Universidade Federal de Pernambuco

Centro de Tecnologia e Geociências

Programa de Pós-graduação em Engenharia Elétrica

JOHN DOE

TÍTULO DA TESE

Recife

2022

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Tese apresentada ao Programa de Pós-Graduação em Engenharia Elétrica da Universidade Federal de Pernambuco como requisito parcial para obtenção do título de Doutor em Engenharia Elétrica. Área de Concentração: Comunicações.

Orientador: Prof. Dr. John Doe

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Dedicatória.

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> Insisto: na simplicidade do teu trabalho habitual, nos detalhes monótonos de cada dia, tens que descobrir o segredo - para tantos escondido - da grandeza e da novidade: o Amor. (ESCRIVÁ, 2016, n. 489)

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Algebra is generous, she often gives more than is asked of her. (D’ALEMBERT, 1905)

ABSTRACT

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Keywords: quaternions; fractional quaternion discrete Fourier transform; fractional graph shift operator; quaternion graph signal processing.

RESUMO

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Palavras-chave: quatérnios; transformada discreta de Fourier quaterniônica fracionária; operador de deslocamento de grafo fracionário; processamento de sinal de grafo quaterniônico.

LISTA DE ILUSTRAÇÕES

LISTA DE ABREVIATURAS E SIGLAS

DFT	Discrete Fourier Transform.
DSP	Digital Signal Processing.
FIR	Finite Impulse Response.

LISTA DE SÍMBOLOS

$\mathbf{A}$	Matriz de adjacência ponderada do grafo.
$\mathbf{L}$	Matriz Laplaciana do grafo.
$\mathbf{\Lambda}, \mathbf{J}$	Respectivamente a matriz de autovalores (se existir) e a matriz de Jordan da matriz de adjacência do grafo.
$\mathbf{V}$	A matriz de autovetores (possivelmente generalizada) da matriz de adjacência do grafo.
$\mathcal{R}e\{x\}$	Parte real do número complexo x.
$\mathcal{I}m\{x\}$	Parte imaginária do número complexo x.
$\overline{x}$	Conjugado do número complexo (ou quaterniônico) x. Se vetores ou matrizes forem usados em vez de x, a conjugação é realizada em cada uma de suas entradas.
$\mathbf{M}^T$	Transposta da matriz M.
$\mathbf{M}^H$	Transposta conjugada da matriz M.
$\mathbb{R}$, $\mathbb{C}$ e $\mathbb{H}$	Respectivamente, o conjunto dos números reais, complexos e (hamiltonianos) quaterniônicos. Por associação, também podemos nos referir ao respectivo *skew field*.

SUMÁRIO

1 INTRODUCTION

Signal processing often interweaves pure mathematics and engineering. One of its concerns is the *representation* of signals (functions) and how different representations may be explored to better manipulate such signals. The spectral analysis via Fourier and similar transforms, for instance, aims to project the signal onto a domain in which the energy support is more compact (compression), or in which some frequencies are easier to be removed (filtering), or yet in which some relevant features may be created (feature engineering for machine learning problems), among others (OPPENHEIM, 1999; RABINER; SCHAFER, 2010; GRAF et al., 2015; VERGIN; O'SHAUGHNESSY; FARHAT, 1999).

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2 A REVIEW ON QUATERNION ALGEBRA AND ITS FOURIER TRANSFORM

In 1833, at the age of 28, Willian Rowan Hamilton presented to the Royal Irish Academy (RIA) a work in which complex numbers were treated as ordered pairs of real numbers, given the appropriate definition of operations.[1] In the following years, he struggled to extend the complex field into a normed division algebra over triples, but soon realized that, as much as his attempts were inventive, the resulting algebra[2] was not closed under multiplication. We can see this through a simple example (SANTOS; FERREIRA, 2011): let it be the set $\mathbb{F} = \{a + b\boldsymbol{i} + c\boldsymbol{j} \mid (a, b, c) \in \mathbb{R}^3\}$, with $\boldsymbol{i}^2 = \boldsymbol{j}^2 = -1$ and $\boldsymbol{i} \neq \boldsymbol{j}$. Since $\boldsymbol{i}, \boldsymbol{j} \in \mathbb{F}$, so there should exist $x, y, z \in \mathbb{R}$ so that

$$\boldsymbol{i}\boldsymbol{j} = x + y\boldsymbol{i} + z\boldsymbol{j}. \tag{1}$$

Multiplying by $\boldsymbol{i}$ both sides of the equation,

$$\boldsymbol{i}^2\boldsymbol{j} = \boldsymbol{i}x + \boldsymbol{i}^2 y + z(\boldsymbol{i}\boldsymbol{j}), \tag{2}$$

and using (1) it yields,

$$-\boldsymbol{j} = \boldsymbol{i}x - y + z(x + y\boldsymbol{i} + z\boldsymbol{j}) \iff (zx - y) + \boldsymbol{i}(x + zy) + \boldsymbol{j}(z^2 + 1) = 0. \tag{3}$$

That is: $z \notin \mathbb{R}$ and $\boldsymbol{i}\boldsymbol{j} \notin \mathbb{F}$, proving that such algebraic structure is not closed under multiplication.

Only a decade later, in 1843, while walking by the roads in Dublin toward the RIA, "an electric circuit seemed to close, and a spark flashed forth," as he would say. He had conceived the four-dimensional structure required to the desired algebra, creating the quaternions. Moved by excitement, he craved on the stone below Broome Bridge, in Cabra (Dublin), the equations that define the relations between the canonical basis elements of quaternions.[3] This creation, made possible by an insight in 1843, is found accross most of this work. The following sections lead the reader through the foundations of quaternion algebra and quaternion signal analysis.

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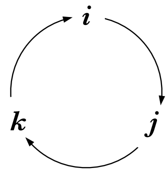
[1] The results were published in 1837, in the paper *Theory of Conjugate Functions, or Algebraic Couples; with a Preliminary and Elementary Essay on Algebra as the Science of Pure Time* (HAMILTON, 1837).

[2] An *algebra over a field*, or simply *algebra*, is a vector space over a field with a bilinear multiplication (that is, the multiplication distributes over the addition and the associativity is valid for multiplication) (SCHAFER, 1955).

[3] Close to the original site of the inscriptions, the RIA placed a commemorative plaque in 1958, with the same writings.

Figura 1 – Illustration of the multiplication rule between the imaginary units $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$.



Source: the author (2022).

2.1 INTRODUCTION TO THE QUATERNION ALGEBRA

Quaternions are numbers $q \in \mathbb{H}$ in the form

$$q = a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k}, \tag{4}$$

in which $a, b, c, d \in \mathbb{R}$, holding true the fundamental relations:

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1. \tag{5}$$

The multiplication rules between $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ follow directly from (5), resembling those between orthonormal basis vectors from $\mathbb{R}^3$ and the vector product: the product between two of them yields the third, the sign being determined from the operands order. For instance, to find the result of $\boldsymbol{ij}$ one may start from (5) and write

$$\begin{aligned} \boldsymbol{ijk} &= -1 \\ \boldsymbol{ij}\underbrace{\boldsymbol{kk}}_{=-1} &= -\boldsymbol{k} \\ \boldsymbol{ij} &= \boldsymbol{k}. \end{aligned} \tag{6}$$

Similarly, to find $\boldsymbol{ji}$,

$$\begin{aligned} \boldsymbol{ijk} &= -1 \\ \boldsymbol{iijk} &= -\boldsymbol{i} \\ -\boldsymbol{jk} &= -\boldsymbol{i} \\ \boldsymbol{jjk} &= \boldsymbol{ji} \\ -\boldsymbol{k} &= \boldsymbol{ji}. \end{aligned} \tag{7}$$

Fig. 1 depicts the order in which the product between any pair in the triplet $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ yields the third one, with positive sign. All three units commute with real numbers. The most relevant consequence, therefore, of (5), is that the quaternion product is *noncommutative*. In fact, it is the first example of noncommutative normed division algebra in history (KLEINER, 2007).

3 CONCLUSION

> Quaternions came from Hamilton after his really good work had been done; and, though beautifully ingenious, have been an unmixed evil to those who have touched them in any way [...]. (ALTMANN, 1989, quoting directly Lord Kelvin, in letter to Hatward, 1892)

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APÊNDICE A – MAPEANDO EQUAÇÕES DE $\mathbb{H}$ PARA $\mathbb{C}$

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