Matrix representation of graphs.

Representating graphs by means of matrices, we can study the structual properties of graph from algebraic point of view. There are various ways of representating graphs by matrices.

A) Adjacency matrix & (B) Incidence matrix:

(A) Adjacency matrix:

(i) Representation of Undirected graph:

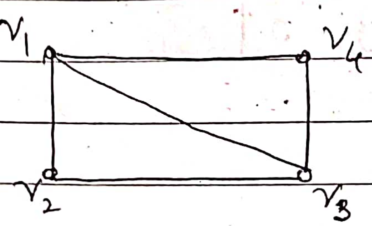
Let $G = (V, E)$ be a simple graph with n vertices. Then an $n \times n$ matrix $A = (a_{ij})$ defined by

$$a_{ij} = \begin{cases} 1, & \text{if } (v_i, v_j) \text{ is an edge} \\ 0, & \text{otherwise} \end{cases}$$

is called the adjacency matrix. Hence adjacency matrix is a bit matrix or a Boolean matrix as the elements are either 0 or 1.

Q1. Represent the following graphs using adjacency matrix:

i)



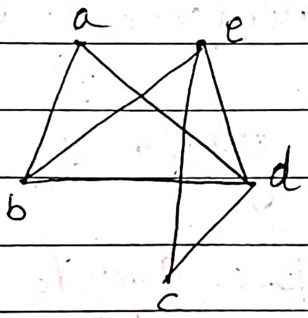
Sol: Here we order the vertices as v_1, v_2, v_3, v_4
∴ there are four vertices, the adjacency matrix representing the graph will be a square matrix of order 4 (4×4 matrix).

The reqd. adjacency matrix is

The required adjacency matrix is :

$$A = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 1 & 1 & 1 \\ v_2 & 1 & 0 & 1 & 0 \\ v_3 & 1 & 1 & 0 & 1 \\ v_4 & 1 & 0 & 1 & 0 \end{array}$$

(ii)

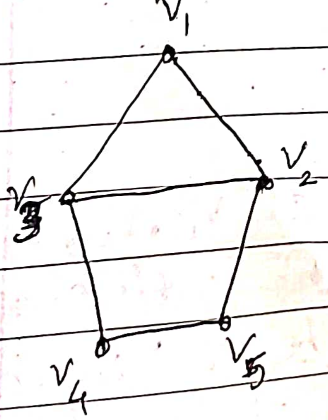


Sol^n: Here we order the ~~vertices~~ vertices as a, b, c, d, e. Since there are five vertices, the adjacency matrix representing the graph will be a square matrix of order 5 (5×5 matrix).

The reqd. adjacency matrix is :

$$A = \begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a & 0 & 1 & 0 & 1 & 0 \\ b & 1 & 0 & 0 & 1 & 1 \\ c & 0 & 0 & 0 & 1 & 1 \\ d & 1 & 1 & 1 & 0 & 1 \\ e & 0 & 1 & 1 & 1 & 0 \end{array}$$

(iii)



Sol. Here we order the vertices as v_1, v_2, v_3, v_4, v_5. Since there are five vertices the adjacency matrix representing the graph will be a square matrix of order 5 (5×5 matrix).
The reqd. adjacency matrix is :

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

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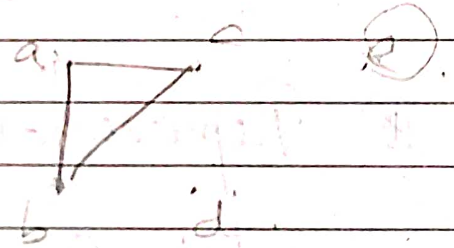
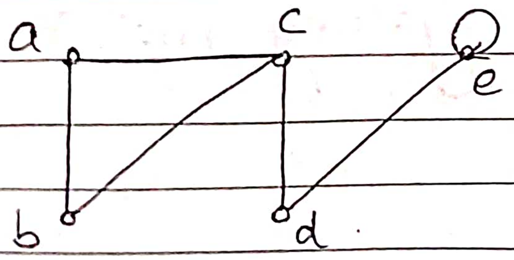
Q2. Draw an undirected graph representing the following adjacency matrices.

(a)

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

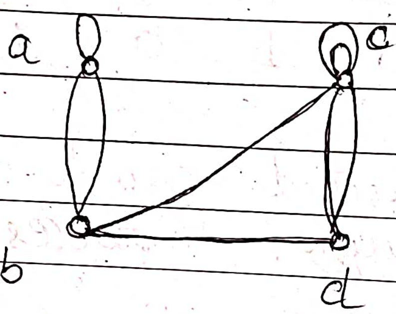


Sol. The given adjacency matrix is a square matrix of order 5×5. Hence the graph will have 5 vertices say a, b, c, d, e. The required graph for the given adjacency matrix is :



(b)
$$B = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 2 & 0 & 0 \\ b & 2 & 0 & 1 & 1 \\ c & 0 & 1 & 2 & 2 \\ d & 0 & 1 & 2 & 0 \end{array}$$

Solⁿ The given adjacency matrix is a square matrix of order 4×4. Hence the graph will have 4 vertices say a, b, c, d.
The required graph for the given adjacency matrix is



ii) Representation of Digraph :

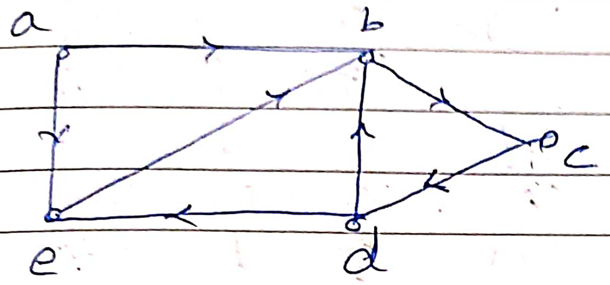
Let $G=(V, E)$ be a simple graph with n vertices. Then an $n\times n$ matrix $A=(a_{ij})$ defined by

$$a_{ij} = \begin{cases} 1 ; & \text{if there is an edge from } v_i \text{ to } v_j \\ 0 ; & \text{Otherwise.} \end{cases}$$

is called the adjacency matrix.

Q3. Represent the following digraphs by adjacency matrix.

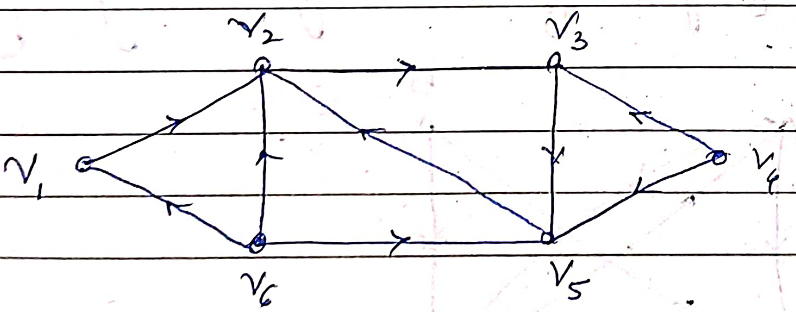
i)



Soln. Here we order the vertices as a, b, c, d, e. Since there are five vertices, the adjacency matrix representing the graph will be a square matrix of order 5 (5×5 matrix). The required matrix is:

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

(ii)



Sol. Here we order the vertices as $v_1, v_2, v_3, v_4, v_5, v_6$. Since there are 6 vertices, the adjacency matrix representing the graph will be a square matrix of order 6 (6×6 matrix). The required adjacency matrix is:

$$A = \begin{array}{c|cccccc|} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ \hline v_1 & 0 & 1 & 0 & 0 & 0 & 0 \\ v_2 & 0 & 0 & 1 & 0 & 0 & 0 \\ v_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ v_4 & 0 & 0 & 1 & 0 & 1 & 0 \\ v_5 & 0 & 1 & 0 & 0 & 0 & 0 \\ v_6 & 1 & 1 & 0 & 0 & 1 & 0 \end{array}$$

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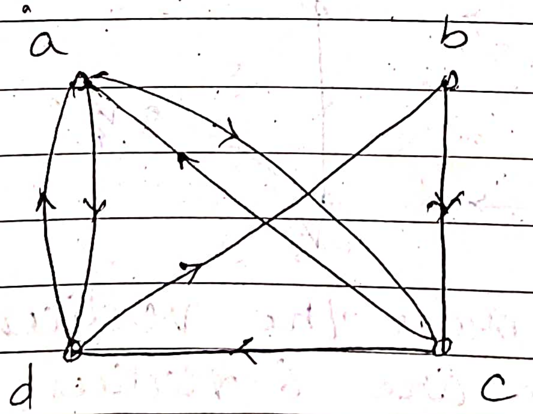
Q4. Draw a digraph corresponding to the following adjacency matrix A.

$$A = \begin{array}{c|cccc|} & a & b & c & d \\ \hline a & 0 & 0 & 1 & 1 \\ b & 0 & 0 & 1 & 0 \\ c & 1 & 0 & 0 & 1 \\ d & 1 & 1 & 0 & 0 \end{array}$$

Sol: The given matrix is a square matrix of order 4x4. ~~matri~~ Hence the graph will have 4 vertices say a, b, c, d.
The reqd. digraph for the given adjacency matrix is :



Incidence Matrix :

(i) Representation of Undirected Graph :

Let G = (V, E) be an undirected graph with n vertices and m edges. The incidence matrix $B = (b_{ij})$, is an $n \times m$ matrix whose elements are given by

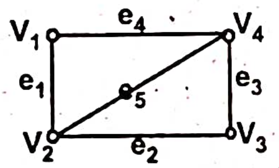
$$b_{ij} = \begin{cases} 1 \; ; & \text{if edge } e_j \text{ is incident with vertex } v_i \\ 0 \; ; & \textit{otherwise} \end{cases}$$

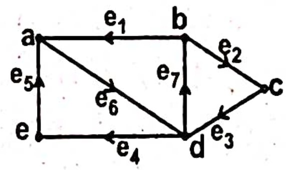
(ii) Representation of Digraph :

Let G = (V, E) be a digraph with n vertices and m edges. The incidence matrix $B = (b_{ij})$ is an $n \times m$ matrix whose elements are given by

$$b_{ij} = \begin{cases} -1; & \text{if arc } j \text{ is } \textit{directed } \underline{\textbf{towards}} \textit{ vertex } v_i \\ 1; & \text{if arc } j \text{ is directed } \underline{\textbf{away}} \text{ from vertex } v_i \\ 0; & \text{otherwise} \end{cases}$$

Example (5) : Represent the following graphs by means of incidence matrix.

(i) 

(ii) 

Solution : (i) Here we order the vertices as v_1, v_2, v_3, v_4 and edges as e_1, e_2, e_3, e_4, e_5. There are 4 vertices and 5 edges. Hence the incidence matrix will have order 4×5. The required incidence matrix is

$$B = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

(ii) Here we order the vertices as a, b, c, d, e and edges as $e_1, e_2, e_3, e_4, e_5, e_6, e_7$, . There are 5 vertices and 7 edges. Hence the incidence matrix will have order 5 x 7. The required incidence matrix is

$$B = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

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Isomorphic graphs :

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be any two graphs. Then G_1 and G_2 are said to be isomorphic if there exists a function $f: V_1 \to V_2$ such that

i) f is one-one and onto
ii) $(a, b) \in E_1 \Rightarrow (f(a), f(b)) \in E_2$, $\forall\, a, b \in V_1$

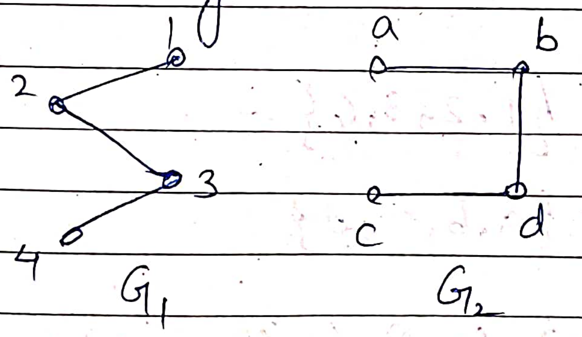
If G_1 is isomorphic to G_2 then it is denoted by $G_1 \approx G_2$ or $G_1 \cong G_2$.

Note:
i) If the number of vertices in G_1 and G_2 are unequal, then G_1 and G_2 cannot be isomorphic.

ii) If the number of edges in G_1 and G_2 are unequal, then G_1 and G_2 cannot be isomorphic.

Q6.(a) Find which of the following graphs are isomorphic.

i)

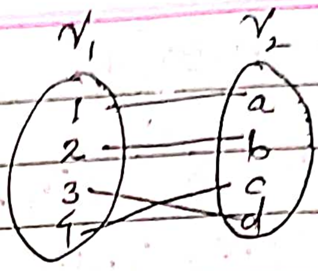


Sol^n: Here $V_1 = V(G_1) = \{1, 2, 3, 4\}$

$V_2 = V(G_2) = \{a, b, c, d\}$

$E_1 = E(G_1) = \{(1,2), (2,3), (3,4)\}$

$E_2 = E(G_2) = \{(a,b), (b,d), (d,c)\}$



Let $f : V_1 \to V_2$

$f(1) = a$
$f(2) = b$
$f(3) = d$
$f(4) = c$

from the figure it is clear that $f : V_1 \to V_2$ is both one-one & onto.

Also $(1,2) \in E_1 \Rightarrow (f(1), f(2)) = (a,b) \in E_2$

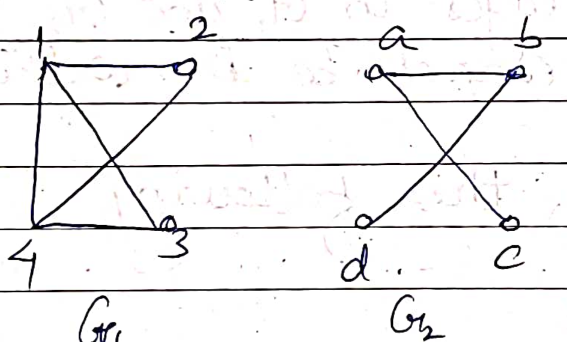
Similarly $(2,3) \in E_1 \Rightarrow (f(2), f(3)) = (b,d) \in E_2$

& $(3,4) \in E_1 \Rightarrow (f(3), f(4)) = (d,c) \in E_2$.

$\therefore (a,b) \in E_1 \Rightarrow (f(a), f(b)) \in E_2, \; \forall a,b \in V_1$

$\therefore G_1 \cong G_2$

ii)



Sol^n Here $V_1 = V(G_1) = \{1, 2, 3, 4\}$

$V_2 = V(G_2) = \{a, b, c, d\}$

$E_1 = E(G_1) = \{(1,2), (1,3), (1,4), (2,4), (3,4)\}$

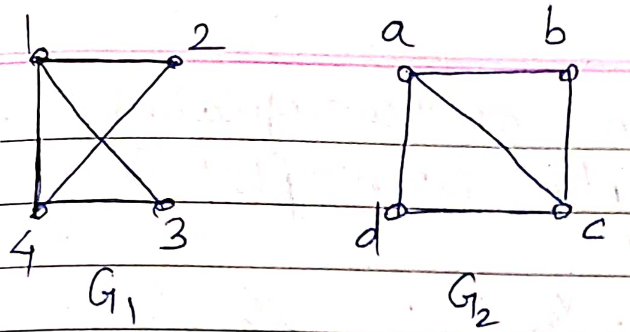
$E_2 = E(G_2) = \{(a,b), (a,c), (b,d)\}$

$n(E_1) = 5, \quad n(E_2) = 3$

$\therefore n(E_1) \neq n(E_2)$

$\therefore G_1 \not\cong G_2$

(iii)



G_1 G_2

Sol^n: Here $V_1 = V(G_1) = \{1, 2, 3, 4\}$

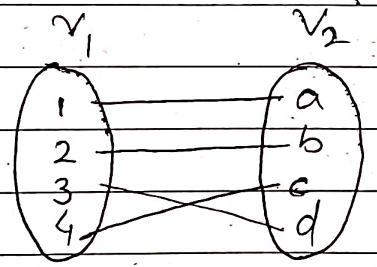
$V_2 = V(G_2) = \{a, b, c, d\}$

$E_1 = E(G_1) = \{(1,2), (1,3), (1,4), (2,4), (3,4)\}$

$E_2 = E(G_2) = \{(a,b), (a,c), (a,d), (b,c), (c,d)\}$

Let $f : V_1 \to V_2$

From the figure it is clear that $f : V_1 \to V_2$ is both one one & onto



$f(1) = a$
$f(2) = b$
$f(3) = d$
$f(4) = c$

Also $(1,2) \in E_1 \Rightarrow (f(1), f(2)) = (a,b) \in E_2$

$(1,3) \in E_1 \Rightarrow (f(1), f(3)) = (a,d) \in E_2$

$(1,4) \in E_1 \Rightarrow (f(1), f(4)) = (a,c) \in E_2$

$(2,4) \in E_1 \Rightarrow (f(2), f(4)) = (b,d) \in E_2$

$(3,4) \in E_1 \Rightarrow (f(3), f(4)) = (d,c) \in E_2$

$\therefore (a,b) \in E_1 \Rightarrow (f(a), f(b)) \in E_2 \quad \forall\, a, b \in V_1$

$\therefore G_1 \cong G_2$

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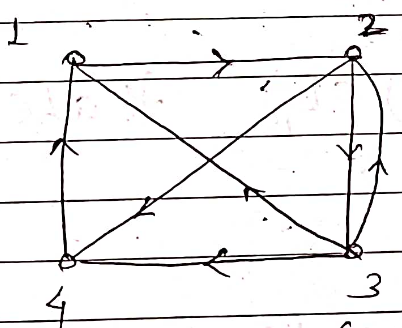
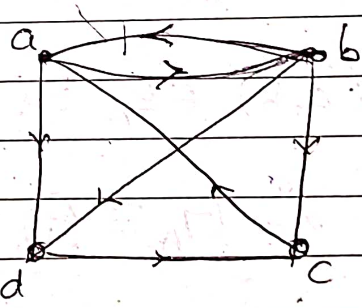
Q7. Draw the digraphs of the following adjacency matrices A and B, Are these graphs isomorphic? Also draw the digraph corresponding to A^T and B^T. Are these graph isomorphic?

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Solⁿ Since the order of matrices A and B is 4×4 hence their digraphs will have 4 vertices say 1, 2, 3, 4 & a, b, c, d respectively. The required digraphs are:



graph of A (G_A)



graph of B (G_B)

Here $V_A = V(G_A) = \{1, 2, 3, 4\}$

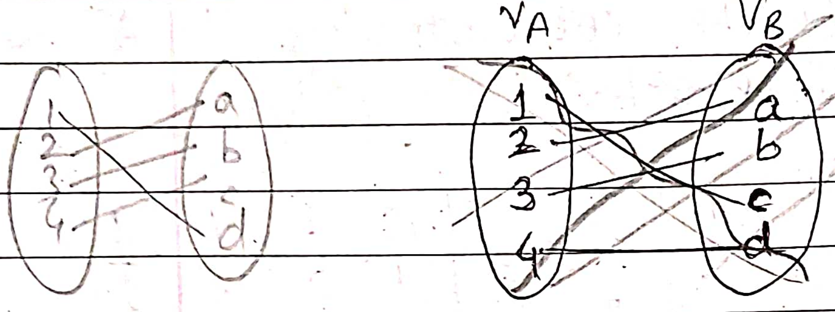
$V_B = V(G_B) = \{a, b, c, d\}$

$E_A = E(G_A) = \{(1,2), (2,3), (2,4), (3,1), (3,2), (3,4), (4,1)\}$

$E_B = E(G_B) = \{(a,b), (a,d), (b,a), (b,c), (b,d), (c,a), (d,c)\}$

Let $f : V_A \rightarrow V_B$

From the fig. it is clear that $f: V_A \to V_B$ ~~such~~ is both one-one and onto.



$f(1) = \cancel{a} = d$
$f(2) = \cancel{a} \quad a$
$f(3) = \cancel{b} \quad b$
$f(4) = \cancel{d} \quad c$

Also. $(2,3) \in E_A \Rightarrow (f(2), f(3)) = (a, b) \in E_B$
$(3,1) \in E_A \Rightarrow (f(3), f(1)) = (b, d) \in E_B$
$(3,4) \in E_A \Rightarrow (f(3), f(4)) = (b, c) \in E_B$

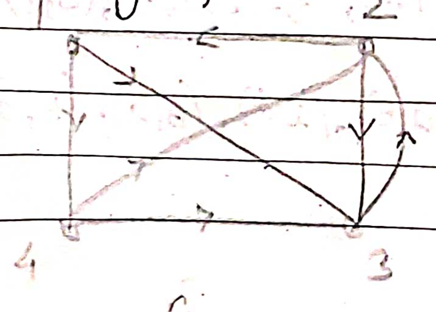
$\therefore (a,b) \in E_A \Rightarrow (f(a), f(b)) \in E_B \quad \forall\, a, b \in E_A$.

$\therefore G_A \cong G_B$.

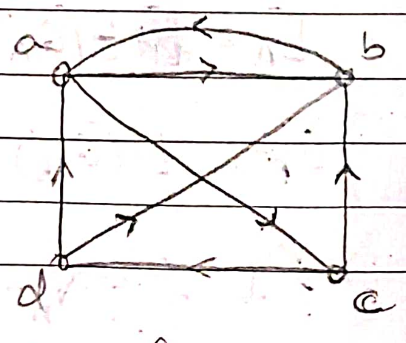
Now

$$A^T = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}, \quad B^T = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

(ii) The diagraphs of A^T and B^T are drawn below:



G_{A^T}



G_{B^T}

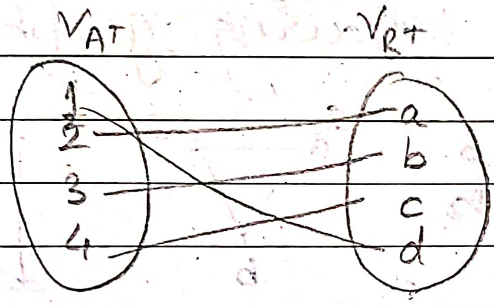
Here, $V_{AT} = V(G_{AT}) = \{1, 2, 3, 4\}$

$V_{BT} = V(G_{BT}) = \{a, b, c, d\}$.

$E_{AT} = E(G_{AT}) = \{(1,3), (1,4), (2,1), (2,3), (3,2), (4,2), (4,3)\}$

$E_{BT} = E(G_{BT}) = \{(a,b), (a,c), (b,a), (c,b), (c,d), (d,a), (d,b)\}$

Let $f : V_{AT} \rightarrow V_{BT}$.



$f(1) = d$
$f(2) = a$
$f(3) = b$
$f(4) = c$.

From the fig., it is clear that $f : V_A \rightarrow V_B$ is both one-one & onto.

Also, $(1,3) \in E_{AT} \Rightarrow (f(1), f(3)) = (d, b) \in E_{BT}$
$(2,3) \in E_{AT} \Rightarrow (f(2), f(3)) = (a, b) \in E_{BT}$.
$(4,3) \in E_{AT} \Rightarrow (f(4), f(3)) = (c, b) \in E_{BT}$.

$\therefore (a, b) \in E_{AT} \Rightarrow (f(a), f(b)) \in E_B \quad \forall a, b \in E_A$

$\therefore G_1 \cong G_B$.