# Dynamic Programming Optimizations - Codeforces

Several recent problems on Codeforces concerned dynamic programming optimization techniques.

The following table summarizes methods known to me.

| Name | Original Recurrence | Sufficient Condition of Applicability | Original Complexity | Optimized Complexity | Links |
|---|---|---|---|---|---|
| Convex Hull Optimization1 | $dp[i] = min_{j<i}\{dp[j] + b[j] \star a[i]\}$ | $b[j] \geq b[j+1]$<br>~~optionally~~ $a[i] \leq a[i+1]$ | $O(n^2)$ | $O(n)$ | 1<br>p1 |
| Convex Hull Optimization2 | $dp[i][j] = min_{k<j}\{dp[i-1][k] + b[k] * a[j]\}$ | $b[k] \geq b[k+1]$<br>~~optionally~~ $a[j] \leq a[j+1]$ | $O(kn^2)$ | $O(kn)$ | 1<br>p1 p2 |
| Divide and Conquer Optimization | $dp[i][j] = min_{k<j}\{dp[i-1][k] + C[k][j]\}$ | $A[i][j] \leq A[i][j+1]$ | $O(kn^2)$ | $O(knlogn)$ | 1<br>p1 |
| Knuth Optimization | $dp[i][j] = min_{i<k<j}\{dp[i][k] + dp[k][j]\} + C[i][j]$ | $A[i, j-1] \leq A[i, j] \leq A[i+1, j]$ | $O(n^3)$ | $O(n^2)$ | 1 2<br>p1 |

Notes:

- $A[i][j]$ — the smallest k that gives optimal answer, for example in $dp[i][j] = dp[i-1][k] + C[k][j]$
- $C[i][j]$ — some given cost function
- We can generalize a bit in the following way: $dp[i] = min_{j<i}\{F[j] + b[j] * a[i]\}$, where $F[j]$ is computed from $dp[j]$ in constant time.
- It looks like **Convex Hull Optimization2** is a special case of **Divide and Conquer Optimization**.
- It is claimed (in the references) that **Knuth Optimization** is applicable if $C[i][j]$ satisfies the following 2 conditions:
- **quadrangle inequality**:

$$C[a][c] + C[b][d] \leq C[a][d] + C[b][c],\ a \leq b \leq c \leq d$$

- **monotonicity**:

$$C[b][c] \leq C[a][d],\ a \leq b \leq c \leq d$$

- It is claimed (in the references) that the recurrence $dp[j] = min_{i<j}\{dp[i] + C[i][j]\}$ can be solved in $O(nlogn)$ (and even $O(n)$) if $C[i][j]$ satisfies **quadrangle inequality**. WJMZBMR described how to solve some case of this problem.

Open questions:

1. Are there any other optimization techniques?
2. What is the sufficient condition of applying **Divide and Conquer Optimization** in terms of function $C[i][j]$? Answered

References:


- *"Efficient dynamic programming using quadrangle inequalities" by F. Frances Yao.* find
- *"Speed-Up in Dynamic Programming" by F. Frances Yao.* find
- *"The Least Weight Subsequence Problem" by D. S. Hirschberg, L. L. Larmore.* find
- *"Dynamic programming with convexity, concavity and sparsity" by Zvi Galil, Kunsoo Park.* find
- *"A Linear-Time Algorithm for Concave One-Dimensional Dynamic Programming" by Zvi Galil, Kunsoo Park.* find


Please, share your knowledge and links on the topic.