CSC/ECE/OR 579:
Introduction to Computer Performance Evaluation
Homework #1.

Question 1.

St. Mary's and St. Malachy's are two local high schools with excellent water polo teams that have competed against each other for many decades. Going into this year's final match, the historical record give St. Mary's a 55% chance of winning. Two prominent sports commentators like to get in on the action. The first who is right 9 times out of 10 says that St. Mary's will win whereas the second who is right 4 times out of 5 says that St. Malachy's will win. Taking into account the opinions of the commentators, what is the probability that St. Mary's will once again come out the victor?

Question 2.

Let the joint probability density function of two continuous random variables X and Y be given by

$$f_{X,Y}(x,y) = \begin{cases} \alpha, & a < x \leq b, \quad 0 < y \leq c, \\ 0 & \text{otherwise.} \end{cases}$$

Find the value of α and the marginal density functions of X and Y. Are X and Y independent random variables?

Question 3.

The cumulative distribution function of a continuous random variable X is given as

$$F_X(x) = \begin{cases} 1 - e^{-2x}, & 0 < x < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Find the probability density function and expectation of the derived random variable $Y = e^X$.

Question 4.

Consider a store that sells television sets. If at the end of the day there is one or zero sets left, then that evening, after the store has closed, the shopkeeper brings in enough new sets so that the number of sets in stock for the next day is equal to five. This means that each morning, at store opening time, there are between two and five television sets available for sale. Such a policy is said to be an (s, S) inventory control policy. Here we have assigned the values $s = 1$, $S = 5$. The shopkeeper knows from experience that the probabilities of selling 0 through 5 sets on any given day are $0.4,\ 0.3,\ 0.15,\ 0.15,\ 0.0$, and 0.0.

Explain how this scenario may be modeled by a Markov chain $\{X_n,\ n = 1, 2, \ldots\}$, where X_n is the random variable that defines the number of television sets left at the end of the n^{th} day. Write down and explain the structure of the transition probability matrix.

Question 5.

The transition probability matrix of a discrete-time Markov chain is given by

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 1.0 \\ 0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0 & 1.0 & 0 \\ 0 & 0.8 & 0.2 & 0 & 0 \\ 0.4 & 0 & 0.6 & 0 & 0 \end{pmatrix}.$$

Draw all sample paths of length 4 that begin in state 1. What is the probability of being in each of the states 1 through 5 after four steps beginning in state 1?

Question 6.

A Markov chain with two states a and b has the following conditional probabilities: If it is in state a at time step n, $n = 0, 1, 2, \ldots$, then it stays in state a with probability $0.5(0.5)^n$. If it is in state b at time step n, then it stays in state b with probability $0.75(0.25)^n$. If the Markov chain begins in state a at time step $n = 0$, compute the probabilities of the following sample paths:

$$a \longrightarrow b \longrightarrow a \longrightarrow b \quad \text{and} \quad a \longrightarrow a \longrightarrow b \longrightarrow b.$$

Question 7.

The transition probability matrix of an embedded Markov chain is

$$P^E = \begin{pmatrix} 0.0 & 0.3 & 0.4 & 0.3 \\ 0.1 & 0.0 & 0.2 & 0.7 \\ 0.3 & 0.2 & 0.0 & 0.5 \\ 0.4 & 0.4 & 0.2 & 0.0 \end{pmatrix}.$$

Given that the homogeneous, discrete-time Markov chain P from which this embedded chain is extracted, spends on average, 1 time unit in state 1, 2 time units in state 2, 4 time units in state 3 and 2.5 time units in state 4, derive the original Markov chain, P.

Question 8.

The following matrix is the single-step transition probability matrix of a discrete time Markov chain which describes the weather. State 1 represents a sunny day, state 2 a cloudy day, and state 3 a rainy day.

$$P = \begin{array}{c} \\ Sunny \\ Cloudy \\ Rainy \end{array} \begin{array}{c} \begin{array}{ccc} Sunny & Cloudy & Rainy \end{array} \\ \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.6 & 0.2 \end{pmatrix} \end{array}$$

(a) What is the probability of a sunny day being followed by two cloudy days?

(b) Given that today is rainy, what is the probability that the sun will shine the day after tomorrow?

(c) What is the mean length of a rainy period?

Question 9.

Consider the four-state discrete-time Markov chain whose transition probability matrix at time step $n,\ n = 0, 1, \ldots$, is

$$P(n) = \begin{pmatrix} 0 & 0.6 & 0.4 & 0 \\ 0.8 & 0 & 0 & 0.2 \\ 0 & 0 & 0.5(0.5)^n & 1 - 0.5(0.5)^n \\ 0 & 0 & 0.8(0.8)^n & 1 - 0.8(0.8)^n \end{pmatrix}.$$

What is the probability distribution after two steps if the Markov chain is initiated in (a) state 1; (b) state 4?

Question 10.

William, the collector, enjoys collecting the toys in McDonald's Happy Meals. And now McDonalds has come out with a new collection containing 5 toy warriors. Each Happy Meal includes one randomly chosen warrior. Naturally William has to collect all five different types.

(a) Use a discrete time Markov chain to represent the process which William will go through to collect all 5 warriors and draw the state transition diagram.

(b) Construct the stochastic transition probability matrix for this discrete-time Markov chain and compute the probability distribution after William has eaten 3 happy meals.

(c) Let T denote the total number of Happy Meals that William will eat to enable him to get all 5 warriors. Compute $E[T]$ and $Var[T]$.

Question 11. [15 points]

Give as precise a classification as possible to each of the states of the discrete-time Markov chain whose transition probability matrix is

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & .3 & 0 & .7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .6 & 0 & 0 & .4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .5 & 0 & 0 & .5 & 0 \\ 0 & 0 & 0 & 0 & 0 & .2 & 0 & 0 & 0 & .8 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & .9 & .1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & .8 & 0 & 0 & .2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Question 12.

Find the mean recurrence time of state 2 and the mean first passage time from state 1 to state 2 in the discrete-time Markov chain whose transition probability matrix is

$$P = \begin{pmatrix} 0.1 & 0.3 & 0.6 \\ 0.2 & 0.5 & 0.3 \\ 0.4 & 0.4 & 0.2 \end{pmatrix}.$$

Hint: Use Matlab.

Question 13.

In the Markov chain whose transition probability matrix P is given below, identify examples of the following (if they exist):

(a) a return state,
(b) a nonreturn state,
(c) an absorbing state,
(d) a closed communicating class,
(e) an open communicating class,
(f) a closed communicating class containing recurrent states,
(g) an open communicating class containing recurrent states,
(h) a closed communicating class containing transient states,
(i) an open communicating class containing transient states, and
(j) a communicating class with both transient and recurrent states.

$$P = \begin{pmatrix} 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0 \\ 0 & 0 & 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.0 & 0 \end{pmatrix}.$$

Question 14.

In the Markov chain whose transition probability matrix P is given below, what are the communicating classes to which the following states belong: (a) state 2, (b) state 3, (c) state 4 and (d) state 5.

$$P = \begin{pmatrix} 0.5 & 0.25 & 0.25 & 0 & 0 & 0 & 0 \\ 0.25 & 0.5 & 0 & 0.25 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.25 & 0.25 \\ 0 & 0 & 0 & 0 & 0.25 & 0.5 & 0.25 \\ 0 & 0 & 0 & 0 & 0.25 & 0.25 & 0.5 \end{pmatrix}.$$

Question 15.

Prove that if one state in a communicating class C is transient, then all states in C are transient.

Question 16.

Let C be a non-closed communicating class. Show that no state in C can be recurrent. (This means that every recurrent class is closed.)

Question 17.

Compute the potential matrix of the Markov chain whose transition probability matrix is

$$P = \begin{pmatrix} .5 & .5 & 0 & 0 \\ .5 & 0 & .5 & 0 \\ .5 & 0 & 0 & .5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and from it find (a) the expected number of visits to state 3 beginning from state 1, (b) the expected number of visits to state 1 beginning in state 3, and (c) the expected number of visits to state 2 beginning in state 4.

Question 18.

Consider the gambler's ruin problem in which a gambler begins with \$50, wins \$10 on each play with probability $p = 0.45$, or loses \$10 with probability $q = 0.55$. The gambler will quit once he doubles his money or has nothing left of his original \$50.

(a) What is the expected number of times he has \$90 before quitting?
(b) What is the expected number of times he has \$50 before quitting?
(c) What is the expected number of times he has \$10 before quitting?

Question 19.

Consider a discrete-time Markov chains whose transition probability matrix is

$$P = \begin{pmatrix} .2 & .8 & 0 & 0 & 0 \\ 0 & .4 & .6 & 0 & 0 \\ 0 & 0 & .6 & .4 & 0 \\ 0 & 0 & 0 & .8 & .2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Compute the mean and variance of the time to absorption from all transient starting states.

Question 20.

The transition probability matrix of a discrete-time Markov chain is

$$P = \begin{pmatrix} .6 & .4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .2 & .6 & 0 & .2 & 0 & 0 & 0 \\ .2 & 0 & .3 & 0 & 0 & 0 & 0 & .5 \\ 0 & 0 & 0 & 0 & 0 & .9 & 0 & .1 \\ 0 & 0 & 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .5 & 0 & .5 & 0 \\ 0 & 0 & 0 & 0 & 1.0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0 \end{pmatrix}.$$

Assume that the initial state is 2 with probability 0.4 and 4 with probability 0.6. Compute

(a) the mean and variance of the number of times the Markov chain visits state 1 before absorption;
(b) the mean and variance of the total number of steps prior to absorption.

Question 21.

Consider a discrete-time Markov chain whose transition probability matrix is

$$P = \begin{pmatrix} .2 & .8 & 0 & 0 & 0 \\ 0 & .4 & .6 & 0 & 0 \\ 0 & 0 & .6 & .4 & 0 \\ .2 & 0 & 0 & .6 & .2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(a) Compute the part of the reachability matrix that corresponds to transitions among transient states (the matrix called H in the text).
(b) Beginning in state 1, what is the probability of going to state 4 exactly four times?
(c) Beginning in state 1, how many different states are visited, on average, prior to absorption?

Question 22.

In the gambler's ruin scenario of Question 19.

(a) What is the mean number of plays he makes (starting with \$50) before stopping?

(b) How many different amounts of money does he have during the time he is playing?

Question 23.

Compute the *absorbing chain* and from it, the matrix of absorbing probabilities, of the Markov chain whose transition probability matrix is

$$P = \begin{pmatrix} .6 & .4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .2 & .6 & 0 & .2 & 0 & 0 & 0 \\ .2 & 0 & .3 & 0 & 0 & 0 & 0 & .5 \\ 0 & 0 & 0 & 0 & 0 & .9 & 0 & .1 \\ 0 & 0 & 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .5 & 0 & .5 & 0 \\ 0 & 0 & 0 & 0 & 1.0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0 \end{pmatrix}.$$

What are the probabilities of being absorbed into state 8 from states 1, 2, and 3?

Question 24.

Random walk on the integers $0, \pm 1, \pm 2, \ldots$.
Consider the Markov chain whose transition probability matrix is

$$P = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ \cdots & 0 & p & 0 & 0 & 0 & 0 & \cdots \\ \cdots & q & 0 & p & 0 & 0 & 0 & \cdots \\ \cdots & 0 & q & 0 & p & 0 & 0 & \cdots \\ \cdots & 0 & 0 & q & 0 & p & 0 & \cdots \\ \cdots & 0 & 0 & 0 & q & 0 & p & \cdots \\ \cdots & 0 & 0 & 0 & 0 & q & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Show that

$$p_{00}^{(2n+1)} = 0 \quad \text{and} \quad p_{00}^{(2n)} = \binom{2n}{n} p^n q^n, \quad \text{for} \quad n = 1, 2, 3, \ldots.$$

Use Sterling's formula, $n! \approx n^{n+1/2} e^{-n} \sqrt{2\pi}$, to show that $(2n)!/(n!n!) = 4^n/\sqrt{n\pi}$ and hence that

$$p_{00}^{(2n)} \approx \frac{(4pq)^n}{\sqrt{n\pi}}.$$

Write an expression for $\sum_{n=1}^{\infty} p_{00}^{(n)}$ and show that this is infinite if and only if $p = 1/2$, and hence that the Markov chain is recurrent if and only if $p = 1/2$ and is transient for all other values of p.

Question 25.

Consider a Markov chain whose transition probability matrix is given by

$$P = \begin{pmatrix} q & p & 0 & \cdots & & \\ q & 0 & p & 0 & \cdots & \\ 0 & q & 0 & p & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix},$$

where $p > 0,\ q > 0$ and $p + q = 1$. This is a variant of the random walk problem with Bernoulli boundary. By considering the system of equations $z = zP$ with $ze = 1$ and Theorem 9.7.1, find conditions under which the states of the Markov chain are ergodic and, assuming that these conditions hold, find the value of z_k for $k = 0, 1, \ldots$.

Question 26.

On each play in roulette, a player wins \$10 with probability 18/38 and loses \$10 with probability 20/38. Kathie has \$100 in her purse but really needs \$250 to purchase the pair of shoes she has just seen in the shop next to the casino. What is the probability of her converting her \$100 into \$250 at the roulette table?

Question 27.

A square matrix is said to be *stochastic* if each of its elements lies in the interval $[0, 1]$ and the sum of the elements in each row is equal to 1. It is said to be *doubly stochastic* if, in addition, the sum of elements in each *column* is equal to 1. Show that each element of the stationary distribution of an irreducible, finite, K-state Markov chain whose transition probability matrix is doubly stochastic is equal to $1/K$. Is this stationary distribution unique? Does this same result hold when the Markov chain is reducible? Explain your answer. Can an irreducible Markov chain whose transition probability matrix is doubly stochastic have transient states? Explain your answer. Does the *limiting distribution* necessarily exist for an irreducible Markov chain whose transition probability matrix is doubly stochastic? Explain your answer.

Question 28.

In the scenario of the weather at Belfast,

(a) What is the unconditional probability of having a sunny day?

(b) What is the mean number of rainy days in a month of 31 days?

(c) What is the mean recurrence time of sunny days?

Question 29.

For each of the following Markov chains (A, B, C, and D), represented by transition probability matrices given below, state whether it has (i) a stationary distribution, (ii) a limiting distribution, (iii) a steady-state distribution. If your answer is negative, explain why. If your answer is positive, give the distribution.

$$P_A = \begin{pmatrix} 0.5 & 0.5 & 0.0 \\ 0.0 & 0.5 & 0.5 \\ 0.5 & 0.0 & 0.5 \end{pmatrix}, \quad P_B = \begin{pmatrix} 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \\ 1.0 & 0.0 & 0.0 \end{pmatrix}, \quad P_C = \begin{pmatrix} 0.5 & 0.5 & 0.0 & 0.0 \\ 0.5 & 0.5 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.5 & 0.5 \\ 0.0 & 0.0 & 0.5 & 0.5 \end{pmatrix},$$

$$P_D(n) = \begin{pmatrix} 1/n & (n-1)/n \\ (n-1)/n & 1/n \end{pmatrix}, \quad n = 1, 2, \ldots.$$

Question 30.

Consider a machine that at the start of any particular day is either broken down or in operating condition. Assume that if the machine is broken down at the start of the n^{th} day, the probability that it will be successfully repaired and in operating condition at the start of the $(n+1)^{\text{th}}$ day is p. Assume also that if the machine is in operating condition at the start of the n^{th} day, the probability that it will fail and be broken down at the start of the $(n+1)^{\text{th}}$ day is q. Let $\pi_0(0)$ denote the probability that the machine is broken down initially.

(a) Find the following probabilities

$$\text{Prob}\{\ X_{n+1} = 1 \mid X_n = 0\ \},\quad \text{Prob}\{\ X_{n+1} = 0 \mid X_n = 1\ \},\quad \text{Prob}\{\ X_{n+1} = 0 \mid X_n = 0\ \},$$

$$\text{Prob}\{\ X_{n+1} = 1 \mid X_n = 1\ \}\quad \text{and}\quad \text{Prob}\{\ X_0 = 1\ \}.$$

(b) Compute $\text{Prob}\{X_n = 0\}$ and $\text{Prob}\{X_n = 1\}$ in terms of p, q, and $\pi_0(0)$.

(c) Find the steady-state distribution $\lim_{n\to\infty} \text{Prob}\{X_n = 0\}$ and $\lim_{n\to\infty} \text{Prob}\{X_n = 1\}$.