Stability and Incremental Improvement of Suboptimal MPC Without Terminal Constraints

Knut Graichen and Andreas Kugi

***Abstract*—The stability of suboptimal model predictive control (MPC) without terminal constraints is investigated for continuous-time nonlinear systems under input constraints. Exponential stability and decay of the optimization error are guaranteed if the number of optimization steps in each sampling instant satisfies a lower bound that depends on the convergence ratio of the underlying optimization algorithm. The decay of the optimization error shows the incremental improvement of the suboptimal MPC scheme.**

***Index Terms*—Control Lyapunov function (CLF), model predictive control (MPC), optimal control problem (OCP).**

I. Introduction

Model predictive control (MPC) relies on the solution of an optimal control problem (OCP) to predict the system behavior over a time horizon, see, e.g., [1]–[5]. Although the methodology of MPC is naturally suited to handle constraints and multiple-input systems, the iterative solution of the underlying OCP is in general computationally expensive.

An intuitive way to reduce the computational burden in MPC is to compute a suboptimal solution in each sampling step. Several suboptimal MPC approaches exist in the literature both for continuous-time and discrete-time systems. For instance, it is shown in [5]–[8] that an initial feasible trajectory together with a descent condition are sufficient to achieve stability of a suboptimal MPC scheme. The sufficient descent condition can be satisfied if a terminal cost is used that represents a (local) control Lyapunov function (CLF) [9].

In practice, the suboptimal trajectories are often improved by using a number of iterations of an optimization algorithm in each MPC step. For instance, [10] addresses nonlinear systems with bounded inputs and describes an algorithm to compute feasible solutions that achieves asymptotic stability based on a terminal set constraint. Moreover, the real-time iteration approach for nonlinear discrete-time systems described in [11] uses a Newton-type iteration scheme and a terminal equality constraint to achieve asymptotic stability.

Motivated by the above results, the aim of this technical note is to investigate stability of a suboptimal MPC scheme for input-constrained continuous-time nonlinear systems based on a general optimization algorithm with a linear rate of convergence. The MPC scheme does not rely on terminal constraints in order to combine the computational advantages of a suboptimal solution strategy and a free end point formulation. Similar to previous results on MPC without terminal constraints [9], [12], [13], a local CLF on a compact set is assumed to be known.

The first part of the technical note describes the MPC strategy as well as the optimal MPC case. To investigate the suboptimal case, the existence of a general optimization algorithm with a linear rate of convergence is presumed. The optimization error of the algorithm represents a measure of suboptimality in each MPC step that in addition reduces the domain of attraction compared to the optimal MPC case. Based on contraction estimates of the optimal cost value and the optimization error, it is shown that the suboptimal MPC scheme is exponentially stable if the number of iterations performed in each MPC step satisfies a lower bound that depends on the convergence rate of the optimization algorithm. Moreover, the exponential decay of the optimization error shows the self-refining character of the suboptimal MPC scheme.

Several norms are utilized throughout the technical note. The Euclidean norm of a vector $p \in \mathbb{R}^q$ is denoted by $\|p\|$. For time (vector) functions $p(t) \in \mathbb{R}^q$ defined on $t \in [0,T]$ the standard norms $L_q^i(0,T)$ with $i = 1, 2, \infty$ denoted by $\|p\|_{L_q^i(0,T)}$ are used.

II. Problem Statement and Optimal MPC Case

Consider a nonlinear continuous—time system of the form

$$\dot{x} = f(x,u), \quad u(t) \in U \tag{1}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, and (vector valued) pointwise-in-time constraints defined by the compact and convex set U. It is assumed that $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is sufficiently smooth in its arguments and that the origin is an equilibrium of the system for $u = 0$, i.e., $f(0,0) = 0$.

For a given sampling time Δt and the sampling time points $t_k = k\Delta t, k \in \mathbb{N}_0^+$, the goal of the MPC strategy is to determine a stabilizing feedback control law for the nonlinear system (1) based on the solution of the optimal control problem (OCP)

$$\min \quad J(x_k, \bar{u}) := V(\bar{x}(T)) + \int_0^T l(\bar{x}(\tau), \bar{u}(\tau)) \mathrm{d}\tau \tag{2}$$

$$\text{s.t.} \quad \dot{\bar{x}}(\tau) = f(\bar{x}(\tau), \bar{u}(\tau)), \quad \bar{x}(0) = x_k \tag{3}$$

$$\bar{u}(\tau) \in U, \quad \tau \in [0,T] \tag{4}$$

where $x(t_k) = x_k$ is the state of the system (1) at time $t = t_k$. The bar denotes internal variables with respect to the internal time $\tau \in [0,T]$ and the horizon length $T \geq \Delta t$. The integral cost function $l : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_0^+$ and the terminal cost $V : \mathbb{R}^n \to \mathbb{R}_0^+$ are sufficiently smooth in their arguments and satisfy the quadratic bounds $m_l(\|x\|^2 + \|u\|^2) \leq l(x,u) \leq M_l(\|x\|^2 + \|u\|^2)$ and $m_V\|x\|^2 \leq V(x) \leq M_V\|x\|^2$ for some constants m_l, $M_l > 0$ and m_V, $M_V > 0$. By defining the admissible input space as $\mathcal{U}_T = \{\bar{u} \in L_m^\infty(0,T) : \bar{u}(\tau) \in U, \tau \in [0,T]\}$, the OCP (2)–(4) can be expressed in the compact form $\min_{\bar{u} \in \mathcal{U}_T} J(x_k, \bar{u})$.

In the optimal MPC case it is usually assumed that the solution of (2)–(4), i.e., the optimal trajectories $\bar{u}_k^*(\tau)$, $\bar{x}_k^*(\tau)$, $\tau \in [0,T]$ and the optimal cost $J^*(x_k) := J(x_k, \bar{u}_k^*)$ are exactly known. The first part of the optimal control $\bar{u}_k^*(\tau)$ is then used as control input for the system (1) on the sampling interval $[t_k, t_{k+1})$ that corresponds to a nonlinear "sampled" control law of the form

$$u(t_k + \tau) = \bar{u}_k^*(\tau) =: \kappa(\bar{x}_k^*(\tau), x_k), \quad \tau \in [0, \Delta t). \tag{5}$$

In the next MPC step t_{k+1}, the OCP (2)–(4) is solved again with the new initial condition $\bar{x}(0) = x_{k+1}$ that—in the nominal case—is determined by $x_{k+1} = \bar{x}_k^*(\Delta t)$.

A. Basic Assumptions

A.1 There exists an open and non-empty set $\Gamma \subset \mathbb{R}^n$ with $0 \in \Gamma$ s.t. $\forall\ x_k \in \Gamma$, the OCP (2)–(4) has a solution $(\bar{u}_k^*, \bar{x}_k^*)$ and $J^*(x_k)$ is continuous over Γ.

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K. Graichen is with the Institute of Measurement, Control and Microtechnology, University of Ulm, Ulm 89069, Germany (e-mail: knut.graichen@uni-ulm.de).

A. Kugi is with the Automation and Control Institute, Vienna University of Technology, Vienna 1040, Austria (e-mail: kugi@acin.tuwien.ac.at).

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Assumption A.1 concerns the basic domain for the stability considerations. Note that in the case of state and/or terminal constraints (not considered here), the continuity assumption in A.1 can be a delicate matter [14].

There exist various approaches to achieve stability in MPC, e.g., by means of terminal equality constraints [11], [15], [16], terminal set constraints [7], [8], [17], [18], contractive constraints [3], [19] or without terminal constraints [9], [12], [13], [20]. In the latter case, it is standard to assume the existence of a local control Lyapunov function (CLF) that is used as terminal cost V:

A.2 There exists a feedback law $u = q(x) \in U$ and a non-empty compact set $\Omega_\beta = \{x \in \mathbb{R}^n : V(x) \leq \beta\} \subset \Gamma$ such that $\dot{V}(x, q(x)) + l(x, q(x)) \leq 0 \; \forall \, x \in \Omega_\beta$ with $\dot{V} := (\partial V/\partial x) f$.

If the linearization of the system (1) about the origin is stabilizable, then a common choice is $V(x) = x^T P x$, where $P>0$ follows from a Lyapunov or Riccati equation. This results in a linear feedback law of the form $q(x) = -Kx$, which stabilizes the nonlinear system on a (possibly small) set Ω_β [6], [12], [17].

The next assumption is based on the level set of $J^*(x)$ that will appear throughout the technical note

$$\Gamma_\alpha = \{x \in \Gamma : J^*(x) \leq \alpha\}, \; \alpha := \beta \left(1 + \frac{m_l}{M_V} T\right). \tag{6}$$

A.3 For all $u \in \mathcal{U}_T$ and $x_k \in \Gamma_\alpha$, the state trajectory $x(\cdot) := x(\cdot; x_k, u)$ is upper bounded, i.e., there exists a compact set $X \subset \mathbb{R}^n$, s.t. $x(t) \in X \; \forall \, t \in [0, T]$. Moreover, f and $\kappa(\cdot, x_k)$ are Lipschitz on X, i.e., $\exists L_x, L_u, L_\kappa > 0$ s.t $\|f(x, u) - f(y, v)\| \leq L_x\|x - y\| + L_u\|u - v\|$ and $\|\kappa(x, x_k) - \kappa(y, x_k)\| \leq L_\kappa\|x - y\| \; \forall \, u, v \in U, x, y \in X, x_k \in \Gamma_\alpha$.

The Lipschitz assumptions in A.3 is required to derive suitable bounds in various proofs. By means of A.1 and A.3 it can be shown that there exist constants $m_J, M_J > 0$ such that

$$m_J\|x_k\|^2 \leq J^*(x_k) \leq M_J\|x_k\|^2 \quad \forall \, x_k \in \Gamma_\alpha. \tag{7}$$

B. Stability Results in the Optimal MPC Case

An important requirement for the stability of an MPC scheme without terminal constraints is to ensure that the end point of the optimal state trajectory $\bar{x}_k^*(T)$ reaches the set Ω_β of the local CLF. This condition can be used to define the domain of attraction in the optimal MPC scheme. The following lemma extends the results of [9] to continuous-time systems. The proof follows the lines of [9] and uses Gronwall's inequality based on Assumption A.3.

Lemma 1: Suppose that A.1–A.3 hold. Then, $\bar{x}_k^*(T) \in \Omega_\beta$ for all $x_k \in \Gamma_\alpha$. Moreover, $\Omega_\beta \subseteq \Gamma_\alpha$.

Lemma 1 and the definition of Γ_α in (6) reveal that the region of attraction Γ_α can be enlarged by increasing the horizon length T. This property is a well known observation in MPC. Theorem 1 states the stability results of the closed-loop system in the optimal MPC case:

Theorem 1: Suppose that A.1–A.3 hold. Then, for all $x_0 \in \Gamma_\alpha$, the origin of the closed-loop system resulting from the optimal MPC scheme with control (5) is exponentially stable.

Proof: Lemma 1 guarantees that $\bar{x}_k^*(t) \in \Omega_\beta$ for all $x_k \in \Gamma_\alpha$ which is the basis for the well-known result $J^*(x_{k+1}) \leq J^*(x_k) - \int_0^{\Delta t} l(\bar{x}_k^*(\tau), \bar{u}_k^*(\tau)) d\tau$ that relies on the CLF inequality in A.2, see, e.g., [13]. With the bound (22) in the Appendix it is easy to verify that the optimal cost decreases exponentially. In addition, $\|\bar{x}_k^*(\tau)\| \leq e^{\hat{L}\Delta t}\|x_k\|, \tau \in [0, \Delta t)$ following from (20) and $m_J\|x_k\|^2 \leq J^*(x_k)$ show that there exist constants $v_0, v_1 > 0$ such that the closed-loop trajectory $x(t_k + \tau) = \bar{x}_k^*(\tau), \tau \in [0, \Delta t)$ is bounded by $\|x(t)\| \leq v_0\|x_0\|e^{-v_1 t}$ for all $x_0 \in \Gamma_\alpha$ and $t = t_k + \tau \geq 0$, which shows the exponential stability of the origin on Γ_α. ■

III. Suboptimal MPC Strategy

The following suboptimal MPC strategy avoids the computation of the optimal solution $(\bar{u}_k^*, \bar{x}_k^*)$ by performing a fixed number of numerical iterations in each sampling step. This section derives the necessary modifications and differences of the suboptimal MPC strategy compared to the optimal MPC case in order to prove exponential stability.

A. Class of Optimization Algorithms and Incremental Implementation

The basis for the following considerations is the existence of an optimization algorithm that computes a control update $\bar{u}_k^{(j)} \in \mathcal{U}_T$ in each iteration j for an initial state x_k of the system (1). In addition, it is assumed that the algorithm exhibits a linear rate of convergence

$$J\left(x_k, \bar{u}_k^{(j+1)}\right) - J^*(x_k) \leq p\left(J\left(x_k, \bar{u}_k^{(j)}\right) - J^*(x_k)\right), j \in \mathbb{N}_0^+ \tag{8}$$

with ratio $p \in (0, 1)$ and that the optimal cost $J^*(x_k)$ is reached in the limit, i.e., $\lim_{j\to\infty} J\left(x_k, \bar{u}_k^{(j)}\right) = J^*(x_k)$. For instance, a well known and simple optimization algorithm in optimal control is the gradient (projection) method, see, e.g., [21]–[23]. In particular, a linear rate of convergence of the form (8) is proven in [21] under certain system and optimality assumptions.

The optimization algorithm is used in an incremental strategy with a fixed number of iterations in each time step t_k. After the r-th iteration, the first part of the predicted control trajectory $\bar{u}_k^{(r)}(\tau), \tau \in [0, T]$ is used as control input

$$u(t_k + \tau) = \bar{u}_k^{(r)}(\tau), \quad \tau \in [0, \Delta t), \quad k \in \mathbb{N}_0^+. \tag{9}$$

In the nominal case, the system is steered to the next point $x_{k+1} = \bar{x}_k^{(r)}(\Delta t)$, where $\bar{x}_k^{(r)}(\tau)$ with $\bar{x}_k^{(r)}(0) = x_k$ denotes the state trajectory corresponding to $\bar{u}_k^{(r)}(\tau), \tau \in [0, T]$. In the next MPC step, the new initial control is constructed by

$$\bar{u}_{k+1}^{(0)}(\tau) = \begin{cases} \bar{u}_k^{(r)}(\tau + \Delta t) & \text{if} \quad \tau \in [0, T - \Delta t) \\ \bar{u}^q(\tau - T + \Delta t) & \text{if} \quad \tau \in [T - \Delta t, T] \end{cases} \tag{10}$$

where $\bar{u}^q(\tau) = q(\bar{x}^q(\tau)), \tau \in [0, \Delta t]$ and $\bar{x}^q(\tau)$ with $\bar{x}^q(0) = \bar{x}_k^{(r)}(T)$ follow from the CLF feedback law in Assumption A.2. In view of (8) and provided that $\bar{u}_k^{(r)}(\tau)$ is feasible in the sense that $\bar{x}_k^{(r)}(T) \in \Omega_\beta$, it is easy to show that

$$\begin{aligned} J\left(x_{k+1}, \bar{u}_{k+1}^{(r)}\right) &\leq J\left(x_{k+1}, \bar{u}_{k+1}^{(0)}\right) \\ &\leq J\left(x_k, \bar{u}_k^{(r)}\right) \\ &\quad - \int_0^{\Delta t} l\left(\bar{x}_k^{(r)}(\tau), \bar{u}_k^{(r)}(\tau)\right) d\tau. \end{aligned} \tag{11}$$

This property of "feasiblity implies stability" [7] shows that a stabilizing suboptimal MPC scheme can already be obtained by means of the previous (shifted) control trajectory and the CLF law [5], [9]. However, the proof of stability for the MPC scheme in this technical note is more delicate, since the r—step optimization algorithm that improves $\bar{u}_{k+1}^{(0)}(\tau)$ has the side effect that the resulting control trajectory $\bar{u}_{k+1}^{(r)}(\tau)$ does not rely on the CLF law any longer. The remainder of the technical note addresses this case in order to derive conditions for exponential stability.

B. Additional Assumptions

Some further assumptions are required for the stability analysis in the suboptimal MPC case:

A.4 The problem $\min_{\hat{u} \in \mathcal{U}_{\Delta t}} \int_0^{\Delta t} l(\hat{x}(\tau; x_k, \hat{u}), \; \hat{u}(\tau)) d\tau$ has a solution $(\hat{u}_k^*, \hat{x}_k^*) \; \forall \; x_k \in \Gamma_\alpha$. Moreover, the optimal feedback law $\hat{u}_k^*(\tau) =: \hat{\kappa}(\hat{x}_k^*(\tau), x_k)$ is Lipschitz, i.e., $\exists L_{\hat{\kappa}} > 0$ s.t. $\|\hat{\kappa}(x, x_k) - \hat{\kappa}(y, x_k)\| \le L_{\hat{\kappa}}\|x - y\| \;\forall\, x$, $y \in X$ and $\forall\, x_k \in \Gamma_\alpha$.

A.5 The set $\hat{X}$ defined by $\hat{X} = \{x \in \mathbb{R}^n : \|x - x_k\| \le \max_{u \in \mathcal{U}_{\Delta t}} \|x(\Delta t; x_k, u) - x_k\| \;\forall x_k \in \Gamma_\alpha\}$ is contained in Γ. Moreover, $\exists B > 0$ s.t. $\|\nabla J^*(x)\| \le B\|x\|$ and $\|\nabla^2 J^*(x)\| \le B \;\forall\, x \in \hat{X}$.

A.6 The cost functional $J(x_k, u)$ in (2) satisfies the quadratic growth condition $D\|u - u^*\|^2_{L^2_m(0,T)} \le J(x_k, u) - J^*(x_k) \;\forall\, u \in \mathcal{U}_T$, $x_k \in \Gamma_\alpha$ for some constant $D > 0$.

Assumptions A.4 and A.5 are required to derive additional bounds. Assumption A.6 is a quadratic growth condition that relates cost function values to the L^2—norm of the respective arguments. Note that A.6 is always satisfied for linear systems with quadratic cost functional.

C. Stability Results in the Suboptimal MPC Case

It is obvious that the stability results from the optimal MPC case (Section II-B) have to be adapted to account for the suboptimality of the trajectories $\left(\bar{u}_k^{(r)}, \bar{x}_k^{(r)}\right)$. A measure for the suboptimality in each step k is the optimization error between the actual and the optimal cost defined by $\Delta J^{(r)}(x_k) := J\left(x_k, \bar{u}_k^{(r)}\right) - J^*(x_k)$. In the following, both quantities $\Delta J^{(r)}(x_k)$ and $J^*(x_k)$ will be used to characterize the behavior of the suboptimal MPC scheme. The next lemma reveals the interdependency of $J^*(x_k)$ and $\Delta J^{(r)}(x_k)$.

Lemma 2: Suppose that A.1–A.6 are satisfied and that $x_k \in \Gamma_\alpha$ and $\bar{u}_k^{(r)} \in \mathcal{U}_T$ are such that $\bar{x}_k^{(r)}(T) \in \Omega_\beta$. Then, there exist constants $0 < b \le 1$ and $c, d > 0$ such that

$$J^*(x_{k+1}) \le (1 - b)J^*(x_k) + \Delta J^{(r)}(x_k) \tag{12}$$
$$\Delta J^{(r)}(x_{k+1}) \le p^r(1 + c)\Delta J^{(r)}(x_k) + p^r d J^*(x_k). \tag{13}$$

Proof: To prove (12), note that the relation $J^*(x_{k+1}) \le J\left(x_{k+1}, \bar{u}_{k+1}^{(0)}\right)$ and (11) lead to

$$J^*(x_{k+1}) \le J^*(x_k) - \int_0^{\Delta t} l\left(\bar{x}_k^{(r)}(\tau), \bar{u}_k^{(r)}(\tau)\right) d\tau + \Delta J^{(r)}(x_k).$$

Similar to the derivation of (22) in the Appendix (with $\hat{x}_k^*(t)$ instead of $\bar{x}_k^*(t)$), the above integral term can be bounded by

$$\int_0^{\Delta t} l\left(\bar{x}_k^{(r)}(\tau), \bar{u}_k^{(r)}(\tau)\right) d\tau \ge \int_0^{\Delta t} l\left(\hat{x}_k^*(\tau), \hat{u}_k^*(\tau)\right) d\tau \ge bJ^*(x_k)$$

for some constant $0 < b \le 1$. The proof of (13) is based on the overall convergence ratio of the optimization algorithm in step $k + 1$, i.e.

$$\Delta J^{(r)}(x_{k+1}) \le p^r\left(J\left(x_{k+1}, \bar{u}_{k+1}^{(0)}\right) - J^*(x_{k+1})\right) \tag{14}$$

following from (8). The optimal cost $J^*(x_{k+1})$ in (14) is estimated using A.5

$$\begin{aligned} J^*(x_{k+1}) &= J^*(x_k) + \int_0^1 \nabla J^*(x_k + s\Delta x_k)\Delta x_k ds \\ &= J^*(x_k) + \int_0^1 \Bigg[\nabla J^*(x_k) + \\ &\qquad \int_0^s \nabla^2 J^*(x_k + s_2\Delta x_k)\Delta x_k ds_2\Bigg]\Delta x_k ds \\ &\ge J^*(x_k) - B\|x_k\|\|\Delta x_k\| - \frac{1}{2}B\|\Delta x_k\|^2 \end{aligned} \tag{15}$$

where Δx_k is defined as $\Delta x_k := x_{k+1} - x_k$. With (15) and $J\left(x_{k+1}, \bar{u}_{k+1}^{(0)}\right) \le J\left(x_k, \bar{u}_k^{(r)}\right)$ following from (11), the relation (14) can be put into the form

$$\Delta J^{(r)}(x_{k+1}) \le p^r \Delta J^{(r)}(x_k) + p^r\left(B\|x_k\|\,\|\Delta x_k\| + \tfrac{1}{2}B\|\Delta x_k\|^2\right).$$

The estimates (23), (24) in the Appendix finally lead to(13) with c, $d > 0$ following from the respective estimation constants. ∎

In contrast to the optimal case, the relation (12) now depends on the optimization error $\Delta J^{(r)}(x_k)$. This shows that the achievable decrease of the optimal cost in the suboptimal case is naturally smaller than in the optimal MPC case, as it is intuitively expected from theory. In (13) for the optimization error, the right hand side contains the contraction term for the optimization error that is impaired by the optimal cost term $p^r dJ^*(x_k)$.

The stability of the suboptimal MPC scheme is not directly visible due to the coupling of the two inequalities (12) and (13). Evidently, the number of optimization steps r will have a direct influence on the behavior and stability of the suboptimal MPC scheme as it reduces the overall convergence ratio $p^r \in (0, 1)$ of the optimization algorithm for larger r.

However, the investigation of stability requires a deeper look into the reachability properties of the terminal set Ω_β. To re-use Lemma 1, the next lemma states a useful property that relates $\bar{x}_k^{(r)}(T)$ to the end point $\bar{x}_k^*(T)$ of the corresponding optimal trajectory.

Lemma 3: Suppose that Assumptions A.1, A.3, and A.6 are satisfied. Then, there exists a constant $g > 0$ such that for all $x_k \in \Gamma_\alpha$ and $\bar{u}_k^{(r)} \in \mathcal{U}_T$ the end point of the state trajectory $\bar{x}_k^{(r)}(\tau)$ satisfies

$$\left\|\bar{x}_k^{(r)}(T)\right\| \le \|\bar{x}_k^*(T)\| + g\sqrt{\Delta J^{(r)}(x_k)}. \tag{16}$$

Proof: With Assumption A.3 and Gronwall's inequality, we have

$$\begin{aligned} &\left\|\bar{x}_k^{(r)}(\tau) - \bar{x}_k^*(\tau)\right\| \\ &\le \int_0^\tau \left\|f\left(\bar{x}_k^{(r)}(s), \bar{u}_k^{(r)}(s)\right) - f\left(\bar{x}_k^*(s), \bar{u}_k^*(s)\right)\right\| ds \\ &\le L_u e^{L_x \tau}\left\|\bar{u}_k^{(r)} - \bar{u}_k^*\right\|_{L^1_m(0,T)}. \end{aligned} \tag{17}$$

The triangle inequality and $\tau = T$ lead to $\left\|\bar{x}_k^{(r)}(T)\right\| \le \|\bar{x}_k^*(T)\| + L_u e^{L_x T}\left\|\bar{u}_k^{(r)} - \bar{u}_k^*\right\|_{L^1_m(0,T)}$. Finally, applying Hölder's inequality to obtain $\left\|\bar{u}_k^{(r)} - \bar{u}_k^*\right\|_{L^1_m(0,T)} \le \sqrt{T}\left\|\bar{u}_k^{(r)} - \bar{u}_k^*\right\|_{L^2_m(0,T)}$ and using A.6 proves the relation (16) with $g := L_u e^{L_x T}\sqrt{T/D}$. ∎

Lemma 4: Suppose that Assumptions A.1–A.6 hold. Consider $x_k \in \Gamma_{\hat{\alpha}}$ with $\hat{\alpha} := m_V/(4M_V)\alpha$ and let $\bar{u}_k^{(r)} \in \mathcal{U}_T$ be such that the optimization error satisfies $\Delta J^{(r)}(x_k) \le \beta/(4g^2 M_V)$. Then, $\bar{x}_k^{(r)}(T) \in \Omega_\beta$. Moreover, if the horizon length satisfies $T \ge (4M_V/m_V - 1)M_V/m_l$, then $\Omega_\beta \subseteq \Gamma_{\hat{\alpha}}$.

Proof: Note that Lemma 1 equally holds if α and β are replaced by $\hat{\alpha} = \gamma\alpha$ and $\hat{\beta} = \gamma\beta$ with $\gamma \in (0, 1]$. Thus, $x_k \in \Gamma_{\hat{\alpha}}$ implies $\bar{x}_k^*(T) \in \Omega_{\hat{\beta}}$ with $\gamma = m_V/(4M_V) < 1$. Using $m_V\|x\|^2 \le V(x)$, an upper bound on $\bar{x}_k^*(T)$ is given by $\|\bar{x}_k^*(T)\| \le \sqrt{\gamma\beta/m_V} = \sqrt{\beta/(4M_V)}$. Inserting the bounds on $\|\bar{x}_k^*(T)\|$ and $\Delta J^{(r)}(x_k)$ (given in the lemma) in (16) shows that $\left\|\bar{x}_k^{(r)}(T)\right\| \le \sqrt{\beta/(4M_V)} + g\sqrt{\beta/(4g^2 M_V)} = \sqrt{\beta/M_V}$, which guarantees that $\bar{x}_k^{(r)}(T) \in \Omega_\beta$ holds. To prove that $\Gamma_{\hat{\alpha}}$ contains the terminal set Ω_β, note that the bound on T given in Lemma 4 and the definition of α in (6) yields $\hat{\alpha} = m_V/(4M_V)\alpha \ge \beta$. Moreover, consider $x_k \in \Omega_\beta$ and the optimal cost $J^*(x_k) \le V(\bar{x}^q(T)) + \int_0^T l(\bar{x}^q(\tau), \bar{u}^q(\tau))d\tau$, where $\bar{u}^q(\tau)$, $\bar{x}^q(\tau)$ with $\bar{x}^q(0) = x_k \in \Omega_\beta$ follow from the CLF feedback law in Assumption A.2. Integration of the CLF inequality leads to $V(\bar{x}^q(T)) + \int_0^T l(\bar{x}^q(\tau), \bar{u}^q(\tau))d\tau \le V(x_k)$, which shows that $J^*(x_k) \le V(x_k) \le \beta \le \hat{\alpha}$ for all $x_k \in \Omega_\beta$. This proves that $\Gamma_{\hat{\alpha}}$ contains Ω_β. ∎

Lemma 4 restricts the original region of attraction Γ_α to its subset $\Gamma_{\hat{\alpha}}$ as a trade-off for a suboptimality margin of the optimization error

$\Delta J^{(r)}(x_k)$. Nevertheless, $\Gamma_{\hat{\alpha}}$ still contains the terminal region Ω_β if the horizon length T is sufficiently large.

The results of Lemma 4 provide the legitimation to use the bounds (12) and (13) on the optimal cost $J^*(x_k)$ and optimization error $\Delta J^{(r)}(x_k)$ for the stability analysis. The following theorem involves these previous results to state conditions for the exponential stability of the r—step suboptimal MPC strategy.

Theorem 2: Suppose that Assumptions A.1–A.6 hold and that the number of iterations in each MPC step satisfies

$$r > \log_p \left(\frac{\Delta \hat{J}}{(b + bc + d)\hat{\alpha}} \right)$$

with

$$\hat{\alpha} := \frac{m_V}{4M_V}\alpha, \quad \Delta \hat{J} := \min\left\{ \frac{\beta}{4g^2 M_V}, \frac{b m_V}{4M_V}\alpha \right\}.$$

Then, for all initial states $x_0 \in \Gamma_{\hat{\alpha}}$ (i.e., $J^*(x_0) \leq \hat{\alpha}$) and initial control trajectories $\bar{u}_0^{(0)} \in \mathcal{U}_T$ satisfying $\Delta J^{(0)}(x_0) = J\left(x_0, \bar{u}_0^{(0)}\right) - J^*(x_0) \leq \Delta\hat{J}/p^r$ the origin of the closed-loop system resulting from the suboptimal MPC strategy with control (9) is exponentially stable.

Proof: In the first step, r initial optimization iterates lead to $\Delta J^{(r)}(x_0) \leq \Delta \hat{J}$ by repeatedly applying (8). With $\Delta \hat{J} \leq \beta/(4g^2 M_V)$, Lemma 4 leads to $\bar{x}_0^{(r)}(T) \in \Omega_\beta$, which justifies to use the estimates (12) and (13) for the initial point x_0. With $J^*(x_0) \leq \hat{\alpha}$ and $\Delta J^{(r)}(x_0) \leq b\hat{\alpha}$, the bound (12) becomes $J^*(x_1) \leq (1-b)\hat{\alpha} + b\hat{\alpha} = \hat{\alpha}$. To proceed with $\Delta J^{(r)}(x_1)$, we restate the given condition on r as an upper bound on the convergence ratio

$$p^r < \frac{\Delta \hat{J}}{(1+c)b\hat{\alpha} + d\hat{\alpha}} \leq \frac{\Delta \hat{J}}{(1+c)\Delta\hat{J} + d\hat{\alpha}} \tag{18}$$

with $b\hat{\alpha} \geq \Delta\hat{J}$. Using this estimate in (13) together with $J^*(x_0) \leq \hat{\alpha}$ yields $\Delta J^{(r)}(x_1) \leq p^r((1+c)\Delta\hat{J} + d\hat{\alpha}) \leq \Delta\hat{J}$. Hence, $x_1 \in \Gamma_{\hat{\alpha}}$ and $\Delta J^{(r)}(x_1) \leq \Delta\hat{J}$ again imply that $\bar{x}_1^{(r)}(T)$ reaches Ω_β (Lemma 4), which is required to use the estimates (12) and (13). Following the above reasoning for $k = 1, 2, \ldots$ therefore shows that $J^*(x_k) \leq \hat{\alpha}$ and $\Delta J^{(r)}(x_k) \leq \Delta\hat{J}$ for all $k \in \mathbb{N}_0^+$. The exponential stability is proven using the linear discrete-time system

$$\begin{pmatrix} \hat{J}^*_{k+1} \\ \Delta\hat{J}_{k+1} \end{pmatrix} = \begin{pmatrix} 1-b & 1 \\ p^r d & p^r(1+c) \end{pmatrix} \begin{pmatrix} \hat{J}^*_k \\ \Delta\hat{J}_k \end{pmatrix} \tag{19}$$

with $\hat{J}^*_0 = \hat{\alpha}$ and $\Delta\hat{J}_0 = \Delta\hat{J}$. The system yields an upper bound on $J^*(x_k) \leq \hat{J}^*_k$ and $\Delta J^{(r)}(x_k) \leq \Delta\hat{J}_k$ for all $k \in \mathbb{N}_0^+$. The system (19) is asymptotically stable for $0 < b \leq 1$, $c, d > 0$ and $p^r < b/(b + bc + d)$, which can be checked via the eigenvalues or Jury's criterion. The bound on p^r is satisfied in view of (18) and $\Delta\hat{J} \leq b\hat{\alpha}$, which proves the exponential decay of $J^*(x_k)$ and $\Delta J^{(r)}(x_k)$. Moreover, $\left\|\bar{x}_k^{(r)}(\tau)\right\| \leq \|\bar{x}_k^*(\tau)\| + g\sqrt{\Delta J^{(r)}(x_k)}$ (see proof of Lemma 3) together with $m_J\|x_k\|^2 \leq J^*(x_k)$ and $\|\bar{x}_k^*(\tau)\| \leq e^{\hat{L}\Delta t}\|x_k\|$, $\tau \in [0, \Delta t)$ from (20) shows that there exist constants $w_0, w_1 > 0$ such that the closed-loop trajectory $x(t_k+\tau) = \bar{x}_k^{(r)}(\tau), \tau \in [0, \Delta t)$ is bounded by $\|x(t)\| \leq w_0\|x_0\|e^{-w_1 t}, t = t_k + \tau \geq 0$ for all $x_0 \in \Gamma_{\hat{\alpha}}$ and $\Delta J^{(0)} \leq \Delta\hat{J}/p^r$, which shows the exponential stability of the origin on the set $\Gamma_{\hat{\alpha}}$. ■

The proof shows that the min-function for $\Delta\hat{J}$ in Theorem 2 accounts for the boundedness of the optimal cost $J^*(x_k)$ and the optimization error $\Delta J^*(x_k)$ as well as for the reachability of the terminal set Ω_β based on the results of Lemma 4. As mentioned before, the reachability of Ω_β also reduces the domain of attraction to the subset $\Gamma_{\hat{\alpha}}$ of Γ_α, although in practice $\Gamma_{\hat{\alpha}}$ will typically be larger than the conservative estimate with $\hat{\alpha} = m_V/(4M_V)\alpha$. As part of the stability results, the exponential decay of the optimization error also shows the incremental refinement of the suboptimal MPC scheme.

IV. Conclusion

The main advantage of suboptimal MPC schemes is the reduced computational complexity. By initializing an optimization algorithm with the previous solution, the MPC solution is incrementally improved from one sampling step to another. The joint consideration of optimal cost value and optimization error allows to investigate this property in more detail. The number of iterations in each sampling step turns out to be a key parameter for exponential stability and may be used to increase the overall performance of the suboptimal approach. Since no terminal constraints are considered in the proposed MPC strategy, an ideal choice of optimization algorithms are gradient methods, which are particularly suited for free end point problems. This is demonstrated in [24] for a laboratory crane with a sampling time of 2 ms.

Appendix

Some Useful Bounds*:* An upper bound on the optimal predicted state trajectory $\bar{x}_k^*(\tau)$, $\tau \in [0, T]$ can be obtained by using the Lipschitz constants in Assumption A.3 with $f(0,0) = 0$ and $\kappa(0, x_k) = 0 \; \forall \, x_k \in \Gamma_\alpha$

$$\begin{aligned} \|\bar{x}_k^*(\tau)\| &\leq \|x_k\| + \int_0^\tau \|f(\bar{x}_k^*(s), \kappa(\bar{x}_k^*(s), x_k))\| \, ds \\ &\leq \|x_k\| + \hat{L}\int_0^\tau \|\bar{x}_k^*(s)\| \, ds \leq \|x_k\| e^{\hat{L}\tau} \end{aligned} \tag{20}$$

for all $\tau \in [0, T]$ and with $\hat{L} = L_x + L_u L_\kappa$. The second line follows from Gronwall's inequality. A lower bound on $\bar{x}_k^*(\tau)$, $\tau \in [0, T]$ follows from:

$$\begin{aligned} \|\bar{x}_k^*(\tau)\| &\geq \|x_k\| - \int_0^\tau \|f(\bar{x}_k^*(s), \kappa(\bar{x}_k^*(s), x_k))\| \, ds \\ &\geq \|x_k\| - \hat{L}\int_0^\tau \|\bar{x}_k^*(s)\| \, ds \geq \|x_k\| e^{-\hat{L}\tau} \end{aligned} \tag{21}$$

by using an inverse formulation of Gronwall's inequality [25]. In addition, the bound (21) together with (7) and $l(x,u) \geq m_l\|x\|^2$ can be used to establish a lower bound on the integral term

$$\int_0^{\Delta t} l\left(\bar{x}_k^*(\tau), \bar{u}_k^*(\tau)\right) d\tau \geq aJ^*(x_k), 0 < a \leq 1 \tag{22}$$

with $a := m_l/(2\hat{L}M_J)(1 - e^{-2\hat{L}\Delta t})$. Note that $a < 1$ holds if $\|\bar{x}_k^*(\Delta t)\| > 0$ since in this case $J^*(x_k) > \int_0^{\Delta t} l(\bar{x}_k^*(\tau), \bar{u}_k^*(\tau)) \mathrm{d}\tau \geq aJ^*(x_k)$.

Two further estimates are derived in the following lines that are used in the proof of Lemma 2. To this end, define $\Delta\bar{x}(\tau) := \bar{x}_k^{(r)}(\tau) - x_k$ with $\Delta\bar{x}(\Delta t) =: \Delta\bar{x}_k$ and use Assumptions A.3 and A.6 as well as Gronwall's inequality to obtain

$$\begin{aligned} \|\Delta\bar{x}_k\| &\leq \int_0^{\Delta t} \left\| f\left(x_k + \Delta\bar{x}(\tau), \bar{u}_k^{(r)}(\tau)\right)\right\| d\tau \\ &\leq \left(L_x \Delta t\|x_k\| + L_u \left\|\bar{u}_k^{(r)}\right\|_{L^1_m(0,T)}\right) e^{L_x \Delta t} \\ &\leq \mu_1\|x_k\| + \mu_2\sqrt{\Delta J^{(r)}(x_k)} \end{aligned} \tag{23}$$

with some (not explicitly stated) constants $\mu_1, \mu_2 > 0$. In the last line, we made use of the triangle inequality, Hölder's inequality and A.6 to obtain

$$\left\|\bar{u}_k^{(r)}\right\|_{L^1_m(0,T)} \leq \|\bar{u}_k^*\|_{L^1_m(0,T)} + \sqrt{\frac{T}{D}}\sqrt{\Delta J^{(r)}(x_k)}.$$

The second relation used in the proof of Lemma 2 is

$$\begin{aligned}\|x_k\|\sqrt{\Delta J^{(r)}(x_k)} &\le \frac{1}{\sqrt{m_J}}\sqrt{J^*(x_k)\left(J^{(r)}(x_k)-J^*(x_k)\right)}\\ &\le \frac{1}{\sqrt{m_J}}J^{(r)}(x_k)\\ &= \frac{1}{\sqrt{m_J}}\left(J^*(x_k)+\Delta J^{(r)}(x_k)\right). \qquad (24)\end{aligned}$$

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Sensor Deployment for Network-Like Environments

Luca Greco, *Member, IEEE*, Matteo Gaeta, and Benedetto Piccoli

***Abstract*—This technical note considers the problem of optimally deploying omnidirectional sensors, with potentially limited sensing radius, in a network-like environment. This model provides a compact and effective description of complex environments as well as a proper representation of road or river networks. We present a two-step procedure based on a discrete-time gradient ascent algorithm to find a local optimum for this problem. The first step performs a coarse optimization where sensors are allowed to be clustered, to move in the plane, to vary their sensing radius and to make use of a reduced model of the environment called collapsed network. The sensors' positions found in the first step are then projected on the network and used in the second finer optimization, where sensors are constrained to move only on the network. The second step can be performed online, in a distributed fashion, by sensors moving in the real environment, and can make use of the full network as well as of the collapsed one. The adoption of a less constrained initial optimization has the merit of reducing the negative impact of the presence of a large number of local optima. The effectiveness of the presented procedure is illustrated by a simulated deployment problem in an airport environment.**

***Index Terms*—Gradient-like algorithm, network-like environment, optimal deployment, sensor networks.**

I. Introduction

There is a great number of situations that would greatly enjoy the use of network of sensors. They range from surveillance to habitat and environment monitoring, from wild fire detection to search and rescue operations, from exploration to intruder detection. Sensors' network can be static or dynamic (involving moving sensors) and both need to be deployed in the environment. Sensors' deployment problems are strictly related to facilities location-allocation problems, which are the subject of the locational optimization discipline [1]. In locational optimization, objective functions are used to describe the interactions between users and facilities and among them. Users may find facilities desirable [1] or undesirable [2]. Two well known problems, involving

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L. Greco and M. Gaeta are with the DIIMA, University of Salerno, Fisciano (SA) 84084, Italy (e-mail: lgreco@ieee.org; gaeta@crmpa.unisa.it).

B. Piccoli is with the Istituto per le Applicazioni del Calcolo "Mauro Picone," Roma 00185, Italy; (e-mail: b.piccoli@iac.cnr.it).

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