Unit V.

Graph Theory

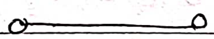
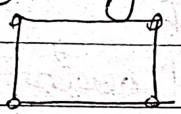
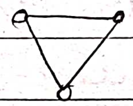
- Basic concepts of graph theory
- Digraphs
- Basic definitions
- Paths and circuits
- Reachability & Connectedness
- Matrix representation of graphs
- Subgraphs & Quotient graphs
- Isomorphic diggraphs & Transitive closure digraph
- Euler's path & circuit (only def^n & eg)
- Trees
- Binary tree
- Labelled trees
- Undirected trees
- Spanning trees of Connected Relations
- Prim's Algorithm to construct spanning trees
- Weighted graphs
- Minimal spanning trees by Prim's Algorithm and Kruskal's Algorithm

Basic concepts of graph theory

Graph

A graph $G = (V, E)$ consists of a non-empty set V, whose elements are called nodes or vertices or points and E is the set of edges whose elements are pairs of elements of V.

$\therefore$ $G = (V, E)$ ~~are~~ is a graph ~~where~~
where V = set of vertices or nodes or points
E = set of edges.

For eg. 1)  2)  3) 

Note: If (u, v) is an edge in ~~eds~~ a graph $G = (V, E)$, then u and v are said to be a adjacent nodes.

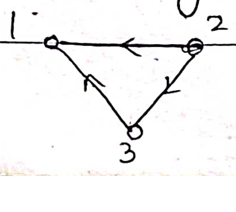
Definition:

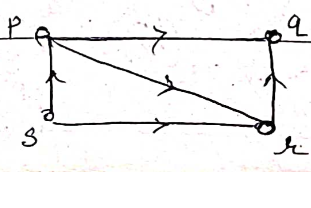
CSE W06 W08 S11 IT S09 i) **Undirected graph:**

Let $G = (V, E)$ be a graph. If the elements of E are represented by unordered pairs of vertices of G, then G is called a undirected graph.
The above two graphs (2) & (3) are undirected graphs.

CSE W06 W08 S11 IT S09 (ii) **Directed graph**

Let $G = (V, E)$ be a graph. If the elements of E are represented by ordered pairs of vertices of G then G is called a directed graph.
Following are the examples of directed graphs (or digraphs)



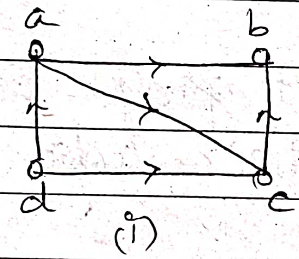


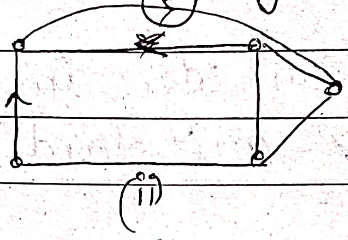
The edge (u, v) of a digraph $G = (V, E)$ is an ordered pair where u is called the initial node (or vertex) and v is called the terminal node (or vertex)

3) Mixed graph :

A graph $G = (V, E)$ is said to be a mixed graph if it is neither a digraph nor an undirected graph.

for eg. Consider the following graph

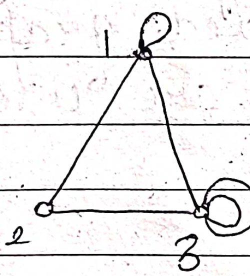




4) Loop :

Let $G = (V, E)$ be a graph. An edge of G, which joins a node v to itself is called a loop

for eg. Consider the following graph.

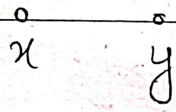


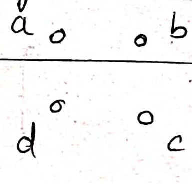
Here node 1 has one loop, node 3 has 2 loops & node 2 does not have a loop.

5) Null or Void graph.

Let $G = (V, E)$ be a graph. If the set of edges E is empty, then the graph G is said to be null graph or void graph

for eg. Consider the following graph.

(a) 

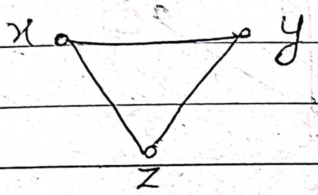
(b) 

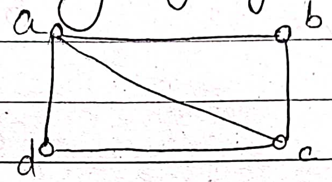
6) Simple graph :

Let $G = (V, E)$ be a graph. Then G is said to be a simple graph if

1) G has no loops.
2) There is exactly one edge between any given pair of vertices.

for eg. Consider the following graph

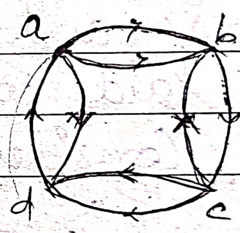
i) 

ii) 

7) Parallel Edges :

Let $G = (V, E)$ be a graph. Two edges of G are ~~said~~ said to parallel if they have same beginning and ending point.

for eg. Consider the following graph



Here (a, b) is a parallel edge.
Also (d, c) is a parallel edge.
But (a, d) & (b, c) are not parallel edges.

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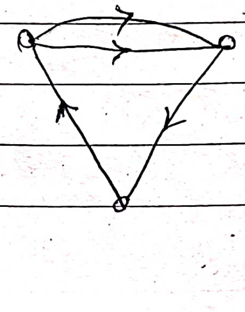
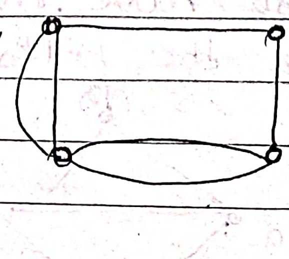
8) Multigraph :

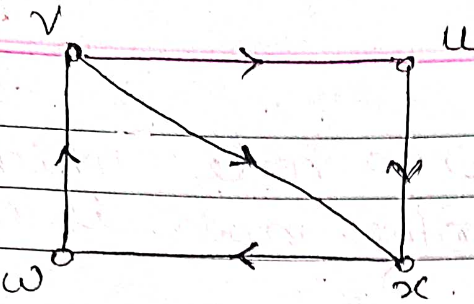
Let $G = (V, E)$ be a graph. Then G is said to be a multigraph if it has some multiple edges.

for eg. Consider the following graph.

i) 

(ii) 

→ o Indegree



Vertex	Indegree	Outdegree	Total degree
u	1	1	2
v	1	2	3
w	1	1	2
x	2	1	3

A vertex whose indegree is zero is called a source & a vertex whose outdegree is zero called a sink.

Note: In a digraph, a loop contributes 1 to indegree and 1 to outdegree of the vertex.

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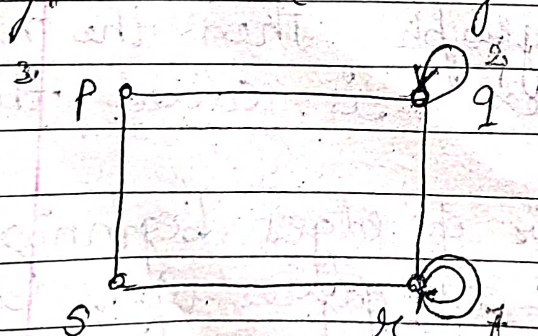
14) Degree of a vertex in an undirected graph.
Let $G = (V, E)$ be an undirected graph. Then the number of edges incident with a vertex v with self loops counted twice is called the degree of the vertex v.

For eg. Consider the following graph.



Vertex	Degree
p	~~3~~ 2
q	4
r	~~4~~ 6
s	2 2

A vertex whose degree is zero is called an isolated vertex. A vertex whose degree is one is called a pendant vertex.

Matrix representation of graphs

Representating graphs by means of matrices, we can study the structual properties of graph from algebraic point of view. There are various ways of representing graphs by matrices.

A) Adjacency matrix & (B) Incidence matrix:

(A) Adjacency matrix:

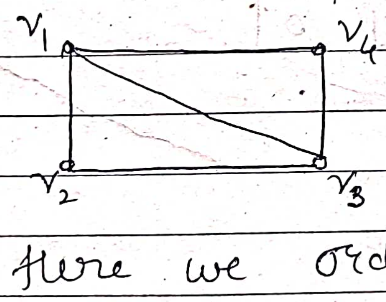
(i) Representation of Undirected graph:

Let $G=(V,E)$ be a simple graph with n vertices. Then an $n\times n$ matrix $A=(a_{ij})$ defined by

$$a_{ij} = \begin{cases} 1, & \text{if } (v_i, v_j) \text{ is an edge} \\ 0, & \text{otherwise} \end{cases}$$

is called the adjacency matrix. Hence adjacency matrix is a bit matrix or a Boolean matrix as the elements are either 0 or 1.

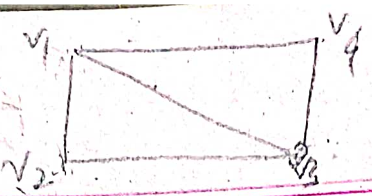
Q1. Represent the following graphs using adjacency matrix:

i) 

Sol: Here we order the vertices as v_1, v_2, v_3, v_4

∴ there are four vertices, the adjacency matrix representing the graph will be a square matrix of order 4 (4 × 4 matrix).

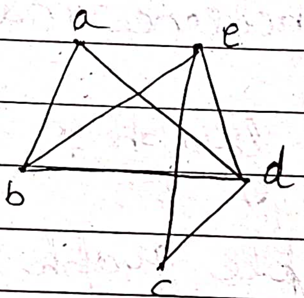
The reqd. adjacency matrix is



The required adjacency matrix is :

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

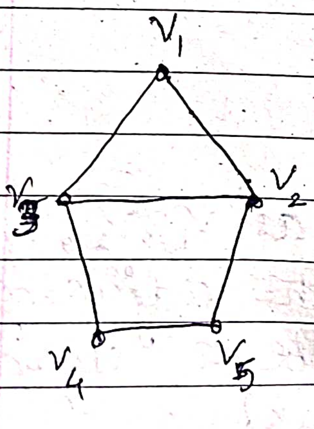
(ii)



Soln: Here we order the ~~vertices~~ vertices as a, b, c, d, e. Since there are five vertices, the adjacency matrix representing the graph will be a square matrix of order 5 (5×5 matrix).
The reqd. adjacency matrix is :

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

(iii)



Solⁿ. Here we order the vertices as v_1, v_2, v_3, v_4, v_5.
Since there are five vertices the adjacency matrix representing the graph will be a square matrix of order 5 (5×5 matrix).
The reqd. adjacency matrix is :

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

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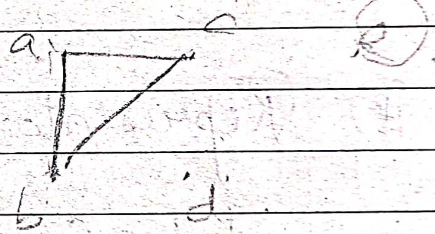
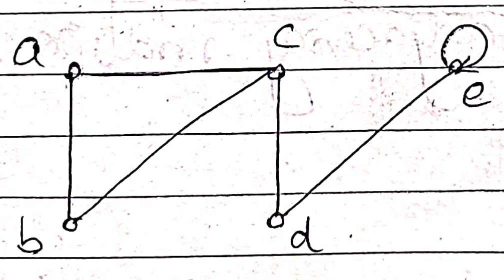
Q2. Draw an undirected graph representing the following adjacency matrices.

(a)

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

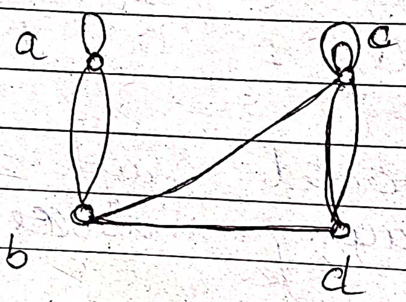


Solⁿ. The given adjacency matrix is a square matrix of order 5×50. Hence the graph will have 5 vertices say a, b, c, d, e. The required graph for the given adjacency matrix is:



(b)
$$B = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

Solⁿ: The given adjacency matrix is a square matrix of order 4×4. Hence the graph will have 4 vertices say a, b, c, d.
The required graph for the given adjacency matrix is



ii) Representation of Digraph :

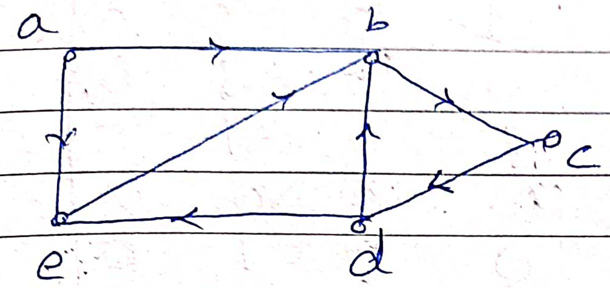
Let $G = (V, E)$ be a simple graph with n vertices. Then an $n \times n$ matrix $A = (a_{ij})$ defined by

$$a_{ij} = \begin{cases} 1 \; ; & \text{if there is an edge from } v_i \text{ to } v_j \\ 0 \; ; & \text{Otherwise.} \end{cases}$$

is called the adjacency matrix.

Q3. Represent the following digraphs by adjacency matrix.

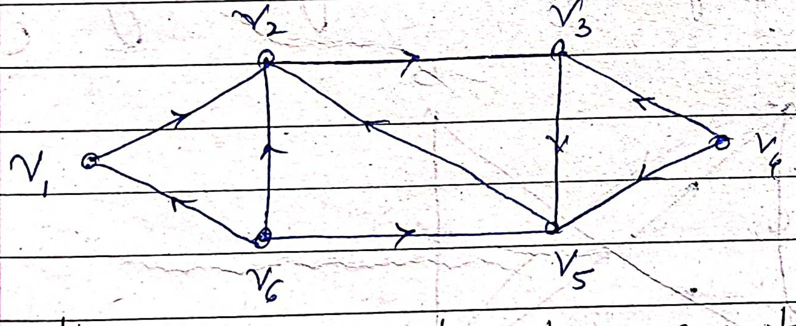
i)



Sol^n. Here we order the vertices as a, b, c, d, e. Since there are five vertices, the adjacency matrix representing the graph will be a square matrix of order 5 (5×5 matrix). The required matrix is :

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \left[\begin{array}{ccccc} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right] \end{array}$$

(ii)



Sol^n. Here we order the vertices as $v_1, v_2, v_3, v_4, v_5, v_6$. Since there are 6 vertices, the adjacency matrix representing the graph will be a square matrix of order 6 (6×6 matrix). The required adjacency matrix is :

$$A = \begin{array}{c|cccccc|} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ \hline v_1 & 0 & 1 & 0 & 0 & 0 & 0 \\ v_2 & 0 & 0 & 1 & 0 & 0 & 0 \\ v_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ v_4 & 0 & 0 & 1 & 0 & 1 & 0 \\ v_5 & 0 & 1 & 0 & 0 & 0 & 0 \\ v_6 & 1 & 1 & 0 & 0 & 1 & 0 \end{array}$$

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Q4. Draw a digraph corresponding to the following adjacency matrix A.

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Sol. The given matrix is a square matrix of order 4x4. ~~mat~~ Hence the graph will have 4 vertices say a, b, c, d.
The reqd. digraph for the given adjacency matrix is :