in Section III. Simulation examples are presented in Section IV, and Section V concludes the paper. Throughout the paper, $\Re[y]$ and $\Im[y]$ denote the real and imaginary parts, respectively, of a complex quantity y, and $[\mathbf{z}]^T$ and $[\mathbf{z}]^{-1}$ denote the transposition and inverse, respectively, of the vector/matrix $\mathbf{z}$, superscript $*$ is the complex conjugate, E is the expectation operator, $\delta(\cdot)$ is the Kronecker delta, $N \sim (\mu, \sigma^2)$ denotes a real-valued, Gaussian random variable with mean μ and variance σ^2, $\angle$ is the angle notation, $\mathbf{1} = [1 \;\; 1 \;\; \cdots ; 1]^T$ and $j = \sqrt{-1}$.

II. Problem Formulation

The purpose is to recover accurately the frequency of a single sinusoid observed in AWGN, where, the complex, discrete-time, received observation signals $\{r(k)\}$ are given by

$$r(k) = Ae^{j(\omega k+\theta)} + n(k), \quad k = 0, 1, 2, \cdots \tag{1}$$

where A is the transmitted single-sinusoid signal amplitude, which is assumed to be known when necessary (e.g., it is known in Tretter's model [15]), and the AWGN $\{n(k)\}$ is a sequence of zero-mean, circularly symmetric, independent, identically distributed, complex Gaussian random variables with covariance function $E[n(k)n^*(k-j)] = \sigma^2\delta(j)$. The unknown frequency ω and the unknown phase θ are modeled as random parameters with uniform probability density function (PDF) $\frac{1}{2\pi}$ over the interval $[-\pi, \pi)$, i.e., the modulo-2π reduced ω and the modulo-2π reduced θ can take any value over $[-\pi, \pi)$. We further assume that ω and θ are statistically independent of each other, and of the AWGN $\{n(k)\}$. The SNR is defined as A^2/σ^2. It has been shown in [Figs. 2 and 3, 21] that the unwrapped received signal phase, $\angle r(k)$, can be represented as the sum of the transmitted signal phase $\omega k + \theta$ and an AOPN component, $\epsilon(k)$, caused by the AWGN $n(k)$, i.e., we have

$$\angle r(k) = \omega k + \theta + \epsilon(k) \tag{2}$$

where $\angle r(k)$ is obtained from the principal argument of the complex phasor $r(k)$ in (1) by phase unwrapping [20, 21].

III. Sample-Autocorrelation-Function-Based Frequency Estimators

Given a block of N received data samples $\{r(k)\}_{k=0}^{N-1}$, the sample autocorrelation function of $r(k)$ is defined as

$$R(l) = \sum_{i=l}^{N-1} r(i)r^*(i-l), \quad l = 1, 2, \cdots, N-1 \tag{3}$$

which, by making use of (1) and (2), can be further expressed in different forms as

$$R(l) = \sum_{i=l}^{N-1} |r(i)||r^*(i-l)|e^{j\left[\angle r(i) - \angle r(i-l)\right]} \tag{4}$$

$$= \sum_{i=l}^{N-1} |r(i)||r^*(i-l)|e^{j\left[\omega l + \epsilon(i) - \epsilon(i-l)\right]} \tag{5}$$

$$= \sum_{i=l}^{N-1} \left[Ae^{j(\omega i+\theta)} + n(i)\right]\left[Ae^{-j\left(\omega(i-l)+\theta\right)} + n^*(i-l)\right]$$

$$= A\sum_{i=l}^{N-1} \left[Ae^{j\omega l} + e^{j(\omega i+\theta)}n^*(i-l) + e^{-j(\omega(i-l)+\theta)}n(i) + \frac{n(i)n^*(i-l)}{A}\right] \tag{6}$$

Several new frequency estimation algorithms can be developed in terms of the sample autocorrelation function $R(l)$. The first algorithm is obtained by differentiating the square of the periodogram $\left|\sum_{l=0}^{N-1} r(l)e^{-j\omega l}\right|$ of ML estimation [5] with respect to ω, and then setting the result to zero[2], i.e., $\sum_{l=0}^{N-1}\sum_{i=0}^{N-1}(l-i)r(l)r^*(i)e^{-j\omega(l-i)} = 0$. By expansion and reassembling, we have, after neglecting the zero terms

$$\Im\left[\sum_{l=1}^{N-1} lR(l)e^{-j\omega l}\right] = 0 \tag{7}$$

Simplifying (7) leads to

$$\sum_{l=1}^{N-1} l\,\Im\left[|R(l)|e^{j\left(\angle R(l) - \omega l\right)}\right] = 0,$$

$$\text{or} \quad \sum_{l=1}^{N-1} l|R(l)|\sin\left[\angle R(l) - \omega l\right] = 0 \tag{8}$$

Due to highly non-linear sine function, (8) does not admit closed-form solution. A simple, approximate solution can be obtained by assuming that the SNR A^2/σ^2 is reasonably high, so that the quantities $\{\angle R(l) - \omega l\}$ in (8) are small for most time. This assumption enables us to use the mathematical approximation: $\sin x \approx x$, for small x, in (8) which then leads to the explicit, approximate, closed-form estimate $\widehat{\omega}$ given by

$$\widehat{\omega} = \sum_{l=1}^{N-1} w(l)\angle R(l) = \sum_{l=1}^{N-1} \frac{l|R(l)|}{\sum_{m=1}^{N-1} m^2|R(m)|}\angle R(l) \tag{9}$$

where $w(l) = \frac{l|R(l)|}{\sum_{m=1}^{N-1} m^2|R(m)|}$ is the weight component. The result (9) states that the frequency estimate $\widehat{\omega}$ is obtained as a linear sum of the weighted angular components of the sample autocorrelation functions $\{R(l)\}$. The weight $w(l)$ depends on the magnitudes $\{|R(l)|\}$. This reveals that the frequency estimator (9) makes use of both angle information and magnitude information of the sample autocorrelation functions. Simulation results presented in Section IV show that the estimator (9) provides an improved performance over the PFE proposed in [22] and [23] where the weight is given by $w(l) = \frac{6l}{N(N+1)(2N+1)}$, which is independent of the magnitudes $\{|R(l)|\}$. For the special case that only the first-order sample autocorrelation function $R(1)$ is utilized for frequency estimation, the estimator (9) and the PFE [22] reduce to

$$\widehat{\omega} = \angle R(1) = \angle\left[\sum_{i=1}^{N-1} r(i)r^*(i-1)\right] \tag{10}$$

which is identical to the linear prediction estimator (LPE) proposed in [16]. Therefore, the LPE [16] can be viewed as a special case of the estimator (9) and the PFE.

[2]Since $|f(x)|^2$ is a monotonically increasing function of $|f(x)|$, maximizing $|f(x)|$ over x is equivalent to finding $\max_x |f(x)|^2$.

Since the unwrapped angle $\angle R(l)$ in (9) is obtained from the principal argument of the complex phasor $R(l)$, it is seen from (5) that phase unwrapping is necessary when the autocorrelation order l is large. To avoid phase unwrapping, a constraint $|\omega J| < \pi$ has been imposed in [22], where the quantity J is a design parameter [22, eq.(13)]. However, this will limit the estimation range for ω. To remove this constraint, the phase unwrapping algorithm proposed in [20] can be modified here to use in the estimator (9), where the frequency estimation and phase unwrapping can be implemented recursively. The algorithm works as follows. We use the first-order implementation of (9), i.e., the LPE (10), to obtain an initial estimate of ω, say, $\widehat{\omega}_1$. We then take $2\widehat{\omega}_1$ as the prediction of the angle of the second-order, transmitted signal autocorrelation, i.e., the phase of $\sum_{i=2}^{N-1}\left[Ae^{j(\omega i+\theta)}\right]\left[Ae^{-j(\omega(i-2)+\theta)}\right]$. The angle of $R(2)$ is then unwrapped within a 2π-interval centered around the predicted value $2\widehat{\omega}_1$, i.e., the value of $\angle R(2)$ chosen is the one lying in the interval $[2\widehat{\omega}_1-\pi,\ 2\widehat{\omega}_1+\pi)$. This can be done by adding multiples of $\pm 2\pi$ to the principal value of $\angle R(2)$ when the absolute difference between $2\widehat{\omega}_1$ and the principal value of $\angle R(2)$ is greater than π. Next, the unwrapped angle $\angle R(2)$ is substituted into (9) to calculate the second estimate of ω, denoted as $\widehat{\omega}_2$, given by $\widehat{\omega}_2 = \widehat{\omega}_1 + w(2)\angle R(2)$. We then make use of $3\widehat{\omega}_2$ to unwrap $\angle R(3)$, and put the result back into (9) to compute the third estimate $\widehat{\omega}_3 = \widehat{\omega}_2 + w(3)\angle R(3)$. The above procedure is repeated until $R(N-1)$ has been used, where the final estimate is given by $\widehat{\omega}_{N-1} = \widehat{\omega}_{N-2} + w(N-1)\angle R(N-1)$. The recursive implementation of the estimator (9) and its phase unwrapping algorithm is shown in Fig. 1.

The second estimation algorithm is obtained by noting that if we neglect the term $\frac{n(i)n^*(i-l)}{A}$ in (6), which is small relative to the dominant noise term $e^{j(\omega i+\theta)}n^*(i-l)+e^{-j\left(\omega(i-l)+\theta\right)}n(i)$ for the SNR of practical interest, and normalize $R(l)$ by dividing through A and $N-l$, the normalized sample autocorrelation function, denoted as $R'(l)$, is obtained as

$$R'(l) = \frac{R(l)/A}{N-l} = Ae^{j\omega l} + n'(l) \tag{11}$$

where the quantity $n'(l)$ represents the normalized dominant noise component, and it is given by

$$n'(l) = \frac{1}{N-l}\sum_{i=l}^{N-1}\left[e^{j(\omega i+\theta)}n^*(i-l)+e^{-j\left(\omega(i-l)+\theta\right)}n(i)\right] \tag{12}$$

which is a zero-mean, complex Gaussian random variable.

Comparing (11) with (1), we see that the normalized autocorrelation function model $R'(l) = Ae^{j\omega l} + n'(l)$ parallels the received signal model $r(k) = Ae^{j(\omega k+\theta)} + n(k)$, namely, both are given by the sum of a single sinusoid and a Gaussian noise. Thus, in comparison with the ITDML estimator [21, eq.(16)] where $\{r(k)\}_{k=0}^{N-1}$ have been used to provide a linear solution to recover ω and θ, the estimator (9) can be interpreted as a linear estimator to recover ω by using $\{R(l)\}_{l=1}^{N-1}$. Furthermore, along with the WPA estimator [16] that uses the differenced phase $\angle r(k) - \angle r(k-1)$ via the differential operation $r(k)r^*(k-1)$ so that phase unwrapping can be avoided, a similar frequency estimator can be developed to use the angle increment $\angle R'(l) - \angle R'(l-1)$ of the sample autocorrelation function via the differential operation $R'(l)R'^*(l-1)$. Specifically, in view of (11), the inputs to the frequency estimator are the increments $\{\Lambda(l) = \angle R'(l) - \angle R'(l-1)\}_{l=2}^{N-1}$, which, in analogy with (2), can be written as

$$\begin{aligned}\Lambda(l) &= \angle R'(l) - \angle R'(l-1) = \omega + \epsilon'(l) - \epsilon'(l-1)\\ &= \angle\Big[R'(l)R'^*(l-1)\Big]\end{aligned} \tag{13}$$

where $\epsilon'(l)$ represents the AOPN component in the normalized autocorrelation angle $\angle R'(l)$, i.e., we have $\angle R'(l) = \omega l + \epsilon'(l)$, which can be treated as a counterpart of the AOPN $\epsilon(k)$ in $\angle r(k)$ of (2). Note that unlike $\epsilon(k)$ which is caused by the AWGN component $n(k)$, the AOPN $\epsilon'(l)$ is not exactly attributed to the Gaussian noise $n'(l)$ since the term $\frac{n(i)n^*(i-l)}{A}$ has been neglected in (12). The geometric interpretation of $\epsilon'(l)$ can be drawn in a similar way to that of $\epsilon(k)$ [21]. Specifically, Fig. 2 gives a geometric phasor representation of $Ae^{j\omega l}$, $n'(l)$ and $R'(l)$ of (11) in the in-phase-and-quadrature (I-Q) coordinate complex plane. The Gaussian noise phasor $n'(l)$ can be decomposed into two orthogonal components, namely, $n'_I(l)$ and $n'_Q(l)$ in the newly formed I'-Q'-coordinate system, with $n'_I(l)$ being parallel to, and $n'_Q(l)$ perpendicular to the phasor $Ae^{j\omega l}$. The components $n'_I(l)$ and $n'_Q(l)$ can be obtained from $\Re[n'(l)]$ and $\Im[n'(l)]$ in the I-Q-coordinate system through a cartesian coordinate rotation, and they have the relationship given by

$$\begin{aligned}\Re[n'(l)] &= n'_I(l)\cos\omega l - n'_Q(l)\sin\omega l\\ \Im[n'(l)] &= n'_I(l)\sin\omega l + n'_Q(l)\cos\omega l\end{aligned} \tag{14}$$

Note that unlike the complex, circularly symmetric, AWGN $n(k)$ in (1), careful examination shows that the in-phase component $\Re[n'(l)]$ and the quadrature phase component $\Im[n'(l)]$ are correlated, and they have different variances whose values depend on the frequency ω to be estimated. This means that $n'(l)$ is no longer circularly symmetric and its statistics are unknown to the estimator even if we know the PDF of $n(k)$. Thus, to obtain explicitly the statistical distribution $p\big(n'_I(l), n'_Q(l)\big)$ is highly involved as the mathematical manipulation of (12) and (14) is very tedious for the order l of practical interest (e.g., the design parameter l can be set at as large as 64 in data-aided frequency estimation [25]). Nevertheless, it follows from the geometric view on $\epsilon'(l)$ in Fig. 2 that, by making use of Tretter's model [15], $\epsilon'(l)$ can be approximated as

$$\epsilon'_1(l) = \frac{n'_Q(l)}{A} \tag{15}$$

and by making use of geometric model proposed in [20] and [21], $\epsilon'(l)$ can be approximated as

$$\epsilon'_2(l) = \frac{n'_Q(l)}{|R'(l)|} \tag{16}$$

where $n'_Q(l) \sim N\big(0,\ \sigma_Q^2(l,\omega)\big)$, and $\sigma_Q^2(l,\omega)$ denotes the variance of $n'_Q(l)$. Accordingly, we have

$$\epsilon'_1(l) \sim N\left(0,\ \frac{\sigma_Q^2(l,\omega)}{A^2}\right)$$

and conditioned on knowing $|R'(l)|$, which is computed by (3) and (11) from the received signal $\{r(k)\}$,

$$\epsilon'_2(l) \sim N\left(0,\ \frac{\sigma_Q^2(l,\omega)}{|R'(l)|^2}\right)$$

The frequency estimation algorithm can then be obtained as

$$\begin{aligned}\hat{\omega} &= \mathbf{w}'\mathbf{\Lambda} = \frac{\mathbf{1}^T\mathbf{C}'^{-1}}{\mathbf{1}^T\mathbf{C}'^{-1}\mathbf{1}}\mathbf{\Lambda} = \sum_{l=2}^{N-1} w'(l)\Lambda(l) \\ &= \sum_{l=2}^{N-1} w'(l)\angle\Big[R(l)R^*(l-1)\Big] \end{aligned} \quad (17)$$

where $\mathbf{\Lambda} = [\Lambda(2)\ \Lambda(3)\ \cdots\ \Lambda(N-1)]$ is an $(N-2)$-dimensional column vector that holds the the autocorrelation function angle increments, $\mathbf{w}' = \frac{\mathbf{1}^T\mathbf{C}'^{-1}}{\mathbf{1}^T\mathbf{C}'^{-1}\mathbf{1}}$ is the estimator window function [16] (also called the smoothing function [23]) and $\mathbf{C}'$ is an $(N-2)$-by-$(N-2)$ covariance matrix of the noise vector $[\epsilon'(2)-\epsilon'(1)\ \ \epsilon'(3)-\epsilon'(2)\ \ \cdots\ \ \epsilon'(N-1)-\epsilon'(N-2)]$. Clearly, unlike the covariance matrix $\mathbf{C}$ in [16, eq.(12)], $\mathbf{C}'$ in (17) does not take on the tridiagonal form any more. Evaluating $\mathbf{C}'$ explicitly would become even more cumbersome as all components in the noise vector are complicatedly correlated. A crude approximate, simplified smoothing function has been obtained in [25] using the results given in [22].

The third algorithm is obtained by making use of (4). Specifically, in view of (4) and (8), we have

$$\sum_{l=1}^{N-1} l\,\Im\left\{\left[\sum_{i=l}^{N-1}|r(i)||r(i-l)|e^{j\left(\angle r(i)-\angle r(i-l)\right)}\right]e^{-j\omega l}\right\} = 0 \quad (18)$$

or equivalently,

$$\sum_{l=1}^{N-1} l\sum_{i=l}^{N-1}|r(i)||r(i-l)|\sin\Big[\angle r(i)-\angle r(i-l)-\omega l\Big] = 0 \quad (19)$$

Using, again, the approximation: $\sin x \approx x$, we have

$$\widehat{\omega} = \frac{\sum\limits_{l=1}^{N-1} l\sum\limits_{i=l}^{N-1}|r(i)||r(i-l)|\Big[\angle r(i)-\angle r(i-l)\Big]}{\sum\limits_{l=1}^{N-1} l^2\sum\limits_{i=l}^{N-1}|r(i)||r(i-l)|} \quad (20)$$

The simulation results presented in Scetion IV show that the estimator (20) has the same performance as the ITDML estimator proposed in [20, 21]. In this sense, the estimator (20) presents an alternative form to the ITDML estimator.

IV. Simulation Examples

This section presents simulation examples to compare the estimation performance of the sample-autocorrelation-function-based frequency estimator (9), the PFE [22, eq.(12)] and the ITDML estimator [21, eq.(16)]. The simulation assumptions are as follows. To ensure the estimation accuracy, the number of simulation run is set to 10^5. The actual values of ω and θ are set to 0.1 and 0.15π, respectively. The performance is measured by the inverse estimation variance achievable versus the SNR with the specified number of the data samples $N = 10$ as the parameter. The results are presented in Fig. 3. As a basis for comparison, the inverse CRLB (ICRLB) is also plotted. The CRLB for the frequency ω is given by [5]

$$\mathrm{CRLB}_\omega = \frac{\sigma^2}{A^2}\frac{6}{N(N+1)(N-1)}, \quad (21)$$

As a figure of merit by which we can compare the performance, we define the estimation threshold SNR as the value of SNR at which its inverse variance curve dips by 1 dB from the ICRLB curve, as is common in the literature. Several conclusions can be drawn from Fig. 3. First, it is verified and confirmed that the estimator (20) and the ITDML estimator [21, eq.(16)] are identical. Second, it is seen that the PFE [22, eq.(12)], or equivalently, the sample-autocorrelation-function-based estimator (9) with $w(l) = \frac{6l}{N(N-1)(2N-1)}$ in which the weight is a constant independent of the autocorrelation magnitude $|R(l)|$, cannot achieve the CRLB, even at high SNR region. In fact, although the PFE [22, eq.(12)] outperforms the LPE [16, eq.(19)], they have the same property, namely, no noticeable estimation threshold SNR can be seen. Third, the sample-autocorrelation-function-based frequency estimator (9) with $w(l) = \frac{l|R(l)|}{\sum_{m=1}^{N-1} m^2|R(m)|}$ has better overall performance than the the ITDML estimator [21, eq.(16)], especially at low SNR. For high SNR, both estimators can attain the CRLB. This further illustrates that the on-line magnitude information $|r(k)|$ and $|R(l)|$ is important in achieving the CRLB or getting better estimation performance. As discussed in section III, in comparison with the ITDML estimator [21, eq.(16)] which is phase-based, time-domain *linear* estimator of the frequency ω using the observation data model $r(k) = Ae^{j(\omega k+\theta)} + n(k)$ with $\angle r(k) = \omega k + \theta + \epsilon(k)$, the sample-autocorrelation-function-based frequency estimator (9) can be interpreted as phase-based, time-domain *linear* estimator of the frequency ω using the observation data model $R'(l) = Ae^{j\omega l} + n'(l)$ with $\angle R'(l) = \omega l + \epsilon'(l)$. Clearly, their estimation performance will depend on the variances of the AOPN's $\epsilon(k)$ and $\epsilon'(l)$, which in turn depend on the variances of $n(k)$ and $n'(l)$, respectively. Although it is difficult to evaluate the PDF of $n'(l)$ explicitly, an inspection in [22] has shown that $n'(l)$ has a lower variance than $n(k)$, which means that the variance of $\epsilon'(l)$, $E\left[\epsilon'^2(l)\right]$, is smaller than the variance of $\epsilon(k)$, $E\left[\epsilon^2(k)\right]$. Recall that in deriving [21, eq.(16)], the ITDML estimator uses the same type of approximation: $\sin x \approx x$, as that in deriving (9). The difference lies in that in deriving [21, eq.(16)], the actual approximation is given by $\sin[\angle r(k) - (\omega k+\theta)] \approx \angle r(k) - (\omega k+\theta)$ [21, eq.(10)], whereas, in deriving (9), the actual approximation is given by $\sin[\angle R(l) - \omega l] \approx \angle R(l) - \omega l$. It follows from the above observation data model that we have $\angle r(k) - (\omega k+\theta) = \epsilon(k)$ and $\angle R(l) - \omega l = \epsilon'(l)$. This indicates that in order for the approximation $\sin x \approx x$ to be valid in [21, eq.(10)] and (9) simultaneously, the ITDML estimator requires about $10\log_{10}\left(\frac{E\left[\epsilon^2(k)\right]}{E\left[\epsilon'^2(l)\right]}\right)$ dB more SNR than the estimator (9). In other words, it is expected that the ITDML estimator [21, eq.(16)] is approximately $10\log_{10}\left(\frac{E\left[\epsilon^2(k)\right]}{E\left[\epsilon'^2(l)\right]}\right)$ dB poorer in threshold SNR performance than the estimator (9). Of course, this is under the condition that perfect phase unwrapping has been achieved for both estimators.

V. Conclusion

An explicit, sample-autocorrelation-function-based, approximate ML frequency estimator that does not require numerical search and has better performance than the PFE and the ITDML estimator is derived. A new phase unwrapping algorithm is presented to faciliate its recursive implementation. A geometric interpretation on the autocorrelation function is introduced. A further simplified estimator that utilizes autocorrelation function angle increments to avoid phase unwrapping is proposed.

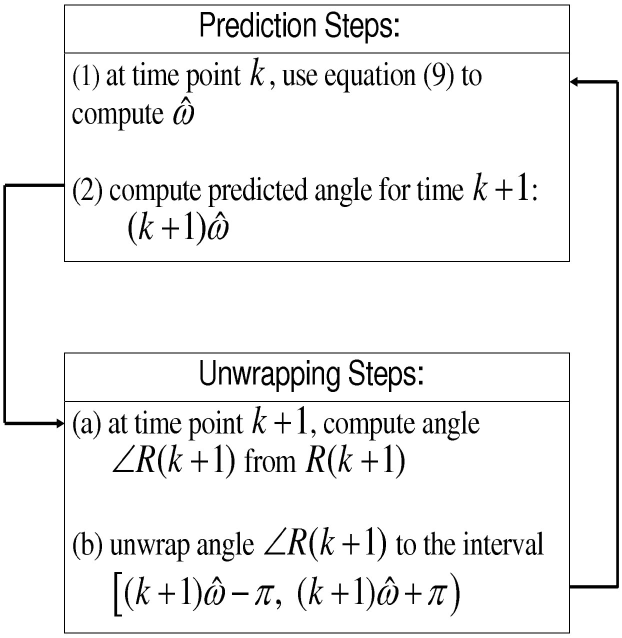

Fig. 1. Recursive implementation of phase unwrapping.

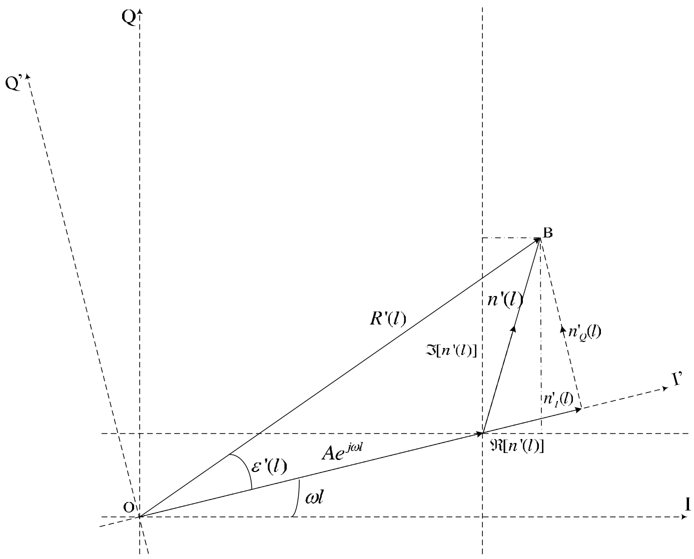

Fig. 2. Geometric representation of $R'(l) = Ae^{j\omega l} + n'(l)$.

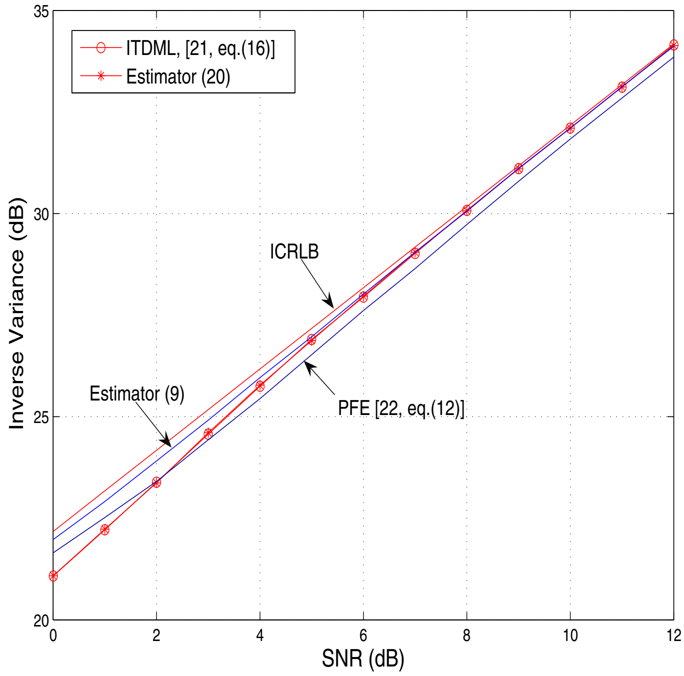

Fig. 3. Performance comparison between the estimator (9), the ITDML estimator [21, eq.(16)] and the PFE [22, eq.(12)] with $N = 10$.

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