Forced harmonic oscillator

$$q - \omega_0^2\,\Delta = \ddot{\Delta}$$

$$\begin{cases} \Delta(t=0) = 0 \\ \dot{\Delta}(t=0) = 0 \end{cases}$$

Initial condition

$$\Downarrow$$

$$\Delta(t) = \frac{q}{\omega_0^2} + C_1\,\cos(\omega_0\,t) + C_2\,\sin(\omega_0\,t)$$

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$$\dot{\Delta}(t=0) = C_2\,\omega_0 \Rightarrow C_2 = 0$$

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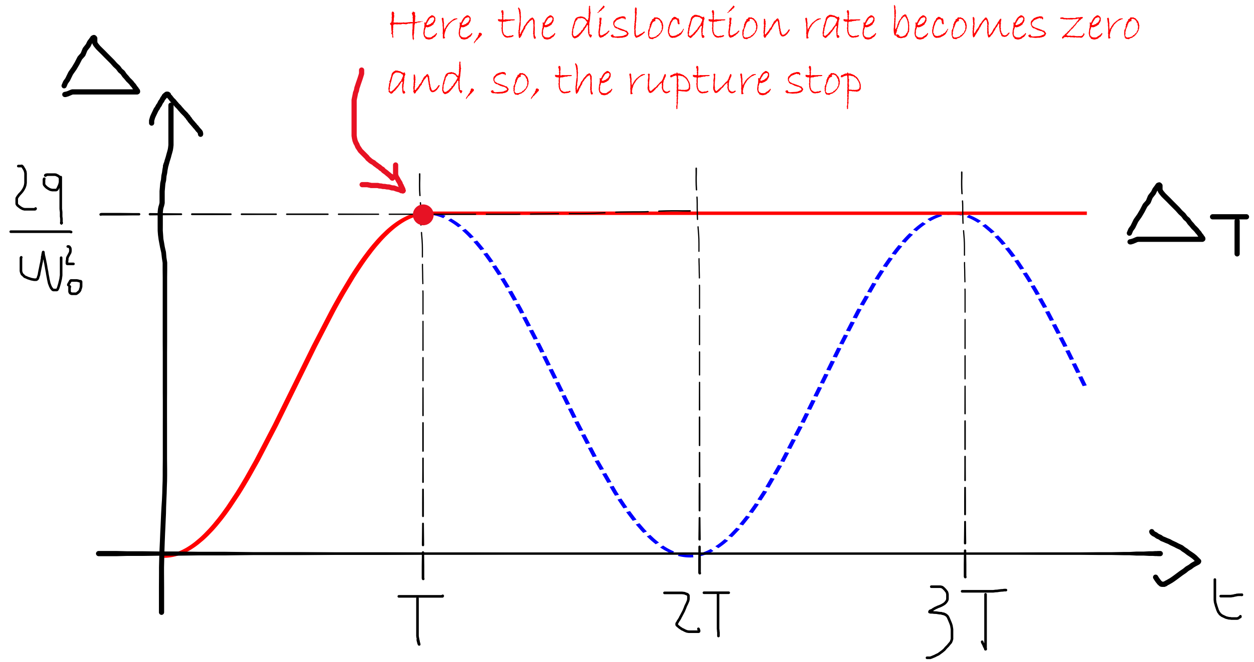

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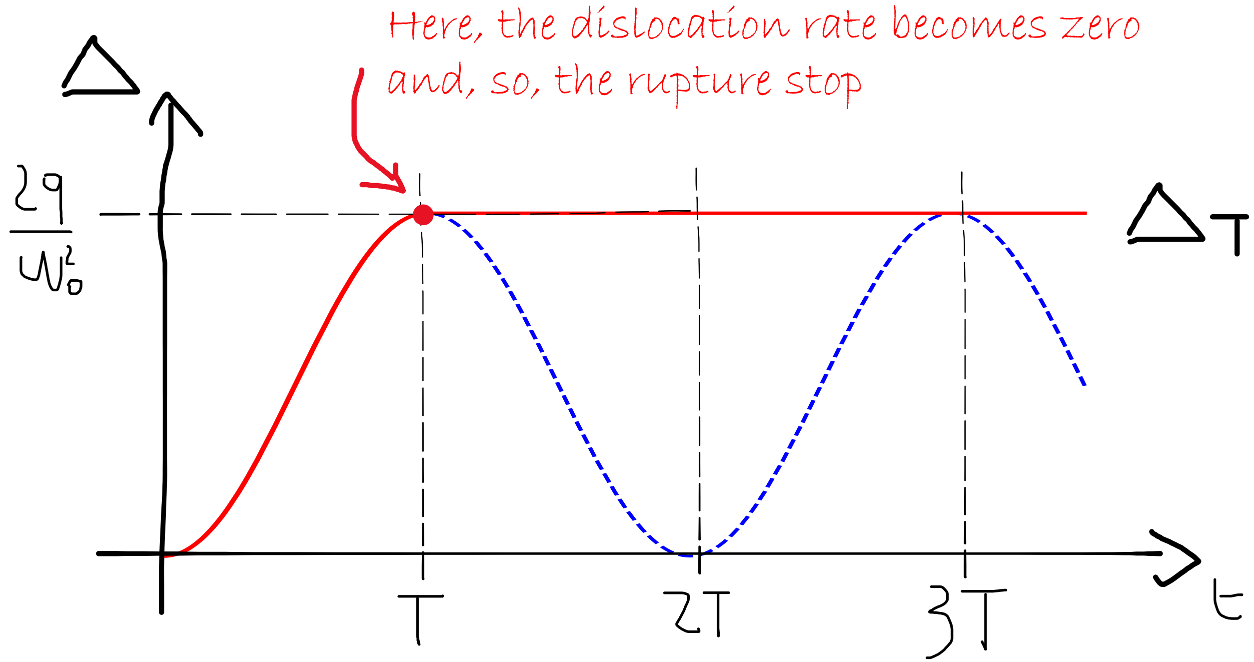
$$\dot{\Delta}(t) = \frac{q}{\omega_0}\,\sin(\omega_0\,t)$$

$$\dot{\Delta}(t=T) = 0$$

$$T = \frac{\pi}{\omega_0} = \frac{\pi}{\sqrt{2}}\,\frac{\sqrt{A}}{\beta}$$

Rupture time



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$$\dot{\Delta}(t=T) = 0$$

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Rupture time

Total slip

$$\Delta_T = \Delta(t=T) = \frac{2\,q}{\omega_0^2} = \frac{4\sqrt{A}\,p\,(f_S - f_D)}{\mu}$$

Forced harmonic oscillator

$$q - \omega_0^2\,\Delta = \ddot{\Delta}$$

$$\begin{cases} \Delta(t=0) = 0 \\ \dot{\Delta}(t=0) = 0 \end{cases}$$

Initial condition

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Rupture time

Total slip

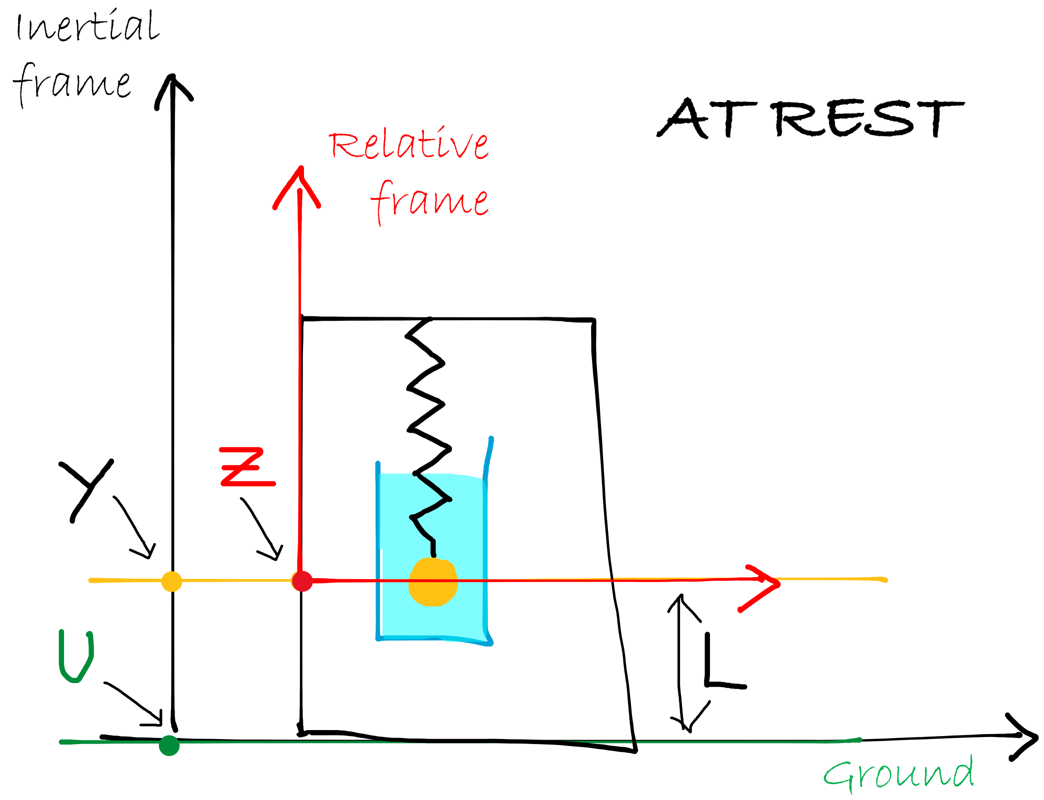
$$\Delta_T = \Delta(t=T) = \frac{2\,q}{\omega_0^2} = \frac{4\sqrt{A}\,p\,(f_S - f_D)}{\mu}$$

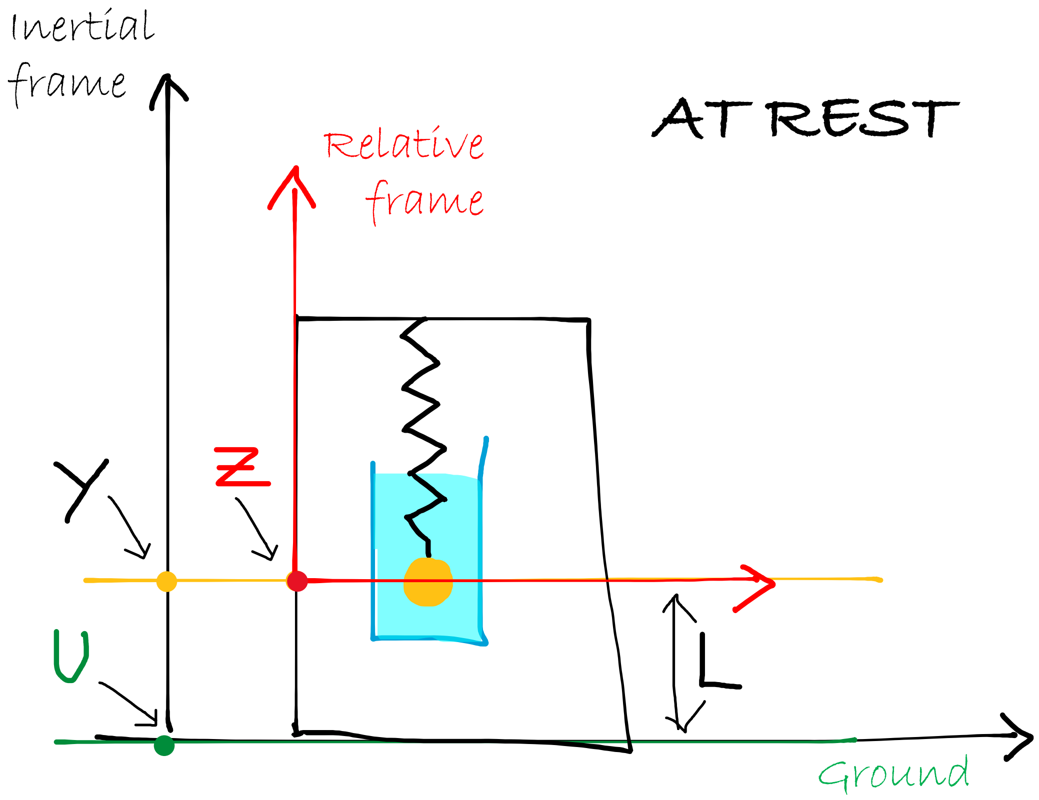
Stress drop

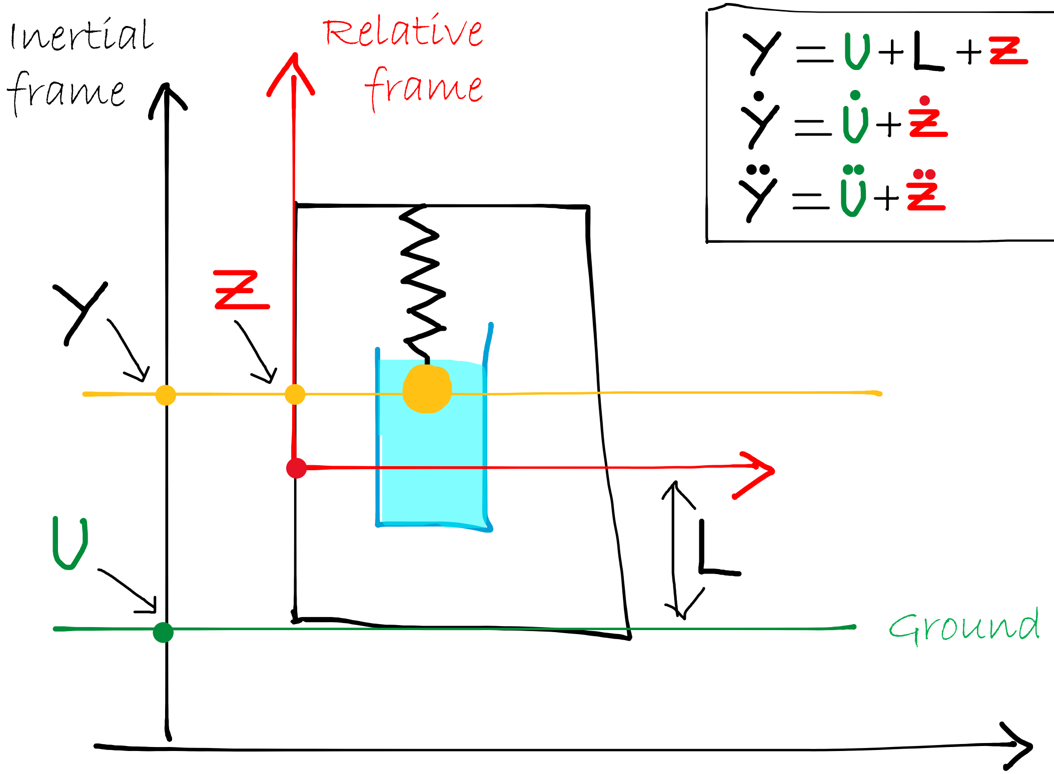
$$\delta t_{\parallel} = t_{\parallel}(t=T) - t_{\parallel}(t=0) = -\frac{\mu\,\Delta_T}{b} = -2\,p\,(f_S - f_D)$$

Seismic moment

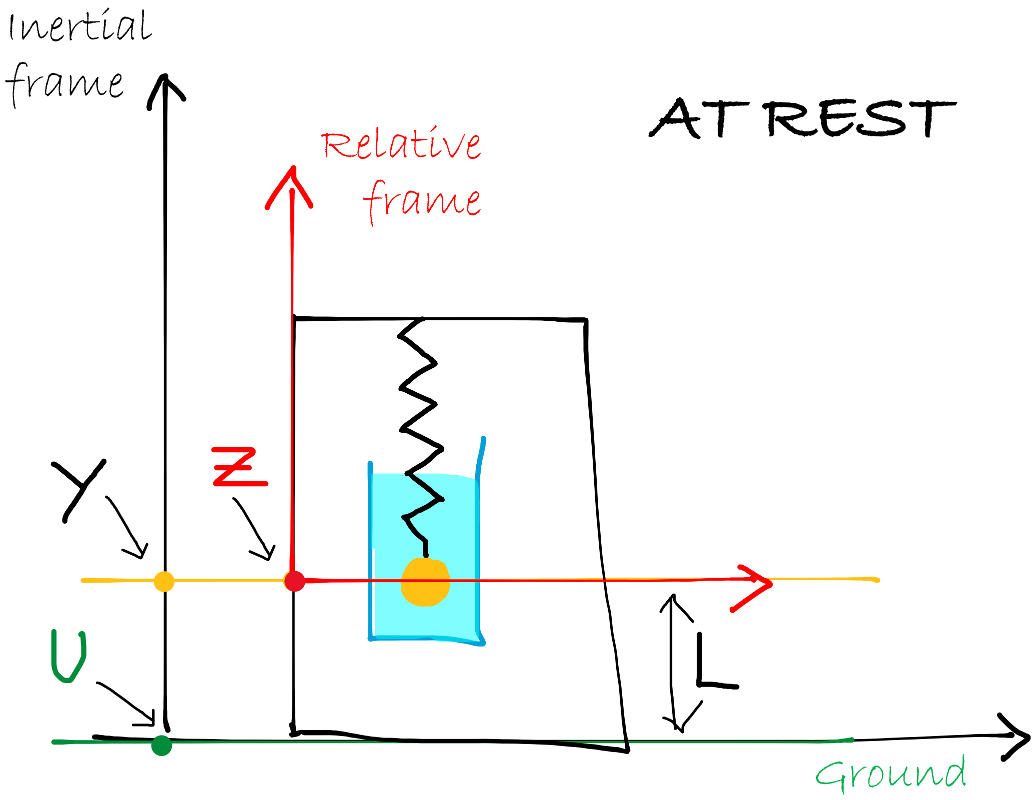
$$M = \mu\,A\,\Delta_T = 4\,A^{\frac{3}{2}}\,p\,(f_S - f_D)$$

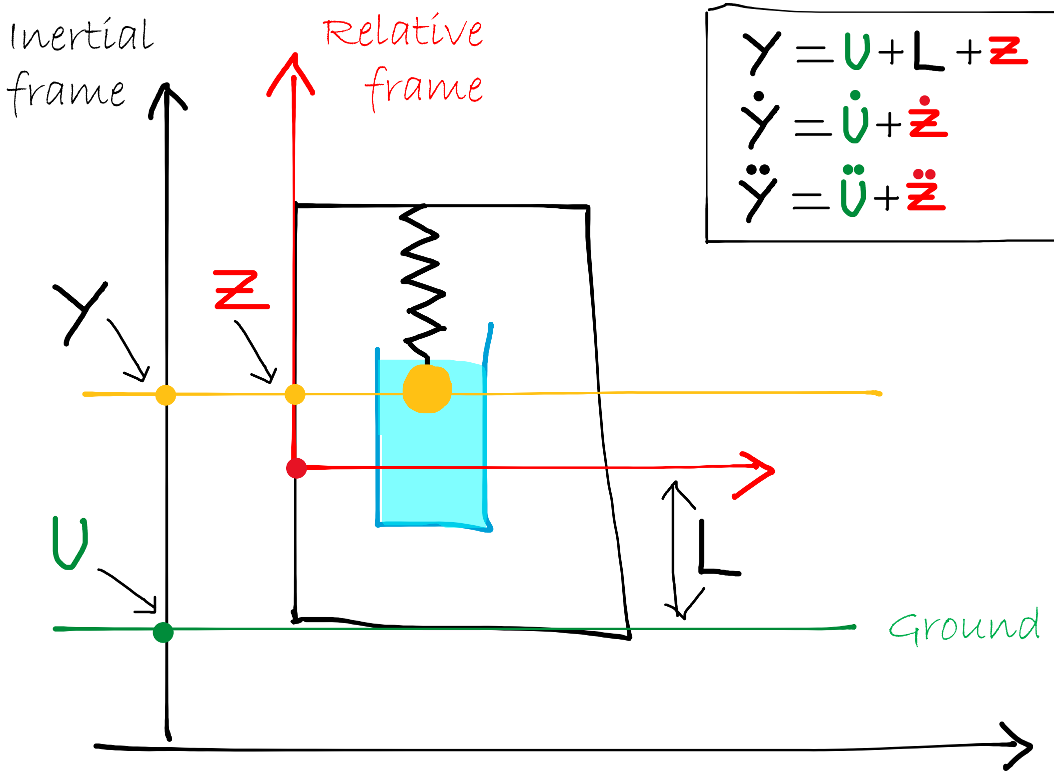

Inertial frame
AT REST
Relative frame
Y
Z
U
L
Ground





$$Y = U + L + Z$$
$$\dot{Y} = \dot{U} + \dot{Z}$$
$$\ddot{Y} = \ddot{U} + \ddot{Z}$$



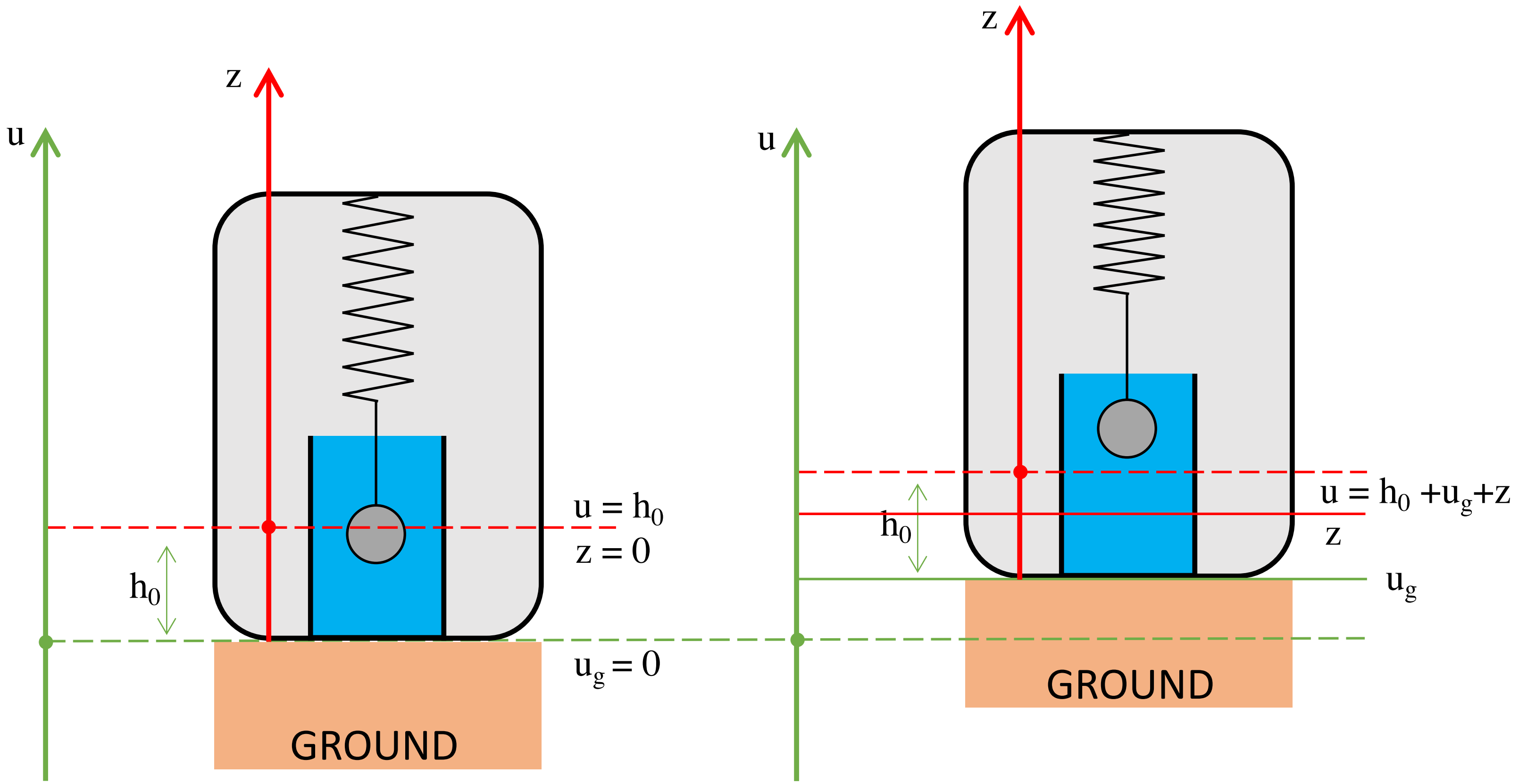


$Y = U + L + Z$
$\dot{Y} = \dot{U} + \dot{Z}$
$\ddot{Y} = \ddot{U} + \ddot{Z}$

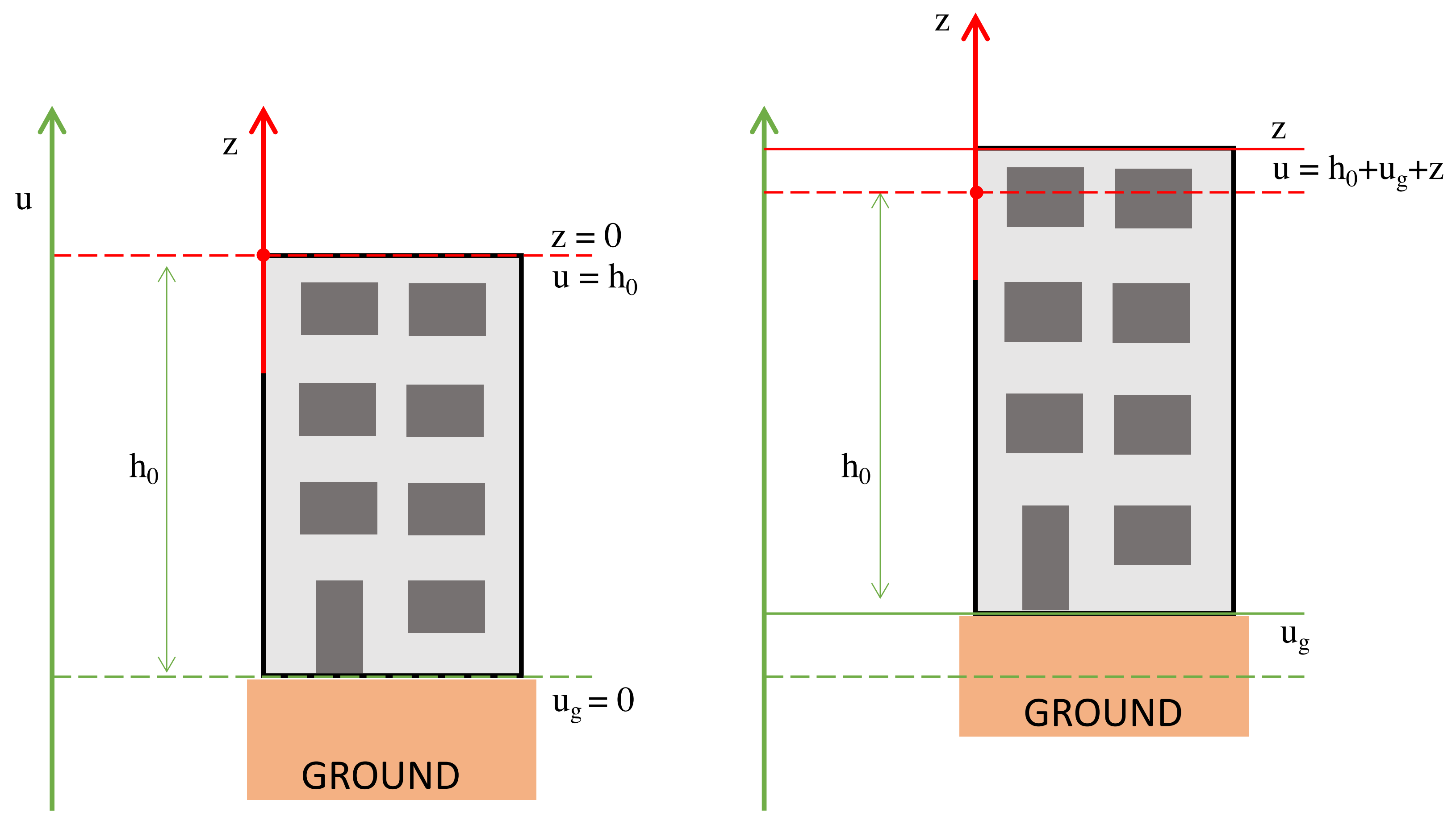
TOTAL FORCE

$$-k\,z - c\,\dot{z} = m\,\ddot{y}$$

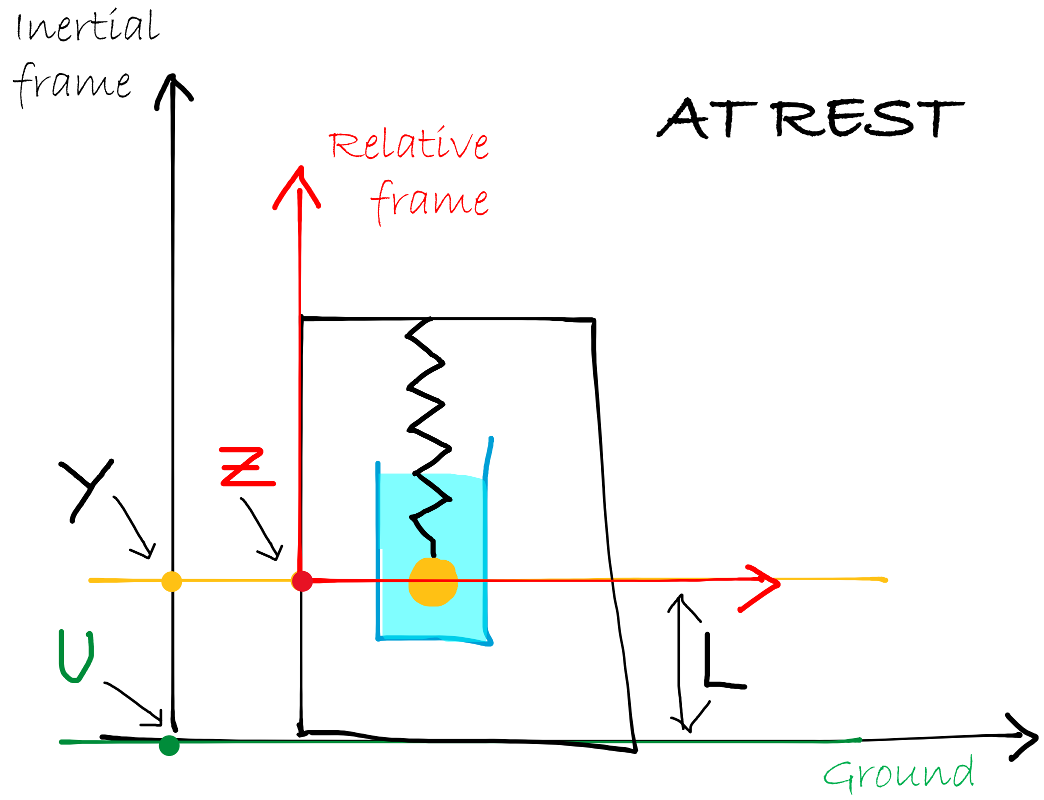
spring dash pot

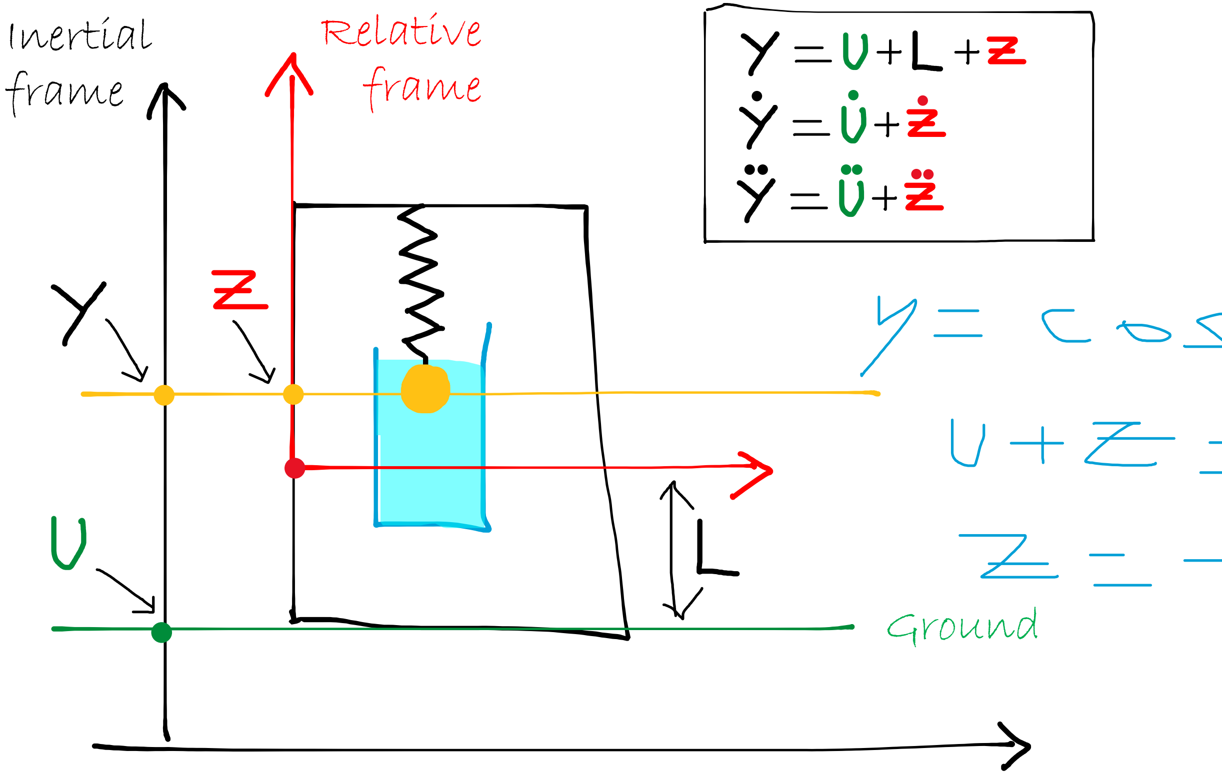


u
z
u = h0
z = 0
h0
ug = 0
GROUND
z
u
u = h0 +ug+z
z
h0
ug
GROUND


u
z
z = 0
u = h0
h0
ug = 0
GROUND
z
z
u = h0+ug+z
h0
ug
GROUND

AT REST





$$Y = U + L + Z$$
$$\dot{Y} = \dot{U} + \dot{Z}$$
$$\ddot{Y} = \ddot{U} + \ddot{Z}$$

TOTAL FORCE

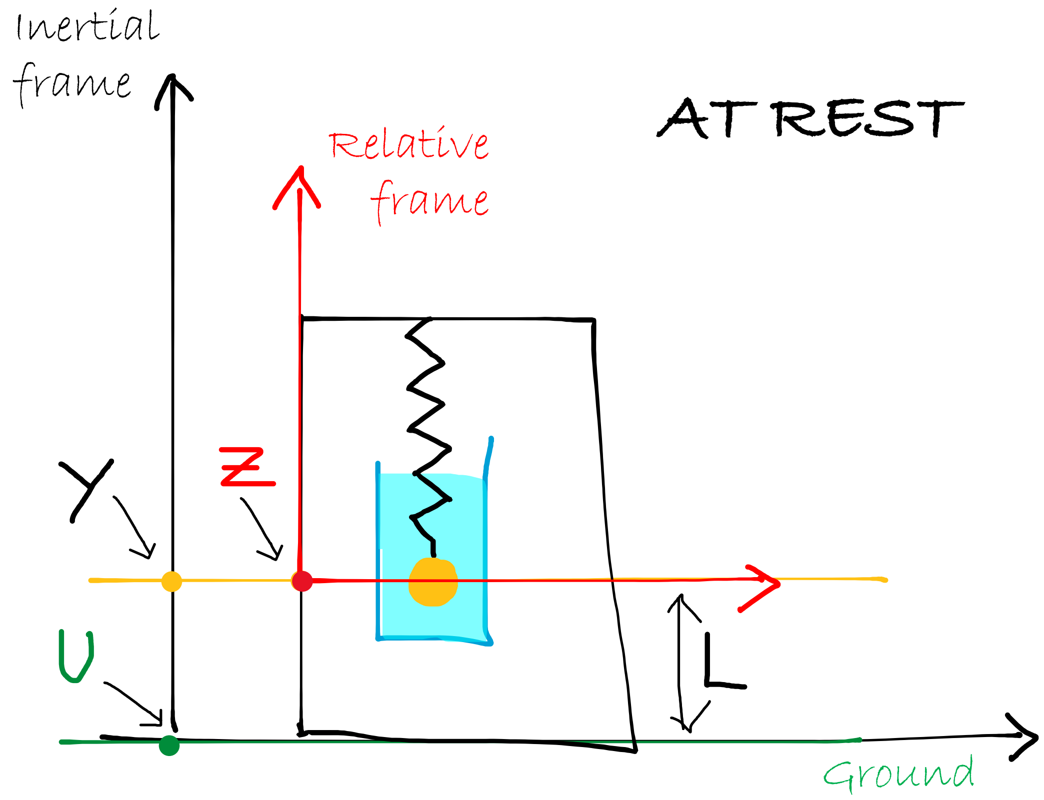
$$-\underbrace{k\,z}_{\text{spring}} - \underbrace{c\,\dot{z}}_{\text{dash pot}} = m\,\ddot{y} \quad \Rightarrow \quad -k\,z - c\,\dot{z} = m\,(\ddot{u} + \ddot{z})$$

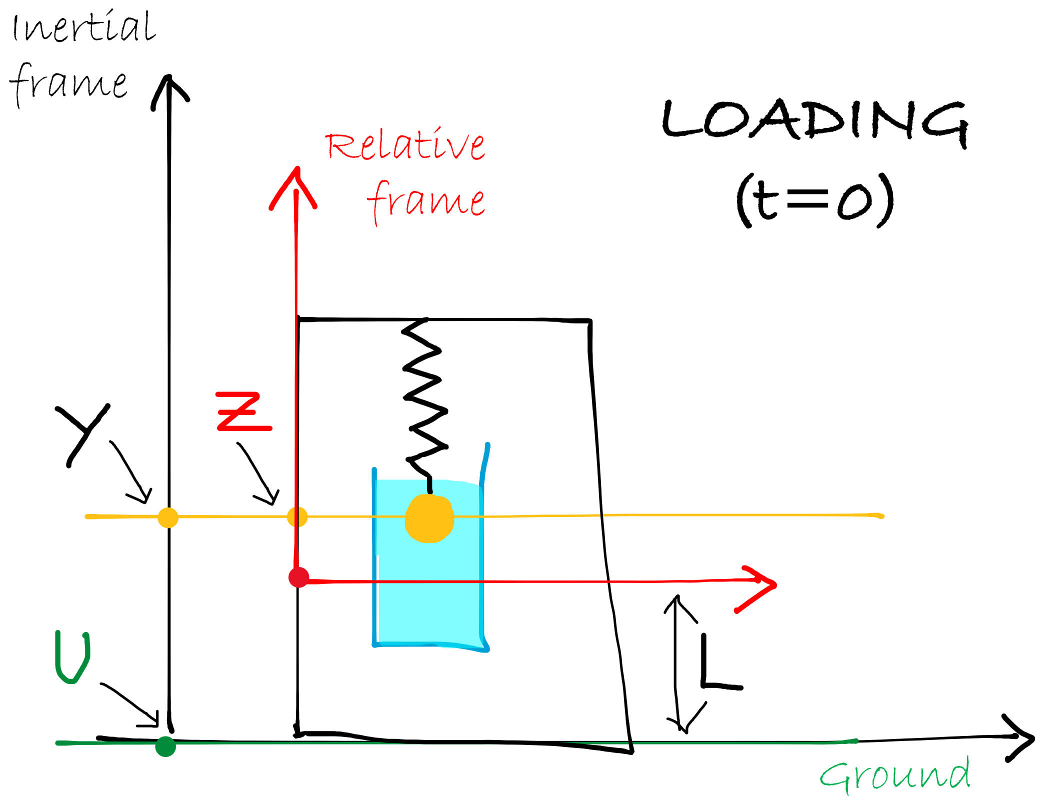
spring constant (k), damping coefficient (c), mass (m), forcing ($\ddot{u}$)

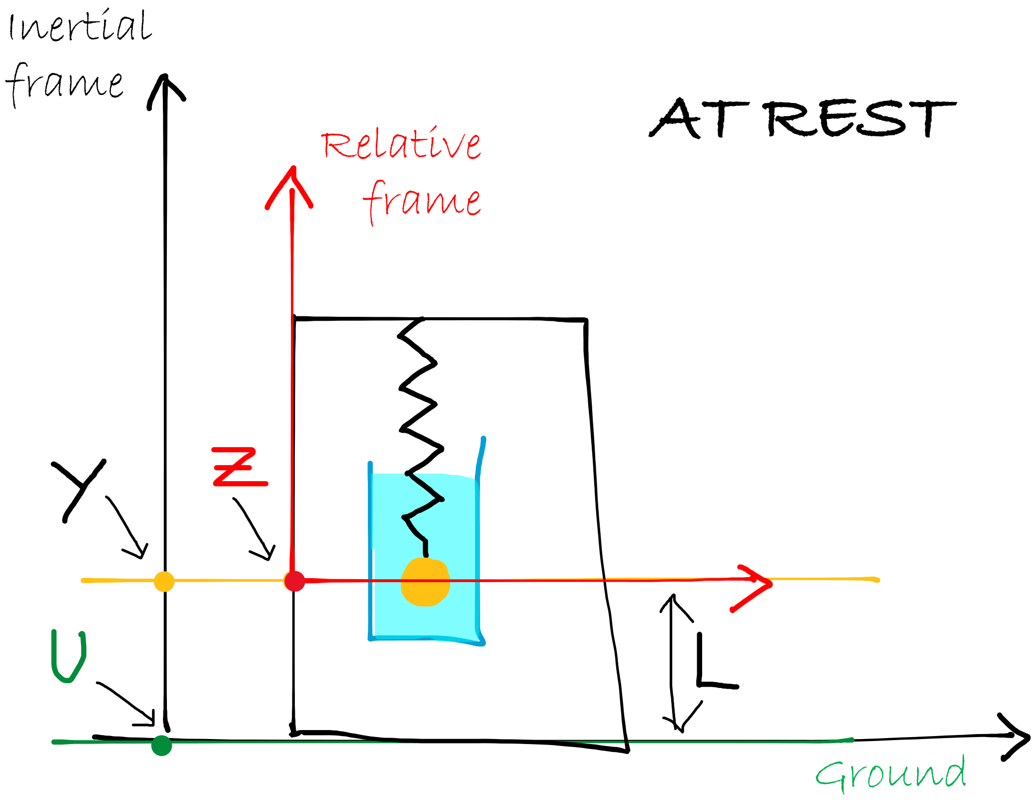
Forced and damped harmonic oscillator

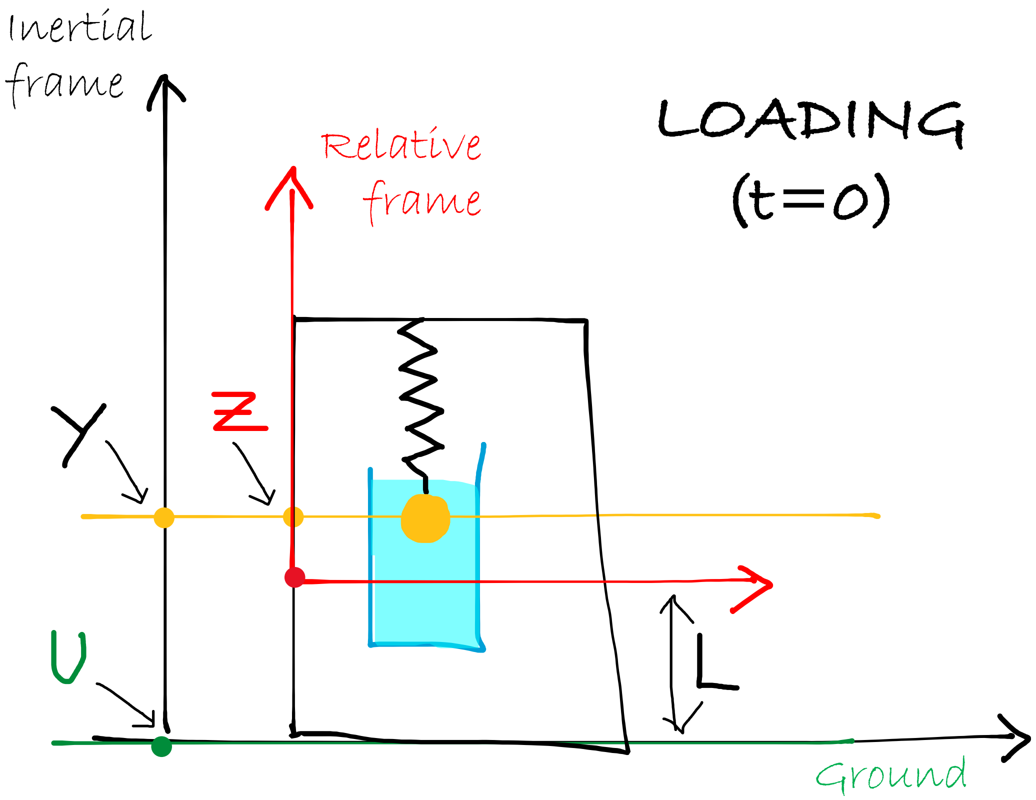
natural angular frequency $\omega_0 = \sqrt{\dfrac{k}{m}}$ damping ratio $\xi = \dfrac{c}{2\,\omega_0\,m}$

$$\Rightarrow \quad \boxed{-\omega_0^2\,z - 2\,\omega_0\,\xi\,\dot{z} = \ddot{z} + \ddot{u}}$$


Inertial frame
AT REST
Relative frame
Y
Z
U
L
Ground


Inertial frame
LOADING (t=0)
Relative frame
Y
Z
U
L
Ground

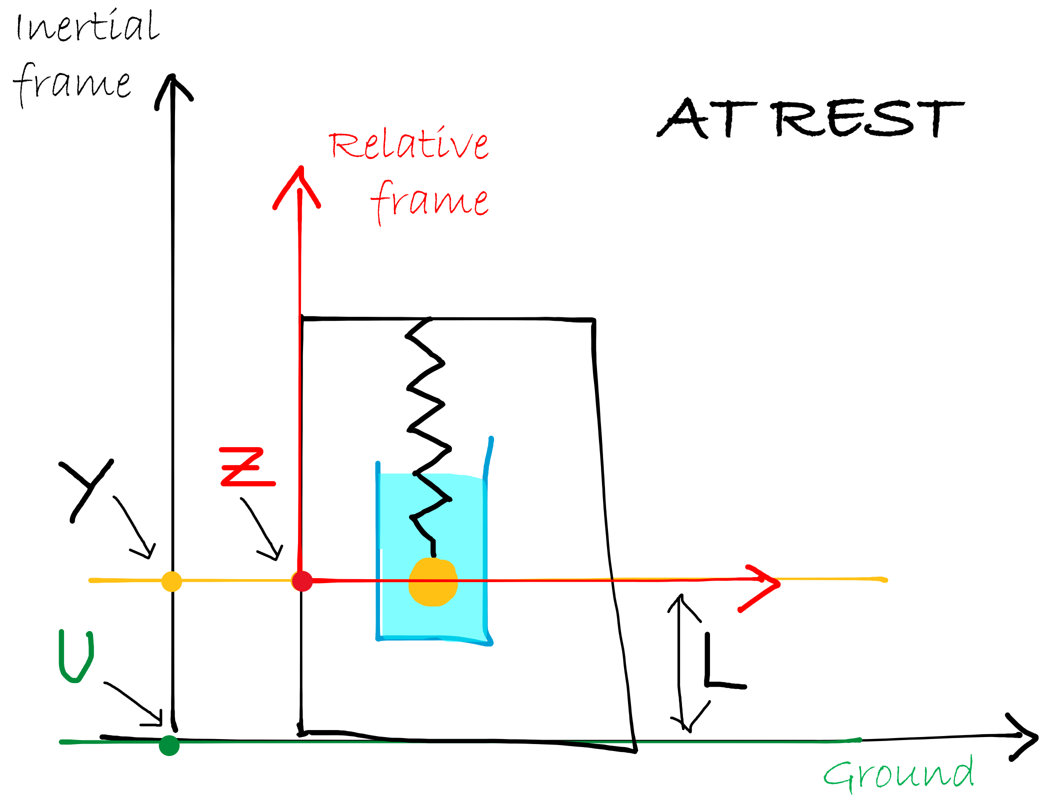


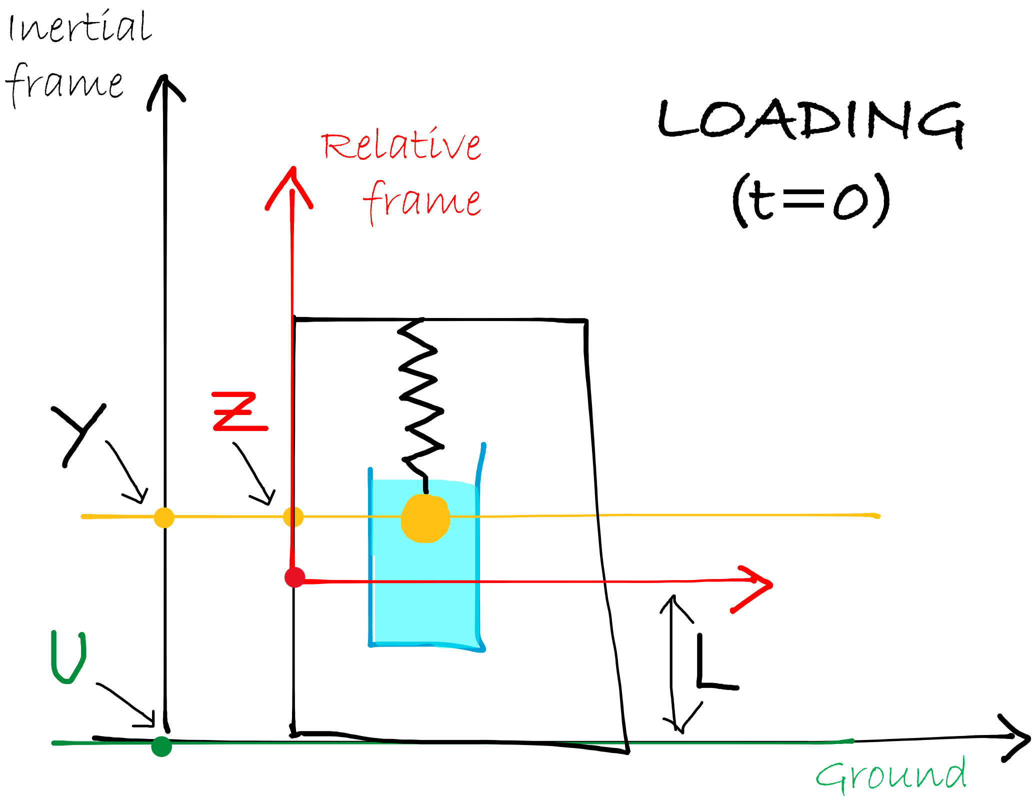


Damped harmonic oscillator

$$-\omega_0^2\, z - 2\,\omega_0\,\xi\,\dot{z} = \ddot{z}$$

Initial conditons
$$\begin{cases} z(0) = z_0 \\ \dot{z}(0) = 0 \end{cases}$$





Damped harmonic oscillator

$$-\omega_0^2\, z - 2\,\omega_0\,\xi\,\dot{z} = \ddot{z}$$

Initial conditons

$$\begin{cases} z(0) = z_0 \\ \dot{z}(0) = 0 \end{cases}$$

solution

$$z = e^{-\omega_0\,\xi\,t}\left(C_+\, e^{\omega_0\,\sqrt{1-\xi^2}\,t} + C_-\, e^{-\omega_0\,\sqrt{1-\xi^2}\,t}\right)$$

... we have still to determine the integration constants (using the initial conditions)

Initial conditons $\begin{cases} z(0) = z_0 \\ \dot{z}(0) = 0 \end{cases}$

Solution

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$$\omega_0\,\sqrt{\xi^2-1} = i\,\omega_d$$

$$\boxed{\omega_d = \omega_0\,\sqrt{1-\xi^2}}$$

$\Rightarrow$

$$z = e^{-\omega_0\,\xi\,t}\,\left(C_+\,e^{i\,\omega_d\,t} + C_+\,e^{-i\,\omega_d\,t}\right)$$

$$z = e^{-\omega_0\,\xi\,t}\,\left(C_c\,\cos(\omega_d\,t) + C_s\,\sin(\omega_d\,t)\right)$$

Initial conditons $\begin{cases} z(0) = z_0 \\ \dot{z}(0) = 0 \end{cases}$

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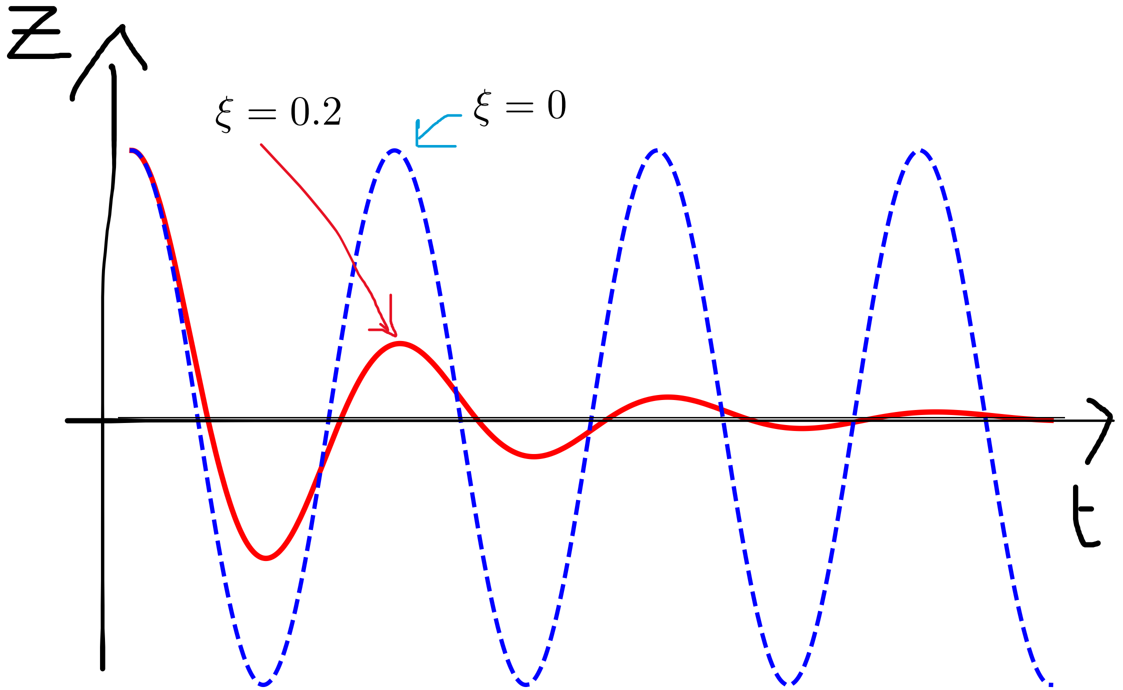
Solution with initial conditions

$$z = z_0\,e^{-\omega_0\,\xi\,t}\,\left(\cos(\omega_d\,t) + \frac{\omega_0\,\xi}{\omega_d}\,\sin(\omega_d\,t)\right)$$



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Solution

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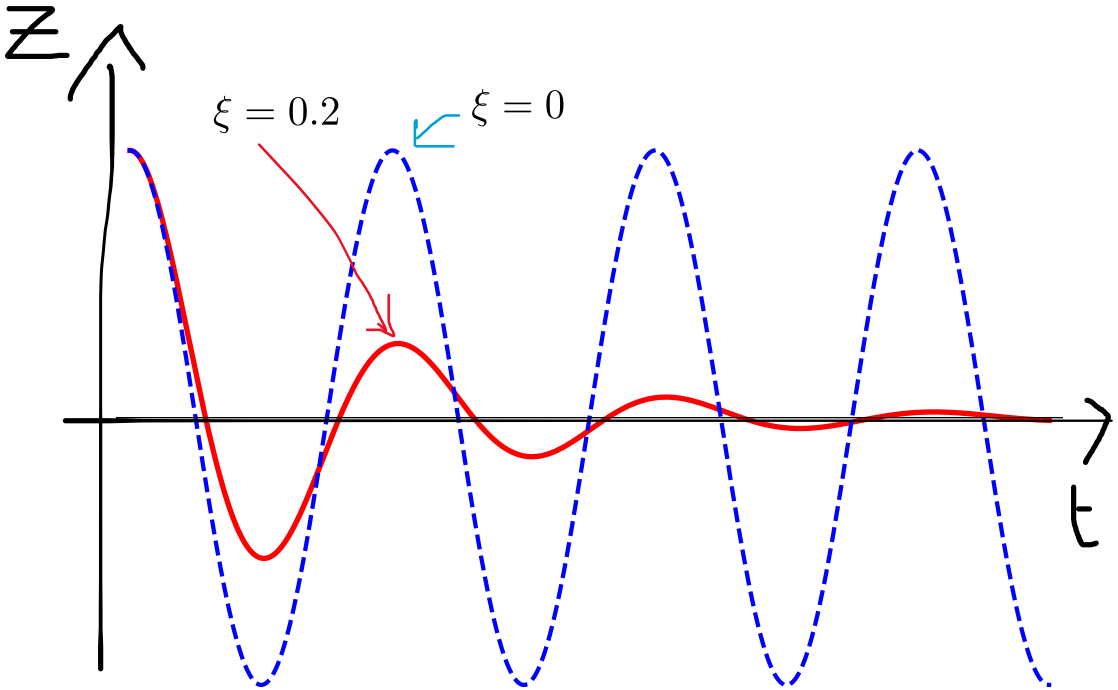
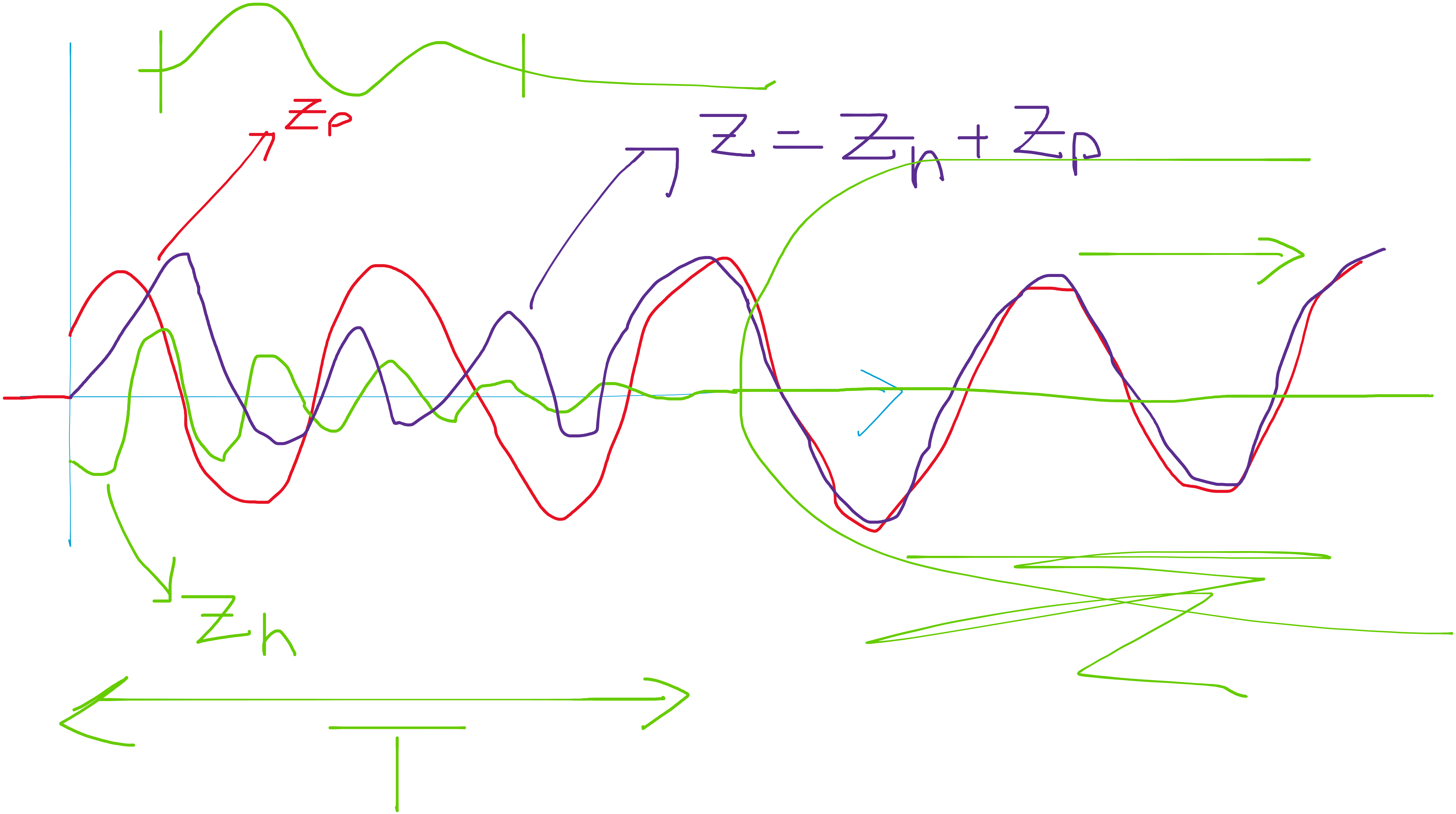


Undamped: $\xi = 0$ Underdamped: $\xi \in (0,1)$

Overdamped: $\xi > 1$ Critically damped: $\xi = 1$



z_p
$z = z_h + z_p$
z_h
T