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A Stabilizing Real-time Implementation of Nonlinear Model Predictive Control

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1 Introduction

Nonlinear model predictive control (NMPC) is a feedback control technique based on the real-time solution of optimal control problems. It has attracted significant interest over the past decades [2, 25, 34, 36], not only from theoreticians but also from practitioners. This interest is mainly motivated by the fact that NMPC allows for a wide flexibility in formulating the objective and the process model, and facilitates to directly take equality and inequality constraints on the states and inputs into account.

One important precondition for the successful application of NMPC is the availability of reliable and efficient numerical dynamic optimization algorithms, since at every sampling time a nonlinear dynamic optimization problem must be solved in real-time. Solving such an optimization problem efficiently and fast is a non trivial task, see e.g. [3, 4, 7, 12, 15, 17, 29, 31, 35, 38–40].

Often classical off-line dynamic optimization algorithms are used to solve the optimization problems arising in NMPC. This is done as fast as possible, and the solution is applied to the system once it is available. For small systems using fast computers the feedback delay due to the required computation time is often negligible if compared to the timescale of the system. In this case the computational delay can be considered as a small disturbance, which NMPC, under certain conditions, is able to reject [21, 24, 30, 37].

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However, is it also possible to guarantee stability of the closed loop if the computational delay for an “exact” solution of the optimal control problem is not negligible? Or could one use approximated solutions which are cheaply available, still achieve stability and good performance, while being able to reject disturbances fast? These are the questions considered in this paper.

One possible approach to take account of the computation time is to predict the state at the time we expect the optimization to be finished and carry out the optimization for this prediction [9, 21], allowing to prove nominal stability. One drawback of this approach is that new information is only incorporated in the feedback once the optimal solution has been found. This can lead to performance decrease if disturbances occuring in between the sampling times drive the system far away from the predicted trajectories.

In contrast to these approaches in the so called “real-time iteration” scheme [7, 12, 15, 17] the sampling times are reduced by using approximated solutions of the optimal control problem. The sampling times and feedback delays are reduced by dovetailing the dynamics of the system with the dynamics of the optimization algorithm. In principle only one optimization iteration is performed per sampling instant, i.e., the solution of the optimal control problem and the control of the system are performed in parallel. Applying only approximated solutions reduces the sampling time significantly, thus being able to react to external disturbances significantly faster. The real-time iteration scheme allows us to efficiently treat large-scale systems [18, 20, 22] or systems with short time scales [13] on standard computers, thus pushing forward the frontier of practical applicability of NMPC.

The principal aim of the paper is to prove that for the real-time iteration scheme the closed loop consisting of the combined system-optimizer dynamics is stable under certain conditions. The investigation combines concepts from both, classical stability theory for NMPC as well as from convergence theory for Newton type optimization methods. In comparison to similar results presented in [16], which employs a shift between iterations, we focus in this paper on a real-time iteration scheme without any shifting between sampling times. This simplifies the presentation and proofs significantly, and brings theory and practical implementations of the real-time iteration scheme closer together. The second aim is to illustrate and underpin the obtained theoretical results with numerical experiments considering the control of a high-purity distillation column. The paper is structured as follows: In Section 2 we shortly review the basics of NMPC and present the considered setup. The real-time iteration scheme is outlined in Section 3. In Section 4 and 5 we derive intermediate results needed for the main contribution, i.e., the proof of stability of the real-time iteration scheme without shift, which is presented in Section 6. Section 7 contains some numerical experiments for the control of a distillation column, which illustrate the derived theoretical results. We conclude in Section 8 with some final remarks.

2 Discrete Time Nonlinear Model Predictive Control

In the following we review the basics of discrete time NMPC. The presentation is kept short on purpose, since the main objective of the paper is to derive stability properties of the real-time iteration scheme. More details on NMPC can be found in [1, 10, 23, 32]. Throughout this paper, we consider the following nonlinear discrete time system:

$$x^{k+1} = f(x^k, u^k), \quad k = 0, 1, 2, \dots, \tag{1}$$

with system states $x^k \in \mathbb{R}^{n_x}$ and controls $u^k \in \mathbb{R}^{n_u}$. We assume that $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$ is twice continuously differentiable, and, without loss of generality, that the origin is a steady state for (1), i.e. $f(0,0) = 0$.

In NMPC the applied controls $u^k = u(x^k)$ depend on the current system state x^k. They are defined such that they are optimal with respect to a specified objective that depends on the predicted system behavior over a horizon of length N. For the derivations in this paper we assume that the objective minimized at every time instant k is given by

$$\sum_{i=0}^{N-1} L(s_i, q_i) + E(s_N).$$

Here $s_i,\ i = 0, \ldots, N$ are the predicted states over the fixed prediction horizon $N \in \mathbb{N}$ starting from x^k considering a predicted input sequence $(q_0, q_1, \ldots, q_{N-1})$:

$$s_{i+1} = f(s_i, q_i),\ i = 0, \ldots, N-1, \qquad s_0 = x^k.$$

We use s_i and q_i for the predicted states and controls to distinguish them clearly from the states x^k and controls u^k of the real system (1). The functions L and E weigh the system state and input applied to the system. They can be motivated by economical or practical interests and might represent asset values. With respect to the stage cost function L we assume that

ASSUMPTION 1. *The stage cost $L : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}$, and the terminal penalty $E : \mathbb{R}^{n_x} \to \mathbb{R}$ are twice continuously differentiable, $L(0,0) = 0$ and $E(0) = 0$, and there exists a positive constant $m > 0$ such that*

$$L(x,u) \geq m\|x\|^2, \quad \forall\, x \in \mathbb{R}^{n_x}, u \in \mathbb{R}^{n_u}. \tag{2}$$

Often L is quadratic, e.g. $L(x,u) = x^T Q x + u^T R u$, with positive definite matrices Q and R.

The input applied in NMPC is defined as the first input q_0^* of the optimal (optimal values are in the following denoted by a star) predicted input sequence $(q_0^*, \ldots, q_{N-1}^*)$:

$$u(x^k) := q_0^*(x^k). \tag{3}$$

Thus the closed loop system obeys the "ideal NMPC dynamics"

$$x^{k+1} = f(x^k, u(x^k)). \tag{4}$$

We denote this as the ideal NMPC dynamics, since no external disturbances nor any numerical errors with respect to the optimality of the applied input are considered.

Summarizing, in an ideal NMPC setting the applied input is given by the solution of a sequence of optimization problems $P(x^k)$ of the form:

DEFINITION 2.1 ($P(x)$).

$$\min_{\substack{s_0, \ldots, s_N, \\ q_0, \ldots, q_{N-1}}} \quad \sum_{i=0}^{N-1} L(s_i, q_i) + E(s_N) \tag{5a}$$

subject to
$$x - s_0 = 0, \tag{5b}$$
$$f(s_i, q_i) - s_{i+1} = 0, \quad i = 0, \ldots, N-1, \tag{5c}$$

Besides the problem of having to solve this optimal control problem efficiently and online for an practical implementation of NMPC, one central question in NMPC is if the closed loop system (4) is stable. This question has been examined extensively over recent years and a variety of NMPC schemes exist that can guarantee stability, see e.g. [1, 10, 23, 33]. For the purpose of this paper we consider a specific setup of NMPC that ensures stability without enforcing the last predicted state to lie in a certain region (a so called final region constraint) or even to reach the origin (a so called zero terminal constraint) [1, 10, 23, 33]. Namely we require that for a certain region of attraction X the end penalty term $E(s_N)$ bounds from above the cost occuring from applying a stabilizing control law $k(x)$ over an infinite time. Since we do not consider any additional path or stability constraints, the setup is close to the setup presented for continuous time systems in [26].

Before we come to the assumptions ensuring nominal stability of the ideal NMPC setup considered, we state the following assumption, which is required to ensure the existence of unique solutions to problem $P(x)$:

ASSUMPTION 2. *There exists an open set $X \subset \mathbb{R}^{n_x}$ such that for all initial values $x \in X$ problem $P(x)$ has a unique optimal solution $(s_0^*(x), \ldots, s_N^*(x), q_0^*(x), \ldots, q_{N-1}^*(x))$, the value function $V(x)$ which is defined via the optimal cost for every x by*

$$V(x) := \sum_{i=0}^{N-1} L(s_i^*(x), q_i^*(x)) + E(s_N^*(x)) \tag{6}$$

is continuous on the set X, and that there exists a (possibly large) $M > 0$ such that $V(x) \leq M\|x\|^2 \; \forall x \in X$.

Note that the steady state trajectory $(0, 0, \ldots, 0)$ is the solution of $P(0)$ and has optimal cost $V(0) = 0$, and that because of $V(x) \geq L(x, q_0^*(x)) \geq m\|x\|^2$ we also have $V(x) > 0, \; \forall x \in X \backslash \{0\}$.

In the remainder of this paper we are not interested in the set X, but rather in an compact level set of V that is contained in X. Thus in the following we consider a fixed $\alpha > 0$ such that
$$X_\alpha := \{x \in X | V(x) \leq \alpha\} \subset X, \tag{7}$$
and assume that the set X_α is compact. Clearly, X_α contains a neighborhood of the origin.

We now state the following theorem that ensures stability of the closed loop under ideal NMPC. It is an slight adaptation of the results contained in [1, 10, 32].

THEOREM 1 (Nominal Stability for Ideal NMPC). *Let Assumption 1 and 2 hold. Then the closed-loop system $x^{k+1} = f(x^k, u(x^k)), \; k = 0, 1, \ldots$ defined by the ideal NMPC feedback law* (3) *is asymptotically stable with a region of attraction of at least the size of X_α, if*
(a) $E(x) > 0, \; \forall x \in X_\alpha \setminus \{0\}$

(b) there exists a local control law $k(x)$ *defined on* X_α, *such that*
$f(x, k(x)) \in X_\alpha, \ \forall x \in X_\alpha$
(c) $E(f(x, k(x)) - E(x) \leq -L(x, k(x)), \forall x \in X_\alpha$

The proof follows from the results presented in [1, 10, 32]. The assumptions required are standard assumptions used in NMPC. The main difference to other approaches is that a strict terminal constraint for the last predicted state (s_N) is avoided, to make the presentations and derivations easier. Note, also, that terminal equality constraints requiring the final state to be fixed are difficult to satisfy in practical applications to large scale systems.

Assumptions (a) and (c) ensure that E is a local control Lyapunov function in X_α. Assumption (b) ensures that the region X_α is positive invariant under the virtual control law $k(x)$, which is never applied in practice, and guarantees feasibility at the next time instance. A number of methods to determine a terminal penalty term and a terminal region exist, see e.g. [1, 10, 32].

Results on the "robustness" of Lyapunov functions and results on the stability of NMPC under perturbations suggest that the ideal NMPC controller has some inherent robustness properties with respect to disturbances under the stated assumptions (in particular because V is continuous). This is based on the fact that along solution trajectories of the closed loop under ideal NMPC the following inequality holds:

$$V(f(x, u(x))) \leq V(x) - L(x, u(x)), \tag{8}$$

Basically $-L(x, u(x))$ on the right hand side provides some robustness with respect to disturbances that might lead to a lower decrease – but no increase – of the value function V from time step to time step [23, 24, 27, 37]. Thus, considering the error of an approximate feedback compared to the ideal NMPC input $u(x^k)$ as a disturbance, it can be hoped that under certain conditions the closed loop is stable. We build on somewhat similar arguments in the proof of the main result of this paper.

Remark: In practical applications, inequality path constraints of the form $h(x_i, q_i) \geq 0$, like bounds on controls or states, are of major interest, and should be included in the formulation of the optimization problems $P(x)$. For the derivations of this paper, however, we leave such constraints unconsidered. Note that in the practical implementation of the real-time iteration scheme they are included.

2.1 Online Solution of NMPC: Interconnection of System and Optimizer Dynamics

In ideal NMPC it is assumed that the feedback $u(x^k)$ is available instantaneously at every sampling time k. However, in practice the numerical solution of $P(x^k)$ requires a non negligible computation time and involves some numerical errors. Typically the initial value x^k is only known at the time k when the corresponding control u^k is already required for implementation. Thus, instead of implementing the ideal NMPC control $u(x^k)$, often some quickly available, but sufficiently good approximation $\tilde{u}(x^k, w^k)$ is used, where the additional argument w^k indicates a data vector $w^k \in \mathbb{R}^n$ (to be defined later) that we use to parameterize the control approximation. These are generated by an online optimization algorithm, and they may be updated from one time step to the next one, according to the law $w^{k+1} = F(x^k, w^k)$, where the argument x^k takes account of the fact that the update shall of course depend on the current system state. To achieve this the computations performed are thus divided in two parts

1. Preparation: computation of $w^k = F(x^{k-1}, w^{k-1})$, and generation of the feedback approximation function $\tilde{u}(\cdot, w^k)$, during the transition of the system from state x^{k-1} to x^k.

2. Feedback Response: At time k, give the feedback approximation $u^k := \tilde{u}(x^k, w^k)$ to the system, which then evolves according to $x^{k+1} = f(x^k, u^k)$.

From a system theoretic point of view, instead of the ideal NMPC dynamics (4), we have to investigate the combined system-optimizer dynamics

$$x^{k+1} = f(x^k, \tilde{u}(x^k, w^k)), \tag{9a}$$
$$w^{k+1} = F(x^k, w^k). \tag{9b}$$

The difficulty in the analysis of the closed-loop behavior of this system stems from the fact that the two subsystems mutually depend on each other, compare Figure 1. The real-

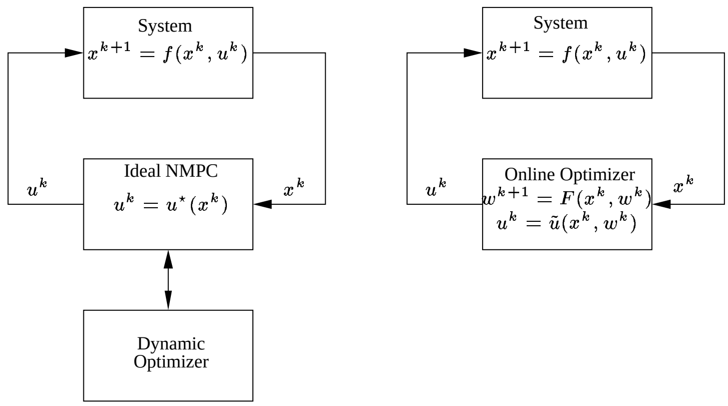

Figure 1. *Ideal NMPC (left) and interconnection of system and optimizer dynamics in the real-time iteration scheme (right).*

time iteration scheme investigated in this paper is one specific approach for an practical implementation of NMPC, where the data vector w^k is essentially a guess for the optimal solution trajectory of $P(x^k)$. The data update law $w^{k+1} = F(x^k, w^k)$ shall provide iteratively refined solution guesses, and is derived from a Newton type optimization scheme. The approximate feedback law $\tilde{u}(x^k, w^k)$ can be considered a by-product of this Newton type iteration scheme.

3 The Real-Time Iteration Scheme

In order to characterize the solution of the optimization problem $P(x)$ we introduce the Lagrange multipliers $\lambda_0, \ldots, \lambda_N$ for the constraints (5b) and (5c), and define the Lagrangian function $\mathcal{L}_x(\lambda_0, s_0, q_0, \ldots)$ of problem $P(x)$ as

$$\mathcal{L}_x(w) := \textstyle\sum_{i=0}^{N-1} L(s_i, q_i) + E(s_N) + {\lambda_0}^T(x - s_0) + \sum_{i=0}^{N-1} {\lambda_{i+1}}^T(f(s_i, q_i) - s_{i+1})$$

where all variables are summarized in the (data) vector

$$w := (\lambda_0, s_0, q_0, \ldots, \lambda_N, s_N) \in \mathbb{R}^n. \tag{10}$$

Note that under the stated assumptions it follows that $\mathcal{L}_x$ is twice continuously differentiable in its arguments over the considered regions. The necessary optimality conditions of first order for $P(x)$ are:

$$\nabla_w \mathcal{L}_x(w) = \begin{bmatrix} x - s_0 \\ \nabla_s L(s_0, q_0) + \frac{\partial f}{\partial s}(s_0, q_0)^T \lambda_1 - \lambda_0 \\ \nabla_q L(s_0, q_0) + \frac{\partial f}{\partial q}(s_0, q_0)^T \lambda_1 \\ \vdots \\ f(s_{N-1}, q_{N-1}) - s_N \\ \nabla_s E(s_N) \end{bmatrix} = 0. \tag{11}$$

One possible solution method for this set of nonlinear equations is to use Newton type iterations, as outlined in the following.

3.1 Review of Newton Type Iterations

The Newton type methods investigated in this paper aim to find the solution w^* of the necessary optimality conditions (11) of $P(x)$, $\nabla_w \mathcal{L}_x(w^*) = 0$. Starting at some guess w for w^*, they compute an iterate $w' = w + \Delta w(x, w)$ with

$$\Delta w(x, w) := -J(w)^{-1} \ \nabla_w \mathcal{L}_x(w), \tag{12}$$

where $J(w)$ is an approximation of the second derivative $\nabla_w^2 \mathcal{L}_x(w)$. We will use this definition of $\Delta w(x, w)$ throughout the paper, and often also refer to its subvectors as

$$\Delta w(x, w) = (\Delta\lambda_0(x, w), \Delta s_0(x, w), \Delta q_0(x, w), \ldots). \tag{13}$$

Note that $\nabla_w^2 \mathcal{L}_x$ – often called *Karush-Kuhn-Tucker (KKT)* matrix – is independent of the initial value x, which enters the Lagrangian $\mathcal{L}_x(w)$ only linearly. The index argument x can therefore be omitted for the KKT matrix, i.e., we will write $\nabla_w^2 \mathcal{L}(w)$ in the sequel. Of course, its approximation $J(w)$ shall also be independent of x, and we assume in the following that $J(w)$ is continuous over the considered regions. Moreover, the steps shall have the property that $s_0 + \Delta s_0(x, w) = x$, i.e., that the linear initial value constraint (5b), $x - s_0 = 0$, is satisfied after one Newton type iteration. This is easily accomplished by noting that the first n_x rows of $\nabla_w^2 \mathcal{L}(w)$ are constant, and choosing them to also be the first n_x rows of $J(w)$.

The Constrained Gauss-Newton Method

An important special case of the Newton type methods considered in this paper is the constrained Gauss-Newton method [5], which is applicable for problems with a least squares form of the objective function, i.e.,

$$L(s_i, q_i) = \frac{1}{2} \|l(s_i, q_i)\|_2^2.$$

For this case, the Hessian blocks $\nabla^2_{(s_i,q_i)}\mathcal{L}(w)$ within the KKT matrix $\nabla^2_w\mathcal{L}(w)$ can cheaply be approximated by

$$\left[\frac{\partial l(s_i,q_i)}{\partial(s_i,q_i)}\right]^T \frac{\partial l(s_i,q_i)}{\partial(s_i,q_i)} = \nabla^2_{(s_i,q_i)}\mathcal{L}(w) + O(\|\lambda_{i+1}\| + \|l(s_i,q_i)\|)$$

neglecting terms that become small when the multipliers and the least squares residuals are small. In the constrained Gauss-Newton method, this approximation of the Hessian blocks is used within the iteration matrix $J(w)$, while all remaining parts of $J(w)$ are identical to the exact KKT matrix $\nabla^2_w\mathcal{L}(w)$. It is interesting to note that this iteration matrix $J(w)$ does not depend on the values of the Lagrange multipliers. In the numerical experiments in Section 7 we use the constrained Gauss-Newton method. An alternative to the Gauss-Newton method might be to also approximate the constraint jacobians. In this case adjoint techniques can be used for computing the gradient of the lagrangian efficiently, and each iteration becomes considerably cheaper [6].

3.2 The Real-Time Iteration Algorithm

Assume that during the transition from one sampling instant to the next we only have time to perform one Newton type iteration. After an initial disturbance it subsequently delivers approximations u^k for the optimal feedback control that allow to steer the system close to the desired steady state, as will be shown in Section 6, under suitable conditions.

The real-time iteration scheme (without shift strategy, compare [16]) proceeds now as follows:

1. Preparation: Based on the current guess $w^k = (\lambda_0^k, s_0^k, q_0^k, \lambda_1^k, s_1^k, q_1^k, \ldots, \lambda_N^k, s_N^k)$ compute all components of the vector $\nabla_w\mathcal{L}_{x^k}(w^k)$ apart from the first one, and compute the matrix $J(w^k)$. Prepare the computation of $J(w^k)^{-1}\nabla_w\mathcal{L}_{x^k}(w^k)$ as much as possible without knowledge of the value of x^k (a detailed description how this can be achieved is given in [15] or [12]).

2. Feedback Response: At the time k, when x^k measured, compute the feedback approximation $u^k = \tilde{u}(x^k, w^k) := q_0^k + \Delta q_0(x^k, w^k)$ and apply the control u^k immediately to the real system.

3. Transition: Compute the next initial guess w^{k+1} by adding the step vector Δw^k to w^k
$$w^{k+1} = w^k + \Delta w^k = w^k - J(w^k)^{-1}\ \nabla_w\mathcal{L}_{x^k}(w^k).$$
Continue by setting $k = k+1$ and going to 1.

In contrast to the ideal NMPC feedback closed loop (4), in the real-time iteration scheme we have to regard combined system-optimizer dynamics of the form (9), which are given by

$$x^{k+1} = f\left(\ x^k,\ \ q_0^k + \Delta q_0(x^k, w^k)\ \right) \tag{14a}$$
$$w^{k+1} = \ w^k + \Delta w(x^k, w^k). \tag{14b}$$

In the remainder of the paper we concentrate on investigating the nominal stability of these system-optimizer dynamics.

4 Local Convergence of Newton Type Optimization

In a first step we review a local convergence result of Newton type optimization for the solution of one fixed optimization problem, i.e., we regard only the optimizer dynamics (14b) for $x^k = x$, with fixed $x \in X_\alpha$. We denote in the following by w_0 an (arbitrary) initial guess for the primal-dual variables of problem $P(x)$. A standard Newton type scheme for the solution of (11), $\nabla_w \mathcal{L}_x(w) = 0$, proceeds by computing iterates $w_1, w_2, \ldots$ according to

$$w_{i+1} = w_i + \Delta w_i, \quad \Delta w_i := \Delta w(x, w_i).$$

The following theorem states a well known affinely invariant condition for convergence of Newton type methods that is due to Bock [5].

THEOREM 2 (Local Convergence of Newton Type Optimization [5, 16].).
Assume that $J(w)$ is invertible for all $w \in D$, where $D \subset \mathbb{R}^n$. Furthermore, assume:

(i) there exist constants $\kappa < 1, \omega < \infty$ such that for all $w', w \in D$, $\Delta w = w' - w$ and all $t \in [0, 1]$

$$\left\| J(w')^{-1} \left(J(w + t\Delta w) - \nabla_w^2 \mathcal{L}(w + t\Delta w) \right) \Delta w \right\| \leq \kappa \|\Delta w\|, \tag{15a}$$

$$\left\| J(w')^{-1} \left(J(w + t\Delta w) - J(w) \right) \Delta w \right\| \leq \omega t \|\Delta w\|^2, \tag{15b}$$

(ii) the first step Δw_0 is sufficiently small, such that

$$\delta_0 := \kappa + \frac{\omega}{2}\|\Delta w_0\| < 1, \quad \textit{and} \tag{15c}$$

(iii) the ball $B_0 := \left\{ w \in \mathbb{R}^n \,|\, \|w - w_0\| \leq \frac{\|\Delta w_0\|}{1-\delta_0} \right\}$ is completely contained in D.

Then the Newton type iterates $w_0, w_1, \ldots$ are well-defined, stay in the ball B_0, and converge towards a point $w^* \in B_0$ satisfying $\nabla_w \mathcal{L}_x(w^*) = 0$. In addition, the iterates satisfy the contraction property

$$\|\Delta w_{i+1}\| \leq \left(\kappa + \frac{\omega}{2}\|\Delta w_i\| \right) \|\Delta w_i\| \leq \delta_0 \|\Delta w_i\|$$

and

$$\|w_i - w^*\| \leq \frac{\|\Delta w_i\|}{1 - \delta_0}.$$

Remark 1: Equations (15a) and (15b) are well known affine invariant assumptions for the convergence of Newton type methods [5, 11]. Equation (15a) is an assumption on the quality of J as an approximation of the second derivative $\nabla_w^2 \mathcal{L}$ (and is satisfied with $\kappa = 0$ for an exact Newton's method), while (15b) is an assumption on the Lipschitz continuity of J and is satisfied for any twice differentiable J on a compact set. It is in general difficult to determine κ and ω a priori for a given problem, but a posteriori estimates can be obtained while the Newton type iterations are carried out.

Remark 2: It is interesting to note that the assumptions of the above Theorem imply that not only $J(w)$, but also the exact KKT matrix $\nabla_w^2 \mathcal{L}(w)$ is non-singular, as can be shown by contradiction. Let us for this aim assume that there was, at a point w' in the interior of D, a singular direction $\Delta w \neq 0$ such that $\nabla_w^2 \mathcal{L}(w')\Delta w = 0$. Without loss of generality, assume that Δw is small enough such that $w = w' - \Delta w \in D$. By setting $t = 1$ in Eq. (15a), we obtain the contradiction $\|\Delta w\| = \left\| J(w')^{-1} \left(J(w') - \nabla_w^2 \mathcal{L}(w') \right) \Delta w \right\| \leq \kappa \|\Delta w\| < \|\Delta w\|$.

4.1 Local Convergence of Newton Type Methods for NMPC

We tailor the above results to the NMPC problem. For this purpose we define two sets $D_C \subset D_{2C}$ which are defined in terms of a fixed $C > 0$

$$D_C := \{w \in \mathbb{R}^n \,| \exists x \in X_\alpha, \quad \|w - w^*(x)\| \leq C\} \tag{16}$$
$$D_{2C} := \{w \in \mathbb{R}^n \,| \exists x \in X_\alpha, \quad \|w - w^*(x)\| \leq 2C\}, \tag{17}$$

where $w^*(x)$ is the primal-dual solution of problem $P(x)$, and where X_α is the maximum level set of V in X as introduced in (7). Given these sets we can now state the technical assumptions necessary for the following corollary.

ASSUMPTION 3.

(i) For each $x \in X_\alpha$ the solution $w^*(x)$ is unique in D_{2C}, i.e., it is the only point $w \in D_{2C}$ satisfying $\nabla_w \mathcal{L}_x(w) = 0$.

(ii) For each $w \in D_{2C}$ the matrix $J(w)$ is invertible.

(iii) There exist constants $\omega < \infty$, $\kappa < 1$ such that for all $w', w \in D_{2C}$, $\Delta w = w' - w$ and all $t \in [0, 1]$

$$\left\| J(w')^{-1} \left(J(w + t\Delta w) - \nabla_w^2 \mathcal{L}(w + t\Delta w) \right) \Delta w \right\| \leq \kappa \|\Delta w\| \tag{18a}$$
$$\left\| J(w')^{-1} \left(J(w + t\Delta w) - J(w) \right) \Delta w \right\| \leq \omega t \|\Delta w\|^2. \tag{18b}$$

The following two scalars d and δ are used throughout the paper.

DEFINITION 4.1. *Given a fixed $C > 0$ such that Assumption 3 holds, we define the positive scalars*

$$d := \frac{C(1-\kappa)}{1 + \frac{\omega}{2}C} \quad \textit{and} \quad \delta := \kappa + \frac{\omega}{2} d. \tag{19}$$

Note that $\delta = \frac{\kappa + \frac{\omega}{2}C}{1 + \frac{\omega}{2}C} < 1$. Now we can state the following corollary that provides conditions for the convergence of Newton type methods for NMPC:

COROLLARY 3 (Local Convergence of Newton type methods for NMPC). *Suppose Assumption 3 holds. If for some $x \in X_\alpha$ and some $w_0 \in D_C$ it holds that $\|\Delta w(x, w_0)\| \leq$*

d, then the Newton type iterates w_i for the solution of $\nabla_w \mathcal{L}_x(w) = 0$, initialized with the initial guess w_0, converge towards the solution $w^*(x)$. Furthermore, the iterates remain in D_C, and satisfy $\|\Delta w_{i+1}\| \leq \delta \|\Delta w_i\|$ as well as $\|w_{i+1} - w^*(x)\| \leq \frac{\delta}{1-\delta}\|\Delta w_i\|$.

Proof: We start by noting that $C = \frac{d}{1-\delta}$. The ball B_0 defined in Theorem 2 is contained in the ball $\{w' \in \mathbb{R}^n | \, \|w' - w_0\| \leq C\}$, which itself is contained in the set D_{2C}, as $w_0 \in D_C$. Therefore, there is a solution $w^* \in D_{2C}$ satisfying $\nabla_w \mathcal{L}_x(w^*) = 0$, which must be equal to $w^*(x)$ due to uniqueness. Furthermore, the distance of iterate w_i from $w^*(x)$ is bounded by

$$\|w_i - w^*(x)\| \leq \frac{\|\Delta w_0\|}{1-\delta_0} \leq \frac{d}{1-\delta} = C, \tag{20}$$

i.e., $w_i \in D_C$. The remaining two properties immediately follow from Theorem 2. □

In the remainder of the paper we consider fixed values for α and C and assume that Assumption 3 is satisfied. Furthermore, we often refer to the set Ξ, which is defined as follows:

DEFINITION 4.2 (Set Ξ). *The set Ξ is defined in the following by*

$$\Xi := \{(x, w) \in \mathbb{R}^{n_x} \times \mathbb{R}^n \,|x \in X_\alpha, w \in D_C, \|\Delta w(x, w)\| \leq d\} \tag{21}$$

This set Ξ contains all pairs (x, w) for which Corollary 3 ensures numerical solvability. Note that Ξ is nonempty, as it contains at least the points $(x, w^*(x))$, $\forall x \in X_\alpha$, and their neighborhoods.

5 Contractivity of the Real-Time Iterations

Before being able to prove stability of the real-time iteration scheme in Section 6 we need to establish contraction properties of the Newton type iterations in the real-time iteration scheme. To investigate the stability of the combined system-optimizer dynamics (14) we establish in this section a bound on the size of the steps $\Delta w^k := \Delta w(x^k, w^k)$. For this aim assume the following:

ASSUMPTION 4. *There exist constants $\tilde{\sigma} > 0$, $\eta > 0$, such that for all $w \in D_C$ and all x, u with $\|u - u^*(x)\| \leq \frac{\delta d}{1-\delta}$*

$$\left\| J(w)^{-1} \begin{bmatrix} f(x,u)-x \\ 0 \\ \vdots \end{bmatrix} \right\| \leq \eta \|x\| + \tilde{\sigma} \|u - u(x)\|. \tag{22}$$

The constant $\tilde{\sigma}$ satisfies $\tilde{\sigma} < \frac{(1-\delta)^2}{\delta}$, and for $\eta > 0$ it holds with $\sigma := \frac{\delta}{1-\delta}\tilde{\sigma}$ that

$$\eta \leq \max \left\{ \frac{1}{2}\sqrt{\frac{m}{\alpha}}(1-(\delta+\sigma))d, \;\; \frac{1}{2}\frac{m(1-(\delta+\sigma))}{\sqrt{32(M+m)\mu}} \right\}. \tag{23}$$

Remark 1: From this assumption follows that for all $(x,w) \in \Xi$, $w' = w + \Delta w(x,w)$

$$\left\| J(w')^{-1} \begin{bmatrix} f(s_0',q_0')-s_0' \\ 0 \\ \vdots \end{bmatrix} \right\| \le \eta \|x\| + \sigma \|\Delta w(x,w)\| \tag{24}$$

(with $\sigma = \frac{\delta}{1-\delta}\tilde{\sigma}$), since $s_0' = x$ and $\|q_0' - u(x)\| \le \|w' - w^*(x)\| \le \frac{\delta}{1-\delta}\|\Delta w(x,w)\|$. It also follows that $\delta + \sigma < 1$. We will use this and Inequality (24) in the following.

Remark 2: In particular the fact that η might be required to be quite small deserves some discussion. For this aim we first observe that under suitable differentiability assumptions on f and u and because of $f(0,0) = 0$ and $u(0) = 0$ we first have $f(x,u(x)) - x = O(\|x\|)$. We also remark that if $f(x,u)$ is the transition function over time ΔT of a continuous time system, cf. Section 7, then $f(x,u) = x + O(\Delta T)$, and we could conclude that $f(x,u(x)) - x = O(\Delta T \|x\|)$. These considerations can be seen as a cautious motivation for feasibility of Assumption 4.

LEMMA 4 (Stepsize Contraction for Real-Time Iterations).
Suppose Assumptions 3 and 4 are satisfied. Furthermore, assume that $(x^k, w^k) \in \Xi$ *and* $x^{k+1} := f(x^k, q_0^k + \Delta q_0(x^k, w^k)) \in X_\alpha$. *Then, using* $\Delta w^k = \Delta w(x^k, w^k)$ *and* $w^{k+1} = w^k + \Delta w^k$, *the following holds*

$$\|\Delta w(x^{k+1}, w^{k+1})\| \le (\delta + \sigma)\|\Delta w^k\| + \eta\|x^k\|. \tag{25}$$

In particular, $\|\Delta w(x^{k+1}, w^{k+1})\| \le d$, *i.e.,* $(x^{k+1}, w^{k+1}) \in \Xi$.

Proof: First note that for any $w = (\lambda_0, s_0, q_0, \ldots) \in \mathbb{R}^n$ and any $x_1, x_2 \in \mathbb{R}^{n_x}$

$$\nabla_w \mathcal{L}_{x_1}(w) = \nabla_w \mathcal{L}_{x_2}(w) + \begin{bmatrix} x_1 - x_2 \\ 0 \\ \vdots \end{bmatrix}.$$

Furthermore, note that $x^{k+1} = f(x^k, q_0^k + \Delta q_0^k) = f(s_0^{k+1}, q_0^{k+1})$. Therefore, we can deduce

$$\begin{aligned}
\|\Delta w(x^{k+1}, w^{k+1})\| &= \|\Delta w(f(s_0^{k+1}, q_0^{k+1}), w^{k+1})\| \\
&= \|J(w^{k+1})^{-1} \cdot \nabla_w \mathcal{L}_{f(s_0^{k+1}, q_0^{k+1})}(w^{k+1})\| \\
&= \left\| J(w^{k+1})^{-1} \cdot \left(\nabla_w \mathcal{L}_{x^k}(w^{k+1}) + \begin{bmatrix} f(s_0^{k+1}, q_0^{k+1}) - s_0^{k+1} \\ 0 \\ \vdots \end{bmatrix} \right) \right\| \\
&\le \|J(w^{k+1})^{-1} \cdot \nabla_w \mathcal{L}_{x^k}(w^{k+1})\| \quad + \quad \eta\|x^k\| + \sigma\|\Delta w^k\| \\
&\le \qquad \delta \quad \|\Delta w^k\| \qquad\qquad + \quad \eta\|x^k\| + \sigma\|\Delta w^k\| \\
&= (\delta + \sigma)\,\|\Delta w^k\| + \eta\|x^k\|,
\end{aligned}$$

where we have made use of Assumption 4 in the 4th transformation and of Assumption 3 and Corollary 3 in the 5th. From $m\|x\|^2 \le V(x) \le \alpha$ it follows for all $x \in X_\alpha$ that $\|x\| \le \sqrt{\frac{\alpha}{m}}$, and from $\|\Delta w^k\| \le d$ and from the first upper bound of η in Inequality (23), we can finally deduce that $\|\Delta w(x^{k+1}, w^{k+1})\| \le (\delta + \sigma)\, d + \eta\sqrt{\frac{\alpha}{m}} \le (\delta + \sigma)\, d + \frac{1}{2}\sqrt{\frac{m}{\alpha}}(1 - (\delta + \sigma))d\sqrt{\frac{\alpha}{m}} \le d$. ❑

The lemma allows us to conclude the following contraction property for the real-time iterations (x^k, w^k) as defined in (14), which we use in the following section.

COROLLARY 5 (Shrinking Stepsize for Real-Time Iterations). *Let in addition to Assumptions 2-4 assume that the real-time iterations start with an initialization $(x^0, w^0) \in \Xi$, and that for a given $\alpha_0 \le \alpha$ and for some $k_0 > 0$ we have for all $k \le k_0$ that $x^k \in X_{\alpha_0}$. Then $(x^k, w^k) \in \Xi$ for all $k \le k_0$ and*

$$\|\Delta w^k\| \le (\delta+\sigma)^k \|\Delta w^0\| + \frac{\rho}{2}\sqrt{\alpha_0} \quad, \forall k \le k_0, \quad \textit{with} \quad \rho := \frac{2\eta}{\sqrt{m}(1-(\delta+\sigma))}. \tag{26}$$

Proof: Inductively applying Lemma 4 to the iterates (x^{k+1}, w^{k+1}), we immediately obtain that $(x^k, w^k) \in \Xi$, for $k = 1, 2, \ldots, k_0$. Similarly, from the contraction inequality (25), from $\|\Delta w^{k+1}\| \le (\delta+\sigma)\|\Delta w^k\| + \eta\|x^k\|$, and from the fact that $\|x^k\| \le \sqrt{\frac{\alpha_0}{m}}$ one obtains inductively that

$$\|\Delta w^k\| \le (\delta+\sigma)^k \|\Delta w^0\| + \eta\sqrt{\frac{\alpha_0}{m}} \sum_{i=0}^{k-1} (\delta+\sigma)^i \le (\delta+\sigma)^k \|\Delta w^0\| + \frac{\eta\sqrt{\frac{\alpha_0}{m}}}{1-(\delta+\sigma)}. \quad \square$$

We may furthermore ask how many iterations we need to reduce the stepsize such that it becomes smaller than a given level. However, considering Corollary 5 we must expect that they will not become smaller than the constant $\frac{\rho}{2}\sqrt{\alpha_0}$ in Eq. (26). But how many iterations do we need, for example, to push the stepsize under twice that level?

COROLLARY 6. *Let us in addition to Assumptions 2-4 assume that the real-time iterations start with an initialization $(x^0, w^0) \in \Xi$, and that for a given $\alpha_0 \le \alpha$ and for some*

$$k_0 \ge \log_{\delta+\sigma}\left(\frac{\rho\sqrt{\alpha_0}}{2\|\Delta w^0\|}\right). \tag{27}$$

we have for all $k \le k_0$ that $x^k \in X_{\alpha_0}$. Then

$$\|\Delta w^{k_0}\| \le \rho\sqrt{\alpha_0}. \tag{28}$$

The proof of this corollary trivially follows from (27), since $(\delta+\sigma)^{k_0}\|\Delta w^0\| \le \frac{\rho}{2}\sqrt{\alpha_0}$. This together with (26) yields (28).

6 Stability of the Real-Time Iteration Scheme

In this section we will finally give a proof of asymptotic stability of the combined system-optimizer dynamics (14) of the real-time iteration scheme without shift. The line of proof is very similar to the one used in [16] for the real-time iteration scheme with shift.

So far we have already examined the properties of the optimizer iterates w^k, but under the crucial assumption that the system states x^k remain in a certain level set X_{α_0}. In this section we will show how this can be assured, such that we will finally be able to prove convergence of the combined iterates (x^k, w^k) towards the origin.

For ideal NMPC, the optimal value function can only decrease due to (8), and positive invariance of a level set X_{α_0} would easily follow. In the real-time iteration scheme, however, instead of applying the ideal NMPC control $u(x) := q_0^*(x)$ to the plant, we employ a feedback approximation $\tilde{u}(x, w) := q_0 + \Delta q_0(x, w)$ that depends not only on the system state x but also on the current optimizer parameter vector $w = (\lambda_0, s_0, q_0, \ldots)$. This may result in an increase of the value function.

6.1 Bounding the Error of Feedback Approximations

Fortunately, under the assumptions of Theorem 1 the ideal NMPC feedback inherits under some additional conditions robustness properties as already outlined in Section 2. We do not go into details here and instead refer to [16, 37]. We assume the following.

ASSUMPTION 5. *There exists a constant $\tilde{\mu}$ such that for all $x \in X_\alpha$ and all u with $\|u - u(x)\| \leq \frac{\delta}{1-\delta} d$ the following holds:*

$$V(f(x, u)) \leq V(x) - L(x, u) + \tilde{\mu}\|u - u(x)\|^2.$$

Remark: The assumption can be motivated by the following informal considerations. For the optimal value function V holds the following equality

$$V(x) = \min_u \left(L(x, u) + \tilde{V}(f(x, u)) \right)$$

where $\tilde{V}$ is the optimal value function for a shrunk optimization problem similar to (5), but with only $N - 1$ time steps in the horizon. The minimizer is given by the ideal NMPC law $u(x)$. Assuming suitable differentiability of L, V, and $\tilde{V}$, and using optimality of $u(x)$ we therefore have for u close to $u(x)$

$$L(x, u) + \tilde{V}(f(x, u)) = V(x) + O(\|u - u(x)\|^2).$$

Moreover, from the assumptions of Theorem 1 one could conclude that $V(x) \leq \tilde{V}(x)$ for all $x \in X_\alpha$, so that we obtain

$$L(x, u) + V(f(x, u)) \leq L(x, u) + \tilde{V}(f(x, u)) = V(x) + O(\|u - u(x)\|^2),$$

which is a statement of the form of Assumption 5.

The following theorem states that the error with respect to the ideal NMPC descent property (8) is small if the Newton type step size $\|\Delta w(x, w)\|$ is small. It trivially follows from Assumption 5 by observing that for each $(x, w) \in \Xi$ it holds that $\|q_0 + \Delta q_0(x, w) - u(x)\| \leq \frac{\delta}{1-\delta}\|\Delta w(x, w)\|$.

THEOREM 7 (Error Bound for Approximate Feedback).
Suppose Assumptions 2, 3 and 5 hold. Then for each $(x, w) \in \Xi$

$$V(f(x, q_0 + \Delta q_0(x, w))) \leq V(x) - L(x, q_0 + \Delta q_0(x, w)) + \mu\|\Delta w(x, w)\|^2$$

with $\mu := \frac{\delta^2}{(1-\delta)^2}\tilde{\mu}$.

6.2 Combining Error Bound and Contractivity

Equipped with the error bound from Section 6.1 and the contractivity of the real-time iterations from Section 5 we can finally prove nominal stability of the real-time iteration closed-loop scheme. However, since the error in the decrease in the value function depends on the real-time iteration stepsize $\|\Delta w^k\|$, we have to investigate two competing effects: on the one hand, the feedback errors may allow an increase in $V(x^k)$, instead of the desired decrease that was needed to prove nominal stability for ideal NMPC in Theorem 1. On the other hand, we know that the stepsizes $\|\Delta w^k\|$ shrink during the iterations, and thus we also expect the errors to become smaller. Since an increase in the value function might imply that we leave the level set X_α, we will not be able to stabilize with the real-time iteration scheme the whole set X_α (at least not if Δw^0 is too large). Thus, we have to provide a safety margin to allow an increase in the value function without leaving X_α until Δw^k is small enough to guarantee a decrease of the value function. For this reason we will distinguish two phases:

- In the first phase we may have an increase in the value function $V(x^k)$, therefore we must allow for a safety back-off. However, the stepsizes $\|\Delta w^k\|$ can be shown to shrink.

- In the second phase, finally, the numerical errors are small enough to guarantee a decrease of both, $V(x^k)$ and $\|\Delta w^k\|$ and we can prove convergence of the iterates (x^k, w^k) towards the origin $(0, 0)$.

6.3 Phase 1: Increase in Objective, but Decrease in Stepsize

Exploiting Corollary 6, define the number k_α of iterations that are at maximum needed for reduction of the stepsize under the value $\rho\sqrt{\alpha}$ if all iterates stay in the level set X_α.

DEFINITION 6.1 (k_α and Ξ_{attr}). *We define k_α to be the smallest integer such that*

$$k_\alpha \geq \log_{\delta+\sigma}\left(\frac{\rho\sqrt{\alpha}}{2d}\right). \tag{29}$$

Furthermore, we define our safety back-off set as the set

$$\Xi_{\mathrm{attr}} := \left\{(x, w) \in \Xi \,\middle|\, V(x) \leq \alpha - k_\alpha \mu d^2 \right\}. \tag{30}$$

LEMMA 8 (Increase in Objective, Decrease in Stepsize). *Assume that Assumptions 1-5 hold and that $(x^0, w^0) \in \Xi_{\mathrm{attr}}$. Then for $k = 0, \ldots k_\alpha$ it holds that $(x^k, w^k) \in \Xi$. Furthermore, $\|\Delta w^{k_\alpha}\| \leq \rho\sqrt{\alpha}$.*

Proof: We make use of Corollary 5 and 6. To apply them, we first observe that $(x^0, w^0) \in \Xi$. It remains to be shown that $x^0, \ldots, x^{k_\alpha} \in X_\alpha$. We do this inductively, and show: if for some $k \leq k_\alpha$ it holds that $(x^k, w^k) \in \Xi$ and $V(x^k) \leq \alpha + (k - k_\alpha)\mu d^2$ then also $(x^{k+1}, w^{k+1}) \in \Xi$ and $V(x^{k+1}) \leq \alpha + (k + 1 - k_\alpha)\mu d^2$. To show this we first note that $\|\Delta w^k\| \leq d$ as an immediate consequence of Corollary 5. Now from Theorem 7 we know that

$$V(x^{k+1}) \leq V(x^k) - L(x^k, u^k) + \mu d^2$$

from which we conclude

$$V(x^{k+1}) \leq V(x^k) + \mu d^2 \leq \alpha + (k - k_\alpha)\mu d^2 + \mu d^2 = \alpha + (k + 1 - k_\alpha)\mu d^2.$$

□

Remark: The restriction of the initial system state x^0 to the level set $\left\{x \in X \,\middle|\, V(x) \leq \alpha - k_\alpha \mu d^2\right\}$ is unnecessarily restrictive. On the one hand we neglected the decrease $-L(x^k, u^k)$ in each step; and on the other hand an initial stepsize $\|\Delta w(x^0, w^0)\|$ considerably smaller than d would allow the errors in the decrease condition be considerably smaller than μd^2. Note in particular that an initial iterate (x^0, w^0) where the optimizer is initialized so well that $\|\Delta w(x^0, w^0)\| \leq \rho\sqrt{\alpha}$ directly qualifies for Phase 2, if only $V(x^0) \leq \alpha$, without requiring any safety back-off at all. However, to keep the discussion as simple as possible, we chose to stick to our above definition of the set Ξ_{attr} of states attracted by the origin.

6.4 Phase 2: Convergence towards the Origin

We now show that the real-time iterations – once the errors have become small enough – not only remain in their level sets, but moreover, are attracted by even smaller level sets. For convenient formulation of the results of this subsection we first define two constant integers.

DEFINITION 6.2 (k_1 and k_2). *Define the constants k_1 and k_2 to be the smallest integers that satisfy*

$$k_1 \geq \frac{6(M+m)}{m} \quad \text{and} \quad k_2 \geq \log_{\delta+\sigma}\left(\frac{1}{4}\right).$$

LEMMA 9 (Objective and Stepsize Reduction).
Let us in addition to Assumptions 1-5 assume that for an $\alpha_0 \leq \alpha$ and a $k_0 \geq 0$ it holds that

$$V(x^{k_0}) \leq \alpha_0 \quad \text{and} \quad \|\Delta w^{k_0}\| \leq \rho\sqrt{\alpha_0}.$$

Then all iterates $k \geq k_0$ are well-defined and also satisfy $V(x^k) \leq \alpha_0$ and $\|\Delta w^k\| \leq \rho\sqrt{\alpha_0}$. Moreover, for $k \geq k_0 + k_1 + k_2$

$$V(x^k) \leq \frac{1}{4}\alpha_0 \quad \text{and} \quad \|\Delta w^k\| \leq \rho\sqrt{\frac{1}{4}\alpha_0}.$$

Proof: We prove the lemma in three steps: invariance of level sets, attractivity of a small level set for x^k and reduction of the Newton steps $\|\Delta w^k\|$.

Step 1: Well-definedness of all iterates and invariance of the level sets.

The proof is by induction. We assume that for some $k \geq k_0$ it holds that $V(x^k) \leq \alpha_0$ and $\|\Delta w^k\| \leq \rho\sqrt{\alpha_0}$. We will show that then the next real-time iterate is well-defined and remains in the level sets, i.e., $V(x^{k+1}) \leq \alpha_0$ and $\|\Delta w^{k+1}\| \leq \rho\sqrt{\alpha_0}$.

First note that by the definition of ρ in (26), by $\alpha_0 \leq \alpha$, and by the first upper bound of η in Inequality (23)

$$\|\Delta w^k\| \leq \frac{2\eta}{\sqrt{m}(1-(\delta+\sigma))}\sqrt{\alpha} \leq \frac{2\frac{1}{2}\sqrt{\frac{m}{\alpha}}(1-(\delta+\sigma))d}{\sqrt{m}(1-(\delta+\sigma))}\sqrt{\alpha} = d,$$

i.e., $(x^k, w^k) \in \Xi$ and the real-time iterate is well-defined. Now, due to Assumption 2 $\|x^k\| \leq \sqrt{\frac{\alpha_0}{m}}$. By Lemma 4 we know that if $x^{k+1} \in X_\alpha$ then

$$\|\Delta w^{k+1}\| \leq (\delta+\sigma)\|\Delta w^k\| + \eta\|x^k\| \leq (\delta+\sigma)\rho\sqrt{\alpha_0} + \eta\sqrt{\frac{\alpha_0}{m}}$$

and therefore, using the definition of ρ in (26),

$$\|\Delta w^{k+1}\| \leq \rho\sqrt{\alpha_0}\left((\delta+\sigma) + \frac{1}{2}(1-(\delta+\sigma))\right) = \rho\sqrt{\alpha_0}\frac{1+\delta+\sigma}{2} \leq \rho\sqrt{\alpha_0}.$$

It remains to be shown that $x^{k+1} \in X_{\alpha_0} \subset X_\alpha$. To show this we first observe that due to the second upper bound of η in Inequality (23) in Assumption 4 we have

$$\rho \leq \sqrt{\frac{m}{8(M+m)\mu}}$$

and therefore

$$\mu\|\Delta w^{k_0}\|^2 \leq \frac{m}{8(M+m)}\alpha_0 =: \epsilon_0. \tag{31}$$

By Theorem 7 we obtain $V(x^{k+1}) \leq V(x^k) + \epsilon_0 - m\|x_k\|^2$. We now easily verify that $V(x^{k+1}) \leq \alpha_0$ (i.e., $x^{k+1} \in X_{\alpha_0}$), by distinguishing the following two cases:

a) $m\|x^k\|^2 \geq 2\epsilon_0$: we have $V(x^{k+1}) \leq V(x^k) - \epsilon_0 \leq \alpha_0 - \epsilon_0$. $\surd$

b) $m\|x^k\|^2 \leq 2\epsilon_0$: because of $V(x^k) \leq M\|x^k\|^2$ we have that $V(x^k) \leq 2\frac{M}{m}\epsilon_0$ and therefore $V(x^{k+1}) \leq 2\frac{M}{m}\epsilon_0 + \epsilon_0 = \frac{1}{4}\alpha_0$ by the definition of ϵ_0 in (31). $\surd$

This completes the first step of the proof.

Step 2: Attraction of the states x^k for $k \geq k_0 + k_1$ by the level set $X_{\frac{1}{4}\alpha_0}$.

We already showed that all iterates are well-defined and satisfy $V(x^k) \leq \alpha_0$ and $\|\Delta w^k\| \leq \rho\sqrt{\alpha_0}$, and furthermore that $\mu\|\Delta w^k\|^2 \leq \epsilon_0$.

To prove the stronger result that the states x^k with $k \geq k_0 + k_1$ are in the reduced level set $X_{\frac{1}{4}\alpha_0} = \{x \in X | V(x) \leq \frac{1}{4}\alpha_0\}$, we first show that if $V(x^{k'}) \leq \frac{1}{4}\alpha_0$ then also $V(x^{k'+1}) \leq \frac{1}{4}\alpha_0$. We do this again by checking the two cases:

a) $m\|x^{k'}\|^2 \geq 2\epsilon_0$: we have $V(x^{k'+1}) \leq V(x^{k'}) - \epsilon_0 \leq V(x^{k'}) \leq \frac{1}{4}\alpha_0$. $\surd$

b) $m\|x^{k'}\|^2 \leq 2\epsilon_0$: as before, we have $V(x^{k'+1}) \leq \frac{1}{4}\alpha_0$. $\surd$

So let us see how many state iterates x^k can at maximum remain outside $X_{\frac{1}{4}\alpha_0}$. First note that if $V(x^k) \geq \frac{1}{4}\alpha_0$ we also have $M\|x^k\|^2 \geq \frac{1}{4}\alpha_0 = 2\frac{M+m}{m}\epsilon_0 \geq 2\frac{M}{m}\epsilon_0$, i.e., $m\|x^k\|^2 \geq 2\epsilon_0$. Therefore, for every iterate that remains outside $X_{\frac{1}{4}\alpha_0}$, case a) holds, and $V(x^{k+1}) \leq V(x^k) - \epsilon_0$. We deduce that $V(x^{k_0+\Delta k}) \leq \alpha_0 - \Delta k \epsilon_0$, and therefore for $k \geq k_0 + \frac{6(M+m)}{m}$ that $V(x^k) \leq \alpha_0 - \frac{6(M+m)}{m}\epsilon_0 = \frac{1}{4}\alpha_0$.

Step 3: Reduction of the steps $\|\Delta w^k\|$ for $k \geq k_0 + k_1 + k_2$ under the level $\rho\sqrt{\frac{1}{4}\alpha_0}$.

We already know that all iterates $k \geq k_0 + k_1$ satisfy $V(x^k) \leq \frac{1}{4}\alpha_0$ and $\|\Delta w^k\| \leq \rho\sqrt{\alpha_0}$. We can now use Corollary 6 with $\|\Delta w^0\|$ replaced by $\rho\sqrt{\alpha_0}$, α_0 replaced by $\frac{1}{4}\alpha_0$, and k_0 replaced by $k - (k_0 + k_1)$, which yields the proposition:

$$\text{If} \quad k - (k_0 + k_1) \geq \log_{\delta+\sigma}\left(\frac{\rho\sqrt{\frac{1}{4}\alpha_0}}{2\rho\sqrt{\alpha_0}}\right) \quad \text{then} \quad \|\Delta w^k\| \leq \rho\sqrt{\frac{1}{4}\alpha_0}.$$

By definition of k_2 this implies $\|\Delta w^k\| \leq \rho\sqrt{\frac{1}{4}\alpha_0}$ for all $k \geq k_0 + k_1 + k_2$. $\square$

6.5 Nominal Stability of Real-Time Iterations Without Shift

Lemma 9 allows us to conclude that each $k_1 + k_2$ iterations, the level of the objective is reduced by a factor of $\frac{1}{4}$. This allows us to state the main result of this paper, nominal stability of the real-time iteration scheme without shift.

THEOREM 10 (Nominal Stability of the Real-Time Iteration Scheme without Shift). *Suppose Assumptions 1-5 and assume that $(x^0, w^0) \in \Xi_{\mathrm{attr}}$. Then all system-optimizer states generated by the combination* (14) *of the nominal system dynamics and the real-time iteration scheme without shift are well-defined, i.e., satisfy $(x^k, w^k) \in \Xi$, and for all integers $p \geq 0$ and $k \geq k_\alpha + p(k_1+k_2)$ it holds that $V(x^k) \leq \alpha\frac{1}{4^p}$ and $\|\Delta w^k\| \leq \rho\sqrt{\alpha}\frac{1}{2^p}$. Therefore, $(x^k, w^k) \to (0,0)$.*

Proof: The theorem is an immediate consequence of Lemma 8 followed by an inductive application of Lemma 9. In Phase 1, Lemma 8 ensures that the first k_α system-optimizer iterates remain in Ξ, and that the optimizer step size shrinks under the level $\|\Delta w^{k_\alpha}\| \leq \rho\sqrt{\alpha}$. This allows us to apply, at the start of Phase 2, Lemma 9 with the constants α_0 and k_0 set to $\alpha_0 := \alpha$ and $k_0 := k_\alpha$. The lemma ensures that after $(k_1 + k_2)$ iterates more, both the objective and the optimizer step are reduced, to levels under $\frac{1}{4}\alpha$ and $\rho\sqrt{\frac{1}{4}\alpha}$, respectively. This allows us to apply the same Lemma 9 again, with the constants chosen to be $\alpha_0 := \frac{1}{4}\alpha$ and $k_0 := k_\alpha + (k_1 + k_2)$. The result is a further decrease of the objective to $\frac{1}{4}$ of its previous value, and the corresponding decrease in the optimizer stepsize. Repeating this procedure inductively yields the desired bounds $V(x^k) \leq \alpha \frac{1}{4^p}$ and $\|\Delta w^k\| \leq \rho\sqrt{\alpha}\frac{1}{2^p}$ for all $k \geq k_\alpha + p(k_1 + k_2)$. Convergence of x^k towards the origin follows from $m\|x^k\|^2 \leq V(x^k) \leq \alpha\frac{1}{4^p}$. □

Remark: From a practical point of view, the derived result can be interpreted as follows: whenever the system state is subject to a disturbance, but such that after the disturbance the combined system-optimizer state is in the region Ξ_{attr}, the subsequent closed-loop response will lead the system towards the origin with a linear convergence rate. The proof should not be seen as a construction rule for designing suitable real-time iteration schemes. Instead it gives a theoretical underpinning of the real-time iteration scheme.

Similar convergence results as for the real-time iteration scheme would also hold true for numerical schemes where more than one Newton type iteration is performed per sampling time, sacrificing, however, the instantaneous feedback of the real-time iteration scheme. In the limit of infinitely many iterations per optimization problem, the set Ξ_{attr} would approach the set Ξ and the whole region of attraction of the ideal NMPC controller would be recovered, cf. Theorem 1.

7 Numerical Experiments: Distillation Control

In order to illustrate the investigated method, tests with a nontrivial process control example, namely a high purity distillation column differential-algebraic equation (DAE) model with 82 differential and 122 algebraic states are performed. For details of the distillation model and the control problem we refer to [12, 18], where also experimental tests of the real-time iteration scheme at a pilot plant distillation column in Stuttgart are presented. The column has two inputs $u = (L_{\mathrm{vol}}, Q)^T$, reflux L_{vol} and heating Q, and shall track two tray temperatures, T_{14} and T_{28}, at the values 88 °C and 70 °C (that are part of the algebraic states).

Though the DAE model is in continuous time, it is straightforward to apply the discrete time setting investigated in the previous sections. For this purpose the time axis is divided into intervals $[t_k, t_{k+1}]$ of length ΔT (in the experiments ΔT is set to the different values 60, 120, 300, 600, and 1200 seconds). On these intervals $[t_k, t_{k+1}]$ we leave the control inputs u^k constant, and compute the trajectories $x_k(t)$ and $z_k(t)$ of differential and algebraic states as the solution of an initial value problem:

$$\dot{x}_k(t) = f_{\mathrm{dae}}(x_k(t), z_k(t), u^k) \tag{32a}$$

$$0 = g_{\mathrm{dae}}(x_k(t), z_k(t), u^k) \tag{32b}$$

$$x_k(t_k) = x^k \tag{32c}$$

Note that the trajectory $x_k(t)$ on the interval $[t_k, t_{k+1}]$ depends on the initial value x^k and the control input u^k only, so that we can refer to it as $x_k(t; x^k, u^k)$. The discrete time system function from Eq. (1) is then simply defined by

$$f(x^k, u^k) := x_k(t_{k+1}; x^k, u^k). \tag{33}$$

The distillation column is open loop stable, but takes very long to reach the steady state (more than 10 000 seconds), which corresponds to controls $u_S = (4.18, 2.49)^T$.

7.1 Optimal Control Problem Formulation

The control horizon is chosen to $T_c = 1200$ seconds. By dividing it by the interval length, we obtain the number of control intervals in the optimal control problem (5) as $N = \frac{T_c}{\Delta T}$.

The terminal penalty term $E(s_N)$ appearing in (5) is obtained by a prediction interval $[T_c, T_p]$ on which the controls are fixed to the setpoint values u_S. The objective contribution of the prediction interval provides an upper bound of the neglected future costs that are due after the end of the control horizon, if T_p is chosen sufficiently long. For the numerical experiments we use a length of $T_p - T_c = 36\,000$ seconds that we consider long enough for approximating the infinite horizon. Thus, we practically satisfy the Assumptions (b) and (c) of Theorem 1, with the local control law $k(x)$ being just the constant setpoint controls u_S. The resulting open-loop NMPC optimal control problem to solve at each sampling instant t_k is given by:

$$\min_{u(\cdot),x(\cdot)} \int_0^{T_p} \left\{ \left\| \begin{bmatrix} T_{14}(\tau)-88 \\ T_{28}(\tau)-70 \end{bmatrix} \right\|_2^2 + 0.01\,\|u - u_S\|_2^2 \right\} \; d\tau \tag{34}$$

subject to the model DAE

$$\left.\begin{array}{rcl} \dot{x}(\tau) & = & f_{\mathrm{dae}}(x(\tau), z(\tau), u(\tau)) \\ 0 & = & g_{\mathrm{dae}}(x(\tau), z(\tau), u(\tau)) \end{array}\right. \qquad \text{for } \tau \in [0, T_p].$$

The initial value is given by: $x(0) = x^k$. We furthermore require that

$$u(\tau) = q_i \quad \text{for all} \quad \tau \in [i\Delta T, (i+1)\Delta T], \quad i = 0, \ldots, N-1,$$

and on the long prediction interval the steady state control u_S is applied, i.e. $u(\tau) = u_S \quad$ for $\tau \in [T_c, T_p]$. This implicitly defines the stage costs $L(s_i, q_i)$ and the final penalty term $E(s_N)$ in (5). It is easy to argue that Assumption 1, $L(s_i, q_i) \geq m\|s_i\|^2$, is satisfied even though we penalize only two of the states in the least squares integrals (34). We only have to assume that the continuous time system dynamics with constant controls will during one sampling time only be able to keep both penalized temperatures at their setpoint values if all components of the initial state s_i are already at the desired setpoint. This is a reasonable assumption that is strongly related to observability of the system with respect to the two temperatures.

In contrast to the simplified problem (5), we also formulate state and control inequality constraints by

$$h(x(i\Delta T), z(i\Delta T), u(i\Delta T) \geq 0 \quad i = 0, \ldots, N,$$

where

$$h(x,z,u) := \begin{bmatrix} D(x,z,u) - D_{\min} \\ B(x,z,u) - B_{\min} \\ u - u_{\min} \\ u_{\max} - u \end{bmatrix}$$

define the lower bounds for the fluxes $D(x,z,u)$ and $B(x,z,u)$ that are determined according to the model assumptions, which should always maintain small positive values, and lower and upper bounds for the controls.

Overall, we claim that all assumptions of Theorem 1 are practically satisfied in this NMPC setup, such that nominal stability of the ideal NMPC scheme would be ensured.

Direct Multiple Shooting for DAE

The practical implementation of the real-time iteration scheme is based on the direct multiple shooting method [8] for *differential algebraic equation (DAE)* models (see [28]), which is very briefly – and in a simplified form – reviewed here. The main idea is as follows: to transform the continuous time problem into an NLP of the form (5), we introduce the node values s_i at times $i\Delta T$, $i = 0, \ldots, N-1$, and compute on each subinterval $\tau \in [i\Delta T, (i+1)\Delta T]$, and also for $\tau \in [T_c, T_p]$, the trajectories $x(\tau)$ and $z(\tau)$ as the solution of an initial value problem, according to (32).

The integral part of the cost function is then evaluated on each interval independently:

$$L_i(s_i, q_i) := \int_{i\Delta T}^{(i+1)\Delta T} \left\| \begin{bmatrix} T_{14}(\tau)-88 \\ T_{28}(\tau)-70 \end{bmatrix} \right\|_2^2 + 0.01 \, \|q_i - u_S\|_2^2 \;\; d\tau \qquad (35)$$

as well as

$$E(s_N) := \int_{T_c}^{T_p} \left\| \begin{bmatrix} T_{14}(\tau)-88 \\ T_{28}(\tau)-70 \end{bmatrix} \right\|_2^2 \;\; d\tau \qquad (36)$$

because $u(\tau) = u_S$ for $\tau \in [T_c, T_p]$. In Figure 2 we show the content of the optimizer for an arbitrary sample iteration.

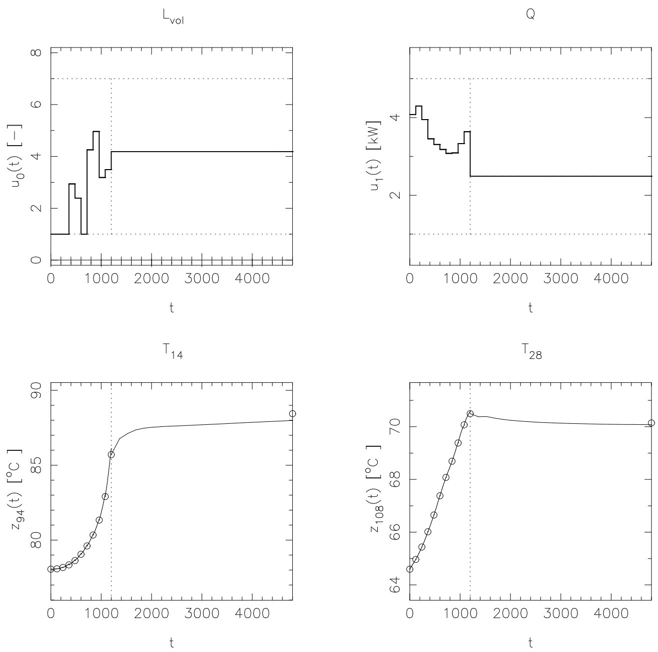

Figure 2. *Snapshot of the optimizer content w^k during a simulation scenario, for a sampling time of $\Delta T = 120$ seconds. The number of multiple shooting nodes is $N = \frac{T_c}{\Delta T} = 10$. The prediction interval at the end of 36 000 seconds length is only partly shown.*

Table 1. *CPU times for the preparation phase in each real-time iteration. The feedback phase was always more than a factor of 100 shorter.*

ΔT [s]	60	120	300	600	1200
N	20	10	4	2	1
CPU [s]	20-45	14-27	11-18	7-14	8-12

For more details on direct multiple shooting and the practical implementation of the real-time iteration scheme we refer to [7, 8, 12, 15, 17, 28].

7.2 Simulation Results and Discussion

Several closed loop simulations with different sampling times ΔT over 3000 seconds are performed. In all scenarios, the initial value x^0 is largely disturbed from steady state (to the value that results after applying a control vector of $u = (2.18, 2.49)^T$ with reduced reflux for 1500 seconds). The real-time iterations are initialized with variables w^0 corresponding to the constant steady state solution. The result of the simulations are given in Figure 3, for ΔT = 60, 120, 300, 600, and 1200 (=T_c) seconds sampling time. One can see that convergence of the closed loop system occurs in every scenario, but that the closed loop performance deteriorates considerably with increasing interval length. This is partly due to the fact that larger intervals mean less freedom for optimization, and partly due to the fact that the nonlinear optimization procedure is more strongly disturbed by larger changes in the initial values, cf. Assumption 4. It is important to note that Assumption 4 might be violated if the state perturbation $\|x\|$ is large and at the same time the sampling time is too long. We claim, however, that for a sufficiently short choice of sampling time ΔT, Assumption 4 can be satisfied even for large variations. This is because the crucial disturbance term in Assumption 4, $f(x, u(x)) - x$, is of $O(\Delta T\|x\|)$, cf. the second remark after Assumption 4. In the limiting case of infinitesimal sampling times, we would recover the ideal NMPC feedback, as the time needed to converge towards the exact NMPC solutions would become infinitely short.

The necessary CPU time for the preparation phase of each real-time iteration was in all scenarios below 60 seconds on an Intel Pentium 4 processor, cf. Table 1. The CPU time is decreasing with the number N of multiple shooting intervals on the control horizon T_c. The feedback phase was always more than two orders of magnitude shorter, with at maximum 200 milliseconds CPU time (for $\Delta T = 60$s).

8 Summary and Conclusions

We have presented a Newton type method for optimization in NMPC – the real-time iteration scheme without shift – and have proven nominal stability of the resulting system-optimizer dynamics. This scheme is characterized by a dovetailing of the dynamics of the system with those of the optimizer, resulting in an efficient online optimization algorithm which, however, shows intricate dynamics that do not allow to apply readily available stan-

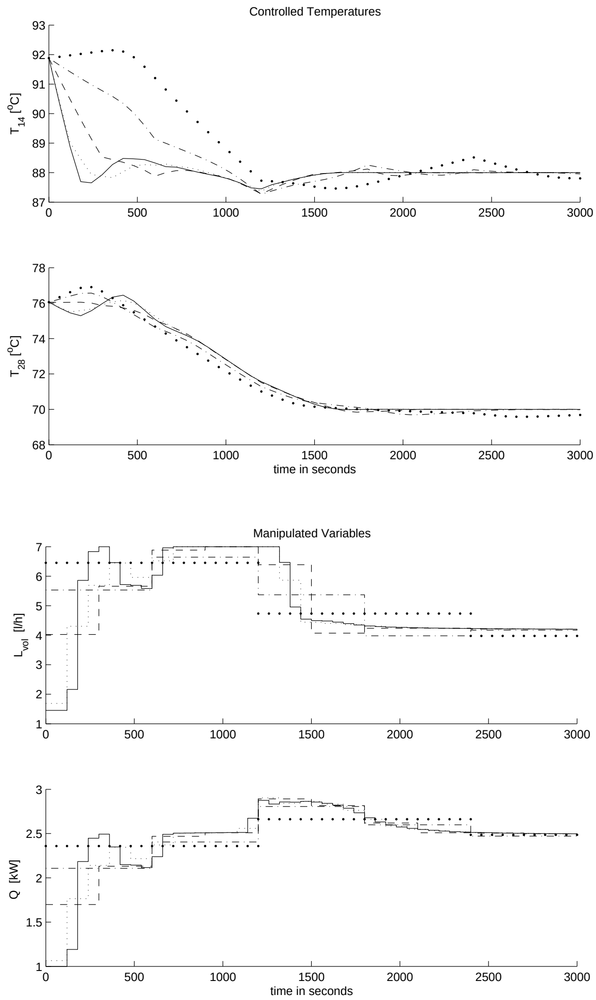

Figure 3. *Comparison of simulation runs with sampling times of $\Delta T = 60$ ($-$), 120 ($\cdots$), 300 ($--$), 600 ($-\cdot-$), and 1200 ($\cdot$) seconds, for application of the real-time iteration scheme to a distillation column model. The optimizer was initialized with the steady state solution.*

dard stability results from NMPC.

The proof of nominal stability makes use of results from both, classical stability theory for NMPC as well as from convergence theory for Newton type optimization methods.

The derived result gives a theoretical underpinning of the real-time iteration scheme, which has already successfully been applied to several example systems, among them a real pilot-plant distillation column [18, 20]. Our numerical results illustrate the practical experience that the real-time iteration scheme is able to bring the system-optimizer dynamics back into the region of attraction even after rather large disturbances, which even holds in the case of strongly unstable systems (cf. [13, 14, 19]).

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