Problem set (Exam).

This is the revised version of the problem set, which replaces the previous one. It includes additional exercises, as mentioned in the last class, since the earlier set was incomplete, and it also corrects several typos.

Exercise 1. Consider the following points: $A=(1,0)$, $B=(5,0)$ and $C=\left(3,\ \frac{1}{2}\right)$. Let $\overline{AC}$ be the equation of the line passing through the points A and C, and $\overline{BC}$ the equation of the line passing through B and C.

i) Find the lines $\overline{AC}$ and $\overline{BC}$. Obtain the expression for the function $f(x)$ where $f(x)=\overline{AC}$ for $1 \le x \le 3$, $f(x)=\overline{BC}$ for $3 < x \le 5$, and $f(x)=0$ in other cases.

ii) Plot the graph of $f(x)$ and verify that it is a pdf.

Exercise 2. The previous exercise is related to the *Triangular* (Tr) *distribution*, denoted as $X \sim Tr(a,b,c)$ where $a < c < b$, with the points: $A=(a,0)$, $B=(b,0)$ and $C=\left(c,\ \frac{2}{b-a}\right)$. Answer the questions:

i) Obtain $f(x)$ and $F(x)$ corresponding to the pdf and cdf of $X \sim Tr(a,b,c)$.

ii) Compute $\mu_X = E(X)$ and $\sigma_X^2 = V(X)$.

iii) Find the expression of $X = F^{-1}(U)$ where $U \sim U(0,1)$. Generate 10,000 draws of $X \sim Tr(1,5,3)$. Plot the graph of $f(x)$, already obtained in Exercise 1, joint with the histogram of the draws.

Hint: See https://en.wikipedia.org/wiki/Triangular_distribution

Exercise 3. Consider the random variables of stock returns R_1 and R_2 with bivariate pdf:

$$f(r_1,r_2)=\begin{cases}\alpha+\beta r_1+\gamma r_2, & -1\le r_1\le 2,\ -2\le r_2\le 2.5\\ 0, & \text{elsewhere}\end{cases} \qquad \text{(I)}$$

where $\alpha=0.0642, \beta=0.0049, \gamma=0.0296$. Given some results from the problem set with title "Introduction to bivariate copulas", answer the questions:

i) Find $E\left(R_1 \mid R_1 \le F_1^{-1}(p)\right)$ with $p=0.01$. Obtain $E\left(R_1 \mid R_2=r_2\right)=\int_{-1}^{2} r_1\, f\left(r_1 \mid r_2\right) dr_1$, and also plot its graph.

ii) Obtain $C^{\mathrm{I}}\left(u_1 \mid u_2\right)$, which is the conditional copula $C\left(u_1 \mid u_2\right)=P\left(U_1 \le u_1 \mid U_2=u_2\right)$ from the bivariate pdf (I).

iii) Generate $N=10,000$ draws $\left\{\left(v_{1,k},v_{2,k}\right)\right\}_{k=1}^{N}$ of independent $V_i \sim U(0,1)$ where $i=1,2$ to obtain N draws $\left\{\left(u_{1,k},u_{2,k}\right)\right\}_{k=1}^{N}$ of dependent $U_i \sim U(0,1)$ such that $v_{1,k}=C^{\mathrm{I}}\left(u_{1,k} \mid v_{2,k}\right)$ in the previous section and $u_{2,k}=v_{2,k}$. Make the scatterplots for the simulated independent and dependent $U(0,1)$ points.

iv) Use the "conditional sampling" for draws of (R_1, R_2) with bivariate pdf (I) to generate *N* draws of (R_1, R_2) by changing the implied marginal distribution with $R_i \sim Tr(a_i, b_i, c_i)$ in Exercise 2, such that for R_1 we set the values of $a_1 = -1,\ b_1 = 2$ and $c_1 = 3\mu_1 - b_1 - a_1$ with $\mu_1 = 0.55$; and for R_2 the values of $a_2 = -2,\ b_2 = 2.5$ and $c_2 = 3\mu_2 - b_2 - a_2$ with $\mu_2 = 0.93$. Compute both the sample mean and standard deviation of portfolio $\boldsymbol{A}$ returns, containing 25% of stock 1 and 75% of stock 2 under the above procedure, and also the VaR values at 1% and 5%.

v) Plot the histogram of the simulated returns of portfolio $\boldsymbol{A}$ joint with the *Normal distribution* fit (you can use *hisfit* function in Matlab), and compare with the *hisfit* by running the Matlab program "bivaMC.m" to build portfolio ***B*** returns, which are the simulated ones with the same weights as in portfolio $\boldsymbol{A}$ under the bivariate pdf (I). Use a QQ-plot for the comparison of the quantiles of portfolio $\boldsymbol{A}$ returns against portfolio $\boldsymbol{B}$ returns. Obtain the same QQ-plot but considering now standardized portfolio $\boldsymbol{A}$ and $\boldsymbol{B}$ returns.

<u>**Hint**</u>: Given program "bivaMC.m", it can be easily adapted to set $R_i \sim Tr(a_i, b_i, c_i)$ as new marginal distribution.

Exercise 4. Consider the random variables of stock returns R_1 and R_2 with bivariate pdf given by $f(r_1, r_2) = c(F_1(r_1), F_2(r_2)) f_1(r_1) f_2(r_2)$ such that the marginal distributions are those from the bivariate pdf (I). Obtain $f(r_1, r_2)$ under alternative bivariate copulas in the following cases:

i) The *bivariate standardized Gaussian* (G) *copula* pdf with ρ as the correlation coefficient is given by $c^G(u_1, u_2; \rho) = |\Psi|^{-1/2} \exp\left(-\frac{1}{2}\eta'(\Psi^{-1} - I_2)\eta\right)$, where $\eta = (\Phi^{-1}(u_1), \Phi^{-1}(u_2))'$, Ψ is the correlation matrix, I_2 is the identity matrix of order 2, and $\Phi^{-1}(\bullet)$ is the inverse of the $N(0,1)$ cdf.

ii) The *bivariate FGM copula* pdf is given by $c^{FGM}(u_1, u_2; \lambda) = 1 + \lambda(2u_1 - 1)(2u_2 - 1)$ where $\lambda \in [-1, 1]$.

Exercise 5. Consider the bivariate FGM copula cdf given by $C^{FGM}(u_1, u_2; \lambda) = u_1 u_2 \left[1 + \lambda(1 - u_1)(1 - u_2)\right]$ where $\lambda \in [-1, 1]$. Check that the following results are verified:

i) $\partial^2 C^{FGM}(u_1, u_2; \lambda) / \partial u_1 \partial u_2$ is equal to the FGM copula pdf in section ii) in Exercise 4.

ii) $E(U_1 | U_2 = u_2)$ is linear in u_2.

iii) $C^{FGM}(u_i | u_j) = u_i \left[1 + \lambda(1 - u_i)(1 - 2u_j)\right]$.

<u>**Hint**</u>: $C(u_i | u_j) = P(U_i \le u_i | U_j = u_j) = \partial C(u_1, u_2) / \partial u_j, \quad i, j \in \{1, 2\}; \quad i \ne j$

Exercise 6. Given the results in Exercise 5 for the bivariate FGM copula, try to answer the following questions:

i) Generate $N=10,000$ draws $\left\{\left(v_{1,k},v_{2,k}\right)\right\}_{k=1}^{N}$ of independent $V_i \sim U\left(0,1\right)$ where $i=1,2$ to obtain *N* draws $\left\{\left(u_{1,k},u_{2,k}\right)\right\}_{k=1}^{N}$ of dependent $U_i \sim U\left(0,1\right)$ such that $v_{1,k}=C^{FGM}\left(u_{1,k}\middle|v_{2,k}\right)$ with $\lambda=1/2$ and $u_{2,k}=v_{2,k}$. Make two scatterplots for the simulated independent and dependent $U\left(0,1\right)$ points.

ii) Consider the marginal distributions of R_1 and R_2 from the bivariate pdf (I). Generate $N=10,000$ draws of $\left(R_1,R_2\right)=\left(F_1^{-1}\left(U_1\right),F_2^{-1}\left(U_2\right)\right)$ where $\left(U_1,U_2\right)$ is a bivariate uniform obtained under the "conditional sampling" in section iii) in Exercise 5. Compute the sample mean and standard deviation of the *N* simulated portfolio $\boldsymbol{C}$ returns, containing 25% of stock 1 and 75% of stock 2 under the above procedure, and also the VaR values at 1% and 5%. Plot the histogram of the portfolio $\boldsymbol{C}$ returns joint with the *Normal distribution* fit (you can use *hisfit* function in Matlab).

Exercise 7. Let $\left(U_1,U_2\right)$ be a bivariate uniform generated from the G copula with $\rho=-0.7$ as correlation coefficient (standardized Gaussian copula), see section i) in Exercise 4. Repeat the same analysis as in Exercise 6 by obtaining $N=10,000$ draws of $\left(R_1,R_2\right)$ with marginal distributions of R_i from the bivariate pdf (I). The cdf of U_1 given $U_2=u_2$ in the *G* copula (or conditional *G* copula) is obtained from the cdf of the G copula, $\Phi_2\left(\bullet\right)$, then

$$C^G\left(u_1\middle|u_2\right)=\frac{\partial}{\partial u_2}\Phi_2\left(\Phi^{-1}\left(u_1\right),\Phi^{-1}\left(u_2\right)\right)=\Phi\left(\frac{\Phi^{-1}\left(u_1\right)-\rho\,\Phi^{-1}\left(u_2\right)}{\sqrt{1-\rho^2}}\right),$$

where $\Phi\left(\bullet\right)$ and $\Phi^{-1}\left(\bullet\right)$ are, respectively, the $N\left(0,1\right)$ cdf and its inverse. Obtain both the sample mean and standard deviation of the portfolio $\boldsymbol{D}$ returns, containing 25% of stock 1 and 75% of stock 2 under the above procedure, and also the VaR values at 1% and 5%.

Exercise 8. According to Exercise 7, the *q* quantile curve under the *G* copula when the marginals are arbitrary distributions, with cdf's $F_i\left(\bullet\right)$ $i=1,2$, is given by $C^G\left(u_1\middle|u_2\right)=q$, then $u_2=\Phi\left(\rho\,\Phi^{-1}\left(u_1\right)+\sqrt{1-\rho^2}\Phi^{-1}\left(q\right)\right)$ where $u_i=F_i\left(r_i\right)$. Definitively, $r_2=F_2^{-1}\left[\Phi\left(\rho\,\Phi^{-1}\left(F_1\left(r_1\right)\right)+\sqrt{1-\rho^2}\Phi^{-1}\left(q\right)\right)\right]$. Consider as marginals those obtained from the bivariate pdf (I). Make a plot of the *q* quantile curves for $q=0.05, 0.25, 0.5, 0.75, 0.95$.

Exercise 9. Repeat Exercise 8 where $u_i = \Phi(r_i)$ $i = 1,2$, then $r_2 = \rho r_1 + \sqrt{1-\rho^2}\Phi^{-1}(q)$. Make again a plot of the *q* quantile curves for $q = 0.05, 0.25, 0.5, 0.75, 0.95$.

Exercise 10. The cdf of the conditional Clayton copula is

$$C^{Cla}(u_1|u_2) = \frac{\partial}{\partial u_2}\left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-1/\theta} = u_1^{-(1+\theta)}\left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-(1+\theta)/\theta},$$

such that $\theta > 0$. Given the equation $C^{Cla}(u_1|u_2) = q$, the *q* quantile for the Clayton copula is obtained as $u_2 = \left(1 + u_1^{-\theta}\left(q^{-\theta/(1+\theta)} - 1\right)\right)^{-1/\theta}$ where $u_i = F_i(r_i)$. If we set the marginals those from the bivariate pdf (I), then $r_2 = F_2^{-1}\left[\left(1 + F_1(r_1)^{-\theta}\left(q^{-\theta/(1+\theta)} - 1\right)\right)^{-1/\theta}\right]$. Draw the *q* quantile curves for $q = 0.05, 0.25, 0.5, 0.75, 0.95$.

Exercise 11. The four steps to simulate under a bivariate *G* copula with an alternative procedure are: **(a)** Find the *Cholesky* decomposition L of correlation matrix $\Psi = L'L$; **(b)** Generate draws (v_1, v_2) of indep. $V_i \sim U(0,1)$ to obtain draws $(z_1, z_2) = \left(\Phi^{-1}(v_1), \Phi^{-1}(v_2)\right)$ of indep. $Z_i \sim N(0,1)$; **(c)** Generate draws $x = Lz$ with $z = (z_1, z_2)'$ of the bivariate $X = LZ \sim N(0,\Psi)$; **(d)** Set $u_i = \Phi(x_i)$ which are draws of the bivariate uniform (U_1, U_2) from the *G* copula. The expression of L is given by $L = \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix}$. Repeat Exercise 7 by using the above method.

Exercise 12. Using the Cholesky decomposition in Exercise 11 is easy to simulate draws from the bivariate Student *t* copula (hereafter, *T* copula). The steps **(a)** and **(b)** are the same; **(c)** Simulate a draw *s* from the random variate χ_υ^2 independent of Z; **(d)** Generate draws $y = \sqrt{(\upsilon/s)}\,x$ where $x = Lz$ with $z = (z_1, z_2)'$ of the bivariate $X = LZ \sim N(0,\Psi)$; **(e)** Set $u_i = T_\upsilon(y_i)$ where T_υ denotes the cdf of the univariate Student *t* distribution. Definitively, $u_i = T_\upsilon(y_i)$ are draws of the bivariate uniform (U_1, U_2) from the *T* copula. Repeat Exercise 7 by using the *T* copula instead of the *G* copula with marginals those from the bivariate pdf (I). Consider $\rho = -0.7$ and $\upsilon = 5$. Obtain both the sample mean and standard deviation of the portfolio $\boldsymbol{E}$ returns, containing 25% of stock 1 and 75% of stock 2 under the above procedure, and also the VaR values at 1% and 5%

Exercise 13. Compute the sample covariance and correlation matrices of the portfolio return series $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ and $\boldsymbol{E}$ already obtained in Exercises 3, 6, 7 and 12. Obtain also the skewness and kurtosis of each of them. Build an equally weighted (*EW*) portfolio series containing the five portfolio return series. Compute the mean, variance, skewness and kurtosis of the *EW* return series. Obtain the QQ plot of the standardized *EW* return series.

Exercise 14. Consider Exercise 11 for the trivariate *G* copula with correlation matrix $\Psi = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{bmatrix}$. Show that $X = LZ \sim N(0, \Psi)$ where L is the Cholesky decomposition of $\Psi = L'L$ given by

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \rho_{12} & \sqrt{1-\rho_{12}^2} & 0 \\ \rho_{13} & \dfrac{\rho_{23} - \rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}} & \sqrt{1-\rho_{13}^2 - \dfrac{(\rho_{23} - \rho_{12}\rho_{13})^2}{1-\rho_{12}^2}} \end{bmatrix}.$$

Exercise 15. Consider the three daily return series (*Apple*, *Amazon* and *Pfizer*) in file "stocks.xlsx", each with length $T = 3{,}892$. Answer the following questions:

i) Plot the empirical distribution function (EDF) for each standardized return series (you can use *cdfplot* in Matlab). **Hint 1**: See EDF in https://en.wikipedia.org/wiki/Empirical_distribution_function

ii) Assume a trivariate G copula pdf, see section i) in Exercise 4 for the bivariate case. Estimate the correlation matrix Ψ under maximum likelihood (ML) with input series: $\hat{\eta}_t = \left(\Phi^{-1}(\hat{u}_{1t}), \Phi^{-1}(\hat{u}_{2t}), \Phi^{-1}(\hat{u}_{3t})\right)'$, $t = 1, \ldots, T$ such that $\hat{u}_{it}$ is EDF_i. **Hint 2:** The ML estimation of Ψ is given by $\hat{\Psi}_{ML} = T^{-1}\sum_{t=1}^{T} \hat{\eta}_t \hat{\eta}_t'$ with $\hat{\eta}_{it} = \Phi^{-1}(\hat{u}_{it})$. The correlation matrix estimation can be *renormalized* so that all diagonal elements equal one.

iii) Generate by Monte Carlo an equally weighted (EW) portfolio series of length $N = 10{,}000$ containing the three stocks under the G copula with $\hat{\Psi}_{ML}$ obtained in section ii), and marginal distribution for the stock return *i* given by $R_i = \mu_i + \sigma_i Z_i$; $Z_i \sim ST_{\upsilon_i}$; $E(Z_i) = 0$; $E(Z_i^2) = 1$, with $ST_{\upsilon_i} = \sqrt{\dfrac{\upsilon_i - 2}{\upsilon_i}}\, T_{\upsilon_i}$ as the *standardized Student-T distribution* with degrees of freedom $\upsilon_i > 2$. The estimation of υ_i is based on either ML or MM (method-of-moments), the estimation of μ_i and σ_i are the sample mean and standard deviation of the stock return *i*. Compute the sample mean, standard deviation, skewness, kurtosis and VaR (1% and 5%) for the EW portfolio return series.

Exercise 16. The cdf of the conditional copula for a bivariate Normal mixture (NM) copula is easy to derive from the bivariate *G* copula in Exercise 7:

$$C^{NM}\left(u_1 \middle| u_2\right) = \pi \times \Phi\left(\frac{\Phi^{-1}\left(u_1\right) - \rho_1\, \Phi^{-1}\left(u_2\right)}{\sqrt{1-\rho_1^2}}\right) + \left(1-\pi\right) \times \Phi\left(\frac{\Phi^{-1}\left(u_1\right) - \rho_2\, \Phi^{-1}\left(u_2\right)}{\sqrt{1-\rho_2^2}}\right),$$

where $0 < \pi < 1$. However, this time the *q* quantile curves cannot be expressed as an explicit function and numerical methods need to be used to back out a value of u_2 from $C^{NM}\left(u_1 \middle| u_2\right) = q$ from each u_1 and q. If we set the marginals $F_i(\bullet)$ $i=1,2$ those from the bivariate pdf (I), i.e., $r_i = F_i^{-1}\left(u_i\right)$. Draw the *q* quantile curves for $q = 0.05, 0.25, 0.5, 0.75, 0.95$ when $\pi = 0.3, \rho_1 = -0.7, \rho_2 = 0.4$.

Exercise 17. Repeat the same analysis as in section iv) in Exercise 3 but considering as marginal distributions for the returns the *Truncated Normal* (*TN*) *distribution*, $R_i \sim TN\left(a_i, b_i, \tilde{\mu}_i, \tilde{\sigma}_i\right)$ with pdf $f_i\left(r_i\right) = \dfrac{1}{\tilde{\sigma}_i} \dfrac{\phi\left(\left(r_i - \tilde{\mu}_i\right)/\tilde{\sigma}_i\right)}{\Phi\left(\tilde{b}_i\right) - \Phi\left(\tilde{a}_i\right)}$ such that $a_i < r_i < b_i$, $\tilde{\sigma}_i > 0$, $\tilde{a}_i = \left(a_i - \tilde{\mu}_i\right)/\tilde{\sigma}_i$ and $\tilde{b}_i = \left(b_i - \tilde{\mu}_i\right)/\tilde{\sigma}_i$. Both mean and standard deviation of R_i, denoted as μ_i and σ_i, are: $\mu_i = \tilde{\mu}_i - \tilde{\sigma}_i B_i$; $\sigma_i = \tilde{\sigma}_i \sqrt{1 - A_i - B_i^2}$; $A_i = \dfrac{\tilde{b}_i \phi\left(\tilde{b}_i\right) - \tilde{a}_i \phi\left(\tilde{a}_i\right)}{\Phi\left(\tilde{b}_i\right) - \Phi\left(\tilde{a}_i\right)}$; $B_i = \dfrac{\phi\left(\tilde{b}_i\right) - \phi\left(\tilde{a}_i\right)}{\Phi\left(\tilde{b}_i\right) - \Phi\left(\tilde{a}_i\right)}$. By using the "inverse transform method" for simulated random variables, we have:

$$R_i = \tilde{\mu}_i + \tilde{\sigma}_i \cdot \Phi^{-1}\left(\Phi\left(\tilde{a}_i\right) + U_i \cdot \left[\Phi\left(\tilde{b}_i\right) - \Phi\left(\tilde{a}_i\right)\right]\right); \qquad i = 1,2$$

where $\phi(\bullet)$ and $\Phi(\bullet)$ denote, respectively, pdf and cdf of the $N(0,1)$ distribution, and $\left(U_1, U_2\right)$ is a bivariate uniform random vector under the "conditional sampling" in section iv) in Exercise 3. The values of $\left(a_i, b_i\right)$ are the same as those in Exercise 3, while μ_i and σ_i are set to be equal to those from the implied marginal distribution of R_i in the pdf (I). Thus, $\left(\mu_1, \sigma_1\right) = \left(0.55, 0.86\right)$ and $\left(\mu_2, \sigma_2\right) = \left(0.93, 1.11\right)$. Compute both the sample mean and standard deviation of portfolio $\boldsymbol{F}$ returns, containing 25% of stock 1 and 75% of stock 2 under the above procedure, and also the VaR values at 1% and 5%. Compare these VaR values of portfolio $\boldsymbol{F}$ with those from portfolios $\boldsymbol{A}$ and $\boldsymbol{B}$.

Hint: See https://en.wikipedia.org/wiki/Truncated_normal_distribution.