

Pricing Policies for Selling Indivisible Storable Goods to Strategic Consumers

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Abstract

We study the dynamic pricing problem faced by a monopolistic retailer who sells a storable product to forward-looking consumers. In this framework, the two major pricing policies (or mechanisms) studied in the literature are the preannounced (commitment) pricing policy and the contingent (threat or history dependent) pricing policy. We analyse and compare these pricing policies in the setting where the good can be purchased along a finite time horizon in indivisible atomic quantities. First, we show that, given linear storage costs, the retailer can compute an optimal preannounced pricing policy in polynomial time by solving a dynamic program. Moreover, under such a policy, we show that consumers do not need to store units in order to anticipate price rises. Second, under the contingent pricing policy rather than the preannounced pricing mechanism, (i) prices could be lower, (ii) retailer revenues could be higher, and (iii) consumer surplus could be higher. This result is surprising, in that these three facts are in complete contrast to the case of a retailer selling divisible storable goods [Dudine et al., 2006]. Third, we quantify exactly how much more profitable a contingent policy could be with respect to a preannounced policy. Specifically, for a market with N consumers, a contingent policy can produce a multiplicative factor of $\Omega(\log N)$ more revenues than a preannounced policy, and this bound is tight.

1 Introduction

The problem of determining an optimal pricing policy for a monopolistic retailer has attracted the attention of many researchers from Economics and Operations Research. This interplay

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between a retailer that dynamically sets up prices and consumers who anticipate future price reductions and/or future prices increases may sometimes lead to surprising equilibria. Some relevant questions in this topic are: What does the price path look like in an equilibrium? How profitable can the retailer be, and what pricing policy should it use? Is it possible to compute the optimal pricing policy efficiently? In this paper we study the dynamic pricing problem faced by a retailer (a monopolist) who sells storable goods to forward-looking consumers along a finite time horizon. Storable goods are goods that can be stored for a long period but then loses all value after its first use. Prominent examples of storable goods include non-perishable grocery items and many natural commodities, for example, oil and natural gas. A less conspicuous example concerns digital goods. Digital goods are durable, but the technology often exists to make them perishable and, thus, convert them into storable goods.¹

Dynamic pricing, a practice which is very common in most industries, can be seen as a weak form of price discrimination. The retailer does not dictate how consumers are segmented, but rather consumers segment themselves by self-selecting the time (and corresponding price) at which they prefer to buy the item. A classic and well studied dynamic pricing problem occurs when a monopolist sells a durable good to strategic consumers.² In this setting, Ronald Coase [1972] conjectured that a durable good monopolist who lacks price commitment cannot sell the good above the competitive price. Stokey [1979] then showed that the optimal preannounced pricing policy consists of setting a fixed price.

The Coase conjecture was first proven by Gul et al. [1986] under an infinite time horizon model with non-atomic consumers. However, the conjecture relied upon several assumptions, such as that the time between periods approaches zero and that there is an continuum of consumers. Since then, there have been several studies regarding the performance of different pricing policies for various economic models for durable goods, perishable goods, under different time horizons. Güth and Ritzberger [1998] studied a finite horizon model with a continuum of consumers (i.e. non-atomic demand) and showed that when consumer valuations follow a uniform distribution, there exists a subgame perfect equilibrium, as period lengths approach zero, in which the monopolist profits converge to the static monopoly profits discounted to the beginning of the game. Bagnoli et al. [1989] studied the durable good monopoly problem in which there are finite (atomic) consumers (i.e. discrete demand). When the time horizon is infinite, they proved a surprising result: the existence of a subgame perfect Nash equilibrium in which the duopolist extracts all the consumer surplus. This equilibrium refutes the Coase conjecture when there is a finite set of atomic buyers and an infinite time horizon. Indeed, it shows there exists a subgame perfect Nash equilibrium where duopoly profits exceed static monopoly profits by an unbounded factor. Berbeglia et al. [2014] studied the same model

¹A firm may wish to reduce the lifespan of a durable good to avoid a Coasian outcome, that is, one where the firm loses all monopoly power. Indeed, Amazon has recently been granted a US patent (8,364,595) for a second-hand market in digital goods that restricts the number of times a good can be resold, that is, limits the longevity of the goods. Apple has applied for a similar patent (US 20130060616 A1).

²A *durable good* is a long-lasting good that can be “stored” and consumed repeatedly over time. Examples include digital goods, land and, to a large degree, housing, metals, electronic goods, etc.

as [Bagnoli et al. \[1989\]](#) and proved that when there is a finite time horizon the monopolist profits cannot exceed twice the static monopoly profits. [Dasu and Tong \[2010\]](#) analyzed pre-announced pricing policies for a retailer for the case in which consumers are finite and the number of items are limited (i.e. there is an availability constraint). The authors showed that dynamic pricing (through preannounced pricing) is beneficial for the retailer, but the benefit decreases as the market size increases. [Su \[2007\]](#) studied contingent (subgame perfect or threat) pricing policies in a finite time horizon model where consumers, who arrive continuously, are heterogeneous in their valuation and in their patience. The author characterized the structure of optimal pricing policies and showed that, depending on what population type dominates the market, prices are increasing or decreasing. [Aviv and Pazgal \[2008\]](#) studied contingent pricing and preannounced pricing equilibria for a related two-period model of a retailer selling a finite number of items to strategic consumers that arrive under a Poisson process. The authors proved that a preannounced policy, with respect to contingent pricing, can increase monopoly profits by up to eight percent. [Li \[2001\]](#) consider optimal pricing policies for another model where items are also limited inspired by the revenue management problem faced by airlines. The potential benefit of strategic capacity rationing under preannounced pricing was analyzed by [Liu and van Ryzin \[2008\]](#) in a two period pricing model of a durable good. Their main result is that rationing can be beneficial for the retailer but only when consumers are risk-adverse. Recently, [Correa et al. \[2016\]](#) studied a new type of pricing mechanism so called *contingent preannounced pricing* in which the retailer commits to a price menu which states what would be the future price as a function of the number of available inventory. The authors showed that under this pricing policy, there exists a unique equilibrium when there is a single unit of inventory, but multiple equilibria may exist otherwise. Computational experiments showed that the contingent preannounced pricing outperforms previously proposed pricing mechanisms. Other models of durable good markets that incorporate multi-generation products and the effects of price sensitivity has been studied by [Kuo and Huang \[2012\]](#) and [Liberali et al. \[2011\]](#) respectively.

A related setting in which there is a complex dynamic interaction between a retailer and the consumers is when the good is storable – a *storable good* is a good that can be stored prior to consumption but it is not durable as it perishes upon consumption. Consequently, as stated, such a good can be stored for a long period but then loses all value after its first use. Prominent examples of storable goods include non-perishable grocery items and many natural commodities, for example, oil and natural gas. A less conspicuous example concerns digital goods. As with durable good markets, the consumers in a storable good market can strategically time their purchases by taking into account current and expected future prices. In a storable good market, a consumer who observes a low price promotion sale, may increase her purchases not only to augment the current consumption level (the *consumption effect*) but also to stockpile the good for future consumption (the *stockpiling effect*). There is a large amount of evidence from empirical work showing that the stockpiling effect – also known as *demand anticipation* – is non-negligible in markets for storable goods. For example, [Pesendorfer \[2002\]](#) investigated supermarket price reductions for ketchup products in Springfield, Missouri. His

main finding is that demand at low prices depends on past prices, which suggests the existence of a stockpiling effect. Similarly, [Hendel and Nevo \[2006\]](#) examined the price and demand dynamics of soft drinks, laundry detergents, and yogurt from nine supermarkets in the US. This analysis also showed that demand increases during price reductions are, in part, due to the stockpiling effect. To capture the potential for stockpiling, the models of storable good markets assume that consumers have access to inventory space for an associated cost.³

[Dudine et al. \[2006\]](#) studied a market where the retailer sells a storable good to consumers with time-dependent demand over an arbitrary (but finite) number of time periods. They proved that, under some mild assumptions on consumer demand, if the retailer uses a preannounced price mechanism then consumers will not stockpile any goods in equilibrium. They also showed that (i) prices are higher, (ii) profits are lower, and (iii) consumer surplus is lower if the retailer uses a contingent pricing mechanism rather than a preannounced price mechanism. These results also suggest, unlike the classical results that proved the Coase conjecture in durable good settings, that monopoly power in storable good markets leads to several inefficiencies. The results of [Dudine et al. \[2006\]](#) have later been extended to other settings of a storable good market, namely a two-level vertically integrated monopoly [[Nie, 2009a](#)], and a duopoly market model [[Nie, 2009b](#)]. Other models for storable goods studied in the literature include a continuous time stochastic inventory model for a commodity traded in a spot market [[Chiarolla et al., 2015](#)] and a model where the storable good has a fixed shelf-life [[Herbon, 2014](#)].

1.1 Contributions

A major assumption in the storable good market model of [Dudine et al. \[2006\]](#) is that the good is infinitesimally divisible. In addition, they consider that either there is a single consumer in the market who can always obtain some positive additional utility by consuming an additional fraction of the good (no matter how small that fraction is), or that there is a continuum of non-atomic buyers. In this paper we analyze the market of a retailer selling storable goods in which either (1) there is a finite (possibly very large) number of buyers with a unitary demand per period or (2) there is a single buyer with an arbitrary demand per period but who can only obtain an utility from an integral number of items. Equivalently, the items are *indivisible*.

Our main contributions are as follows. We first characterize the optimal preannounced pricing strategy for the retailer. We then show that it can be computed efficiently and that, under this preannounced price mechanism, the consumers will not stockpile the good. We next show that it is possible for (i) prices to be lower, (ii) retailer’s profits to be higher and (iii) the consumer surplus to be higher, if the retailer uses a contingent pricing mechanism rather than

³These storable goods models are also referred to as *dynamic inventory models*.

a preannounced price mechanism. These last three results are surprising, since they are in complete contrast to the case of a retailer for divisible storable goods [Dudine et al., 2006].

We then quantitatively compare the profitability of the preannounced price and contingent pricing mechanisms. Specifically, for a market with N consumers, a contingent pricing mechanism can generate $\Omega(\log N)$ times more profits than an optimal preannounced price mechanism, and this bound is tight. This is notable because previous models with a finite number of periods have the property that the contingent pricing mechanism cannot significantly outperform the preannounced price mechanism. In particular, we explain how our result contrasts with related results for a durable good retailer. Finally, we examine the setting where inventory costs are concave. Interestingly, the consumers may now utilise storage when the retailer uses a preannounced price mechanism.

2 The Model

Consider a retailer facing consumers with demands for a consumable good over T time periods. The retailer can produce units of the good at a unitary cost u and wishes to select prices $\{p_1, p_2, \dots, p_T\}$ to produce sale quantities $\{x_1, x_2, \dots, x_T\}$ that maximize profits, $\sum_{t=1}^T p_t \cdot q_t$. We assume, without loss of generality, that $u = 0$ and consequently, profit and revenue are interchangeable in this setting. For a perishable good with a short shelf-life, the corresponding optimization problem is straightforward as the firm can optimize in each time period independently. The problem, however, becomes more complex and interesting as the shelf-life grows. In particular, we examine the case of storable goods. Storability offers the consumers the opportunity to purchase the good *before* they actually wish to use it. Indeed, this option is desirable if the resultant price savings exceed storage costs. In accounting for this possibility, the retailer's optimization problem now becomes non-separable.

We consider two cases for this problem: the *single-buyer* and the *multi-buyer* settings. In the former, all the demand comes from a single buyer whose (time-depended) utility depends on the quantity of indivisible items consumed. In the latter, there is a finite number of consumers, each having a single (indivisible) unitary demand per period. In previous studies, the results obtained from these two settings are essentially identical. However, for an indivisible good, these two settings are **not equivalent** and may produce different outcomes. Specifically, in Section 4, we show an example where, in the case of a single buyer with multiple units of demand, the retailer can extract more profit than in the case where there are multiple buyers with unitary demand. Furthermore, in Section 5, we show that the two settings have different outcomes with respect to the use of storage under concave storage costs.

Our interest lies in analysing and comparing two standard pricing mechanisms: the *preannounced price mechanism* and the *contingent pricing mechanism*. In the preannounced price

mechanism, the retailer publicly announces in advance (that is, at time $t = 0$) a price schedule $\{p_1, p_2, \dots, p_T\}$ where p_t is the per unit price in period t . The consumers can then make their purchasing and storage decisions based upon these prices. In the contingent pricing mechanism, the retailer announces prices sequentially over time. Thus, the announced price $p_t = p(\mathcal{H}, t)$ may depend not only on the current time period t , but also on past history \mathcal{H} . Once the price at period p_t is announced, the consumer (or consumers) must decide how many units to purchase and store without knowledge of future prices. For such a sequential game, the solutions we examine are pure subgame perfect Nash equilibria (SPNE). In particular, this restricts our attention to *credible* contingent pricing mechanisms.

2.1 The Single-Buyer Case.

We begin with case where the retailer faces a unique buyer (monopsonist) for its good. Let the utility the buyer receives from consuming x units of the good at period t be given by the function $U(x, t)$. Units are assumed to be indivisible, and therefore the domain of the utility function $U(x, t)$ is $\mathbb{N} \times [T]$. Assume the buyer faces a set of prices $\{p_1, \dots, p_T\}$ and that the storage cost is $c \geq 0$ dollars per unit per time period. The consumer problem then consists of choosing the consumption quantities $\{x_1, x_2, \dots, x_T\}$, the number of units purchased per period $\{q_1, \dots, q_T\}$, and the storage levels $\{S_1, S_2, \dots, S_T\}$. These are chosen to maximize her total utility

$$\sum_{t=1}^T (U(x_t, t) - p_t \cdot q_t - c \cdot S_t)$$

subject to the storage constraints that $S_t = S_{t-1} + q_t - x_t$, for all $t = 1, \dots, T$ where $S_0 = 0$.

The retailer selects the prices $\{p_1, \dots, p_T\}$ in order to maximise the resulting profits $\sum_{t=1}^T p_t \cdot q_t$. As stated, for the preannounced price mechanism the prices are chosen in advance and for the contingent pricing mechanism the prices are chosen sequentially. In analyzing these two mechanisms we make the following standard assumptions. First we assume the buyer's utility function $U(x, t)$ is non-decreasing in x . Second, we assume the marginal utility function is non-increasing in x . Specifically, let $V(x, t) = U(x, t) - U(x - 1, t)$ represent the marginal utility of consuming x units of the good at period t . Then we have that $V(x, t)$ is non-increasing in x . We also assume that there exists $H \in \mathbb{N}$ such that $V(H, t) = 0$ for all $t \in T$. No other restrictions are imposed on the consumer utilities. For notational purposes, we set $V(0, t) = L$ for all $t \in [T]$ where L is a huge number.

2.2 The Multi-Buyer Case.

Next consider the set-up for the multi-buyer case. We now have a set of N consumers each seeking to maximize their own utility. Each consumer i has a demand of at most one unit per period and the value she obtains if she consumes the unit at period t is $u_{i,t}$. A consumer i may, however, purchase multiple units $q_{i,t}$ of the indivisible good in any time period and store them for future consumption. Again, there is a storage cost of c per period and per unit stored. This time the retailer chooses prices $\{p_1, \dots, p_T\}$ to maximize $\sum_{t=1}^T p_t \cdot q_t$, where $q_t = \sum_i q_{i,t}$ is the total number of items sold in period t .

For the preannounced price mechanism, the consumers make their purchasing/storage decisions based upon the entire set of prices. For the contingent pricing mechanism, in each time period, the consumers decide (simultaneously) their purchasing/storage quantities for that period, given the current price. As we shall see, these decisions impose constraints on the retailer's optimization problem. In particular, for the contingent pricing mechanism these constraints force solutions to correspond to pure subgame perfect Nash equilibria (SPNE).

To simplify notation, let $\sigma_t : [N] \rightarrow [N]$ denote the permutation that ranks the N consumers in decreasing values of their utilities in period t and let $v_{i,t} = u_{\sigma^{-1}(i),t}$ for all $i \in [N]$ and all $t \in [T]$. Again, we set $v_{0,t} = L$ where L is a huge number.

2.3 An Example

In this section, we present a simple example to illustrate the proposed model and the concepts involved. Then, in Section 2.4, we consider the continuous version of the same example (i.e. the case in which the goods are divisible) to provide an insight of why the indivisibility of the goods alters some of the results obtained by [Dudine et al. \[2006\]](#).

Consider a two-period game with a per unit storage cost of $c = 1$ and with consumer valuations given in Table 2. Here we assume that we are in the multi-buyer case with $N = 2$; however, it is straightforward to check that our analysis also applies if there is a single-buyer whose marginal utilities for each extra unit are described by the same table.

Table 1: A small example

Consumer	Value at t=1	Value at t=2
1	17	15
2	10	4

We first calculate the optimal *preannounced pricing strategy* for this game. Let p_1 and p_2

denote an optimal sequence of prices under the preannounced pricing policy for periods 1 and 2 respectively. We claim that no consumer will ever buy a unit at period 1 and then store it for consumption in period 2. Otherwise we must have $p_2 > p_1 + 1$ for it to be beneficial to buy early and pay the storage cost. But this cannot be optimal for the retailer as it would then benefit from setting $p_2 = p_1 + 1$. It follows that the optimal choice for p_1 must either be $p_1 = 17$ in which case only consumer 1 buys, or be $p_1 = 10$ in which case both consumers buy. In the former case, the optimal choice for p_2 would be $p_2 = 15$ yielding a total profit of $17 \cdot 1 + 15 \cdot 1 = 32$. In the latter case, the optimal period two price p_2 cannot be higher than 11, otherwise consumer 1 would use storage. Consequently, the optimal price is $p_2 = 11$, which yields the retailer a profit of $10 \cdot 2 + 11 \cdot 1 = 31$. Therefore the optimal preannounced price strategy yields a profit of 32 and consists of setting prices $p_1 = 17$ and $p_2 = 15$.

Next we calculate the optimal *contingent pricing mechanism*. Because prices and actions are chosen sequentially we can compute the optimal contingent pricing mechanism by reasoning backwards from the final time period. So consider time $t = 2$. In order to select p_2 the retailer will base its decision on the set of items currently stored. If consumer 1 has not stored an item from period 1 to 2 then the retailer will set $p_2 = 15$, regardless of the prior decisions made by consumer 2. Knowing this, consumer 1 will purchase two items in period 1 provided $p_1 + c \leq 15$, that is $p_1 \leq 14$. This observation leads to the following four possibilities. (i) The retailer sets $p_1 = 17$. Consumer 1 then purchases one item in each time period giving a profit of $17 \cdot 1 + 15 \cdot 1 = 32$. (ii) The retailer sets $p_1 = 14$ and consumer 1 purchases two units. Given this, the retailer then charges $p_2 = 4$ and consumer 2 buys one unit in the last period. This gives a profit of $14 \cdot 2 + 4 \cdot 1 = 32$. (iii) The retailer sets $p_1 = 10$ and consumer 1 purchases two units and consumer 2 purchases one unit. Again, the retailer then charges $p_2 = 4$ and consumer 2 buys one unit in the last period. This gives a profit of $10 \cdot 3 + 4 \cdot 1 = 34$. (iv) In order to induce consumer 2 to purchase two units in period 1 the retailer will have to charge $p_1 \leq 3$ giving profits of at most $3 \cdot 4 = 12$. Note that option (iii) is a credible threat. If consumer 1 refuses to buy two units in period 1 her utility will fall as the price would then rise to $p_2 = 15$. If consumer 2 refuses to buy one unit in period 1 then his utility will be unchanged. Thus, the optimal contingent pricing mechanism gives total profits of 34.

We now make some observations about the example described. First, observe that the equilibrium under the contingent pricing mechanism involves storage whereas the preannounced pricing solution does not. This observation is in fact general in our model (and it also holds under the model of [Dudine et al. \[2006\]](#)). We prove in [Section 3](#) that regardless of the number of time periods, the valuation functions, or whether the game is single-buyer or multi-buyer, there is an optimal preannounced pricing strategy that induces no storage. Thus, in terms of storage costs, the preannounced pricing mechanism is more efficient than the contingent pricing mechanism. Second, the prices under the preannounced pricing policy are higher than under the contingent pricing mechanism in each period. Observe as well that the retailer profits are lower under preannounced pricing (32) than contingent pricing (34). Finally, the consumer surplus (the total utility obtained by consumers minus money transfers to the retailer and storage costs) is lower under the preannounced pricing policy (0) than with the

contingent pricing mechanism (11). The last three observations are surprising as [Dudine et al. \[2006\]](#) prove that none of those three statements can be true if the retailer is selling a divisible storable good. The following section provides insights about why it makes a difference whether the goods are divisible or indivisible.

2.4 The Impact of Item Divisibility

In order to provide an insight of what is the impact of good divisibility, we consider the model of [Dudine et al. \[2006\]](#) and analyze an example that corresponds to the one described in the previous section. In [Dudine et al. \[2006\]](#) model, there is a unit measure of consumers, and the utility of consuming an amount $x \in [0, 1]$ at period t is $u_t(x)$, where $u_t(x)$ is an increasing and continuous function in $[0, 1]$. For this example, the utility functions $u_t(x)$ are obtained by setting $u_t(x) = \int_0^x U_t(x)dx$ where the functions $U_t(x)$ are defined as follows:

$$U_{t=1}(x) = \begin{cases} 17 & \text{if } 0 \leq x \leq 0.5 \\ 10 & \text{if } 0.5 < x \leq 1 \end{cases}$$

$$U_{t=2}(x) = \begin{cases} 15 & \text{if } 0 \leq x \leq 0.5 \\ 4 & \text{if } 0.5 < x \leq 1 \end{cases}$$

Under the *preannounced pricing policy* it is easy to verify that the optimal prices and quantities bought are the same as in the divisible good settings. Specifically, at period 1, $p_1 = 17$, and a measure of 0.5 consumers buy and consume; and at period 2, $p_2 = 15$ and a 0.5 measure of consumers from period 2 buy and consume. Again here, no storage is used, and the retailer revenue is $\frac{17}{2} + \frac{15}{2} = 16$ ⁴.

Next consider *contingent pricing*. We now show that the price contingent equilibrium we found under the indivisible model cannot be maintained in the [Dudine et al. \[2006\]](#) model. The potential equilibrium that corresponds to the equilibrium for the divisible good model is that at $t = 1$, $p_1 = 10$ and all the complete unit measure of consumers from $t = 1$ buy, as well as 0.5 measure of consumers from period 2 (which they store for later consumption at period 2). Then, at $t = 2$, we have $p_2 = 4$ and the remaining 0.5 measure of consumers buy. The reason why this is not an equilibrium is because the measure of period 2 consumers who store the good from $t = 1$ to $t = 2$ is so large that the retailer will then set a price $p_2 = 4$. To understand this, let $y \leq 0.5$ denote the measure of consumers who buy at $t = 1$ for price

⁴The reason the profit is not 32 as in the case where goods are not divisible is simply because of the scaling: the demand in this example is scaled to a unit per period, which is half of the example from section 2.3

10 and store the good for $t = 2$. Then, the price at $t = 2$ would still be 4 if and only if the retailer obtains more profit by charging 4 dollars to the remaining measure consumers instead of charging 15 dollars to the remaining measure of consumers whose valuation is 15. Therefore, we have that $4 \cdot (1 - y) \geq 15 \cdot (0.5 - y)$. Solving this gives $y \geq \frac{7}{22}$.

Thus, if the retailer sets price $p_1 = 10$ at $t = 1$, all consumers from period 1 buy and a measure of $\frac{7}{22}$ consumers of $t = 2$ buy and store. Then, at period $t = 2$, the retailer sets price $p_2 = 4$ and all remaining consumers buy. The retailer revenue under this specific contingent pricing policy is then $10 \cdot (1 + \frac{7}{22}) + 4 \cdot \frac{15}{22} = \frac{175}{11} \simeq 15.91 < 16$. It can be shown that the optimal contingent pricing strategy is to set $p_1 = 17$ and $p_2 = 15$ so the retailer revenue under contingent pricing in this example is the same as the preannounced pricing policy (see [Dudine et al. \[2006\]](#) for the general characterization of both pricing policies under this model). Table 2 summarizes the differences between the two models.

Table 2: Differences between Divisible and Indivisible Goods: An Example

Mechanism-Model	Retailer Profit	Consumer surplus	Prices
PA/Indivisible	32	0	$p_1 = 17$ $p_2 = 15$
CP/Indivisible	34	11	$p_1 = 10$ $p_2 = 4$
PA/Divisible	16	0	$p_1 = 17$ $p_2 = 15$
CP/Divisible	16	0	$p_1 = 17$ $p_2 = 15$

In essence; using a contingent pricing policy; the retailer can sometimes extract more revenue when the goods are indivisible by performing some price discrimination that the consumer cannot overcome (under the indivisible model one unit was bought at 10, whereas the other unit was bought at 4). If the retailer tries to perform a similar price discrimination under the divisible goods setting, the consumer can finely optimize the (continuous) quantities bought, which makes the price discrimination less effective.

3 Optimal Preannounced Pricing Mechanisms

In this section we derive an efficient procedure to calculate an optimal sequence of prices $\{p_1, \dots, p_T\}$ (price at period t is p_t) that the retailer could announce to maximize its revenue under preannounced pricing. Furthermore, the optimal price sequence is the same regardless of whether we are in the single-buyer or multi-buyer case.

We begin with the following simple observation that holds for the multi-buyer and the single buyer settings.

Observation 1. *Let $\{p_1, \dots, p_T\}$ be any sequence of preannounced prices. If sales at period t are positive then p_t is finite and $p_t \leq p_{t'} + (t - t') \cdot c$ for all $t' < t$.*

The following lemma shows that under preannounced pricing there is no wasteful storage. This “storage-proof” property of the preannounced pricing mechanism extends the result of [Dudine et al. \[2006\]](#) to the case of indivisible demand.

Lemma 3.1. *There always exists a revenue maximising preannounced pricing strategy in which there is no storage. This pricing strategy is the same for the single-buyer and the multiple-buyer settings.*

Now recall that in the multiple-buyer setting, $v_{i,t}$ is the utility of the i^{th} consumer (ranked in decreasing order of their utility values in time period t). In the single-buyer setting, in order to unify notation, we let $v_{i,t} = V(i, t)$ (that is the marginal utility of consuming x units of the good at period t) and let the value of $H \in \mathbb{N}$ such that $V(H, t) = 0$ for all $t \in T$ be equal to $N + 1$.

Lemma 3.1 allows us to write the retailer’s problem as follows. The formulation is valid for the single-buyer model as well as for the multiple-buyer setting.

$$\begin{aligned} \max \quad & \sum_{t=1}^T p_t \cdot q_t \\ \text{s.t.} \quad & p_t \leq v_{q_t, t} \quad \forall t \\ & p_t \leq p_{t'} + (t - t') \cdot c \quad \forall t' < t \quad \text{if } q_t > 0 \\ & q_t \in \mathbb{Z}^+ \quad \forall t \end{aligned}$$

Here q_t is a non-negative integer representing the number of items sold at time t . For the consumer to desire this amount, the price at time t must be at most the corresponding marginal utility – this gives the first constraint. The second constraint states that if there are sales in period t then the price p_t cannot be more than the total cost of purchasing a good in an earlier period and storing it until time t . Recall that we set $v_{0,t} = L$. Thus, if $q_t = 0$ we may set $p_t = L$; this cannot decrease the objective function as it can only weaken the restrictions imposed by the second set of constraints.

The following theorem is crucial for the construction of the polynomial-time algorithm to compute an optimal preannounced pricing strategy.

Theorem 3.2. *There exists optimal preannounced pricing sequence p_1, \dots, p_T such that for each t , we have $p_t = v_{j,s} + (t - s) \cdot c$ for some $s \leq t$ and some $0 \leq j \leq N$.*

Proof. Take an optimal sequence of preannounced pricing sequence (p_1, \dots, p_T) that do not require storage. If these do not satisfy the requirements of the theorem then take smallest time t such that $p_t \neq v_{j,s} + (t - s) \cdot c$ for any $s \leq t$ and for any $j \leq N$. Now, if $q_t = 0$ then set $\hat{p}_t = v_{0,t} = L$. It is easy to verify that the new prices $\{p_1, \dots, \hat{p}_t, \dots, p_T\}$ induce the same behaviour by the consumer or consumers and, thus, produce the same revenue. Moreover,

the period t now satisfies the requirements of the theorem for $s = t$ and $j = 0$. So suppose $q_t \geq 1$. We now set

$$\hat{p}_t = \min\{v_{j,s} + (t - s) \cdot c : s \leq t; 0 \leq j \leq N; v_{j,s} + (t - s) \cdot c > p_t\}$$

We claim $(p_1, \dots, \hat{p}_t, \dots, p_T)$ is a more profitable sequence of preannounced prices. To do this we again desire that the consumer (or consumers) behaviour does not change. If so, the price increase at time t will result in additional revenue. First, the price increase at time t will not affect consumption by the consumer(s) in any period $t' < t$. Second the price increase at time t will not alter consumption in periods $t'' > t$; making additional purchases at time t and storing them for future consumption is even less desirable than before. It remains to consider sales in period t . Recall there were originally q_t sales in period t . Because there was no storage, it must, by consumer rationality, be the case that $v_{q_t+1,t} < p_t \leq v_{q_t,t}$.

Thus, $p_t < v_{q_t,t}$, otherwise p_t does satisfy the requirements of the theorem. It follows that $v_{q_t,t}$ was a feasible option when selecting \hat{p}_t . Hence, $\hat{p}_t \leq v_{q_t,t}$. Thus the price of the item at time t is still at most the marginal value the q_t^{th} unit (in the multi-buyer case, \hat{p}_t is at most the utility at period t of the consumer with the q_t^{th} highest utility in that period). But does the price increase from p_t to \hat{p}_t now induce the consumer or consumers to now purchase early and store for consumption at time t ? This is not the case. Since $q_t > 0$, we know by Observation 1 that $p_t \leq p_s + (t - s) \cdot c$ for all $s < t$. Since by assumption $p_t \neq v_{j,s} + (t - s) \cdot c$ for any $s \leq t$ and for any $j \leq N$, we have that $p_t < p_s + (t - s) \cdot c$. Now, by the minimality of t we have that $p_s = v_{j',r} + (s - r) \cdot c$ for some $r \leq s$ and $0 \leq j' \leq N$. Hence

$$p_t < p_s + (t - s) \cdot c = v_{j',r} + (s - r) \cdot c + (t - s) \cdot c = v_{j',r} + (t - r) \cdot c$$

It then follows, by the definition of \hat{p}_t , that $\hat{p}_t \leq p_s + (t - s) \cdot c$. Consequently, it is not beneficial for the consumer or consumers to use storage and $\{p_1, \dots, \hat{p}_t, \dots, p_T\}$ is a more profitable solution for the retailer. Iterating the argument produces a sequence of preannounced prices that satisfy the requirements of the theorem for every period t . \square

The structure provided by Theorem 3.2 allows us to construct an optimal preannounced pricing strategy in polynomial time via a dynamic program. The key is that we search for prices of the form $p_t = v_{j,s} + (t - s) \cdot c$, for some $s \leq t$ and some $0 \leq j \leq N$. Intuitively, this means we are matching a price p_t with a value $v_{j,s}$. Of course, this correspondence has to satisfy numerous constraints in order to be rational and profit-maximizing.

To explain how to find the optimal correspondence, we begin with some definitions. For each demand utility $v_{i,t}$ we associate a 2D *contour* $\gamma_{i,t}$ to the demand of the i th item in period t . (Recall that demands are sorted by decreasing value in each period, that is, $v_{i,t} \geq v_{i+1,t}$.) We then define a total ordering \preceq over all the contours. Specifically, we say $\gamma_{i,t} \preceq \gamma_{j,r}$ if $v_{i,t} + c \cdot (r - t) \leq v_{j,r}$ and $t \leq r$. To motivate this, assume the consumer is debating whether to purchase and consume j items in period r or to purchase in period $t < r$ and then store

the goods until period r . The best choice is determined by the ordering of the contours $\gamma_{i,t}$ and $\gamma_{j,r}$.

We denote by $\mathcal{C}_t = \{\gamma_{i,t} : i \in [N]\}$ the set of contours associated with demands in period t . Further, we set \mathcal{C}_0 to contain a dummy contour, $\mathcal{C}_0 = \{\gamma^*\}$ where $\gamma \preceq \gamma^*$ for all $\gamma \in \bigcup_{t=1}^T \mathcal{C}_t$. We will view the selection of the dummy contour in time t as the assignment $p_t = L$ for some huge L .

Now suppose we select a contour $\gamma_{j,s}$ at time t . As described, this produces the price $p_t = v_{j,s} + (t - s) \cdot c$. In addition, this choice constrains the price choices for all periods $r > t$. Specifically, such a price p_r must correspond to a contour no higher than $\gamma_{j,s}$. Hence, the choice of contours (excluding dummy contours) is non-increasing over time. To formalize this for the dynamic program we use the following notation. Define $\mathcal{F}_t(\gamma_{i,y}) = \{\gamma \in \mathcal{C}_t \cup \{\gamma_{i,y}\} : \gamma \preceq \gamma_{i,y}\}$ where $\mathcal{F}_t(\gamma_{i,y})$ is the *feasible set* of contours for period t , given that the lowest contour chosen so far is $\gamma_{i,y}$. Given a contour $\gamma_{i,s}$ and a time period $t \geq s$, we define $p_t(\gamma_{i,s}) = v_{i,s} + c \cdot (t - s)$. We then set $q_t(\gamma_{i,s}) = \#\{j \geq 1 : v_{j,t} \geq p_t(\gamma_{i,s})\}$ to be the number of items in period t with value at least $p_t(\gamma_{i,s})$. Note that if there is no storage then $q_t(\gamma_{i,s})$ is exactly the number of sales in period t when prices are $p_t(\gamma_{i,s})$. We are now ready to present a dynamic program that calculates the optimal sequence of preannounced prices in polynomial time.

ALGORITHM 1: Optimal preannounced pricing mechanism

```

 $\mathcal{R}(T+1, \gamma) \leftarrow 0$  for all  $\gamma \in \bigcup_{\tau=1}^T \mathcal{C}_\tau$ .
for  $t = T \rightarrow 1$  do
  for each  $\gamma_{i,s} \in \bigcup_{\tau=0}^{t-1} \mathcal{C}_\tau$  do
     $\mathcal{R}(t, \gamma_{i,s}) \leftarrow \max\{q_t(\gamma_{j,r}) \cdot p_t(\gamma_{i,r}) + \mathcal{R}(t+1, \gamma_{j,r}) : \gamma_{j,r} \in \mathcal{F}_t(\gamma_{i,s})\}$ 
     $\mathcal{S}(t, \gamma_{i,s}) \leftarrow \arg \max_{\gamma_{j,r}} \{q_t(\gamma_{j,r}) \cdot p_t(\gamma_{i,r}) + \mathcal{R}(t+1, \gamma_{j,r}) : \gamma_{j,r} \in \mathcal{F}_t(\gamma_{i,s})\}$ 
  end for
end for
return  $\{p_1(\mathcal{S}(1, \gamma^*)), p_2(\mathcal{S}(2, \mathcal{S}(1, \gamma^*))), \dots\}$ 

```

Theorem 3.3. *The dynamic program computes an optimal preannounce pricing strategy.*

Proof. We want to show that $\mathcal{R}(t, \gamma_{i,s})$ computes the optimal revenue obtainable in period t to T , given that the maximum price we may use in period t is induced by the contour $\gamma_{i,s}$. That is, the maximum price p_t is at most $v_{i,s} + (t - s) \cdot c$. We prove this by backwards induction from the last period. For $t = T$, we have that $\mathcal{R}(t, \gamma_{i,s})$ maximises the quantity times price given the upper bound on price implied by $\gamma_{i,s}$, as desired. Now consider $\mathcal{R}(t, \gamma_{i,s})$, for $t < T$. Again, by Theorem 3.2 there exists an optimal pricing strategy in which the price at period t is equal to $p_t(\gamma_{i,r})$ for some $\gamma_{i,r} \in \bigcup_{\tau=1}^t \mathcal{C}_\tau$. To maximise profits we wish to select a contour in \mathcal{C}_t ; however, the corresponding price must not be above $\gamma_{i,s}$. Thus, the optimal choice, $\mathcal{S}(t, \gamma_{i,s})$, must either be $\gamma_{i,s}$ itself or a lower contour in \mathcal{C}_t . The objective is to maximise revenue in period t plus revenues from periods $t + 1$ to T . The correctness of the procedure follows as the latter has been calculated inductively. \square

The following lemma shows that the algorithm presented above is very efficient in terms of time requirement.

Lemma 3.4. *An optimal sequence $(p_1 \dots, p_T)$ of preannounced pricing can be computed in time $O(T \cdot D^2)$, where T is the number of periods and D is the total number of items in demand ($D \leq N \cdot T$).*

Proof. The outer loop covers T periods. The inner loop takes examines at most D counters (one for each demand item). The two variable assignments require $O(D)$ operations. Together this takes time $O(T \cdot D^2)$. \square

We end this section by noting that for the model with divisible goods, [Dudine et al. \[2006\]](#) provided another dynamic program to compute efficiently the sequence of preannounced prices. One of appealing features of the algorithm presented here is the flexibility to work under any values for the consumer utilities.⁵

4 A Relationship between Preannounced Pricing Profits and Contingent Pricing Profits

Recall, in the divisible storable good [\[Dudine et al., 2006\]](#), retailer profits under a preannounced pricing mechanism are always higher than or equal to those obtained under a contingent pricing mechanism. We saw an example in [Section 2.3](#) showing this need not be the case for an indivisible good. In this section, we quantify exactly how much more profits a contingent pricing mechanism can generate than a preannounced pricing mechanism for an indivisible good. Specifically, let $\Pi^{CP}(\mathcal{G})$ and $\Pi^{PA}(\mathcal{G})$ denote the retailer revenue in the game \mathcal{G} under contingent pricing and preannounced pricing respectively. Our main result is that, for the multi-buyer case with $N(\mathcal{G})$ consumers, there exists an example for which $\Pi^{CP}(\mathcal{G}) = \Omega(\log(N(\mathcal{G})) \cdot \Pi^{PA}(\mathcal{G}))$. Furthermore this result is tight, for any game \mathcal{G} , we have $\Pi^{CP}(\mathcal{G}) = O(\log(N(\mathcal{G})) \cdot \Pi^{PA}(\mathcal{G}))$. Before presenting these results, we give a simple example to show that in the single-buyer case there are examples where the retailer can obtain arbitrarily higher profits using a contingent pricing mechanism rather than a preannounced pricing mechanism.

⁵We remark that the model in [Dudine et al. \[2006\]](#) contains additional assumptions that are not required here, such as the retailer revenue being a concave function of the price in each time period.

Table 3: Single-Buyer Example with Profits arbitrarily Higher under Contingent Pricing than under Preannounced Pricing

Item	1	2	3	...	i	...	N
Marginal Utility in Period $t = 1$	0	0	0	...	0	...	0
Marginal Utility in Period $t = 2$	$1+\epsilon$	$\frac{1}{2}$	$\frac{1}{3}$...	$\frac{1}{i}$...	$\frac{1}{N}$

4.1 The Single-Buyer Case.

Here is a single-buyer example where profits for the contingent pricing mechanism are arbitrarily higher than for the preannounced pricing mechanism

Theorem 4.1. *For any $k \in \mathbb{N}$, there exists a single-buyer storable good game \mathcal{G} such that $\Pi^{CP}(\mathcal{G}) > k \cdot \Pi^{PA}(\mathcal{G})$.*

Proof. We create a game \mathcal{G} with $T = 2$ periods. The buyer receives zero utility from consumption in the first period. In the second period, the buyer has a marginal utility of $1 + \epsilon$ for the first unit of consumption (for some small $\epsilon > 0$), and a marginal utility of $\frac{1}{i}$ for the i th unit of consumption, for $2 \leq i \leq N$. This is summarized in Table 3. The storage costs are assumed to be negligible, that is, $c = 0$.

Using a preannounced pricing strategy the maximum profit obtainable by the retailer is clearly $\Pi^{PA}(\mathcal{G}) = 1 + \epsilon$. On the other hand, the retailer can do far better by using a contingent pricing mechanism. To see this, first consider what happens if the consumer refuses to buy any items in the first period. The price in the second period will then be the static monopoly price of $1 + \epsilon$; the consumer surplus would then be zero. So suppose the retailer announces a price of $p_1 = H_N + \frac{1}{N} - 1 + \epsilon$ in the first period (where $H_N = \sum_{i=1}^N \frac{1}{i}$). As discussed, the consumer generates no surplus if she refuses to buy in period 1. What happens if the consumer buys one unit at price p_1 and stores it for consumption in period 2? Then, in the second period, if the consumer buys $k < N$ additional items, her marginal utility for that final unit of consumption is $\frac{1}{k+1}$. Therefore, given that the consumer bought exactly one unit in period 1, the optimal price for the retailer is $p_2 = \frac{1}{N}$ in the second period. Consumer surplus over the two periods is then

$$\left(1 + \epsilon + \sum_{i=2}^N \frac{1}{i}\right) - (1 \cdot p_1 + (N-1) \cdot p_2) = (H_N + \epsilon) - \left(H_N + \frac{1}{N} - 1 + \epsilon + \frac{N-1}{N}\right) = 0$$

Hence the consumer is no worse off by purchasing one item in the first period at p_1 than if she waits and buys only in the second period. The consumer has negative utility if she purchases more than one unit in period one. Thus, we have an equilibrium where the consumer purchases

exactly one unit in period one with retailer's profits of

$$\Pi^{CP}(\mathcal{G}) = 1 \cdot p_1 + (N - 1) \cdot p_2 = H_N + \frac{1}{N} - 1 + \epsilon + \frac{N - 1}{N} = H_N + \epsilon$$

It is well-known that $H_N > \log N + \frac{1}{2}$. Consequently, we have $\Pi^{CP} \geq \log N \cdot \Pi^{PA}$, as $\epsilon \rightarrow 0$. \square

4.2 The Multi-Buyer Case.

Is it also the case that retailer profits can be arbitrarily higher under contingent pricing mechanisms than under preannounced pricing mechanisms when there are a large number of consumers? We remark that the example used to prove Theorem 4.1 does not extend to the multi-buyer case. To see this, suppose we interpret Table 3 as a game with $N(\mathcal{G}) = N$ buyers each with a unit demand. The optimal prices using the contingent pricing mechanism would then be $p_1 = 1 + \epsilon$ (one consumer buys) and $p_2 = 1/N$ (and the remaining $N - 1$ consumers buy). But this only yields a total profit that is less than $2 \cdot \Pi^{PA}$. Intuitively, the reason why higher profits cannot be extracted is because the N buyers do not cooperate. In contrast, in the single-buyer setting, the retailer is able to exploit the fact that the buyer is willing to buy at a “loss” in period 1 in order to obtain a surplus in period 2.

Nonetheless, we now show, via a far more complex example, that the retailer in the multi-buyer setting can generate a multiplicative factor $\log(N(\mathcal{G}))$ more profits by using a contingent pricing mechanism. This is interesting, and perhaps surprising, in light of the results of [Berbeglia et al. \[2014\]](#) who examined a model of a durable good monopoly problem with N atomic consumers and finite time horizon originally proposed by [Bagnoli et al. \[1989\]](#). In that model, [Berbeglia et al. \[2014\]](#) proved that retailer profits under a contingent pricing mechanism are always within an additive constant (and within at most a multiplicative factor of two) of profits under a preannounced pricing strategy. The model proposed by [Bagnoli et al. \[1989\]](#) corresponds to a special case of the storable good problem studied here. Specifically, to capture their model we simply assume each consumer obtains a positive utility only in the final period, that is, $v_{i,t} = 0$ for all $1 \leq i \leq N$ and for all $1 \leq t \leq T - 1$.⁶ This additive constant bound does not extend to the more general storable goods problem studied in this paper. Again, as in the single-buyer case, the ratio between the profits under a contingent pricing mechanism over the profits under a preannounced pricing strategy can be unbounded.

Theorem 4.2. *There exists an infinite sequence of games $\mathcal{G}_1, \mathcal{G}_2, \dots$, such that the number of consumers is strictly increasing and that $\frac{\Pi^{CP}(\mathcal{G}_i)}{\Pi^{PA}(\mathcal{G}_i)} = \Omega(\log(N(\mathcal{G}_i))) = \Omega(\log T)$.*

⁶Essentially, we can view the all the consumers in the [Bagnoli et al. \[1989\]](#) model as only wishing to consume the durable good in the final period. In those circumstances, it does not matter whether the good is durable or simply storable.

Table 4: Block Sizes and Valuations in the Multi-Buyer Game

Block Number	1	2	3	...	s	...	n
Time Length	2^{n-1}	2^{n-2}	2^{n-3}	...	2^{n-s}	...	$2^{n-n} = 1$
Consumer Value	1	2^1	2^2	...	2^{s-1}	...	2^{n-1}

Proof. Take any integer $n > 2$. We create a game with $T = \sum_{i=1}^n 2^{n-i} = 2^n - 1$ periods. In each period $t \in [T]$, there is exactly one consumer with positive utility for the item; that is, $v_{1,t} > 0$ and $v_{i,t} = 0$, for all $1 \leq t \leq T$ and all $2 \leq i \leq n$. Furthermore, each consumer has a positive utility for the good in only one period. Specifically, we partition the time periods into n blocks, $\{\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_n\}$. Here block \mathcal{B}_k consists of 2^{n-k} consecutive time periods. Note that $\sum_{k=1}^n |\mathcal{B}_k| = 2^n - 1$ and so there are indeed $T = 2^n - 1$ time periods in total. It remains to assign, at each period t , a valuation to the only consumer who wishes to consume in that period. If consumer i is the unique buyer with positive utility in period t then we say that i is the consumer of period t , or that i belongs to period t . (Given a set S of time periods, we say that a consumer belongs to S if the consumer belongs to any period in S .) The consumer belonging to any period in block \mathcal{B}_k , for each $1 \leq k \leq n$, is given a valuation of $v_k = 2^{k-1}$. Again, we assume that storage costs are negligible, that is $c = 0$. Table 4.2 summarizes this construction.

Let's first calculate the retailer profit, $\Pi^{PA}(\mathcal{G})$, under preannounced pricing. To do this, we claim that there is an optimal preannounced pricing strategy in which prices do not change over time. Let $\{p_1, p_2, \dots, p_T\}$ denote an optimal preannounced pricing strategy. Consider the largest t such that $p_t < p_{t+1}$. Because there are no storage costs no consumer in any period s with $s > t$ would buy at price p_{t+1} . Thus, setting a price $p'_{t+1} = p_t$ at period $t + 1$ will not cause the retailer's profits to decrease. On the other hand, take the largest t such that $p_t > p_{t+1}$. If no consumer buys in period t then decreasing the price in period t down to $p'_t = p_{t+1}$ can only increase profits. If at least one consumer does buy in period t at price p_t then that must be the consumer that belongs to period t ; any other consumer will do better by waiting to purchase in period $t + 1$. This means that the consumer from period t has a valuation at least p_t . By the construction of the game, this in turn implies that every consumer belonging to a period s , with $s > t$, also has a valuation at least p_t . Therefore, even with a fixed price of p_t between periods $t + 1$ to T all consumers in periods $t + 1$ to T would still buy the item. This increases monopoly profits. It follows that there is a fixed price strategy that yields optimal profits. Therefore, to compute $\Pi^{PA}(\mathcal{G})$ we can restrict attention to fixed price strategies. It is then easy to verify that the optimal fixed price is 1. Every consumer will then purchase the good giving the retailer a profit of 1 in each time period, and thus of $T = \sum_{i=0}^{n-1} 2^i = 2^n - 1$ in total. (Note also that $N(\mathcal{G}) = T = 2^n - 1$). Hence $\Pi^{PA}(\mathcal{G}) = 2^n - 1$.

Next consider contingent pricing mechanisms. We will show that there is a subgame perfect

equilibrium in which the retailer extracts almost all the consumer surplus. This equilibrium is composed of the following pair of strategies. The strategy of the retailer is to announce a price equal to the value of the highest consumer yet to buy. The strategy of each consumer is to buy the item in the earliest period t where the price is at most her valuation, provided the consumer belongs to a period $s \geq t$. Following Bagnoli et al. [1989], who studied a similar strategy pairing in a different setting, we call the retailer's strategy *pacman* and the consumers strategies *get-it-while-you-can*.

To analyse this we denote by $\mathcal{G}_{i,k}$ a subgame in which the remaining consumers belong to the set of blocks $\{\mathcal{B}_i, \mathcal{B}_{i+1}, \dots, \mathcal{B}_k\}$. That is, the consumers belong to a subset $S \subseteq \bigcup_{j=i}^k \mathcal{B}_j$. Without loss of generality, we may assume that S contains at least one consumer of the block \mathcal{B}_k and that the subgame begins at a period t associated with block \mathcal{B}_i . We wish to prove that, in any subgame $\mathcal{G}_{i,k}$, the *pacman* strategy is a best response to the consumer strategy *get-it-while-you-can*. We proceed by induction on $k - i$. If $k - i = 0$ then all the remaining consumers have a fixed value. It follows immediately that the *pacman* strategy is a best response to *get-it-while-you-can*. Suppose that there are two blocks, that is, $k - i = 1$. Let $|\tilde{\mathcal{B}}_i|$, respectively $|\tilde{\mathcal{B}}_k|$, denote the number of active consumers in block i , respectively k . Note that $|\tilde{\mathcal{B}}_j| \leq |\mathcal{B}_j|$ as the set S in the subgame $\mathcal{G}_{i,k}$ need contain only a subset of the original collection of customers in block \mathcal{B}_j . Using the *pacman* strategy then yields profits at least:

$$|\tilde{\mathcal{B}}_k| \cdot v_k + (|\tilde{\mathcal{B}}_i| - 1) \cdot v_i = |\tilde{\mathcal{B}}_k| \cdot 2 \cdot v_i + (|\tilde{\mathcal{B}}_i| - 1) \cdot v_i = (2 \cdot |\tilde{\mathcal{B}}_k| + |\tilde{\mathcal{B}}_i| - 1) \cdot v_i$$

Observe that in period 2, of the subgame $\mathcal{G}_{i,k}$, we may make only $|\tilde{\mathcal{B}}_i| - 1$ sales as one of the lower value consumers may only wish to purchase in the first period t of the subgame. The only other logical strategy against *get-it-while-you-can* is for the retailer to announce a price of v_i in the first period t . This would cause each consumer to buy in the first time period giving a profit of $(|\tilde{\mathcal{B}}_k| + |\tilde{\mathcal{B}}_i|) \cdot v_i$. But $2 \cdot |\tilde{\mathcal{B}}_k| + |\tilde{\mathcal{B}}_i| - 1 \geq |\tilde{\mathcal{B}}_k| + |\tilde{\mathcal{B}}_i|$, and so the *pacman* strategy is at least as profitable as this.

Before proceeding with the inductive step, denote by τ^P the number of periods required by the *pacman* strategy to sell to (almost) every active consumer in $\mathcal{G}_{i,k}$. Observe that $\tau^P \leq k - i + 1$ but that some consumers may not be able to buy the good. Specifically, the consumers (if any) that do not buy all belong to the first $k - i$ time periods, that is from period t up to period $t + k - i - 1$. All those periods are associated with blocks \mathcal{B}_i and \mathcal{B}_{i+1} , since the number of periods in these two blocks adds up to at least $1 + 2^{k-i-1}$, which is not less than $k - i$ for all $k \geq 2$.

Now if the retailer announces a first price of $v_\alpha = 2^{\alpha-1}$, for some $\alpha < k$, then the active consumers of blocks \mathcal{B}_j with $j \geq \alpha$ will buy due to the *get-it-while-you-can* strategy. Period $t + 1$ then begins a subgame with at least one fewer block in which, by induction, *pacman* is the best response. Let $\Pi^\alpha(\mathcal{G}_{i,k})$ denote the retailer revenue obtained by announcing a first period price of v_α ($\alpha < k$). We can now write necessary and sufficient conditions for the *pacman* strategy to provide a higher or equal revenue than any such deviation. Let τ^α be

the number of periods needed to sell to all the consumers (that buy) if the retailer announces a first period price of v_α and then follows the pacman strategy. Thus, $\tau^\alpha \geq \tau^P - (k - \alpha)$. The corresponding additional surplus generated by selling to potentially more consumers is at most

$$(k - \alpha) \cdot \min[v_\alpha, v_{i+1}]$$

To see this, recall that these $k - \alpha$ consumers belong to blocks \mathcal{B}_i and \mathcal{B}_{i+1} .

On the other hand, by setting an initial price of v_α the retailer cannot obtain full value from the consumers in the block \mathcal{B}_k . There is at least one such consumer, so this causes the retailer to lose at least $v_k - v_\alpha$. To prove that pacman is a best response, it then suffices to show that the potential gains of announcing a price v_α do not compensate the losses. To show this, we have

$$\begin{aligned} \Pi^{pacman}(\mathcal{G}_{i,k}) - \Pi^\alpha(\mathcal{G}_{i,k}) &\geq (v_k - v_\alpha) - (k - \alpha) \cdot \min[v_\alpha, v_{i+1}] \\ &= (2^{k-\alpha} - k + \alpha - 1) \cdot v_\alpha \\ &\geq 0 \quad [\text{as } k - \alpha \geq 1] \end{aligned}$$

It follows that we have a subgame perfect Nash equilibrium. In this equilibrium, the retailer achieves near perfect price discrimination. Every consumer except (possibly) the first $n - 1$ consumers buy the good at exactly their valuation. The revenue obtained is therefore

$$\Pi^{CP} = \sum_{i=1}^n 2^i \cdot 2^{n-i} - (n - 1) \cdot 1 = n \cdot 2^n - (n - 1) \geq n \cdot (2^n - 1) = n \cdot \Pi^{PA}$$

Recall that the number of consumers in this game is $N(\mathcal{G}_i) = T = 2^n - 1$. Thus

$$\Pi^{CP} \geq \log N(\mathcal{G}_i) \cdot \Pi^{PA}$$

as desired. □

The following lemma shows that Theorem 4.2 is tight.

Lemma 4.3. *For any game \mathcal{G} , $\frac{\Pi^{CP}(\mathcal{G})}{\Pi^{PA}(\mathcal{G})} = O(\log(N(\mathcal{G})) + \log T)$.*

Proof. Let $\Pi^F(\mathcal{G})$ denote the optimal profit that can be obtained by the retailer in \mathcal{G} by setting a fixed price along the game. Clearly, $\Pi^{PA}(\mathcal{G}) \geq \Pi^F(\mathcal{G})$. Let v_1, v_2, \dots, v_ℓ denote, in decreasing order, the set of all consumer valuations over all time periods. Observe that $\ell \leq N(\mathcal{G}) \cdot T$. The contingent pricing mechanism cannot extract more profit than a perfect price discrimination mechanism. Thus

$$\Pi^{CP}(\mathcal{G}) \leq \sum_{k=1}^{\ell} v_k$$

Thus, it suffices to show the following inequality:

$$\Pi^F(\mathcal{G}) \cdot H_\ell \geq \sum_{k=1}^{\ell} v_k, \quad (1)$$

where $H_\ell = \sum_{i=1}^{\ell} \frac{1}{i} = O(\log(N(\mathcal{G}) \cdot T)) = O(\log(N(\mathcal{G}) + \log T))$. To do this, set $j^* = \arg \max_j \{j \cdot v_j : j \in [l]\}$. Thus, a retailer that announces a static price of v_{j^*} would obtain a profit of $j^* \cdot v_{j^*} = \Pi^F(\mathcal{G})$. Without loss of generality, scale the consumer valuations such that $j^* \cdot v_{j^*} = 1$. We then have, for all $i \in [\ell]$, that $i \cdot v_i \leq 1$. Hence, $v_i \leq \frac{1}{i}$. Inequality (1) follows. \square

5 Concave Storage Costs

We now examine our model under the more general setting of concave storage costs. Here the marginal cost of storage is non-decreasing function in quantity. Let $\mathcal{C} : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ be an arbitrary concave function such that $\mathcal{C}(q)$ denotes the cost of storing q units of the good for one period. (Here $\mathcal{C}(0) = 0$.) Because linear costs are a special case of concave costs, the example in Theorem 4.2 shows the contingent pricing mechanism can produce an $\Omega(\log N)$ factor more profits than the optimal preannounced pricing mechanism in the case of concave costs. Furthermore, it can be seen that the $O(\log N)$ upper bound of Lemma 4.3 also applies in the case of concave storage cost. We now examine the question of whether storage occurs under preannounced pricing mechanisms. The answer differs for the single-buyer and multi-buyer settings.

5.1 The Single-Buyer Case.

We extend Theorem 3.1 from linear to concave storage costs in the single-buyer setting. Specifically, we show that there exists an optimal price-commitment strategy under which no storage occurs.

Theorem 5.1. *In the single-buyer setting where storage costs are concave, there always exists a revenue maximising price commitment strategy that induces no storage.*

Proof. Let \mathcal{D}_1 denote any optimal price-commitment strategy for the retailer, i.e the sequence of T prices, and let \mathcal{D}_2 denote the buyer best response to \mathcal{D}_1 , i.e. an optimal sequence of the number of items bought in each time period. Let the storage levels under \mathcal{D}_2 be S_1, S_2, \dots, S_{T-1} , where S_t denotes the number of units stored from period t to period $t+1$ for all $t \in [T-1]$. Let $Q(\mathcal{D}_2) = \sum_{t=1}^{T-1} S_t$ be the total amount of storage. Without loss

of generality, we assume that given two purchasing schedules that yield the same utility the consumer picks the one with the lowest total storage. Note that the total storage cost incurred by the consumer is $\sum_{t=1}^{T-1} \mathcal{C}(S_t)$.

Let $\Delta\mathcal{C}(q) = \mathcal{C}(q+1) - \mathcal{C}(q)$. Thus, $\Delta\mathcal{C}(q)$ represents the additional cost of storing one extra unit, given that $x-1$ items are already stored. To conduct our analysis, we assign the storage costs incurred over the units as follows. We order the items in increasing order of purchase time; we arbitrarily order any units purchased in the same period. We now assume that units are consumed in a *first-in-first-out* (FIFO) fashion; given a choice, the buyer consumes the unit purchased earliest. Observe that the storage cost at period t is

$$\mathcal{C}(S_t) = \sum_{\ell=1}^{S_t} \Delta\mathcal{C}(\ell-1)$$

As we have an ordering on the items, we may assign to the ℓ th of these S_t units the storage cost $\Delta\mathcal{C}(\ell-1)$.

Assume that $Q(\mathcal{D}_2) > 0$. Take an interval \mathcal{I} consisting of the periods $\{i, i+1, \dots, i+k = \tau\}$, such that $S_i = S_\tau = 0$ and $S_t > 0$ for all $i < t < \tau$. By adding a dummy time period 0 with $S_0 = 0$, such an interval must exist as $S_T = 0$. Next, let α denote the last period in the $\mathcal{I} \setminus \{\tau\}$ at which the consumer bought an item. Thus $q_\alpha > 0$ and $q_t = 0$ for all $\alpha < t < \tau$.

Because $S_{\tau-1} > 0$ and $S_\tau = 0$ we have that consumption x_τ at time τ is positive and strictly exceeds the number of items q_τ purchased in that period. Moreover, we claim that every item consumed at period τ was bought in period α . Suppose not. Then by the FIFO policy every item purchased at time α is consumed in period τ , and at least one item purchased in period β is consumed in period τ , where $\beta < \alpha$ is the last period before α at which there was a purchase. Instead of purchasing that item at time β for consumption at time τ , the consumer could have purchased it at time α . By viewing this as the last of the resultant $S_\alpha + 1$ items stored at time α , we see that this does not affect the storage cost of any other item bought in any time period. Thus, by consumer rationality, the price paid at time β plus the storage cost for the item over the interval $[\beta, \alpha]$ must be less than the price announced at period α . Specifically

$$p_\beta + \Delta\mathcal{C}(S_\beta - 1) + \Delta\mathcal{C}(S_{\beta+1} - 1) + \dots + \Delta\mathcal{C}(S_{\alpha-1} - 1) < p_\alpha$$

But then, by concavity of the storage costs, it is better for the consumer to purchase all of the items currently bought in period α , in period β instead. Thus, as claimed, every item consumed in period τ was bought in period α .

Now suppose the retailer modifies her price commitment strategy by changing the price at period τ to

$$p'_\tau = p_\alpha + \sum_{j=\alpha}^{\tau-1} \Delta\mathcal{C}(|S_j| - 1).$$

Call the resulting price-commitment strategy for the retailer \mathcal{D}'_1 . Faced now with \mathcal{D}'_1 , we claim the consumer utility cannot decrease if she purchases as under \mathcal{D}_1 except that she buys $S_{\tau-1}$ items less in period α and $S_{\tau-1}$ items more in period τ . Under \mathcal{S} , the price and storage costs associated with the last item bought at period α is

$$p_\alpha + \sum_{j=\alpha}^{\tau-1} \Delta\mathcal{C}(|S_j| - 1)$$

But this is exactly the price p'_τ . By FIFO and the assignment rule for storage costs, postponing the purchase of that item until time τ will not alter the storage costs assigned to any other item (note that storage quantities after period τ remain unchanged). Thus the consumer is indifferent to postponing the purchase of the item until time τ when the retailer uses \mathcal{D}'_1 . But by the concavity of the storage costs, a similar argument implies that the consumer may postpone the purchase of every item bought in period α (that is, $S_{\tau-1}$) until period τ without a reduction in utility under \mathcal{S}'

In addition, we claim that the consumer cannot obtain a higher utility using another strategy against the new prices. If so, an improvement was also possible against the old prices, a contradiction. To see this, recall that the only storage modification induced by the retailer change of strategy from \mathcal{D}_1 to \mathcal{D}'_1 was a reduction in storage from period α to period τ of $S_{\tau-1}$. By FIFO and the assignment rule for storage costs, this storage reduction can only contribute to an increase in the storage costs for a potential new item bought before period τ , and for items already bought before period τ the storage costs remain the same. On the other hand, assume that it is now beneficial to buy at period τ , for price p'_τ , an additional item or items for consumption at time τ or later. But then, by concavity, it was also beneficial to do so under the old prices. Thus, the postponement strategy is optimal against the new prices.

To conclude, observe that the new prices yield a higher profit for the retailer than the old prices as $p'_\tau > p_\alpha$. This contradicts the optimality of \mathcal{D}_1 . Thus, there is an optimal solution without storage. \square

5.2 The Multi-Buyer Case.

Unlike in the single-buyer case, Theorem 3.1 cannot be extended to the more general setting of concave costs when there are multiple buyers.

Theorem 5.2. *There are instances of the multi-buyer setting where storage costs are concave in which every revenue maximising price commitment strategy induces some storage.*

Proof. We provide a simple example with $T = 3$ time periods. Let the storage cost function $\mathcal{C} : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ be defined as:

$$\mathcal{C}(q) := \begin{cases} 1.5 & \text{if } q = 1 \\ 1.5 + \epsilon & \text{if } q = 2 \end{cases}$$

Here ϵ is a tiny positive number. Now let there be $N = n_1 + n_2 + 1$ buyers where n_1 and n_2 are very large numbers. The first n_1 consumers receive an utility of 1 in the first period and zero in later periods. The second n_2 consumers receive no utility in the first two periods and an utility of 4 in the last period. Finally the last consumer obtains an utility of 2.75 in the second period, an utility of 3 in the third period and no utility in the first period.

One strategy for the retailer is to set $p_1 = 1, p_2 = \infty, p_3 = 4$. In that case, n_1 consumers would buy and consume at $t = 1$, n_2 consumers would buy and consumer at $t = 3$, and the remaining consumer would buy at $t = 1$ two units incurring in a storage cost of $1.5 + 1.5 + \epsilon = 3 + \epsilon$. This pricing strategy yields a profit for the retailer of $(n_1 + 2) \cdot 1 + n_2 \cdot 4$. Observe that is almost optimal in the sense that all except one consumer pay their total valuation.

Suppose there exists another pricing strategy (p'_1, p'_2, p'_3) that is at least as profitable as the one mentioned above, but that induces no storage. Since n_1 and n_2 are very large numbers, any such price commitment strategy requires that the first n_1 buyers pay a price of 1 and the n_2 consumers pay a price of 4. Thus $p'_1 = 1$ and $p'_3 = 4$.

If the remaining consumer buys at period 1, she obtains an utility of $2.75 + 3 - 2 \cdot 1 - (1.5 + \epsilon) - 1.5 = 0.75 - \epsilon$. Thus in order for the consumer to buy at period 2 (and not use storage) at a price p'_2 her utility has to be at least equal to $0.75 - \epsilon$. If the consumer buys one unit at $t = 2$ we have that $2.75 - p'_2 \geq 0.75 - \epsilon$. Therefore, $p'_2 \leq 2 + \epsilon$. On the other hand, if the consumer buys two units at $t = 2$ we have that $2.75 + 3 - 2 \cdot p'_2 - 1.5 \geq 0.75 - \epsilon$. This implies that $p'_2 \leq 1.875 + \epsilon/2$. Observe now than in any of these cases, p'_2 needs to be so low that the n_2 consumers would always prefer to buy at $t = 2$ and store rather than to buy at $t = 3$. As a result, we proved that in this game every optimal price commitment strategy induces some storage. \square

6 Conclusions and future research

In this paper we studied the dynamic pricing problem of a monopolistic retailer who sells indivisible storable goods to strategic consumers under a finite time horizon. The case where storable goods are arbitrarily divisible has been studied by [Dudine et al. \[2006\]](#). We focused in the understanding of two major pricing mechanisms: (1) *preannounced pricing*, where the retailer announces at the beginning of the game the price to be charged at each time period; (2) *contingent pricing*, the retailer announces prices sequentially at each time period, which means that the price announced is allowed to be contingent to game history.

First, we showed that computing the optimal preannounced pricing policy can be done in polynomial time (specifically in $O(N^2T^3)$ time). This result is in line with divisible goods model where the authors have shown that finding the optimal preannounced pricing can be done also efficiently by solving a different dynamic program (see supplemental appendix B in [Dudine et al. \[2006\]](#)).

We constructed a simple example to show that under the contingent pricing policy rather than the preannounced pricing mechanism, (i) prices can be lower, (ii) retailer revenues can be higher, and (iii) consumer surplus can be higher. Surprisingly, these three facts are in complete contrast to the case of a retailer selling divisible storable goods [[Dudine et al., 2006](#)]. Intuitively, under contingent pricing, the retailer can sometimes extract more revenue when the goods are indivisible by performing some price discrimination difficult to overcome for the consumer. When the retailer tries to perform a similar price discrimination under the divisible goods setting, the consumer can finely optimize the (continuous) quantities bought, making the price discrimination less effective. The example illustrates precisely this phenomenon. We then proceed to quantify how much more profitable can the contingent pricing mechanism be with respect to preannounced pricing. We showed that for a market with N consumers, a contingent policy can produce a multiplicative factor of $\Omega(\log N)$ more revenues than a preannounced policy. This is surprising, since to our knowledge, all previous monopoly models with a finite number of periods have the property that the contingent pricing mechanism cannot significantly outperform the preannounced pricing mechanism. Since preannounced pricing strategies are typically easy to implement and popular with consumers, this means there is little practical benefit for a retailer to explore the use of more complex contingent pricing mechanisms. Our results, however, provide a setting in which it may indeed be beneficial for the retailer to explore such options.

This paper leaves interesting questions for future research. One of them is to extend the preannounced pricing algorithm of this paper to more general settings such as when storage costs are concave or convex. Given that under concave costs, consumers might store goods under preannounced pricing, the construction of such algorithm might not be simple. While our results show that there are cases where the retailer revenue is much larger using contingent pricing rather than preannounced pricing, it is unclear whether the opposite is also true. That is, are there cases where the revenue obtained using a preannounced pricing is arbitrarily larger than by using a contingent pricing mechanism? Finally, another research direction is to characterize the (subgame perfect) Nash equilibrium of the price contingent mechanism. [Berbeglia et al. \[2014\]](#) characterized the subgame perfect equilibrium for a durable good pricing problem (which is special case of the problem studied here) and showed that such equilibrium can be computed in polynomial time. Extending such characterization or providing a hardness result for the problem studied in this paper is an interesting challenge.

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References

- Y. Aviv and A. Pazgal. Optimal pricing of seasonal products in the presence of forward-looking consumers. *Manufacturing & Service Operations Management*, 10(3):339–359, 2008.
- M. Bagnoli, S. Salant, and J. Swierzbinski. Durable-goods monopoly with discrete demand. *Journal of Political Economy*, 97:1459–1478, 1989.
- G. Berbeglia, P. Sloan, and A. Vetta. Bounds on the profitability of a durable good monopolist. In *Web and Internet Economics*, pages 292–293. Springer, 2014.
- M. Chiarolla, G. Ferrari, and G. Stabile. Optimal dynamic procurement policies for a storable commodity with Lévy prices and convex holding costs. *European Journal of Operational Research*, 2015.
- R. Coase. Durability and monopoly. *Journal of Law and Economics*, 15:143–149, 1972.
- José Correa, Ricardo Montoya, and Charles Thraves. Contingent preannounced pricing policies with strategic consumers. *Operations Research*, 64:251–272, 2016.
- S. Dasu and C. Tong. Dynamic pricing when consumers are strategic: Analysis of posted and contingent pricing schemes. *European Journal of Operational Research*, 204(3):662–671, 2010.
- P. Dudine, I. Hendel, and A. Lizzeri. Storable good monopoly: The role of commitment. *American Economic Review*, 96(5):1706–1719, 2006.
- F. Gul, H. Sonnenschein, and R. Wilson. Foundations of dynamic monopoly and the coase conjecture. *Journal of Economic Theory*, 39(1):155–190, 1986.
- W. Güth and K. Ritzberger. On durable goods monopolies and the coase-conjecture. *Review of Economic Design*, 3(3):215–236, 1998.
- I. Hendel and A. Nevo. Sales and consumer inventory. *RAND Journal of Economics*, 37(3): 543–561, 2006.
- A. Herbon. Optimal piecewise-constant price under heterogeneous sensitivity to product freshness. *International Journal of Production Research*, (ahead-of-print):1–21, 2014.

- C. Kuo and K. Huang. Dynamic pricing of limited inventories for multi-generation products. *European Journal of Operational Research*, 217(2):394–403, 2012.
- M. Li. Pricing non-storable perishable goods by using a purchase restriction with an application to airline fare pricing. *European Journal of Operational Research*, 134(3):631–647, 2001.
- G. Liberali, T. Gruca, and W. Nique. The effects of sensitization and habituation in durable goods markets. *European Journal of Operational Research*, 212(2):398–410, 2011.
- Q. Liu and G. van Ryzin. Strategic capacity rationing to induce early purchases. *Management Science*, 54(6):1115–1131, 2008.
- P. Nie. Commitment for storable goods under vertical integration. *Economic Modelling*, 26(2):414–417, 2009a.
- P. Nie. Commitment for storable good duopoly. *Nonlinear Analysis: Real World Applications*, 10(3):1838–1845, 2009b.
- M. Pesendorfer. Retail sales: A study of pricing behavior in supermarkets. *Journal of Business*, 75(1):33–66, 2002.
- N. Stokey. Intertemporal price discrimination. *Quarterly Journal of Economics*, 93:355–371, 1979.
- X. Su. Intertemporal pricing with strategic customer behavior. *Management Science*, 53(5):726–741, 2007.

Appendix 1: Proofs omitted from main text.

Observation 1. *Let $\{p_1, \dots, p_T\}$ be any sequence of preannounced prices. If sales at period t are positive then p_t is finite and $p_t \leq p_{t'} + (t - t') \cdot c$ for all $t' < t$.*

Proof. Suppose not. Let $q_t > 0$ and $p_t > p_{t'} + (t - t') \cdot c$ for some $t' < t$. This contradicts the rationality of the consumer (single-buyer case) or the consumers who buy at period t (multi-buyer case), as she/they can increase their utility by bringing forward the period t purchases to period t' and then storing the good. \square

Lemma 6.1 (Lemma 3.1). *There always exists a revenue maximising preannounced pricing strategy in which there is no storage. This pricing strategy is the same for the single-buyer and the multiple-buyer settings.*

Proof. Take any optimal preannounced pricing strategy $\{p_1, \dots, p_T\}$ in which the consumer (or consumers) use the storage option. We show that there is another optimal preannounced pricing strategy that requires less storage. The theorem will follow. Let t be the first time period where $q_t > x_t$. Thus a consumer stores at least one unit of the good between period t and period $t + 1$. We may assume in the case of a tie that the consumer(s) selects an option with the least storage. By the rationality of each consumer, we then have that $p_{t+1} > p_t + c$. By Observation 1, we then have that $q_{t+1} = 0$. Now suppose the retailer modifies its preannounced policy by setting $p_{t+1} = p_t + c$. Observe first that consumption in periods 1 to t will not change. Second, in the single-buyer case the consumer now prefers to purchase $q_t - x_t$ units in period $t + 1$ rather than buying them earlier and storing them. In the multi-buyer case there are $q_t - x_t$ consumers who previously bought a unit to store at period t and now prefer to purchase it in period $t + 1$. Thirdly, the purchases made from period $t + 2$ are unaffected by this price change; if it is now preferable to buy additional units in period $t + 1$ then each consumer would have originally benefited by making such purchases in period t . This modified strategy pricing cannot decrease the retailer profits (if $c > 0$ then profits strictly increase). \square