Network Dynamics and Learning, Homework 2

Giovanni Cadau *Politecnico di Torino*
Turin, Italy
s304861@studenti.polito.it

December 11, 2022

1 Exercise 1 - Markov chain

In a a continuous-time Markov chain $X(t)$ a random process describes the evolution of a state variable x inside a discrete state space $\mathcal{X}$ with a graph structure.

Fig 1 shows a network $\mathcal{G} = (\mathcal{X}, \Lambda)$ with nodes $\mathcal{X}$ and weight matrix Λ describing possible transitions between nodes/states. $\mathcal{X}$ and Λ are reported below:

$$\Lambda = \begin{array}{c} \begin{matrix} o & a & b & c & d \end{matrix} \\ \begin{pmatrix} 0 & 2/5 & 1/5 & 0 & 0 \\ 0 & 0 & 3/4 & 1/4 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1/3 & 0 & 2/3 \\ 0 & 1/3 & 0 & 1/3 & 0 \end{pmatrix} \end{array} \begin{matrix} o \\ a \\ b \\ c \\ d \end{matrix}$$

$$\mathcal{X} = \begin{pmatrix} o & a & b & c & d \end{pmatrix}$$

Simulations have been used in order to study a single particle performing a continuous-time random walk in the network $\mathcal{G}$ with transition rate matrix Λ.

Transitions happen at random time instants decided by the tick of a Poisson clock, characterized by the property that the time elapsed between any two of its consecutive ticks is an independent random variable with exponential distribution with a specified rate.

To simulate a Poisson clock with rate r, the time between two consecutive ticks must be simulated, denoted by t_{next}:

$$t_{next} = -\frac{\ln(u)}{r}$$

with u random variable with uniform distribution, i.e.: $u \in \mathcal{U}(0, 1)$.

Two different approaches have been used:

- A unique global Poisson clock with a rate $\omega^* = \max_i(\omega_i)$ is defined, where $w = \Lambda\mathbf{1}$.

  When the particle is at node i and the global clock ticks, either it jumps to a neighbor j with probability $Q_{ij} = \frac{\Lambda_{ij}}{\omega_*}$, $i \neq j$ or it stays in the same node (no transition) with probability $Q_{ii} = 1 - \sum_{i \neq j} Q_{ij}$.

- Each node i is equipped with its own Poisson clock with rate $\omega_i = \sum_j \Lambda_{ij}$.

  When the particle is at node i and the clock of that node ticks, it jumps to a neighbor j with probability $P_{ij} = \frac{\Lambda_{ij}}{\omega_i}$.

The probability distribution $\bar{\pi}(t)$ of the continuous-time Markov chain $X(t)$ with transition rate matrix Λ, defined as in 1

$$\bar{\pi}_i(t) = P(X(t) = i), \quad i \in \mathcal{X}\,, \tag{1}$$

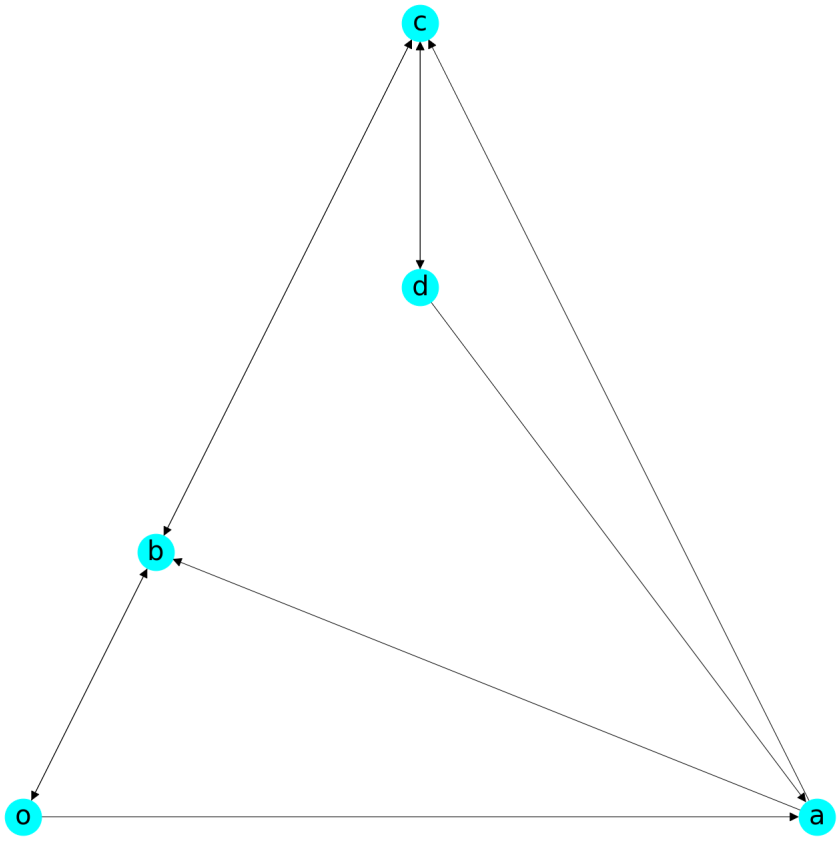

Figure 1: Network of exercise 1

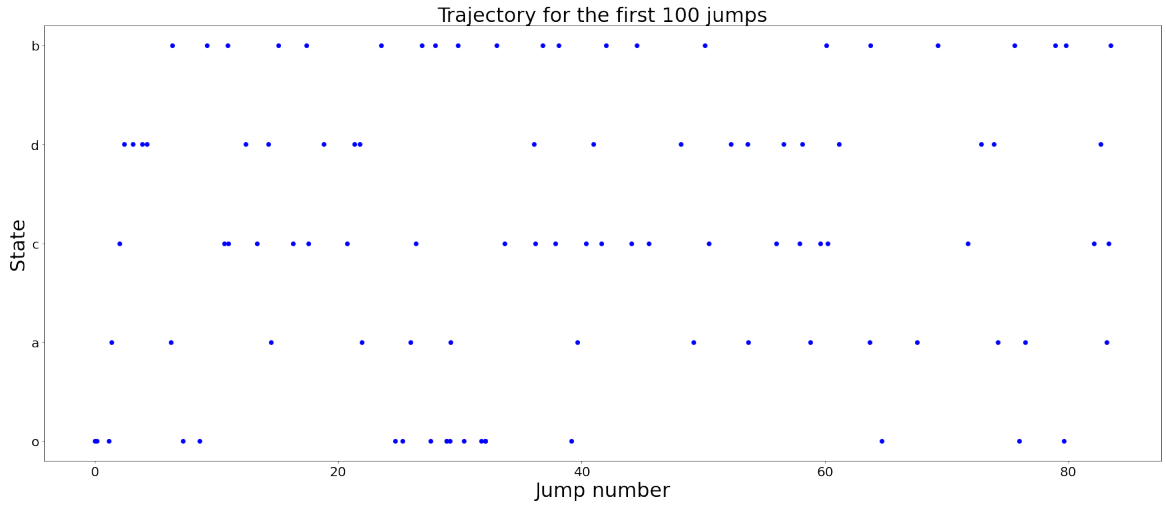

Figure 2: Trajectory for the first 100 jumps (1st aprroach)

and evolving according to 2

$$\frac{d}{dt}\bar{\pi}(t) = -L'\bar{\pi}(t) \tag{2}$$

where $L = diag(w) - \Lambda$,
is the left dominant eigenvector of Q.
With Λ related to graph $\mathcal{G}$, the left dominant eigenvector of Q computed is the following:

$$\bar{\pi} = (0.185 \quad 0.148 \quad 0.222 \quad 0.222 \quad 0.222)$$

Following the first approach, a simulation has been run for 1000000 steps (i.e.: transitions made by the particles), starting from node o.

Figure 2 shows the trajectory of the particle (for the sake of clarity, only the first 100 jumps have been considered).

Given time istants at which transition happen and the corresponding states for the particle, it is also possible to compute the time spent on each state by the particle. If divided by the total time of the process, it is possible to have an extimation of the invariant distribution $\bar{\pi}$:

$$\bar{\pi}_{estimation} = (0.185 \quad 0.148 \quad 0.222 \quad 0.223 \quad 0.222)$$

As expected, $\bar{\pi}$ and $\bar{\pi}_{estimation}$ are really close. This happens because a very high number of transitions has been specifically chosen to simulate the behaviour of the particle for $t \in [0, \infty[$.

Following the second approach, a simulation has been run for 1000000 steps (i.e.: transitions made by the particles), starting from node o.

Figure 3 shows the trajectory of the particle (for the sake of clarity, only the first 100 jumps have been considered).

In this second case, the extimation of the invariant distribution $\bar{\pi}$ is the following:

$$\bar{\pi}_{estimation} = (0.184 \quad 0.148 \quad 0.222 \quad 0.223 \quad 0.223)$$

As expected, also in this case $\bar{\pi}$ and $\bar{\pi}_{estimation}$ are really close.

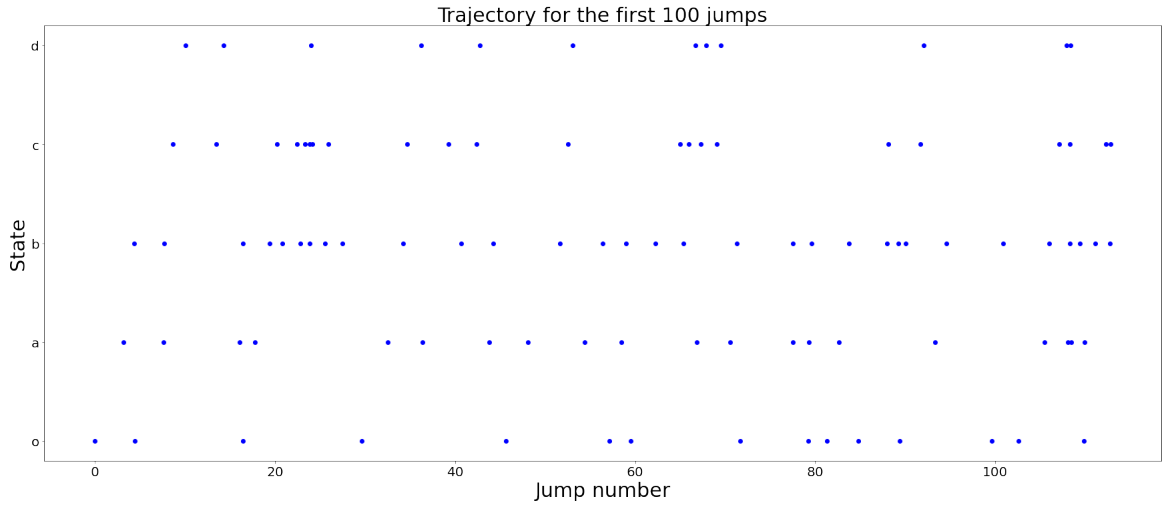

Figure 3: Trajectory for the first 100 jumps (2nd aprroach)

1.1 Point c-d

The expected hitting time $\hat{x} = (E_i[T_S])_{i \in R}$ for the set S and for all nodes $i \in R = \mathcal{V} \setminus S$ is the expected value of T_s for the random walk $X(t)$, assuming that the random walk starts in i, with hitting time:

$$T_s = \min\{t \geq 0 \mid X(t) \in S\}$$

It can be computed by solving the system of equations

$$\hat{x} = \hat{w_{inv}} + \hat{P}\hat{x}, \tag{3}$$

where $\hat{P}$ is obtained from P (the normalized weight matrix of the graph) by removing the rows and columns corresponding to the nodes in the set S and w_{inv} such that $w_{inv_i} = 1/w_i$ ($w = \Lambda\mathbf{1}$)

More explicitly, the expected hitting times can be expressed as

$$\hat{x} = (I - \hat{P})^{-1}\hat{w_{inv}}$$

$(I-\hat{P})$ is invertible only if $V \setminus S$ has at least a link pointing to S. Indeed, if $(I-\hat{P})$ is not invertible, the random walk starting from nodes in $V \setminus S$ cannot hit nodes in S, and the hitting times diverge.

With P related to graph $\mathcal{G}$, if S contains one node at a time, $(I - \hat{P})$ is invertible for each S and the solution of 3 is the following:

$$\begin{array}{rccccc} & o & a & b & c & d \\ \hat{x} = & (0 & 8 & 6 & 10 & 10.5) \end{array} \quad \text{S}=\{o\}$$

$$\begin{array}{rccccc} & o & a & b & c & d \\ \hat{x} = & (3.571 & 0 & 5.714 & 5.857 & 4.428) \end{array} \quad \text{S}=\{a\}$$

$$\begin{array}{rccccc} & o & a & b & c & d \\ \hat{x} = & (3 & 2 & 0 & 4 & 4.5) \end{array} \quad \text{S}=\{b\}$$

$$\begin{array}{rccccc} & o & a & b & c & d \\ \hat{x} = & (5.428 & 3.785 & 3.714 & 0 & 3.392) \end{array} \quad \text{S}=\{c\}$$

$$\begin{array}{rccccc} & o & a & b & c & d \\ \hat{x} = & (8.786 & 7.142 & 7.071 & 3.357 & 0) \end{array} \quad \text{S}=\{d\}$$

Note that, according to the definition, $\hat{x}$ has been extended by adding 0 elements if the position i is such that $i \in S$.

In particular, if $S = \{d\}$ and $i = o$ the theoretical hitting time $E_o[T_d]$ is 8.7857 (Note that $(I - \hat{P})$ is invertible since $V \setminus \{d\}$ has at least a link pointing to $\{d\}$: the value is finite).

The computed value can be compared with the average time it takes to move from node o to node d.

To compute such value, following the first approach, 1000000 simulations have been run starting from node o until the particle reach node d. The average value of the time instants at which the particle reachs node d between different simulations is 8.78322.

1000000 simulations have been run also following the second approach. The average value computed is 8.78103.

The following is the comparison between values computed analytically and through simulations.

- 1st approach. Error: $2.490806e-3$
- 2nd approach. Error: $4.683170e-3$

1.2 Point a-b

The expected return time $E_i[T_i^+]$ is the expected value of T_i^+ for the random walk $X(t)$, assuming that the random walk starts in i, with return time:

$$T_i^+ = \inf\{t \geq 0 \mid X(t) = i \ \wedge X(s) \neq i \ s \in (0,t)\}$$

It can be computed thanks to the following relations:

$$E_i[T_i^+] = \frac{1}{w_i} + \sum_j P_{ij} E_j[T_i] \tag{4}$$

where $E_j[T_i]$ is the expected hitting time to the set $S = \{i\}$ starting from j.

With P related to graph $\mathcal{G}$, the solution of 4 is the following:

$$\hat{y} = \begin{matrix} o & a & b & c & d \\ (9 & 6.75 & 4.5 & 4.5 & 6.75) \end{matrix}$$

The expected return time $E_i[T_i^+]$ can also be computed thanks to the following formula:

$$E_i[T_i^+] = \frac{1}{w_i \bar{\pi}_i} \tag{5}$$

The solution of 5 is the following:

$$\hat{y'} = \begin{matrix} o & a & b & c & d \\ (9 & 6.75 & 4.5 & 4.5 & 6.75) \end{matrix}$$

As can be seen, 4 and 5 lead to the same result.

In particular, if $i = a$ the theoretical return time $E_i[T_i^+]$ is 6.75.

The computed value can be compared with the average time it takes a particle that starts in node a to leave the node and then return to it.

To compute such value, following the first approach, 1000000 simulations have been run starting from node a until the particle reach again, for the first time, node a. The average value of the time instants at which the particle reachs node a between different simulations is 6.75468.

1000000 simulations have been run also following the second approach. The average value computed is 6.74810.

The following is the comparison between values computed analytically and through simulations.

- 1st approach. Error: $4.683772e-3$
- 2nd approach. Error: $1.898082e-3$

2 Exercise 1 - Opinion dynamics

Fig 1 shows a network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \Lambda)$ with nodes $\mathcal{V}$, edges $\mathcal{E}$ and weight matrix Λ. $\mathcal{V}$ and Λ are reported below:

$$\Lambda = \begin{array}{c} \begin{matrix} o & a & b & c & d \end{matrix} \\ \begin{pmatrix} 0 & 2/5 & 1/5 & 0 & 0 \\ 0 & 0 & 3/4 & 1/4 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1/3 & 0 & 2/3 \\ 0 & 1/3 & 0 & 1/3 & 0 \end{pmatrix} \end{array} \begin{matrix} o \\ a \\ b \\ c \\ d \end{matrix}$$

$$\mathcal{V} = \begin{pmatrix} o & a & b & c & d \end{pmatrix}$$

Let $x(t) \in \mathrm{R}^{\mathcal{V}}$ denote the state of the nodes of the graph.

The dynamics of $x(t)$ reads

$$x(t+1) = Px(t),$$

where P is the normalized adjacency matrix.

If x_i is interpreted as the opinion of node i, the resulting dynamics is known as French - De Groot.

By convention it is assumed that the opinion of node i is influenced by the opinion of node j if $P_{ij} > 0$, i.e., the link (i, j) has to be interpreted as i watching j and updating her opinion based on opinion of j.

Simulations have been used in order to study the dynamics on the network $\mathcal{G}$ with different initial conditions $x_i(0)$.

2.1 Point e

5 different simulation have been run with different arbitrary initial conditions $x_i(0)$, randomly chosen each simulation. Each simulation is runned from t = 0 to t = 100, since has been noticed that $x_i(0)$ don't change for higher values of t.

The following are the states of the nodes for the simulations:

$$x(0) = \begin{matrix} o & a & b & c & d \\ (2.604 & 0.36 & 2.476 & 1.416 & 2.350) \end{matrix}$$

$$x(100) = \alpha \mathbf{1}, \alpha = 1.826$$

$$x(0) = \begin{matrix} o & a & b & c & d \\ (0.950 & 1.140 & 1.870 & 1.570 & 1.016) \end{matrix}$$

$$x(100) = \alpha \mathbf{1}, \alpha = 1.412$$

$$x(0) = \begin{matrix} o & a & b & c & d \\ (2.389 & 0.970 & 1.146 & 2.553 & 2.350) \end{matrix}$$

$$x(100) = \alpha \mathbf{1}, \alpha = 1.854$$

$$x(0) = \begin{matrix} o & a & b & c & d \\ (2.298 & 2.093 & 0.228 & 1.826 & 0.603) \end{matrix}$$

$$x(100) = \alpha \mathbf{1}, \alpha = 1.304$$

$$x(0) = \begin{matrix} o & a & b & c & d \\ (0.487 & 2.796 & 2.462 & 0.166 & 0.404) \end{matrix}$$

$$x(100) = \alpha \mathbf{1}, \alpha = 1.306$$

In order to study the convergence of the states of the nodes, it is useful to compute the condensation graph of $\mathcal{G}$.

The graph $\mathcal{G}$ is strongly connected: its condensation has only 1 component (which obviously is a sink component).

1 sink in the condensation graph: 0, $\mathcal{W}_0 = \{b, d, o, a, c\}$: set of nodes of component 0

Since $\mathcal{G}$ has only 1 sink component, its invariant distribution is unique. Let denote with π the invariant distribution:

$$\pi = \begin{matrix} o & a & b & c & d \\ (0.130 & 0.174 & 0.260 & 0.260 & 0.174) \end{matrix}$$

π is such that $\pi_i > 0$ if and ony if $i \in \mathcal{W}_0$. ($\pi_i > 0, \forall i$ since $\mathcal{G}$ is strongly connected).

Additionally, the sink component 0 is aperiodic.

Since:

- the condensation graph of $\mathcal{G}$ has 1 sink
- the sink component of the graph is aperiodic,

$$\lim_{t \to +\infty} x(t) = \alpha \mathbf{1}, \alpha = \pi' x(0)$$

i.e., the dynamics converges to a common state (known as consensous) for every initial condition $x(0)$.

All simulations above presented confirm what stated.

2.2 Point f

If initial conditions are chosen such that $x_i(0) = \xi_i$ with $i \in \mathcal{V}$ and ξ_i i.i.d. random variable with variance σ^2,

the consensous value $\alpha = \pi' x(0)$ is such that its variance is:

$$\sigma_\alpha^2 = \sigma^2 \sum_i \pi_i^2$$

since ξ_i are independent random variables.

Since $\mathcal{G}$ is strongly connected, $\sum_i \pi_i^2 < \sum_i \pi_i = 1$ and:

$$\sigma_\alpha^2 < \sigma^2. \tag{6}$$

The result presented in 6 can be confirmed with numerical simulations.

In particular, $x_i(0)$ are chosen as independent random variables uniformly distributed in $\mathcal{U}(a, b)$, with $a = 0$ and $b = 1$. It implies that:

$$\sigma^2 = \frac{(b-a)^2}{12} = \frac{1}{12} = 0.08333, \quad \sigma_\alpha^2 = \sigma^2 \sum_i = \pi_i^2 = 0.017800 \tag{7}$$

which confirm what stated in 6.

To verify 7, 1000000 simulations have been run with $x_i(0)$ chosen as independent random variables uniformly distributed in $\mathcal{U}(a, b)$, with $a = 0$ and $b = 1$. The average variance of the consensous state between simulations found is ${\sigma_\alpha}^2_{simul} = 0.017815$. The difference between ${\sigma_\alpha}^2_{simul}$ and σ_α^2 is 1.48283e-5.

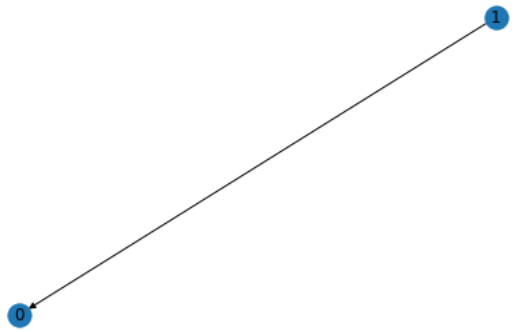

Figure 4: Condensation graph of $\mathcal{G}$, after removal of edges (d, a) and (d, c)

2.3 Point g

Edges (d, a) and (d, c) are removed from $\mathcal{E}$.

To study the convergence of the states of the nodes, it is useful to compute the condensation graph of $\mathcal{G}$ after the edges removal.

The condensation graph of $\mathcal{G}$ is shown in 4: the condensation graph has 2 components, 1 and 0. The component 0 is a sink component.

1 sink in the condensation graph: 0, $\mathcal{W}_0 = \{d\}$: set of nodes of component 0

Since $\mathcal{G}$ has only 1 sink component, its invariant distribution is unique. Let denote with π the invariant distribution:

$$\pi = \begin{matrix} o & a & b & c & d \\ (0 & 0 & 0 & 0 & 1) \end{matrix}$$

π is such that $\pi_i > 0$ if and ony if $i \in \mathcal{W}_0$.

Additionally, the sink component 0 is aperiodic.

Since:

- the condensation graph of $\mathcal{G}$ has 1 sink
- the sink component of the graph is aperiodic,

$$\lim_{t \to +\infty} x(t) = \alpha \mathbf{1}, \alpha = \pi' x(0)$$

i.e., the dynamics converges to consensous for every initial condition $x(0)$.

Since α is such that:

$$\alpha = \pi' x(0),$$

the consensus value α is the weighted average of the initial conditions of the nodes, where the weights are given by the invariant distribution π. The invariant distribution is unique since the condensation graph has only 1 sink. Since the invariant distribution π is such that $\pi_i > 0$ if and ony if i belongs to a sink component, the initial condition of the nodes not belonging to the sink are negligible for the consensus value. In addition, the sink component has only one node (d, in the 5^{th} position of the vector representing nodes): π is such that $\pi_i = 0$ if $i \neq 5$, $\pi_i = 1$ if $i = 5$ (since $\sum_i \pi_i = 1$).

As a consequence,

$$\alpha = \pi_5 x_5(0) = x_5(0)$$

5 different simulations have been run with different arbitrary initial conditions $x_i(0)$, randomly chosen each simulation. Each simulation is runned from t = 0 to t = 100, since has been noticed that $x_i(0)$ don't change for higher values of t. Here is reported only the first, the others are available in the attached code.

$$x(0) = \begin{matrix} o & a & b & c & d \\ (0.528 & 2.821 & 0.588 & 0.734 & 2.900) \end{matrix}$$

$$x(100) = \alpha \mathbf{1}, \alpha = x_d(0) = 2.900$$

All simulations confirm what stated.

If initial conditions are chosen such that $x_i(0) = \xi_i$ with $i \in \mathcal{V}$ and ξ_i i.i.d. random variable with variance σ^2,

the consensous value $\alpha = \pi' x(0)$ is such that its variance is:

$$\sigma_\alpha^2 = \sigma^2 \sum_i \pi_i^2$$

since ξ_i are independent random variables.

Since π is such that $\pi_i = 0$ if $i \neq 5$, $\pi_i = 1$ if $i = 5$ (since $\sum_i \pi_i = 1$), $\sum_i \pi_i^2 = \sum_i \pi_i = 1$ and:

$$\sigma_\alpha^2 = \sigma^2. \tag{8}$$

This happens because the graph has a unique sink node.

The result presented in 8 can be confirmed with numerical simulations.

In particular, $x_i(0)$ are chosen as independent random variables uniformly distributed in $\mathcal{U}(a, b)$, with $a = 0$ and $b = 1$. It implies that:

$$\sigma^2 = \frac{(b-a)^2}{12} = \frac{1}{12} = 0.08333, \quad \sigma_\alpha^2 = \sigma^2 \sum_i = \pi_i = 0.08333 \tag{9}$$

which confirm what stated in 8.

To verify 9, 1000000 simulations have been run with $x_i(0)$ chosen as independent random variables uniformly distributed in $\mathcal{U}(a, b)$, with $a = 0$ and $b = 1$. The average variance of the consensous state between simulations found is ${\sigma_\alpha}^2_{simul} = 0.083349$. The difference between ${\sigma_\alpha}^2_{simul}$ and σ_α^2 is 1.56237e-5.

2.4 Point h

Edges (c, d) and (d, a) are removed from the original edge set $\mathcal{E}$.

To study the convergence of the states of the nodes, it is useful to compute the condensation graph of $\mathcal{G}$ after the edges removal.

The condensation graph of $\mathcal{G}$ is shown in 5: the condensation graph has 2 components, 1 and 0. The component 0 is a sink component. It is shown in 6.

1 sink in the condensation graph: 0, $\mathcal{W}_0 = \{d, c\}$: set of nodes of component 0

Since $\mathcal{G}$ has only 1 sink component, its invariant distribution is unique. Let denote with π the invariant distribution:

$$\pi = \begin{matrix} o & a & b & c & d \\ (0 & 0 & 0 & 0.5 & 0.5) \end{matrix}$$

π is such that $\pi_i > 0$ if and ony if $i \in \mathcal{W}_0$.

Additionally, the sink component 0 is periodic.

To sum up:

- the condensation graph of $\mathcal{G}$ has 1 sink
- the sink component of the graph is not aperiodic.

Since the sink component is not aperiodic, the dynamics does not converges to a consensous state.

In addition, let $\lambda := \max\{\lambda_2, |\lambda_n|\}$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of P.

For the graph $\mathcal{G}$ the eigenvalues of P are the following: 1, 0.718, -0.359+0.469j, -0.359-0.469j, -1. Since $\lambda_n = -1$ (thus $\lambda = 1$), then the convergence to consensus is not achieved.

Additionally, the following considerations can be made.

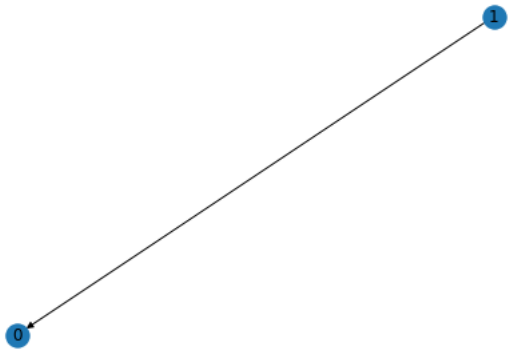

Figure 5: Condensation graph of $\mathcal{G}$, after removal of edges (c, b) and (d, a)

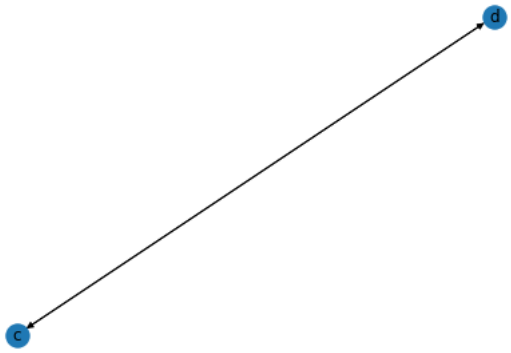

Figure 6: Sink component of the condensation graph of $\mathcal{G}$, after removal of edges (c, b) and (d, a)

Whatever initial condition $x(0)$ is chosen, after a small number of time instants, the state of each node oscillates in 2 consecutive time istants between 2 values and continues to oscillate up to infinity. The teorical consensous value (to which each node would converge if the sink component would not be periodic) can be computed as before as:

$$\alpha = \pi' x(0)$$

Since π such that $\pi_i \neq 0$ only if $i \in \{3, 4\}$ (nodes belonging to the sink component) and $\pi_3 = \pi_4 = \frac{1}{2}$, α is equals to the mean between the initial state of nodes 3 and 4. The state of each node oscillates between 2 values whose average is α. In particular:

- The state of nodes 3 and 4 oscillates between their initial values
- The state of nodes $0, 1, 2$ oscillates between values close to α. The amplitude of the oscillation depends on the initial states and it is, in general higher for node 2.

For istance, in the first simulation run, $\alpha = 2.522$. Node 4 state oscillate between 2.246 (its initial state) and 2.799 (node 4 initial state) and the same happen for node 3, starting from its initial state. Node 0 state, starting from 1.5, oscillates, after some time istants, between 2.501 and 2.544. Node 1 state, starting from 2.406, oscillates, after some time istants, between 2.480 and 2.565. Node 2 state, starting from 1.333, oscillates, after some time istants, between 2.374 and 2.671.

The trajectory of the different nodes is reported in 7.

If the initial condition $x(0)$ is chosen such that nodes belonging to the periodic sink component (i.e., nodes 3 and 4) have the same initial condition the state nodes converges to consensous value α

$$\alpha = \pi' x(0) = x_3(0) = x_4(0)$$

since $\pi_3 = \pi_4 = \frac{1}{2}$

For istance, in the second simulation run, $\alpha = 0.964$ and each node converges to such value.

The trajectory of the different nodes is reported in 8.

3 Exercise 2

In the same network of Section 1, simulations have been used in order to study multiple particles performing a continuous-time random walk. In particular, the number of particles during simulation is chosen as 100.

The system is simulated from 2 perspectives: a node perspective and a particle perspective.

3.1 Particle perspective

To simulate the system from particle perspective a single, system-wide Poisson clock with rate equals the number of particles has been used. At every tick of the system-wide clock the particle to move is randomly selected. The particle selected is moved to a neighbour node based on the transition probability matrix Q previously defined.

A simulation has been run for 10000 steps (i.e.: transitions made by the particles), with all particles starting from node o.

Figure 9 and 10 show the trajectory of the first 2 particles. Other trajectories for other particles are available in the attached code. (For the sake of clarity, only the first 100 jumps have been considered).

If all the particles start in node a, to compute the average time for a particle to return to node a, 10000 simulations have been run starting from node a. For each simulation the average value, between all particles, of the time instant at which a particle return to the starting node has been computed. The average value of such values between different simulations is 6.745644. The time computed with only 1 particle is 6.75. The difference between these 2 values is 0.0043550.

The 2 times computed are close each other. This can be explained taking into account the following consideration. With only 1 particle, the time a particle has to wait for the next transition (which can also lead to the same node) is a random variable with exponential distribution with rate $w_* = 1$. With multiple particles, the time a general particle has to wait for the next transition (which can also lead to the same node) is a random variable with exponential distribution with rate n, where n is the number of particles. However, a single specific particle has a probability of being extracted as particle which will make the transition equals $1/n$.

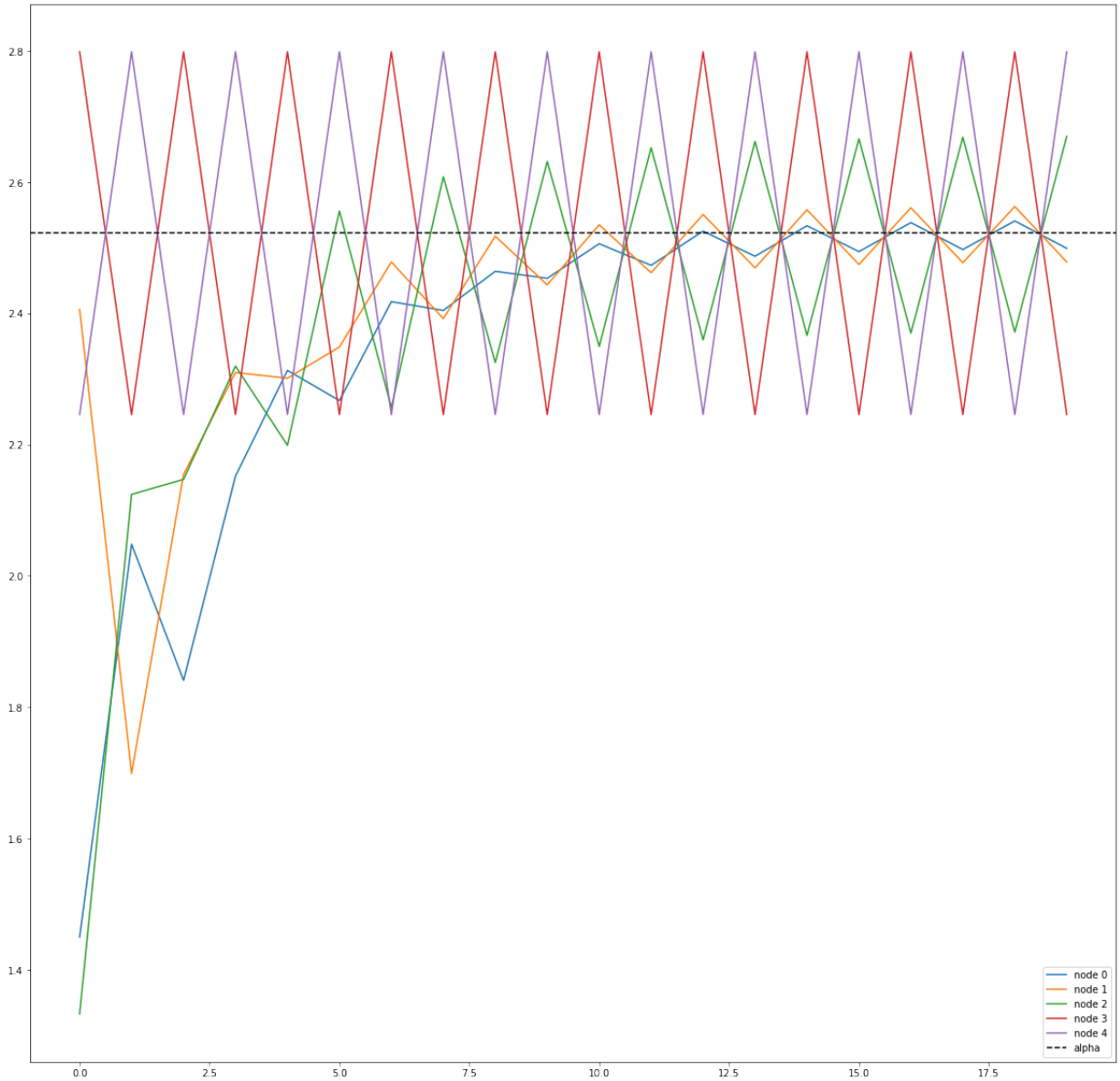

Figure 7: Trajectory for different nodes at different time istants, reported on x axis. y axis shows the value of the state for each node

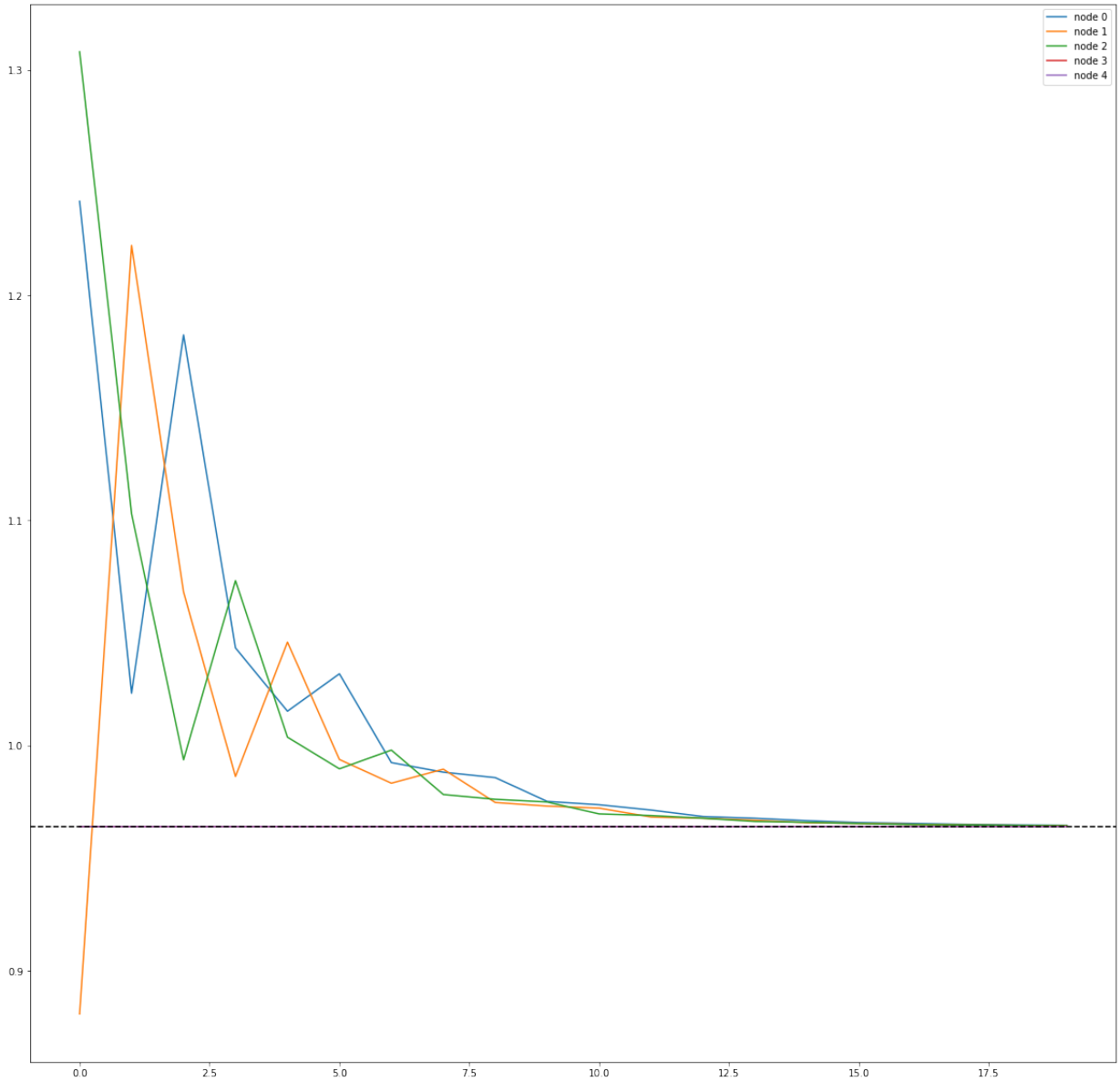

Figure 8: Trajectory for different nodes at different time istants, reported on x axis. y axis shows the value of the state for each node

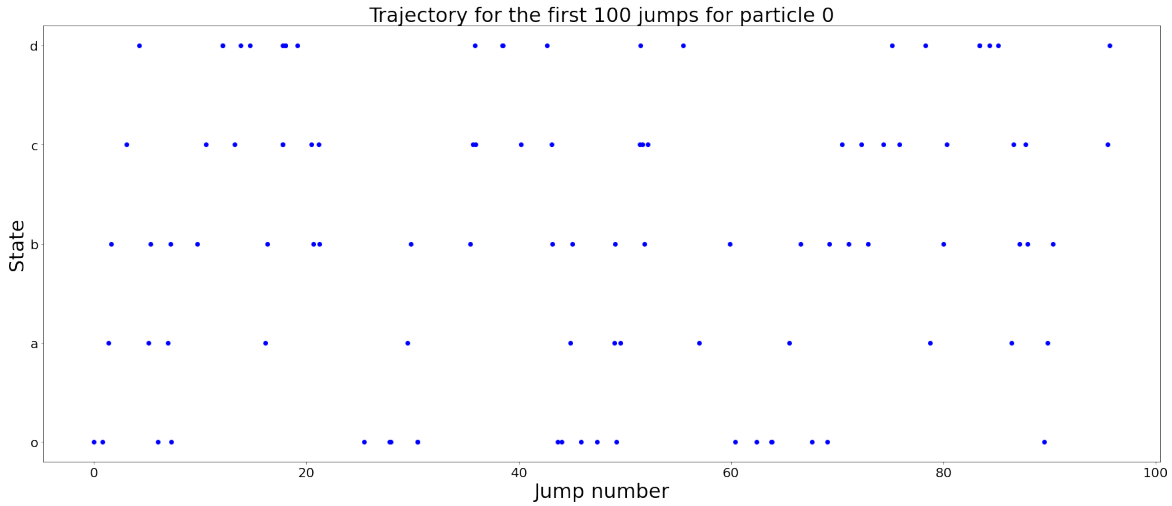

Figure 9: Trajectory for the first 100 jumps (particle 0)

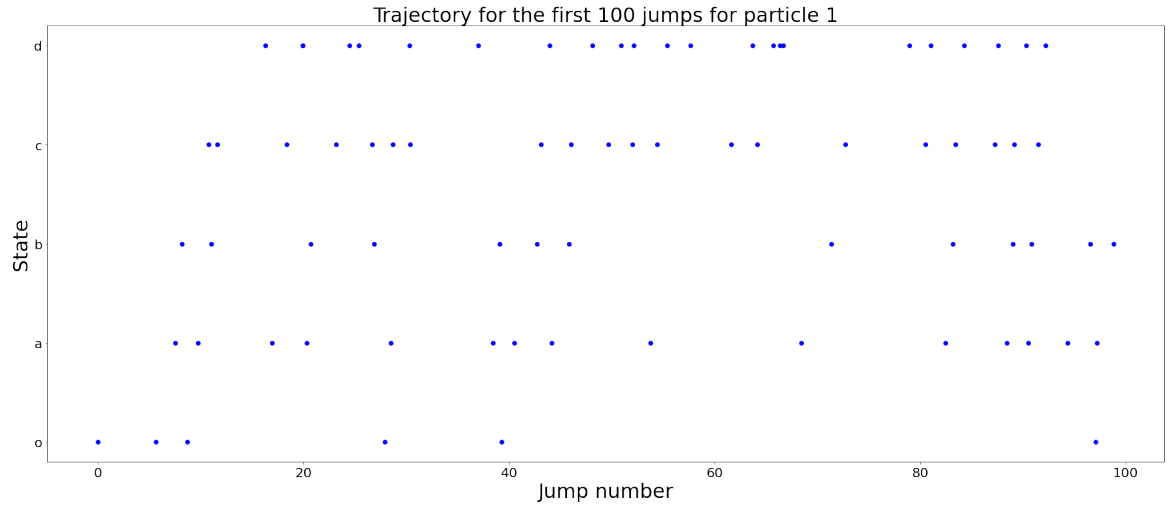

Figure 10: Trajectory for the first 100 jumps (particle 1)

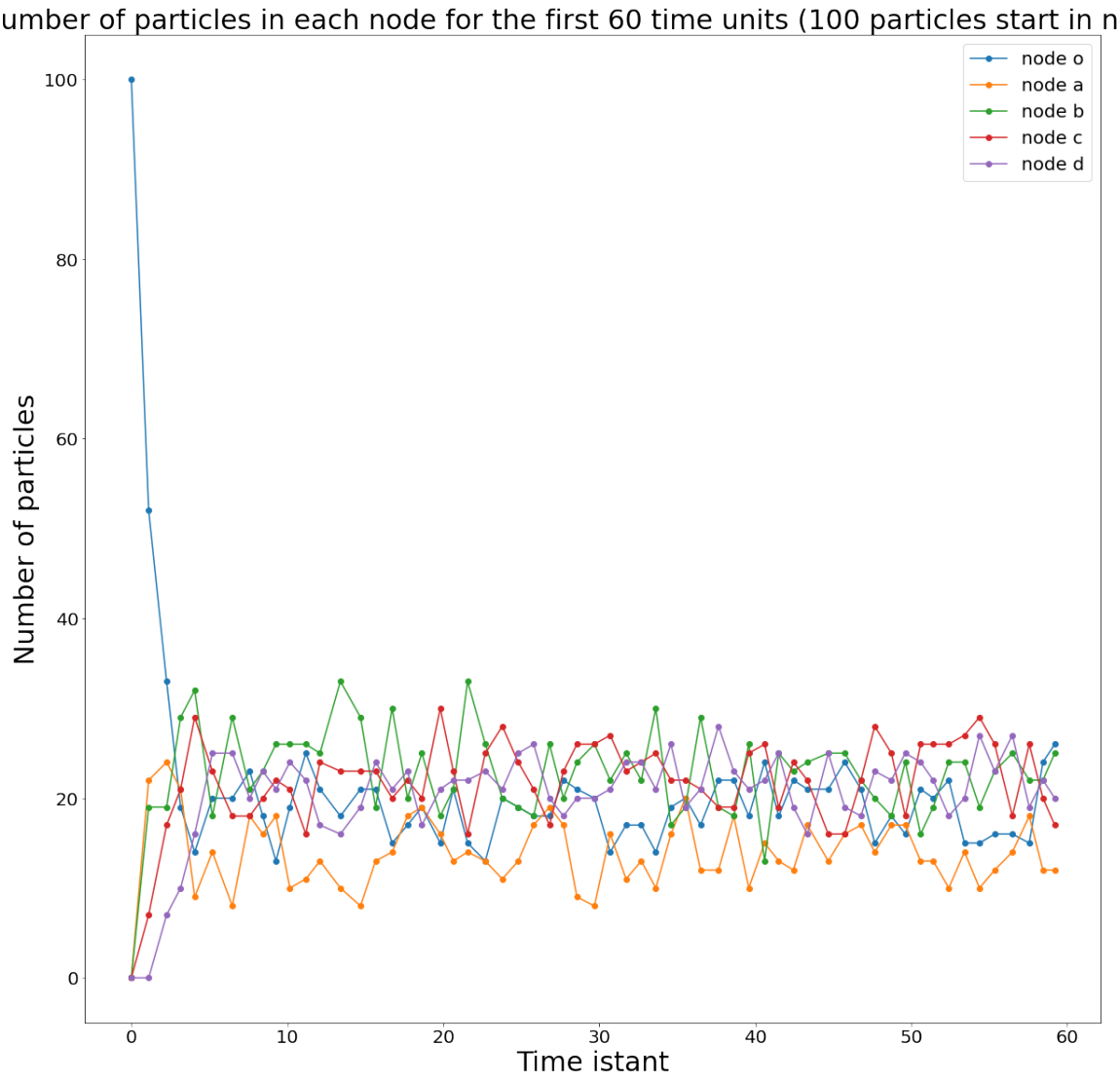

Figure 11: Number of particles in each node for the first 60 time units

3.2 Node perspective

To simulate the system from node perspective a single, system-wide Poisson clock with rate equals the number of particles has been used. At every tick of the system-wide clock, a node is randomly, and proportionally to the number of particles in the different nodes, selected. A particle from the selected node is moved to a neighbour node based on the transition probability matrix Q previously defined.

Thus, if at time t the number of particles in node i is $n_i(t)$, it will pass along particles at a rate of $n_i(t)w_i$, with w previously defined.

A simulation has been run for 60 time units, with all particles starting from node o.

Figure 11 shows the number of particles in each node for the first 60 time units. (For visualization purposes, only 1 point each 100 is plotted).

If all the particles start in node o, to compute the average number of particles in the different nodes after 60 time units, 20000 simulations have been run. The average number of particles, between simulations, in the different nodes after 60 time units is:

$$\begin{array}{cccccc} & o & a & b & c & d \\ p = & (18.5226 & 14.7806 & 22.2456 & 22.2406 & 22.2107) \end{array}$$

The average number of particles in the different nodes after 60 time units, if normalized, p_{norm}

can be compared with the stationary distribution of the continuous-time random walk followed by the single particles previously computed $\bar{\pi}$:

$$\bar{\pi} = \begin{pmatrix} 0.185 & 0.148 & 0.222 & 0.222 & 0.222 \end{pmatrix}$$

The norm of the difference between p_{norm} and $\bar{\pi}$ is $4.7e-4$.

4 Exercise 3

Fig 12 shows an open network $\mathcal{G} = (\mathcal{X}, \Lambda)$ with nodes $\mathcal{X}$ and weight matrix Λ describing possible transitions between nodes/states. $\mathcal{X}$ and Λ are reported below:

$$\Lambda = \begin{array}{c} \begin{matrix} o & a & b & c & d \end{matrix} \\ \begin{pmatrix} 0 & 3/4 & 3/8 & 0 & 0 \\ 0 & 0 & 1/4 & 1/4 & 2/4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \begin{matrix} o \\ a \\ b \\ c \\ d \end{matrix}$$

$$\mathcal{X} = \begin{pmatrix} o & a & b & c & d \end{pmatrix}$$

Particles enter the system at node o according to a Poisson process with input rate $\lambda = 1$ and exit the system from node d.

Simulations have been used in order to study how different particles performing a continuous-time random walk in the network $\mathcal{G}$ with transition rate matrix Λ affect each other.

The system is simulated from a node perspective according to two different scenarios that differ by what rate the nodes will pass along particles: proportional rate and fixed rate.

Let $w = \Lambda \mathbf{1}$ and let $N(t)$ denote the vector of number of particles in each node at time t. Since node d does not have a node to send its particles to, it is assumed that $w_d = 2$. When the Poisson clock ticks for this node, the number of particles in the node is decreased by 1 (if there are any particles in the node).

4.1 Proportional rate

To simulate the system from node perspective according to proportional rate scenario, a system-wide Poisson clock with rate, at time t, equals $N(t) \cdot w$ (i.e., the sum over nodes of the number of particles in node i times the out-degree of the node i) has been used. At every tick of the system-wide clock, a node is randomly, and proportionally to $N_i(t)w_i$ (i.e., the number of particles in node i times the out-degree of the node i, for each node), selected. A particle from the selected node is moved to a neighbour node based on the transition probability matrix P previously defined.

Thus, each node i will pass along particles according to a Poisson process with rate equal to the number of particles in the node times the out-degree of the node, i.e., at a rate of $N_i(t)w_i$.

For what concerns particles entering in node o, an independent Poisson clock with rate equals λ has been used. At every tick of this clock, a particle enters in node o.

A simulation has been run for 60 time units.

Figure 13 shows the number of particles in each node for the first 60 time units (for visualization purposes, only 1 point each 15 is plotted).

4.2 Fixed rate

To simulate the system from node perspective according to fixed rate scenario, a system-wide Poisson clock with rate equals $w \cdot \mathbf{1}'$ (i.e., the sum over nodes of the out-degree of the node i) has been used. At every tick of the system-wide clock, a node is randomly, and proportionally to w_i (i.e., the out-degree

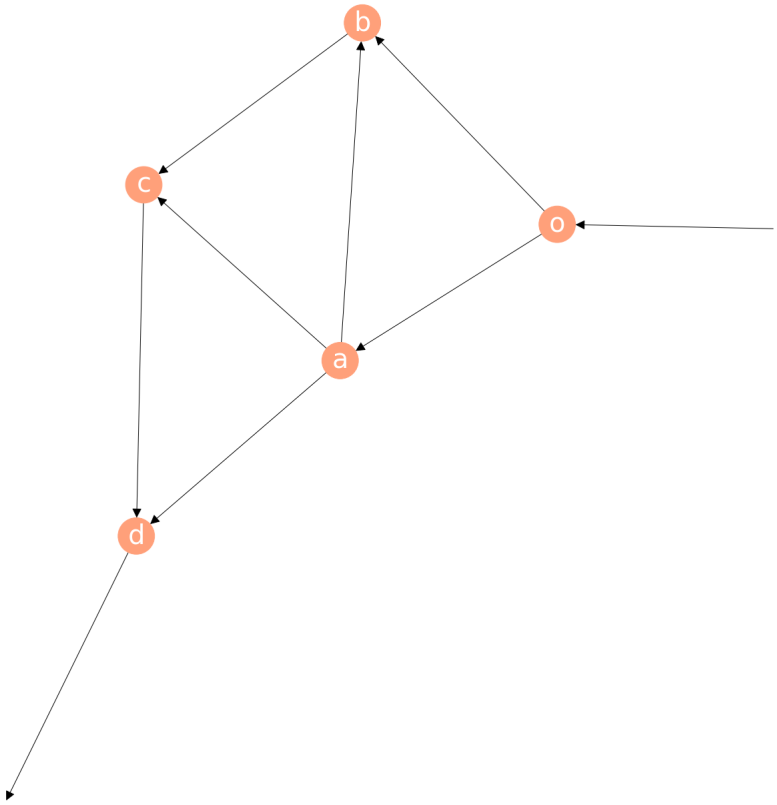

Figure 12: Network of exercise 3

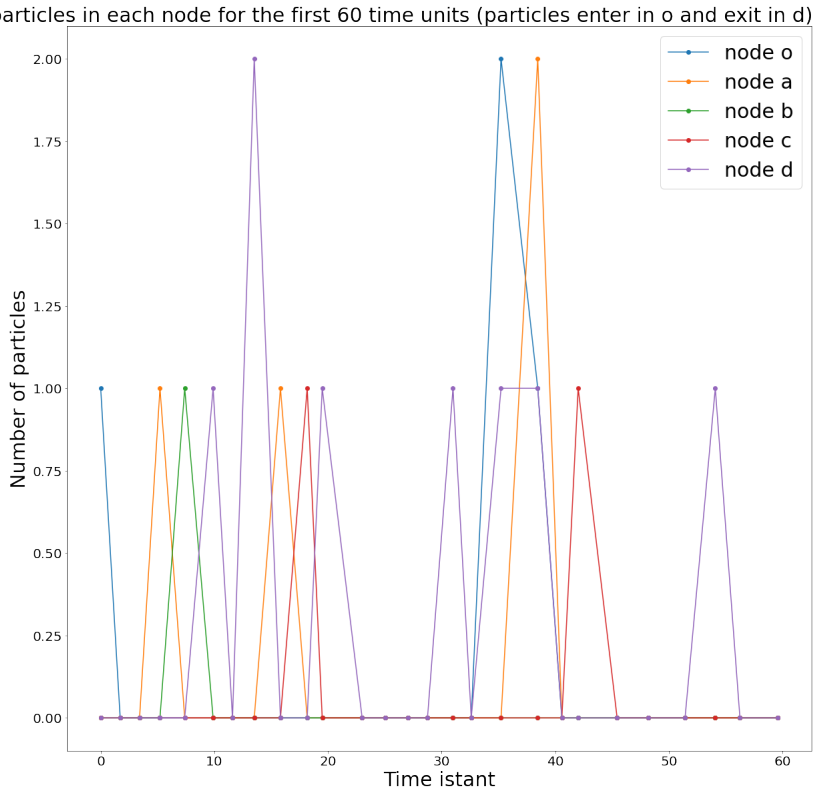

Figure 13: Number of particles in each node at the 60-th time unit instant

of the node i, for each node), selected. A particle from the selected node is moved to a neighbour node based on the transition probability matrix P previously defined.

Thus, each node i will pass along particles according to a Poisson process with fixed rate equal to the out-degree of the node, i.e., at a rate of w_i.

For what concerns particles entering in node o, an independent Poisson clock with rate equals λ has been used. At every tick of this clock, a particle enters in node o.

A simulation has been run for 60 time units.

Figure 14 shows the number of particles in each node for the first 60 time units (for visualization purposes, only 1 point each 15 is plotted).

4.3 Largest input rate

To compute the largest input rate λ the system can handle without blowing up, it must be taken into account the number of particles entering in the open network and exiting from it in a given time interval.

The time instants at which a particle enter in the network (from node o) are computed according to a Poisson clock with fixed rate λ.

The time instants at which a particle exit from the network (from node d) are determined by the rate with which node d pass along particles. The latter is equals $w_d N_d(t)$ in the proportional rate scenario, and w_d in the fixed rate scenario.

If, on average, the number of particles entering in the network is higher than the number of particles exiting from it, the system will blow up.

To study it, different simulations have been run, each with a different λ, from 0.05 to 3.00, with a step of 0.05, both for the proportional rate scenario and the fixed rate scenario. For each value of λ, 1000 simulations are run and their average value is considered. In each simulation it is computed the average value of the number of particles in the network at a specific time instant t (i.e., the number of particles exited from the network subtracted from the number of particles entered in it from the beginning) between all time instants at which transitions happen, starting from the 30-th time unit

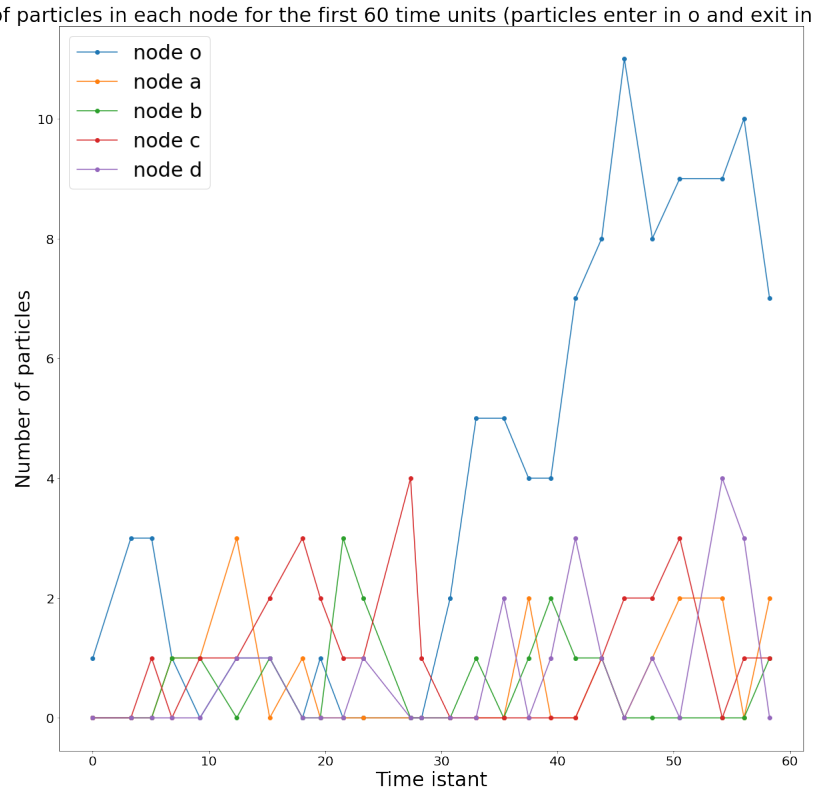

Figure 14: Number of particles in each node at the 60-th time unit instant

instant (i.e., after thousand of transitions).

Figure 15 shows the average number of particles in the network for the corresponding input rate λ in the proportional rate scenario.

Figure 16 shows the average number of particles in the network for the corresponding input rate λ in the fixed rate scenario.

As can be seen from 15, in the proportional rate scenario an input rate approximately equal to 0.65 will avoid system blowing up, since in this case the average number of particles in the network is with high probability lower than 1, meaning that, on average, the number of particles entering in the system and the number of particles exiting are similar. A greater input rate will produce a number of entering particles, on average, greater than the number of exiting ones, leading to system blowing up. From simulation, it can also be seen that the number of particles will increase when t increases for values greater than 60 time units.

From what concerns the fixed rate scenario, as can be seen from 16, in the proportional rate scenario an appropriate input rate is 0.2. From the graph can also be seen how the number of particles increases much faster with respect to the previous case when λ increases.

The difference between the max value of λ in the different cases can be explained by taking into account that, while the rate of the Poisson process determining the particles entering in node o is fixed, the rate of the Poisson process determining the particles exiting from node d is fixed (and equals w_d) only in the fixed rate scenario, while in the proportional rate scenario it also depends on the current number of particles at node d at a given time instant t, $N_d(t)$.

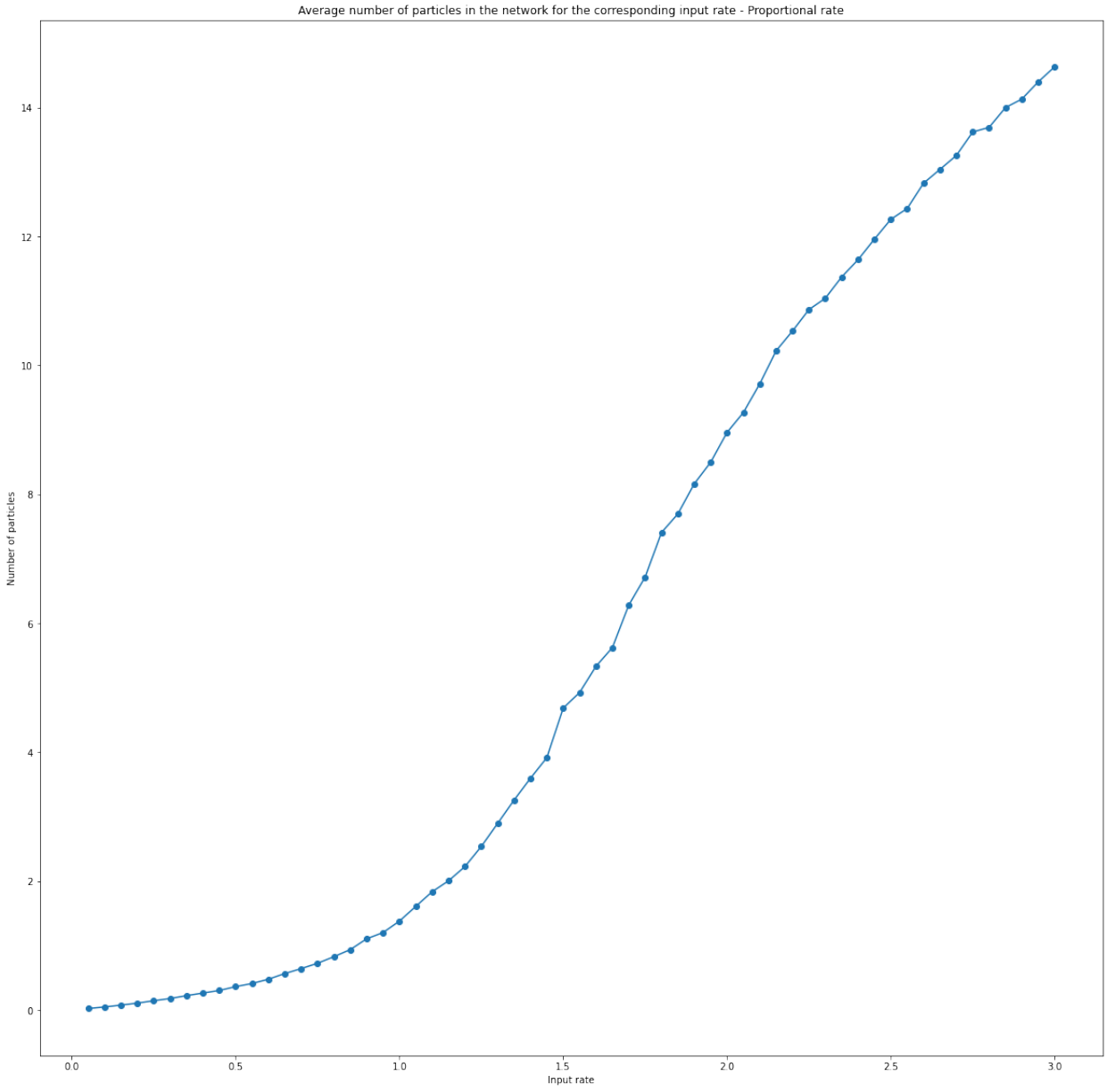

Figure 15: Average number of particles in the network for the corresponding input rate λ

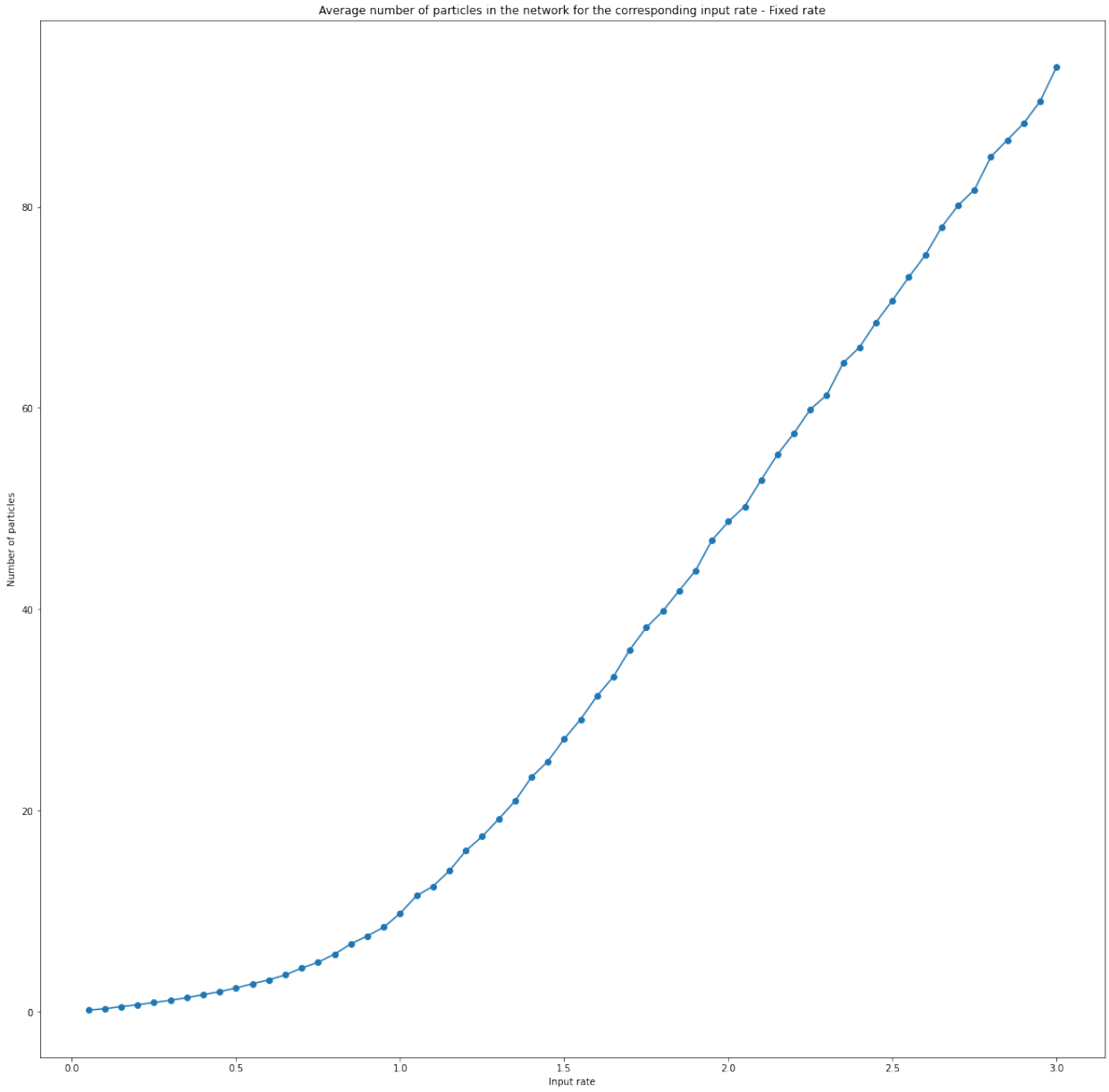

Figure 16: Average number of particles in the network for the corresponding input rate λ