Recitation 1
Background

10-301/10-601: Introduction to Machine Learning

05/19/2022

1 NumPy and Workflow

NumPy Notebook   install numpy via pip

Workflow Presentation

Logging Notebook

2 Vectors, Matrices, and Geometry

1. **Inner Product:** $\mathbf{u} = \begin{bmatrix} 6 & \overset{3}{1} & 2 \end{bmatrix}^T$, $\mathbf{v} = \begin{bmatrix} 3 & \overset{3}{-10} & -2 \end{bmatrix}^T$, what is the inner product of $\mathbf{u}$ and $\mathbf{v}$? What is the geometric interpretation? → proportional to projecting v in direction v

   $U \cdot V \mid U^T V = \sum_k U_k V_k$ $18 + -10 + -4 = -4$

2. **Cauchy-Schwarz inequality** (Optional): Given $\mathbf{u} = \begin{bmatrix} 3 & 1 & 2 \end{bmatrix}^T$, $\mathbf{v} = \begin{bmatrix} 3 & -1 & 4 \end{bmatrix}^T$, what is $||\mathbf{u}||_2$ and $||\mathbf{v}||_2$? What is $\mathbf{u} \cdot \mathbf{v}$? How do $\mathbf{u} \cdot \mathbf{v}$ and $||\mathbf{u}||_2 ||\mathbf{v}||_2$ compare? Is this always true?

   $||U||_2 = U \cdot U$ $U \cdot V \leq ||U||_2 ||V||_2$

   $= (\sum_k U_k^2)^{\frac{1}{2}}$ always true

   M=P

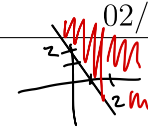
3. **Matrix algebra.** Generally, if $\mathbf{A} \in \mathbb{R}^{\underset{\sim}{M} \times N}$ and $\mathbf{B} \in \mathbb{R}^{N \times \underset{\sim}{P}}$, then $\mathbf{AB} \in \mathbb{R}^{M \times P}$ and $(AB)_{ij} = \sum_k A_{ik} B_{kj}$.

   Given $\mathbf{A} = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 4 & -3 & 2 \\ 1 & 1 & -1 \\ 3 & -2 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. $AB = \begin{bmatrix} 21 & -11 & 10 \\ 8 & -2 & 4 \\ 12 & -8 & 8 \end{bmatrix}$

   - What is $\mathbf{AB}$? Does $\underset{\text{false}}{\mathbf{BA}} = \underset{\text{generally}}{\mathbf{AB}}$? What is $\mathbf{Bu}$?
   - What is rank of $\mathbf{A}$? # of linearly independent columns of matrix → gaussian elimination 3
   - What is $\mathbf{A}^T$? $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 5 & 2 & 4 \end{bmatrix}$
   - Calculate $\mathbf{uv}^T$.
   - What are the eigenvalues of $\mathbf{A}$?

   $\det(A - \lambda I) = 0$ $A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 5 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 4-\lambda \end{bmatrix}$ $\det(A - \lambda I) =$

   $Av = \lambda v$ $\lambda = 1, 2, 4$



4. **Geometry:** Given a line $2x + y = 2$ in the two-dimensional plane,
   - If a given point (α, β) satisfies $2\alpha + \beta > 2$, where does it lie relative to the line?
   - What is the relationship of vector $\mathbf{v} = [2, 1]^T$ to this line?
   - What is the distance from origin to this line?

$a = [-1, 2]$

$a \cdot v = 0$
perpendicular
orthogonal

$$\frac{|A\alpha + B\beta + c|}{\sqrt{A^2 + B^2}} = \frac{|-2|}{\sqrt{2^2 + 1^2}} \quad 2/\sqrt{5}$$

3 CS Fundamentals

1. For each (f, g) functions below, is $f(n) \in \mathcal{O}(g(n))$ or $g(n) \in \mathcal{O}(f(n))$ or both?
   - $f(n) = \log_2(n)$, $g(n) = \log_3(n)$ both
   - $f(n) = 2^n$, $g(n) = 3^n$ $f(n) \in O(g(n))$
   - $f(n) = \frac{n}{50}$, $g(n) = \log_{10}(n)$ $g(n) \in O(f(n))$
   - $f(n) = n^2$, $\in g(n) = 2^n$

     $\log \in O(\text{poly}) \in O(\text{expo})$

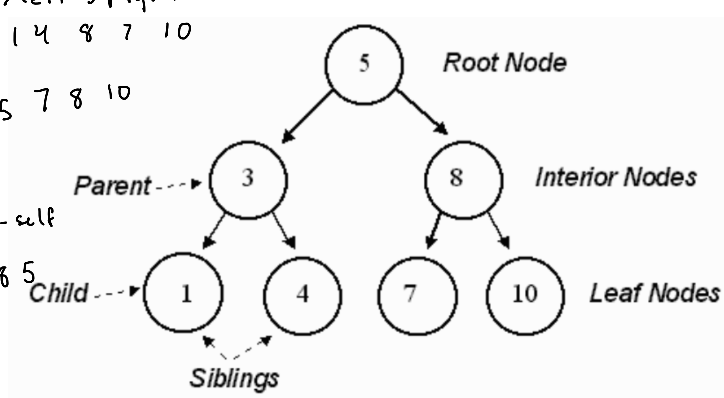
2. Find the DFS Pre-Order, In-Order, Post-Order and BFS traversal of the following binary tree. What are the time complexities of the traversals?

pre order: self → left → right
5 3 1 4 8 7 10

in-order
1 3 4 5 7 8 10

post-order
left-right-self
1 4 3 7 10 8 5

5 3 8 1 4 7 10



4 Calculus

1. If $f(x) = x^3 e^x$, find $f'(x)$. power rule $\frac{d(x^n)}{dx} = nx^{n-1}$

   $$h(x)^{\to x^3} g(x)^{e^x} = h'(x)g(x) + g'(x)h'(x) = 3x^2e^x + e^x x^3$$

2. If $f(x) = e^x$, $g(x) = 4x^2 + 2$, find $h'(x)$, where $h(x) = f(g(x))$.

   $$\frac{df(g(x))}{d(x)} = f'(g(x))\,g'(x) \quad h(x) = e^{4x^3+2} = e^{4x^2+2} \quad 8x$$

3. If $f(x, y) = y \log(1 - x) + (1 - y) \log(x)$, $x \in (0, 1)$, evaluate $\frac{\partial f(x,y)}{\partial x}$ at the point $(\frac{1}{2}, \frac{1}{2})$.

   difrentiation of sumations

   $$f'(x) = \frac{y}{(1-x)}(-1) + \frac{(1-y)}{x}(1) = 0$$ substitute

4. Find $\frac{\partial}{\partial w_j}\mathbf{x}^T\mathbf{w}$, where $\mathbf{x}$ and $\mathbf{w}$ are M-dimensional real-valued vectors and $1 \leq j \leq M$.

   $$x^T w = \sum_{k \in M} x_k w_k = x_1 w_1 + \ldots x_j w_j + \ldots x_m w_m = x_j$$

5 Probability and Statistics

You should be familiar with event notations for probabilities, i.e. $P(A \cup B)$ and $P(A \cap B)$, where A and B are binary events.

In this class, however, we will mainly be dealing with random variable notations, where A and B are random variables that can take on different states, i.e. a_1, a_2, and b_1, b_2, respectively. Below are some notation equivalents, as well as basic probability rules to keep in mind.

- $P(A = a_1 \cap B = b_1) = P(A = a_1, B = b_1) = p(a_1, b_1)$ joint probability
- $P(A = a_1 \cup B = b_1) = \sum_{b \in B} p(a_1, b) + \sum_{a \in A} p(a, b_1) - p(a_1, b_1)$ or probability $\Pr\{A = a_1\} + \Pr\{B = b_1\} - \Pr\{A = a_1, B = b_1\}$
- $p(a_1 \mid b_1) = \frac{p(a_1, b_1)}{p(b_1)}$ given
- $p(a_1) = \sum_{b \in B} p(a_1, b)$

1. Two random variables, A and B, each can take on two values, a_1, a_2, and b_1, b_2, respectively. a_1 and b_2 are considered disjoint (mutually exclusive). $P(A = a_1) = 0.5$, $P(B = b_2) = 0.5$.

   - What is $p(a_1, b_2)$? disjoint a_1 b_2
   - What is $p(a_1, b_1)$? $p\{A = a_1, B = b_1\} = \Pr\{A = a_1 \mid B = b_1\} \cdot \Pr\{B = b_1\} = \Pr\{B = b_1 \mid A = a_1\} \cdot \Pr\{A = a_1\} = 0.5$ (1, 0.5)
   - What is $p(a_1 \mid b_2)$?

     $= 0$

     disjoint

2. Now, instead, a_1 and b_2 are not disjoint, but the two random variables A and B are independent.
   - What is $p(a_1, b_2)$? $= p(a_1) \cdot p(b_2) = (0.5)^2$
   - What is $p(a_1, b_1)$? $= (0.5)^2$
   - What is $p(a_1 \mid b_2)$? $P(a_1) = 0.5$
3. A student is looking at her activity tracker (Fitbit/Apple Watch) data and she notices that she seems to sleep better on days that she exercises. They observe the following:

Exercise	Good Sleep	Probability
yes	yes	0.3
yes	no	0.2
no	no	0.4
no	yes	0.1

   - What is the $P(GoodSleep = yes \mid Exercise = yes)$? 3/5
   - Why doesn't $P(GoodSleep = yes, Exercise = yes) = P(GoodSleep = yes) \cdot P(Exercise = yes)$? two variables not independent $p\{A=a, B=b\} = Pr\{A=a \mid B=b\} \cdot Pr\{B=b\}$
   - The student merges her activity tracker data with her food logs and finds that the $P(Eatwell = yes \mid Exercise = yes, GoodSleep = yes)$ is 0.25. What is the probability of all three happening on the same day?

     $Pr\{A=a \mid B=b \text{ and } C=c\} \cdot Pr\{B=b \text{ and } C=c\} = Pr\{A=a, B=b, C=c\}$

4. What is the expectation of X where X is a single roll of a fair 6-sided dice ($S = \{1, 2, 3, 4, 5, 6\}$)? What is the variance of X? variance $= E[x^2] - (E[x])^2$

   $\left(\frac{1}{6} \cdot \sum_{i=1}^{6} i\right) = 3.5$ $E[x^2] = \frac{1}{6} \cdot \sum_{i=1}^{6} i^2$

5. Imagine that we had a new dice where the sides were $S = \{3, 4, 5, 6, 7, 8\}$. How do the expectation and the variance compare to our original dice?

   expectation $\Rightarrow$ inc by 2 $y = x+2$

   variance = same

   $var(x+2) = var(x)$

   $E[x+2] = E[x] + 2$