CSC/ECE/OR 579:
Introduction to Computer Performance Evaluation
Homework #2.

Question 1.

The transition probability matrices of a number of Markov chains are given below. For each different P, find the transition probability matrix of the reversed chain.

$$P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} .4 & .2 & .4 \\ .1 & .3 & .6 \\ .5 & .5 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & .6 & .4 \\ 0 & 0 & 0 & 1 \\ .6 & .4 & 0 & 0 \end{pmatrix}.$$

Question 2.

Which of the transition probability matrices given below belong to reversible Markov chains? Give reasons for your answer.

$$P_1 = \begin{pmatrix} .25 & .25 & .25 & .25 \\ .25 & .50 & .25 & 0.0 \\ .50 & 0.0 & .25 & .25 \\ .10 & .20 & .30 & .40 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 3/4 & 1/4 & 0 \\ 1/3 & 0 & 4/9 & 2/9 \\ 1/10 & 4/10 & 0 & 5/10 \\ 0 & 2/7 & 5/7 & 0 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} .1 & 0 & 0 & .9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ .4 & 0 & 0 & 0 & .2 & .4 \\ 0 & .5 & .3 & .2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad P_4 = \begin{pmatrix} .5 & .1 & .4 \\ .2 & .4 & .4 \\ .3 & .2 & .5 \end{pmatrix}.$$

Question 3.

A triple redundancy system consists of three computing units and a“voter.” The purpose of the voter is to compare the results produced by the three computing units and to select the answer given by at least two of them. The computing units as well as the voter are all susceptible to failure. The rate at which a computing unit fails is taken to be λ failures per hour and the rate at which the voter fails is ξ failures per hour. All items are repairable: computing units are repaired at a rate of μ per hour and the voter at a rate of ν per hour. During the repair of the voter, each computing unit is inspected and repaired if necessary. The time for repair of the computing units in this case is included in the $1/\nu$ hours needed to repair the voter. This system is to be represented as a continuous-time Markov chain.

(a) What conditions are imposed by representing this system as a continuous-time Markov chain?
(b) What is a suitable state descriptor for this Markov chain?
(c) Draw the state transition-rate diagram.
(d) Write down the infinitesimal generator for this Markov chain.

Question 4.

Suppose a single repairman has been assigned the responsibility of maintaining three machines. For each machine, the probability distribution of up time (machine is functioning properly) before a breakdown is exponential with a mean of nine hours. The repair time is also exponentially distributed with a mean of two hours. Calculate the steady-state probability distribution and the expected number of machines that are not running.

Question 5.

Consider a continuous-time Markov chain with transition-rate matrix

$$Q = \begin{pmatrix} 0 & 0 & & & & \\ 2 & -3 & 1 & & & \\ & 4 & -6 & 2 & & \\ & & 6 & -9 & 3 & \\ & & & 8 & -12 & 4 \\ & & & & 0 & 0 \end{pmatrix}.$$

Use Matlab and its matrix exponential function to compute the probability of being in state 1, state 4, and state 6 at times $t = 0.01,\ 0.1,\ 0.5,\ 1.0,\ 2.0$, and 5.0, given that the system begins in state 4.

Question 6.

Compute the stationary probability distribution of the continuous-time Markov chain whose infinitesimal generator Q is shown below

(a) from the stationary distribution of its embedded chain, and

(b) by solving the system of equation $\pi Q = 0$.

$$Q = \begin{pmatrix} -1 & 1 & & & & \\ & -2 & 2 & & & \\ & & -3 & 3 & & \\ & & & -4 & 4 & \\ & & & & -5 & 5 \\ 6 & & & & & -6 \end{pmatrix}.$$

Question 7.

State Gerschgorin's theorem and use it to prove that the eigenvalues of a stochastic matrix cannot exceed 1 in magnitude.

Question 8.

Use Gaussian elimination to compute the stationary distribution of the discrete-time Markov chain whose transition probability matrix P is

$$P = \begin{pmatrix} 0 & 0 & .6 & 0 & 0 & .4 \\ 0 & .3 & 0 & 0 & 0 & .7 \\ 0 & 0 & 0 & .4 & .6 & 0 \\ 0 & .5 & 0 & .5 & 0 & 0 \\ .6 & .4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .8 & 0 & .2 \end{pmatrix}.$$

Is this stationary distribution a limiting distribution? Explain your answer.
Hint: Draw a picture.

Question 9.

The stochastic transition probability matrix of a three-state Markov chain is given by

$$P = \begin{pmatrix} 0.0 & 0.8 & 0.2 \\ 0.0 & 0.1 & 0.9 \\ 0.6 & 0.0 & 0.4 \end{pmatrix}.$$

Carry out three iterations of the power method and the method of Gauss-Seidel using $(0.5, 0.25, 0.25)$ as an initial approximation to the solution. Construct the Gauss-Seidel iteration matrix and compute the number of iterations needed to achieve ten decimal places of accuracy.