Algorithms and Data Structures

Homework 2, Problem 2

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f is of order at most g, written $f(n)$ is $O(g(n))$ or $f(n) \in O(g(n))$ if and only if there exists a positive real k and nonnegative real number N such that

$$f(n) \leqslant k\, g(n) \quad \text{for all real numbers } n > N.$$

Intuitively, the definition that $f(n)$ is $O(g(n))$ says that $f(n)$ grows slower than some fixed multiple of $g(n)$ as n grows without bound. We can also use $f \in O(g)$ if and only if $\lim\limits_{n \to \infty} \frac{f(n)}{g(n)} = c$ where c is finite.

The constants N and k in the definition above are called witnesses to the relationship $f(n)$ is $O(g(n))$.

① $7n \leqslant n^7$ for all $n > 2$; witnesses are $N = 2$ and $k = 1$. Also

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{7n}{n^7} = 0, \text{ finite. Thus } 7n \in O(n^7) \text{ or}$$

$$f(n) \in O(g(n))$$

(2) $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{n^2}{n \cdot \log_2 n} = \frac{n}{\log_2 n}$ (Then apply L'Hopital's rule) $= \lim_{n\to\infty} \left(\frac{1}{\frac{1}{n \ln 2}} \right) = \lim_{n\to\infty} n \ln(2) = \ln 2 \cdot \lim_{n\to\infty} (n) = \infty$.

Thus, $n^2/\log_2 n \notin O(n)$ but $n \in O(n^2/\log_2 n)$ or $g(n) \in O(f(n))$.

(3) $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{27 \cdot \log_2(n)}{(\log_2(4n))^2} = \frac{27 \log_2(n)}{(\log_2(n)+2)^2} = \frac{27 \log_2(n)}{(\log_2(n))^2 + 4\log_2 n + 4}$

$= 27 \cdot \lim_{n\to\infty} \left(\frac{\log_2(n)}{(\log_2(n))^2 + 4\log_2(n) + 4} \right) = 27 \lim_{n\to\infty} \left(\frac{\frac{1}{\log_2(n)}}{\left(1 + \frac{4}{\log_2(n)} + \frac{4}{(\log_2(n))^2}\right)} \right)$

$= 27 \cdot \frac{\lim_{n\to\infty} \left(\frac{1}{\log_2(n)} \right)}{\lim_{n\to\infty} \left(1 + \frac{4}{\log(n)} + \frac{4}{(\log_2(n))^2} \right)}$

$$\boxed{\begin{aligned} &\lim_{n\to\infty} \left(\frac{1}{\log_2(n)} \right) = 0; \\ &\lim_{n\to\infty} \left(1 + \frac{4}{\log_2(n)} + \frac{4}{(\log_2(n))^2} \right) = 1 \end{aligned}}$$

$= 27 \cdot \frac{0}{1} = 0$. Thus $f(n) \in O(g(n))$.

④ $$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{\sqrt{n}}{5\sqrt{n}+7n^{0.3}+11\log_2 n}$$

$$= \lim_{n\to\infty}\left(\frac{\frac{\sqrt{n}}{\sqrt{n}}\ 1}{5+\frac{7}{n^{0.2}}+\frac{11\log_2 n}{\sqrt{n}}}\right) = \frac{\lim_{n\to\infty}(1)}{\lim_{n\to\infty}\left(5+\frac{7}{n^{0.2}}+\frac{11\log_2 n}{\sqrt{n}}\right)} = \frac{1}{5}, \text{ finite}$$

Thus $f(n)\in O(g(n))$ and also $g(n)\in O(f(n))$