

# Assortment optimization under a multinomial logit model with position bias and social influence

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**Abstract** Motivated by applications in retail, online advertising, and cultural markets, this paper studies the problem of finding an optimal assortment and positioning of products subject to a capacity constraint in a setting where consumers preferences can be modeled as a discrete choice under a multinomial logit model that captures the intrinsic product appeal, position biases, and social influence. For the static problem, we prove that the optimal assortment and positioning can be found in polynomial time. This is despite the fact that adding a product to the assortment may increase the probability of selecting the no-choice option, a phenomenon not observed in almost all models studied in the literature. We then consider the dynamics of such a market, where consumers are influenced by the aggregate past purchases. In this dynamic setting, we provide a small example to show that the natural and often used policy known as popularity ranking, that ranks products in decreasing order of the number of purchases, can reduce the expected profit as times goes by. We then prove that a greedy policy that applies the static optimal assortment and positioning at each period, always benefits from the popularity signal and outperforms any policy where consumers cannot observe the number of past purchases (in expectation).

**Keywords** Assortment optimization · Marketing · Social influence

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### 1 Introduction

One of the most studied problems in the area of revenue management is the optimal assortment problem. Informally, the problem consists in selecting a subset of products to offer to consumers so that the expected profit is maximized. Such optimal subset depends on the profits obtained by selling a unit of each of these products, as well as the purchasing behavior of the consumers, which is represented using a discrete choice model. Most of the discrete choice models of practical significance are special cases of the Random Utility Model (RUM) (Block and Marschak 1960).

Some of the most studied discrete choice models are the Multinomial Logit (MNL) (Luce 1965), the Nested Multinomial Logit (NMNL) (Williams 1977) and the Mixed Multinomial Logit (MMNL) (Daly and Zachary 1978). When consumers choose products according to the MNL model, the optimal assortment problem can be solved efficiently (Talluri and Van Ryzin 2004). This result was later extended by Rusmevichientong et al. (2010a) to the case in which there is a constraint on the maximum number of products that can be offered. However, when consumers follow either an NMNL or an MMNL model, the optimal assortment problem is NP-hard (Davis et al. 2014; Rusmevichientong et al. 2010b; Bront et al. 2009).

The underlying assumption in the vast literature that studies the assortment problem is that the consumer choice behavior is solely affected by the subset of products being offered. Specifically, consumers are not assumed to be influenced by the order or way in which products are displayed, nor by social signals such as the past purchases. Those assumptions, however, are violated in many real-world situations. For instance, supermarket profits are considerably affected not only by which products they place in shelves, but also by how those products are allocated among the shelves (Dreze et al. 1995). Position effects are also seen in search engines, where top ranked results receive more clicks than lower ranked ones (see e.g. Buscher et al. 2009; Craswell et al. 2008), in online markets (Kempe and Mahdian 2008; Maillé et al. 2012; Hummel and McAfee 2014), and in online recommendations systems (Lerman and Hogg 2014). The effect of social signals (such as the display of the number of past purchases) in consumer behaviour has also been observed in a diverse range of online stores (Krumme et al. 2012; Salganik et al. 2006; Tucker and Zhang 2011; Engstrom and Forsell 2014).

Given the importance of the bias effects present in retail, online advertising, and cultural markets, we study a model to find the optimal assortment of products to offer to consumers who are affected by position bias and social influence. More specifically, our setting considers that consumers choices can be described by a multinomial logic model where product perceived utilities are a combination of their intrinsic utility, a position bias factor, and a (positive and nondecreasing) function of the aggregate past purchases, which captures the social influence. Our contributions are as follows.

1. We show that the static problem of finding the optimal assortment and positioning for this multinomial logit model is polynomial-time solvable. This holds true, even with a capacity constraint which imposes a maximum number of products that can be displayed.

2. For a dynamic version of this model, we study the problem of choosing a policy which selects a ranking depending on the current time and the past purchases in order to maximize the expected profit. Surprisingly, we provide an small example that shows that the policy, used by many online firms, of ranking products in decreasing order of their number of purchases can reduce the expected profit as times goes by.
3. We then propose a greedy policy that applies the optimal assortment and positioning at every period and prove that it is beneficial to reveal the aggregate past purchases to consumers in order to maximize the expected profit. Specifically, we prove that this policy achieves a higher expected profit than the optimal profit achieved when consumers cannot observe the number of aggregate past purchases.

The first contribution generalizes a result by [Rusmevichientong et al. \(2010a\)](#) to include position bias and to study the role of social influence. The second and third contributions shed new light on social influence. Indeed, most studies focus on showing the negative side of social influence, i.e., the increased unpredictability and inequalities it creates. This paper shows its main positive aspect: its ability to improve the efficiency of the market.

The rest of the paper is organized as follows. In Sects. 2 and 3, we present the related work and problem specification. A detailed review of previous results that are required for understanding are given in Sect. 4. In Sect. 5, we prove that maximizing the expected profit in the proposed model can be performed in polynomial time. The benefits of social influence are presented in Sect. 6. Finally, Sect. 7 presents some conclusions and future work.

## 2 Related literature

The problem studied in the paper is motivated by the seminal study of social influence in the MUSICLAB ([Salganik et al. 2006](#)) and the quantitative model introduced in ([Krumme et al. 2012](#)) that tries to understand those experiments. In the MUSICLAB, participants enter a cultural market and are presented with a number of songs. A market participant chooses a song and, after listening to the song, is given the opportunity to download it. The number of downloads of each song were shown as a social signal to some participants, while no social signal was displayed to others. The experiments were used to demonstrate the unpredictability of cultural markets in the presence of social influence.

The model for the MUSICLAB introduced in ([Krumme et al. 2012](#)) is defined in terms of a market composed of  $n$  songs. Each song  $i \in \{1, \dots, n\}$  is characterized by two values: Its *appeal*  $A_i$  which represents the inherent preference of listening to song  $i$  based only on its name and its band; Its *quality*  $q_i$  which represents the conditional probability of downloading song  $i$  given that it was sampled. The MUSICLAB experiments present each participant with a playlist  $\pi$ , i.e., a permutation of  $\{1, \dots, n\}$ . Each position  $p$  in the playlist is characterized by its *visibility*  $v_p$  which is the inherent probability of sampling a song in position  $p$ . The model specifies the probability of listening to song  $i$  at time  $k$  given a playlist  $\sigma$  as

$$p_{i,k}(\sigma) = \frac{v_{\sigma_i} (\alpha A_i + D_{i,k})}{\sum_{j=1}^n v_{\sigma_j} (\alpha A_j + D_{j,k})},$$

where  $D_{i,k}$  is the number of downloads of song  $i$  at time  $k$  and  $\alpha > 0$  is a scaling factor which is the same for all songs. Observe that the probability of sampling a song is in fact a Multinomial Logit Model with position bias effects.

This paper generalizes the MUSICLAB model by introducing a limit on how many products can be selected for display, and by embedding the social signal in a non-decreasing, positive social influence function  $f$ . This paper also generalizes some of our own results in (Abeliuk et al. 2015) by showing that in this more general setting, the expected profit can also be optimized in polynomial time and that social influence is also beneficial in maximizing expected profit.

In addition to the MUSICLAB, this paper is also related to the sponsored search literature. In a sponsored search market, a publisher selects items (typically ads) to place in a number of slots on a Web page. The *Click-Through Rate* (CTR) for a link  $l$  is the probability that  $l$  receives a click. This probability may depend on a combination of factors, the most significant ones being the relevance of the content and the positioning of the links. The simplest model, which is pervasive in the sponsored search literature (Kempe and Mahdian 2008; Maillé et al. 2012), assumes that the CTRs are independent. More precisely, that model assumes that the CTR of link  $l$  is the product of a position effect  $\theta_k$  and a relevance effect  $q_l$ . This simplification makes the model attractive both from theoretical and practical standpoint in online advertising, since the optimal allocation is simply obtained by sorting the advertisements by decreasing  $\theta_k q_l$ . However, the independence assumption of CTRs is not always justified. Experimental analysis using eye-tracking (Joachims et al. 2005; Buscher et al. 2009) has inspired cascade models, first introduced by Craswell et al. (2008) and subsequently generalized. Informally speaking, the cascade model captures a sequential search, where users consider links from top to bottom and only look at the next link if the previous link was not selected. An axiomatic model for predicting click-through rates when the probability an item receives a click is affected by the qualities of the items by the other positions is studied in Hummel and McAfee (2014). The authors present the MNL model with position bias as a specific case of their general model and provide an algorithm to compute the optimal position allocation. However, no bound on its time complexity were given in the paper. The complexity was by fact settled in Abeliuk et al. (2015) for a more general setting with social influence.

The model studied in this paper generalizes these results further by considering the capacitated assortment version of the MNL model. The resulting model includes position bias, utilities, revenues, social influence, and a capacity constraint on the number of products presented to customers. After submission of this paper, we became aware of a preprint by Davis et al. (2013) where it is shown how to solve the static assortment problem under a multinomial logit model with a set of totally unimodular constraints via a linear program. The authors proved that many static assortment optimization problems can fit into their setting including one in which there is a position bias as in our model.

### 3 Problem specification

Consider a market with  $N$  products  $\mathcal{P} = \{1, \dots, N\}$ , where only  $c \leq N$  products in  $\mathcal{P}$  can be displayed. In the MNL model, each product  $i \in \mathcal{P}$  has an utility  $q_i + \epsilon_i$ , where  $q_i$  is a constant representing the inherent quality of product  $i$  and  $\epsilon_i$  is a random variable following a Gumbel distribution with zero mean and representing the error term. The no-purchase option, denoted as the product 0, is always available to consumers, and without loss of generality its quality is set to zero (i.e.  $q_0 = 0$ ). Given a subset  $S \subseteq \mathcal{P}$ , the probability that a consumer chooses product  $i \in S$  is

$$P_i(S) = \frac{e^{q_i}}{\sum_{j \in S} e^{q_j} + 1}.$$

In this paper, we consider the optimal assortment problem in which consumers follow the MNL but each product must be displayed in one of  $N$  positions, each of which has a visibility  $\theta_i \geq 0$  ( $1 \leq i \leq N$ ). Without loss of generality, we assume  $\theta_1 \geq \theta_2 \geq \dots \geq \theta_N \geq 0$ . A position assignment is an injective function  $\sigma : S \rightarrow \{1, \dots, N\}$  that assigns a position to each product in an assortment  $S \subseteq \mathcal{P}$ . The intrinsic utility  $u_i$  of product  $i$  when displayed in position  $j$  is shifted by  $\ln \theta_j$ . Since

$$e^{q_i + \ln \theta_j} = e^{\ln \theta_j} e^{q_i} = \theta_j e^{q_i},$$

the probability of selecting product  $i$  becomes

$$P_i(S, \sigma) = \frac{\theta_{\sigma_i} u_i}{\sum_{j \in S} \theta_{\sigma_j} u_j + 1}.$$

where  $u_i = e^{q_i}$  for notational simplicity. Note that, for any assortment  $S$  and position assignment  $\sigma$ , we have

$$\sum_{i \in S} P_i(S, \sigma) + P_0(S, \sigma) = 1,$$

where  $P_0(S, \sigma)$  is the no-choice option. For a given position assignment  $\sigma$ , the term  $\theta_{\sigma_i} u_i$  can be interpreted as the position bias correction of the utility of product  $i$  in a equivalent MNL model with no position bias.

Let  $r_i$  denote the marginal profit of product  $i$  and assume that the no-purchase option has no profit, i.e.,  $r_0 = 0$ . The expected profit  $U(S, \sigma)$  of displaying an assortment  $S$  using a position assignment  $\sigma$  is

$$U(S, \sigma) = \sum_{i \in S} r_i P_i(S, \sigma).$$

**Definition 1** (CMLAPP) With the above notations, the Capacitated Multinomial Logit Assortment and Positioning Problem (CMLAPP) is defined as an assortment  $S$  and a position assignment  $\sigma$  that solves

$$Z^* = \max \{U(S, \sigma) \mid S \subseteq \mathcal{P} \wedge |S| \leq c \wedge \sigma : S \rightarrow \{1, \dots, N\}\}.$$

Throughout the paper, the special case of the CMLAPP in which  $c = N$  is called the Unconstrained Multinomial Logit Assortment and Positioning Problem (UMLAPP).

## 4 Capacitated multinomial logit assortment

In this section, we review some algorithmic concepts required for the understanding of the main result of Sect. 5. [Rusmevichientong et al. \(2010a\)](#) considered a special case of the CMLAPP where there is no position bias, i.e.,  $\theta_i = 1$  for all  $(i = 1, \dots, c)$  and no social influence. In other words, they consider the following problem.

**Definition 2 (CMLAP)** The Capacitated Multinomial Logit Assortment Problem (CMLAP) amounts to determining

$$Z^* = \max \{U(S) \mid S \subseteq \mathcal{P} \wedge |S| \leq c\}.$$

where the profit of an assortment  $S$  is defined as  $U(S) = \frac{\sum_{i \in S} r_i P_i(S)}{\sum_{j \in S} u_j + 1}$ , and the probability to select product  $i \in S$  is given by  $P_i(S) = \frac{u_i r_i}{\sum_{j \in S} u_j + 1}$ .

The special case where  $c = N$  is called the Unconstrained Multinomial Logit Assortment Problem (UMLAP).

One of the main results of their paper is to show that the CMLAP can be solved in polynomial time and we now review some of the key concepts and intuition underlying their results. The optimal profit can be characterized in terms of a parameter  $\lambda^*$  such that

$$\begin{aligned} \lambda^* &= \max \{\lambda \in \mathbb{R} \mid \exists S \subseteq \mathcal{P} : |S| \leq c \wedge U(S) \geq \lambda\} \\ &= \max \left\{ \lambda \in \mathbb{R} \mid \exists S \subseteq \mathcal{P} : |S| \leq c \wedge \frac{\sum_{i \in S} u_i r_i}{\sum_{j \in S} u_j + 1} \geq \lambda \right\} \\ &= \max \left\{ \lambda \in \mathbb{R} \mid \exists S \subseteq \mathcal{P} : |S| \leq c \wedge \sum_{i \in S} u_i (r_i - \lambda) \geq \lambda \right\}. \end{aligned}$$

Observe that, for a specific  $\lambda$ , it suffices to rank the expressions  $u_i(r_i - \lambda)$  ( $i \in \mathcal{P}$ ) to find the subset  $S$  maximizing the term  $\sum_{i \in X} u_i(r_i - \lambda)$ . We can then define a function  $A : \mathbb{R} \rightarrow \{S \subseteq \mathcal{P} : |S| \leq c\}$  as

$$A(\lambda) = \arg\text{-max}_{S: |S| \leq c} \sum_{i \in S} u_i (r_i - \lambda)$$

where ties are broken arbitrarily. The optimal profit can be shown to be equivalent to

$$Z^* = \max \{U(A(\lambda)) \mid \lambda \in \mathbb{R}\}.$$

It is valid to drop the condition  $\sum_{i \in S} u_i(r_i - \lambda) \geq \lambda$  due to dominance properties: When the condition is violated for some  $\lambda$ , there exists a value  $\lambda' < \lambda$  such that  $U(A(\lambda)) < U(A(\lambda'))$ . Therefore, the optimal assortment is in the collection of

assortments  $\{A(\lambda) : \lambda \in \mathbb{R}\}$ , which can be interpreted as a nondominated set (Talluri and Van Ryzin 2004) among all subsets of size  $c$  or less. The key observation by Rusmevichientong et al. (2010a) is that there are at most  $O(N^2)$  values to consider for  $\lambda$  and hence at most  $O(N^2)$  subsets of  $\mathcal{P}$  to consider for finding the optimal assortment.

Suppose, without loss of generality, that products are indexed in decreasing order of profit, i.e.,  $r_1 \geq r_2 \geq \dots \geq r_N$ . A *profit-ordered assortment*<sup>1</sup> is a subset of  $\mathcal{P}$  that contains exactly the products whose profit is at least  $r$  for some  $r \in \mathbb{R}$ . It can be shown that in the UMLAP, the family of all profit-ordered assortments contains always an optimal set (Talluri and Van Ryzin 2004). Indeed, if there are no constraints on the capacity, the set  $A(\lambda)$  contains all products  $i$  such that  $u_i(r_i - \lambda) \geq 0$  or equivalently  $r_i \geq \lambda$ , and consequently we have

$$\{A(\lambda) : \lambda \in \mathbb{R}\} = \{\emptyset, \{1\}, \{1, 2\}, \dots, \{1, \dots, N\}\}.$$

Hence, at most  $N$  subsets have to be considered to find an optimal assortment in the uncapacitated version of the problem.

## 5 Capacitated multinomial logit assortment and positioning

This section shows that the CMLAPP can be solved in polynomial time by building on the techniques used for the CMLAP. The intuition behind this result is that the products in the optimal assortment for the CMLAP also have the highest measure of  $u_i(r_i - \lambda)$ , which is a joint measure of utility and profitability. With position effects, it makes sense to assign these same products in decreasing order of  $u_i(r_i - \lambda)$  to maximize the total market share. The key insight is that the CMLAPP only needs to consider the candidate subsets of the CMLAP.

In the problem definition, a position assignment of an assortment could contain ‘gaps’ between products. By a ‘gap’, we mean that there is at least one empty position between two products in the assignment of products to positions. For example, if there are 3 positions available and only 2 products are displayed, then a position assignment contains a gap if either the first or second position is left empty. Suppose the latter occurs, that is product  $a$  is assigned to position 1 and product  $b$  is assigned to position 3. If this position assignment yields a higher profit than the one achieved when product  $b$  is assigned to position 2 instead of 3, then it seems intuitive that not displaying at all product  $b$  is even more profitable. We now show that there is never a benefit of choosing a position assignment that has gaps.

**Definition 3** (*Gap-Free Position Assignment*) Given an assortment  $S$ , a position assignment  $\sigma$  is gap-free if  $\{\sigma_i \mid i \in S\} = \{1, \dots, |S|\}$ .

**Lemma 1** *The CMLAPP always admits an optimal solution that is gap-free.*

*Proof* Let  $\sigma$  be a position assignment with gaps. Assume that no product is assigned to position  $k$  and let  $l$  be the first product assigned to a position higher than  $k$ , i.e.,

<sup>1</sup> Profit-ordered assortments are sometimes referred as revenue-ordered assortments.

$\sigma_l = \min\{\sigma_i \mid i \in S \wedge \sigma_i > k\}$ . Since  $\theta_k \geq \theta_{\sigma_l}$ , moving product  $l$  from  $\sigma_l$  to  $k$  increases its visibility. If this move is not profitable, it must be that

$$\frac{\sum_{i \in S} \theta_{\sigma_i} u_i r_i + (\theta_k - \theta_{\sigma_l}) u_l r_l}{\sum_{j \in S} \theta_{\sigma_j} u_j + (\theta_k - \theta_{\sigma_l}) u_l + 1} \leq \frac{\sum_{i \in S} \theta_{\sigma_i} u_i r_i}{\sum_{j \in S} \theta_{\sigma_j} u_j + 1}.$$

Let  $R = \sum_{i \in S} \theta_{\sigma_i} u_i r_i$  and  $Q = \sum_{i \in S} \theta_{\sigma_i} u_i + 1$ . The above inequality becomes

$$\begin{aligned} \frac{R + (\theta_k - \theta_{\sigma_l}) u_l r_l}{Q + (\theta_k - \theta_{\sigma_l}) u_l} &\leq \frac{R}{Q} \\ \Leftrightarrow QR + Q(\theta_k - \theta_{\sigma_l}) u_l r_l &\leq RQ + R(\theta_k - \theta_{\sigma_l}) u_l \\ \Leftrightarrow r_l &\leq \frac{R}{Q}. \end{aligned}$$

The last inequality states that the marginal profit of product  $l$  is smaller or equal to the expected profit. We now show that we can remove product  $l$  without degrading the profit. Indeed,  $r_l \leq \frac{R}{Q}$  is equivalent to

$$\begin{aligned} -R\theta_{\sigma_l} u_l &\leq -Q\theta_{\sigma_l} u_l r_l \\ \Leftrightarrow RQ - R\theta_{\sigma_l} u_l &\leq RQ - Q\theta_{\sigma_l} u_l r_l \\ \Leftrightarrow \frac{R}{Q} &\leq \frac{R - \theta_{\sigma_l} u_l r_l}{Q - \theta_{\sigma_l} u_l}. \end{aligned}$$

where the first condition is obtained by multiplying both sides by  $-\theta_{\sigma_l} u_l$  and the second by adding  $RQ$  to both sides. The result follows.  $\square$

Lemma 1 indicates that the position assignment of an assortment  $S$  only needs to consider positions  $1, 2, \dots, |S|$ . Let  $[n] = \{1, 2, \dots, n\}$ ,  $n \in \mathbb{N}^+$ . Hence, the position assignment is a bijection from  $S$  to  $[|S|]$ . As in the CMLAP, we define a function  $B : \mathbb{R} \rightarrow \{X \subseteq \mathcal{P} : |X| \leq c\}$  as

$$B(\lambda) = \arg\max_{S: |S| \leq c} \max_{\sigma: S \rightarrow [|S|]} \sum_{i \in S} \theta_{\sigma_i} u_i (r_i - \lambda) \quad (5.1)$$

which, given a value  $\lambda$ , specifies the assortment producing the best profit for some optimal position assignment. The optimal position assignment  $\sigma_S^\lambda$  for assortment  $S$  and value  $\lambda$  is defined as

$$\sigma_S^\lambda = \arg\max_{\sigma: S \rightarrow [|S|]} \sum_{i \in S} \theta_{\sigma_i} u_i (r_i - \lambda),$$

Ties are broken arbitrarily in these two expressions. The optimal profit of the CMLAPP can be then reformulated as

$$Z^* = \max_{\lambda} \left\{ U \left( B(\lambda), \sigma_{B(\lambda)}^\lambda \right) \mid \lambda \in \mathbb{R} \right\}.$$



The optimal position assignment  $\sigma_S^\lambda$  can be computed easily thanks to a rearrangement inequality. Indeed, as the following lemma states,  $\sigma_S^\lambda$  assigns products  $i \in S$  in decreasing order of  $u_i (r_i - \lambda)$ .

**Lemma 2** (Rearrangement Inequality) (Hardy et al. 1952), Section 10.2, Theorem 368) *Let  $x_1, x_2, \dots, x_n$  and  $y_1, y_2, \dots, y_n$  be real numbers (not necessarily positive) with  $x_1 \leq x_2 \leq \dots \leq x_n$  and  $y_1 \leq y_2 \leq \dots \leq y_n$ , and let  $\pi$  be any permutation of  $\{1, 2, \dots, n\}$ . Then the following inequality holds:*

$$x_1 y_{\pi_1} + x_2 y_{\pi_2} + \dots + x_n y_{\pi_n} \leq x_1 y_1 + x_2 y_2 + \dots + x_n y_n.$$

The optimal position assignment must thus satisfy

$$u_{\pi_1} (r_{\pi_1} - \lambda) \geq \dots \geq u_{\pi_{|S|}} (r_{\pi_{|S|}} - \lambda)$$

for some ranking assignment  $\pi : [|S|] \rightarrow S$  and the optimal position assignment can be defined as the inverse of  $\pi$ . We now show that the CMLAPP only needs to consider the assortments considered by the CMLAP. In other words, the optimal assortment and positioning can be found by using the nondominated set  $\{A(\lambda) : \lambda \in \mathbb{R}\}$  defined in Sect. 2.

**Lemma 3** *Let  $\mathcal{B} = \{B(\lambda) : \lambda \in \mathbb{R}\}$  and  $\mathcal{A} = \{A(\lambda) : \lambda \in \mathbb{R}\}$ . Then,  $\mathcal{B} \subseteq \mathcal{A}$ .*

*Proof* Consider  $\lambda \in \mathbb{R}$ ,  $S \in \mathcal{B}(\lambda)$ , and  $\pi$  be the inverse of  $\sigma_S^\lambda$ .  $\pi$  satisfies

$$u_{\pi_1} (r_{\pi_1} - \lambda) \geq \dots \geq u_{\pi_{|S|}} (r_{\pi_{|S|}} - \lambda).$$

Consider first the case where  $|S| < c$ . Then, by optimality of  $B(\lambda)$  and  $\theta_k \geq 0$ , it must be the case that  $u_i (r_i - \lambda) \leq 0$  for all  $i \in \mathcal{P} \setminus S$ . When  $|S| = c$ , it must be the case that

$$u_i (r_i - \lambda) \geq u_j (r_j - \lambda) \quad (i \in S \text{ and } j \in \mathcal{P} \setminus S)$$

since otherwise swapping  $i$  and  $j$  would increase the profit. As a result, for any  $S' \subseteq \mathcal{P}$  such that  $|S'| \leq c$ , we have that

$$\sum_{i \in S} u_i (r_i - \lambda) \geq \sum_{j \in S'} u_j (r_j - \lambda).$$

Hence  $S \in \mathcal{A}$ . □

We are now in position to state that the CMLAPP can be solved in polynomial time.

**Theorem 1** *CMLAPP can be solved in polynomial time.*

*Proof* The result follows from Lemmas 1–3 and the fact that  $\mathcal{A}$  can be computed in  $O(N^2)$  time (Rusmevichientong et al. 2010a). □

This result is perhaps surprising given the fact that the choice model of the CMLAPP violates a property known as the regularity axiom (Block and Marschak 1960). This axiom states that the addition of an element to any choice set  $X$  should not increase the probability of selecting an element in the original set  $X$ , nor the probability of choosing the no-choice option. The regularity axiom is satisfied by a large class of discrete choice models including all discrete choice models based on random utility (Berbeglia and Joret 2015) and it is defined for discrete choice models in terms of choice sets rather than a combination of choice set and positioning. An extension of the definition of regularity for our model taking into account positioning is given below.

**Definition 4** (*Regularity Axiom*) Let  $X \subseteq Y \subseteq \mathcal{P}$ . The regularity axiom for the CMLAPP problem states that

$$P_i(X, \sigma^X) \geq P_i(Y, \sigma^Y) \quad \forall i \in X \cup \{0\},$$

where  $\sigma^S$  is the optimal position assignment for assortment  $S \subseteq \mathcal{P}$ .

**Lemma 4** *The CMLAPP violates the regularity axiom.*

*Proof* Consider the following example. There are only two products with  $u_1 = 0.25$ ,  $r_1 = 2$ ,  $u_2 = 0.75$ ,  $r_2 = 1$ , and visibilities  $\theta_1 = 1$ ,  $\theta_2 = 0.5$ . In the table below, for each assortment  $S$  and position assignment  $\sigma$ , we provide the optimal expected revenue  $Z$  and the probabilities of selecting each of the elements  $P_i$  with  $i = 0, 1, 2$ .

$S$	$\sigma$	$Z$	$P_0$	$P_1$	$P_2$
{1}	[1]	0.4	0.8	0.2	0
{2}	[1]	0.4286	0.5714	0	0.4286
{1, 2}	[1, 2]	0.5385	0.6154	0.1538	0.2308
{1, 2}	[2, 1]	0.5333	0.5333	0.06667	0.4

The optimal position assignment for each value of  $S$  is given by

$S$	{1}	{2}	{1, 2}
Optimal position assignment ( $\sigma^S$ )	[1]	[1]	[1, 2]

When  $S = \{2\}$ , product 2, which has the highest utility, is assigned to the first position and has a position bias corrected utility of  $\theta_1 u_2 = 0.75$ . When product 1, which has the highest marginal profit, is added to the assortment, the optimal positioning assigns product 1 to the first position and product 2 to the second position. Thus, when  $S = \{1, 2\}$ , product 2 has a position bias corrected utility of  $\theta_2 u_2 = 0.375$  and  $\theta_1 u_1 = 0.25$  for product 1. Observe that the sum of the corrected utilities has decreased

from assortment  $\{2\}$  to  $\{1, 2\}$  and hence the probability of selecting the no-purchase option increases when we move from the choice set  $\{2\}$  to the choice set  $\{1, 2\}$ , i.e.:

$$P_0(\{2\}, \sigma^{\{2\}}) < P_0(\{1, 2\}, \sigma^{\{1, 2\}}),$$

where  $\sigma^{\{2\}} = [1]$  and  $\sigma^{\{1, 2\}} = [1, 2]$  are the optimal position assignments for assortments  $\{2\}$  and  $\{1, 2\}$  respectively. This inequality violates the regularity axiom.  $\square$

Consider now the family of profit-ordered assortments

$$\{\{1\}, \{1, 2\}, \dots, \{1, \dots, N\}\},$$

(where again products are indexed in decreasing order of profit, i.e.,  $r_1 \geq r_2 \geq \dots \geq r_N$ ). A natural extension when dealing with the CMLAPP is to consider *profit-ordered assortments and positionings*, where both the assortment and the position allocation are in decreasing order of profit. It follows from Lemma 3 and the optimality of profit-ordered assortments in the UMLAP (Gallego et al. 2004; Liu and van Ryzin 2008) that the optimal assortment in the UMLAPP must be a profit-ordered set. However, the optimal position assignment is not necessarily in decreasing order of profits as we show in Example 1.

**Example 1** Consider two products with  $u_1 = 0.25, r_1 = 2, u_2 = 1, r_2 = 1$  and visibilities  $\theta_1 = 1, \theta_2 = 0.5$ . In the table below, for each value of the capacity  $c$ , we provide the optimal assortment  $S^*$ , the optimal position assignment  $\sigma^*$ , the optimal expected revenue  $Z^*$ , and the probabilities of selecting each of the elements  $P_i$  with  $i = 0, 1, 2$ . When  $c = 2$ , there are no capacity constraints and the optimal assortment is  $S = \{1, 2\}$  which is a profit-ordered set; yet the optimal position assignment  $\sigma = [2, 1]$  assigns products in increasing order of profit  $r_i$ . Note also that the optimal assortment for the CMLAPP is not always a profit-ordered set as shown in the example when  $c = 1$ .

$c$	$S^*$	$\sigma^*$	$Z^*$	$P_0$	$P_1$	$P_2$
1	$\{2\}$	$[1]$	0.5	0.5	0	0.5
2	$\{1, 2\}$	$[2, 1]$	0.5882	0.4706	0.0588	0.4706

Although *profit-ordered assortments and positionings* are sub-optimal for the CMLAPP in general, we now provide sufficient conditions under which they are optimal for the CMLAPP.

**Definition 5** An instance of the CMLAPP is positively correlated if, for any pair of products  $i, j \in [1, \dots, N]$  such that  $u_i \geq u_j$ , we have  $r_i \geq r_j$ .

**Lemma 5** For positively correlated instances of the CMLAPP, the optimal assortments and position assignments are ordered by profit.

*Proof* By Lemma 3, the optimal assortment  $S^* = B(\lambda^*)$  and optimal ranking assignment  $\pi^*$  satisfies

$$u_{\pi_1^*}(r_{\pi_1^*} - \lambda^*) \geq \dots \geq u_{\pi_{|S^*|}^*}(r_{\pi_{|S^*|}^*} - \lambda^*) \geq 0.$$

For any two items  $i, j$  such that  $\pi_i^* < \pi_j^*$ , we have that  $u_{\pi_i^*}(r_{\pi_i^*} - \lambda^*) \geq u_{\pi_j^*}(r_{\pi_j^*} - \lambda^*) \geq 0$ . If  $u_{\pi_i^*} < u_{\pi_j^*}$ , this implies that  $r_{\pi_i^*} - \lambda^* > r_{\pi_j^*} - \lambda^* \Leftrightarrow r_{\pi_i^*} > r_{\pi_j^*}$ , but this violates our assumption of positive correlation. Therefore, the only possible case is  $u_{\pi_i^*} \geq u_{\pi_j^*}$  which, under the assumption, yields  $r_{\pi_i^*} \geq r_{\pi_j^*}$ . Hence  $\pi^*$  is ordered by decreasing values of profit and  $S^*$  is a profit-ordered set.  $\square$

## 6 The benefits of social influence

In this section, we consider a multi-period, dynamic market where consumers arrive sequentially, one per time period. Upon arrival, a consumer is able to observe the aggregate purchase decisions of her predecessors. We analyze how purchase decisions are affected by social influence and position bias.

The perceived utility  $u'_i$  is given by a combination of the inherent utility of, and a social influence signal for, product  $i$ . Denote by  $d_i$  the aggregate number of purchases or selections (e.g., the number of downloads in a music market or the number of clicks on a website) of product  $i$ . Let  $d = \langle d_1, d_2, \dots, d_N \rangle$ . We define the perceived utility of product  $i$  to be

$$u'_i(d) = u_i + f(d_i),$$

where  $f : \mathbb{N}^+ \rightarrow \mathbb{R}^+$  is a non-decreasing and positive social influence function.<sup>2</sup> We assume that  $f(0) = 0$ , so that in the case when all products have zero purchases, social influence plays no role, i.e., for all  $i \in N$ :  $u'_i(\langle 0, \dots, 0 \rangle) = u_i$ . Denote by  $D_{i,t}$  the total number of consumers who have purchased product  $i$  from period 1 until the beginning of period  $t$ . Let  $D_t = \langle D_{1,t}, D_{2,t}, \dots, D_{N,t} \rangle$ . This dynamic market can be formulated as Markov Decision Process where the state of the system is defined by the period  $t$  and the total number of purchases  $D_t$ . The state depends exclusively on the state and decision taken in the previous period. The decision variables at period  $t$  are the assortment  $S$  to be displayed and the position assignment  $\sigma$  to be used. A policy  $\Pi : (t, D_t) \rightarrow (S, \sigma)$  is a function that given a state returns the assortment and the position assignment to be displayed. Denote by  $d = D_t$  the probability that the consumer arriving at period  $t$  will purchase one unit of product  $i$  if assortment  $S$  is displayed using a position assignment  $\sigma$  is given by

$$P_i(S, \sigma, d) = \frac{\theta_{\sigma_i}(u_i + f(d_i))}{\sum_{j \in S} (\theta_{\sigma_j}(u_j + f(d_j))) + 1} = \frac{\theta_{\sigma_i} u'_i(d)}{\sum_{j \in S} \theta_{\sigma_j} u'_j(d) + 1}.$$

In a dynamic market with  $T$  periods, the expected profit can be defined by the following recurrence:

<sup>2</sup> Parameter  $d$ , which is the social signal, only appears as an argument of the social influence function, which is an arbitrary positive and non-decreasing function. Hence, all results hold for any social signal that is a positive and non-decreasing function of the number of purchases.

$$\begin{aligned}
 U_t(d) &= \max_{S, \sigma} \left( \sum_{i \in S} P_i(S, \sigma, d)(r_i + U_{t+1}(d + e_i)) \right. \\
 &\quad \left. + (1 - \sum_{i \in S} P_i(S, \sigma, d)) U_{t+1}(d) \right) \quad (t = 1, \dots, T) \\
 U_{T+1}(d) &= 0
 \end{aligned}$$

where  $e_i$  denotes the  $i$ th unit vector. The total expected profit from period 1 until period  $T$  is given by  $U_1((0, \dots, 0))$ .

We say that the market has no social influence if for any  $x \in \mathbb{N}^+ : f(x) = 0$ , or equivalently, when consumers cannot observe the number of past purchases. Therefore, when there is no social influence, the probability that a consumer arriving at period  $t$  will purchase product  $i$  is given by

$$P_i(S, \sigma) = \frac{\theta_{\sigma_i} u_i}{\sum_{j \in S} (\theta_{\sigma_j} u_j) + 1}.$$

Observe that this probability does not depend on the purchase history and therefore the optimization problem at each period  $t$  is independent of the state. Hence, in the absence of social influence, the market becomes static and the optimal policy corresponds to the optimal solution of the CMLAPP, which is the same at every period.

By contrast, finding the optimal policy of the dynamic market with social influence becomes cumbersome. Indeed, expanding the recurrence function of the expected utility yields  $N^T$  possible configurations for the state vector  $d$ . To circumvent this problem, we resort to analyze the greedy myopic policy. Our goal here is to determine the effect of social influence on this myopic policy in a dynamic market. Denote by  $\Pi^*$  the policy that applies the optimal CMLAPP solution at each period. This section proves that, under social influence, the per-period profit increases over time when using policy  $\Pi^*$ . This holds regardless of the number of products, their utilities, the visibilities and the (positive and nondecreasing) social influence function.

The motivation behind the main result of this section is based on the observation that, in a dynamic market, social influence can be detrimental to the profit of the market if the incorrect assortment and ranking policy is used. For instance, a pervasive trend in Web sites is to rank items by their popularity (e.g., a music market may rank songs according to their number of downloads), yet its effect may be detrimental to the market as we show in Example 2. Specifically, we show that the popularity ranking can reduce the expected profit over time. On the other hand, the example shows that, when the optimal assortment and position assignment of the CMLAPP is used, the profit of the market increases.

We define the value function  $V_1((S, \sigma) | d)$  to be the expected profit at period  $t$  if assortment  $S$  is displayed using a position assignment  $\sigma$  and conditional to  $D_t = d$ , i.e.,

$$V_1((S, \sigma) | d) = \sum_{i \in S} P_i(S, \sigma, d) r_i.$$

**Example 2** Consider two positions with visibilities  $\theta_1 = 1$ ,  $\theta_2 = 0.5$ , two products to be displayed with intrinsic utilities  $u_1 = 8$ ,  $u_2 = 16$  and marginal profits  $r_1 = 1$ ,  $r_2 = 0.9$ . The social influence function is defined to be the identity function on the number of purchases. Hence, if product  $i$  is purchased, its perceived utility  $u'_i$  will

increase by one at the next period, i.e.,  $u'_i = u_i + 1$ . We consider three possible states,  $D_2 = \{\langle 0, 0 \rangle, \langle 1, 0 \rangle, \langle 0, 1 \rangle\}$ , which represent the number of purchases each product can have on the second period assuming that the initial condition is  $D_1 = \langle 0, 0 \rangle$ . In the table below, we provide the expected profit  $V_1((S, \sigma) \mid d)$  for each assortment  $S$  and position assignment  $\sigma$  conditional to the number of purchases  $d$ . The maximum of each column is highlighted in bold.

$S$	$\sigma$	$d = \langle 0, 0 \rangle$ $V_1((S, \sigma) \mid d)$	$d = \langle 1, 0 \rangle$ $V_1((S, \sigma) \mid d)$	$d = \langle 0, 1 \rangle$ $V_1((S, \sigma) \mid d)$
$\{1\}$	$[1]$	0.889	<b>0.9</b>	0.889
$\{2\}$	$[1]$	0.847	0.847	0.85
$\{1, 2\}$	$[1, 2]$	<b>0.894</b>	<b>0.9</b>	<b>0.894</b>
$\{1, 2\}$	$[2, 1]$	0.876	0.879	0.877

Below we compare two assortment and positioning policies and show their expected profits for two consecutive periods assuming that the initial condition is  $D_1 = \langle 0, 0 \rangle$ . The first policy is  $\Pi^*$ , that applies the optimal assortment and position assignment of the CMLAPP at each period, i.e., at period  $t$  and given  $d$ , policy  $\Pi^*$  solves  $\arg\max_{S, \sigma} V_1((S, \sigma) \mid d)$ . Let  $d_0 = \langle 0, 0 \rangle$  and  $(S^*, \sigma^*) = (\{1, 2\}, [1, 2]) = \arg\max_{S, \sigma} V_1((S, \sigma) \mid d_0)$ . Thus, the expected profit of  $\Pi^*$  at period  $t = 1$  is  $V_1((S^*, \sigma^*) \mid d_0) = 0.894$  and the expected profit at period  $t = 2$  is given by

$$\begin{aligned} & \sum_{i=1}^2 P_i(S^*, \sigma^*, d_0) \left( \max_{S, \sigma} V_1((S, \sigma) \mid d_0 + e_i) \right) \\ & + \left( 1 - \sum_{i=1}^2 P_i(S^*, \sigma^*, d_0) \right) V_1((S^*, \sigma^*) \mid d_0) = 0.897. \end{aligned}$$

The second policy consists of ranking products in decreasing order of their number of purchases, which is called the *popularity* ranking. We assume that at period  $t = 1$  and  $d = \langle 0, 0 \rangle$ , the *popularity* ranking starts with the same assortment and position assignment  $(S^*, \sigma^*)$  as policy  $\Pi^*$ . Thus, the expected profit of the *popularity* ranking at period  $t = 1$  is 0.894. Let  $\sigma_1 = [1, 2]$  and  $\sigma_2 = [2, 1]$ . Observe also that  $S^* = \{1, 2\}$  is an optimal assortment in the three possible cases. The expected profit at period  $t = 2$  is given by

$$\begin{aligned} & \sum_{i=1}^2 P_i(S^*, \sigma^*, d_0) V_1((S^*, \sigma_i) \mid d_0 + e_i) \\ & + \left( 1 - \sum_{i=1}^2 P_i(S^*, \sigma^*, d_0) \right) V_1((S^*, \sigma^*) \mid d_0) = 0.889. \end{aligned}$$

The table below summarizes the example with the expected profits at each of the two periods for both policies. The example shows that policy  $\Pi^*$  exhibits an increas-

ing expected profit over time, whereas the expected profit of the *popularity* ranking decreases over time. This indicates that it may not be advantageous to show popularity signal to consumers when the popularity ranking is used.

Period	$\Pi^*$	Popularity
1	0.894	0.894
2	0.897	0.889

We now show that the optimal assortment and position assignment of the CMLAPP always yields an increasing expected profit over time when there is social influence in a dynamic market. The key to the proof is Lemma 6 which uses the following notation:  $V_2(S, \sigma \mid d)$  is the expected profit at period  $t + 1$  conditional to  $D_t = d$  when assortment  $S$  and ranking  $\sigma$  are used at periods  $t$  and  $t + 1$ , i.e.,

$$V_2((S, \sigma) \mid d) = \sum_{i \in S} (P_i(S, \sigma, d) V_1((S, \sigma) \mid d + e_i)) + \left(1 - \sum_{i \in S} P_i(S, \sigma, d)\right) V_1((S, \sigma) \mid d).$$

The right term captures the case where no product is purchased at period  $t$ , while the left term captures the cases where a product  $i$  is purchased, which increases its perceived utility for the next period.

**Lemma 6** *Let  $(S^*, \sigma^*)$  be the optimal assortment and position assignment of the CMLAPP at period  $t$  given any  $D_t = d$ . We have*

$$V_2((S^*, \sigma^*) \mid d) \geq V_1((S^*, \sigma^*) \mid d).$$

*Proof* Since  $f$  is non-decreasing, we define  $\epsilon_i(d) = f(d_i + 1) - f(d_i) \geq 0$  such that the perceived utility of product  $i$  can be rewritten as  $u'_i(d + e_i) = u'_i(d) + \epsilon_i(d) = u_i + f(d_i) + \epsilon_i(d)$ . Thus, the expected profit at period  $t + 1$  conditional to  $D_t = d$  in this context is given by

$$\begin{aligned} & V_2((S^*, \sigma^*) \mid d) \\ &= \sum_{j \in S^*} \left( P_j(S^*, \sigma^*) \frac{\sum_{i \in S^*: i \neq j} \theta_{\sigma_i^*} u'_i(d) r_i + \theta_{\sigma_j^*} (u'_j(d) + \epsilon_j(d)) r_j}{\sum_{i \in S^*: i \neq j} \theta_{\sigma_i^*} u'_i(d) + \theta_{\sigma_j^*} (u'_j(d) + \epsilon_j(d)) + 1} \right) \\ &+ \left( 1 - \sum_{i \in S^*} P_{i,t}(S^*, \sigma^*) \right) \cdot V_1((S^*, \sigma^*) \mid d). \end{aligned}$$

For ease of notation, let  $u = u_1, \dots, u_n$  represent the perceived utility  $u'(d)$  at period  $t$  and let  $\epsilon_i(d)$  be abbreviated to  $\epsilon_i$ . Without loss of generality, we can rename the

products so that  $\sigma_i^* = i$  and drop the ranking subscript  $\sigma^*$ . We also omit writing the inclusion in  $S^*$  in the summations. Let  $\lambda^*$  denote the optimal expected profit at period  $t$  given  $D_t = d$ , i.e.,

$$V_1((S^*, \sigma^*) | d) = \frac{\sum_i \theta_i u_i r_i}{\sum_i \theta_i u_i + 1} = \lambda^*.$$

The expected profit  $V_2((S^*, \sigma^*) | d)$  at period  $t + 1$  conditional to period  $t$ , which we abbreviate to  $V_2^*$ , is given by

$$\begin{aligned} V_2^* &= \sum_j \left( \frac{\theta_j u_j}{\sum_i \theta_i u_i + 1} \cdot \frac{\sum_{i \neq j} \theta_i u_i r_i + \theta_j(u_j + \epsilon_j)r_j}{\sum_{i \neq j} \theta_i u_i + \theta_j(u_j + \epsilon_j) + 1} \right) \\ &\quad + \left( 1 - \frac{\sum_i \theta_i u_i}{\sum_i \theta_i u_i + 1} \right) \cdot \frac{\sum_i \theta_i u_i r_i}{\sum_i \theta_i u_i + 1} \\ &= \sum_j \left( \frac{\theta_j u_j}{\sum_i \theta_i u_i + 1} \cdot \frac{\sum_i \theta_i u_i r_i + \epsilon_j \theta_j r_j}{\sum_i \theta_i u_i + \epsilon_j \theta_j + 1} \right) + \left( 1 - \frac{\sum_j \theta_j u_j}{\sum_i \theta_i u_i + 1} \right) \cdot \lambda^*. \end{aligned}$$

We need to prove that

$$V_2^* \geq \lambda^*, \quad (6.1)$$

which amounts to showing that

$$\sum_j \left( \frac{\theta_j u_j}{\sum_i \theta_i u_i + 1} \cdot \frac{\sum_i \theta_i u_i r_i + \epsilon_j \theta_j r_j}{\sum_i \theta_i u_i + \epsilon_j \theta_j + 1} \right) + \left( 1 - \frac{\sum_j \theta_j u_j}{\sum_i \theta_i u_i + 1} \right) \cdot \lambda^* \geq \lambda^*,$$

which reduces to proving

$$\frac{1}{\sum_i \theta_i u_i + 1} \sum_j \left[ \frac{\theta_j^2 u_j \epsilon_j}{\sum_i \theta_i u_i + \epsilon_j \theta_j + 1} (r_j - \lambda^*) \right] \geq 0$$

or, equivalently,

$$\sum_j \left[ \frac{\theta_j^2 u_j \epsilon_j}{\sum_i \theta_i u_i + \epsilon_j \theta_j + 1} (r_j - \lambda^*) \right] \geq 0.$$

By definition of  $B$  (defined in Eq. 5.1), the optimality of  $S^*$  implies that  $S^* \in B(\lambda^*)$ . Hence, by optimality of  $S^*$ ,  $(r_i - \lambda^*) \geq 0$  for all  $i \in S$ . The result follows since, for all  $i$ ,  $\frac{\theta_j^2 u_j \epsilon_j}{\sum_i \theta_i u_i + \epsilon_j \theta_j + 1} \geq 0$ .  $\square$

We are now in position to state the main result of this section. Lemma 6 states that, if the optimal assortment and position assignment of the CMLAPP in period  $t$  is used at



period  $t + 1$ , the expected profit in period  $t + 1$  is at least as high as the expected profit at period  $t$ . Clearly, re-optimizing at period  $t + 1$ , by using the optimal assortment and position assignment of the CMLAPP for period  $t + 1$ , can only increase the expected profit. Together with Lemma 6, this observation leads to the following theorem.

**Theorem 2** *Under social influence, the expected profit per period is non-decreasing over time when the optimal assortment and ranking of the CMLAPP are used at every period.*

*Proof* For any  $t \in N$  and  $D_t = d$ , let  $(S^*, \sigma^*)$  be the optimal CMLAPP solution in period  $t$  given  $D_t = d$ . By Lemma 6, we know that  $V_2((S^*, \sigma^*) | d) \geq V_1(S^*, \sigma^* | d)$ . By definition,  $V_2((S^*, \sigma^*) | d)$  is defined as

$$\begin{aligned} & \sum_{i \in S} (P_i(S^*, \sigma^*, d) V_1((S^*, \sigma^*) | d + e_i)) \\ & + \left(1 - \sum_{i \in S} P_i(S^*, \sigma^*, d)\right) V_1((S^*, \sigma^*) | d) \end{aligned}$$

while the greedy myopic policy computes

$$\begin{aligned} & \sum_{i \in S} \left( P_i(S^*, \sigma^*, d) \max_{S, \sigma} V_1((S, \sigma) | d + e_i) \right) \\ & + \left(1 - \sum_{i \in S} P_i(S^*, \sigma^*, d)\right) V_1((S^*, \sigma^*) | d) \end{aligned}$$

The result follows.  $\square$

We have shown that the expected profit per period of  $\Pi^*$  in a dynamic market with social influence is non-decreasing over time. Moreover, Theorem 2 entails an interesting corollary about  $\Pi^*$  and the impact of social influence in the market. Indeed, when there is no social influence, the market is static and hence the optimal policy is  $\Pi^*$  since the optimization problem is the same for every period. The expected per-period profit of policy  $\Pi^*$  when there is no social influence is

$$\sum_{i \in S^*} P_i(S^*, \sigma^*) r_i = V_1(S^*, \sigma^* | \langle 0, \dots, 0 \rangle),$$

which corresponds to the profit of the first period in a dynamic market with social influence when the initial conditions are  $D_1 = \langle 0, \dots, 0 \rangle$ . Thus, by Lemma 6, the expected per-period profit of policy  $\Pi^*$  under social influence is never worse than when consumers cannot observe the number of aggregate past purchases.

**Corollary 1** *In expectation, the profit of policy  $\Pi^*$  in the presence of social influence, is at least the optimal profit achieved in the absence of social influence, i.e., when consumers cannot observe the number of past purchases.*

As shown in Example 2, the popularity ranking does not comply with Corollary 1. For that policy, social influence can be detrimental to the profit of the market. However, our results show that social influence if it is used correctly, it can improve the efficiency of the market.

## 7 Conclusion

Motivated by practical applications in retail, e-Commerce and cultural markets, this paper studied the optimal assortment and positioning problem under a multinomial logit model of discrete choice where consumers have position biases and their behavior is affected by the past purchases. We showed that this problem can be solved in polynomial time even when there is a maximum limit on the number of products that the assortment can have. In addition, we analyzed the dynamics of the market when consumers arrive sequentially, and the firm displays an optimal assortment and positioning at each arrival. In this setting, we proved the expected profit can only increase if the firm displays the social influence signal (past purchases).

There are at least five potential research directions which we believe are worth pursuing. The first one is to generalize the results to the case where market participants have more complex preferences, such as for example when they follow a Mixed MNL. The second one is to study the market dynamics when the social signal is not necessarily the number of the past purchases. As an example, it would be very interesting to analyze the market dynamics when the social signal is the market share of all the past purchases. The third one is to study the optimal policy in the dynamic market when there are discount factors over time and compare the discounted profits of different policies. The fourth one is to analyze the expected profit of the popularity ranking which is a pervasive trend in Web sites. Finally, it would be interesting to study the tradeoffs between revenues and customer satisfaction. L'Ecuyer et al. (2015) studied such a tradeoff in another model: They analyzed a separable CTR model in which an online platform ranks items taking into account the short term expected profits and the quality of the results, as both objectives are not necessarily aligned.

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