Using Finite Volume Method to solve the forward problem in ECT Process Tomography

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Introduction

The basic principles of Electrical Capacitance Tomography (ECT) are to take multiple measurements at the border of a process vessel by placing electrodes on the process vessel. Then these are combined to provide information on the electrical properties of the process volume. One of its applications is the measurement of flow of fluids in industrial pipes. Figure 1 shows an example of an assembly of electrodes on a process vessel.

Figure 1: Electrodes on a pipe

Discretization of the FVM scheme

The Poisson equation for ECT process tomography is

$$\nabla.[\epsilon(\overrightarrow{x})\nabla\phi(\overrightarrow{x})] = 0 \tag{1}$$

where:
$\epsilon(\overrightarrow{x})$ is the dielectric constant
$\phi(\overrightarrow{x})$ is the electric potential which is unknown
$\rho(\overrightarrow{x})$ is the charge density within the sensor
$\overrightarrow{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$
If the charge within the pipeline is zero, then the potential distribution will be,

$$\nabla.[\epsilon(\overrightarrow{x})\nabla\phi(\overrightarrow{x})] = 0 \tag{2}$$

The charge density has been set to 5V for the purpose of this experiment.

Step 1: Take volume integral

$$\int_v \nabla.[\epsilon(\overrightarrow{x})\nabla\phi(\overrightarrow{x})]dv = 0 \tag{3}$$

Step 2: Take surface integral using the gauss divergence theorem

$$\int_s [\epsilon(\overrightarrow{x})\nabla\phi(\overrightarrow{x})]ndS = 0 \tag{4}$$

Step 3: Make piecewise continuous

$$\sum_L \int_{s_L} [\epsilon(\overrightarrow{x})\nabla\phi(\overrightarrow{x})]n_L dS = 0 \tag{5}$$

L is the set of edges in the control volume, it should be noted that the mesh is in 3D, therefore it contains regular tetrahedrons(4 nodes). $\epsilon(\overrightarrow{x})$ is stored at the vertices. We make use of the centroid dual control volume scheme. We can use the midpoint approximation rule to discretize the integrals. Let x_L be the midpoint of L and L_L the length of an edge in L

$$\sum_L [\epsilon(x_L)\nabla\phi(x_L)]n_L L_L = 0 \tag{6}$$

Furthermore, we can use Central difference approximation to approximate $\nabla\phi(x_L)$.

$$\sum_L \left[\frac{\phi_j - \phi_i}{2L_{ij}}\epsilon(x_L)n_L L_L\right] = 0 \tag{7}$$

Matrix assembly

Consider a control volume i, with surrounding tetrahedrons A, B, C, D with nodes, i, j, k, l, m, n, o. We represent $\epsilon(x_L)$ as E for simplicity. Table 1 shows the surrounding tetrahedrons for control volume i, the nodes in the tetrahedrons and the edges that belong to the control volume. Then from (7), we have the

Table 1: Control Volume i

Tetrahedrons	Nodes	Edges in control volume i
A	i,j,k,l	ij, ik, il
B	i,m,k,l	im, ik, il
C	i,m,n,o	in, im, io
D	i,m,l,o	io, im, il

following,

$$\begin{aligned}\frac{\phi_k-\phi_i}{2L_{ki}}E_{ki}L_{BA}+\frac{\phi_l-\phi_i}{2L_{li}}E_{li}L_{BA}+\frac{\phi_m-\phi_i}{2L_{mi}}E_{mi}L_{DB}+\\ \frac{\phi_k-\phi_i}{2L_{ki}}E_{ki}L_{BA}+\frac{\phi_m-\phi_i}{2L_{mi}}E_{mi}L_{CD}+\frac{\phi_o-\phi_i}{2L_{oi}}E_{oi}L_{CD}+\\ \frac{\phi_o-\phi_i}{2L_{oi}}E_{oi}L_{CD}+\frac{\phi_m-\phi_i}{2L_{mi}}E_{mi}L_{DB}+\frac{\phi_l-\phi_i}{2L_{li}}E_{li}L_{DB}\end{aligned} \tag{8}$$

Simplifying further,

$$\begin{aligned}\frac{E_{ki}L_{BA}}{2L_{ki}}(2\phi_k-2\phi_i)+\frac{E_{li}L_{BA}}{2L_{li}}(\phi_l-\phi_i)+\frac{E_{li}L_{DB}}{2L_{li}}(\phi_l-\phi_i)+\\ \frac{E_{mi}L_{DB}}{2L_{mi}}(2\phi_m-2\phi_i)+\frac{E_{mi}L_{CD}}{2L_{mi}}(\phi_m-\phi_i)+\frac{E_{oi}L_{CD}}{2L_{oi}}(2\phi_o-2\phi_i)\end{aligned} \tag{9}$$

Naming coefficients, we have

$$\begin{aligned}a_k(2\phi_k-2\phi_i)+a_{l1}(\phi_l-\phi_i)+a_{l2}(\phi_l-\phi_i)+\\ a_{m1}(2\phi_m-2\phi_i)+a_{m2}(\phi_m-\phi_i)+a_o(2\phi_o-2\phi_i)\end{aligned} \tag{10}$$

From the mathematical derivations above, we can generate a linear system of equations

$$A\phi = b \tag{11}$$

The matrices are assembled as follows:

$$\begin{bmatrix} a_j & a_k & (a_{l1}+a_{l2}) & (a_{m1}+a_{m2}) & a_n & a_o \\ \\ \\ \\ \end{bmatrix} * \begin{bmatrix} \phi_j \\ 2\phi_k-2\phi_i \\ \phi_l-\phi_i \\ (2\phi_m-2\phi_i) \\ (\phi_m-\phi_i) \\ \phi_n \\ 2\phi_o-2\phi_i \end{bmatrix} + \begin{bmatrix} \end{bmatrix} = \begin{bmatrix} b_1 \\ . \\ . \\ . \\ . \\ . \\ b_n \end{bmatrix}$$

Therefore, we have

$$A(i,i) = A(i,i) - \frac{L_L\epsilon(x_L)}{2L_{ij}} \tag{12}$$

$$A(i,j) = A(i,j) + \frac{L_L\epsilon(x_L)}{2L_{ij}} \tag{13}$$

$$b(i) = 5 \tag{14}$$

The system of equations should then be solved for ϕ, however, the RHS is set to 5V for nodes beneath the electrodes and 0 otherwise.

Results

The matrices were assembled as stated above to calculate the ϕ, the electric potential for each of the 32 electrodes. The following figures show the electric fields for Electrode 1, 2, 30, 18.

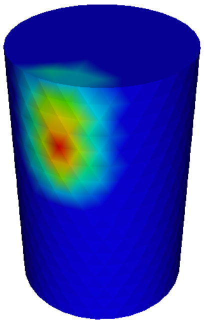

Figure 2: Electrode 1

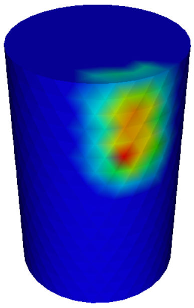

Figure 3: Electrode 2

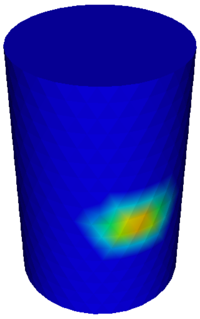

Figure 4: Electrode 18

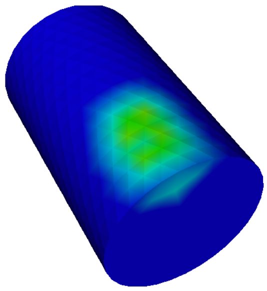

Figure 5: Electrode 30