Griffiths Intro to Quantum Mechanics, Self-Study

Selected Solutions for Griffiths' Intro to Quantum Mechanics (3rd)

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Preface

TODO blah blah

Chapter 1

The Wave Equation

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Exercise 1.4
Given positive constants A, a, and a:

$$\Psi(x,0) = \begin{cases} A(x/a), & \text{if } 0 \le x \le a, \\ A(b-x)/(b-a), & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

- Normalize Ψ.

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \; dx = 1$$

$$A^2 \left(\int_0^a \frac{x^2}{a^2} \; dx + \int_a^b \frac{(b-x)^2}{(b-a)^2} \; dx \right) = 1$$

$$A^2 \left(\frac{a}{3} + \frac{b-a}{3} \right) = 1 \implies A = \sqrt[2]{\frac{3}{b}}$$

$$\Psi(x,0) = \begin{cases} \sqrt[2]{\frac{3}{b}}(x/a), & \text{if } 0 \le x \le a, \\ \sqrt[2]{\frac{3}{b}}(b-x)/(b-a), & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

- Where is particle most likely to be found at $t = 0$? Based on plots, you will see it is most likely at position a.

- Probablility of finding particle to the left of a? Check with b = a and b = 2a.

$$\int_0^a \left| \sqrt[2]{\frac{3}{b}}(x/a) \right|^2 \; dx$$

$$\int_0^a \frac{3}{b}(x^2/a^2) \; dx = \frac{a}{b}$$

- What is the first moment (expected value) of x?

$$\langle x \rangle = \int_0^b x\Psi(x,t)dx = \int_0^a \sqrt[2]{\frac{3}{b}}(x/a) \; dx + \int_a^b \sqrt[2]{\frac{3}{b}}(b-x)/(b-a) \; dx$$
$$= \frac{b+2a}{4}$$

Exercise 1.5MOD
Given positive, real constants A, λ, ω:

$$\Psi(x,t) = Ae^{-\lambda|x| - i\omega t}$$

- Normalize Ψ.

$$\begin{aligned}
\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \; dx &= \int_{-\infty}^{\infty} \Psi^*\Psi = 1 \\
\int_{-\infty}^{\infty} A^2 e^{-2\lambda|x|} e^{-i\omega t} e^{i\omega t} \; dx &= 1 \\
A^2 \int_{-\infty}^{\infty} e^{-2\lambda|x|} \; dx &= 1 \\
A^2 \left(\int_{-\infty}^{0} e^{-2\lambda\cdot(-x)} \; dx + \int_{0}^{\infty} e^{-2\lambda\cdot(x)} \right) &= 1 \\
A^2 \left(\frac{1}{2\lambda} e^{2\lambda x} \right)\Bigg|_{-\infty}^{0} - A^2 \left(\frac{1}{2\lambda} e^{-2\lambda x} \right)\Bigg|_{0}^{\infty} &= 1 \\
\frac{A^2}{\lambda} = 1 \implies A &= \sqrt[2]{\lambda} \\
\Psi(x,t) &= \sqrt{\lambda} e^{-\lambda|x| - i\omega t} \\
|\Psi(x,t)|^2 = \Psi^*\Psi &= \lambda e^{-\lambda|x| - i\omega t} e^{-\lambda|x| + i\omega t} = \lambda e^{-2\lambda|x|}
\end{aligned}$$

- Find the n^{th} moment.

$$\begin{aligned}
\langle x^n \rangle &= \int_{\mathbb{R}} x^n \lambda e^{-2\lambda|x|} \; dx \\
&= \lambda \left(\int_{-\infty}^{0} x^n e^{2\lambda x} + \int_{0}^{\infty} x^n e^{-2\lambda x} \right) \; dx
\end{aligned}$$

Now note the following is smells like the gamma function,

$$I_{n_1} = \int_{-\infty}^{0} x^n e^{2\lambda x} \; dx = -\int_{0}^{-\infty} x^n e^{2\lambda x} \; dx$$

By using substitution of the type $u = -2\lambda x$ we get,

$$I_{n_1} = \frac{-1}{2\lambda(-2\lambda)^n} \int_{0}^{\infty} e^{-u} u^n du = \frac{(-1)^n}{(2\lambda)^{n+1}} \Gamma(n+1), \;\; \mathrm{Re}(n) > -1$$

where the last equation can be used to show the required base case of $I_0 = \frac{1}{2\lambda}$. A similar analysis for the second integrand gives us the combined relation

$$\begin{aligned}
\langle x^n \rangle &= \lambda I_n = \lambda (I_{n_1} + I_{n_2}) \\
&= \lambda \left(\frac{(-1)^n + 1}{(2\lambda)^{n+1}} \right) \Gamma(n+1)
\end{aligned}$$

For practical purposes, we see that the first few moments give

$$\begin{aligned}
\langle x \rangle &= 0 \\
\langle x^2 \rangle &= \frac{2\lambda}{8\lambda^3} \Gamma(3) = \frac{1}{2\lambda^2} \\
\langle x^3 \rangle &= 0 \\
\langle x^4 \rangle &= \frac{2\lambda}{32\lambda^5} \Gamma(4) = \frac{3}{8\lambda^4}
\end{aligned}$$

- Find standard deviation. Compute probability particle is outside one standard deviation from the mean.

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 \implies \sigma = \frac{1}{\lambda\sqrt{2}}$$
$$|\Psi(0 \pm \sigma, t)|^2 = |A|^2 e^{-2\lambda\sigma} = \lambda e^{-\sqrt{2}}$$
$$P_{outside} = 1 - P_{inside} = 1 - \int_{-\sigma}^{\sigma} |\Psi|^2 \; dx = 1 - |A|^2 \int_{-\sigma}^{\sigma} e^{-2\lambda|x|} \; dx = 2\lambda \int_{\sigma}^{\infty} e^{-2\lambda x} \; dx = e^{-\sqrt{2}}$$

Exercise 1.6
Why can't you do integration-by-parts (IBP) directly in the middle expression of Equation 1.29 – pull the time derivative over into x, note that $\frac{\partial x}{\partial t} = 0$, and conclude that $\frac{\langle x \rangle}{dx} = 0$?

Well, you could but this would not allow us to do IBP over some domain D:

$$\frac{\partial x|\Psi|^2}{\partial t} = \frac{\partial x}{\partial t}|\Psi|^2 + x\frac{\partial|\Psi|^2}{\partial t} = x\frac{\partial|\Psi|^2}{\partial t}$$
$$\int_{\partial D} x\frac{\partial|\Psi|^2}{\partial t}dx = \int_{\partial D} \frac{\partial(x|\Psi|^2)}{\partial t}dx \neq (x|\Psi|^2)|_{\partial D}$$

Exercise 1.7
Calculate $\frac{d\langle p \rangle}{dt}$.

By Ehrenfest's theorem, expectation values are goverened by classical laws: $\langle p \rangle = m\langle v \rangle = m\frac{d\langle x \rangle}{dt}$. Recall the time derivatives for the conjugate pairs or derive it yourself. Also note that interchange of differentiaiton to integration (Leibnitz integral rule) implicitly assumes the (wave) function and its first partial derivative are continuous in time and space (both) in the open neighborhood of $\{x\} \times [a, b]$ for any continuous and differentiable functions a, b. Text assumes all partials continuous, and by extent differentiable (converse not necessarily true).

$$\frac{\partial \Psi^* \frac{\partial \Psi}{\partial x}}{\partial t} =$$