A New Upper Bound for the Normalized Detection Threshold of the FFT-Based Summation Detector

Sichun Wang
Defence R&D Canada-Ottawa
sichun.wang@drdc-rddc.gc.ca

François Patenaude
Communications Research Centre Canada
francois.patenaude@crc.gc.ca

Robert Inkol
Defence R&D Canada-Ottawa
robert.inkol@drdc-rddc.gc.ca

***Abstract*—The FFT summation detector, a detection scheme extensively used in applications such as civilian spectrum monitoring and military radio surveillance, is also applicable to cognitive radio spectrum sensing applications. The practical implementation of the FFT summation detector depends on the reliable numerical computation of the normalized detection threshold, T_n. Although this problem has been the subject of considerable study, the computation of T_n still presents difficulties. Existing approaches depend on numerical procedures such as the Newton-Raphson and the golden section search algorithms, which often fail to converge when the number of input data blocks, L, or the number of FFT bins, N, used for channel power estimation is large. To circumvent such numerical difficulties, easily computable lower and upper bounds for T_n are of great interest as they are essential for verifying the accuracy of approximations to T_n when the true value of T_n is not available. Although several lower bounds for T_n have been proposed, few good upper bounds for T_n have been derived. This paper proposes a new upper bound for T_n, which is easily computed for larger values of the probability of false alarm, P_{fa}, using the Imhof integral formula.**

***Index Terms*—FFT filter bank, detection and estimation, spectrum monitoring, spectrum sensing, cognitive radios, constant false alarm rate, normalized detection threshold, Newton-Raphson algorithm, golden section search algorithm**

I. The Normalized Detection Threshold T_n for the FFT summation detector

The FFT summation detector depicted in Fig. 1 has been widely used in many applications such as sonar, spectrum monitoring, radio surveillance and instrumentation [1] - [3]. In recent years, the FFT summation detector and its variations have been considered for cognitive radio spectrum sensing applications [4]-[8].

A detailed formulation of the FFT summation detector is given in [1], [9], [10], whose assumptions and definitions, including the additive white Gaussian channel noise model, will be adopted here. Assume that the variance of the additive zero-mean circular complex white Gaussian channel noise, denoted by σ^2 here, is known. Let L be the number of input data blocks processed by the FFT filter bank (with input data blocks possibly overlapped with overlap ratio γ, $0 \leq \gamma \leq 1/2$), N the number of FFT bins used for channel power estimation and $T > 0$ the detection threshold [9]. The probability of false alarm, P_{fa}, of the FFT summation detector is then computed by (c.f. Eqns. (8), (9) of [10])

$$P_{fa} = \Pr\{Q \geq T_n\} \tag{1}$$

where $T_n = \frac{T}{\sigma^2}$ is called the normalized detection threshold [1] and

$$Q = \begin{cases} \sum_{l=1}^{LN} \frac{\lambda_l}{2}\chi_2^2(l) & 0 < \gamma \leq 1/2 \\ \sum_{l=1}^{N} \frac{\mu_l}{2}\chi_{2L}^2(l) & \gamma = 0 \end{cases} \tag{2}$$

with the sequences, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{LN} > 0$, and, $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_N > 0$, being, respectively, the eigenvalue sequences of the Hermitian matrices $\mathbf{H}$ and $\mathbf{A}$ defined by equations (6) and (7) in [9]. Here $\chi_2^2(l)$, $1 \leq l \leq LN$, are independent central χ^2 random variables each of 2 degrees of freedom and $\chi_{2L}^2(l)$, $1 \leq l \leq N$, are independent central χ^2 random variables each of $2L$ degrees of freedom.

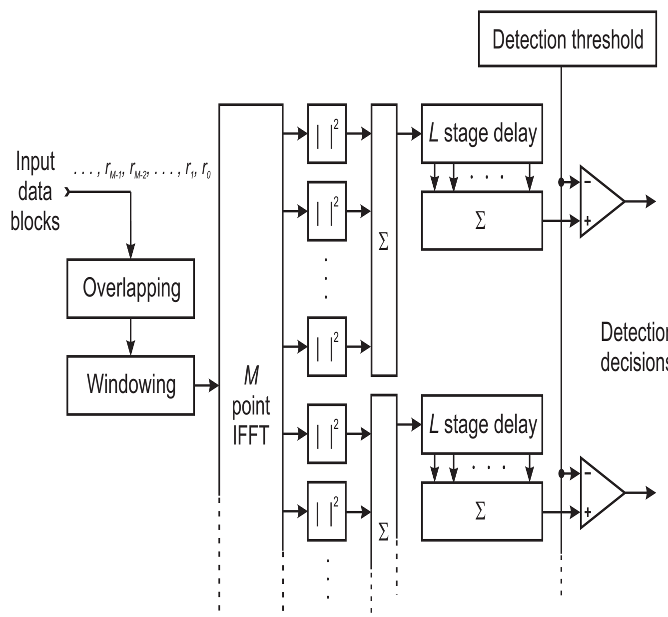

Fig. 1. L-block FFT summation detector.

[1]Note that in the literature T_n is sometimes further normalized by, LN, yielding, $T/(\sigma^2 LN)$, which is also called the normalized detection threshold. As long as it is clear from the context which definition of the normalized detection threshold is used, there should be no danger of confusion.

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It can be shown [9] that

$$P_{fa}=\sum_{m=1}^{LN}\frac{\lambda_m^{LN-1}}{\prod\limits_{1\le l\le LN, l\ne m}(\lambda_m-\lambda_l)}\; e^{-\frac{T_n}{\lambda_m}},\quad 0<\gamma\le 1/2 \quad (3)$$

and

$$P_{fa}=\sum_{m=1}^{N}\sum_{k=1}^{L}A_{mk}\sum_{t=0}^{k-1}\frac{\left(\frac{T_n}{\mu_m}\right)^t}{t!}\; e^{-\frac{T_n}{\mu_m}},\quad \gamma=0 \qquad (4)$$

where A_{mk} in (4) are defined by Eqn. (14) in [9]. Thus, for a given P_{fa}, the normalized detection threshold, $T_n = T/\sigma^2$, can be computed by solving the nonlinear equation (3) or (4), using iterative procedures such as the Newton-Raphson or the golden section search algorithm [11]. However, these approaches often fail due to numerical difficulties, which typically result from the presence of very small values in the sequences $\{\lambda_l\}_{l=1}^{LN}$ and $\{\mu_k\}_{k=1}^{N}$ when L or N is large.

To circumvent numerical difficulties, one useful technique is to truncate the small values in the sequences $\{\lambda_l\}_{l=1}^{LN}$ and $\{\mu_k\}_{k=1}^{N}$ and compute lower bounds for T_n [12]. Extensive numerical tests have shown that this method yields tight lower bounds for typical values of L and N [12]. However, the truncation method does not fully work for arbitrarily large values of L and N. Thus, robust procedures for the computation of T_n continue to be of interest.

In this paper, a new sharp upper bound for T_n is proposed. Using the proposed upper bound and the lower bounds developed in [12], it is then verified that the Pearson approximation to T_n proposed in [13] is very close to T_n even though the true value of T_n may not be available. Since the Pearson approximation to T_n is very easy to compute, it can be used in practical implementations of the FFT summation detector when L or N is very large.

II. A New Upper Bound for T_n

In this section, we derive the proposed upper bound for T_n. First, we start with some definitions and notation.

Definition 1. Let the window used in the FFT filter bank shown in Fig.1 be defined by $\mathbf{W} = [w_0, \cdots, w_{M-1}]^t$, where t denotes vector transposition and M is the length of the FFT transform (i.e., window length). The window sequence $\mathbf{W}$ is called normalized if $\sum_{m=0}^{M-1} w_m^2 = 1$.

Definition 2. Assume that the window $\mathbf{W}$ used in the FFT filter bank shown in Fig.1 is normalized. For any positive number ϵ with $0 < \epsilon < 1$, the random variable Q_ϵ is defined as follows. If $0 < \gamma \le 1/2$, let the index l_ϵ be defined by the requirement that $\lambda_{l_\epsilon} \ge \epsilon$ and $\lambda_{(l_\epsilon+1)} < \epsilon$. Then Q_ϵ is defined by setting

$$Q_\epsilon=\sum_{l=1}^{l_\epsilon}\frac{\lambda_l}{2}\chi_2^2(l)+\sum_{l=1+l_\epsilon}^{LN}\frac{\lambda_{l_\epsilon}}{2}\chi_2^2(l) \qquad (5)$$

i.e., Q_ϵ is obtained by replacing the last $(LN-l_\epsilon)$ eigenvalues in $\{\lambda_l\}_{l=1}^{LN}$ by λ_{l_ϵ} in the summation defining Q (c.f. the first summation in (2)). If $\gamma = 0$, let the index l_ϵ be defined by the requirement that $\mu_{l_\epsilon} \ge \epsilon$ and $\mu_{(l_\epsilon+1)} < \epsilon$. Q_ϵ is then defined by setting

$$Q_\epsilon=\sum_{l=1}^{l_\epsilon}\frac{\mu_l}{2}\chi_{2L}^2(l)+\sum_{l=1+l_\epsilon}^{N}\frac{\mu_{l_\epsilon}}{2}\chi_{2L}^2(l) \qquad (6)$$

i.e., Q_ϵ is obtained by replacing the last $(N-l_\epsilon)$ eigenvalues in $\{\mu_l\}_{l=1}^{N}$ by μ_{l_ϵ} (c.f. the second summation in (2)).

Definition 3. Let $\epsilon \in (0,1)$ be fixed. For any positive number $z > 0$, the probability that the random variable Q_ϵ exceeds the threshold z is denoted by $G_\epsilon(z)$, i.e.,

$$G_\epsilon(z)=\Pr\{Q_\epsilon\ge z\} \qquad (7)$$

Since the function G_ϵ is a strictly decreasing function of z on the interval $(0,\infty)$ and assumes values in the open unit interval $(0,1)$, it has an inverse function, which shall be denoted by H_ϵ, i.e., $H_\epsilon = (G_\epsilon)^{-1}$.

Theorem 1. Let $\epsilon \in (0,1)$ be fixed. For any given probability of false alarm $P_{fa} \in (0,1)$, the value of the function H_ϵ computed at P_{fa}, denoted by $T(\epsilon, P_{fa}) = H_\epsilon(P_{fa})$, is an upper bound for T_n.

The proof of **Theorem 1** is based on the following proposition:

Lemma 1. (Lemma 1, [14]) Suppose X_k, Y_k, $1 \le k \le m$, are mutually independent continuous random variables with distribution functions F_{X_k}, F_{Y_k}, $1 \le k \le m$, and

$$F_{X_k}(z)\le F_{Y_k}(z),\quad \forall z \qquad (8)$$

then

$$F_{S_X}(z)\le F_{S_Y}(z),\quad \forall z \qquad (9)$$

where F_{S_X} and F_{S_Y} are, respectively, the distribution functions of the random variables S_X and S_Y defined by $S_X = \sum_{k=1}^m X_k$ and $S_Y = \sum_{k=1}^m Y_k$.

We shall only consider the case $0 < \gamma \le 1/2$, for the proof for the case $\gamma = 0$ is very similar.

For any positive real number $z > 0$, since $\lambda_l \le \lambda_{l_\epsilon}$ for $l \ge l_\epsilon$, we have

$$\Pr\left\{\sum_{l=1+l_\epsilon}^{LN}\frac{\lambda_{l_\epsilon}}{2}\chi_2^2(l)\le z\right\}\le \Pr\left\{\sum_{l=1+l_\epsilon}^{LN}\frac{\lambda_l}{2}\chi_2^2(l)\le z\right\} \quad (10)$$

From **Lemma 1** it then follows that

$$\begin{aligned}\Pr\{Q_\epsilon\le z\}&=\Pr\left\{\sum_{l=1}^{l_\epsilon}\frac{\lambda_l}{2}\chi_2^2(l)+\sum_{l=1+l_\epsilon}^{LN}\frac{\lambda_{l_\epsilon}}{2}\chi_2^2(l)\le z\right\}\\ &\le \Pr\left\{\sum_{l=1}^{l_\epsilon}\frac{\lambda_l}{2}\chi_2^2(l)+\sum_{l=1+l_\epsilon}^{LN}\frac{\lambda_l}{2}\chi_2^2(l)\le z\right\}\\ &=\Pr\{Q\le z\} \qquad (11)\end{aligned}$$

and therefore

$$\begin{aligned}G_\epsilon(z)&=\Pr\{Q_\epsilon\ge z\}=1-\Pr\{Q_\epsilon\le z\}\\ &\ge 1-\Pr\{Q\le z\}=\Pr\{Q\ge z\} \qquad (12)\end{aligned}$$

Thus for any given P_{fa}, we have

$$P_{fa} = \Pr\{Q \geq T_n\} \leq G_\epsilon(T_n) \tag{13}$$

Since $G_\epsilon(T(\epsilon, P_{fa})) = P_{fa}$, we obtain the inequality:

$$G_\epsilon(T(\epsilon, P_{fa})) \leq G_\epsilon(T_n) \tag{14}$$

Since the function G_ϵ is strictly decreasing, this implies that

$$T(\epsilon, P_{fa}) \geq T_n \tag{15}$$

i.e., $T(\epsilon, P_{fa})$ is an upper bound for the normalized detection threshold T_n.

III. Numerical Computation of the Upper Bound of T_n Using the Imhof Formula

The Imhof formula is an integral formula for computing the cumulative distribution function of a weighted sum of χ^2 random variables [15]. Let S be the random variable defined by:

$$S = \sum_{r=1}^{m} \beta_r \chi^2_{h_r} \tag{16}$$

where $\chi^2_{h_r}$, $1 \leq r \leq m$, are independent central χ^2 random variables with h_r degrees of freedom and $\beta_r \neq 0$, $1 \leq r \leq m$, are a real-valued weight sequence. Then, for any real number x, it is known that (see Eq. (3.2) of [15])

$$P\{S > x\} = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\sin\theta(u)}{u\rho(u)}\, du \tag{17}$$

where

$$\begin{cases} \theta(u) &= \dfrac{1}{2} \displaystyle\sum_{r=1}^{m} h_r \tan^{-1}(\beta_r u) - \dfrac{1}{2} x u \\ \rho(u) &= \displaystyle\prod_{r=1}^{m} (1 + \beta_r^2 u^2)^{\frac{h_r}{4}} \end{cases} \tag{18}$$

The identity (17), which is a powerful tool for computing the distribution functions of sums of weighted central χ^2 random variables, is called the Imhof formula in the literature. It can be used to compute the normalized detection threshold T_n.

In this paper, the upper bound $T(\epsilon, P_{fa})$ will be computed by using the Imhof formula (17) to compute the values of G_ϵ. For brevity, $T(\epsilon, P_{fa})$ shall be simply called the Imhof upper bound for T_n.

IV. Lower bounds of T_n Computed by truncating small eigenvalues

As previously noted, tight lower bounds for T_n can be obtained by truncating small eigenvalues [12]. For the convenience of readers, the mathematical definitions are reproduced here.

Definition 4. Let $0 < \gamma \leq 1/2$ and let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{LN} > 0$ be the LN eigenvalues of the Hermitian matrix $\mathbf{H}$ defined by Eqn. (6) in [9]. Let $0 < \epsilon < 1$ and let the index l_ϵ be defined by the requirement that $\lambda_{l_\epsilon} \geq \epsilon$ and $\lambda_{(l_\epsilon+1)} < \epsilon$. The solution of the following equation in $z > 0$ is denoted by $T_o(\epsilon, P_{fa})$:

$$P_{fa} = \Pr\left\{\sum_{l=1}^{l_\epsilon} \frac{\lambda_l}{2} \chi^2_2(l) \geq z\right\} \tag{19}$$

where $\chi^2_2(l)$, $1 \leq l \leq l_\epsilon$, are independent central χ^2 random variables, each of 2 degrees of freedom.

Definition 5 Let $\gamma = 0$ and let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_N > 0$ be the N eigenvalues of the Hermitian matrix $\mathbf{A}$ defined by Eqn. (7) in [9]. Let $0 < \epsilon < 1$ and let the index l_ϵ be defined by the requirement that $\mu_{l_\epsilon} \geq \epsilon$ and $\mu_{(l_\epsilon+1)} < \epsilon$. The solution of the following equation in $z > 0$ is denoted by $T_{no}(\epsilon, P_{fa})$:

$$P_{fa} = \Pr\left\{\sum_{l=1}^{l_\epsilon} \frac{\mu_l}{2} \chi^2_{2L}(l) \geq z\right\} \tag{20}$$

where $\chi^2_{2L}(l)$, $1 \leq l \leq l_\epsilon$, are independent central χ^2 random variables, each of $2L$ degrees of freedom.

It is known [9], [12] that $T_o(\tau, P_{fa})$ is a lower bound for T_n for overlapped input data ($0 < \gamma \leq 1/2$) and $T_{no}(\tau, P_{fa})$ is a lower bound for T_n for nonoverlapped input data ($\gamma = 0$). These two lower bounds for T_n, since they are both computed in this paper using the Imhof formula (17), shall be called the Imhof lower bounds for T_n.

V. Pearson Approximation to T_n

The Pearson approximation provides a simple approximation to T_n for both overlapped and nonoverlapped input data [13]. Since, using known methods, T_n cannot be computed accurately and reliably for many choices of relatively large values of L and N, a performance comparison of the Pearson approximation with the Imhof lower and upper bounds discussed in the previous sections is of practical interest. Due to space constraints, the precise formulation of the Pearson approximation to T_n is omitted here. The reader can consult [13] for more details.

VI. Numerical Results

Typical numerical results are plotted in Figs. 2-5 for several combinations of L and N. The results for overlapped ($\gamma = 1/2$) and nonoverlapped ($\gamma = 0$) data blocks are denoted by 'o' and 'no', respectively. Note that these results were obtained using a normalized window and an eigenvalue sequence cut-off threshold, ϵ, set to 0.01. The Imhof lower and upper bounds are very close to each other and the Pearson approximation. Since L and N are relatively small in Fig. 2, the true value of T_n can be computed for both overlapped and nonoverlapped input data, in addition to the Imhof lower and upper bounds and the Pearson approximation. Here, the results obtained for the upper and lower bounds are consistent with the true values. In Figs. 3-5, the true value of T_n was not available for either overlapped or nonoverlapped input data due to numerical problems. Nevertheless, the computed lower and upper bounds confirm that the Pearson approximation is very close to the true value of T_n.

Finally, it is important to note that the computation of the Imhof upper bound can present difficulties; the computation of the improper integral in (17) challenges the Matlab numerical integration routine for very small values of P_{fa}. Consequently, further research is needed to address this issue. Nevertheless, the results presented here provide further confirmation of the viability of the Pearson approximation for T_n.

VII. Conclusions

A new upper bound for the normalized detection threshold, T_n, for the FFT summation detector has been developed. Comparisons with lower bounds obtained by small eigenvalue truncation and the Pearson approximation confirm that: (1) this new upper bound is close to the true value of T_n; (2) the Pearson approximation can be used in practical implementations of the FFT summation detector when the true value of T_n cannot be reliably computed.

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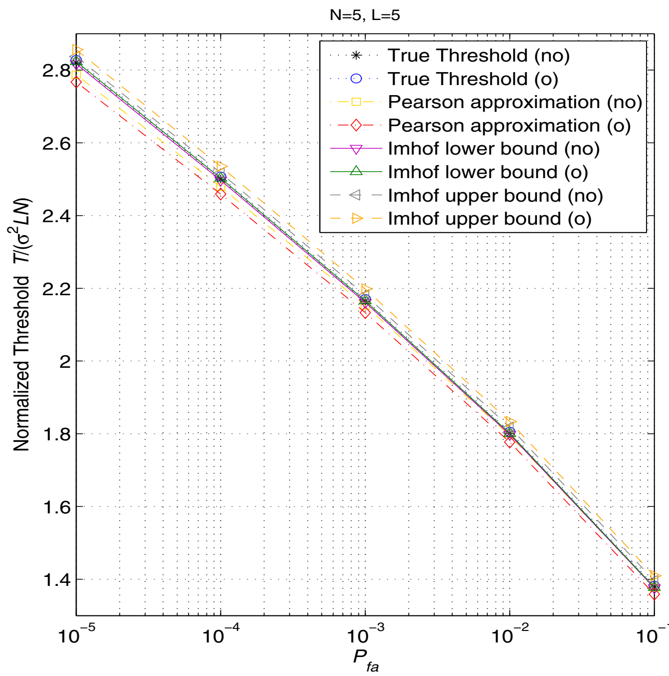

Fig. 2. Normalized Blackman window, $N = 5, L = 5, \epsilon = 0.01$.

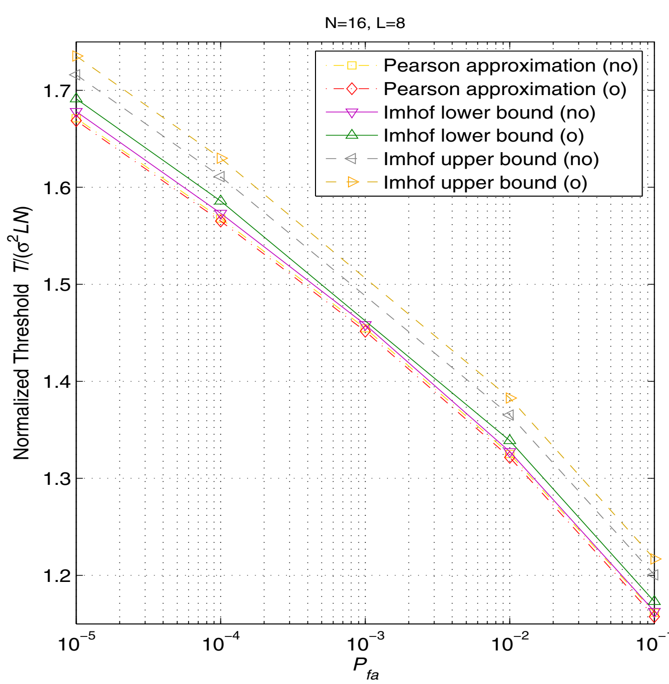

Fig. 3. Normalized Blackman window, $N = 16, L = 8, \epsilon = 0.01$.

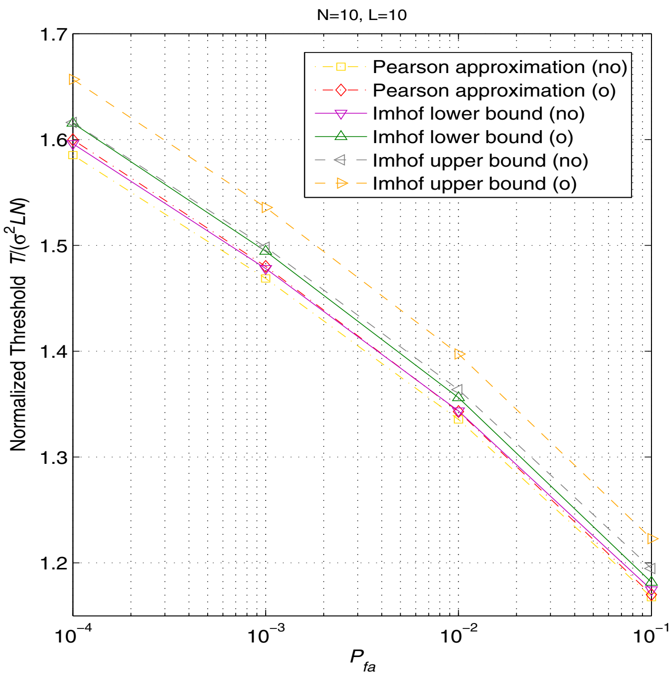

Fig. 4. Normalized Hann window, $N = 10, L = 10$, $\epsilon = 0.01$.

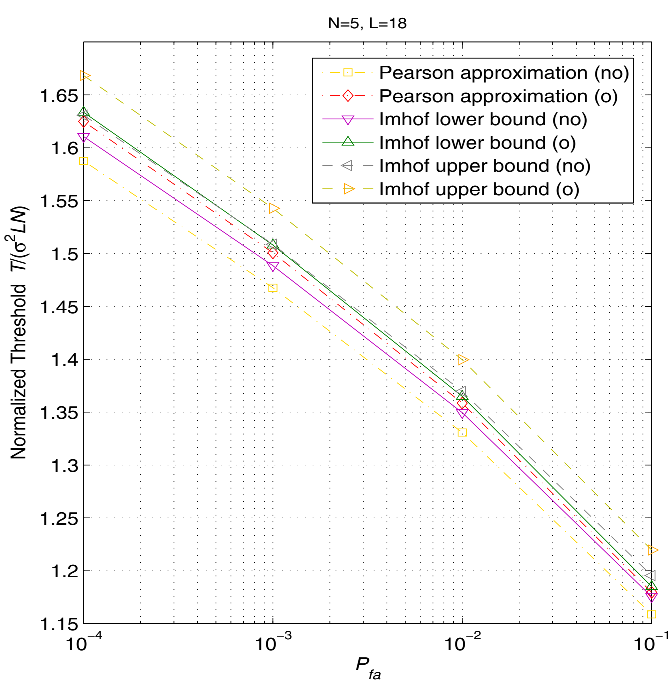

Fig. 5. Normalized Hann window, $N = 5, L = 18$, $\epsilon = 0.01$.