Homework 1 Advanced Analytics and Metaheuristics

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1. To examine this story, we create a table:

Baby Yoda Tells Truth	gurump = yes	pvlork = yes	Possible
T	T	T	N
T	T	F	N
T	F	T	N
T	F	F	N
F	T	T	N
F	T	F	Y
F	F	T	Y
F	F	F	N

   I will assume that a sinister lying creature always lies (even though we know this too not be true eg. politicians).

   Let's examine each with our information. We were told that the words meant yes or no so we assume that they cannot both mean the same thing. This eliminates half of our truth table. Next, we see that if Jedi would tell the truth, the statement cannot be. You would not answer yes to a wrong question. Similarly you cannot have the third entry either as you will tell the truth. We move on to sith liars. If gurump means No and asked if it means yes, a liar would reply in the affirmative. This means the logic on the sixth one is possible. We see the seventh is also possible. Since gurump is yes and the creature lies, we would answer no or pvlork. Thus we see that Baby Yoda is a liar. We cannot however determine the meaning of gurump and pvlork.

2. I am going to state the problem here

A portfolio manager in charge of a bank portfolio has \$10 million to invest. The securities available for purchase, as well as their respective quality ratings, maturities, and yields, are shown in Table

Name	Type	QS Moody's	QS Banks	Years to M	Yield to m	After-tax yield
A	Municipal	Aa	2	9	4.3%	4.3%
B	Agency	Aa	2	15	5.4	2.7
C	Government	Aaa	1	4	5.0	2.5
D	Government	Aaa	1	3	4.4	2.2
E	Municipal	Ba	5	2	4.5	4.5

The bank places the following policy limitations on the portfolio managers actions:

(a) Government and agency bonds must total at least \$4 million.

(b) The average quality of the portfolio cannot exceed 1.4 on the banks quality scale. (Note that a low number on this scale means a high-quality bond.)

(c) The average years to maturity of the portfolio must not exceed 5 years.

Here are the questions we answer below:

(a) Assuming that the objective of the portfolio manager is to maximize after-tax earnings and that the tax rate is 50 percent, what bonds should he purchase?

(b) If it became possible to borrow up to \$1 million at 5.5 percent before taxes, how should his selection be changed?

(a) We'll start by stating the objective function, the return on investment after taxes:

$$P(\vec{x}) = 0.043x_A + 0.027x_B + 0.025x_C + 0.022x_D + 0.045x_E$$

This is the function that we wish to maximize.

Next we examine each of the constraints. There is a total of 10 million to invest

$$\sum x_i \leq 10000000$$

We need a total of at least 4 million in government and agency bonds so

$$x_B + x_C + x_D \geq 4000000$$

Then we want the average on the banks quality scale to not exceed 1.4 so

$$\frac{2x_A + 2x_B + x_C + x_D + 5x_E}{\sum x_i} \leq 1.4$$

We need to make this linear for AMPL (I learned after only a few minutes of face to keyboard)

$$2x_A + 2x_B + x_C + x_D + 5x_E - 1.4\sum x_i \leq 0$$

Which can be simplified to (but was not required in AMPL)

$$0.6x_A + 0.6x_B - 0.4x_C - 0.4x_D + 3.6x_E \leq 0$$

Lastly we wanted the average years to maturity to be less than 5 so

$$\frac{9x_A + 15x_B + 4x_C + 3x_D + 2x_E}{\sum x_i} \leq 5$$

This simplifies to

$$4x_A + 10x_B - x_C - 2x_D - 3x_E \leq 0$$

With all of this we code it into AMPL and arrive at the output:

```
Console
AMPL
ampl: model hw1p2a.mod
CPLEX 20.1.0.0: optimal solution; objective 298363.6364
3 dual simplex iterations (1 in phase I)
Solution:
Return on Investment: 298364
Investment in Each Bond =
invest [*] :=
A  2181820
B        0
C  7363640
D        0
E   454545
;
```

(b) For the second part of the problem, we add the additional condition that we can take out a loan of up to 1 million at 5.5%. This will change our objective function in significant ways and add an extra variable x_{loan}. We examine the new objective, we will use the before tax rates and recognize that Municipal bonds grow tax free. We also note that our total interest is subtracted from our tax liability

$$\begin{aligned} z = {} & interestEarned - interestPaid - taxesPaid \\ = {} & \textstyle\sum_i BeforeTaxRates_i x_i - 0.055x_{loan} - 0.5(taxableIncome) \\ = {} & 0.043x_A + 0.054x_B + 0.05x_C + 0.044x_D + 0.045x_E - 0.055x_{loan} \\ & - 0.5\left(0.054x_B + 0.05x_C + 0.044x_D - 0.055x_{loan}\right) \end{aligned}$$

The total amount invested must also change

$$\sum x_i \leq 10000000 + x_{loan}$$

Of course there are the restrictions on the loan too

$$0 \leq x_{loan} \leq 1000000$$

The rest of the constraints remain unchanged. Running it through AMPL arrives at the solution:

```
Console
AMPL
ampl: model hw1p2b.mod
CPLEX 20.1.0.0: optimal solution; objective 300700
5 dual simplex iterations (3 in phase I)
Solution:
Return on Investment: 300700
Amount Borrowed: 1e+06
Investment in Each Bond =
invest [*] :=
A  2400000
B        0
C  8100000
D        0
E    5e+05
;
```

3. I am going to state the problem here:

   This exercise starts with a two-variable linear program similar in structure to the one of Sec- tions 1.1 and 1.2, but with a quite different story behind it.

   (a) You are in charge of an advertising campaign for a new product, with a budget of \$1 million. You can advertise on TV or in magazines. One minute of TV time costs \$20,000 and reaches 1.8 million potential customers; a magazine page costs \$10,000 and reaches 1 million. You must sign up for at least 10 minutes of TV time. How should you spend your budget to maximize your audience? Formulate the problem in AMPL and solve it. Check the solution by hand using at least one of the approaches described in Section 1.1.

   (b) It takes creative talent to create effective advertising; in your organization, it takes three person-weeks to create a magazine page, and one person-week to create a TV minute. You have only 100 person-weeks available. Add this constraint to the model and determine how you should now spend your budget.

(c) Radio advertising reaches a quarter million people per minute, costs \$2,000 per minute, and requires only 1 person-day of time. How does this medium affect your solutions?

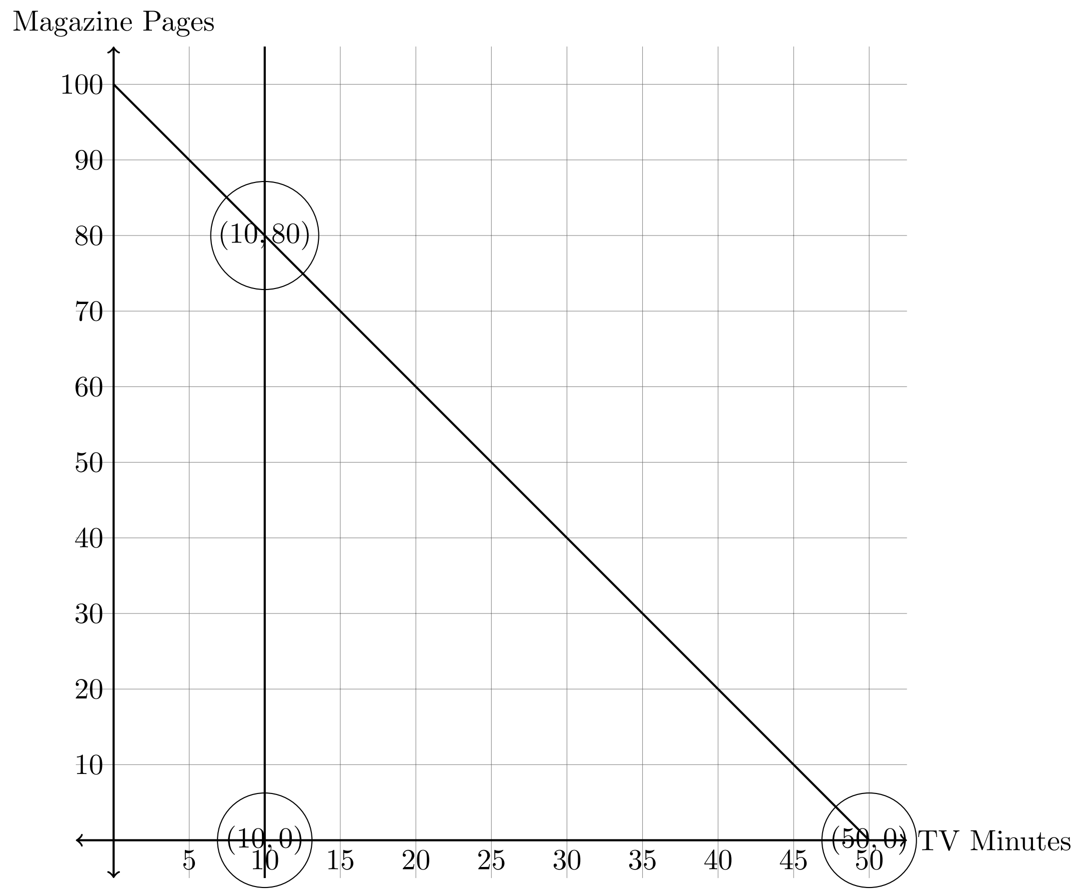
(d) How does the solution change if you have to sign up for at least two magazine pages? A maximum of 120 minutes of radio?

(a) We begin by stating the objective function to maximize, eyeballs:

$$z = 1.8x + 1y$$

Where x is minutes of TV time, y is pages in a magazine and z is million viewers. Next we set the two constraints, at least 10 minutes of TV and no more than \$1 million spent.

$$\begin{aligned} &x \geq 10 \\ &20000x + 10000y \leq 1000000 \end{aligned}$$



We see the corner points of the feasible set as possible solutions.
We are left to find the maximum:

Point	z	Optimizer
$(10, 80)$	18+80 = 98	Max
$(10, 0)$	18+0 = 18	Min
$(50, 0)$	90+0 = 90	

We repeat the same system in AMPL and arrive at the same solution

```
Console
AMPL
ampl: model hw1p3.mod
CPLEX 20.1.0.0: optimal solution; objective 98
0 dual simplex iterations (0 in phase I)
Solution:
Total Audience in Millions of Viewers: 98
Advertising Budget Left Over in Dollars: 0.0001
Amount of Advertising Purchased per Type of Media:
time [*] :=
 TV  10
mag  80
;
```

(b) For this part, we repeat the exercise adding the additional requirement of creative time

$$1x + 3y \leq 100$$

We see this changes our maximum, to 92 million people.

```
Console
AMPL
ampl: model hw1p3b.mod
CPLEX 20.1.0.0: optimal solution; objective 92
2 dual simplex iterations (1 in phase I)
Solution:
Total Audience in Millions of Viewers: 92
Creative Person-Weeks Left Over: 0.04
Amount of Advertising Purchased per Type of Media:
time [*] :=
 TV  40
mag  20
;
```

(c) For the next question we add radio into the advertising mix. This will add a new variable and change all the constraints and objective. We also note that the 2D graphical method will no longer be available to us to check our solution.

$$z = 1.8x_{TV} + 1x_{magazine} + 0.25x_{radio}$$

$$\begin{aligned}
&x_{TV} \geq 10 \\
&20000x_{TV} + 10000x_{magazine} + 2000x_{radio} \leq 1000000 \\
&1x_{TV} + 3x_{magazine} + \tfrac{1}{7}x_{radio} + \leq 100
\end{aligned}$$

We change notation here too, applying subscripts to be more descriptive of our vector x.

```
Console
AMPL
ampl: model group3_HW1_p3c.mod
CPLEX 20.1.0.0: optimal solution; objective 118
1 dual simplex iterations (1 in phase I)
Solution:
Total Audience in Millions of Viewers: 118
Creative Person-Weeks Left Over: 32.88
Amount of Advertising Purchased per Type of Media:
time [*] :=
   TV   10
  mag    0
radio  400
;

ampl:
```

We see an increase in viewers and note that there is a lot of slack in the creative hours.

(d) To add the condition of at lease two pages of magazine, we add the condition

$$x_{magazine} \geq 2$$

and

$$x_{radio} \leq 120$$

```
Console
AMPL
ampl: model hw1p3d.mod
CPLEX 20.1.0.0: optimal solution; objective 100.1942857
3 dual simplex iterations (2 in phase I)
Solution:
Total Audience in Millions of Viewers: 100.194
Creative Person-Weeks Left Over: 0
Amount of Advertising Purchased per Type of Media:
time [*] :=
   TV   29.0286
  mag   17.9429
radio  120
;
```

We should note that viewership numbers are high with this model but there is a strangeness of fractional magazine pages and TV minutes. Both seem plausible but we doubt the prices remain consistent if you start slicing up the page or the minute.

4. We'll repeat the question here

   The steel model of this chapter can be further modified to reflect various changes in production requirements. For each part below, explain the modifications to Figures 1-6a and 1-6b that would be required to achieve the desired changes. (Make each change separately, rather than accumulating the changes from one part to the next.)

(a) How would you change the constraints so that total hours used by all products must equal the total hours available for each stage? Solve the linear program with this change, and verify that you get the same results. Explain why, in this case, there is no difference in the solution.

(b) How would you add to the model to restrict the total weight of all products to be less than a new parameter, max_weight? Solve the linear program for a weight limit of 6500 tons, and explain how this extra restriction changes the results.

(c) The incentive system for mill managers may tend to encourage them to produce as many tons as possible. How would you change the objective function to maximize total tons? For the data of our example, does this make a difference to the optimal solution?

(d) Suppose that instead of the lower bounds represented by commit[p] in our model, we want to require that each product represent a certain share of the total tons produced. In the algebraic notation of Figure 1-1, this new constraint might be represented as

$$X_j \geq s_j \sum_{k \in P} X_k, \forall j \in P$$

where s j is the minimum share associated with project j. How would you change the AMPL model to use this constraint in place of the lower bounds commit[p]? If the minimum shares are 0.4 for bands and plate, and 0.1 for coils, what is the solution? Verify that if you change the minimum shares to 0.5 for bands and plate, and 0.1 for coils, the linear program gives an optimal solution that produces nothing, at zero profit. Explain why this makes sense.

(e) Suppose there is an additional finishing stage for plates only, with a capacity of 20 hours and a rate of 150 tons per hour. Explain how you could modify the data, without changing the model, to incorporate this new stage.

(a) To make the constraint from a less than to equal, we simply change the constraint in AMPL to == instead of <=. We see a screen shot of this in the code base:

```
group3_HW1_p4a.mod    group3_HW1_p4.dat

reset;

set PROD; # products
set STAGE; # stages

param rate {PROD,STAGE} > 0; # tons per hour in each stage
param avail {STAGE} >= 0; # hours available/week in each stage
param profit {PROD}; # profit per ton
param commit {PROD} >= 0; # lower limit on tons sold in week
param market {PROD} >= 0; # upper limit on tons sold in week

var Make {p in PROD} >= commit[p], <= market[p]; # tons produced


maximize Total_Profit: sum {p in PROD} profit[p] * Make[p];
# Objective: total profits from all products
subject to Time {s in STAGE}: sum {p in PROD} (1/rate[p,s]) * Make[p] == avail[s];
#subject to Time2 {s in STAGE}: sum {p in PROD} (1/rate[p,s]) * Make[p] >= avail[s];
# In each stage: total of hours used by all
# products may not exceed hours available

data group3_HW1_p4.dat;

solve;

display Make;

display Time;
```

We see that the output agrees with the values reported in the text.

```
Console
AMPL
ampl: model group3_HW1_p4a.mod
CPLEX 20.1.0.0: optimal solution; objective 190071.4286
2 dual simplex iterations (0 in phase I)
Make [*] :=
bands  3357.14
coils   500
plate  3142.86
;

Time [*] :=
reheat  1800
  roll  3200
;

ampl:
```

The reason for this agreement even with a new condition is that we were already using all available hours. It did not matter that we required equality as it was already achieved in the optimizer.

(b) Adding the constraint of max_weight will result in an extra constraint. Using the language of text, we will require that

$$\sum_{p \in PROD} Make[p] \leq max\_weight$$

When added to our model we see the following results

```
Console
AMPL
ampl: model hw1p4b.mod
CPLEX 20.1.0.0: optimal solution; objective 183791.6667
3 dual simplex iterations (0 in phase I)
Solution:
Number of Each Type of Part to Make:
Make [*] :=
bands  1541.67
coils  1458.33
plate  3500
;

Lbs Left Over in Weight Limit: 0
```

We note that the objective has dropped with this production schedule. We can also see that the original optimizer was over this weight limit which is why we now have a new result.

(c) To examine this question, we change the objective function. We now which to maximize the total weight produced rather than the profit. The new objective is simply

$$z = \sum_{i \in PROD} x_i$$

We make this the sole objective in the system in AMPL and provide the Total_Profit for comparison.

```
Console
AMPL
ampl: model hw1p4c.mod
CPLEX 20.1.0.0: optimal solution; objective 7000
1 dual simplex iterations (0 in phase I)
Solution:
Number of Each Type of Part to Make:
Make [*] :=
bands  5750
coils   500
plate   750
;

Total Profit when Maximizing Weight:
sum{p in PROD} profit[p]*Make[p] = 180500
```

We note that the profit has dropped (as expected) but the total weight did not actually change from the original result, both are 7000 tonnes.

(d) To tackle this question, we first restate the question in our own terms. Here we desire not a baseline production but a baseline production percentage. In reality what we desire is a minimum percentage of the total to be in each product. We will make this a separate constraint called Production_Percentage and eliminate the commit minimums. We write the percentage as a new param and write the constraint as

$$\text{subject to} \qquad \text{percentage}\{p \in PROD\} :$$
$$Make[p] \geq percent[p] * (sum\{r \in PROD\} Make[r]) \, ;$$

Output is included below and shows that constraint decreases the optimal profit.

```
Console
AMPL
ampl: model hw1p4d.mod
CPLEX 20.1.0.0: optimal solution; objective 189700
5 dual simplex iterations (0 in phase I)
Number of Each Type of Part to Make:
Make [*] :=
bands  3500
coils   700
plate  2800
;

Number of Hours Used by Each Product
Time [*] :=
reheat  1762.86
  roll  3200
;
```

If we change the percentages to 50% for two products and 10% for the remaining we indeed produce nothing. Afterall the only way give 110% is to have started with nothing...

```
Console
AMPL
ampl: model group3_HW1_p4d.mod
CPLEX 20.1.0.0: optimal solution; objective 0
4 dual simplex iterations (0 in phase I)
Make [*] :=
bands  0
coils  0
plate  0
;

Time [*] :=
reheat  0
  roll  0
;

ampl: |
```

(e) We could also just simply change the market demand for plates. Currently the demand is 3500 but with this new finishing requirement only 3000 can be made. If you simply reduce the market data for this one entry, we will get an equivalent result if we added the finishing constraint.

5. We restate the question here:

Consider the bond-portfolio problem formulated in Section 1.3 (in this assignment question 2). Reformulate the problem restricting the bonds available only to bonds A and D. Further add a constraint that the holdings of municipal bonds must be less than or equal to $3 million.

(a) What is the optimal solution?

(b) What is the shadow price on the municipal limit?

(c) How much can the municipal limit be relaxed before it becomes a nonbinding constraint?

(d) Below what interest rate is it favorable to borrow funds to increase the overall size of the portfolio?

(e) Why is this rate less than the earnings rate on the portfolio as a whole?

(a) Since bond A was the only remaining Municipal bond, the only new constraint is

$$invest[A] \leq 3000000$$

We copy/pasta the data file from question 2 and delete all mentions to bond B, C, and E as well. Output is shown below.

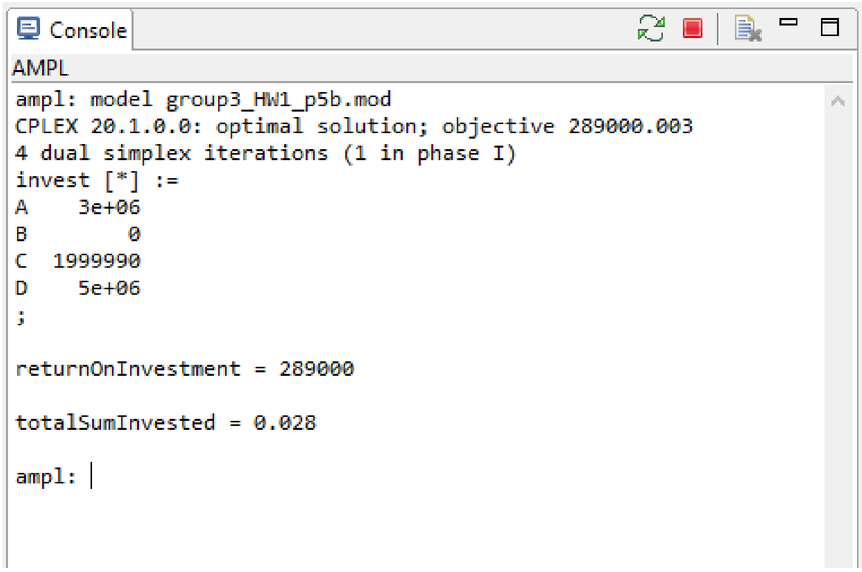
```
Console
AMPL
ampl: model group3_HW1_p5a.mod
CPLEX 20.1.0.0: optimal solution; objective 283000
0 dual simplex iterations (0 in phase I)
invest [*] :=
A  3e+06
D  7e+06
;

returnOnInvestment = 283000

ampl:
```

(b) To find the shadow price on the municipal limit we simply increase the limit by $1 and see how the optimal value changes. This change in the optimal value is the shadow price. We see in the output below some interesting features by this slight change. First off, the objective (profit) has increased by $0.003 but the investment strategy changed the amount invested in Bond C decreasing it by $10. We are unsure where that extra money was to be invested due to other outputs expressed in scientific notation.

This shadow price was also observed to be the same if we added another dollar to the constraint or subtracted a dollar!



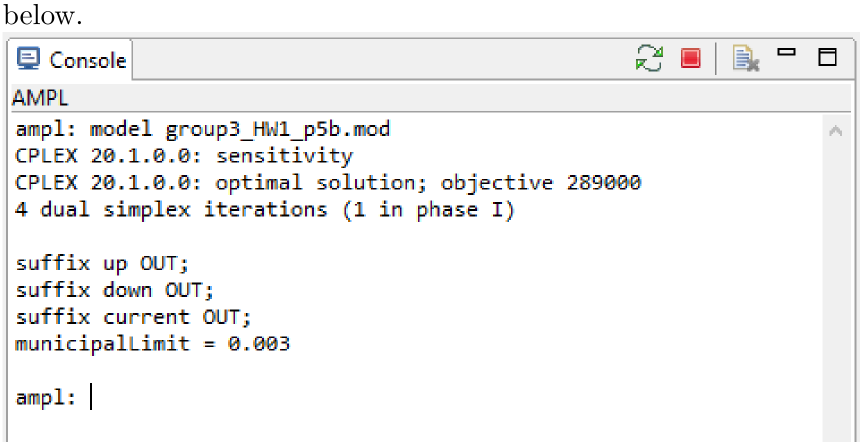
```
Console
AMPL
ampl: model group3_HW1_p5b.mod
CPLEX 20.1.0.0: optimal solution; objective 289000.003
4 dual simplex iterations (1 in phase I)
invest [*] :=
A    3e+06
B        0
C  1999990
D    5e+06
;

returnOnInvestment = 289000

totalSumInvested = 0.028

ampl: 
```

We note that after videos dropped explaining how to compute shadow limit with AMPL, we repeated this exercise using the cplex_options 'sensitivity' and found the same result presented below.

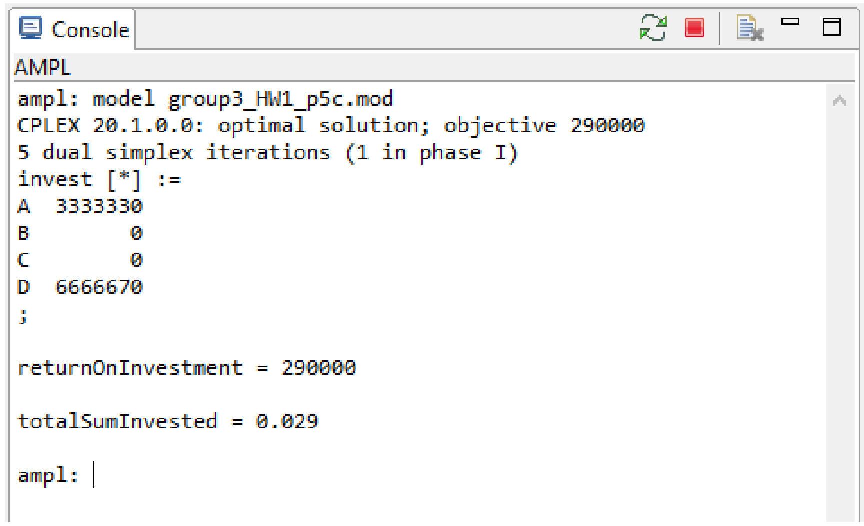


```
Console
AMPL
ampl: model group3_HW1_p5b.mod
CPLEX 20.1.0.0: sensitivity
CPLEX 20.1.0.0: optimal solution; objective 289000
4 dual simplex iterations (1 in phase I)

suffix up OUT;
suffix down OUT;
suffix current OUT;
municipalLimit = 0.003

ampl: 
```

(c) To see how much this municipal constraint can be relaxed before becoming non-binding, we simply increased the limit to \$4 000 000. When we run the code with that limit, we see that we do not achieve that limit but instead have a limit in bond A of \$ 3 333 330. We wonder if the actual value should be \$ 3 333 333.3$\overline{3}$.



```
Console
AMPL
ampl: model group3_HW1_p5c.mod
CPLEX 20.1.0.0: optimal solution; objective 290000
5 dual simplex iterations (1 in phase I)
invest [*] :=
A  3333330
B        0
C        0
D  6666670
;

returnOnInvestment = 290000

totalSumInvested = 0.029

ampl: 
```

(d) Considering a 50% tax rate and deducting you interest from your total tax liability, we see that at 4.4% we would not take the loan but at any rate below we would. We get this from the after-tax return on bond D as the limit on Municipal bonds has been met. We state this as

$$r < 4.4\%$$

We iterated in AMPL to see this appears to be the cuttoff point in our model.

(e) We see the return on the entire investment package at 2.83%. We don't like the question and refuse to answer.