Common Poisson Shock Models: Applications to Insurance and Credit Risk Modelling

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1. Introduction

The following report analyzes using common Poisson shock processes to model dependent event frequencies within two contexts: insurance loss modeling and credit risk modeling. Poisson shock models have been used in reliability to model failure dependence of different system components caused by independent shocks. The common shock model is a natural approach for modelling dependencies in loss frequencies and loss severities in both contexts. The report summarizes the model and its applications in real world examples.

2. Shock in reliability applied to insurance and credit risk modelling

In reliability literature, shock or events that cause failure of different kinds of system components may be fatal or not-necessarily-fatal. Fatal shock models are used when shock always destroys the component. Non-fatal shock models, or not-necessarily-fatal shock models are used when components have a chance of surviving the shock. In either case, the desire is to be able to analyze the behavior of the total number of failures that occur from different types of shocks.

The purpose is to address the interest of modelling different types of losses over a given amount of time. The frequency of different types of losses are assumed to be dependent and that the losses would be those that occur from independent shock processes. In insurance, shock may be natural catastrophes and in credit risk modelling, shock may be various economic events like recessions. A single occurrence of a shock may cause different types of losses and the number of each type of loss is dependent due to the common shock.

The objective of this report is to investigate the use of common Poisson shock models to determine important properties like risk in insurance and credit risk applications. Since variance measures the dispersion or volatility from an average and volatility is a measure of risk, variance is helpful in determining the profit provision for insurance contracts and risk obligors have when they purchase specific assets for their portfolios. Poisson shock models provide insight into the nature of the dependence implied by common shock and how measuring important properties is at the mercy of the sensitivity to the specification of the model parameters.

3. Theoretical Framework

It is assumed that the different types of shocks arrive as independent Poisson processes and the counting processes different types of losses can also then be assumed Poisson.

The following notation defines the general model will be used throughout the report. For different types of shocks or events e = 1,…,m, the Poisson process

$$\{N^{(e)}(t),\ t \geq 0\} \tag{1}$$

counts the number of events until time t and has intensity $\lambda^{(e)}$. For losses j = 1,…,n,

$$\{N_j(t),\ t \geq 0\} \tag{2}$$

is a Poisson process that counts the number of losses until time t and has intensityλ_j.

For each number of occurrences of shock of type e, the Bernoulli random variable $I_{j,r}^{(e)}$ indicates whether a loss of type j occurs or not. Overall, if there is a shock of type e, there could be n different types of losses associated with that shock. This means that over a given time t, there exists $N^{(e)}(t)$ shocks that occur, so therefore there are $N^{(e)}(t)$ Bernoulli variables for the j^th loss. At one single shock, losses j = 1,..n can occur and the Bernoulli random variable equals 1 if it occurs and 0 otherwise. For occurrences $r = 1, \dots, N^{(e)}(t)$, the vectors

$$\boldsymbol{I}_r^{(e)} = (I_{1,r}^{(e)}, \dots, I_{n,r}^{(e)}) \tag{3}$$

are independent and identically distributed with a multivariate Bernoulli distribution.
The loss processes are obtained by super positioning m independent Poisson shock event processes represented in the following equation

$$N_j(t) = \sum_{e=1}^{m} \sum_{r=1}^{N^{(e)}(t)} I_{j,r}^{(e)} \tag{4}$$

The p-dimensional marginal probabilities of the multivariate Bernoulli distribution of vectors $\boldsymbol{I}_r^{(e)}$ is indicated by

$$P(I_{j_1}^{(e)} = i_{j_1}^{(e)}, \dots, I_{j_p}^{(e)} = i_{j_p}^{(e)}) = p_{j_1,\dots,j_p}^{(e)}(i_{j_1}^{(e)}, \dots, i_{j_p}^{(e)}) \tag{5}$$

where $i_{j_1}, \dots i_{j_p} \in \{0,1\}$. In the case of two-dimensional marginals(bivariate indicators) equation (5) derives the following combinations:

$$P(I_{j_1}^{(e)} = 0, I_{j_2}^{(e)} = 0) = p_{j_{j_1},j_2}^{(e)}(0,0) \tag{6}$$

$$P(I_{j_1}^{(e)} = 1, I_{j_2}^{(e)} = 0) = p_{j_{j_1},j_2}^{(e)}(1,0) \tag{7}$$

$$P(I_{j_1}^{(e)} = 0, I_{j_2}^{(e)} = 1) = p_{j_{j_1},j_2}^{(e)}(0,1) \tag{8}$$

$$P(I_{j_1}^{(e)} = 1, I_{j_2}^{(e)} = 1) = p_{j_{j_1},j_2}^{(e)}(1,1) \tag{9}$$

In the independent case, equations (6)-(9) take the form,

$$P(I_{j_1}^{(e)} = i_{j_1}^{(e)}, I_{j_2}^{(e)} = i_{j_2}^{(e)}) = p_{j_1,\dots,j_p}^{(e)}(i_{j_1}^{(e)}, i_{j_2}^{(e)}) = p_{j_1}^{(e)} p_{j_2}^{(e)} \tag{10}$$

which is the product of one-dimensional marginal probabilities. Equation (9) can be generalized for p-dimensional independent marginal probabilities as the product of p one-dimensional marginal probabilities.

The total number of losses is considered a compound Poisson process or the sum of m independent compound Poission distributed random variables represented from the second equality by

$$N(t) = \sum_{j=1}^{n} N_j(t) = \sum_{e=1}^{m} \sum_{r=1}^{N^{(e)}(t)} \sum_{j=1}^{n} I_{j,r}^{(e)} \tag{11}$$

It is easy now to consider adding dependent severities associated with each loss that occurs. A severity could be considered a loss size. At the r^{th} occurrence of shock of type *e*, if a potential loss of type *j* with severity $X_{j,r}^{(e)}$ occurs, then the product $I_{j,r}^{(e)}\ X_{j,r}^{(e)}$ accounts for the loss size where $I_{j,r}^{(e)}$ is independent of $X_{j,r}^{(e)}$. For all occurrences of event type e and for all event types, $X_{j,r}^{(e)}$ random variables are independent and identically distributed with distribution F_j and if different types of potential losses caused by the same event are dependent, then the vector $(X_{1,r}^{(e)}, \ldots, X_{n,r}^{(e)})'$ is assumed to have a single multivariate distribution *F*.

In adding the elements of severity, an aggregate loss process is formed for losses of type j is a compound Poisson process:

$$Z_j(t) = \sum_{e=1}^{m} \sum_{r=1}^{N^{(e)}(t)} I_{j,r}^{(e)}\ X_{j,r}^{(e)} \tag{12}$$

Aggregate loss process caused by losses of all types is a sum of m independent compound Poisson distributed random variables and therefore itself compound Poisson distributed

$$Z(t) = \sum_{e=1}^{m} \sum_{r=1}^{N^{(e)}(t)} \sum_{j=1}^{n} I_{j,r}^{(e)} X_{j,r}^{(e)} = \sum_{e=1}^{m} \sum_{r=1}^{N^{(e)}(t)} \boldsymbol{I_{j,r}^{(e)\prime}}\ \boldsymbol{X_{j,r}^{(e)}} \tag{13}$$

3.1 Effect of Dependent Loss Frequencies

In order to determine the effect of the random vector $(N_j(t), \ldots, N_n(t))$, first it is important to consider the behavior of a single loss frequency and of bivariate dependent loss frequencies. Proposition 1 addresses the expectation of a single loss process in the vector $(N_j(t), \ldots, N_n(t))$, bivariate marginal probabilities, and the covariance and correlation structures.

3.1.1 Proposition 1

1. $\left\{\left(N_j(t), \ldots, N_n(t)\right)', t \geq 0\right\}$ is a multivariate Poisson process with

$$E\left(N_j(t)\right) = t \sum_{e=1}^{m} \lambda^{(e)} p_j^{(e)}$$

2. The two-dimensional marginal distributions are given by

$$P\left(N_j(t) = n_j, N_k(t) = n_k\right)$$
$$= e^{-\lambda t(p_{j,k}(1,1)+p_{j,k}(1,0)+p_{j,k}(0,1))} x \sum_{i=0}^{\min\{n_j,n_k\}} \frac{(\lambda t p_{j,k}(1,1))^i (\lambda t p_{j,k}(1,0))^{n_j-i} (\lambda t p_{j,k}(0,1))^{n_k-i}}{i!\,(n_j-i)!\,(n_k-i)!}$$

Where $\lambda = \sum_{e=1}^{m} \lambda^{(e)}$ and

$$p_{j,k}(i_j, i_k) = \lambda^{-1} \sum_{e=1}^{m} \lambda^{(e)} p_{j,k}^{(e)}(i_j, i_k), i_j, i_k \in \{0,1\}$$

which is referenced from Barlow and Proschan (1975).

4. The covariance and correlation structure is given by

$$cov\big(N_j(t), N_k(t)\big) = t\sum_{e=1}^{m} \lambda^{(e)} p_{j,k}^{(e)}(1,1)$$

And

$$\rho\left(N_j(t), N_k(t)\right) = \frac{\sum_{e=1}^{m} \lambda^{(e)} p_{j,k}^{(e)}(1,1)}{\sqrt{\left(\sum_{e=1}^{m} \lambda^{(e)} p_j^{(e)}\right)\left(\sum_{e=1}^{m} \lambda^{(e)} p_k^{(e)}\right)}}$$

If two types of loss processes are independent, then it must be impossible for both types of losses to be caused by the same event and $p_{j,k}^{(e)}(1,1) = 0$. However, if for at least one event it is possible that both loss types occur, then we have positive correlation between loss numbers and losses are dependent.

Corollary 2.
$N_j(t)$and $N_k(t)$ are independent if and only if $p_{j,k}^{(e)}(1,1) = 0$ for all e. Note that if $p_{j,k}^{(e)}(1,1) = 0$ for all j, k with $j \neq k$ then

$$P\left(I_r^{(e)} = 0\right) = 1 - P(\bigcup_{j=1}^{n} \{ I_{j,r}^{(e)} = 1\})$$
$$= 1 - (\sum_{j=1}^{n} p_j^{(e)} - \sum_{j<k} p_{j,k}^{(e)}(1,1) + \cdots + (-1)^{n-1} p_{1,\ldots n}^{(e)}(1, \ldots, 1)) = 1 - \sum_{j=1}^{n} p_j^{(e)}$$

Corollary 3. If $\sum_{j=1}^{n} p_j^{(e)}$>1 for some e, the $N_j(t), \ldots, N_n(t)$ are not independent.

Corollaries 2 and 3 imply that if $p_{j,k}^{(e)}(1,1) > 0$ for some $j \neq k$, then $N_j(t), \ldots, N_n(t)$ are dependent.

Although the expected number of total losses is the same in both cases of independence and dependence, the total loss number in the dependence case has a higher variance.

3.2 Insurance Example

The following is a simple insurance example that evaluates the different cases of modelling independence and dependence loss frequencies through univariate and bivariate loss indicator probabilities. First, the example considers the case of no common shock and demonstrates its unlikelihood. Then, it considers specifying the modelling with independent indicators and dependent indicators. The example specifies the number of losses n = 2 and the number of shocks e = 3.

Problem parameters:

- t = 5 years
- $N_1(t)$ counts number of windstorm losses in France with $\lambda_1 = 5$ times per year
- $N_2(t)$ counts number of windstorm losses in Germany with $\lambda_2 = 6$ times per year
- $N^{(1)}(t)$ counts west European windstorms with $\lambda^{(1)}$= 4

    - Likely to cause French losses but no German losses
- $N^{(2)}(t)$ counts central European windstorms with $\lambda^{(2)}$= 3
    - Likely to cause German losses but no French losses
- $N^{(3)}(t)$ counts pan European windstorms with $\lambda^{(3)}$= 3
    - Likely to cause both French and German losses
- $p_1^{(1)} = \frac{1}{2}, p_2^{(1)} = \frac{1}{4}, p_1^{(2)} = \frac{1}{6}, p_2^{(2)} = \frac{5}{6}, p_1^{(3)} = \frac{5}{6}, p_2^{(3)} = \frac{5}{6}$
    - Assumed to be estimated by empirical evidence and expert judgement

The example considers three models for the dependence between loss frequencies.
Case 1: No common shocks
In this case, $N(5) = N_1(5) + N_2(5)$ has a Poisson distribution with intensity $\lambda = \lambda_1 + \lambda_2 = 5 + 6$.
However, by Corollary 3 and noting that $p_1^{(3)} + p_2^{(3)} > 1$, the assumption of no common shocks is clearly impossible and unrealistic.
We can check that the loss frequencies and indicator probabilities added up to the assessment of loss frequencies by evaluating

$$\lambda_1 = \lambda^{(1)}p_1^{(1)} + \lambda^{(2)}p_1^{(2)} + \lambda^{(3)}p_1^{(3)} = 4 * \frac{1}{2} + 3 * \frac{1}{6} + 3 * \frac{5}{6} = 5$$
$$\lambda_2 = \lambda^{(1)}p_2^{(1)} + \lambda^{(2)}p_2^{(2)} + \lambda^{(3)}p_2^{(3)} = 4 * \frac{1}{4} + 3 * \frac{5}{6} + 3 * \frac{5}{6} = 6$$

However, the univariate indicator probabilities are insufficient to completely specify the model. To fully specify the model, the dependence structure of the bivariate indicators $(I_1^{(e)}, I_2^{(e)})$ need to be fixed. Two dependence structures are considered.

Case 2: Independent Indicators

$$p_{1,2}^{(e)}(1,1) = p_1^{(e)}p_2^{(e)} \text{for e = 1,2,3} \tag{14}$$

Case 3: Positively dependent indicators

$$p_{1,2}^{(e)}(1,1) = \min\{p_1^{(e)}, p_2^{(e)}\} \geq p_1^{(e)}p_2^{(e)} \tag{15}$$

This specification demonstrates the strongest possible dependence between the indicators or the worst case scenario. Specifically, in the insurance example it demonstrates the following situations:

- $P\left(I_1^{(1)} = 1 \middle| I_2^{(1)} = 1\right) = 1$ or if a west European windstorm causes a German loss, then with certainty causes a French loss
- $P\left(I_2^{(2)} = 1 \middle| I_1^{(2)} = 1\right) = 1$ or if a central European windstorm causes a French loss, then with certainty causes a German loss
- $P\left(I_1^{(3)} = 1 \middle| I_2^{(3)} = 1\right) = 1$, $P\left(I_2^{(3)} = 1 \middle| I_1^{(3)} = 1\right) = 1$ or if a pan-European windstorm causes one kind of loss, then with certainty it causes the other kind of loss.

Given the conclusions following Proposition 1, we expect the variance of the three different cases to be

$$var(N_{Case\ 1}(t)) \leq var(N_{Case\ 2}(t)) \leq var(N_{Case\ 3}(t))$$

The simulation in the next section shows the dependence structure in more detail in order to evaluate case 2 and 3 and the variances of total loss number that result from both cases.

3.3 Simulation

Here, I have provided a simplified version of applying the Poisson shock model to determine the effect of dependent event frequencies similar to the insurance example. In this example, there is only one type of event and an individual may provide a rate for a single shock process. Additionally, it evaluates the process in the case that there could be only two different types of losses that could occur from the single type of event. Therefore, one should provide probabilities for losses of two different types.

In the independent case, the two different loss processes are evaluated independently by generating two uniform random variables for determining if the losses occurred or not at the single occurrence of an event. If the first generated random variable was less than the first loss probability provided then loss of the first type occurred and was recorded for that occurrence of the event. The same procedure was used for the second loss type at the same event occurrence. In this way, both indicators are evaluated independently.

In the dependent case, the independence structure must be broken to determine the dependent bivariate conditional probabilities. The two-dimensional marginal probabilities for the indicator variables are evaluated again using equations (15) which is the strongest possible dependence between the indicators.
We can calculate the additional marginal probabilities given the specification and from the definition of the discrete joint PMFs by using the formulas

$$p_1\left(I_1^{(e)} = 1\right) = \sum_{i_2=0}^{1} p(1, i_2) = p_{1,2}^{(e)}(1,0) + p_{1,2}^{(e)}(1,1)$$

$$p_1\left(I_1^{(e)} = 0\right) = \sum_{i_2=0}^{1} p(0, i_2) = p_{1,2}^{(e)}(0,0) + p_{1,2}^{(e)}(0,1)$$

$$p_1\left(I_2^{(e)} = 1\right) = \sum_{i_1=0}^{1} p(i_1, 1) = p_{1,2}^{(e)}(0,1) + p_{1,2}^{(e)}(1,1)$$

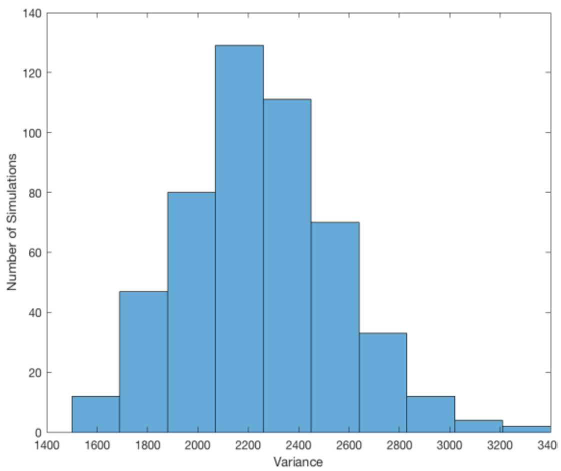
$$p_1\left(I_2^{(e)} = 0\right) = \sum_{i_1=0}^{1} p(i_1, 0) = p_{1,2}^{(e)}(0,0) + p_{1,2}^{(e)}(1,0)$$

$$p_{1,2}^{(e)}(0,0) + p_{1,2}^{(e)}(1,0) + p_{1,2}^{(e)}(0,1) + p_{1,2}^{(e)}(1,1) = 1$$

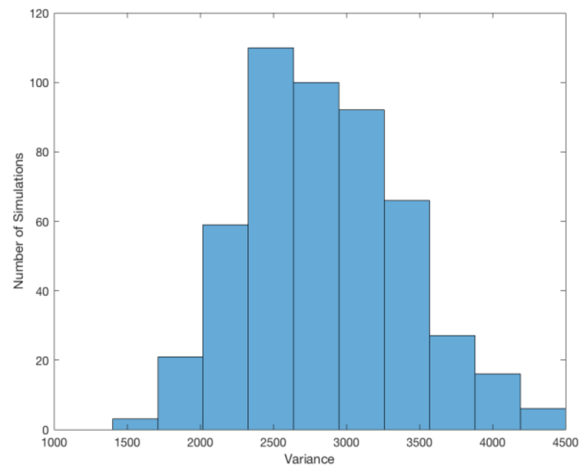
The marginal probabilities were used for generating the values of the discrete indicators $(I_1^{(e)}, I_2^{(e)})$. After generating a random number U that is uniformly distributed over (0,1), we determined if the loss occurred $(I_1^{(e)}, I_2^{(e)} \;= 1)$ or not $(I_1^{(e)}, I_2^{(e)} \;= 0)$ by finding the interval that U lies in based on the two-dimensional marginal probabilities. Hence, there were four intervals to consider and losses were accumulated based on which interval U fell into.

After accumulating loss number of both loss types, the total loss number was calculated using equation (11) with e = 1, n = 2, and for the generated number of event occurrences using the *poissrnd()* function in Matlab.

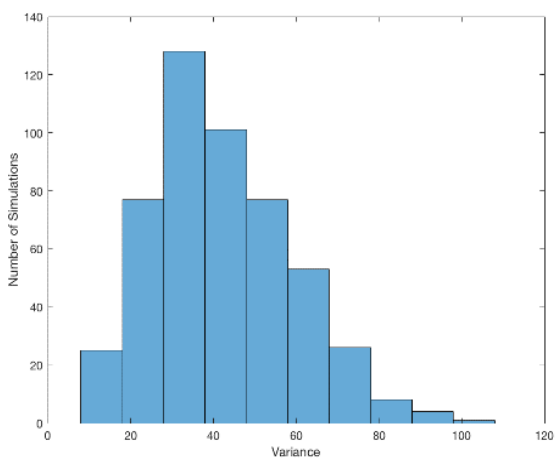
The following histograms represent the simulation of 500 replications calculating the variance of total loss process with losses of two different types. The first chart shows the variance trend with independent indicator probabilities and the second chart shows the variance trend with positive dependent indicator probabilities. For both cases, event intensities are in units of times per year. Histograms are shown for when the time horizon is set to T = 10 years and 100 years while event intensity is fixed at λ=3. Also, histograms are shown for when λ=10 while T=10 is fixed. For all simulations univariate loss probabilities we fixed at $p_1^{(e)} = \frac{1}{4}$ and $p_2^{(e)} = \frac{1}{3}$.



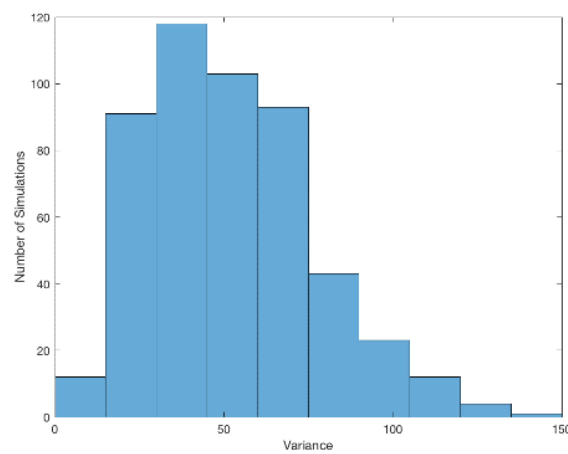
Figure[1]: Histogram of 500 simulations of N(100) with λ=3 for the independent case



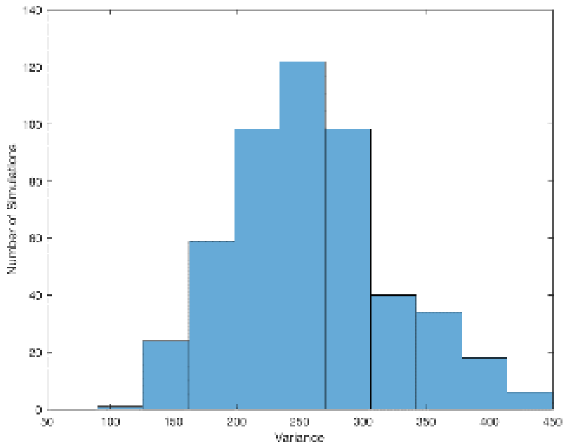
Figure[2]: Histogram of 500 simulations of N(10) with λ=3 for the dependent case



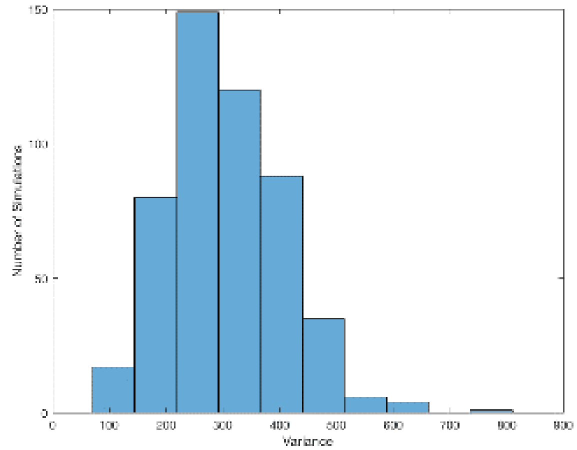
Figure[3]: Histogram of 500 simulations of N(10) with λ=3 for the independent case



Figure[4]: Histogram of 500 simulations of N(10) with λ=3 for the dependent case



Figure[5]: Histogram of 500 simulations of N(10) with λ=10 for the independent case



Figure[6]: Histogram of 500 simulations of N(10) with λ=10 for the dependent case

From the histograms where the event intensity is fixed at 3, it can be seen that the variance of the total loss numbers in the dependence case tends to be higher than the variance of the total loss numbers in the independent case. If we increase the event intensity to 10, the variance in the dependent case again tends to be larger than that of the independent case. From these results, it can be seen that the model is sensitive to dependent parameter specifications.

4 Applying Poisson Shock Models to Portfolio Credit Risk

Poisson shock models can also be used to model dependent defaults. The challenge is to quantify risk in large portfolios of defaultable assets that are dependent. Other industry solutions have been produced by KMV Corporation and Credit Suisse Financial Products. The goal is to accurately evaluate overall portfolio credit risk with default intensity models.

A Poisson shock model for defaults can be consider with the simplest example of a loan portfolio. As mentioned earlier, shocks in the context of credit risk may be various economic events. Four different types of shock events are defined: 1)global shock events that affect all counterparties, 2)sector shock events that affect only certain kinds of companies (ie. Companies in geographical area or companies concentrated in a certain industry), 3)idiosyncratic shock events that affect individual counterparties (ie. Bad management) and 4)endogenous shock events which result from default of primary companies affecting other companies.

In order to set up the model, we consider the times to default or survival times for n individual counterparties in the portfolio to be represented by the vector T = $(T_1, \ldots, T_n)'$, which are the first events in a series of dependent Poisson processes and have a multivariate exponential distribution. It is assumed that nothing is recovered from defaulted firms and that interest rates are neglected. For t in years and exposures (loan sizes) $e_1, \ldots, e_n$, the overall portfolio loss is given by

$$L = \sum_{i=1}^{n} e_i 1\{T_i \leq 1\}$$

where L is fully determined by the set of individual default probabilities $\{p_i = P(T_i \leq 1),\ i = 1, \ldots, n\}$ and the copula C of the vector T. The copula is the more determining factor of the tail of the overall portfolio loss distribution and other risk measures. Common shocks imply a Marshall-Olkin copula to describe the dependence of the survival times. Survival times are conditionally exponential.

5. Conclusion

Poisson shock models present a natural solution to modelling dependent loss frequencies with ease in adding dependent loss severities. The models provide a way for calculating important statistics like variance or tail probabilities for measuring the risk in insurance and risk applications.

Fixing the structure of the indicator probabilities is important for accurately specifying the model and the effect of common shocks. The simulation demonstrates that independent indicators in the model causes a smaller variance than positively dependent indicators, which can be of extreme important in insurance and credit risk applications. Ultimately, the dependent case implies if the event causes the loss with a smaller probability to occur, than with certainty, it will cause the loss the higher probability to occur. Conversely, if the loss with a higher probability does not occur, then with certainty the loss with the lower probability will not occur.

With further investigation, the equivalent fatal shock model is helpful for counting specifically loss-causing shocks as opposed to all shocks from which we can draw a number of non-trivial conclusions about risk measurements. Additionally, with further look into modelling dependent event severities under common shock, it is possible to see that adding dependence between severities can change the performance of the tail behavior of the total aggregate loss numbers (equation(13)).

Finally, it would be important to consider that current theory is moving away from dispersion and more toward tail risk in insurance and credit risk applications. For the insurance example, it is possible to evaluate the tail probabilities analytically the using the bivariate frequency function in Proposition 1 part 2.

6. References

[1] Lindskog, F., McNeil, A.J.(2003) Common Poisson Shock Models: Applications to Insurance and Credit Risk Modelling. *Astin Bulletin*, 33, 209-238.
[2] Bertsekas, D. P., & Tsitsiklis, J. N. (2008). *Introduction to probability*. Belmont (Mass.): Athena Scientific.