a.) $f(x) = \sin(2x)$ and $x_0 = 0$

$f(x) = \sin(2x)$, $f(0) = 0$

$f'(x) = 2\cos(2x)$, $f'(0) = 2$

$f''(x) = -4\sin(2x)$, $f''(0) = 0$

$f'''(x) = -8\cos(2x)$, $f'''(0) = -8$

$f''''(x) = 16\sin(2x)$, $f''''(0) = 0$

$$\sin(2x) = \frac{f(0)x^0}{0!} + \frac{f'(0)x^1}{1!} + \frac{f''(0)x^2}{2!} \cdots\cdots$$

$$\sin(2x) = 0 + \frac{2x^1}{1!} + \frac{0x^2}{2!} - \frac{8x^3}{3!} + \frac{0x^4}{4!} + \cdots$$

$$\underline{\underline{\sin(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{(2n+1)}}{(2n+1)!}}}, \quad L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

$$L = \lim_{n\to\infty}\left|\left(\frac{(-1)^{(n+1)}(2x)^{\overset{(2n+3)}{(2(n+1)+1)}}}{\underset{(2n+3)!}{(2(n+1)+1)!}}\right)\left(\frac{(2n+1)!}{(-1)^n (2x)^{(2n+1)}}\right)\right|$$

$$= \lim_{n\to\infty}\left|\frac{-1(2x)^2\,(2n+1)!}{(2n+3)!}\right| = \left|-1(2x)^2\right| \lim_{n\to\infty}\left|\frac{(2n+1)!}{(2n+3)!}\right|$$

$$= \left|-4x^2\right| \lim_{n\to\infty}\left|\frac{\cancel{(2n+1)!}}{(2n+3)(2n+2)\cancel{(2n+1)!}}\right| = 4x^2(0)$$

$\underline{\underline{\text{Radius of convergence is } \infty}}$

b) $f(x) = \ln(2x)$ and $x_0 = 1$,

$f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$, $f'''(x) = \frac{2}{x^3}$, $f''''(x) = -\frac{6}{x^4}$,

$f(1) = \ln(2)$, $f'(1) = 1$, $f''(1) = -1$, $f'''(1) = 2$, $f''''(1) = -6$,

$$\ln(2x) = \ln(2) + x - \frac{x^2}{2!} + \frac{2x^3}{3!} - \frac{6x^4}{4!} \dots$$

$$\ln(2x) = \ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{(n+1)} (x-1)^n}{n},$$

$$\lim_{n\to\infty} \left| \left(\frac{(-1)^{((n+1)+1)} (x-1)^{(n+1)}}{n+1} \right) \left(\frac{n}{(-1)^{n+1} (x-1)^n} \right) \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)(x-1)n}{n+1} \right| = |(1-x)|$$

$|(1-x)| < 1$ is the radius of convergence.

C.) $f(x) = e^{2x}$ and $x_0 = 1$, $f'(x) = 2e^{2x}$, $f''(x) = 4e^{2x}$,

$f'''(x) = 8e^{2x}$, $f''''(x) = 16e^{2x}$, $f(1) = e^2$, $f'(1) = 2e^2$

$f''(1)\ 4e^2$, $f'''(1) = 8e^2$, $f''''(1) = 16e^2$

$$e^{2x} = e^2 + 2e^2(x-1) + \frac{4e^2(x-1)^2}{2!} + \frac{8e^2(x-1)^3}{3!} + \frac{16e^2(x-1)^4}{4!} + \dots$$

$$= e^2 + 2e^2(x-1) + 2e^2(x-1)^2 + \frac{4e^2(x-1)^3}{3} + \frac{2e^2(x-1)^4}{3} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{2^n \cdot e^2(x-1)^n}{n!}.$$

$$\lim_{n\to\infty} \left(\frac{2^{(n+1)}\ e^2(x-1)^{(n+1)}}{(n+1)!}\right)\left(\frac{n!}{2^n\ e^2(x-1)^n}\right)$$

$$= \lim_{n\to\infty} \left(\frac{2e^2(x-1)}{(n+1)}\right) = 2e^2(x-1)(0)$$

Radius of convergence is ∞

d.) $f(x) = 3x^2 - 2x + 5$ and $x_0 = 0$; $f(0) = 5$

$f'(x) = 6x - 2$, $f'(0) = -2$

$f''(x) = 6$, $f''(0) = 6$

$f'''(x) = 0$ $f'''(0) = 0$

$$3x^2 - 2x + 5 = 5 - 2x + \frac{6x^2}{2!} = \underline{\underline{3x^2 - 2x + 5}}.$$

$|x| < 1$ is the radius of convergence.

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e.) $f(x) = 3x^2 - 2x + 5$ and $x_0 = 1$, $f(1) = 6$

$f'(1) = 4$, $f''(1) = 6$, $f'''(1) = 0$,

$$3x^2 - 2x + 5 = 6 + 4(x-1) + 3(x-1)^2 = 6 + 4x - 4 + 3x^2 - 6x + 3$$

$$= \underline{\underline{3x^2 - 2x + 5}}.$$

$|x-1| < 1$ is the radius of convergence.

f.) $f(x) = (3x^2 - 2x + 5)^{-1}$ and $x_0 = 1$, $f(1) = 1/6$

$$f'(x) = \frac{-6x+2}{(3x^2-2x+5)^2}, \quad f'(1) = \frac{-4}{36} = -\frac{1}{9}$$

$$f''(x) = \frac{2(27x^2 - 18x - 11)}{(3x^2-2x+5)^3}, \quad f''(1) = -\frac{1}{54}$$

$$f'''(x) = \frac{-24(3x-1)(9x^2-6x-13)}{(3x^2-2x+5)^4}, \quad f'''(1) = \frac{10}{27}$$

$$f(x) = \frac{1}{6} - \frac{1}{9}(x-1) - \frac{1}{108}(x-1)^2 + \frac{5}{81}(x-1)^3 + \dots$$

$$= \sum_{n=0}^{\infty} c \frac{(x-1)^n}{n!} .$$

$$c \left[\lim_{n \to \infty} \left| \left(\frac{(x-1)^{(n+1)}}{(n+1)!} \right) \left(\frac{n!}{(x-1)^n} \right) \right| \right]$$

$$= c \left[\lim_{n \to \infty} \left| \frac{x-1}{n+1} \right| \right] = c(x-1)(0)$$

radius of convergence is ∞.

9.) $f(x) = \cosh(x-3)$ and $x_0 = 1$; $f(1) = \cosh(2)$

$f'(x) = \sinh(x-3)$, $f'(1) = -\sinh(2)$,

$f''(x) = \cosh(x-3)$, $f''(1) = \cosh(2)$,

$f'''(x) = \sinh(x-3)$, $f'''(1) = -\sinh(2)$,

$f''''(x) = \cosh(x-3)$, $f''''(1) = \cosh(2)$,

$$f(x) = \cosh(2) - \sinh(2)(x-1) + \frac{\cosh(2)(x-1)^2}{2!} - \frac{\sinh(2)(x-1)^3}{3!} + \frac{\cosh(2)(x-1)^4}{4!}$$

$$= \sum_{n=0}^{\infty} \frac{\cosh(2)(x-1)^{(2n)}}{(2n)!} - \frac{\sinh(2)(x-1)^{(2n+1)}}{(2n+1)!}.$$

$$\cosh(2)\sum \frac{(x-1)^{(2n)}}{(2n)!} - \sinh(2)\sum \frac{(x-1)^{(2n+1)}}{(2n+1)!}$$

$$\left(\frac{(x+1)^{2(n+1)}}{(2(n+1))!}\right)\left(\frac{(2n)!}{(x-1)^{(2n)}}\right) = \frac{(x-1)^2}{(2n+2)(2n+1)}, \quad \lim_{n\to\infty} = 0$$

$$\left(\frac{(x-1)^{(2(n+1))+1}}{((2(n+1))+1)!}\right)\left(\frac{(2n+1)!}{(x-1)^{(2n+1)}}\right) = \frac{(x-1)^2}{(2n+3)(2n+2)}, \quad \lim_{n\to\infty} = 0$$

$(2n+3)!$

Radius of convergence is ∞.

h.) $f(x)$ and $x_0 = a$.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)\,(x-a)^n}{n!},$$

$$\lim_{n\to\infty} \left| \frac{(x-a)^{(n+1)}}{(n+1)!} \frac{n!}{(x-a)^n} \right| = \lim_{n\to\infty} \left| \frac{x-a}{(n+1)} \right| = 0$$

radius of convergence is ∞.

i.) $f(a)$ and $x_0 = x$,

$$f(a) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)\,(x-x)^n}{n!} = 0.$$

j.) $f(a+h)$ and $x_0 = a$,

$$\frac{(x-a)^{(n+1)}}{(n+1)!} \frac{n!}{(x-a)^n} = \frac{x-a}{n+1}$$

$$f(a+h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a+h)\,(x-a)^n}{n!}$$

radius of convergence is ∞.