Sum Rate of p-Sphere Encoding for MIMO Broadcast Channels with Reduced Peak Power

Mahmood Mazrouei-Sebdani, Witold A. Krzymień

University of Alberta / TRLabs, Edmonton, AB, Canada, T6G 2V4
Email: {mazrouei,wak}@ece.ualberta.ca

***Abstract*—Vector perturbation is a sum-capacity approaching non-linear precoding technique. Vector perturbation uses sphere encoding to perturb data such that the unscaled sum power is minimized. Unfortunately, this minimization does not guarantee that the peak to average power ratio (PAPR) remains in the acceptable range for the transmitter. A p-sphere encoder was proposed in literature to reduce the PAPR by using p-norm minimization instead of the 2-norm one used typically in vector perturbation and its performance was analyzed in terms of bit error rate (BER). In this paper we focus on the spectral efficiency of the p-sphere encoding and investigate its sum rate for multiple-input multiple-output broadcast channel (MIMO-BC) with multiple-antenna users and uniformly distributed input. The results show that for $p = 5$, the PAPR is reduced by 25%, while the sum rate is just slightly less than that of vector perturbation employing 2-norm sphere encoding (only 1% reduced sum rate).**

***Index Terms*—Vector perturbation, multiple-input multiple-output broadcast channel (MIMO-BC), p-sphere encoding, peak to average power ratio (PAPR) reduction.**

I. Introduction

Multiple-input multiple-output (MIMO) antenna techniques enhance the spectral efficiency of not only point-point [1], but also multi-user wireless communications [2]. This characteristic results from the spatial multiplexing capability of MIMO techniques. To enable non-collocated users to separate the spatial streams transmitted on the downlink and detect their own signal, some suitable precoding needs to be done at the transmitter. The well-known dirty paper coding (DPC) technique [3]–[5] represents capacity-achieving optimal precoding, but unfortunately its implementation is extremely complex and requires non-causal knowledge of the transmitted signals.

With lattice encoding employed in vector perturbation, the system sum rate can approach approximately that achievable by DPC [6]. Vector perturbation adds a Gaussian (complex) integer [7] offset to data in order to minimize the unscaled average sum power. However, this minimization not necessarily leads to the reduction of the peak to average power ratio (PAPR).

A technique of p-sphere encoding was suggested in [8] to cope with the high PAPR of vector perturbation for MIMO broadcast channel (MIMO-BC) with single-antenna users. It is known that vector perturbation minimizes 2-norm of the unscaled transmitted signal, which translates into minimization of the average power. [8] suggested that the p-norm minimization be used instead of the 2-norm one in order to overcome the undesired effects of the high PAPR. $p = \infty$ minimizes the maximum magnitude of the elements of the unscaled transmitted signal, which implies PAPR reduction. [8] considered bit error rate (BER) as the performance metric for analysis and demonstrated that as p increased, the BER increased, but lower PAPR was achieved.

In this paper, we investigate the sum rate of the p-sphere encoding for MIMO-BC with multiple-antenna users. By deriving mutual information for the case of uniformly distributed input, [9] obtained the sum rate for MIMO-BC with vector perturbation and single-antenna users. [10] extended the approach of [9] to multi-cell MIMO with multiple-antenna users. We apply the approach of [9] to the p-sphere encoder and obtain the sum rate. We also compare the complexity, PAPR and the average power of the p-sphere-encoded signal for different values of p. Results show that $p = 5$ can be a practical choice, which provides acceptable sum rate, PAPR and complexity. The study presented in this paper is applicable to highly spectrally efficient systems with limited transmitter dynamic range, in which rate allocation and user scheduling additionally can be performed in order to maximize the sum rate.

The notation used in this paper is as follows. We use $(.)^T$ for matrix transposition, $(.)^*$ for matrix conjugate transposition and $(.)^+$ for Moore-Penrose pseudoinverse. The notation $\mathrm{diag}(x_1, x_2, ..., x_K)$ represents a diagonal matrix with diagonal elements $x_1, x_2, ..., x_K$. We use $\det(\mathbf{A})$ to represent the determinant of matrix $\mathbf{A}$, $\mathbb{Z}[j]$ to represent a set of Gaussian integers, $\lfloor . \rfloor$ to represent the floor function and $\lfloor . \rceil$ to represent rounding to the nearest integer, which are applied separately to the real part and imaginary part of a complex number. $\|\mathbf{x}\|_p$ stands for the p-norm of vector $\mathbf{x}$.

II. System Model

We consider a MIMO broadcast channel (MIMO-BC) with N_t transmit antennas and K users with N_r receive antennas each. Downlink channel matrix from the transmitter to user k is represented by $\mathbf{H}_k \in \mathcal{C}^{N_r \times N_t}$, whose elements are i.i.d zero-mean Gaussian random variables with unit variance. We assume that the front-end linear precoder for user k is defined by $\mathbf{T}_k \in \mathcal{C}^{N_t \times N_r}$. We discuss how to obtain this precoder later in this section. Let $\mathbf{x}_k \in \mathcal{C}^{N_r \times 1}$ denote the transmitted signal vector for user k. With the foregoing assumptions, the received signal vector $\mathbf{y}_k \in \mathcal{C}^{N_r \times 1}$ at user k can be expressed

978-1-4673-0990-5/12/$31.00 ©2012 IEEE

as

$$\mathbf{y}_k = \frac{1}{\sqrt{\gamma_p}} \mathbf{H}_k \sum_{i=1}^{K} \mathbf{T}_i \mathbf{x}_i + \mathbf{n}_k, \tag{1}$$

where $\mathbf{n}_k \in \mathcal{C}^{N_r \times 1}$ is the zero-mean white Gaussian noise vector at user k with $\mathrm{E}\{\mathbf{n}_k \mathbf{n}_k^*\} = \mathbf{I}_{N_r}$. We assume the average sum power constraint and denote the maximum average power at the transmitter by P_t. Therefore, we can write the power constraint as

$$\mathrm{E}\| \sum_{i=1}^{K} \mathbf{T}_i \mathbf{x}_i \|_2^2 \leq P_t, \tag{2}$$

where the above average is taken over the distribution of the input data. γ_p in (1) is the power scaling factor for the p-sphere encoding, which is given by

$$\gamma_p = \frac{\mathrm{E}\| \sum_{i=1}^{K} \mathbf{T}_i \mathbf{x}_i \|_2^2}{P_t}. \tag{3}$$

There are several different criteria for the design of the front-end precoder in vector perturbation such as the minimum mean square error (MMSE), zero forcing (ZF) and BER minimization (see [11], [12] for detailed discussion). In this paper we consider ZF criterion and since users have multiple antennas we use block diagonalization (BD) as the front-end linear precoding [10]. Perturbation is applied first to the transmitted data and then the perturbed data is precoded by BD to remove inter-user interference. Let $\tilde{\mathbf{H}}_k \in \mathcal{C}^{(K-1)N_r \times N_t}$ denote the aggregate interference channel for user k, which is defined as

$$\tilde{\mathbf{H}}_k = [\mathbf{H}_1^T \cdots \mathbf{H}_{k-1}^T \mathbf{H}_{k+1}^T \cdots \mathbf{H}_K^T]^T. \tag{4}$$

To remove other-user interference for user k, its precoder $\mathbf{T}_k$ has to lie in the null space of $\tilde{\mathbf{H}}_k$. Singular value decomposition can provide the basis of this null space. Let us write the singular value decomposition (SVD) of $\tilde{\mathbf{H}}_k$ as

$$\tilde{\mathbf{H}}_k = \tilde{\mathbf{U}}_k \tilde{\mathbf{\Lambda}}_k [\tilde{\mathbf{V}}_{k,1} \tilde{\mathbf{V}}_{k,0}]^*, \tag{5}$$

where the $N_t - \mathrm{rank}(\tilde{\mathbf{H}}_k)$ columns of $\tilde{\mathbf{V}}_{k,0}$ form the orthogonal basis of $\tilde{\mathbf{H}}_k$. It is sufficient to choose N_r columns to construct the precoder $\mathbf{T}_k$. With BD precoding the received signal becomes

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{T}_k \mathbf{x}_k + \mathbf{n}_k. \tag{6}$$

Next step is to parallelize the effective channel $\mathbf{H}_{k,e} = \mathbf{H}_k \mathbf{T}_k$ of user k in order to separate its streams. This can be achieved using SVD of $\mathbf{H}_{k,e}$. The SVD of $\mathbf{H}_{k,e}$ is given by

$$\mathbf{H}_{k,e} = \mathbf{U}_k \mathbf{\Lambda}_k \mathbf{V}_k^*, \tag{7}$$

where $\mathbf{U}_k$ and $\mathbf{V}_k$ are the $N_r \times N_r$ left unitary and $N_r \times N_r$ right unitary matrices, respectively. $\mathbf{\Lambda}_k = \mathrm{diag}(\lambda_{k,1}, \cdots, \lambda_{k,N_r})$ is the $N_r \times N_r$ matrix of singular values. We use $\mathbf{V}_k$ at the transmitter and $\mathbf{U}_k$ at the receiver. Consequently we have N_r parallel channels with the gains equal to the eigenvalues of the effective channel. The effective precoder for user k now becomes

$$\mathbf{T}_{k,e} = \mathbf{T}_k \mathbf{V}_k. \tag{8}$$

Note that due to BD precoding, there is a limit on the total number of receive antennas, which has to be less than the total number of transmit antennas, i.e. $K \leq \lfloor \frac{BN_t}{N_r} \rfloor$ [13], [14]. This constraint can be satisfied with user scheduling.

III. p-SPHERE ENCODING

In this section, we discuss the p-sphere encoding. Based on the system model introduced in Section II, the transmitted signal vector for user k is given by

$$\mathbf{x}_k = \mathbf{V}_k \mathbf{d}_k, \tag{9}$$

where $\mathbf{d}_k$ is the perturbed data vector. Let $\mathbf{a}$ denote the aggregate data vector of all users:

$$\mathbf{a} = [\mathbf{a}_1^T, \mathbf{a}_2^T, \cdots, \mathbf{a}_K^T]^T. \tag{10}$$

The component $\mathbf{a}_k$ is the $N_r \times 1$ data vector of user k. The elements of $\mathbf{a}$ are independent and uniformly distributed over the following set:

$$\mathbf{A} = \{a : |\mathrm{Re}\{a\}| < 0.5, |\mathrm{Im}\{a\}| < 0.5\}. \tag{11}$$

Let $\mathbf{z}_p = [\mathbf{z}_1^T, \mathbf{z}_2^T, \cdots, \mathbf{z}_K^T]^T$ be the perturbing vector of the p-norm sphere encoding, which is a Gaussian integer vector found by the p-sphere encoding through the following minimization [8]:

$$\mathbf{z}_p = \underset{\mathbf{q} \in \mathbb{Z}[j]^{KN_r}}{\mathrm{argmin}} \; \|\mathbf{T}_e(\mathbf{a} + \mathbf{q})\|_p^2, \tag{12}$$

where $\mathbf{T}_e$ denotes the aggregate (over all users) effective precoding matrix

$$\mathbf{T}_e = [\mathbf{T}_1 \mathbf{V}_1, \mathbf{T}_2 \mathbf{V}_2, \cdots, \mathbf{T}_K \mathbf{V}_K]. \tag{13}$$

The minimization in (12) is the closest point search in an infinite lattice with p-norm. When $p = 2$ the minimization is the conventional sphere encoding problem, which attempts to find an integer offset in order to minimize the sum power. The algorithm proposed in [15] can be used for sphere encoding. It divides lattice into sub-layers and decreases the radius of search area until it can find any node with a better 2-norm distance. Unfortunately for $p > 2$ the problem is more complicated than for $p = 2$, since QR-decomposition is not applicable to the p-norm closest point search [8]. [8] proposed a technique to solve this problem. It seeks the closest point in a 2-norm ball with the smallest possible radius. In the algorithm of [8], whenever a node with lower p-norm distance is visited, the radius of the search area decreases.
The perturbed data vector is given by

$$\mathbf{d} = \mathbf{a} + \mathbf{z}_p. \tag{14}$$

At the receiver of user k, we have

$$\mathbf{r}_k = \mathbf{U}_k^* \mathbf{y}_k = \frac{1}{\sqrt{\gamma_p}} \mathbf{\Lambda}_k \mathbf{d}_k + \mathbf{U}_k^* \mathbf{n}_k = \frac{1}{\sqrt{\gamma_p}} \mathbf{\Lambda}_k \mathbf{d}_k + \mathbf{w}_k, \tag{15}$$

and from (15) we have

$$r_{k,i} = \frac{1}{\sqrt{\gamma_p}}\lambda_{k,i}d_{k,i} + w_{k,i}, \tag{16}$$

where $r_{k,i}$ denotes the ith stream of user k after being processed by $\mathbf{U}_k^*$. Using modulo function at the receiver side, the ith stream of user k is detected as [6]

$$\begin{aligned}\hat{a}_{k,i} &= [(\frac{\lambda_{k,i}}{\sqrt{\gamma}_p})^{-1}r_{k,i}]_{\text{mod }\mathbf{A}} \\ &= [a_{k,i} + z_{k,i} + (\frac{\lambda_{k,i}}{\sqrt{\gamma}_p})^{-1}w_{k,i}]_{\text{mod }\mathbf{A}} \\ &= [a_{k,i} + \eta_{k,i}]_{\text{mod }\mathbf{A}}, \end{aligned} \tag{17}$$

where $\eta_{k,i} \triangleq (\frac{\lambda_{k,i}}{\sqrt{\gamma}_p})^{-1}w_{k,i}$ is the effective noise for the ith stream of user k with variance $E\{|\eta_{k,i}|^2\} = (\frac{\lambda_{k,i}}{\sqrt{\gamma}_p})^{-2}$. The function $[.]_{\text{mod }\mathbf{A}}$ is a modulo function, i.e. for an arbitrary complex number ψ, $[\psi]_{\text{mod }\mathbf{A}} = \psi - \lfloor\psi\rceil$. The input data comes from the set $\mathbf{A}$ defined in (11) and the modulo function applied to the real and imaginary parts independently forces the decoded data always to lie inside the set $\mathbf{A}$ [9].

The PAPR is defined as [8]

$$\text{PAPR} = \frac{\|\mathbf{T}_e(\mathbf{a}+\mathbf{z}_p)\|_\infty^2}{\text{E}_{\mathbf{a}}\|\mathbf{T}_e(\mathbf{a}+\mathbf{z}_p)\|_2^2/N_t}, \tag{18}$$

where $\text{E}_{\mathbf{a}}(.)$ represents the expectation over the data vector $\mathbf{a}$. On the one hand, the ∞-norm encoding minimizes the peak power and on the other hand it maximizes the unscaled sum power, which gives us the best solution in the category of p-sphere encoding from the PAPR reduction perspective. In the next section, we derive the sum rate and show its relation to the unscaled sum power. We will observe that from the sum rate maximization perspective, ∞-norm provides lower sum rate than the 2-norm encoding, but the difference is small.

IV. Sum rate of p-sphere encoding

Following the discussion in [9], the maximum mutual information $I(\hat{a}_{k,i}; a_{k,i})$ with the assumption of restricted values for the input data is obtained by uniformly distributed input and is given by [9]

$$I(\hat{a}_{k,i}; a_{k,i}) = -\log 2\pi e\beta_{k,i} + 2\Omega(\beta_{k,i}), \tag{19}$$

where

$$\beta_{k,i} = \frac{1}{2}E\{|\eta_{k,i}|^2\} = \frac{1}{2}(\frac{\lambda_{k,i}}{\sqrt{\gamma}_p})^{-2}, \tag{20}$$

where $\Omega(.)$ is a function, which is defined as

$$\Omega(\beta) = \frac{1}{2} + \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{s=-\infty}^{\infty} \frac{1}{\sqrt{2\pi\beta}} e^{-\frac{-|\xi-s|^2}{2\beta}} \left[\log \sum_{t=-\infty}^{\infty} e^{-\frac{-|\xi-t|^2}{2\beta}}\right] d\xi. \tag{21}$$

$\Omega(\beta)$ captures the nonlinearity of the modulo function within the mutual information. Now similarly to discussion in [9], [10] the sum rate of p-sphere encoding with uniformly distributed input given the aggregate channel matrix $\mathbf{H} = [\mathbf{H}_1^T, \mathbf{H}_2^T, \cdots, \mathbf{H}_K^T]^T$ and the effective precoding matrix $\mathbf{T}_e$ can be expressed as

$$\begin{aligned} R_p(\mathbf{H}, \mathbf{T}_e) &= \sum_{k=1}^{K}\sum_{i=1}^{N_r} I(\hat{a}_{k,i}; a_{k,i}) \\ &= \sum_{k=1}^{K}\sum_{i=1}^{N_r} -\log 2\pi e\beta_{k,i} + 2\Omega(\beta_{k,i}). \end{aligned} \tag{22}$$

$\Omega(\beta)$ is an increasing function in β and as β tends to zero, $\Omega(\beta)$ tends to zero as well. The term $\Omega(\beta)$ is greater and equal to zero for all values of β, so removing this term from (22) gives us a lower bound on the sum rate, which becomes tighter as β goes to zero or equivalently as P_t increases to infinity. We write this lower bound on the sum rate as

$$R_{p,\text{LB}} \triangleq -KN_r \log \pi e\gamma_p - \sum_{k=1}^{K}\sum_{i=1}^{N_r} \log(\lambda_{k,i})^{-2}. \tag{23}$$

From the property of p-norm mentioned in [8], we can state the following inequalities for $p > 2$

$$\text{E}(\gamma_2) < \text{E}(\gamma_p) < N_t^{1-2/p}\text{E}(\gamma_2), \tag{24}$$

and consequently, we have

$$0 < \text{E}(R_{2,\text{LB}} - R_{p,\text{LB}}) < (1 - \frac{2}{p})KN_r \log N_t, \tag{25}$$

where the expectation is taken over the channel distribution. Fig. 1 shows the ratio $\text{E}(\gamma_p)/\text{E}(\gamma_2)$ ranges from 1 to approximately 1.2 for different values of p, which confirms that p-sphere encoding can exhibit approximately the same performance as vector perturbation.

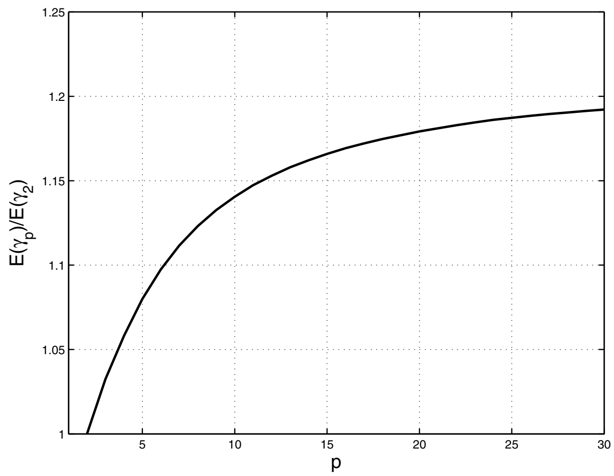

Fig. 1. $\text{E}(\gamma_p)/\text{E}(\gamma_2)$ versus p.

A lower bound on γ_2 has been derived in [16] as follows (see also [9], [10]).

$$\gamma_p > \gamma_2 \geq \frac{KN_r\Gamma(KN_r+1)^{1/KN_r}}{P_t(KN_r+1)\pi} \det(\mathbf{T}_e^*\mathbf{T}_e)^{1/KN_r}, \tag{26}$$

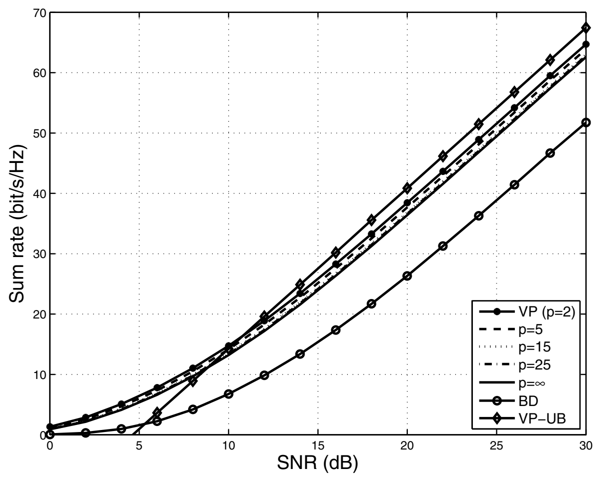

Fig. 2. Average sum rates of vector perturbation (VP), upper-bound with vector perturbation (VP-UB), p-sphere encodin and BD versus SNR ρ.

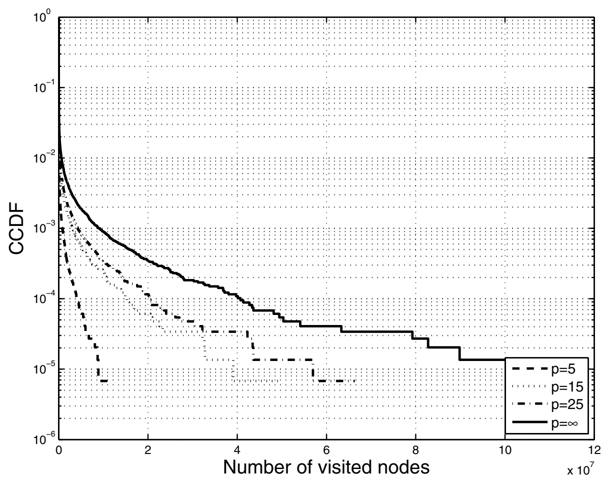

Fig. 3. CCDF characterizing the complexity of the p-sphere encoding for different values of p in terms of the number of visited nodes per channel use.

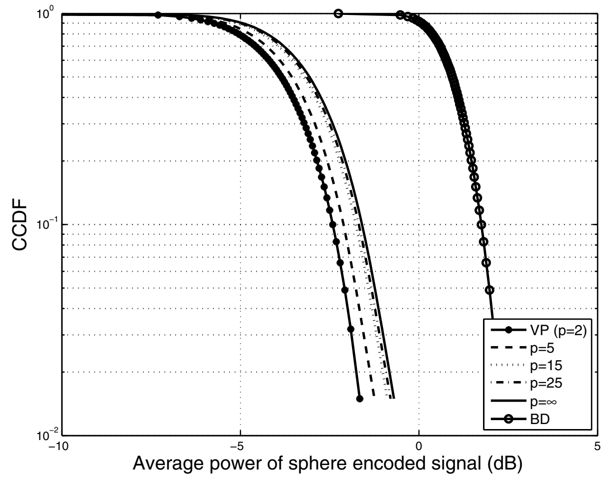

Fig. 4. CCDF of the average power of the sphere encoded signal for vector perturbation (VP), p-sphere encoding and BD.

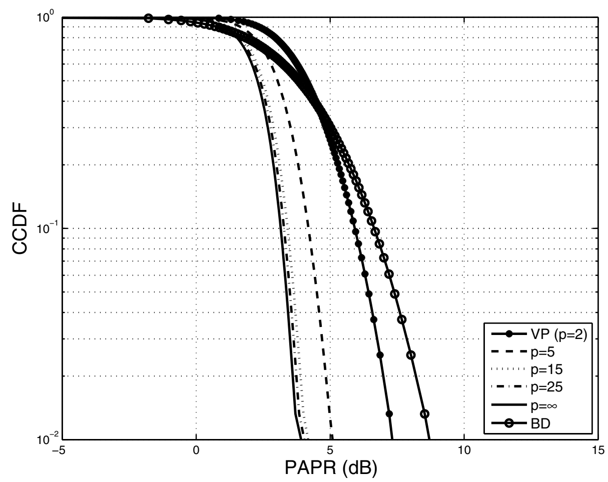

Fig. 5. CCDF of the PAPR for vector perturbation (VP), p-sphere encoding and BD.

Following the discussion in [9], [10], we can obtain the following asymptotic upper bound on the sum rate at high signal to noise ratios (SNRs) for the p-sphere encoding.

$$\lim_{P_t \to \infty} R_{p,\text{LB}} < KN_r \log \frac{P_t(KN_r+1)\det(\mathbf{HH}^*)^{1/KN_r}}{KN_r \Gamma(KN_r+1)^{1/KN_r} e}, \tag{27}$$

V. Simulation Results

We assume a MIMO-BC with a transmitter equipped with $N_t = 8$ transmit antennas serving $K = 4$ users with $N_r = 2$ receive antennas. We use 10000 channel realizations to produce the results. For simplicity, we assume that all users have the same average SNR and the nominal SNR is defined as $\rho = P_t/1$ (the noise power is 1). Fig. 2 depicts the average sum rate versus SNR and reflects the spectral efficiency of the p-sphere encoding. We observe that as p increases, the average sum rate decreases due to the fact that the unscaled power of the encoded signal increases with p (see Fig. 1 as well). For $p = 5$ the sum rate is reduced by just 1% compared to vector perturbation. BD is prone to noise enhancement and the gap between BD and lattice-aided precoding shows that the latter avoids this problem. The plot also includes the asymptotic upper bound curve and as it turns out it is a true upper bound at high SNRs and not in quite low SNRs.

Fig. 3 shows the complimentary cumulative distribution function (CCDF) of the complexity of the algorithm in terms of the number of visited nodes per channel use. As p increases, the algorithm needs to search over a larger area and consequently the complexity increases [8]. In other words, the smallest 2-norm ball that contains all the proper nodes at the beginning of the search becomes larger as p increases. However, Fig. 3 shows that at $p = 5$ the complexity is still reasonable.

Fig. 4 shows the CCDF of the average power of the sphere encoded signal. Vector perturbation ($p = 2$) exhibits the lowest average power and among the p-sphere encoding techniques, ∞-sphere encoder exhibits the highest average power since it focuses on minimizing the maximum power and not the average power.

Fig. 5 depicts the PAPR. The ∞-norm encoder exhibits the lowest PAPR among all the analyzed precoding techniques, but the choice of $p = \infty$ involves quite high complexity. For $p = 5$, the average PAPR is reduced by 25% compared to vector perturbation.

VI. CONCLUSION

The p-sphere encoding has been proposed in the literature as a solution for PAPR reduction of vector perturbation. In this paper we have studied the sum rate of the p-sphere encoding for the case of multiple-antenna users and uniformly distributed input. Block diagonalization is used for front-end linear precoding. We observe that $p = 5$ can be a practical choice for this precoding technique, which ensures acceptable PAPR, sum rate and complexity. Future research can consider other strategies for vector perturbation and the front-end precoder design such as MMSE or an even more complicated approach of maximizing the sum rate while imposing the PAPR constraints.

ACKNOWLEDGEMENT

Funding for this work has been provided by TRLabs, the Rohit Sharma Professorship and the Natural Sciences and Engineering Research Council (NSERC) of Canada. The work has been facilitated by the computing resources of AICT (Academic Information and Communication Technologies) at the University of Alberta, WestGrid and Compute/Calcul Canada.

REFERENCES

[1] E. Telatar, "Capacity of multi-antenna Gaussian channels," *Europ. Trans. Telecommun.*, vol. 10, no. 6, pp. 585–595, Nov./Dec. 1999.

[2] D. Gesbert, M. Kountouris, R. W. Heath Jr., C. Chan-Byoung, and T. Salzer, "From single user to multiuser communications: Shifting the MIMO paradigm," *IEEE Signal Process. Mag.*, vol. 24, no. 5, pp. 36 –46, Sept. 2007.

[3] M. Costa, "Writing on dirty paper," *IEEE Trans. Inf. Theory*, vol. 29, no. 3, pp. 439 – 441, May 1983.

[4] G. Caire and S. Shamai, "On the achievable throughput of a multiantenna Gaussian broadcast channel," *IEEE Trans. Inf. Theory*, vol. 49, no. 7, pp. 1691 – 1706, July 2003.

[5] ——, "On achievable rates in a multi-antenna broadcast downlink," in *Proc. 38^{th} Annu. Allerton Conf. Commun., Control and Comput.*, Monticello, IL, USA, Oct. 2000.

[6] B. M. Hochwald, C. B. Peel, and A. L. Swindlehurst, "A vector-perturbation technique for near-capacity multiantenna multiuser communication-Part II: perturbation," *IEEE Trans. Commun.*, vol. 53, no. 3, pp. 537 – 544, Mar. 2005.

[7] J. Stillwell, *Elements of Number Theory*. New York: Springer-Verlag, 2003.

[8] F. Boccardi and G. Caire, "The p-sphere encoder: Peak-power reduction by lattice precoding for the MIMO Gaussian broadcast channel," *IEEE Trans. Commun.*, vol. 54, no. 9, p. 1703, Sept. 2006.

[9] A. Razi, D. J. Ryan, I. B. Collings, and J. Yuan, "Sum rates, rate allocation, and user scheduling for multi-user MIMO vector perturbation precoding," *IEEE Trans. Wireless Commun.*, vol. 9, no. 1, pp. 356 –365, Jan. 2010.

[10] M. Mazrouei-Sebdani and W. A. Krzymień, "Vector perturbation precoding and user scheduling for network MIMO," in *Proc. IEEE Wireless Commun. and Networking Conf. (WCNC'11)*, Cancun, Mexico, Mar. 2011, pp. 203–208.

[11] W. Yao, S. Chen, and L. Hanzo, "Generalized MBER-based vector precoding design for multiuser transmission," *IEEE Trans. Veh. Technol.*, vol. 60, no. 2, pp. 739 –745, Feb. 2011.

[12] D. A. Schmidt, M. Joham, and W. Utschick, "Minimum mean square error vector precoding," in *Proc. IEEE 16^{th} Int. Symp. on Personal, Indoor and Mobile Radio Commun. (PIMRC'05)*, vol. 1, Sept. 2005, pp. 107 –111.

[13] Q. H. Spencer, A. L. Swindlehurst, and M. Haardt, "Zero-forcing methods for downlink spatial multiplexing in multiuser MIMO channels," *IEEE Trans. Signal Process.*, vol. 52, no. 2, pp. 461 – 471, Feb. 2004.

[14] J. Zhang, R. Chen, J. G. Andrews, A. Ghosh, and R. W. Heath Jr., "Networked MIMO with clustered linear precoding," *IEEE Trans. Wireless Commun.*, vol. 8, no. 4, pp. 1910 –1921, April 2009.

[15] E. Agrell, T. Eriksson, A. Vardy, and K. Zeger, "Closest point search in lattices," *IEEE Trans. Inf. Theory*, vol. 48, no. 8, pp. 2201 – 2214, Aug. 2002.

[16] D. J. Ryan, I. B. Collings, I. V. L. Clarkson, and R. W. Heath Jr., "Performance of vector perturbation multiuser MIMO systems with limited feedback," *IEEE Trans. Commun.*, vol. 57, no. 9, pp. 2633 – 2644, Sept. 2009.