The Finite Horizon, Undiscounted, Durable Goods Monopoly Problem with Finitely Many Consumers *

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Abstract. We study the uncommitted durable goods monopoly problem when there are finitely many consumers, a finite horizon, and no discounting. In particular we characterize the set of strong-Markov subgame perfect equilibria that satisfy the skimming property. We show that in any such equilibrium the profits are not less than static monopoly profits; and at most the static monopoly profits plus the monopoly price. When each consumer is small relative to the market, profits are then approximately the same as those of a static monopolist which sets a single price. Finally, we extend the equilibrium characterization to games with an arbitrary discount factor.

Keywords: durable goods monopoly, discrete buyers, profit bounds, inter-temporal price discrimination, skimming property.

1 Introduction

We study a standard durable-goods monopoly game, where a seller sets a price in every period but cannot commit to its pricing strategy, while each consumer leaves the market once it has purchased one unit. It is well known that with an infinite horizon and no discounting any sharing is possible in a subgame perfect Nash equilibria (SPNE). The literature has therefore typically dealt with an infinite horizon and discounting. However, results depend crucially on how demand is modeled.

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In 1972, Nobel recipient Ronald Coase made the startling conjecture that a durapolist (a monopolist in the market of a durable good) has no monopoly power at all. Specifically, a durapolist who lacks commitment power cannot sell the good above the competitive price if the time between periods approaches zero (Coase, 1972). The intuition behind the Coase conjecture is that if the monopolist charges a high price then consumers anticipate a future price reduction (as they expect the durapolist to later target lower value consumers) and therefore they prefer to wait. The durapolist, anticipating this consumer behavior, will then drop prices down to the competitive level. In essence, the argument is that a durapolist is not a monopolist at all: the firm does face stiff competition – not from other firms but, rather, from future incarnations of itself. This is known as the *commitment problem*: the durapolist cannot credibly commit to charging a high price.

The Coase conjecture was first proven by Gul et al. (1986) under an infinite time horizon model with non-atomic consumers. They showed that if buyers strategies are stationary then, as period length goes to zero, the durapolist's first price offer converges to the lowest consumer valuation or the marginal cost, whichever is higher. Ausubel and Deneckere (1989) later showed that if the stationary condition is relaxed, the durapolists profits at subgame perfect equilibria can range from Coasian profits to the static monopoly profit.[1] Stokey (1979) studied pricing mechanisms for durapolists that *possess* commitment power in a continuous time model. She showed that durapolists can then attain the static monopoly profit by committing to a fixed price; all sales are then made at the beginning of the game. McAfee and Wiseman (2008) examined the Coase conjecture in a model where there is small cost for production capacity which can be augmented at each period. In this setting, the authors showed that the monopoly profits are equal to those that can be obtained if she could commit ex ante to a fixed capacity. Recently, Ortner (2017) studied a model where the durapolist incurs a stochastic cost.

In contrast to the results mentioned above, full extraction of economic surplus may arise if demand is instead composed of a finite number of consumers. Indeed, Bagnoli et al. (1989) proved the existence of a subgame perfect Nash equilibrium in which the durapolist extracts all the economic surplus if the demand is atomic and the time horizon is infinite. To obtain this, they considered the following pair of strategies. The durapolist strategy, dubbed *Pacman*, is to announce at each time period, a price equal to the valuation of the consumer with the highest value who has yet to buy. The strategy of each consumer, dubbed *get-it-while-you-can*, is to buy the first time it induces a non-negative utility. This equilibrium refutes the Coase conjecture. Indeed, it suggests that a durapolist may have perfect price discriminatory power. Moreover, it shows there exist subgame perfect Nash equilibria where durapoly profits exceed the static

[1] However, profits larger than the Coasian value occur only in the case where there is no gap between the lowest consumer value and the marginal cost of production (e.g. for $c = 0$, the lowest consumer value is 0).

monopoly profits by an unbounded factor.[2]

Von der Fehr and Kühn (1995) studied the model of Bagnoli et al. (1989) with an infinite time horizon and showed that under certain conditions Pacman is the only equilibrium. However other equilibria also exist with less than full extraction. Cason and Sharma (2001) considered a different model with atomic buyers where the Pacman equilibrium of Bagnoli et al. (1989) cannot exist. Instead of assuming a durapolist with perfect information, the authors constructed a two-buyer and two-valuation model with infinite time periods in which the durapolist does not know exactly whether a consumer is of high type or of low type. They showed that in these games there exists a unique equilibrium that is Coasian. In his recent study of the durapoly problem with finitely many consumers and infinite horizon, Montez (2013) showed that there are inefficient equilibria where the time at which the market clears does not converge to zero as the length of the trading periods approaches zero.

Given the important role demand plays in the analysis discussed above, it is also relevant to understand the role played by other assumptions as well, such as the role of an infinite horizon and discounting in the analysis. Indeed there are situations where sales need to take place within a bounded period of time (for example, tickets to a running show), discounting is of second order, and a seller is able to make a sequence of offers (with potentially more eager consumers watching the show earlier). Theoretical and empirical evidence of the strong effects of deadlines have been observed in many bargaining contexts such as in contract negotiations and civil case settlements – see, for example, Cramton and Tracy (1992), Williams (1983).

If there is a finite horizon and a continuum of consumers, a feasible action for the durapolist is to decline to sell goods until the final period and then announce the static monopoly price, obtaining the static monopoly profits discounted to the beginning of the game. Although this strategy is not an equilibrium, Güth and Ritzberger (1998) showed that when consumer valuations follow a uniform distribution, there exists a subgame perfect equilibrium, as period lengths approach zero, in which the durapolist profits converge to the static monopoly profits discounted to the beginning of the game.

It is however less well understood what happens with finite buyers and finite time horizon. Bagnoli et al. (1989) has presented some examples with two or three consumers, and showed that the Pacman equilibrium may hold as well under a finite horizon.

In this paper we provide a general treatment of profits under discrete consumers, finite horizon and no discounting for equilibria that satisfy a skimming property. An equilibrium is said to satisfy the skimming property if high-value consumers buy before (or at the same time) than lower-valued consumers. The skimming property is satisfied in many settings of

[2] For example, consider a game with N consumers where buyer i has a valuation of $1/i$. Then, as N gets large, durapoly profits under the Pacman strategy approach $\log N$ whilst static monopoly profits are clearly equal to 1.

the durable good monopoly problem (e.g. Güth and Ritzberger (1998), Gul et al. (1986)) as an indirect consequence of a non-increasing price path. In our setting with a finite number of consumers, however, it is possible that non-skimming equilibria may exist.

First, we prove that there always exists a skimming-property equilibrium and we are able to characterize, in Section 3, the class of all subgame perfect equilibria that satisfy the skimming property [3]. (In Section 5, we are able to extend this characterization to games with discounting.)

Our main result is then that at every such skimming equilibrium the durapoly profits are bounded by below by the static monopoly profits, and by above by the sum of the static monopoly profits and the static monopoly price. These bounds hold regardless of the number of consumers, their values, and the number of time periods. Therefore, if the size of each individual buyer is small, the durapolist will neither make significantly more, nor less, than the static monopoly profits.

The intuition behind the upper bound is as follows. Using an inductive argument, we prove that each buyer has an associated threat price: the price it can get in the end if all buyers with a valuation above her own have already purchased. The argument concludes by showing that the sum of all the consumer threat prices cannot exceed the static monopoly profit plus the static monopoly price. Moreover, we prove that our bound is tight. Indeed, we construct a (infinite) family of examples where durapoly profits approach the static monopoly profit plus the static monopoly price as the number of consumers goes to infinity.

We believe that our main result sheds light into this classical problem in at least four ways. To begin, our main theoretical result concurs with the practical experience that durapolists and static monopolists have comparable profitability (i.e. within a constant multiplicative factor). For example, following a comprehensive study on the practices of durable goods monopolies, Orbach (2004) concludes "Durapolists may collect profits higher than static monopoly profits. In fact, some of the practices durapolists employ to increase profits are not available to perishable-goods monopolists, and, therefore, monopolies over durable goods markets may be more profitable than monopolies over perishable-goods markets." In contrast, previous theoretical works have suggested that the durapolist either has no monopoly power, perfect price discriminatory power, or a multitude of equilibria over all the range in between.

Second, the result that a durapolist can do up to an additive amount better using a threat-based strategy rather than a price-commitment strategy is actually best viewed from the opposite direction. Specifically, a durapolist can obtain almost the optimum profit (losing at most an additive amount equal to the static monopoly price) by mimicking a static monopolist via a price commitment strategy. From a practical perspective this is important because a price-

[3] Bagnoli et al. (1989) state that such a characterization would be extremely interesting.

commitment strategy can generally be implemented by the durapolist very easily, even with limited consumer information. Furthermore, price-commitment strategies can be popular with consumers as they are typically introduced within a money back guarantee or envy-free pricing framework. In contrast, a threat based optimization strategy is harder to implement and can antagonize consumers.

Third, the literature often highlights that the surprising and well-known result of Bagnoli et al. (1989), namely that the durapolist can extract all economic surplus, is due to the assumption of finitely many consumers. Our results show that this is not true in general – their result is also driven by the infinite time horizon. For finite time horizons, the power of a durapolist is sometimes limited (with or without discounting). For some distributions of consumer valuations, the durapoly profits are strictly less than the full extraction of economic surplus (see Sections 4 and 5) [4].

Finally, the main result highlights a distinction on how the time horizon affects bargaining power. With non-atomic consumers, a finite time horizon increases the bargaining power of the durapolist. In Güth and Ritzberger (1998), a finite time-horizon increases durapolist profits from the Coasian result to the static monopoly profits. With finitely many consumers, the finiteness of the time horizon reduces the durapolist's bargaining power for some demand distributions (either with or without discounting). In particular, for the undiscounted case, the durapolist's profits become approximately the static monopoly profits.

2 The Model

We now present the durable good monopoly model of Bagnoli et al. (1989) that we will analyze in the subsequent sections [5]. Consider a durable good market with one seller (a durapolist), a set $[N] = \{1, 2, \dots, N\}$ of N consumers, and a finite horizon of T time periods. The N consumers have valuations $v_1 \geq v_2 \geq \cdots \geq v_N$ [6] and the firm can produce units of the good at a unitary cost of c dollars. Here we assume, without loss of generality, that $c = 0$. Consequently, profit and revenue are interchangeable in this setting.

We can view this as a sequential game over T periods. At time t, $1 \leq t \leq T$, the firm will

[4]Moreover, in Section 3.1 we explain that in games without discounting, it is only under very specific settings, that the Pacman strategy will produce an equilibrium in finite horizons. Even under such settings, the static monopoly profits are approximately equal to (and at least half of) the durapoly profits.

[5]We interpret our model as if the items sold are durable goods in which the utility v obtained by each buyer is the result of a long consumption stream (that may happen after the selling horizon). Another possible interpretation of our model is that each consumer utility v comes instead from the consumption of a good that is only needed once.

[6]We also use notation $v(y_j)$ instead of v_{y_j} in certain cases to avoid nested subscripts

select a price μ_t to charge for the good.[7]

The durapolist seeks a pricing strategy that maximizes her revenue, namely $\sum_{t=1}^{T}(x_t \cdot \mu_t)$, where x_t denotes the number of consumers who buy in period t.

Each consumer i desires at most one item and seeks to maximize her *utility*, which is $v_i - \mu_t$ if she buys the good in period t.[8] The consumers decide simultaneously if they will buy an item for μ_t. The game then proceeds to period $t+1$. If a consumer does not buy an item before the end of period T her utility is zero.

For such a sequential game, the solutions we examine are pure subgame perfect Nash equilibria that satisfy the standard *skimming property* defined below.

Definition 2.1 (Skimming property)**.** *An equilibrium satisfies the* skimming property *if whenever a buyer with value v buys at price μ_t, then every buyer with value $w > v$, who has not yet bought, also buys at this price given the same history.*

For subgame perfect Nash equilibria (SPNE) that satisfy the skimming property, consumers' strategies can be characterized using a cutoff function. Given a history of prices $h_t = (\mu_1, \mu_2, \ldots, \mu_{t-1})$ and the current offered price μ_t, consumers with valuations above cutoff $\kappa(h_t, \mu_t, t)$ buy and consumers with valuations below the cutoff do not buy [9](see Fudenberg and Tirole (1991) for a discussion). When consumers are non-atomic it can be shown that all subgame perfect equilibria satisfy the skimming property (Fudenberg et al., 1985). In the case of atomic consumers and an infinite time horizon, the monopolist can extract all economic surplus using the Pacman strategy, in which case the skimming property is clearly satisfied. Intuitively, the skimming property says that higher value consumers pay a higher (or at least equal) price compared to consumers with a lower valuation.

Ausubel and Deneckere (1989) define two special types of SPNE that are *Markovian* in the sense that they depend only on the most recent information available. A SPNE is a *weak-Markov* equilibrium if consumers' accept/reject decisions depend only on the current price and period. A SPNE is a *strong-Markov* equilibrium if, in addition to the weak Markov property, the durapolist conditions her strategy only on the payoff-relevant part of the history. In the infinite horizon case, this is the set of remaining consumers. In the finite horizon case, it may depend on the number of periods left as well. When consumers are non-atomic, an equilibrium is strong-Markov if and only if the Weak-Markov property is satisfied and $\kappa(\mu_t, t) := \kappa(h_t, \mu_t, t)$ (since

[7] In this paper we omit the study of discriminatory pricing mechanisms in which two or more consumers can be charged different prices in the same time period. Those mechanisms face far less acceptance by society, potentially causing PR issues, and they are even sometimes banned by federal laws (see, e.g., Ross (1984))

[8] Discount factors can easily be introduced into the model, see Section 5.

[9] In order to deal with tie-breaking, when the number of consumers is finite and the cutoff matches the consumer valuation, the decision of buying or waiting can depend on the consumer index (or identity).

consumers' accept/reject decisions do not depend on the price history) is strictly increasing in μ_t (Proposition 1, Fudenberg et al. (1985)). In the atomic case, we can obtain strong-Markov equilibria even if κ is constant over an interval.

When constructing an SPNE in the atomic finite-horizon model, we will restrict the strategy space of the durapolist and consumers so that they satisfy the strong-Markov conditions: the prices the durapolist chooses are a function that depends only on the remaining consumers and the number of time periods left, that is, $\mu : \mathcal{P}([N]) \times T \to \mathcal{R}^+$ and the consumers strategies are such that i buys in period t iff $v_i \geq \kappa(\mu_t, t)$ for some function κ. However, our main result from Section 4 only requires the equilibrium satisfies the skimming property.

Observe that in the model of Bagnoli et al. (1989) which we studied in this paper, the values of each consumer are known to all participants in the market. In Appendix 2 we extend the equilibrium results to a restricted incomplete information setting, where the durapolist knows the distribution of values and the aggregate number of sales per period, but does not know which are the consumers who buy in each round.

2.1 An Example

We now present a small example to illustrate the model and the concepts involved. Consider a two-period game with four consumers, where the consumers' valuations are $\{100, 85, 80, 50\}$, as shown in Table 1.

Table 1: Example of a game with 4 consumers.

Consumer	**Consumer value**
1	100
2	85
3	80
4	50

Denote by Π^D and Π^M the revenue obtainable by the durapolist and the corresponding static

Table 2: Threat prices

Consumer	**Consumer value**	**Threat price**
1	100	80
2	85	80
3	80	50
4	50	50

monopolist, respectively. Then the static monopoly profit Π^M is equal to 240, obtained by selling to the top three consumers for a price of 80. However, the durapolist can, in fact, extract a revenue of 260. Furthermore, the corresponding equilibrium satisfies the skimming property: no consumer will buy earlier than another consumer with a higher valuation. To understand SPNEs in this game, let us begin with a subgame comprising of only the final (second) time period. In such a subgame, it is a dominant strategy for all consumers who have not yet bought to pay any price less than or equal to their value. Consequently, in the final period, it is a dominant strategy for the durapolist to charge the static monopoly price *as calculated with respect to the set of consumers who have not yet bought.* Note that these strategies satisfy the strong-Markov property as everyone remaining with value above the price will buy, everyone else will not buy, and the price depends only on the set of consumers remaining.

Now consider the first time period. If the skimming property is satisfied, then there will be a cut-off point j_1 at which consumers $j \leq j_1$ buy and consumers $j > j_1$ wait until period 2. In order for this to be an equilibrium, the consumer j_1 must prefer buying in period 1 to period 2. Therefore, the durapolist can charge no more than the static monopoly price as calculated if all consumers $j \geq j_1$ wait until period 2. We call this price the *threat price* for consumer j_1. The threat prices are listed in Table 2.1.

The consumers' strategies then correspond to "buy in period 1 if and only if μ_1 is at most their threat price", whilst the durapolist's strategy is to charge the threat price which maximizes the total revenue. The period 2 strategies are the dominant strategies described above: remaining consumers pay up to their value, while the durapolist charges the static monopoly price calculated for the set of consumers that are left.

Charging $\mu_1 = 80$ means that the top two consumers would buy in period 1, while the last two consumers would wait until period 2 and buy at $\mu_2 = 50$ (the static monopoly price for the remaining two consumers is 50). The total profit would therefore be 260. It is easy to see that charging $50 < \mu_1 < 80$ gives a smaller profit. Moreover, if $\mu_1 > 80$ then no consumers will buy in the first period. This would lead to a profit of 240 as the static monopoly price would then be charged in the final period. Finally charging $\mu_1 = 50$ would result in all consumers buying in period 1, as the durapolist is guaranteed to charge at least 50 in period 2. The total profit would then be 200. Thus, for a profit maximizing durapolist we have $\Pi^D = 260$. So durapoly profits are greater than static monopoly profits. For additional comparisons, Coasian profits are $\Pi^C = 200$ since the competitive price is 50, and Price Discriminatory profits are $\Pi^{PD} = 315$, that is, the economic surplus.

Observe that these strategies satisfy the skimming property, as threat prices are monotonically increasing with consumer value, and satisfy the weak-Markov property, as $\kappa(\mu_t, t)$ is the smallest consumer value such that his threat price is larger than μ_t. Note that $\kappa(\mu_1, 1)$ is constant for all $\mu_1 \in (50, 80]$. Since the durapolist's strategy depends only on the values of remaining players in each period (trivially in period 1) it also satisfies the strong-Markov property. Furthermore, it is easy to prove that this is a subgame perfect Nash equilibrium.

3 Subgame Perfect Equilibria

We now characterize the subgame perfect equilibria that satisfy the strong-Markov conditions and maximize durapolist profits. To do this we reason backwards from the final time period T. It is easy to determine the behavior of rational consumers and a profit maximizing durapolist at time T. Given this information, we can determine the behavior of rational consumers at time $T-1$, etc.

To formalize this, let $\mathcal{G}_i$ denote the market consisting of consumers $\{i, i+1, \dots, N\}$, and let $\Pi(i,t)$ denote the maximum profit obtainable in the market $\mathcal{G}_i$ if we begin in time period t. Thus $\Pi^D = \Pi(1,1)$. Now set $\Pi(i, T+1) = 0$ for all consumers i. Let $p(i,t)$ be a profit maximizing price at period t in the market $\mathcal{G}_i$ beginning at time t [10]. First, consider the last period, T. Any consumer i (who has not yet bought the good) will buy in period T if and only if this final price is at most v_i. Therefore, in the market $\mathcal{G}_i$, starting at time T, a profit maximizing durapolist will simply set $p(i,T)$ to be the static monopoly price p_i [11] for the market $\mathcal{G}_i$: $p(i,T) = p_i \equiv v_{j^*(i,T)}$, where

$$j^*(i,T) = \arg\max_{j \geq i} \,(j - i + 1) \cdot v_j.$$

Thus, $j^*(i,T)$ denotes the consumer with the lowest valuation who buys in the market $\mathcal{G}_i$ beginning at period T. The profit is then

$$\Pi(i,T) \quad = \quad (j^*(i,T) - i + 1) \cdot v_{j^*(i,T)}$$

In general we will denote by $j^*(i,t)$, the consumer with the lowest valuation who buys (under our proposed strategy) at period t in the market $\mathcal{G}_i$ beginning at period t. It is possible that not all consumers who have the same value buy the item in the same period. This gives rise to multiple equilibria. Because we focus on equilibria that satisfy the skimming property, we are interested in tie breaking rules that are consistent across time periods. In particular, such tie breaking rules correspond to fixed relabeling of the agents, so that if consumers with indexes $\{x, y\}$ have the same valuation and x buys earlier than y, then $x < y$.

[10]There could be more than one price that maximizes profit due to ties. In that case, we choose the lowest price among all those that are revenue maximizing.

[11] Note that it is possible that $p_i \neq p_j$ even if $v_i = v_j$

Now, suppose we are at time period $T-1$ in the market $\mathcal{G}_i$. If the durapolist at period $T-1$ wishes to sell to consumers $\{i, i+1, \ldots, k\}$, then the announced price has to be at most k's threat price, $p(k,T) \equiv v_{j^*(k,T)}$. To see this, suppose that the price announced at $T-1$ is higher and the durapolist still expects to sell the item to consumers $\{i, i+1, \ldots, k\}$. Then, if consumer k refuses to buy while all consumers above her buy, the durapolist would, in the final time period T be in the market $\mathcal{G}_k$, and announce a price $p(k,T)$, meaning that consumer k would have benefited from deviating. So, the optimal strategy for the durapolist would be to sell to $k-i+1$ consumers at period $T-1$ at price $v_{j^*(k,T)}$, choosing the value of k such that the profits from periods $T-1$ and T are maximized:

$$j^*(i,T-1) = \arg\max_{k \geq i}\{(k-i+1) \cdot p(k,T) + \Pi(k+1,T)\}$$
$$\Pi(i,T-1) = (j^*(i,T-1) - i + 1) \cdot p(j^*(i,T-1),T) + \Pi(j^*(i,T-1)+1,T)$$

The price announced at period $T-1$ can then be written as

$$p(i,T-1) = p(j^*(i,T-1),T)$$

Observe then, that in the final period, we will be in the subgame composed of consumers $\{j^*(i,T-1)+1, \ldots, N\}$.

Iterating this argument backwards in terms of the periods, we have that

$$\begin{aligned} \Pi(i,t) &= (j^*(i,t) - i + 1) \cdot p(j^*(i,t), t+1) + \Pi(j^*(i,t)+1, t+1) \\ j^*(i,t) &= \arg\max_{j \geq i} \left((j-i+1) \cdot p(j,t+1) + \Pi(j+1,t+1)\right) \\ p(i,t) &= p(j^*(i,t), t+1) \end{aligned} \tag{1}$$

In case there are multiple values $j \geq i$ that achieve a maximum in (1), we take the largest value among them. Thus, the monopolist is setting the lowest price among those equally profitable price paths. This ensures that no consumer can benefit by postponing their purchase from the equilibrium.

We can generalize the concept of the threat price from our two-period example using the above recursion. Specifically, we say that the *threat price* $\tau(i,t)$ for consumer i at period $t < T$ under the recursive scheme given in (1) is the price i is offered in the market $\mathcal{G}_i$ starting at period $t+1$, namely $\tau(i,t) := p(i,t+1)$. That is, the price offered if i and all consumers of lower value do not buy in period t.

We can now define the strategy of the durapolist and the consumers in any subgame. Consider a subgame whose remaining consumers are the set S and let there be $T-t+1$ periods remaining (i.e. we are starting in period t). Then by re-indexing the consumer names, the durapolist can treat the subgame as a full game $\mathcal{G}'$ with $T - t + 1$ total periods and S as the set of all consumers. She then calculates, for all i and t, the prices $p_{\mathcal{G}'}(i,t)$ from the recursion relationship (1) and chooses the sales schedule which maximizes her profits for $\mathcal{G}'$. She then charges $\mu_t = p_{\mathcal{G}'}(1,1)$ in period t. The consumers buy if and only if the price is less than or equal to their threat price as calculated for $\mathcal{G}'$. By definition, this price only depends on the payoff-relevant part of the history, as it only looks at the consumers remaining in the subgame. Since the price is always equal to the threat price of one consumer j^*, we can define $\kappa(\mu_t, t)$ to be the value of this critical consumer, v_{j^*}. We can show that this function is monotonically increasing in μ if the threat prices are decreasing in consumer value (a higher price means a higher valued critical consumer). This is proved in Lemma 3.2. Therefore κ is indeed a cutoff function for the given consumer strategies. We conclude that, if this strategy profile is an SPNE, it must satisfy the strong-Markov property.

The reader may have noted that our recursion relationship does not allow the durapolist to refuse to sell any items in a period where there are still consumers left who have not yet bought (if no consumer buys in the market $\mathcal{G}_i$ at period t, $j^*(i,t)$ is undefined). It can be shown that there is a subgame perfect equilibrium in which a sale occurs in each period until either all consumers have bought the item or the final time period is over. Moreover, this equilibrium achieves at least as much profit for the durapolist as any which allows the durapolist to not sell in some periods. A proof of this is included in Appendix 1 as Lemma 6.1.

The following series of lemmas establish basic monotonicity results for static monopoly prices, threat prices, and the prices $p(i,t)$ which form the durapolist's equilibrium strategy. The proofs of these lemmas are given in Appendix 1.

Our first lemma shows that the static monopoly price cannot decrease if we incorporate to the market a consumer with a value higher than all the other consumer valuations.

Lemma 3.1. *The static monopoly prices on the markets $\mathcal{G}_i$ are non-increasing in i: $p_i \geq p_{i+1}$ for $i = 1, \ldots, N-1$.*

The following lemma shows that the consumers' strategies defined above satisfy the skimming property.

Lemma 3.2. *In any game $\mathcal{G}$ with T periods, the threat prices are non-increasing in i: for all $i \leq k$ and all $t < T$, $\tau(i,t) \geq \tau(k,t)$.*

The next two lemmas are required to show that a deviation from a consumer by delaying a purchase or buying early does not yield higher utility.

Lemma 3.3. *Consider two durapoly games, $\mathcal{G}$, with T periods and a set S of consumers, and $\mathcal{G}'$, with T periods and a set S' of consumers such that only the top valued consumer in S and*

S' differs in value, and the top valued consumer in S' has the higher value. If we use $p_{\mathcal{G}}(1,1)$ and $p_{\mathcal{G}'}(1,1)$ to denote the first period prices as calculated by the recursion relationship above, then $p_{\mathcal{G}'}(1,1) \geq p_{\mathcal{G}}(1,1)$.

Lemma 3.4. *In any game $\mathcal{G}$ with T periods, if the durapolist and consumers follow the strategies described above, then prices are non-increasing in time.*

We are now ready to state the following result, whose proof is in Appendix 1.

Theorem 3.1. *The strategies defined above constitute a SPNE.*

To compute such an equilibrium, we can compute $j^*(i,t)$ for each (i,t) going backwards from period T, and choose the sales path x_t which maximizes profit. The prices μ_t are then computed by "passing back" the next period's threat price. We may solve the corresponding dynamic program to find the maximum profit Π^D for the durapolist.

It is easy to see that if there is another strong-Markov SPNE, it cannot result in more revenue for the durapolist. In any proposed SPNE where we sell in every period, any period price cannot be higher than the threat price of any consumer who buys in that period, as otherwise she would earn more profit by waiting one more period. So given a sales schedule, the monopolist can do no better than to charge the threat price of the lowest valued consumer to buy in each period. However, (1) finds the optimal sales schedule in terms of revenue when the durapolist charges threat prices in each period. Lemma 6.1 in the appendix 1 covers the case of a strong-Markov SPNE in which the seller may choose not to sell in one or more periods when consumers remain to buy. The full result we have, then, is

Corollary 3.1. *The optimal revenue Π^D given by the dynamic program derived from (1) is the maximum revenue obtainable by a strong-Markov SPNE.*

3.1 A Pacman Theorem

As discussed, Bagnoli et al. (1989) proved that a durapolist who faces atomic consumers with an infinite time horizon can always extract all economic surplus. They left open the case of finite time horizons. Although such equilibria may still exist under a finite horizon, the conditions required for their existence are very restrictive. Indeed, applying the techniques we have developed, we characterize in this section necessary and sufficient conditions for this phenomenon to happen.

Recall that the durapolist strategy named Pacman is to announce at each time period, a price equal to the valuation of the consumer with the highest value who has yet to buy, and that the consumer strategy known as get-it-while-you-can is to buy the first time it induces a non-negative utility. The next lemma gives sufficient conditions for such strategies to be at equilibrium. Observe that we are not imposing that the number of periods is greater than or equal to the number of different consumer valuations. This makes the argument not trivial.

Lemma 3.5. *If $p_i = v_i$ for all $i \in [N]$, then there exists an equilibrium in which the durapolist uses the Pacman strategy and consumers follow the get-it-while-you-can strategy.*

We are now ready to state the following theorem which provides sufficient and necessary conditions for the existence of an equilibrium that extracts all economic surplus.

Theorem 3.2 (Pacman Theorem)**.** *Consider a durapoly game $\mathcal{G}$ with $M \leq N$ distinct valuations. There exists an equilibrium at which the durapolist extracts all the economic surplus if and only if $M \leq T$ and $v_i = p_i$ for all $i \in [N]$.*

We conclude this section with some observations. Recall, from Footnote 2, that there exist finite time horizon games in which the Pacman solution has profits that are a factor $\log N$ greater than static monopoly profits. On the other hand, the main result of next section is that durapoly profits are approximately the static monopoly profits and never more than twice the static monopoly profits. Thus, when Pacman is an equilibrium, static monopoly profits are at least half of Pacman profits. To see this, observe that the condition $p_i = v_i$ for all $i \in [N]$, implies that each distinct value must be no more than half the next higher value [12]. In these scenarios, $n_1 \cdot v_1$ (where n_1 is the number of consumers with valuation v_1) is at least half of the sum of all consumer valuations.

4 A Relationship between Durapoly Profits and Static Monopoly Profits

In this section, we will prove our main result: the profits of the durapolist in a skimming property-satisfying SPNE of the durapoly game are at least the profits of the corresponding static monopolist, but at most the static monopoly profits plus the static monopoly price. We highlight that this result is not restricted to strong Markov equilibria.

Recall that Π^M and Π^D denote the static monopoly profits and the durapoly profits respectively. Our main result is as follows.

Theorem 4.1. $\Pi^M \leq \Pi^D \leq (\Pi^M + p_1)$

We remark that if the unitary cost c is not normalized to zero, the bound becomes $\Pi^D \leq (\Pi^M + v_1 - c)$. At the end of this section we show that Theorem 4.1 is in fact tight, i.e., durapoly

[12]To be precise, valuations should satisfy that $v_i = v_{i+1}$ or $v_{i+1} \leq \frac{1}{2} v_i$ for all $i \in [N-1]$

profits can be as close as desired to the static monopoly profits plus the static monopoly price. A direct consequence of Theorem 4.1 is that durapoly profits are never more than twice the static monopoly profits. Observe however, that in many situations Theorem 4.1 suggests that durapoly profits can be only slightly higher than static monopoly profits. For example, this happens when the value of the highest value consumer is negligible with respect to the static monopoly profits, i.e. $p_1 \leq v_1 \ll \Pi^M$.

The first inequality of Theorem 4.1 is proven in the Appendix 1 (Proposition 6.1). This inequality is intuitive as one alternative for the durapolist is to charge a very high price in periods 1 to $T-1$ and then charge the monopoly price p_1 in period T. Although such strategy is not subgame perfect, we showed, using an induction argument on the number of time periods, that the durapoly profits under a subgame perfect Nash equilibrium are at least equal to those obtained by a fixed price policy.

The second inequality of Theorem 4.1 is more substantial. The intuition behind the upper bound on durapoly profits stated in Theorem 4.1 goes as follows. In a game with a finite time horizon, consumers have an additional option which is to *wait* until the end. In the final round, the durapolist would set a price equal to the static monopoly price of the remaining consumers. We extend this argument inductively and prove (Lemma 4.2) that in every equilibrium satisfying the standard skimming-property, every consumer is willing to pay at most the static monopoly price from the submarket in which she is the consumer with the highest valuation. Finally, we show in Lemma 4.5 that the sum of all those static monopoly prices cannot be much greater than the static monopoly profits.

We emphasize that the difference between the finite horizon and the infinite horizon outcomes can be arbitrarily large. For example, consider the game with N consumers where buyer i has a valuation of $1/i$. In the finite horizon case, $\Pi^D \leq \Pi^M + p_1 = 2$; whereas in the infinite horizon case $\Pi^D_\infty = \sum_{i=1}^N \frac{1}{i} \approx \log(N)$. Under both scenarios, $\Pi^M = 1$. Thus, the ratio $\frac{\Pi^D - \Pi^M}{\Pi^D_\infty - \Pi^M} \approx \frac{1}{\log(N)-1}$ tends to zero as N tends to infinity.

We prove the second inequality of Theorem 4.1 in two steps. To describe them, recall we have N consumers with valuations $v_1 \geq v_2 \geq \cdots \geq v_N$. Furthermore, $\mathcal{G}_i$ is the market consisting of consumers $\{i, i+1, \ldots, N\}$ and p_i is the static monopoly price for the market $\mathcal{G}_i$. First we show that $\Pi^D \leq \sum_{i=1}^N p_i$, and second we show that $\Pi^M + v_1 \geq \sum_{i=1}^N p_i$.

Lemma 4.1. *The maximum profit of the durapolist satisfies* $\Pi^D \leq \sum_{i=1}^N p_i$.

To prove this we require the following three lemmas.

Lemma 4.2. *In equilibrium, consumer i never pays more than p_i whenever she buys before the last period.*

For the case $T = 2$, the lemma follows by contradiction. Indeed, assuming the lemma is false, we show in the proof that if the lowest value consumer who is supposed to buy at $t = 1$ refuses to buy, then at $t = 2$ the durapolist would charge a lower price and the consumer can obtain a higher profit. We then extend this result for games with arbitrary number of periods using an inductive argument.

Next, we provide an upper bound on durapolist profits based on the static monopoly prices $\{p_1 \ldots, p_n\}$ and the static monopoly profits of each submarket $\mathcal{G}_i$. Note that each p_i is equal to some consumer's value so we will define y_i as the consumer with the smallest index such that $p_i = v(y_i)$.

Lemma 4.3. *The maximum revenue of the durapolist satisfies*

$$\Pi^D \leq \max_{m \leq N} \left((y_m - m + 1) \cdot v(y_m) + \sum_{i=1}^{m-1} p_i \right)$$

The intuition behind this result is that regardless of what is the optimal strategy, the durapolist sells prior to the final round to the top $m - 1$ consumers for some value $m \leq N$. The total profit extracted can then be split into the profit obtained in the last period (first term of the right hand side) and the revenue obtained prior to period T (second term of the right hand side). The result then follows by taking the maximum among all values for m.

The last piece required to prove Lemma 4.1 consists of an upper bound on the static monopoly profits based on the sum of all static monopoly prices.

Lemma 4.4. *The static monopoly revenue for the market $\mathcal{G}_m$ is at most*

$$\sum_{j=m}^{N} p_j$$

The proof is based on an inductive argument, but this time on the number of *distinct* static monopoly prices on the submarkets $\mathcal{G}_i$ with $1 \leq i \leq N$. Lemma 4.1 naturally follows by combining Lemma 4.3 and Lemma 4.4.

It now remains to prove the upper bound on the sum of the static monopoly prices of every submarket $\mathcal{G}_i$.

Lemma 4.5. $\sum_{i=1}^{N} p_i \leq \Pi^M + p_1$

Observe that if there is a single consumer, the lemma is trivially true. The proof is based on an induction argument on the total number of consumers. Finally, the second inequality of Theorem 4.1 follows directly from Lemmas 4.1 and 4.5.

We finish this section by showing that there are examples in which the durapolist can extract the static monopoly profits plus the static monopoly price. To see this, assume that $T = 2$ and set $v_j = v_H$ for $1 \leq j \leq k$ and $v_j = v_L = \frac{1}{n-k+1} \cdot v_H$ for all $k+1 \leq j \leq n$. The optimal solution is to charge V_H in the first period and V_L in the last period. The high value consumers will buy in the first period and the low value consumers in the last period. It can be verified that this solution satisfies the equilibria conditions.

The total revenue is then

$$\begin{aligned} k \cdot v_H + (n-k) \cdot v_L &= k \cdot v_H + \frac{n-k}{n-k+1} \cdot v_H \\ &= \Pi^M + \left(1 - \frac{1}{n-k+1}\right) \cdot v_H \end{aligned}$$

Thus in the limit $n \gg k$ we obtain $\Pi^M + v_H$. As $v_H = p_1$, the durapoly profits exceed static monopoly profits by an additive amount close to static monopoly price. Observe also that in the case $k = 1$ we have that $\Pi^M = v_H$ and, thus, durapoly profits are twice static monopoly profits.

5 The Effect of Discounting

The purpose of this section is: first to extend the characterization provided in Section 3 via a dynamic program to the case where the discount factor is less than 1. Second, we provide a bound on how much each consumer would pay which can then be translated to an upper bound on the durapoly profits. Finally, we show that in the limit when the discount factor tends to 1, we recover the bound provided in Section 4.

In the durable good monopoly problem with discounting, if consumer i buys a product in period t for a price μ she obtains an utility of $\delta^{t-1} \cdot (v_i - \mu)$. Similarly, if the durapolist sells k items at price μ in period t, her profit in that period is $\delta^{t-1} \cdot k\mu$ [13].

We first show how the dynamic program constructed in Section 3 can be extended into this setting. In the last round (i.e. period T) the durapoly and consumers would behave exactly the same as before. Now, following the same notation, suppose we are at time period $T-1$ in the market $\mathcal{G}_i$. If the durapolist at period $T-1$ wishes to sell to consumers $\{i, i+1, \ldots, k\}$, then the announced price has to be at most k's threat price which is now equal to $(1-\delta) \cdot v_k + \delta \cdot p(k,T)$. To see this, suppose that the price announced at $T-1$ is higher and the durapolist still expects to sell the item to consumers $\{i, i+1, \ldots, k\}$. If consumer k buys, her utility is *less* than

$$\begin{aligned} &\delta^{T-2}[v_k - ((1-\delta) \cdot v_k + \delta \cdot p(k,T))] \\ &= \delta^{T-1} v_k - \delta^{T-1} p(k,T) \end{aligned} \tag{2}$$

[13] In the new dynamic program $\Pi(k,t)$ denotes the profit obtained in the submarket $\mathcal{G}_k$ starting at period t in period t money (i.e. the money is not discounted back to period t=1).

Now observe that (2) is exactly the utility consumer k would get if she refuses to buy in period $T-1$ while all consumers above her buy. This is because in the final time period T the durapolist would be in the market $\mathcal{G}_k$, and announce a price $p(k,T)$. Therefore, consumer k would have benefited from deviating.

Thus, the optimal strategy for the durapolist is to sell to $k-i+1$ consumers at period $T-1$ at price $(1-\delta)\cdot v_k+\delta\cdot p(k,T)$, choosing the value of k such that the profits from periods $T-1$ and T are maximized:

$$j^*(i,T-1) = \arg\max_{k\geq i}\{(k-i+1)((1-\delta)\cdot v_k+\delta\cdot p(k,T))+\delta\cdot\Pi(k+1,T)\}$$
$$\Pi(i,T-1) = (j^*(i,T-1)-i+1)\cdot p(j^*(i,T-1),T)+\delta\cdot\Pi(j^*(i,T-1)+1,T)$$

The price announced at period $T-1$ can then be written as

$$p(i,T-1)=(1-\delta)\cdot v_{j^*(i,T-1)}+\delta\cdot p(j^*(i,T-1),T)$$

Iterating this argument backwards in terms of the periods, we have that

$$\begin{aligned}
\Pi(i,t) \quad &= \quad (j^*(i,t)-i+1)\cdot p(j^*(i,t),t+1)+\delta\cdot\Pi(j^*(i,t)+1,t+1)\\
j^*(i,t) \quad &= \quad \arg\max_{j\geq i}\ ((j-i+1)((1-\delta)v_j+\delta p(j,t+1))+\delta\cdot\Pi(j+1,t+1))\\
p(i,t) \quad &= \quad (1-\delta)\cdot v_{j^*(i,t)}+\delta\cdot p(j^*(i,t),t+1)
\end{aligned}\tag{3}$$

Next, we provide an upper bound on the durapoly profits for games with an arbitrary discount factor δ. We begin by generalizing Lemma 4.2 to account for discounting.

Lemma 5.1. *In an equilibrium with the skimming property, if consumer i buys at period $t<T$, she never pays more than $(1-\delta)v_i+\delta^{T-t}p_i$.*

Observe that the upper bound provided by Lemma 5.1 on consumer i buying price is a combination of two terms: her valuation and the static monopoly price (p_i) in the submarket $\mathcal{G}_i$. In games with a large discount factor (i.e. δ close to zero), consumers cannot afford to wait much and they are willing to pay a price close to their valuation. By contrast, when the discount factor is small (i.e. δ close to 1) consumers can afford to wait and therefore the bound is close to the bound provided in Section 4 (i.e. p_i). Indeed, one can easily show that in the limit when δ tends to 1, we recover the bound obtained in Section 4.

Corollary 5.1. *Let Π^D_δ denote the durapoly profits in a game with a common discount factor $\delta<1$. Then, $\lim_{\delta\to1}\Pi^D_\delta\leq\Pi^M+p_1\leq 2\cdot\Pi^M$.*

Table 3: The impact of discounting on durapoly profits

Discount factor	$t = 1$		$t = 2$		Durapoly Profits
	Price	Items sold	Price	Items sold	
$\delta = 1$	80	2	50	2	260
$\delta = 0.9$	80.5	2	50	2	251
$\delta = 0.8$	81	2	50	2	242
$\delta = 0.7$	81.5	2	50	2	233
$\delta = 0.6$	82	2	50	2	224
$\delta = 0.5$	65	3	50	1	220
$\delta = 0.4$	68	3	50	1	224
$\delta = 0.3$	71	3	50	1	228
$\delta = 0.2$	74	3	50	1	232
$\delta = 0.1$	77	3	50	1	236

We finish this section with an analysis on how a discount factor affects the outcome of our original example provided in Section 2.1. Recall our example involves two periods and four consumers whose valuations are $\{100, 85, 80, 50\}$ (see Table 1). Without a discount factor, we previously calculated the durapoly profits to be 260. Those profits were obtained by selling two items in the first period for a price of 80, and selling another two units in period two for a price of 50. Table 5 reports the optimal prices, units sold and durapoly profits under the SPNE characterized in (3) under different discount factors. First, observe that if the consumers wait to purchase at period 2, their utility is diminished as the discount factor gets closer to zero. This fact is exploited by the durapolist who slightly increases the price in period 1 as the discount factor goes from 1 to 0.6. It worth noting that despite that the price increases as δ gets smaller (from 1 to 0.6) the durapoly profits decrease due the diminished profits obtained in the second period. When δ reaches 0.5, the durapoly profits obtained from selling two items in period 2 become too small and then the durapoly is better off by targeting three consumers in the first round. To do so, the period 1 price is reduced to 65, so that the consumer 3 would not strictly prefer to wait and obtain an utility of $0.5 \cdot (80 - 50) = 15$ rather than buying at period 1 for the same utility. When the discount factor gets even smaller, the durapolist is again able to raise the first period price for the same reasons as before: the utility consumers get from period 2 purchases get smaller and smaller as δ decreases. In the limit when δ tends to zero, the utilities obtained from period 2 (for both durapolist and consumers) are negligible and the durapoly profits approach the static monopoly profits.

6 Conclusions

In this paper we studied the durable good monopoly problem, a classical problem in bargaining theory. In our setting, we consider consumers to be atomic and that there is a finite time horizon during which sales occur. We characterized all profit maximizing strong-Markovian equilibria and proved that, in those equilibria, durapolist profits are comparable to those of a static monopolist. This is in contrast with previous results in which durapoly profits are either arbitrary small or arbitrary large compared to those of a static monopolist.

The paper leaves two interesting questions for future research. The first one is to study whether our bounds hold for non-skimming equilibria. For two-period games, we can prove that the durapolist profits are not larger in those non-skimming equilibria than in the skimming equilibria. However, for games with arbitrary (but finite) number of time periods the question remains open.

Finally, there is an interesting open problem with regard to games with a discount factor. In Section 4 we provided tight bounds for durapoly profits without discounting. Later, in Section 5, we analyzed games with discounting and we provided a Nash equilibrium characterization via a dynamic program which extended that of Section 3. Moreover, using a new lemma on consumers' maximum willingness to pay in games with discounting (Lemma 5.1), we showed that in the limit when the discount factor tends to 1, we recover our result that durapoly profits are at most equal to the static monopoly profits plus the static monopoly price (Corollary 5.1). A natural follow-up question, which we leave open, is whether our upper bound provided in Theorem 4.1 also holds for any discount factor.

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Appendix 1: Proofs

Proofs of Section 3

Proof of Lemma 3.1:

Without loss of generality we can restrict to the case $i = 1$. Let $p_1 = v_a$ and $p_2 = v_b$ and therefore $a \geq 1$ and $b \geq 2$. As a first case, consider that $a \leq b$. Given that valuations are non-increasing, it follows that $p_1 = v_a \geq p_2 = v_b$. As the second case, suppose that $a > b \geq 2$. We know that

$$av_a \geq bv_b,$$

by definition of p_1. Now if $a > b$, in the market $\mathcal{G}_2$ the static monopolist has the option of selling to exactly the consumers in $[2, a]$. Therefore, as v_b is the static monopoly price for $\mathcal{G}_2$, the game with consumers $[2, N]$, it follows that

$$(b-1)v_b \geq (a-1)v_a.$$

Combining these two inequalities gives us $v_a \geq v_b$. But $a > b$ implies $v_b \geq v_a$, so we conclude that $v_a = v_b$. In either case, $v_a \geq v_b$. □

Proof of Lemma 3.2:

We proceed by backwards induction on t. For $t = T-1$, $\tau(i,t) = p_i$, the static monopoly price for the market $\mathcal{G}_i$, for all i. By Lemma 3.1, $p_i \geq p_k$ whenever $i \leq k$. Now consider any earlier period t. $p(i, t+1) = p(j^*(i, t+1), t+2)$ for all i. By the inductive hypothesis, if $j^*(i, t+1) \leq j^*(k, t+1)$, then we are done. Recall that $j^*(k, t+1)$ is determined by the sales schedule which maximizes revenue earned in $\mathcal{G}_i$ for the remaining periods. In particular, for all $l \geq j^*(k, t+1)$,

$$\begin{aligned}(j^*(k,t+1) - k + 1) \cdot p(j^*(k,t+1), t+2) + \Pi(j^*(k,t+1)+1, t+2)\\ \geq (l-k+1) \cdot p(l, t+2) + \Pi(l+1, t+2)\end{aligned}$$

Again, by the inductive hypothesis, $p(j^*(k, t+1), t+2) \geq p(l, t+2)$. Now multiply this inequality by $k - i$ (which is non-negative) and add it to the above to get

$$\begin{aligned}(j^*(k,t+1) - i + 1) \cdot p(j^*(k,t+1), t+2) + \Pi(j^*(k,t+1)+1, t+2)\\ \geq (l-i+1) \cdot p(l, t+2) + \Pi(l+1, t+2),\end{aligned}$$

for every $l \geq j^*(k, t+1)$. Since $j^*(i, t+1)$ satisfies

$$j^*(i,t+1) = \arg\max_{j \geq i} \left((j-i+1) \cdot p(j, t+2) + \Pi(j+1, t+2)\right)$$

it follows that $j^*(i, t+1) \leq j^*(k, t+1)$, which gives us our result. □

Proof of Lemma 3.3:

Let x' denote the top valued consumer in S'. It is clear that in the case $T = 1$, the result holds (the price either remains the same or increases to $v_{x'}$). Consider the optimal sales schedule for the game $\mathcal{G}$, and assume this schedule sells to more than one person in period 1. But prices for a schedule which sells to more than one person in period 1 do not depend on the value of the top consumer in S (the price in period 1 depends on the threat price of a lower-valued consumer, and threat prices for a consumer i depend only on consumers valued at or below v_i.). Therefore, we can achieve the same profit from the $\mathcal{G}$-optimal schedule in $\mathcal{G}'$ with the same prices. So in $\mathcal{G}'$, the optimal sales and pricing schedule either is the same as the optimal for $\mathcal{G}$, in which case we are done, or involves selling only to x in the first period. If the durapolist sells to x' in the first period, it is at a price $p^*_{\mathcal{G}'}(1,1)$ equal to x''s threat price. This is, by definition, the same price as the first period price of a game $\mathcal{G}''$ with consumers S' but with $T-1$ periods instead of T periods. By the induction hypothesis, this price is higher than the corresponding optimal first period price p_S for the game with consumers S but with $T-1$ periods. But p_S, by definition, is the threat price for the top consumer in S in $\mathcal{G}$, and therefore at least as high as the period 1 threat price under the optimal sales schedule in $\mathcal{G}$ (if $v_i \geq v_j$, i's threat price is $\geq j$'s threat price: see proof of Lemma 3.2 in this section). But the optimal period 1 threat price is $p^*_{\mathcal{G}}(1,1)$, so $p^*_{\mathcal{G}'}(1,1) \geq p_S \geq p^*_{\mathcal{G}}(1,1)$.

It remains to prove the case where the optimal sales schedule in $\mathcal{G}$ sells to just the top consumer in S. In this case, the price charged in period 1 is p_S. But by the same argument as above, the threat price for x is at least as high as p_S. Therefore if the optimal sales schedule in $\mathcal{G}'$ sells to just x in the first period, the first period price is at least as high as in $\mathcal{G}$. But note that if the durapolist sells to more than one person in period 1 of $\mathcal{G}'$, she achieves the same profit as a sub-optimal sales schedule in $\mathcal{G}$. But she can clearly beat that revenue by selling to x in period 1 and then following the optimal sales schedule in $\mathcal{G}$ from period 2 onwards. Therefore, whatever the optimal sales schedule in $\mathcal{G}'$, it must involve selling exactly one item in period 1. Therefore we have $p^*_{\mathcal{G}'}(1,1) \geq p_S = p^*_{\mathcal{G}}(1,1)$.

We have covered all cases, so the lemma is proved. □

Proof of Lemma 3.4:

If both durapolist and consumer follow the strategies described in Section 3, the durapolist will select an initial sales path $\{x_t\}$, such that, for the last consumer j_t scheduled to buy in period t ($j_t = \sum_{i \leq t} x_i$), we have the recursive relationship:

$$p(j_t + 1, t + 1) = \tau(j_{t+1}, t + 1) = p(j_{t+1}, t + 2)$$

In other words, in period t the durapolist plans to sell to consumers $j_t + 1, j_t + 2, \ldots, j_{t+1}$ at

j_{t+1}'s threat price. By Lemma 3.2, this is less than or equal the threat price of everyone in the set $\{j_t + 1, j_t + 2, \ldots, j_{t+1}\}$. So under the consumer strategies specified, all x_t consumers in $\{j_t + 1, j_t + 2, \ldots, j_{t+1}\}$ buy in period t, and the durapolist's strategy never deviates from the initial sales path. So the price she charges in each period t is $p(j_{t-1} + 1, t)$ (with $j_0 \equiv 0$).

As part of Lemma 3.2 we showed that $j^*(i, t+1) \leq j^*(k, t+1)$ for all $i \leq k$. Using this and the full result of Lemma 3.2, we have

$$\begin{aligned}
p(j_{t-1} + 1, t) &= p(j_t, t+1) \\
&= \tau(j^*(j_t, t+1), t+1) \\
&\geq \tau(j^*(j_t + 1, t+1), t+1) \\
&= \tau(j_{t+1}, t+1) \\
&= p(j_t + 1, t+1),
\end{aligned}$$

where in the fourth line, we use the fact that $j^*(j_t + 1, t+1) = j_{t+1}$ from the definition of the j_t's and the argmax condition of the recursion relation (1). So these prices are non-increasing in time. $\square$

Lemma 6.1. *There is a subgame perfect equilibrium which follows the recursion relationship (1) in which a sale occurs in each period until all consumers have already bought the item, and this equilibrium achieves at least as much profit for the durapolist as any which allows the durapolist to not sell in some periods where there are consumers remaining.*

Proof. The proof will be by induction. In the case $T = 1$, it is clearly a dominant strategy for the durapolist to sell if there are any consumers remaining, as otherwise he will earn nothing. This also holds for the last period of a longer game in a subgame perfect equilibrium.

Now consider $T > 1$, and to start, assume that the claim is false. Then there must be a game $\mathcal{G}$ with a period $t^l < T$ such that the following two conditions hold: (a) there remain consumers who have not bought at the start of period t^l, but the durapolist does not sell any items in this period; (b) the durapolist sells items in every subsequent period of the game. So there is an equilibrium for a game $\mathcal{G}'$, corresponding to the subgame of $\mathcal{G}$ starting at t^l, where the durapolist sells nothing in the first period, and sells at least one item in all subsequent periods, until all consumers have bought, and this equilibrium yields strictly more profit than one which follows (1). Thus, to show our claim is true, we only need to show that equilibria where we sell nothing in the first period followed by sales in every subsequent period (until all consumers have bought) do not yield more profit than those which follow the recursion relationship (1). Furthermore, it is enough to show a sub-optimal sales schedule yields as much profit, as the optimal recursion relationship result must do even better.

Let $\mathcal{G}(N, T)$ be our game with a set of consumers N and T periods. If nothing is sold in the first period, and the durapolist sells at least one item in each subsequent period (until all consumers have bought), the durapolist can achieve profit of at most $\Pi^D_{\mathcal{G}(N,T-1)}$ (see discussion preceeding Corollary 3.1). Let $k \geq 1$ and p be the number of items sold in the first period and the first period price, respectively, of $\mathcal{G}(N, T-1)$ under (1). Consider the following (possibly sub-optimal) strategies for $\mathcal{G}(N, T)$: the durapolist sells at price p in period 1 and follows the equilibrium of (1) for all subsequent periods, while the consumers buy iff the price is less than their threat price. We know that the top consumer will buy in period 1 as she would be offered the same price if everyone refused to buy in period 1. We also know that no consumer $i > k$ will buy as p is equal to k's threat price, $\tau_{\mathcal{G}(N,T-1)}(k, 1)$, but if i and all below her refused to buy in $\mathcal{G}(N, T)$, the price in the second period of $\mathcal{G}(N, T)$ would be at most $\tau_{\mathcal{G}(N,T-1)}(k + 1, 1)$. So some number $1 \leq l \leq k$ buys in the first period. By definition of (1), we sell at least one item in each subsequent period as well, as long as there are consumers remaining to buy. If $l = k$, then, by the induction hypothesis, we make at least $\Pi^D_{\mathcal{G}(N-[k],T-2)}$ with sales in each subsequent period until all consumers are sold to. So our total profit from this possibly sub-optimal scheme is at least

$$\Pi^D_{\mathcal{G}(N-[k],T-2)} + k \cdot p = \Pi^D_{\mathcal{G}(N,T-1)}.$$

If $1 \leq l < k$, we know that $\Pi^D_{\mathcal{G}(N-[l],T-1)}$ is larger than the profit obtained by selling to $k - l$ consumers at price p and then following the equilibrium for $\mathcal{G}(N - [k], T - 2)$ given by (1). So we obtain profit of

$$l \cdot p + \Pi^D_{\mathcal{G}(N-[l],T-1)} \geq l \cdot p + (k - l) \cdot p + \Pi^D_{\mathcal{G}(N-[k],T-2)} = \Pi^D_{\mathcal{G}(N,T-1)}.$$

Therefore under the optimal schedule, the profit is at least $\Pi^D_{\mathcal{G}(N,T-1)}$. Therefore there is no benefit to waiting a period before starting to sell. This proves the claim. $\square$

Proof of Theorem 3.1:

Since we can treat any subgame as an instance of a full game with a different set of consumers, there is no loss of generality in assuming that the deviation occurs in the first period of the full game. Let $j^*(1, 1)$ be the lowest value consumer sold to in equilibrium and assume that a consumer $x \leq j^*(1, 1)$ deviates by not buying in period 1. If $x = j^*(1, 1)$, then x is charged her threat price in the next period, which by definition is $p^*(1, 1)$, so there is no advantage in a deviation. If $x < j^*(1, 1)$, then the remaining consumers for period 2 are $\{x, j^* + 1, ..., N\}$. We know that if the set of consumers was $\{j^*, j^*+1, ..., N\}$, then the price would be $p^*(1,1)$. But by Lemma 3.3, the price with consumers $\{x, j^*+1, ..., N\}$ must be at least as high as $p^*(1,1)$. Therefore x cannot gain by delaying her purchase for one period.

One may wonder whether consumer x could benefit from delaying the purchase by more than one period. But this is not the case. Consider the subgame $\mathcal{G}'$ arrived at after x delays purchase for $t-1$ periods in the full game, and after re-indexing so the remaining consumers are sorted by value from highest to lowest. If $x = j^*(1,t)$ for this subgame, the price at period $t+1$ is $p(j^*(1,t), t+1) = p(j^*(j^*(1,t), t+1), t+2)$. This means that the price at $t+2$ will be the same as the price at $t+1$ if consumer x does not buy and everyone else follows the equilibrium path. If, on the other hand, consumer $x < j^*(1,t)$ for subgame $\mathcal{G}'$, the price at period $t+2$ could only increase or stay equal to $p(j^*(j^*(1,t), t+1), t+2)$ by Lemma 3.3. By repeated use of this argument we conclude that, at equilibrium, no consumer would benefit from delaying its purchase.

It remains to show that no consumer can benefit from buying early. If a consumer deviates from the equilibrium path by buying early, she pays a price $p^*(1,t)$ when she could have bought in period $t' > t$ at price $p^*(k,t')$ for some $k > 0$. But since prices are non-increasing as a function of time along the proposed sales path (Lemma 3.4), she cannot do any better.

It follows that we have a strategy profile which is an equilibrium in every subgame. $\square$

Lemma 6.2. *Given a subset $S \subseteq [N]$, let $p(S)$ be the static monopoly price of the subgame consisting of consumers $S \subseteq [N]$. Then, a game $\mathcal{G}$ satisfies $p_i = v_i$ for all $i \in [N]$ if and only if $p(S) = \max\{v_x : x \in S\}$ for all $S \subseteq [N]$.*

Proof. Suppose there exists a subset $S \subseteq [N]$ such that $p(S) < v_i$ where $i = \arg\max\{v_x : x \in S\}$. Let the valuations of the consumers in S be $v_i \geq v_2' \geq v_3' \geq \ldots \geq v_{|S|}'$. Then, we have $j \cdot v_j' > v_i$ for some $j > 1$. But then $p_i < v_i$ since by setting a price of v_j' in the subgame with consumers $\{i, i+1, \ldots, N\}$ yields a profit of at least $j \cdot v_j' > v_i$ (since in the market $\mathcal{G}_i$ there may be more consumers with valuations between v_j' and v_i). The remaining implication follows directly. $\square$

Proof of Lemma 3.5:

Before we proceed with the proof, we need some notation. Let $w_1 > w_2 > \ldots > w_M$ denote the M *distinct* consumer valuations sorted in decreasing order. Let n_i denote the number of consumers with value w_i. We set $w_i = n_i = 0$ for all $i > M$. The following technical lemma will be required in the proof. In words, the lemma says that the revenue increase of selling to the n_i consumers at price w_i rather than selling to the same consumers at the price w_k (with

$k \geq 2$) is never less than the revenue obtained by selling to the $n_{\beta+i}$ consumers at price $w_{\beta+i}$ (with $\beta \geq k$).

Lemma 6.3. *If $p_i = v_i$ for all $i \in [N]$, the following inequality holds for every natural number $\beta \geq 2$, $k = 2, \ldots, \beta$ and all $i = 1, \ldots, k-1$,*

$$n_i \cdot w_i - n_i \cdot w_k - n_{\beta+i} w_{\beta+i} \geq 0.$$

Proof. The statement is trivially true if $k > M$ as then $w_k = w_{\beta+i} = 0$. Thus, we may assume that $1 \leq i < k \leq M$. First we show that $w_i \geq 2w_k$. By assumption, $p_i = v_i$. Hence

$$1 \cdot w_i \geq (1 + n_{i+1} + \ldots + n_k) \geq 2 \cdot w_k \tag{4}$$

Similarly

$$w_k \geq (1 + n_{k+1} + n_{k+2} + \ldots + n_{\beta+i}) \cdot w_{\beta+i} \geq n_{\beta+i} \cdot w_{\beta+i} \tag{5}$$

Combining (4) and (5) we have

$$\begin{aligned} n_i \cdot w_i - n_i \cdot w_k - n_{\beta+i} w_{\beta+i} &\geq n_i \cdot w_i - (n_i + 1) \cdot w_k \\ &\geq n_i \cdot w_i - (n_i + 1) \cdot \frac{w_i}{2} \\ &\geq 0 \end{aligned}$$

as desired. $\square$

We now proceed with the proof by induction in the number of time periods. For games with a single period (i.e., $T = 1$) the lemma holds because the durapolist announces the optimal static monopoly price, which is p_1. Suppose now that the lemma holds for all games with at most $T - 1$ periods and consider a game with T periods. Let A be the set of consumers that buy at period 1 under an equilibrium $\mathcal{E}$. Observe that by Lemma 6.2 (in this appendix), the subgame that begins at period 2, with consumers $[N] - A$ satisfies $p_i = v_i$ and therefore there exists an equilibrium where the durapolist uses the Pacman strategy from then on. This means that consumers expect zero profits whenever they do not buy in the first time period, and therefore they would buy in the first time period at any price that is not above their valuation. If $M \leq T$ the durapolist may announce at $t = 1$ the price $\mu_1 = v_1$. All consumers with a valuation of v_1 would buy and, by the inductive hypothesis and the fact that the number of different valuations in the remaining game is still less than the time periods left ($M - 1 \leq T - 1$), the durapolist would be able to extract all economic surplus. Thus, Pacman is an optimal strategy. We now analyze the case where $M > T$. Observe that because consumers will buy in the first period if and only if the price is not above their value, the durapolist's strategy space can be restricted, without loss of generality, to announcing a first price equal to the valuation of some consumer.

Let $\Pi(k)$ denote the profits for the whole game if the first price is w_k. Since by induction hypothesis the Pacman strategy is an equilibrium after the first period, we have that

$$\Pi(k) = \sum_{i=1}^{k} n_i \cdot w_k + \sum_{j=k+1}^{T+k-1} n_j \cdot w_j.$$

Now we want to show that

$$\Pi(1) \geq \Pi(k)$$

for all $k = 1, \ldots, M$. Consider first the case where $k \leq T$. By Lemma 6.3, by setting $\beta = T$, we have that for all $k = 1, \ldots, T$ and all $i = 1, \ldots, k-1$:

$$0 \leq n_i \cdot w_i - n_i \cdot w_k - n_{T+i} \cdot w_{T+i}$$

Summing over i we obtain

$$\begin{aligned}
0 &\leq \sum_{i=1}^{k-1} (n_i \cdot w_i - n_i \cdot w_k - n_{T+i} \cdot w_{T+i}) \\
&= \sum_{i=1}^{k-1} n_i \cdot w_i - \sum_{i=1}^{k-1} n_i \cdot w_k - \sum_{j=T+1}^{T+k-1} n_j \cdot w_j \\
&= \sum_{i=1}^{k-1} n_i \cdot w_i + \sum_{i=k}^{T} n_i \cdot w_i - \sum_{i=1}^{k-1} n_i \cdot w_k - \sum_{i=k}^{T} n_i \cdot w_i - \sum_{j=T+1}^{T+k-1} n_j \cdot w_j \\
&= \sum_{i=1}^{T} n_i \cdot w_i - \sum_{i=1}^{k-1} n_i \cdot w_k - \sum_{j=k}^{T+k-1} n_j \cdot w_j \\
&= \sum_{i=1}^{T} n_i \cdot w_i - \sum_{i=1}^{k} n_i \cdot w_k - \sum_{j=k+1}^{T+k-1} n_j \cdot w_j \\
&= \Pi(1) - \Pi(k)
\end{aligned}$$

Thus $\Pi(1) \geq \Pi(k)$. Second, consider the case where $k > T$. By Lemma 6.3, setting $\beta = k > T$, we have that for all $i = 1, \ldots, k-1$:

$$\begin{aligned}
0 &\leq \sum_{i=1}^{T} (n_i \cdot w_i - n_i \cdot w_k - n_{k+i} \cdot w_{k+i}) \\
&= \sum_{i=1}^{T} n_i \cdot w_i - \sum_{i=1}^{T} n_i \cdot w_k - \sum_{j=k+1}^{T+k} n_j \cdot w_j \\
&= \Pi(1) - \Pi(k)
\end{aligned}$$

Thus, again, $\Pi(1) \geq \Pi(k)$. So there exists an equilibrium where the durapolist uses the Pacman strategy. $\square$

Proof of Theorem 3.2:

Take a game $\mathcal{G}$ with $M \leq T$ and $v_i = p_i$ for all $i \in [N]$. By Lemma 3.5 there exists an equilibrium in which the durapolist uses the Pacman strategy. Moreover, since $M \leq T$, under this equilibrium the durapolist obtains all the economic surplus.

The contrapositive also holds. First, assume $M > T$. Then, since the number of time periods is less than the number of different valuations it is impossible for the durapolist to extract the value of every consumer before the end of the game. Second, assume $v_i > p_i$ for some $i \in [N]$. Now take an equilibrium that extracts all the economic surplus. At equilibrium, prices must be non-increasing over time. Moreover, since all economic surplus is extracted, it implies that consumers also purchase in decreasing order of value over time which means that the skimming property holds. If consumer i bought in the last period it means she has the lowest valuation, i.e. $v_i = w_M$, and $p_i = v_i$ which is a contradiction. In the case consumer i buys before the last period, Lemma 4.2 (see next section) implies that consumer i never pays more than p_i, again a contradiction. $\square$

Proofs of Section 4

Proposition 6.1. $\Pi^M \leq \Pi^D$

Proof. The proof is by induction on the number of periods. The base case is trivial. Consider the (possibly sub-optimal) sales schedule where the durapolist sells at p_1 in period 1, and then follows an equilibrium strategy for the remaining subgame $\mathcal{G}'$. Let k_1 be the number of consumers sold to in period 1 under this schedule (note that we do not restrict k_1 to be non-zero). Let $\Pi^M_{\mathcal{G}} = j \cdot v_j \equiv j \cdot p_1$ and $\Pi^M_{\mathcal{G}'} = (k - k_1)v_k$. Since $j = |\{i | v_i \geq v_j\}|$, $j \geq k_1$ (no one with value less than v_j is willing to pay v_j). But $k = \arg\max_{i \geq k_1}(i - k_1)v_i$, therefore $(k - k_1)v_k \geq (j - k_1)v_j$. So

$$\Pi^D \geq k_1 p_1 + \Pi^D_{\mathcal{G}'} \geq k_1 p_1 + \Pi^M_{\mathcal{G}'} \geq k_1 p_1 + (j - k_1)v_j = k_1 p_1 + (j - k_1)p_1 = \Pi^M,$$

where, in the second inequality, we used the induction hypothesis. $\square$

Proof of Lemma 4.2:

We proceed by induction in the number of time periods. For $T = 2$, let A be the set of consumers that buy at $t = 1$. Suppose for the purpose of contradiction that some consumer $i \in A$ pays a price higher than p_i. Because of the skimming property and the relabeling mentioned in Section 3, we have that $A = \{1, \ldots, k\}$ for some k. By Lemma 3.1, this price is also more than p_k, so consumer k pays more than p_k. But if consumer k refuses to buy at $t = 1$, then at $t = 2$ the durapolist would charge p_k which is a contradiction since consumer k would have obtained a higher profit by waiting.

Now suppose that the lemma is true for all games of 1 to T periods and consider a game $\mathcal{G}$ of $T + 1 > 2$ periods. Let E denote a subgame perfect Nash equilibrium with the skimming property in $\mathcal{G}$. For the purpose of contradiction, suppose that at some time period t ($t < T+1$), there is at least one consumer j that pays more than p_j. Among all those consumers, let i denote the one with the lowest valuation. Since it is not true that at period t consumer ℓ pays more than p_ℓ for $\ell > i$, and by Lemma 3.1 $p_i \geq p_\ell$, then consumer i is the lowest valuation consumer that buys at period t. Therefore, if consumer i refuses to buy at period t we end in the market $\mathcal{G}_i$ with $T + 1 - t$ periods. If $T + 1 - t = 1$ (i.e. t was the second to last period), the durapolist will charge the price p_i at the last period. If $T + 1 - t > 1$, it holds by the induction hypothesis that consumer i would never pay more than p_i. Thus, we can conclude that such equilibrium E cannot exist as consumer i would have obtained a higher profit by waiting. $\square$

Proof of Lemma 4.3:

Consider the subgame perfection conditions. In the final time period T, consumer i is willing to pay up to v_i. In periods 1 to $T - 1$, consumer i is willing to pay up $p_i = v(y_i)$, the static monopoly price for the market $\mathcal{G}_i$.

Suppose that in the optimal solution, the durapolist sells to consumers $\{m, m+1, \ldots, M\}$ where $1 \leq m \leq M \leq N$ in the final period T. Since consumer $M = y_m$ buys in the final period, the revenue then is exactly $(y_m - m + 1) \cdot v(y_m)$. By Lemma 4.2, consumers who buy in earlier periods, that is consumers $\{1, 2, \ldots m-1\}$, pay at most their static monopoly prices. Therefore, the maximum revenue is upper bounded by

$$(y_m - m + 1) \cdot v(y_m) + \sum_{i=1}^{m-1} p_i$$

The result follows by taking the maximum over all consumers m. $\square$

Proof of Lemma 4.4:

Without loss of generality, by re-indexing so that $m = 1$, it suffices to show that

$$\Pi^M = y_1 \cdot v(y_1) \leq \sum_{j=1}^{N} p_j \tag{6}$$

Let $C = \{p_j : j = 1, \ldots, N\}$. We proceed by induction on $|C|$, that is, the number of distinct static monopoly prices over all the markets $\mathcal{G}_j$. For the base case, $|C| = 1$, we have $p_1 = p_j$ for all j. Thus $p_1 = p_N = v_N$ and $y_1 = N$. Every consumer then pays v_N and the total revenue is

$$y_1 \cdot v(y_1) = N \cdot v_N = \sum_{j=1}^{N} p_j$$

Assume the proposition holds for $|C| = k-1 \geq 1$. Now take the case $|C| = k$. Let consumer l be the highest index consumer in the original game with $p_l = p_1$. Thus $p_{l+1} < p_1 = v(y_1)$. By the induction hypothesis, applied to the market $\mathcal{G}_{l+1}$ on consumers $\{l+1, l+2, \ldots, N\}$, we have

$$\sum_{i=l+1}^{N} p_i \quad \geq \quad (y_{l+1} - l) \cdot v(y_{l+1}) \tag{7}$$

Consequently,

$$\begin{aligned}
\sum_{i=1}^{N} p_i &= l \cdot p_1 + \sum_{i=l+1}^{N} p_i \\
&\geq l \cdot p_1 + (y_{l+1} - l) \cdot v(y_{l+1}) \\
&\geq l \cdot p_1 + (y_l - l) \cdot v(y_l) \\
&= l \cdot v(y_l) + (y_l - l) \cdot v(y_l) \\
&= y_l \cdot v(y_l) \\
&= y_1 \cdot v(y_1)
\end{aligned}$$

Here the first equality follows by definition of l. The first inequality follows by (7). The second inequality holds as $v(y_{l+1})$ is the static monopoly price for the market $\mathcal{G}_{l+1}$. The final three equalities follow by definition of l. That is $p_1 = p_l$ and so $y_l = y_1$.

This shows that (6) holds as desired. □

Proof of Lemma 4.1:

Combine Lemma 4.3 and Lemma 4.4. □

Proof of Lemma 4.5:

We proceed by induction on N. For games with a single consumer, the statement is trivially true. Recall that $p_i = v(y_i)$ is the static monopoly price for the market $\mathcal{G}_i$ on consumers $\{i, i+1, \ldots, N\}$ and y_i is the index of the lowest value consumer whose value is not less than p_i. Consider now a game $\mathcal{G}$ with $N+1$ consumers. It remains to show that

$$\sum_{i=1}^{N+1} p_i \leq v(y_1) + y_1 \cdot v(y_1).$$

We proceed as follows:

$$
\begin{aligned}
\sum_{i=1}^{N+1} p_i &= v(y_1) + \sum_{i=2}^{N+1} p_i \\
&\leq v(y_1) + v(y_2) + (y_2 - 1) \cdot v(y_2) \qquad (8) \\
&= v(y_1) + y_2 \cdot v(y_2) \\
&\leq v(y_1) + y_1 \cdot v(y_1) \qquad (9)
\end{aligned}
$$

Here equation (8) follows by the induction hypothesis and inequality (9) comes from the fact that $v(y_1)$ is the static monopoly price of the market $\mathcal{G}_1$. □

Proof of Theorem 4.1:

The first inequality follows from Proposition 6.1. The second inequality follows directly from Lemmas 4.1 and 4.5. □

Proofs of Section 5

Proof of Lemma 5.1:

We proceed by induction in the number of time periods. For $T = 2$, let A be the set of consumers that buy at $t = 1$. Suppose for the purpose of contradiction that some consumer $i \in A$ pays a price higher than $(1 - \delta)v_i + \delta^{T-t}p_i$. Because of the skimming property and the relabeling procedure mentioned in Section 3, we have that $A = \{1, \ldots, k\}$ for some k. By Lemma 3.1, this price is also more than $(1 - \delta)v_k + \delta^{T-t}p_k$, so consumer k pays more than $(1 - \delta)v_k + \delta^{T-t}p_k$. Consumer k utility is then *less* than

$$
\begin{aligned}
& v_k - ((1 - \delta) \cdot v_k + \delta \cdot p_k) \\
&= \delta(v_k - p_k) \qquad (10)
\end{aligned}
$$

Now observe that (10) is exactly the utility consumer k would have obtained if she refused to buy at period 1 and bought at period $T = 2$. This is a contradiction.

Now suppose that the lemma is true for all games of 1 to T periods and consider a game $\mathcal{G}$ of $T + 1 > 2$ periods. Let E denote a subgame perfect Nash equilibrium with the skimming property in $\mathcal{G}$. For the purpose of contradiction, suppose that at some time period t ($t < T + 1$), there is at least one consumer j that pays more than $(1 - \delta)v_j + \delta^{T-t}p_j$. Among all those consumers, let i denote the one with the lowest valuation (and higher index). Since it is not true that at period t consumer ℓ pays more than $(1 - \delta)v_\ell + \delta^{T-t}p_\ell$ for $\ell > i$, and by Lemma 3.1 $p_i \geq p_\ell$, then consumer i is the lowest valuation consumer that buys at period t. Therefore, if consumer i refuses to buy at period t we end in the market $\mathcal{G}_i$ with $T+1-t$ periods. If $T+1-t=1$ (i.e. t was the second to last period), the durapolist will charge the price p_i at the last period and consumer i would have obtained a higher profit by waiting. If $T+1-t>1$, it holds by the induction hypothesis that consumer i would never pay more than

$$\begin{aligned}&(1-\delta)v_i+\delta^{T+1-(t+1)}p_i\\&=(1-\delta)v_i+\delta^{T-t}p_i\end{aligned}$$

Thus, we can conclude that such equilibrium E cannot exist as consumer i would have obtained a higher profit by waiting. □

Proof of Corollary 5.1:

Let E denote a subgame perfect Nash equilibrium with the skimming property. If consumer i buys under E, let t_i denote the time period at which the item was bought. Suppose in the last period the remaining consumers are $\{k, k+1, \ldots, n\}$.

Then, by Lemma 5.1, we have

$$\begin{aligned}\Pi_\delta^D &\leq \sum_{i=1}^{k-1}(1-\delta)v_i+\delta^{T-t_i}p_i+\delta^{T-1}\cdot(y_k-k+1)\cdot p_k\\&\leq \sum_{i=1}^{k-1}(1-\delta)v_i+\delta^{T-t_i}p_i+\delta^{T-1}\cdot\sum_{i=k}^{n}p_i \qquad (11)\end{aligned}$$

where the second inequality comes from Lemma 4.4.

In the limit when δ goes to 1 the right hand side in (11) becomes

$$\sum_{i=1}^{N}p_i$$

The result then follows by Lemma 4.5. □

Appendix 2: Incomplete Information

In this appendix, we introduce a restrictive incomplete information setting and show that the SPNE characterization obtained in Section 3 for the complete information setting also applies here.

Consider the setting in which the market participants (durapolist and consumers) *know* the distribution of consumer values and the aggregate number of sales per period, but *do not know* who buys in each period [14]. Since we are interested in studying equilibria that satisfy the skimming property, regardless of the values of consumers who bought at period t, the durapolist off-path belief is that the k consumers who bought in period t are those with the k highest valuations (among those remaining). We will show that the same conditions as in Section 3 characterizing the subgame perfect equilibria apply.

We first define the strategy of the durapolist and the consumers in any subgame under this incomplete information setting. Let $\mathcal{G}_S$ denote the subgame at period t where the $|S|$ remaining consumers have valuations $w_1 \geq w_2 \ldots \geq w_{|S|}$. Due to the off-path belief (i.e., the belief that consumers follow the skimming property), the durapolist would behave as if it were in the market $\mathcal{G}'_1$ with $T - t + 1$ periods in which the consumers are $v'_1 \geq v'_2 \geq \ldots, v'_{|S|}$ where $v'_i = v_{i+N-|S|}$. Observe that $v'_i \leq w_i$ for all $i \in [|S|]$. The monopolist strategy is to then announce the price $p^*_{\mathcal{G}'}(1, 1)$ which is obtained by solving the recursion relationship (1). The consumers strategy remains the same as in the complete information setting, i.e. each of them would buy if and only if the price is less than or equal to their threat price as calculated for $\mathcal{G}'$. We now prove the following result.

Theorem 6.1. *The strategies defined above constitute a SPNE in the incomplete information setting.*

Proof. We consider the subgame $\mathcal{G}' = \mathcal{G}_S$ (of the original game $\mathcal{G}$) that begins at period t in which the remaining consumers consist of the set S. These consumers have valuations $w_1 \geq w_2 \ldots \geq w_{|S|}$. Due to the off-path belief (i.e., the belief that consumers follow the skimming property), the durapolist would behave as if it were in the market $\mathcal{G}'_1$ with $T - t + 1$ periods in which the consumers are $v'_1 \geq v'_2 \geq \ldots, v'_{|S|}$ where $v'_i = v_{i+N-|S|}$. Observe that in this market $\mathcal{G}'_1$, consumer i's real value is actually $w_i \geq v'_i$.

The announced price in $\mathcal{G}'_1$ would then be $p(1, t) = p(j^*(1, t), t + 1)$. Suppose now that some consumer x that was supposed to buy under the proposed equilibrium, i.e. $i \leq x \leq j^*(1, t)$ deviates and chooses not to buy at time t. The number of sales at period t would then be $j^*(1, t) - 1$, i.e., one less than the expected. The durapolist, who observes the total number of sales and assumes consumers follow the skimming property, would then behave as if the remaining subgame starting at $t + 1$ is $\mathcal{G}'_{j^*(1,t)}$. This means that the announced price would be $p(j^*(1, t), t + 1)$ and consumer x would not have benefited from delaying the purchase by one period. One may, again, wonder whether consumer x could benefit from delaying the purchase by more than one period. But this is not possible since the price at period $t+1$ in this subgame is $p(j^*(1, t), t+1) = p(j^*(j^*(1, t), t+1), t+2)$, which means the price will remain constant over

[14]Note this is equivalent to the case where market participants can see who buys in period t, and know the distribution of values (and know their own value), but do not know exactly which consumer has which value.

time, as long as the number of transactions is one less than the expected. By repeated use of this argument we conclude that, at equilibrium, no consumer would benefit from delaying its purchase.

Lastly, observe that no consumer can benefit from buying earlier. If a consumer deviates from the equilibrium path by buying earlier, she pays a price $p^*(1,t)$ when she could have bought in period $t' > t$ at price $p^*(k,t')$ for some $k \geq 1$. But since prices are non-increasing as a function of time along the proposed sales path (Lemma 3.4), she cannot do any better.

So we conclude that we have a strategy profile which is an equilibrium in every subgame. □

Note that since the equilibrium path is the same in both our complete and incomplete information setting, all our results also apply to the incomplete information setting.

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