

The Effect of a Finite Time Horizon in the Durable Good Monopoly Problem with Atomic Consumers *

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Abstract. A durable good is a long-lasting good that can be consumed repeatedly over time, and a duropolist is a monopolist in the market of a durable good. In 1972, Ronald Coase conjectured that a duropolist who lacks commitment power cannot sell the good above the competitive price if the time between periods approaches zero. Coase's counterintuitive conjecture was later proven by Gul et al. (1986) under an infinite time horizon model with non-atomic consumers. Remarkably, the situation changes dramatically for atomic consumers and an infinite time horizon. Bagnoli et al. (1989) showed the existence of a subgame-perfect Nash equilibrium where the duropolist extracts all the consumer surplus.

Observe that, in these cases, duropoly profits are either arbitrarily smaller or arbitrarily larger than the corresponding static monopoly profits – the profit a monopolist for an equivalent consumable good could generate. In this paper we show that the result of Bagnoli et al. (1989) is in fact driven by the infinite time horizon. Indeed, we prove that for finite time horizons and atomic agents, in any equilibrium satisfying the standard skimming property, duropoly profits are at most an additive factor more than static monopoly profits. In particular, duropoly profits are always at least static monopoly profits but never exceed twice the static monopoly profits.

Finally we show that, for atomic consumers, equilibria may exist that do not satisfy the skimming property. For two time periods, we prove that amongst all equilibria that maximize duropoly profits, at least one of them satisfies the skimming property. We conjecture that this is true for any number of time periods.

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1 Introduction

A *durable good* is a long-lasting good that can be consumed repeatedly over time. Theoretically less is known about durable goods than their more well-studied counterparts, consumable and perishable goods. However, on the practical side, durable goods abound and are very familiar to us. For example, many of the most important consumer items are (at least to some extent) durable, such as land, housing, cars, etc. A *duropolist* is a monopolist in the market of a durable good – topically, duropolists include several well-known purveyors of digital goods. Indeed, Amazon has recently been awarded a US patent (8,364,595) for establishing market places for second-hand digital-content items, and Apple has recently applied for a similar patent (20130060616).

Pricing a durable good is not as simple as it may appear. Specifically, whilst durable goods are more desirable to the consumer, it is questionable whether a duropolist has additional monopoly power beyond that of an equivalent monopolist for a perishable good. Indeed, quite the opposite may be true. In 1972, Nobel recipient Ronald Coase made the startling conjecture that, in fact, a duropolist has no monopoly power at all! Specifically, a duropolist who lacks commitment power cannot sell the good above the competitive price if the time between periods approaches zero (Coase, 1972). The intuition behind the Coase conjecture is that if the monopolist charges a high price then consumers anticipate a future price reduction (as they expect the duropolist to later target lower value consumers) and therefore they prefer to wait. The duropolist, anticipating this consumer behaviour, will then drop prices down to the competitive level. In essence, the argument is that a duropolist is not a monopolist at all: the firm does face stiff competition – not from other firms but, rather, from future incarnations of itself! This is known as the *commitment problem*: the duropolist cannot credibly commit to charging a high price.

The Coase conjecture was first proven by Gul et al. (1986) under an infinite time horizon model with non-atomic consumers. They showed that if buyers strategies are stationary (that is, the distribution of consumers after the duropolist announces a price p , lower than all previous prices, is independent of the prior price history) then, as period length goes to zero, the duropolist's first price offer converges to the lowest consumer valuation or the marginal cost, whichever is higher. Ausubel and Deneckere (1989) later showed that if the stationary condition is relaxed, the duropolists profits at subgame perfect equilibria can range from Coasian profits to the static monopoly profit.¹ Stokey (1979) studied pricing mechanisms for duropolists that *possess* commitment power in a continuous time model. She showed that duropolists can then

¹However, profits larger than the Coasian value occur only in the case where there is no gap between the lowest consumer value and the marginal cost of production (e.g. for $c = 0$, the lowest consumer value is 0).

attain the static monopoly profit by committing to a fixed price; all sales are then made at the beginning of the game. McAfee and Wiseman (2008) examined the Coase conjecture in a model where there is small cost for production capacity which can be augmented at each period. In this setting, the authors showed that in the the monopoly profits are equal to those that can be obtained if she could commit ex ante to a fixed capacity.

Underlying the results of Gul et al. (1986) is the assumption of an infinite time horizon. There are nevertheless situations where trade must take place before a hard deadline. For example, consider a TV network selling advertising space a week in advance of a show. Theoretical and empirical evidence of the strong effects of deadlines have been observed in many bargaining contexts such as in contract negotiations and civil case settlements – see, for example, Cramton and Tracy (1992), Williams (1983) and Fuchs and Skrzypacz (2011).

If there is a finite horizon and non-atomic consumers, a feasible action for the duropolist is to decline to sell goods until the final period and then announce the static monopoly price, obtaining the static monopoly profits discounted to the beginning of the game. Although this strategy is not an equilibrium, Guth and Ritzberger (1998) showed that when consumer valuations follow a uniform distribution, there exists a subgame perfect equilibrium, as period lengths approach zero, in which the duropolist profits converge to the static monopoly profits discounted to the beginning of the game.

All these results assumed non-atomic consumers. Less is known about the duropolist problem with atomic consumers. The major difficulty in analyzing equilibria in this setting is that a deviation from a single consumer can modify the equilibrium price path. This characteristic, as we will later show, permits the existence of subgame perfect Nash equilibria where the *skimming property* does **not** hold, i.e. an equilibrium where a buyer with a higher valuation might buy later (and at a lower price) than a buyer with lower valuation. Bagnoli et al. (1989) studied the duropoly problem with atomic consumers. When the time horizon is infinite, they proved another surprising result: the existence of a subgame perfect Nash equilibrium in which the duropolist extracts all the consumer surplus. To obtain this, they considered the following pair of strategies. The duropolist strategy, dubbed *Pacman*, is to announce at each time period, a price equal to the valuation of the consumer with the highest value who has yet to buy. The strategy of each consumer, dubbed *get-it-while-you-can*, is to buy the first time it induces a non-negative utility. If there is an infinite horizon, and the discount factor is large enough, then these strategies are sequential best responses to each other. This equilibrium refutes the Coase conjecture when there is a finite set of atomic buyers and an infinite time horizon. Indeed, it suggests that a duropolist may have perfect price discriminatory power! Moreover, it shows there exist subgame perfect Nash equilibria where duropoly profits exceeds the static monopoly

profits by an unbounded factor.² Later, von der Fehr and Kuhn (1995) showed that under certain conditions Pacman is the only equilibrium.

Another setting, in which the extremely large profits of the Pacman equilibrium of Bagnoli et al. (1989) cannot exist, was proposed by Cason and Sharma (2001). Instead of assuming a duropolist with perfect information, the authors constructed a two-buyer and two-valuation model with infinite time periods in which the duropolist does not know exactly whether a consumer is of high type or of low type. They showed that in these games there exists a unique equilibrium that is Coasian.

Recently, Montez (2013) studied the duopoly problem under infinite horizon with atomic consumers that have two-types (high value consumers and low value consumers) and exactly two consumers are of the high-value type. He showed that there are sometimes inefficient equilibria where the time at which the market clears does not converge to zero as the length of the trading periods approaches zero.

Bagnoli et al. (1989) also examined very small examples with a finite time horizon case. They showed that in these games of two or three consumers, it is possible to obtain subgame perfect equilibria where the duropolist extracts more revenue than the price commitment strategy of Stokey (1979). These examples again refute the Coase conjecture but they also suggest that, even for finite time horizons, the duropolist may have **more** monopoly power than the equivalent static monopolist. This is very interesting because, whilst duropolists are not believed to be powerless in practice, the standard assumption is that duropolists are weaker than monopolists for consumable goods. Indeed this argument has been accepted by the Federal Courts in the United States; a history of duropolies and the law can be found in Orbach (2004).

1.1 Contributions

Several questions arise from the work of Bagnoli et al. (1989). Does this phenomenon (of duopoly profits exceeding static monopoly profits) arise for more natural games where the number of consumers and the number of time periods is much larger than three? If so, how can we characterize the monopolist profit maximizing strategy? Finally, can we quantify *exactly* how much more profit a duropolist can obtain at equilibria in comparison to a static monopolist?

Our main contributions is to answer those three questions. To achieve this, we first characterize, in Section 3, a class of subgame perfect equilibria that satisfy the standard skimming property: high-value consumers buy before lower-valued consumers.³ We do this both in the

²For example, consider a game with N consumers where buyer i has a valuation of $1/i$. Then, as N gets large, duopoly profits under the Pacman strategy approach $\log N$ whilst the static monopoly profit is clearly equal to 1.

³Bagnoli et al. (1989) state that such a characterization would be extremely interesting.

complete information setting as well as a setting where market participants have limited information about other consumer valuations. Our main result, proven in Section 5, is then that, at equilibria, duopoly profits are *at least* static monopoly profits but *at most* static monopoly profits plus the static monopoly price. In particular, duopoly profits are at most twice the static monopoly profits regardless of the number of consumers, their values, and the number of time periods. We also prove that this factor two bound is tight: we construct a (infinite) family of examples where duopoly profits approach double the static monopoly profit as the number of consumers goes to infinity. This construction also gives tightness for the additive bound as the difference between the duopoly profits and the static monopoly price tends to the static monopoly price.

We believe that our main result sheds light into this classical problem in at least four ways. To begin, this is the first theoretical result that concurs with the practical experience that duopolists and static monopolists have comparable profitability. (Recall that previous theoretical works have suggested that the duopolist either has no monopoly power or has perfect price discriminatory power).

Second, the result that a duopolist can do up to an additive amount better using a threat-based strategy rather than a price-commitment strategy is actually best viewed from the opposite direction. Specifically, a duopolist can obtain almost the optimum profit (losing at most an additive amount equal to the static monopoly price) by mimicking a static monopolist via a price commitment strategy. From a practical perspective this is important because a price-commitment strategy can generally be implemented by the duopolist very easily, even with limited consumer information. Furthermore, price-commitment strategies can be popular with consumers as they are typically introduced within a money back guarantee or envy-free pricing framework. In contrast, a threat based optimization strategy is harder to implement and can antagonize consumers.

Thirdly, the standard view in the literature is that the surprising and well-known result of Bagnoli et al. (1989), namely that the duopolist can extract all consumer surplus, is due to the assumption of atomic consumers. Our results show that this is not true in general – their result is driven by the infinite time horizon. For finite time horizons, the power of a duopolist is limited. This is true even when the Pacman strategy is an equilibrium; indeed, we show that the *Pacman* strategy can be an equilibrium in finite time horizon games only under very specific conditions⁴ – see Section 4.

Finally, the main result highlights a distinction in how the time horizon affects bargaining power. With non-atomic consumers, a finite time horizon increases the bargaining power of the

⁴The *Pacman* strategy will not produce an equilibrium in games with finite horizons *unless* consumer valuations decrease exponentially.

duopolist. In Guth and Ritzberger (1998), a finite time-horizon increases duopolist profits from the Coasian result to the static monopoly profits. With atomic consumers, the finiteness of the time horizon, reduces duopolist profits from all consumer surplus to, at most, twice the static monopoly profits.

To conclude the paper, we examine (in Section 6) subgame perfect equilibria in the absence of the skimming-property. We provide the first example of a subgame perfect equilibrium in which the skimming-property does not hold. Furthermore, we conjecture that amongst all equilibria that maximize duopoly profits at least one satisfies the skimming property. We prove this conjecture is true for the case of $T = 2$.

2 The Model

We now present the durable good monopoly model of Bagnoli et al. (1989) that we will analyze in the subsequent sections. Consider a durable good market with one seller (a duopolist), N consumers and a finite horizon of T time periods. The N consumers have valuations $v_1 \geq v_2 \geq \dots \geq v_N$ ⁵ and the firm can produce units of the good at a unitary cost of c dollars. Here we assume, without loss of generality, that $c = 0$. Consequently, profit and revenue are interchangeable in this setting.

We can view this as a sequential game over T periods. At time t , $1 \leq t \leq T$, the firm will select a price μ_t to charge for the good.⁶ The duopolist seeks a pricing strategy that maximises her revenue, namely $\sum_{t=1}^T (x_t \cdot \mu_t)$, where x_t denotes the number of consumers who buy in period t .

Each consumer i desires at most one item and seeks to maximize her *utility*, which is $v_i - \mu_t$ if she buys the good in period t .⁷ The consumers decide simultaneously if they will buy an item for μ_t . The game then proceeds to period $t + 1$. If a consumer doesn't buy an item before the end of period T her utility is zero.

For such a sequential game, the solutions we examine are pure subgame perfect Nash equilibria that satisfy the standard *skimming property* defined below.

Definition 2.1 (Skimming property). *An equilibrium satisfies the skimming property if whenever a buyer with value v is willing to buy at price μ_t , given the previous history of prices h_t , then a buyer with value $w > v$ is also willing to buy at this price given the same history.*

⁵We also use notation $v(y_j)$ instead of v_{y_j} in certain cases to avoid nested subscripts

⁶Note neither the model of Bagnoli et al. (1989), that we analyze in this paper, nor any of the other durable good monopoly models mentioned in the literature review section allow for discriminatory pricing mechanisms in which two or more consumers can be charged different prices in the same time period.

⁷Discount factors can easily be introduced into the model.

For subgame perfect Nash equilibria (SPNE) that satisfy the skimming property, consumers' strategies can be characterized using a cutoff function. Given a history of prices h_t and the current offered price μ_t , consumers with valuations above cutoff $\kappa(h_t, \mu_t, t)$ buy and consumers with valuations below the cutoff do not buy (see Fudenberg and Tirole (1991) for a discussion). When consumers are non-atomic it can be shown that all subgame perfect equilibria satisfy the skimming property (Fudenberg et al., 1985). In the case of atomic consumers and an infinite time horizon, the monopolist can extract all consumer surplus using the *pacman* strategy, in which case the skimming property is clearly satisfied. Intuitively, the skimming property says that higher value consumers pay a higher (or at least equal) price compared to consumers with a lower valuation. In real markets, this phenomenon is widely observed.⁸

Ausubel and Deneckere (1989) define two special types of SPNE that are *Markovian* in the sense that they depend only on the most recent information available. A SPNE is a *weak-Markov* equilibrium if consumers' accept/reject decisions depend only on the current price and period. A SPNE is a *strong-Markov* equilibrium if, in addition to the weak Markov property, the duopolist conditions her strategy only on the payoff-relevant part of the history. In the infinite horizon case, this is the set of remaining consumers. In the finite horizon case, it may depend on the number of periods left as well. When consumers are non-atomic, such equilibria only exist when the Weak-Markov property is satisfied and $\kappa(\mu_t, t)$ is strictly increasing (Fudenberg et al., 1985). In the atomic case, we can obtain strong-Markov equilibria even if κ is constant over an interval.

When constructing an SPNE in the atomic finite-horizon model, we will therefore restrict the strategy space of the duopolist and consumers so that they satisfy the strong-Markov conditions: the prices the duopolist chooses are a function $\mu : \mathcal{P}([N]) \times T \rightarrow \mathcal{R}^+$ and the consumers strategies are such that i buys in period t iff $v_i \geq \kappa(\mu_t, t)$ for some function κ . However, our main result from Section 5 only requires the equilibrium satisfies the skimming property.

We study the model in both a complete and an incomplete information setting. In the complete information setting, the values of each consumer are known to all participants in the market. In our incomplete information setting, the duopolist knows the distribution of values, but not which value corresponds to which consumer. For example, she may know the consumer values are $\{100, 50, 30\}$, but does not know which of these three is the value of Consumer 1. Similarly, each consumer knows the set of values and which is their own value, but does not know which values correspond to the other consumers. Note that, in this model, the N consumers may have N very different types; we do not restrict consumers to be drawn from a common known distribution.

⁸In experimental economics this is tested via screening techniques.

2.1 An Example

We now present an small example to illustrate the model and the concepts involved. Consider a two-period game with the following consumers' valuations.

Table 1: Example of a game with 4 consumers.

Consumer	Consumer value
1	100
2	85
3	80
4	50

Denote by Π^D and Π^M the revenue obtainable by the duopolist and the corresponding static monopolist, respectively. Then the static monopoly profit Π^M is equal to 240, obtained by selling to the top three consumers for a price of 80. However, the duopolist can, in fact, extract a revenue of 260. Furthermore, the corresponding equilibrium satisfies the skimming property: no consumer will buy earlier than another consumer with a higher valuation. To understand SPNEs in this game, let's begin with a subgame comprising of only the final (second) time period. In such a subgame, it is a dominant strategy for all consumers who have not yet bought to pay any price less than or equal to their value. Consequently, in the final period, it is a dominant strategy for the duopolist to charge the static monopoly price *as calculated with respect to the set of consumers who have not yet bought*. Note that these strategies satisfy the strong-Markov property as everyone remaining with value above the price will buy, everyone else will not buy, and the price depends only on the set of consumers remaining.

Now consider the first time period. If the skimming property is satisfied, then there will be a cut-off point j_1 at which consumers $j \leq j_1$ buy and consumers $j > j_1$ wait until period 2. In order for this to be an equilibrium, the consumer j_1 must prefer buying in period 1 to period 2. Therefore, the duopolist can charge no more than the static monopoly price as calculated if all consumers $j \geq j_1$ wait until period 2. We call this price the *threat price* for consumer j_1 . The

Table 2: Threat prices

Consumer	Consumer value	Threat price
1	100	80
2	85	80
3	80	50
4	50	50

threat prices are listed in Table 2.1.

The consumers' strategies then correspond to "buy in period 1 if and only if μ_1 is at most their threat price", whilst the duopolist's strategy is to charge the threat price which maximizes the total revenue. The period 2 strategies are the dominant strategies described above: remaining consumers pay up to their value, while the duopolist charges the static monopoly price calculated for the set of consumers that are left.

Charging $\mu_1 = 80$ means that the top two consumers would buy in period 1, while the last two consumers would wait until period 2 and buy at $\mu_2 = 50$ (the static monopoly price for the remaining two consumers is 50). The total profit would therefore be 260. It is easy to see that charging $50 < \mu_1 < 80$ gives a smaller profit. Moreover, if $\mu_1 > 80$ then no consumers will buy in the first period. This would lead to a profit of 240 as the static monopoly price would then be charged in the final period. Finally charging $\mu_1 = 50$ would result in all consumers buying in period 1, as the duopolist is guaranteed to charge at least 50 in period 2. The total profit would then be 200. Thus, for a profit maximizing duopolist we have $\Pi^D = 260$. So duopoly profits are greater than static monopoly profits. For additional comparisons, Coasian profits are $\Pi^C = 200$ since the competitive price is 50, and Price Discriminatory profits are $\Pi^{PD} = 315$, that is, the consumer surplus.

Observe that these strategies satisfy the skimming property, as threat prices are monotonically increasing with consumer value, and satisfy the weak-Markov property, as $\kappa(\mu_t, t)$ is the smallest consumer value such that his threat price is larger than μ_t . Note that $\kappa(\mu_1, 1)$ is constant for all $\mu_1 \in (50, 80]$. Since the duopolist's strategy depends only on the values of remaining players in each period (trivially in period 1) it also satisfies the strong-Markov property. Furthermore, it is easy to see that this is an equilibrium. Consumers 3 and 4 would be worse off buying in the first period, while both consumers 1 and 2 would still pay 80 if either deviated by waiting until the last period. Similarly, the duopolist would find no buyers if he charged more than 80 in period 1, turning the game into a one-period static game and giving him revenue Π^M , while charging anything between 80 and 50 would yield the same sales schedule but with lower profit. Charging 50 or lower would result in all consumers buying in period 1 for profit at most 200.

We remark that the equilibrium property for the consumers arises from a simple property of static monopoly prices. Take the set of consumers $\{j \geq j_1\}$ and compute the static monopoly price for these consumers. This is j_1 's threat price. Now if we take j_1 and replace him by a higher valued consumer then the static monopoly price can only rise. Hence, as long as μ_1 is j_1 's threat price, any consumer of higher value that refuses to buy in period 1 would be charged at least μ_1 in period 2, ensuring it is a best response to buy in period 1.

In Section 3, we show how these arguments can be extended to give subgame perfect equilibria conditions for games with more than two time periods and explain how, given these constraints, a duopolist can efficiently maximize profits.

3 Sub-Game Perfect Equilibria Conditions

We now characterize the subgame perfect equilibria that satisfy the strong-Markov conditions and maximize duopolist profits. To do this we reason backwards from the final time period T . It is easy to determine the behaviour of rational consumers and a profit maximizing duopolist at time T . Given this information, we can determine the behaviour of rational consumers at time $T - 1$, etc.

We begin with such characterization when there is complete information, that is when all participants know which player has each value and the duopolist can observe who buys in each period. Afterwards, we introduce an incomplete information setting and show that the same SGPNEs characterization applies.

To formalize this, let \mathcal{G}_i denote the subgame consisting of consumers $\{i, i + 1, \dots, N\}$, and let $\Pi(i, t)$ denote the maximum profit obtainable in \mathcal{G}_i if we begin in time period t . Thus $\Pi^D = \Pi(1, 1)$. Now set $\Pi(i, T + 1) = 0$ for all consumers i . Let $p(i, t)$ be the profit maximizing price at period t in the subgame \mathcal{G}_i beginning at time t . First, consider the last period, T . Any consumer i (who has not yet bought the good) will buy in period T if and only if this final price is at most v_i . Therefore, for the subgame \mathcal{G}_i , starting at time T , a profit maximizing duopolist will simply set $\mathbf{p}(i, T)$ to be the static monopoly price p_i for the subgame \mathcal{G}_i : $p(i, T) = p_i \equiv v_{j^*(i, T)}$, where

$$j^*(i, T) = \arg \max_{j \geq i} (j - i + 1) \cdot v_j.$$

Thus, $j^*(i, T)$ denotes the consumer with the lowest valuation who buys in the subgame \mathcal{G}_i beginning at period T . The profit is then

$$\Pi(i, T) = (j^*(i, T) - i + 1) \cdot v_{j^*(i, T)}$$

In general we will denote by $j^*(i, t)$, the consumer with the lowest valuation who buys (under our proposed strategy) at period t in the subgame \mathcal{G}_i beginning at period t . Now, suppose we are at time period $T - 1$ in the subgame \mathcal{G}_i . If the duopolist at period $T - 1$ wishes to sell to consumers $\{i, i + 1, \dots, k\}$, then the announced price has to be at most k 's threat price, $p(k, T) \equiv v_{j^*(k, T)}$. To see this, suppose that the price announced at $T - 1$ is higher and the duopolist still expects to sell the item to consumers $\{i, i + 1, \dots, k\}$. Then, if consumer k refuses to buy while all consumers above her buy, the duopolist would, in the final time period T be in

the subgame \mathcal{G}_k , and announce a price $p(k, T)$, meaning that consumer k would have benefited from deviating. So, the optimal strategy for the duopolist would be to sell to $k - i + 1$ consumers at period $T - 1$ at price $v_{j^*(k, T)}$, choosing the value of k such that the profits from periods $T - 1$ and T are maximized:

$$\begin{aligned} j^*(i, T - 1) &= \arg \max_{k \geq i} \{(k - i + 1) \cdot p(k, T) + \Pi(k + 1, T)\} \\ \Pi(i, T - 1) &= (j^*(i, T - 1) - i + 1) \cdot p(j^*(i, T - 1), T) + \Pi(j^*(i, T - 1) + 1, T) \end{aligned}$$

The price announced at period $T - 1$ can then be written as

$$p(i, T - 1) = p(j^*(i, T - 1), T)$$

Observe then, that in the final period, we will be in the subgame composed of consumers $\{j^*(i, T - 1) + 1, \dots, N\}$.

Iterating this argument backwards in terms of the periods, we have that

$$\begin{aligned} \Pi(i, t) &= (j^*(i, t) - i + 1) \cdot p(j^*(i, t), t + 1) + \Pi(j^*(i, t) + 1, t + 1) \\ j^*(i, t) &= \arg \max_{j \geq i} ((j - i + 1) \cdot p(j, t + 1) + \Pi(j + 1, t + 1)) \\ p(i, t) &= p(j^*(i, t), t + 1) \end{aligned} \tag{1}$$

We can generalize the concept of the threat price from our two-period example using the above recursion. Specifically, we say that the *threat price* $\tau(i, t)$ for consumer i at period $t < T$ under the recursive scheme given in (1) is the price i is offered in the subgame \mathcal{G}_i starting at period $t + 1$, namely $\tau(i, t) := p(i, t + 1)$. That is, the price offered if i and all consumers of lower value do not buy in period t .

We can now define the strategy of the duopolist and the consumers in any subgame. Consider a subgame whose remaining consumers are the set S and let there be $T - t + 1$ periods remaining (i.e. we are starting in period t). Then by re-indexing the consumer names, the duopolist can treat the subgame as a full game \mathcal{G}' with $T - t + 1$ total periods and S as the set of all consumers. She then calculates, for all i and t , the prices $p_{\mathcal{G}'}(i, t)$ from the recursion relationship (1) and chooses the sales schedule which maximizes her profits for \mathcal{G}' . She then charges $\mu_t = p_{\mathcal{G}'}(1, 1)$ in period t . The consumers buy if and only if the price is less than or equal to their threat price as calculated for \mathcal{G}' . By definition, this price only depends on the payoff-relevant part of the history, as it only looks at the consumers remaining in the subgame. Since the price is always equal to the threat price of one consumer j^* , we can define $\kappa(\mu_t, t)$ to be the value of this critical consumer, v_{j^*} . We can show that this function is monotonically increasing in μ if the threat prices are decreasing in consumer value (a higher price means a

higher valued critical consumer). This is proved in Lemma 3.2. Therefore κ is indeed a cutoff function for the given consumer strategies. We conclude that, if this strategy profile is an SPNE, it must satisfy the strong-Markov property.

The reader may have noted that our recursion relationship does not allow the duopolist to refuse to sell any items in a period where there are still consumers left who have not yet bought. It can be shown that there is a subgame perfect equilibrium in which a sale occurs in each period until either all consumers have bought the item or the final time period is over. Moreover, this equilibrium achieves at least as much profit for the duopolist as any which allows the duopolist to not sell in some periods. A proof of this is included in Appendix 1 as Lemma A5.

The following series of lemmas establish basic monotonicity results for static monopoly prices, threat prices, and the prices $p(i, t)$ which form the duopolist's equilibrium strategy. The proofs of these lemmas are give in Appendix 1.

Lemma 3.1. *The static monopoly prices on the subgames \mathcal{G}_i are non-increasing in i : $p_i \geq p_{i+1}$ for $i = 1, \dots, N - 1$.*

The following lemma shows that the consumers' strategies defined above satisfy the skimming property.

Lemma 3.2. *In any game \mathcal{G} with T periods, the threat prices are non-increasing in i : for all $i \leq k$ and all $t < T$, $\tau(i, t) \geq \tau(k, t)$.*

The next two lemmas are required to show that a deviation from a consumer by delaying a purchase or buying early does not yield higher utility.

Lemma 3.3. *Consider two duopoly games, \mathcal{G} , with T periods and a set S of consumers, and \mathcal{G}' , with T periods and a set S' of consumers such that only the top valued consumer in S and S' differ, and the top valued consumer in S' has the higher value. If we use $p_{\mathcal{G}}(1, 1)$ and $p_{\mathcal{G}'}(1, 1)$ to denote the first period prices as calculated by the recursion relationship above, then $p_{\mathcal{G}'}(1, 1) \geq p_{\mathcal{G}}(1, 1)$.*

Lemma 3.4. *In any game \mathcal{G} with T periods, if the duopolist and consumers follow the strategies described above, then prices are non-increasing in time.*

We are now ready to prove the following result.

Theorem 3.1. *The strategies defined above constitute a SPNE.*

Proof. Since we can treat any subgame as an instance of a full game with a different set of consumers, there is no loss of generality in assuming that the deviation occurs in the first period

of the full game. Let $j^*(1, 1)$ be the lowest value consumer sold to in equilibrium and assume that a consumer $x \leq j^*(1, 1)$ deviates by not buying in period 1. If $x = j^*(1, 1)$, then x is charged her threat price in the next period, which by definition is $p^*(1, 1)$, so there is no advantage in a deviation. If $x < j^*(1, 1)$, then the remaining consumers for period 2 are $\{x, j^* + 1, \dots, N\}$. We know that if the set of consumers was $\{j^*, j^* + 1, \dots, N\}$, then the price would be $p^*(1, 1)$. But by Lemma 3.3, the price with consumers $\{x, j^* + 1, \dots, N\}$ must be at least as high as $p^*(1, 1)$. Therefore x cannot gain by delaying her purchase for one period.

One may wonder whether consumer x could benefit from delaying the purchase by more than one period. But this is not the case. Consider the subgame \mathcal{G}' arrived at after x delays purchase for $t - 1$ periods in the full game, and after re-indexing so the remaining consumers are sorted by value from highest to lowest. If $x = j^*(1, t)$ for this subgame, the price at period $t + 1$ is $p(j^*(1, t), t + 1) = p(j^*(j^*(1, t), t + 1), t + 2)$. This means that the price at $t + 2$ will be the same as the price at $t + 1$ if consumer x doesn't buy and everyone else follows the equilibrium path. If, on the other hand, consumer $x < j^*(1, t)$ for subgame \mathcal{G}' , the price at period $t + 2$ could only increase or stay equal to $p(j^*(j^*(1, t), t + 1), t + 2)$ by Lemma 3.3. By repeated use of this argument we conclude that, at equilibrium, no consumer would benefit from delaying its purchase.

It remains to show that no consumer can benefit from buying early. If a consumer deviates from the equilibrium path by buying early, she pays a price $p^*(1, t)$ when she could have bought in period $t' > t$ at price $p^*(k, t')$ for some $k > 0$. But since prices are non-increasing as a function of time along the proposed sales path (Lemma 3.4), she cannot do any better.

It follows that we have a strategy profile which is an equilibrium in every subgame. \square

To compute such an equilibrium, we can compute $j^*(i, t)$ for each (i, t) going backwards from period T , and choose the sales path x_t which maximizes profit. The prices μ_t are then computed by “passing back” the next period's threat price. We may solve the corresponding dynamic program to find the maximum profit Π^D for the duopolist.

It is easy to see that if there is another strong-Markov SPNE, it cannot result in more revenue for the duopolist. In any proposed SPNE where we sell in every period, any period price cannot be higher than the threat price of any consumer who buys in that period, as otherwise she would earn more profit by waiting one more period. So given a sales schedule, the monopolist can do no better than to charge the threat price of the lowest valued consumer to buy in each period. However, (1) finds the optimal sales schedule in terms of revenue when the duopolist charges threat prices in each period. Lemma A5 covers the case where a strong-Markov SPNE may choose not to sell in one or more periods when consumers remain to buy. The full result we have, then, is

Corollary 3.1. *The optimal revenue Π^D given by the dynamic program derived from Equations (1) is the maximum revenue obtainable by a strong-Markov SPNE.*

3.1 Incomplete Information

So far we provided a characterization of subgame perfect equilibria when there is complete information, that is when all participants know which consumer has each value and the duropolist can observe who buys in each period. We now introduce an incomplete information setting and show that the same SPNE characterization applies.

Consider the setting in which the market participants can see who buys in period t , and know the distribution of values (and know their own value), but do not know exactly which consumer has which value.⁹ Since we are interested in studying equilibria that satisfy the skimming property, regardless of the values of consumers who bought at period t , the duropolist off-path belief is that the k consumers who bought in period t are those with the k highest valuations (among those remaining). We will show that the same conditions as in Section 3 characterizing the subgame perfect equilibria apply.

We first define the strategy of the duropolist and the consumers in any subgame under this incomplete information setting. Let \mathcal{G}_S denote the subgame at period t where the $|S|$ remaining consumers have valuations $w_1 \geq w_2 \dots \geq w_{|S|}$. Due to the off-path belief (i.e., the belief that consumers follow the skimming property), the duropolist would behave as if it were in the subgame \mathcal{G}'_1 with $T - t + 1$ periods in which the consumers are $v'_1 \geq v'_2 \geq \dots, v'_{|S|}$ where $v'_i = v_{i+N-|S|}$. Observe that $v'_i \leq w_i$ for all $i \in [|S|]$. The monopolist strategy is to then announce the price $p_{\mathcal{G}'}^*(1, 1)$ which is obtained by solving the recursion relationship (1). The consumers strategy remains the same as in the complete information setting, i.e. each of them would buy if and only if the price is less than or equal to their threat price as calculated for \mathcal{G}' . We now prove the following result.

Theorem 3.2. *The strategies defined above constitute a SPNE in the incomplete information setting.*

Proof. We consider the subgame $\mathcal{G}' = \mathcal{G}_S$ (of the original game \mathcal{G}) that begins at period t in which the remaining consumers consists of the set S . These consumers have valuations $w_1 \geq w_2 \dots \geq w_{|S|}$. Due to the off-path belief (i.e., the belief that consumers follow the skimming property), the duropolist would behave as if it were in the subgame \mathcal{G}'_1 with $T - t + 1$

⁹Note this is equivalent to the participants knowing which consumer has which value, but not seeing who buys, only the total number of sales in each period.

periods in which the consumers are $v'_1 \geq v'_2 \geq \dots, v'_{|S|}$ where $v'_i = v_{i+N-|S|}$. Observe that in this subgame \mathcal{G}'_1 , consumer i 's real value is actually $w_i \geq v'_i$.

The announced price in \mathcal{G}'_1 would then be $p(1, t) = p(j^*(1, t), t + 1)$. Suppose now that some consumer x that was supposed to buy under the proposed equilibrium, i.e. $i \leq x \leq j^*(1, t)$ deviates and chooses not to buy at time t . The number of sales at period t would then be $j^*(1, t) - 1$, i.e., one less than the expected. The duropolist, who observes the total number of sales and assumes consumers follow the skimming property, would then behave as if the remaining subgame starting at $t + 1$ is $\mathcal{G}'_{j^*(1, t)}$. This means that the announced price would be $p(j^*(1, t), t + 1)$ and consumer x would not have benefited from delaying the purchase by one period. One may, again, wonder whether consumer x could benefit from delaying the purchase by more than one period. But this is not possible since the price at period $t + 1$ in this subgame is $p(j^*(1, t), t + 1) = p(j^*(j^*(1, t), t + 1), t + 2)$, which means the price will remain constant over time, as long as the number of transactions is one less than the expected. By repeated use of this argument we conclude that, at equilibrium, no consumer would benefit from delaying its purchase.

Lastly, observe that no consumer can benefit from buying earlier. If a consumer deviates from the equilibrium path by buying earlier, she pays a price $p^*(1, t)$ when she could have bought in period $t' > t$ at price $p^*(k, t')$ for some $k \geq 1$. But since prices are non-increasing as a function of time along the proposed sales path (Lemma 3.4), she cannot do any better.

So we conclude that we have a strategy profile which is an equilibrium in every subgame. \square

Note that since the equilibrium path is the same in both our complete and incomplete information setting, all our results also apply to the incomplete information setting.

4 When can the Duropolist Extract all the Consumer Surplus?

As discussed, Bagnoli et al. (1989) proved that a duropolist who faces atomic consumers with an infinite time horizon can always extract all consumer surplus. They left open the case of finite time horizons. Although such equilibria may still exist under a finite horizon, the conditions required for their existence are very restrictive. Indeed, applying the techniques we have developed, we characterize in this section necessary and sufficient conditions for this phenomenon to happen. We require some definitions and three lemmas.

Let \mathcal{G} be a game with N consumers with valuations $v_1 \geq v_2, \dots, \geq v_N$ and let p_i be the static monopoly price of the subgame consisting of consumers $\{i, i + 1, \dots, N\}$. More generally, given a subset $S \subseteq [N]$ we define $p(S)$ to be the static monopoly price of the subgame consisting of consumers $S \subseteq [N]$.

Lemma 4.1. *A game \mathcal{G} satisfies $p_i = v_i$ for all $i \in [N]$ if and only if $p(S) = \max\{v_x : x \in S\}$ for all $S \subseteq [N]$.*

Proof. Suppose there exists a subset $S \subseteq [N]$ such that $p(S) < v_i$ where $i = \arg \max\{v_x : x \in S\}$. Let the valuations of the consumers in S be $v_i \geq w_2 \geq w_3 \geq \dots \geq w_{|S|}$. Then, we have $j \cdot w_j > v_i$ for some $j > 1$. But then $p_i < v_i$ since by setting a price of w_j in the subgame with consumers $\{i, i+1, \dots, N\}$ yields a profit of at least $j \cdot w_j > v_i$ (since in the subgame \mathcal{G}_i there may be more consumers with valuations between w_j and v_i). The remaining implication follows directly. \square

Let $w_1 > w_2 > \dots > w_M$ denote the M distinct consumer valuations sorted in decreasing order. Let n_i denote the number of consumers with value w_i . The following technical lemma (whose proof is in the appendix) is required to prove the main result of this section. We set $w_i = n_i = 0$ for all $i > M$.

Lemma 4.2. *If $p_i = v_i$ for all $i \in [N]$, the following inequality holds for every natural number $\beta \geq 2$, $k = 2, \dots, \beta$ and all $i = 1, \dots, k-1$,*

$$n_i \cdot w_i - n_i \cdot w_k - n_{\beta+i} w_{\beta+i} \geq 0.$$

Recall that the duropolist strategy named *Pacman* is to announce at each time period, a price equal to the valuation of the consumer with the highest value who has yet to buy, and that the consumer strategy known as *get-it-while-you-can* is to buy the first time it induces a non-negative utility. The following lemma gives sufficient conditions for such strategies to be at equilibrium.

Lemma 4.3. *If $p_i = v_i$ for all $i \in [N]$, then there exists an equilibrium in which the duropolist uses the pacman strategy and consumers follow the get-it-while-you-can strategy.*

Proof. We proceed by induction in the number of time periods. For games with a single period (i.e., $T = 1$) the lemma holds because the duropolist announces the optimal static monopoly price, which is p_1 . Suppose now that the lemma holds for all games with at most $T - 1$ periods and consider a game with T periods. Let A be the set of consumers that buy at period 1 under an equilibrium \mathcal{E} . Observe that by Lemma 4.1, the subgame that begins at period 2, with consumers $[N] - A$ satisfies the that $p_i = v_i$ and therefore there exist an equilibrium where the duropolist uses the Pacman strategy from then on. This means that consumers expect zero profits whenever they don't buy in the first time period, and therefore they would buy in the first time period at any price that is not above their valuation. If $M \leq T$ the duropolist may announce at $t = 1$ the price $\mu_1 = v_1$. All consumers with a valuation of v_1 would buy and, by the inductive hypothesis and the fact that the different valuations in the remaining game is

still less than the time periods left ($M - 1 \leq T - 1$), the duopolist would be able to extract all consumer surplus. Thus, pacman is an optimal strategy. We now analyze the case where $M > T$. Observe that because consumers will buy in the first period if and only if the price is not above their value, the duopolist's strategy space can be restricted, without loss of generality, to announcing a first price equal to the valuation of some consumer. Let $\Pi(k)$ denote the profits for the whole game if the first price is w_k . Since by induction hypothesis the pacman strategy is an equilibrium after the first period, we have that

$$\Pi(k) = \sum_{i=1}^k n_i \cdot w_k + \sum_{j=k+1}^{T+k-1} n_j \cdot w_j.$$

Now we want to show that

$$\Pi(1) \geq \Pi(k)$$

for all $k = 1, \dots, M$. Consider first the case where $k \leq T$. By Lemma 4.2, by setting $\beta = T$, we have that for all $k = 1, \dots, T$ and all $i = 1, \dots, k - 1$:

$$0 \geq n_i \cdot w_i - n_i \cdot w_k - n_{T+i} \cdot w_{T+i}$$

Summing over i we obtain

$$\begin{aligned} 0 &\geq \sum_{i=1}^{k-1} (n_i \cdot w_i - n_i \cdot w_k - n_{T+i} \cdot w_{T+i}) \\ &= \sum_{i=1}^{k-1} n_i \cdot w_i - \sum_{i=1}^{k-1} n_i \cdot w_k - \sum_{j=T+1}^{T+k-1} n_j \cdot w_j \\ &= \sum_{i=1}^{k-1} n_i \cdot w_i + \sum_{i=k}^T n_i \cdot w_i - \sum_{i=1}^k n_i \cdot w_k - \sum_{i=k}^T n_i \cdot w_i - \sum_{j=T+1}^{T+k-1} n_j \cdot w_j \\ &= \sum_{i=1}^T n_i \cdot w_i - \sum_{i=1}^{k-1} n_i \cdot w_k - \sum_{j=k}^{T+k-1} n_j \cdot w_j \\ &= \sum_{i=1}^T n_i \cdot w_i - \sum_{i=1}^k n_i \cdot w_k - \sum_{j=k+1}^{T+k-1} n_j \cdot w_j \\ &= \Pi(1) - \Pi(k) \end{aligned}$$

Thus $\Pi(1) \geq \Pi(k)$. Second, consider the case where $k > T$. By Lemma 4.2, setting $\beta = k > T$,

we have that for all $i = 1, \dots, k-1$:

$$\begin{aligned}
0 &\geq \sum_{i=1}^T (n_i \cdot w_i - n_i \cdot w_k - n_{k+i} \cdot w_{k+i}) \\
&= \sum_{i=1}^T n_i \cdot w_i - \sum_{i=1}^T n_i \cdot w_k - \sum_{j=k+1}^{T+k} n_j \cdot w_j \\
&= \Pi(1) - \Pi(k)
\end{aligned}$$

Thus, again, $\Pi(1) \geq \Pi(k)$. So there exists an equilibrium where the duropolist uses the pacman strategy. \square

We are now ready to prove the following theorem which provides sufficient and necessary conditions for the existence of an equilibrium that extracts all consumer surplus.

Theorem 4.1 (Pacman Theorem). *Consider a duopoly game \mathcal{G} with $M \leq N$ distinct valuations. There exists an equilibrium at which the duropolist extracts all the consumer surplus if and only if $M \leq T$ and $v_i = p_i$ for all $i \in [N]$.*

Proof. Take a game \mathcal{G} with $M \leq T$ and $v_i = p_i$ for all $i \in [N]$. By Lemma 4.3 there exists an equilibrium in which the duropolist uses the pacman strategy. Moreover, since $M \leq T$, under this equilibrium the duropolist obtains all the consumer surplus.

The contrapositive also holds. First, assume $M > T$. Then, since the number of time periods is less than the number of different valuations it is impossible for the duropolist to extract the value of every consumer before the end of the game. Second, assume $v_i > p_i$ for some $i \in [N]$. Now take an equilibrium that extracts all the consumer surplus. At equilibrium, prices must be non-increasing over time. Moreover, since all consumer surplus is extracted, it implies that consumers also purchase in decreasing order of value over time which means that the skimming property holds. If consumer i bought in the last period it means she has the lowest valuation, i.e. $v_i = w_M$, and $p_i = v_i$ which is a contradiction. In the case consumer i buys before the last period, Lemma 5.1 (see next section) implies that consumer i never pays more than p_i , again a contradiction. \square

We conclude this section with some observations. Recall, from Footnote 2, that there exist finite time horizon games in which the pacman solution has revenue that is a factor $\log N$ greater than static monopoly revenue. On the other hand, the main result of next section is that duopoly revenue cannot be more than twice static monopoly revenue. Thus, when pacman is an equilibrium, static monopoly profits are at least half of pacman revenue. To see this, observe that the condition $p_i = v_i$ for all $i \in [N]$, implies that consumer valuations decrease exponentially. In these scenarios, $n_1 \cdot v_1$ (which is static monopoly revenue) is at least half of the sum of all consumer valuations.

5 A Relationship between Duopoly Profits and Static Monopoly Profits

In this section, we will prove our main result: the profits of the duopolist in a skimming property-satisfying SPNE of the duopoly game are at least the profits of the corresponding static monopolist, but at most double. The first claim is straightforward.

Theorem 5.1. $\Pi^M \leq \Pi^D$

Proof. The proof is by induction on the number of periods. The base case is trivial. Consider the (possibly sub-optimal) sales schedule where the duopolist sells at p_1 in period 1, and then follows an equilibrium strategy for the remaining subgame \mathcal{G}' . Let k_1 be the number of consumers sold to in period 1 under this schedule. Let $\Pi_{\mathcal{G}}^M = j \cdot v_j \equiv j \cdot p_1$ and $\Pi_{\mathcal{G}'}^M = (k - k_1)v_k$. Since $j = |\{i | v_i \geq v_j\}|$, $j \geq k_1$ (no one with value less than v_j is willing to pay v_j). But $k = \arg \max_{i \geq k_1} (i - k_1)v_i$, therefore $(k - k_1)v_k \geq (j - k_1)v_j$. So

$$\Pi^D \geq k_1 p_1 + \Pi_{\mathcal{G}'}^D \geq k_1 p_1 + \Pi_{\mathcal{G}'}^M \geq k_1 p_1 + (j - k_1)v_j = k_1 p_1 + (j - k_1)p_1 = \Pi^M,$$

where, in the second inequality, we used the induction hypothesis. \square

The second claim is more substantial.

Theorem 5.2. $\Pi^D \leq (\Pi^M + p_1) \leq 2 \cdot \Pi^M$

In Section 5.1 we show that Theorem 5.2 is in fact tight, i.e., duopoly profits can be as close as desired to twice the static monopoly profits. Observe however, that in many situations Theorem 5.2 suggests that duopoly profits can be only slightly higher than static monopoly profits. For example, this happens when the value of the highest value consumer is negligible with respect to the static monopoly profits, i.e. $p_1 \leq v_1 \ll \Pi^M$.

We prove Theorem 5.2 in two steps. To describe them, recall we have N consumers with valuations $v_1 \geq v_2 \geq \dots \geq v_N$. Furthermore, \mathcal{G}_i is the subgame consisting of consumers $\{i, i+1, \dots, N\}$ and p_i is the static monopoly price for the subgame \mathcal{G}_i . Each p_i is equal to some consumer's value so we will define y_i through the equation $p_i = v(y_i)$. First we show that $\Pi^D \leq \sum_{i=1}^N p_i$, and second we show that $\Pi^M + v_1 \geq \sum_{i=1}^N p_i$.

Theorem 5.3. *The maximum profit of the duopolist satisfies $\Pi^D \leq \sum_{i=1}^N p_i$.*

To prove this we require the following three lemmas.

Lemma 5.1. *In equilibrium, consumer i never pays more than p_i whenever she buys before the last period.*

Proof. We proceed by induction in the number of time periods. For $T = 2$, let A be the set of consumers that buy at $t = 1$. Suppose for the purpose of contradiction that some consumer $i \in A$ pays a price higher than p_i . Because of the skimming property, we have that $A = \{1, \dots, k\}$ for some k . By Lemma 3.1, this price is also more than p_k , so consumer k pays more than p_k . But if consumer k refuses to buy at $t = 1$, then at $t = 2$ the duopolist would charge p_k which is a contradiction since consumer k would have obtained a higher profit by waiting.

Now suppose that the lemma is true for all games of 1 to T periods and consider a game \mathcal{G} of $T + 1 > 2$ periods. Let E denote a subgame perfect Nash equilibrium with the skimming property in \mathcal{G} and let i be smallest value consumer that pays more than p_i , say at some time period t ($t < T + 1$). Again, by Lemma 3.1 consumer i is the lowest valuation consumer that buys at period t . Thus, if consumer i refuses to buy at period t we end in the subgame \mathcal{G}_i with $T + 1 - t$ periods. If $T + 1 - t = 1$ (i.e. t was the second to last period), the duopolist will charge the price p_i at the last period. If $T + 1 - t > 1$, it holds by the induction hypothesis that consumer i would never pay more than p_i . Thus, we can conclude that such equilibrium E cannot exist as consumer i would have obtained a higher profit by waiting. \square

Lemma 5.2. *The maximum revenue of the duopolist satisfies*

$$\Pi^D \leq \max_{m \leq N} \left((y_m - m + 1) \cdot v(y_m) + \sum_{i=1}^{m-1} p_i \right)$$

Proof. Consider the subgame perfection conditions. In the final time period T , consumer i is willing to pay up to v_i . In periods 1 to $T - 1$, consumer i is willing to pay up $p_i = v(y_i)$, the static monopoly price for the subgame \mathcal{G}_i .

Suppose that in the optimal solution, the duopolist sells to consumers $\{m, m + 1, \dots, M\}$ where $1 \leq m \leq M \leq N$ in the final period T . Since consumer $M = y_m$ buys in the final period, the revenue then is exactly $(y_m - m + 1) \cdot v(y_m)$. By Lemma 5.1, consumers who buy in earlier periods, that is consumers $\{1, 2, \dots, m - 1\}$, pay at most their static monopoly prices. Therefore, the maximum revenue is upper bounded by

$$(y_m - m + 1) \cdot v(y_m) + \sum_{i=1}^{m-1} p_i$$

The result follows by taking the maximum over all consumers m . \square

Lemma 5.3. *The static monopoly revenue for the subgame \mathcal{G}_m is at most*

$$\sum_{j=m}^N p_j$$

Proof. Without loss of generality, by re-indexing so that $m = 1$, it suffices to show that

$$\Pi^M = y_1 \cdot v(y_1) \leq \sum_{j=1}^N p_j \quad (2)$$

Let $C = \{p_j : j = 1, \dots, N\}$. We proceed by induction on $|C|$, that is, the number of distinct static monopoly prices over all the subgames \mathcal{G}_j . For the base case, $|C| = 1$, we have $p_1 = p_j$ for all j . Thus $p_1 = p_N = v_N$ and $y_1 = N$. Every consumer then pays v_N and the total revenue is

$$y_1 \cdot v(y_1) = N \cdot v_N = \sum_{j=1}^N p_j$$

Assume the proposition holds for $|C| = k - 1 \geq 1$. Now take the case $|C| = k$. Let consumer l be the highest index consumer in the original game with $p_l = p_1$. Thus $p_{l+1} < p_1 = v(y_1)$. By the induction hypothesis, applied to the subgame \mathcal{G}_{l+1} on consumers $\{l+1, l+2, \dots, N\}$, we have

$$\sum_{i=l+1}^N p_i \geq (y_{l+1} - l) \cdot v(y_{l+1}) \quad (3)$$

Consequently,

$$\begin{aligned} \sum_{i=1}^N p_i &= l \cdot p_1 + \sum_{i=l+1}^N p_i \\ &\geq l \cdot p_1 + (y_{l+1} - l) \cdot v(y_{l+1}) \\ &\geq l \cdot p_1 + (y_l - l) \cdot v(y_l) \\ &= l \cdot v(y_l) + (y_l - l) \cdot v(y_l) \\ &= y_l \cdot v(y_l) \\ &= y_1 \cdot v(y_1) \end{aligned}$$

Here the first equality follows by definition of l . The first inequality follows by (3). The second inequality holds as $v(y_{l+1})$ is the static monopoly price for the subgame \mathcal{G}_{l+1} . The final three equalities follow by definition of l . That is $p_1 = p_l$ and so $y_l = y_1$.

This shows that (2) holds as desired. □

Proof of Theorem 5.3. Combine Lemma 5.2 and Lemma 5.3. □

It remains to prove the upper bound on $\sum_{i=1}^N p_i$.

Theorem 5.4. $\sum_{i=1}^N p_i \leq \Pi^M + p_1$

Proof. We proceed by induction on N . For games with a single consumer, the statement is trivially true. Recall that $p_i = v(y_i)$ is the static monopoly price for the subgame \mathcal{G}_i on consumers $\{i, i+1, \dots, N\}$ and y_i is the index of the lowest value consumer whose value is not less than p_i . Consider now a game \mathcal{G} with $N+1$ consumers. It remains to show that

$$\sum_{i=1}^{N+1} p_i \leq v(y_1) + y_1 \cdot v(y_1).$$

We proceed as follows:

$$\begin{aligned} \sum_{i=1}^{N+1} p_i &= v(y_1) + \sum_{i=2}^{N+1} p_i \\ &\leq v(y_1) + v(y_2) + (y_2 - 1) \cdot v(y_2) \end{aligned} \tag{4}$$

$$\begin{aligned} &= v(y_1) + y_2 \cdot v(y_2) \\ &\leq v(y_1) + y_1 \cdot v(y_1) \end{aligned} \tag{5}$$

Here equation (4) follows by the induction hypothesis and inequality (5) comes from the fact that $v(y_1)$ is the static monopoly price of \mathcal{G}_1 . \square

Proof of Theorem 5.2. The result follows from Theorems 5.3 and 5.4. \square

5.1 A Tight Example

The factor 2 bound is tight. There are examples in which the duopolist can extract twice as much profit as the equivalent static monopolist. To see this, assume that $T = 2$ and set $v_j = v_H$ for $1 \leq j \leq k$ and $v_j = v_L = \frac{1}{n-k+1} \cdot v_H$ for all $k+1 \leq j \leq n$. The optimal solution is to charge V_H in the first period and V_L in the last period. The high value consumers will buy in the first period and the low value consumers in the last period. It can be verified that this solution satisfies the equilibria conditions.

The total revenue is then

$$\begin{aligned} k \cdot v_H + (n - k) \cdot v_L &= k \cdot v_H + \frac{n - k}{n - k + 1} \cdot v_H \\ &= \Pi^M + \left(1 - \frac{1}{n - k + 1}\right) \cdot v_H \end{aligned}$$

Thus in the limit $n \gg k$ we obtain $\Pi^M + v_H$. As $v_H = p_1$, the duopoly profits exceed static monopoly profits by an additive amount approaching static monopoly profits. Moreover, in the case $k = 1$ we have that $\Pi^M = v_H$ and, thus, duopoly profits are twice static monopoly profits.

6 Non-Skimming Equilibria

In this section, we discuss equilibria that do not satisfy the skimming property. For non-atomic consumers, it is easy to show that such equilibria cannot exist. However, for atomic consumers, subgame perfect Nash equilibria that do not satisfy the skimming property can exist. To see this, consider the following three-consumer, two period problem.¹⁰

Table 3: An Example with a Non-Skimming Equilibrium

Consumer valuation	Period 1 Strategy	Period 2 Strategy
80	Buy if $\mu_1 \leq 45$	Buy if $\mu_2 \leq 80$
70	Buy if $\mu_1 \leq 70$	Buy if $\mu_2 \leq 70$
45	Buy if $\mu_1 \leq 45$	Buy if $\mu_2 \leq 45$
Monopolist	$\mu_1 = 70$	Charge Static Monopoly Price (for remaining consumers)

Because period 2 is the last period of the game, all the consumers who did not buy in period 1 have a dominant strategy which is “buy iff $\mu_2 \leq v$ ”. The duopolist’s best response to this is to charge a value μ_2 equal to the static monopoly price for the subgame with all consumers who did not buy in period 1. It is simple to check that all of the consumers’ strategies are mutual best responses. The period 1 price is also the price which maximizes revenue for the duopolist, given the consumer strategies. So this is a subgame perfect Nash equilibrium. However, in this equilibrium, the consumer with value 70 buys in period 1, while the consumer with value 80 buys in period 2 (where $p_2 = 45$). Therefore the skimming property is not satisfied. Also note that the consumer with the second highest value pays his full value, which is significantly more than p_2 , the static monopoly price in the subgame \mathcal{G}_2 ($p_2 = 45$).

Note, however, that if we swapped the period 1 strategies of the top two consumers, we would still have an equilibrium, and one which satisfies the skimming property. We can prove that this is always the case.

Theorem 6.1. *In a 2 period game, for every strategy equilibrium which does not satisfy the skimming property, there is a corresponding equilibrium at the same prices which satisfies the skimming property and earns the same revenue for the duopolist.*

¹⁰One can also show that a non-skimming equilibrium exists in this example if we introduce a discount factor with value strictly less than 1.

To prove this theorem, we first require some results about static monopoly prices.

Claim 6.1. *Let $V = \{v_1, v_2, \dots, v_N\}$ be a set of valuations $v_1 \geq v_2 \dots \geq v_N$. Let $k = \arg \max_i iv_i$ and $p_V^{sm} = v_k$, the static monopoly price calculated for the set V . Now let V' be the set V with some v_j replaced by a value $v_j > v' \geq v_k$, and re-index the set so $v'_1 \geq v'_2 \dots \geq v'_N$. Then $p_{V'}^{sm} = v_k$. If, instead, $v' \geq v_j > v_k$ and we re-index the set so $v'_1 \geq v'_2 \dots \geq v'_N$, then we still have $p_{V'}^{sm} \geq v_k$.*

Proof. First suppose $v' < v_j$. Note that $v'_i \leq v_i$ for every i . Therefore, by definition of k , $kv_k \geq iv_i \geq iv'_i$ for all i . For all i such that $v_i \leq v'$, we still have $v'_i = v_i$, so therefore $v_k = v'_k$ and $kv'_k \geq iv'_i$ for all i . Therefore $k = \arg \max_i iv'_i$ and $p_{V'}^{sm} = v_k$.

Next assume $v' \geq v_j$. Note that, because $v_j > v_k$, we still have $v'_i = v_i$ for every $i \geq k$, and as $kv_k \geq iv_i$ for all $i \geq k$, it follows that the index which maximizes lv_l can be no larger than k . Therefore $p_{V'}^{sm} \geq v'_k = v_k$. \square

Claim 6.2. *Let $V = \{v_1, v_2, \dots, v_N\}$ be a set of valuations $v_1 \geq v_2 \dots \geq v_N$. Let $k = \arg \max_i iv_i$ and $p_V^{sm} = v_k$. Now add a value $v' \geq v_k$ to the set of valuations, call the new set V' and re-index the set so $v'_1 \geq v'_2 \dots \geq v'_{N+1}$. Then $p_V^{sm} \leq p_{V'}^{sm} \leq v'$.*

Proof. Let $v' = v'_l$ (i.e. v' is the l -th highest value in V'). We have that $v'_i = v_{i-1}$ for $i > l$. The result follows immediately as for all $j < l$, $jv_j \leq kv_k < (k+1)v_k = (k+1)v'_{k+1}$, so regardless of what the maximizing index actually is, it cannot be less than l , and therefore $p_{V'}^{sm} \leq v_l = v'$. But we also have that $(k+1)v'_{k+1} = (k+1)v_k \geq (j+1)v_j = (j+1)v'_{j+1}$ for all $j \geq k$. Therefore the maximizing index cannot be more than $k+1$, and thus $v'_{k+1} = v_k = p_V^{sm} \leq p_{V'}^{sm}$. \square

Consider an equilibrium \mathcal{E} which violates the skimming property. The violation must occur in period 1, as in the second period the dominant strategy followed by the consumers ensures that there is a cut-off value above which all remaining consumers buy. We must then have at least one pair of consumers which we denote by their values v, w with $w > v$, where v buys in period 1 and w does not buy in period 1. Without loss of generality, let w be the highest valued consumer who doesn't buy in period 1, and v be the lowest valued consumer who buys with value below w . We denote by E the set of consumers under the equilibrium \mathcal{E} who did not buy in period 1. Then $w \in E$ and $v \notin E$.

The period 2 price, $\mu_2(E)$, is fixed by the fact that it is a dominated strategy to charge the static monopoly price of the remaining consumers. For a consumer $y \notin E$ who buys in period 1, we will also need to compare $\mu_1(E)$ against its threat price, $\mu_2(E^y)$. Here $E^y = E \cup \{y\}$, and so $\mu_2(E^y)$ is the period 2 price y would face if he did not buy.

Lemma 6.1. *At the equilibrium \mathcal{E} , we have*

$$w > v \geq \mu_2(E^v) \geq \mu_1(E) \geq \mu_2(E) \quad (6)$$

Proof. By assumption $w > v$. Consumer v wants to buy in the first period. The equilibrium conditions then imply that (i) $\mu_1(E) \leq v$ and (ii) $\mu_1(E) \leq \mu_2(E^v)$. Consumer w does not want to buy in the first period. But since the first period price is less than his value, he would be willing to buy in the first period if $\mu_2(E) > \mu_1(E)$. The equilibrium conditions then imply that $\mu_2(E) \leq \mu_1(E) < w$ (and w does buy in the second period). It only remains to prove that $v \geq \mu_2(E^v)$. We have seen that $v \geq \mu_2(E)$. But then, by Claim 6.2, we obtain $v \geq \mu_2(E^v)$. \square

To prove Theorem 6.1, we will show that with the same period 1 price, the strategy profile \mathcal{S} which consists of v and w swapping actions from \mathcal{E} (and all other consumer strategies remaining the same) is also an equilibrium. We will also show that the period 2 price stays the same after swapping. By repeated swapping, we will remove all pairs where the lower valued consumer buys before the higher valued consumer, until we reach an equilibrium where the skimming property is satisfied, while the prices are unchanged. So let $S = \{E - w\} \cup \{v\}$ be the set of consumers who did not buy in period 1 under the swapped strategy \mathcal{S} .

Lemma 6.2. *For the strategy profiles \mathcal{E} and \mathcal{S} , we have*

$$w > v \geq \mu_2(E^v) = \mu_2(S^w) \geq \mu_1(E) = \mu_1(S) \geq \mu_2(E) = \mu_2(S) \quad (7)$$

Proof. The inequalities follow from Lemma 6.1. Thus it remains to prove the three equalities. Since S and E differ in exactly the pair $\{w, v\}$ we have that $S^w = E^v$. The equilibrium constraints imply the duopolist applies static monopolist pricing in the final period; thus $\mu_2(S^w) = \mu_2(E^v)$. Next, in the proposed solution corresponding to S , the duopolist is, by choice, setting the same price in period 1 for S as for E ; thus $\mu_1(S) = \mu_1(E)$. Finally, we know that $v \geq \mu_2(E)$. But, then, if we reduce the value of consumer w to v , the static monopoly price will not change by the first case of Claim 6.1; equivalently $\mu_2(S) = \mu_2(E)$. \square

Lemma 6.3. *The solution corresponding to S is an equilibrium.*

Proof. We prove that the strategy profile consisting of v and w swapping actions and all other consumers acting the same way is still an equilibrium by considering the best responses of every consumer separately.

- **Consumer v:** Consumer v buys in the second period for S . If this is a best response strategy then we need $\mu_2(S) \leq \min[v, \mu_1(S)]$. This is true, by Lemma 6.2.
- **Consumer w:** Consumer w buys in the first period for S . If this is a best response strategy then we need $\mu_1(S) \leq \min[w, \mu_2(S^w)]$. This is true, by Lemma 6.2.

• **A consumer $u \neq w$ who buys in period 1:** We wish to show that u still wishes to buy in the first period after we swap: that is, $\mu_1(S) \leq \min[u, \mu_2(S^u)]$.

Since v is the lowest valued consumer who buys in period 1, we must have $u \geq v$. We know, by Lemma 6.2 that $\mu_1(S) \leq v \leq u$. Thus it suffices to prove that $\mu_1(S) \leq \mu_2(S^u)$. We now have two possibilities:

- (a) $u < w$: By Lemma 6.1, we have $\mu_2(E^v) \leq v$. Now set $V = E^v = S^w$ and apply the first case of Claim 6.1 with $v' = u$ and $v_j = w$. Thus, $V' = S^u$ and $\mu_2(E^v) = \mu_2(S^u)$. By the equilibrium conditions, we know $\mu_1(E) \leq \mu_2(E^v)$. Therefore $\mu_1(S) = \mu_1(E) \leq \mu_2(E^v) = \mu_2(S^u)$.
- (b) $u \geq w$: By Lemma 6.2, we have $\mu_2(E) = \mu_2(S) < w$. Now set $V = S$ and apply Claim 6.2 by adding a new consumer with value $v' = w$; thus $V' = S^w$ and $\mu_2(S^w) \leq w$. Applying the second case of Claim 6.1 with $V = S^u$ and $V' = S^w$ and $v' = u$, then gives $\mu_2(S^u) \geq \mu_2(S^w)$. Furthermore, by Lemma 6.2, we know $\mu_1(S) \leq \mu_2(S^w)$; thus $\mu_1(S) \leq \mu_2(S^u)$, as desired.

• **Any other consumer $u \neq \{w, v\}$:** Such a consumer u either buys in period 2 or does not buy at all. In the former case, $\mu_2(E) \leq \min[u, \mu_1(E)]$. In the latter case, $u \leq \mu_2(E) \leq \mu_1(E)$. But, by Lemma 6.2, we have $\mu_1(S) = \mu_1(E)$ and $\mu_2(S) = \mu_2(E)$. Therefore, in either case, the same best response conditions hold for S as they did for E . \square

Proof of Theorem 6.1: By Lemma 6.3, we know that \mathcal{S} gives an equilibrium. From Lemma 6.2, we have that $\mu_1(S) = \mu_1(E)$ and $\mu_2(S) = \mu_2(E)$. Since the same number of consumers buy in each period under \mathcal{S} and \mathcal{E} , the duopolist earns the same revenue in \mathcal{S} as in \mathcal{E} . By repeatedly swapping pairs which are out of order, we obtain an equilibrium with the same revenue which satisfies the skimming property. \square

We conjecture that our two-period result extends to the T -period case, and therefore our bounds would apply not just to skimming-property satisfying equilibria, but to all sub-game perfect Nash equilibria.

Conjecture 6.1. *The profits of the duopolist in any SPNE are at least the profits of the corresponding static monopolist, but at most double.*

7 Conclusions

In this paper we studied the durable good monopoly problem, a classical problem in bargaining theory. In our setting, we consider consumers to be atomic and that there is a finite time

horizon during which sales occur. We characterized all profit maximising strong-Markovian equilibria and proved that, in those equilibria, duopolist profits are comparable to those of a static monopolist. This is in contrast with previous results in which duopoly profits are either arbitrary small or arbitrary large compared to those of a static monopolist.

The paper leaves interesting questions for future research. We provided the first example of a sub-game perfect equilibrium in which the classical skimming-property does not hold. Although such equilibria might be rare in practice, we proved that for two-period games, the duopolist profits are not larger in those non-skimming equilibria than in the skimming equilibria. Whether the duopolist profits can be larger than twice the static monopolist profits under non-skimming equilibria for games with an arbitrary number of time periods is a question we believe worthy of study.

Finally, in this paper we studied the case in which there is no discount factor, neither for the duopolist, or for the consumers. One could easily extend the equilibrium characterization we provided here in order to capture the monopolist and consumers behaviour when there is a discount factor. However, the question on whether duopolist profits could be more than twice as much as static monopolist profits remains open.

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Appendix 1: Proofs omitted from main text.

Lemma A1 (Lemma 3.1). *Let the consumers in \mathcal{G} be ordered by value so $v_i \geq v_{i+1}$ for all i . Then static monopoly prices are non-increasing in i : $p_i \geq p_{i+1}$ for $i = 1, \dots, N - 1$.*

Proof. Without loss of generality we can restrict to the case $i = 1$. Let $p_1 = v_a$ and $p_2 = v_b$ and therefore $a \geq 1$ and $b \geq 2$. As a first case, consider that $a \leq b$. Given that valuations are non-increasing, it follows that $p_1 = v_a \geq p_2 = v_b$. As the second case, suppose that $a > b \geq 2$. We know that

$$av_a \geq bv_b,$$

by definition of p_1 . Now if $a > b$, in the game \mathcal{G}_2 the static monopolist has the option of selling to exactly the consumers in $[2, a]$. Therefore, as v_b is the static monopoly price for \mathcal{G}_2 , the game with consumers $[2, N]$, it follows that

$$(b - 1)v_b \geq (a - 1)v_a.$$

Combining these two inequalities gives us $v_a \geq v_b$. But $a > b$ implies $v_b \geq v_a$, so we conclude that $v_a = v_b$. In either case, $v_a \geq v_b$. □

Lemma A2 (Lemma 3.2). *In any game \mathcal{G} with T periods, for all $i \leq k$ and all $t < T$, $\tau(i, t) \geq \tau(k, t)$.*

Proof. We proceed by backwards induction on t . For $t = T - 1$, $\tau(i, t) = p_i$, the static monopoly price for the game \mathcal{G}_i , for all i . By Lemma 3.1, $p_i \geq p_k$ whenever $i \leq k$. Now consider any earlier period t . $p(i, t + 1) = p(j^*(i, t + 1), t + 2)$ for all i . By the inductive hypothesis, if $j^*(i, t + 1) \leq j^*(k, t + 1)$, then we are done. Recall that $j^*(k, t + 1)$ is determined by the sales schedule which maximizes revenue earned in \mathcal{G}_i for the remaining periods. In particular, for all $l \geq j^*(k, t + 1)$,

$$\begin{aligned} & (j^*(k, t + 1) - k + 1) \cdot p(j^*(k, t + 1), t + 2) + \Pi(j^*(k, t + 1) + 1, t + 2) \\ & \geq (l - k + 1) \cdot p(l, t + 2) + \Pi(l + 1, t + 2) \end{aligned}$$

Again, by the inductive hypothesis, $p(j^*(k, t+1), t+2) \geq p(l, t+2)$. Now multiply this inequality by $k-i$ (which is non-negative) and add it to the above to get

$$\begin{aligned} (j^*(k, t+1) - i + 1) \cdot p(j^*(k, t+1), t+2) + \Pi(j^*(k, t+1) + 1, t+2) \\ \geq (l - i + 1) \cdot p(l, t+2) + \Pi(l+1, t+2), \end{aligned}$$

for every $l \geq j^*(k, t+1)$. Since $j^*(i, t+1)$ satisfies

$$j^*(i, t+1) = \arg \max_{j \geq i} ((j - i + 1) \cdot p(j, t+2) + \Pi(j+1, t+2))$$

it follows that $j^*(i, t+1) \leq j^*(k, t+1)$, which gives us our result. \square

Lemma A3 (Lemma 3.3). *Consider two duopoly games, \mathcal{G} , with T periods and a set S of consumers, and \mathcal{G}' , with T periods and a set S' of consumers such that only the top valued consumers in S and S' differ and the top valued consumer in S' , who we name x , has the higher value. If we use $p_{\mathcal{G}}^*(1, 1)$ and $p_{\mathcal{G}'}^*(1, 1)$ to denote the first period prices as calculated by the recursion relationship in Section 3, then $p_{\mathcal{G}'}^*(1, 1) \geq p_{\mathcal{G}}^*(1, 1)$.*

Proof. It is easily seen that in the case $T = 1$, the result holds (the price either remains the same or increases to v_x). Consider the optimal sales schedule for the game \mathcal{G} , and assume this schedule sells to more than one person in period 1. But prices for a schedule which sells to more than one person in period 1 do not depend on the value of the top consumer in S (the price in period 1 depends on the threat price of a lower-valued consumer, and later prices depend only on the consumers left). Therefore, we can achieve the same profit from such a schedule in \mathcal{G}' with the same prices. So in \mathcal{G}' , the optimal sales and pricing schedule either is the same as the optimal for \mathcal{G} , in which case we are done, or involves selling only to x in the first period. If the durpolist sells to x in the first period, it is at a price $p_{\mathcal{G}'}^*(1, 1)$ equal to x 's threat price. This is, by definition, the same price as for the game \mathcal{G}'' with consumers S' but with $T-1$ periods instead of T periods. By the induction hypothesis, this price is higher than the corresponding optimal first period price p_S for the game with consumers S but with $T-1$ periods. But p_S , by definition, is the threat price for the top consumer in S in \mathcal{G} , and therefore at least as high as the period 1 threat price under the optimal sales schedule in \mathcal{G} (if $v_i \geq v_j$, i 's threat price is $\geq j$'s threat price: see proof of Lemma 3.2 in this section). But the optimal period 1 threat price is $p_{\mathcal{G}}^*(1, 1)$, so $p_{\mathcal{G}'}^*(1, 1) \geq p_S \geq p_{\mathcal{G}}^*(1, 1)$.

It remains to prove the case where the optimal sales schedule in \mathcal{G} sells to just the top consumer in S . In this case, the price charged in period 1 is p_S . But by the same argument as above, the threat price for x is at least as high as p_S . Therefore if the optimal sales schedule in

\mathcal{G}' sells to just x in the first period, the first period price is at least as high as in \mathcal{G} . But note that if the duopolist sells to more than one person in period 1 of \mathcal{G}' , she achieve the same profit as a sub-optimal sales schedule in \mathcal{G} . But she can clearly beat that reveue by selling to x in period 1 and then following the optimal sales schedule in \mathcal{G} from period 2 onwards. Therefore, whatever the optimal sales schedule in \mathcal{G}' , it must involve selling exactly one item in period 1. Therefore we have $p_{\mathcal{G}'}^*(1, 1) \geq p_S = p_{\mathcal{G}}^*(1, 1)$.

We have covered all cases, so the lemma is proved. \square

Lemma A4 (Lemma 3.4). *In any game \mathcal{G} with T periods, if the duopolist and consumers follow the strategies described in Section 3, then prices are non-increasing in time.*

Proof. If both duopolist and consumer follow the strategies described in Section 3, the duopolist will select an initial sales path $\{x_t\}$, such that, for the last consumer j_t scheduled to buy in period t ($j_t = \sum_{i \leq t} x_i$), we have the recursive relationship:

$$p(j_t + 1, t + 1) = \tau(j_{t+1}, t + 1) = p(j_{t+1}, t + 2)$$

In other words, in period t the duopolist plans to sell to consumers $j_t + 1, j_t + 2, \dots, j_{t+1}$ at j_{t+1} 's threat price. By Lemma 3.2, this is less than or equal the threat price of everyone in the set $\{j_t + 1, j_t + 2, \dots, j_{t+1}\}$. So under the consumer strategies specified, all x_t consumers in $\{j_t + 1, j_t + 2, \dots, j_{t+1}\}$ buy in period t , and the duopolist's strategy never deviates from the initial sales path. So the price she charges in each period t is $p(j_{t-1} + 1, t)$ (with $j_0 \equiv 0$).

As part of Lemma 3.2 we showed that $j^*(i, t + 1) \leq j^*(k, t + 1)$ for all $i \leq k$. Using this and the result of Lemma 2, we have

$$\begin{aligned} p(j_{t-1} + 1, t) &= p(j_t, t + 1) \\ &= \tau(j^*(j_t, t + 1), t + 1) \\ &\geq \tau(j^*(j_t + 1, t + 1), t + 1) \\ &= \tau(j_{t+1}, t + 1) \\ &= p(j_t + 1, t + 1), \end{aligned}$$

where in the fourth line, we use the fact that $j^*(j_t + 1, t + 1) = j_{t+1}$ from the definition of the j_t 's and the argmax condition of the recursion relation (1). So these prices are non-increasing in time. \square

Lemma A5. *There is a subgame perfect equilibrium which follows the recursion relationship (1) in which a sale occurs in each period until all consumers have already bought the item, and this*

equilibrium achieves at least as much profit for the duropolist as any which allows the duropolist to not sell in some periods where there are consumers remaining.

Proof. The proof will be by induction. In the case $T = 1$, it is clearly a dominant strategy for the duropolist to sell if there are any consumers remaining, as otherwise he will earn nothing. This also holds for the last period of a longer game in a subgame perfect equilibrium.

Now consider $T > 1$, and to start, assume that the claim is false. Then there must be a game \mathcal{G} with a period $t^l < T$ such that the following two conditions hold: (a) there remain consumers who haven't bought at the start of period t^l , but the duropolist does sell any items in this period; (b) the duropolist sells items in every subsequent period of the game. So there is an equilibrium for a game \mathcal{G}' , corresponding to the subgame of \mathcal{G} starting at t^l , where the duropolist sells nothing in the first period, and sells at least one item in all subsequent periods, until all consumers have bought, and this equilibrium yields strictly more profit than one which follows (1). Thus, to show our claim is true, we only need to show that equilibria where we sell nothing in the first period followed by sales in every subsequent period (until all consumers have bought) do not yield more profit than those which follow the recursion relationship (1). Furthermore, it is enough to show a sub-optimal sales schedule yields as much profit, as the optimal recursion relationship result must do even better.

Let $\mathcal{G}(N, T)$ be our game with a set of consumers N and T periods. If nothing is sold in the first period, and the duropolist sells at least one item in each subsequent period (until all consumers have bought), the duropolist can achieve profit of at most $\Pi_{\mathcal{G}(N, T-1)}^D$ (see discussion preceeding Corollary 3.1). Let $k \geq 1$ and p be the number of items sold in the first period and the first period price, respectively, of $\mathcal{G}(N, T-1)$ under (1). Consider the following (possibly sub-optimal) strategies for $\mathcal{G}(N, T)$: the duropolist sells at price p in period 1 and follows the equilibrium of (1) for all subsequent periods, while the consumers buy iff the price is less than their threat price. We know that the top consumer will buy in period 1 as she would be offered the same price if everyone refused to buy in period 1. We also know that no consumer $i > k$ will buy as p is equal to k 's threat price, $\tau_{\mathcal{G}(N, T-1)}(k, 1)$, but if i and all below her refused to buy in $\mathcal{G}(N, T)$, the price in the second period of $\mathcal{G}(N, T)$ would be at most $\tau_{\mathcal{G}(N, T-1)}(k+1, 1)$. So some number $1 \leq l \leq k$ buys in the first period. By definition of (1), we sell at least one item in each subsequent period as well, as long as there are consumers remaining to buy. If $l = k$, then, by the induction hypothesis, we make at least $\Pi_{\mathcal{G}(N-[k], T-2)}^D$ with sales in each subsequent period until all consumers are sold to. So our total profit from this possibly sub-optimal scheme is at least

$$\Pi_{\mathcal{G}(N-[k], T-2)}^D + k \cdot p = \Pi_{\mathcal{G}(N, T-1)}^D.$$

If $1 \leq l < k$, we know that $\Pi_{\mathcal{G}(N-[l], T-1)}^D$ is larger than the profit obtained by selling to $k-l$ consumers at price p and then following the equilibrium for $\mathcal{G}(N-[k], T-2)$ given by (1). So we obtain profit of

$$l \cdot p + \Pi_{\mathcal{G}(N-[l], T-1)}^D \geq l \cdot p + (k-l) \cdot p + \Pi_{\mathcal{G}(N-[k], T-2)}^D = \Pi_{\mathcal{G}(N, T-1)}^D.$$

Therefore under the optimal schedule, the profit is at least $\Pi_{\mathcal{G}(N, T-1)}^D$. Therefore there is no benefit to waiting a period before starting to sell. This proves the claim. \square

Lemma A6 (Lemma 4.2). *If $p_i = v_i$ for all $i \in [N]$, the following inequality holds for every natural number $\beta \geq 2$, $k = 2, \dots, \beta$ and all $i = 1, \dots, k-1$,*

$$n_i \cdot w_i - n_i \cdot w_k - n_{\beta+i} w_{\beta+i} \geq 0.$$

Proof. The statement is trivially true if $k > M$ as then $w_k = w_{\beta+i} = 0$. Thus, we may assume that $1 \leq i < k \leq M$. First we show that $w_i \geq 2w_k$. By assumption, $p_i = v_i$. Hence

$$1 \cdot w_i \geq (1 + n_{i+1} + \dots + n_k) \cdot 2 \cdot w_k \tag{8}$$

Similarly

$$w_k \geq (1 + n_{k+1} + n_{k+2} + \dots + n_{\beta+i}) \cdot w_{\beta+i} \geq n_{\beta+i} \cdot w_{\beta+i} \tag{9}$$

Combining (8) and (9) we have

$$\begin{aligned} n_i \cdot w_i - n_i \cdot w_k - n_{\beta+i} w_{\beta+i} &\geq n_i \cdot w_i - (n_i + 1) \cdot w_k \\ &\geq n_i \cdot w_i - (n_i + 1) \cdot \frac{w_i}{2} \\ &\geq 0 \end{aligned}$$

as desired. \square