SEISMOLOGY

I

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Chapter 1

Prerequisites

1.1 Vector and tensor algebra

Within the Cartesian reference frame, the vector position can be expressed as

$$\mathbf{x} = x_i \, \hat{\mathbf{x}}_i \tag{1.1}$$

where x_i and $\hat{\mathbf{x}}_i$ are the Cartesian coordinates and unit vectors, respectively.

Vector, $\mathbf{a}$, and tensor, $\mathbf{A}$, fields depend on the vector position and they can be expressed in terms of their Cartesian components

$$\mathbf{a}(\mathbf{x}) = a_i(\mathbf{x}) \, \hat{\mathbf{x}}_i \tag{1.2a}$$

$$\mathbf{A}(\mathbf{x}) = A_{ij}(\mathbf{x}) \, \hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j \tag{1.2b}$$

where $\otimes$ stands for the algebraic product, and $\hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j$ is the dyad identifying the ij-th element of a 2-rank tensor in the matrix notation. Similar, the ijk-th element of a 3-rank tensor can be written in terms of $\hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j \otimes \hat{\mathbf{x}}_k$

$$\mathbf{A}(\mathbf{x}) = A_{ijk}(\mathbf{x})\, \hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j \otimes \hat{\mathbf{x}}_k \tag{1.3}$$

By defining the scalar $(\cdot)$ and cross $(\times)$ products between the Cartesian unit vectors as follows

$$\hat{\mathbf{x}}_i \cdot \hat{\mathbf{x}}_j = \delta_{ij} \tag{1.4}$$

$$\hat{\mathbf{x}}_i \times \hat{\mathbf{x}}_j = \hat{\mathbf{x}}_k\, \epsilon_{kij} \tag{1.5}$$

with δ_{ij} and ϵ_{kij} being the Kronecker and Levi-Civita symbols, we obtain the scalar and cross products between two vectors

$$\mathbf{a} \cdot \mathbf{b} = (a_i\, \hat{\mathbf{x}}_i) \cdot (b_j\, \hat{\mathbf{x}}_j) = a_i\, b_j\, \hat{\mathbf{x}}_i \cdot \hat{\mathbf{x}}_j = a_i\, b_j\, \delta_{ij} = a_i\, b_i \tag{1.6a}$$

$$\mathbf{a} \times \mathbf{b} = (a_i\, \hat{\mathbf{x}}_i) \times (b_j\, \hat{\mathbf{x}}_j) = a_i\, b_j\, \hat{\mathbf{x}}_i \times \hat{\mathbf{x}}_j = \hat{\mathbf{x}}_k\, \epsilon_{kij}\, a_i\, b_j \tag{1.6b}$$

Also, we can define the left and lateral scalar products between vectors and tensors as follows

$$\mathbf{a} \cdot \mathbf{A} = (a_i\, \hat{\mathbf{x}}_i) \cdot (A_{jk}\, \hat{\mathbf{x}}_j \otimes \hat{\mathbf{x}}_k) = a_i\, A_{jk}\, (\hat{\mathbf{x}}_i \cdot \hat{\mathbf{x}}_j)\hat{\mathbf{x}}_k = a_i\, A_{ij}\, \hat{\mathbf{x}}_j \tag{1.7a}$$

$$\mathbf{A} \cdot \mathbf{a} = (A_{jk}\, \hat{\mathbf{x}}_j \otimes \hat{\mathbf{x}}_k) \cdot (a_i\, \hat{\mathbf{x}}_i) = A_{jk}\, a_i\, \hat{\mathbf{x}}_j (\hat{\mathbf{x}}_k \cdot \hat{\mathbf{x}}_i) = A_{ji}\, a_i\, \hat{\mathbf{x}}_j \tag{1.7b}$$

as well as the left and right cross products between vectors and tensor

$$\begin{aligned} \mathbf{a} \times \mathbf{A} &= (a_i\, \hat{\mathbf{x}}_i) \times (A_{jk}\, \hat{\mathbf{x}}_j \otimes \hat{\mathbf{x}}_k) = a_i\, A_{jk}\, (\hat{\mathbf{x}}_i \times \hat{\mathbf{x}}_j) \otimes \hat{\mathbf{x}}_k \\ &= a_i\, A_{jk}\, \epsilon_{pij}\, \hat{\mathbf{x}}_p \otimes \hat{\mathbf{x}}_k \end{aligned} \tag{1.8a}$$

$$\begin{aligned} \mathbf{A} \times \mathbf{a} &= (A_{jk}\, \hat{\mathbf{x}}_j \otimes \hat{\mathbf{x}}_k) \times (a_i\, \hat{\mathbf{x}}_i) = A_{jk}\, a_i\, \hat{\mathbf{x}}_j \otimes (\hat{\mathbf{x}}_k \times \hat{\mathbf{x}}_i) \\ &= A_{jk}\, a_i\, \epsilon_{pki}\, \hat{\mathbf{x}}_j \otimes \hat{\mathbf{x}}_p \end{aligned} \tag{1.8b}$$

Here, we have made use of the following identities

$$\hat{\mathbf{x}}_i \cdot (\hat{\mathbf{x}}_j \otimes \hat{\mathbf{x}}_k) = (\hat{\mathbf{x}}_i \cdot \hat{\mathbf{x}}_j)\, \hat{\mathbf{x}}_k = \delta_{ij}\, \hat{\mathbf{x}}_k \tag{1.9a}$$

$$(\hat{\mathbf{x}}_j \otimes \hat{\mathbf{x}}_k) \cdot \hat{\mathbf{x}}_i = \hat{\mathbf{x}}_j\, (\hat{\mathbf{x}}_k \cdot \hat{\mathbf{x}}_i) = \delta_{ki}\, \hat{\mathbf{x}}_j \tag{1.9b}$$

$$\hat{\mathbf{x}}_i \times (\hat{\mathbf{x}}_j \otimes \hat{\mathbf{x}}_k) = (\hat{\mathbf{x}}_i \times \hat{\mathbf{x}}_j) \otimes \hat{\mathbf{x}}_k = \epsilon_{pij}\, \hat{\mathbf{x}}_p \otimes \hat{\mathbf{x}}_k \tag{1.9c}$$

$$(\hat{\mathbf{x}}_j \otimes \hat{\mathbf{x}}_k) \times \hat{\mathbf{x}}_i = \hat{\mathbf{x}}_j \otimes (\hat{\mathbf{x}}_k \times \hat{\mathbf{x}}_i) = \epsilon_{pki}\, \hat{\mathbf{x}}_j \otimes \hat{\mathbf{x}}_p \tag{1.9d}$$

1.2 The gradient operator

The gradient operator is defined as a vector, the component of which are the partial derivative with respect to the Cartesian coordinates

$$\boldsymbol{\nabla} = \hat{\mathbf{x}}_i \frac{\partial}{\partial x_j} \tag{1.10}$$

When we apply the gradient operator to scalar, a, vector, $\mathbf{a}$, and tensor, $\mathbf{A}$, fields we obtain their gradients

$$\boldsymbol{\nabla} a = \hat{\mathbf{x}}_i \frac{\partial a}{\partial x_i} \tag{1.11a}$$

$$\boldsymbol{\nabla} \otimes \mathbf{a} = \hat{\mathbf{x}}_i \otimes \frac{\partial \mathbf{a}}{\partial x_i} = \frac{\partial a_j}{\partial x_i} \hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j \tag{1.11b}$$

$$\boldsymbol{\nabla} \otimes \mathbf{A} = \hat{\mathbf{x}}_i \otimes \frac{\partial \mathbf{A}}{\partial x_i} = \frac{\partial A_{jk}}{\partial x_i} \hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j \otimes \hat{\mathbf{x}}_k \tag{1.11c}$$

that are vector, tensor and 3-rank tensors, respectively.

The gradient of a scalar (vector or tensor) field allows us to write the Taylor expansion for a given differential $\mathrm{d}\mathbf{x}$ in the following compact form

$$\mathrm{d}f(\mathbf{x}) = f(\mathbf{x} + \mathrm{d}\mathbf{x}) - f(\mathbf{x}) = \mathrm{d}\mathbf{x} \cdot \boldsymbol{\nabla} f(\mathbf{x}) + O(|\mathrm{d}\mathbf{x}|) \tag{1.12}$$

Directional derivatives

Denoting with $\hat{\mathbf{s}}$ and ϵ the (unitary) direction and module of $\mathrm{d}\mathbf{x} = \mathrm{d}x\,\hat{\mathbf{s}}$, we can rewrite eq. (1.12) as follows

$$\frac{f(\mathbf{x} + \epsilon\,\hat{\mathbf{s}}) - f(\mathbf{x})}{\epsilon} = \hat{\mathbf{s}} \cdot \boldsymbol{\nabla} f(\mathbf{x}) + O(1) \tag{1.13}$$

or, in the limit for ϵ going to zero,

$$\frac{\partial f(\mathbf{x})}{\partial \hat{\mathbf{s}}} = \hat{\mathbf{s}} \cdot \boldsymbol{\nabla} f(\mathbf{x}) = \lim_{\epsilon \to 0} \frac{f(\mathbf{x} + \epsilon\, \hat{\mathbf{s}}) - f(\mathbf{x})}{\epsilon} \tag{1.14}$$

that is the definition of directional derivative. We note that when $\hat{\mathbf{s}} = \hat{\mathbf{x}}_i$, we obtain the partial derivative of the field with respect to the j-th Cartesian coordinate

$$\frac{\partial f(\mathbf{x})}{\partial x_i} = \hat{\mathbf{x}}_i \cdot \boldsymbol{\nabla} f(\mathbf{x}) = \lim_{\epsilon \to 0} \frac{f(\mathbf{x} + \epsilon\, \hat{\mathbf{x}}_j) - f(\mathbf{x})}{\epsilon} \tag{1.15}$$

Divergence

We can also define the divergence of vector and tensor fields as their scalar product with the gradient operator

$$\boldsymbol{\nabla} \cdot \mathbf{a} = \hat{\mathbf{x}}_i \cdot \frac{\partial \mathbf{a}}{\partial x_i} = \frac{\partial a_j}{\partial x_i}\, \hat{\mathbf{x}}_i \cdot \hat{\mathbf{x}}_j = \frac{\partial a_i}{\partial x_i} \tag{1.16a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{A} = \hat{\mathbf{x}}_i \cdot \frac{\partial \mathbf{A}}{\partial x_i} = \frac{\partial A_{jk}}{\partial x_i}\, \hat{\mathbf{x}}_i \cdot \hat{\mathbf{x}}_j \otimes \hat{\mathbf{x}}_k = \frac{\partial A_{ik}}{\partial x_i}\, \hat{\mathbf{x}}_k \tag{1.16b}$$

that are scalar and vector fields, respectively.

Curl

We can also define the curl of vector and tensor fields as their cross product with the gradient operator

$$\boldsymbol{\nabla} \times \mathbf{a} = \hat{\mathbf{x}}_i \times \frac{\partial \mathbf{a}}{\partial x_i} = \frac{\partial a_j}{\partial x_i}\, \hat{\mathbf{x}}_i \times \hat{\mathbf{x}}_j = \frac{\partial a_j}{\partial x_i}\, \epsilon_{pij}\, \hat{\mathbf{x}}_p \tag{1.17a}$$

$$\boldsymbol{\nabla} \times \mathbf{A} = \hat{\mathbf{x}}_i \times \frac{\partial \mathbf{A}}{\partial x_i} = \frac{\partial A_{jk}}{\partial x_i}\, \hat{\mathbf{x}}_i \times (\hat{\mathbf{x}}_j \otimes \hat{\mathbf{x}}_k) = \frac{\partial A_{jk}}{\partial x_i}\, \epsilon_{pij}\, \hat{\mathbf{x}}_p \otimes \hat{\mathbf{x}}_k \tag{1.17b}$$

Laplacian

The scalar product of the gradient operator with itself yields the Laplacian operator

$$\nabla^2 = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} = \frac{\partial^2}{\partial x_i \, \partial x_j} \, \hat{\mathbf{x}}_i \cdot \hat{\mathbf{x}}_j = \frac{\partial^2}{\partial x_i \, \partial x_i} \tag{1.18}$$

In the following we shall often use the following identity

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{a} = (\boldsymbol{\nabla} \otimes \boldsymbol{\nabla}) \cdot \mathbf{a} - \nabla^2 \mathbf{a} \tag{1.19}$$

that can be proved by making use of the following property of the Levi-Civita symbol

$$\epsilon_{kij} \, \epsilon_{kpq} = \delta_{ip} \, \delta_{jq} - \delta_{iq} \, \delta_{jp} \tag{1.20}$$

Vector position

The gradient, divergence and curl of the vector position yield

$$\boldsymbol{\nabla} \otimes \mathbf{x} = \frac{\partial x_j}{\partial x_i} \, \hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j = \delta_{ij} \, \hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j = \hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_i = \mathbf{1} \tag{1.21a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{x} = \frac{\partial x_j}{\partial x_i} \, \hat{\mathbf{x}}_i \cdot \hat{\mathbf{x}}_j = \delta_{ij} \, \delta_{ij} = 3 \tag{1.21b}$$

$$\boldsymbol{\nabla} \times \mathbf{x} = \frac{\partial x_j}{\partial x_i} \, \hat{\mathbf{x}}_i \times \hat{\mathbf{x}}_j = \delta_{ij} \, \epsilon_{kij} \, \hat{\mathbf{x}}_k = \mathbf{0} \tag{1.21c}$$

1.3 Spherical symmetry and the gradient operator

Let us consider the radial distance from the origin defined as

$$r = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{x_i\, x_i} \tag{1.22}$$

and its gradient

$$\boldsymbol{\nabla} r = \frac{\partial r}{\partial x_i}\, \hat{\mathbf{x}}_i = \frac{x_i}{r}\, \hat{\mathbf{x}}_i = \gamma_i\, \hat{\mathbf{x}}_i = \hat{\mathbf{r}} = \frac{\mathbf{x}}{r} \tag{1.23}$$

Here, $\hat{\mathbf{r}}$ is the radial unit vector pointing from the origin to the point identified by the vector position $\mathbf{x}$, and γ_i are its cartesian components

$$\hat{\mathbf{r}} = \gamma_i\, \hat{\mathbf{x}}_i \tag{1.24a}$$

$$\gamma_i = \frac{x_i}{r} \tag{1.24b}$$

The components γ_i are the so called direction cosines.

The gradient of scalar and vector fields which depend on the radial distance only yields

$$\boldsymbol{\nabla} a(r) = \frac{\partial a}{\partial x_i}\, \hat{\mathbf{x}}_i = \frac{\partial a}{\partial r}\, \frac{\partial r}{\partial x_i}\, \hat{\mathbf{x}}_i = \frac{\partial a}{\partial r}\, \boldsymbol{\nabla} r = \frac{\partial a}{\partial r}\, \hat{\mathbf{r}} \tag{1.25a}$$

$$\boldsymbol{\nabla} \otimes \mathbf{a}(r) = \frac{\partial a_j}{\partial x_i}\, \hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j = \frac{\partial a_j}{\partial r}\, \frac{\partial r}{\partial x_i}\, \hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j = \hat{\mathbf{r}} \otimes \frac{\partial \mathbf{a}}{\partial r} \tag{1.25b}$$

The divergence of a vector field which depends on the radial distance only, instead, yields

$$\boldsymbol{\nabla} \cdot \mathbf{a}(r) = \frac{\partial a_j}{\partial x_i}\, \hat{\mathbf{x}}_i \cdot \hat{\mathbf{x}}_j = \frac{a_j}{\partial r}\, \frac{\partial r}{\partial x_i}\, \hat{\mathbf{x}}_i \cdot \hat{\mathbf{x}}_j = \hat{\mathbf{r}} \cdot \frac{\partial \mathbf{a}}{\partial r} \tag{1.26}$$

Making use of eqs. (1.23) and (1.25a), we also note that the Laplacian of a scalar function which depends on the radial distance only yields

$$\nabla^2 a(r) = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} a) = \frac{\partial}{\partial x_i} \left(\frac{x_i}{r}\, \frac{\partial a}{\partial r} \right) = \frac{1}{r}\, \frac{\partial a}{\partial r}\, \delta_{ii} + x_i\, \frac{\partial}{\partial r} \left(\frac{1}{r}\, \frac{\partial a}{\partial r} \right) \frac{\partial r}{\partial x_i}$$

$$= \frac{3}{r}\,\frac{\partial a}{\partial r} + \frac{x_i\,x_i}{r}\,\frac{\partial}{\partial r}\left(\frac{1}{r}\,\frac{\partial a}{\partial r}\right) = \frac{3}{r}\,\frac{\partial a}{\partial r} + r\,\frac{\partial}{\partial r}\left(\frac{1}{r}\,\frac{\partial a}{\partial r}\right)$$
$$= \frac{\partial^2 a}{\partial r^2} + \frac{2}{3}\,\frac{\partial a}{\partial r} = \frac{1}{r}\,\frac{\partial^2 (r\,a)}{\partial r^2} \tag{1.27}$$

It is straightforward to generalize eq. (1.27) even to vector and tensor fields

$$\nabla^2 \mathbf{a}(r) = \frac{1}{r}\,\frac{\partial^2 (r\,\mathbf{a})}{\partial r^2} \tag{1.28a}$$
$$\nabla^2 \mathbf{A}(r) = \frac{1}{r}\,\frac{\partial^2 (r\,\mathbf{A})}{\partial r^2} \tag{1.28b}$$

Another useful identity that we shall use in the following chapters for obtaining the elastodynamics Green tensor is

$$\begin{aligned}
\boldsymbol{\nabla}\big(\boldsymbol{\nabla}\cdot\mathbf{a}(r)\big) &= \boldsymbol{\nabla}\left(\frac{\mathbf{x}}{r}\,\frac{\partial \mathbf{a}}{\partial r}\right)\\
&= (\boldsymbol{\nabla}\otimes\mathbf{x})\cdot\left(\frac{1}{r}\,\frac{\partial \mathbf{a}}{\partial r}\right) + \left[\boldsymbol{\nabla}\otimes\left(\frac{1}{r}\,\frac{\partial \mathbf{a}}{\partial r}\right)\right]\cdot\mathbf{x}\\
&= \mathbf{1}\cdot\left(\frac{1}{r}\,\frac{\partial \mathbf{a}}{\partial r}\right) + \left[\hat{\mathbf{r}}\otimes\frac{\partial}{\partial r}\left(\frac{1}{r}\,\frac{\partial \mathbf{a}}{\partial r}\right)\right]\cdot\mathbf{x} =\\
&= (\mathbf{1}-\hat{\mathbf{r}}\otimes\hat{\mathbf{r}})\cdot\left(\frac{1}{r}\,\frac{\partial \mathbf{a}}{\partial r}\right) + \hat{\mathbf{r}}\otimes\hat{\mathbf{r}}\cdot\frac{\partial^2 \mathbf{a}}{\partial r^2}
\end{aligned} \tag{1.29}$$

1.3.1 The 3-dimensional Dirac delta distribution

Let us consider the Laplacian of the inverse of the radial distance. By making use of eq. (1.27) with $a(r)$ given by

$$a(r) = -\frac{1}{4\,\pi\,r} \tag{1.30}$$

we obtain

$$\nabla^2 a(r) = 0 \tag{1.31}$$

everywhere but at the origin $\mathbf{x} = \mathbf{0}$, where the function is singular. In order to better understand the behaviour at the origin, we consider the following integral over an open volume $\mathcal{V}$ containing the origin, $\mathbf{0} \in \mathcal{V}$,

$$\begin{aligned}\int_{\mathcal{V}} \nabla^2 a(r)\,\mathrm{d}V &= -\frac{1}{4\,\pi} \int_{\mathcal{B}} \nabla^2 \frac{1}{r}\,\mathrm{d}V = -\frac{1}{4\,\pi} \int_{\partial\mathcal{B}} \hat{\mathbf{r}} \cdot \boldsymbol{\nabla} \frac{1}{r}\,\mathrm{d}S \\ &= \frac{1}{4\,\pi\,R^2} \int_{\partial\mathcal{B}} \mathrm{d}S = 1\end{aligned} \tag{1.32}$$

Here, $\mathcal{B} \subset \mathcal{V}$ is a sphere of radius R centered at the origin, and $\partial\mathcal{B}$ is its boundary of area $4\,\pi\,R^2$.

It can be shown also that

$$\int_{\mathcal{V}} f(\mathbf{x})\,\nabla^2 a(r)\,\mathrm{d}V = f(\mathbf{0}) \tag{1.33}$$

for any (scalar, vector or tensor) field f that has continuous first and second order partial derivatives. In light of this we can set the following identity

$$\nabla^2 \frac{1}{r} = -4\,\pi\,\delta(\mathbf{x}) \tag{1.34}$$

where $\delta(\mathbf{x})$ is the Dirac delta distribution centered at the origin. Indeed, the Dirac delta distribution is defined by the following two main properties

$$\delta(\mathbf{x} - \boldsymbol{\xi}) = 0 \qquad \forall\,\mathbf{x} \neq \boldsymbol{\xi} \tag{1.35a}$$

$$f(\boldsymbol{\xi}) = \int_{\mathcal{V}} f(\mathbf{x})\,\delta(\mathbf{x} - \boldsymbol{\xi})\,\mathrm{d}V \qquad \forall\,\mathcal{V} : \boldsymbol{\xi} \in \mathcal{V} \tag{1.35b}$$

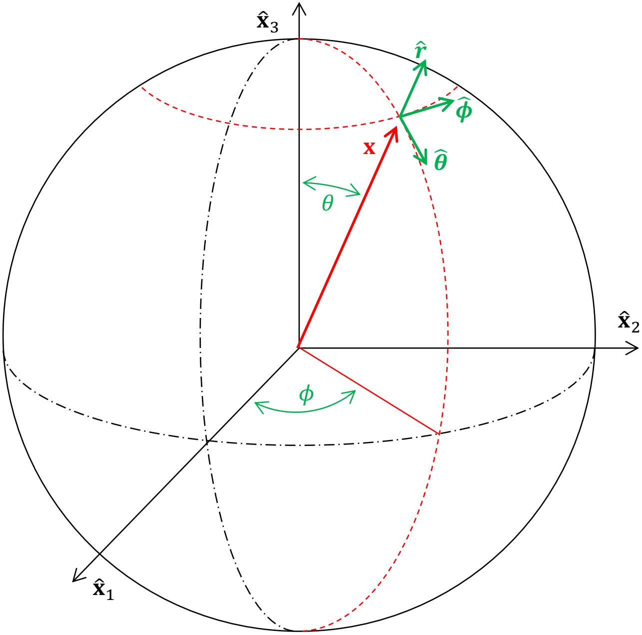

Figure 1.1: Cartoon depicting the spherical coordinate system with the three spherical unit vectors represented at a specific observation point. The radius of the sphere is r.

1.4 Spherical coordinates

Let us introduce the spherical coordinates (r, θ, ϕ) through their relations with the Cartesian coordinates

$$\mathbf{x} = r\,\hat{\mathbf{r}} \tag{1.36}$$

with $\hat{\mathbf{r}}$ being the radial unit vectors defined by

$$\hat{\mathbf{r}} = \sin\theta \left(\cos\phi\,\hat{\mathbf{x}}_1 + \sin\phi\,\hat{\mathbf{x}}_2\right) + \cos\theta\,\hat{\mathbf{x}}_3 \tag{1.37}$$

We note that

$$\frac{\partial \mathbf{r}}{\partial r} = \hat{\mathbf{r}} \tag{1.38a}$$

$$\frac{\partial \mathbf{r}}{\partial \theta} = r\,\hat{\boldsymbol{\theta}} \tag{1.38b}$$

$$\frac{\partial \mathbf{r}}{\partial \phi} = r\,\sin\theta\,\hat{\boldsymbol{\phi}} \tag{1.38c}$$

with $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ being the colatitudinal and longitudinal unit vectors (directed along the meridians and parallels, respectively)

$$\hat{\boldsymbol{\theta}} = \cos\theta\,(\cos\phi\,\hat{\mathbf{x}}_1 + \sin\phi\,\hat{\mathbf{x}}_2) - \sin\theta\,\hat{\mathbf{x}}_3 \tag{1.39a}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}}_1 + \cos\phi\,\hat{\mathbf{x}}_2 \tag{1.39b}$$

Different from the Cartesian unit vectors, the spherical unit vectors vary in space. Indeed, they depend on the angular coordinate (θ, ϕ), but not on the radial distance r.

Making use of eq. (1.38), the differential between the points at (r, θ, ϕ) and $(r + \mathrm{d}r, \theta + \mathrm{d}\theta, \phi + \mathrm{d}\phi)$ reads

$$\mathrm{d}\mathbf{x} = \mathrm{d}r\,\frac{\partial \mathbf{x}}{\partial r} + \mathrm{d}\theta\,\frac{\partial \mathbf{x}}{\partial \theta} + \mathrm{d}\phi\,\frac{\partial \mathbf{x}}{\partial \phi} = \mathrm{d}r\,\hat{\mathbf{r}} + r\left(\mathrm{d}\theta\,\hat{\boldsymbol{\theta}} + \mathrm{d}\phi\,\sin\theta\,\hat{\boldsymbol{\phi}}\right) \tag{1.40}$$

Similar, the differential of the scalar (vector or tensor) function in spherical coordinates reads

$$\mathrm{d}f = \mathrm{d}r\frac{\partial f}{\partial r} + \mathrm{d}\theta\frac{\partial f}{\partial \theta} + \mathrm{d}\phi\frac{\partial f}{\partial \phi} \tag{1.41}$$

In order to obtain the expression for the gradient operator in spherical coordinates, we substitute eq. (1.40) into eq. (1.12)

$$\mathrm{d}f = \mathrm{d}r\,\hat{\mathbf{r}}\cdot\boldsymbol{\nabla} f + r\left(\mathrm{d}\theta\,\hat{\boldsymbol{\theta}}\cdot\boldsymbol{\nabla} f + \mathrm{d}\phi\,\sin\theta\,\hat{\boldsymbol{\phi}}\cdot\boldsymbol{\nabla} f\right) \tag{1.42}$$

and, after comparison with eq. (1.41), we can understand that

$$\boldsymbol{\nabla} = \hat{\mathbf{r}}\,\frac{\partial}{\partial r} + \frac{1}{r}\left(\hat{\boldsymbol{\theta}}\,\frac{\partial}{\partial \theta} + \frac{1}{\sin\theta}\,\hat{\boldsymbol{\phi}}\,\frac{\partial}{\partial \phi}\right) \tag{1.43}$$

It can be shown that the Laplacian in spherical coordinates reads

$$\nabla^2 f = \frac{1}{r}\,\frac{\partial^2(r\,f)}{\partial r^2} + \frac{1}{r^2}\left(\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{\sin^2\theta}\,\frac{\partial^2 f}{\partial \phi^2}\right) \tag{1.44}$$

1.5 Fourier transform

The Fourier transform of a function f depending on time t is defined by

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)\,e^{-i\,\omega\,t}\,\mathrm{d}t \tag{1.45}$$

where ω is the angular frequency.

The inverse Fourier transform results to be defined by

$$f(t) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{i\,\omega\,t}\, \mathrm{d}\omega \tag{1.46}$$

In fact, by substituting eq. (1.49) into eq. (1.46), we obtain

$$f(t) = \int_{-\infty}^{\infty} f(\tau) \left(\int_{-\infty}^{\infty} \frac{e^{i\,\omega\,(t-\tau)}}{2\,\pi}\, \mathrm{d}\omega \right) \mathrm{d}\tau = \int_{-\infty}^{\infty} f(\tau)\, \delta(t-\tau)\, \mathrm{d}\tau \tag{1.47}$$

where we have made use of the following alternative definition of the one-dimensional Dirac delta distribution

$$\delta(t-\tau) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} e^{i\,\omega\,(t-\tau)}\, \mathrm{d}\omega \tag{1.48}$$

Derivatives

The Fourier transform of the n-th order derivative yields the Fourier transform of the function multiplied by $(i\,\omega)^n$

$$\int_{-\infty}^{\infty} \frac{\partial^n f(t)}{\partial t^n}\, e^{-i\,\omega\,t}\, \mathrm{d}t = (i\,\omega)^n\, \tilde{f}(\omega) \tag{1.49}$$

Real-valued function

The Fourier transform of a real-valued function satisfies has the following symmetry

$$\tilde{f}(\omega) = \tilde{f}^*(-\omega) \tag{1.50}$$

where $*$ stands for the complex conjugate. In particular, the real and imaginary parts of the Fourier transform are symmetric and antisymmetric functions, respectively,

$$\Re\big[\tilde{f}(\omega)\big] = \Re\big[\tilde{f}(-\omega)\big] \tag{1.51a}$$

$$\Im\big[\tilde{f}(\omega)\big] = -\Im\big[\tilde{f}(-\omega)\big] \tag{1.51b}$$

where $\Re$ and $\Im$ stand for the operators returning the real and imaginary part of a complex number. Moreover, the inverse Fourier transform of a real-valued function can be cast in the following form

$$f(t) = \frac{1}{\pi}\Re\left[\int_0^\infty \tilde{f}(\omega)\, e^{i\,\omega\,t}\, \mathrm{d}\omega\right] \tag{1.52}$$

where the integral involves only non-negative frequencies, $\omega \geq 0$.

Chapter 2

Momentum equation and linear elasticity

In this chapter we will obtain the momentum equation and introduce the Hooke's law to deal with linear elastic media. Also, we will consider the kinetic and internal energy balance in order to define the strain-energy function. Before doing this, we will provide an intuitive interpretation of the divergence theorem that we shall use often in the present and following chapters.

We will derive all the equations within the Lagrangian approach and a first-order perturbation theory. We denote with $\mathcal{V}$ a region occupied by the undeformed continuum and with $\delta\mathcal{V}$ its boundary, and we identify the initial position of the particle with the vector position $\mathbf{x}$. The current position $\mathbf{r}$ of a particle at the time t is defined by the displacement field $\mathbf{u}$ evaluated at its initial position

$$\mathbf{r}(\mathbf{x}, t) = \mathbf{x} + \mathbf{u}(\mathbf{x}, t) \tag{2.1}$$

In the following, we shall assume that the deformations are infinitesimal in order to develop a first-order perturbations theory and, so, $\mathbf{u}$ can be considered as an infinitesimal term.

2.1 Infinitesimal volume element

As depicted in fig. 2.1, given the vector position $\mathbf{x}$ and three differentials $\mathrm{d}\mathbf{x}^{(1)}$, $\mathrm{d}\mathbf{x}^{(2)}$ and $\mathrm{d}\mathbf{x}^{(3)}$, we can define an infinitesimal parallelepiped. The infinitesimal surface elements of the parallelepiped can be defined by the cross products of two distinct differentials

$$\mathrm{d}\mathbf{x}^{(i)} \times \mathrm{d}\mathbf{x}^{(j)} = \hat{\mathbf{n}}^{(ij)}\, \mathrm{d}S_x^{(ij)} \tag{2.2}$$

where $\mathrm{d}S_x^{(ij)}$ is the area and $\hat{\mathbf{n}}^{(ij)}$ is the unit vector normal to the surface identified by the two differentials.

Also, the scalar product between surface element from two unit vectors and the remaining one yields the infinitesimal volume of the parallelepiped

$$\mathrm{d}V_x = \left|(\mathrm{d}\mathbf{x}^{(1)} \times \mathrm{d}\mathbf{x}^{(2)}) \cdot \mathrm{d}\mathbf{x}^{(3)}\right| = \left|\mathrm{d}x_i^{(1)}\, \mathrm{d}x_j^{(2)}\, \mathrm{d}x_k^{(3)}\, \epsilon_{ijk}\right| \tag{2.3}$$

Let us now consider the deformed parallelepiped due to the displacement field. In this case, the current position of the particles at $\mathbf{x}$ and at the three adjacent vertices $\mathbf{x}^{(i)} = \mathbf{x} + \mathrm{d}\mathbf{x}^{(i)}$ are given by

$$\mathbf{r} = \mathbf{x} + \mathbf{u}(\mathbf{x}, t) \tag{2.4a}$$

$$\begin{aligned}\mathbf{r}^{(i)} &= \mathbf{x}^{(i)} + \mathbf{u}(\mathbf{x}^{(i)}, t) = \mathbf{x} + \mathrm{d}\mathbf{x}^{(i)} + \mathbf{u}(\mathbf{x} + \mathrm{d}\mathbf{x}^{(i)}, t) \\ &= \mathbf{x} + \mathbf{u}(\mathbf{x}, t) + \mathrm{d}\mathbf{x}^{(i)} + \mathrm{d}\mathbf{x}^{(i)} \cdot \boldsymbol{\nabla} \otimes \mathbf{u}(\mathbf{x}, t)\end{aligned}$$

$$= \mathbf{r} + \mathrm{d}\mathbf{r}^{(i)} \tag{2.4b}$$

where we made use of the first order Taylor expansion and $\mathrm{d}\mathbf{r}^{(i)}$ are the differentials that identify the three adjacent vertices of the deformed parallelepiped

$$\mathrm{d}\mathbf{r}^{(i)} = \mathrm{d}\mathbf{x}^{(i)} + \mathrm{d}\mathbf{x}^{(i)} \cdot \boldsymbol{\nabla} \otimes \mathbf{u}(\mathbf{x}, t) \tag{2.5}$$

In light of this, keeping first order terms only, the infinitesimal deformed volume yields

$$\begin{aligned}
\mathrm{d}V &= \left|(\mathrm{d}\mathbf{r}^{(1)} \times \mathrm{d}\mathbf{r}^{(2)}) \cdot \mathrm{d}\mathbf{r}^{(3)}\right| \\
&= \Bigg|\mathrm{d}x_i^{(1)}\,\mathrm{d}x_j^{(2)}\,\mathrm{d}x_k^{(3)}\,\epsilon_{ijk} + \mathrm{d}x_p^{(1)}\,\mathrm{d}x_j^{(2)}\,\mathrm{d}x_k^{(3)}\,\frac{\partial u_i}{\partial x_p}\,\epsilon_{ijk} \\
&\quad + \mathrm{d}x_i^{(1)}\,\mathrm{d}x_p^{(2)}\,\mathrm{d}x_k^{(3)}\,\frac{\partial u_j}{\partial x_p}\,\epsilon_{ijk} + \mathrm{d}x_i^{(1)}\,\mathrm{d}x_j^{(2)}\,\mathrm{d}x_p^{(3)}\,\frac{\partial u_k}{\partial x_p}\,\epsilon_{ijk}\Bigg| \\
&= \left|\mathrm{d}x_i^{(1)}\,\mathrm{d}x_j^{(2)}\,\mathrm{d}x_k^{(3)}\left(\epsilon_{ijk} + \frac{\partial u_p}{\partial x_i}\,\epsilon_{pjk} + \frac{\partial u_p}{\partial x_j}\,\epsilon_{ipk} + \frac{\partial u_p}{\partial x_k}\,\epsilon_{ijp}\right)\right| \\
&= \left|\mathrm{d}x_i^{(1)}\,\mathrm{d}x_j^{(2)}\,\mathrm{d}x_k^{(3)}\,\epsilon_{ijk}\left(1 + \frac{\partial u_p}{\partial x_p}\right)\right| = \mathrm{d}V_x\,(1 + \boldsymbol{\nabla}\cdot\mathbf{u})
\end{aligned} \tag{2.6}$$

where we have made use of the following identity

$$\frac{\partial u_p}{\partial x_i}\,\epsilon_{pjk} + \frac{\partial u_p}{\partial x_j}\,\epsilon_{ipk} + \frac{\partial u_p}{\partial x_k}\,\epsilon_{ijp} = \frac{\partial u_p}{\partial x_p}\,\epsilon_{ijk} \tag{2.7}$$

and the fact that $1 + \boldsymbol{\nabla}\cdot\mathbf{u} > 0$ due to the assumption of first-order perturbation theory. In this respect, the divergence of the displacement field represents the relative volume variations of the infinitesimal volume element

$$\boldsymbol{\nabla}\cdot\mathbf{u} = \frac{\mathrm{d}V - \mathrm{d}V_x}{\mathrm{d}V_x} \tag{2.8}$$

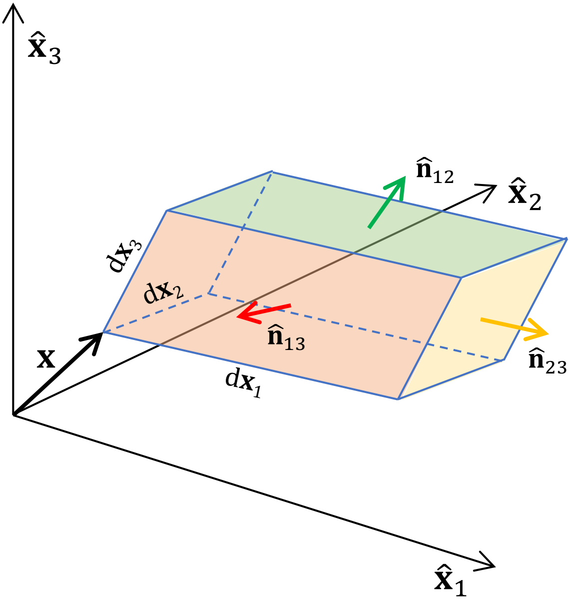

Figure 2.1: Ciao

2.2 Divergence theorem

Let us evaluate the volume $V(t)$ of a region $\mathcal{V}(t)$ subjected to the displacement field $\mathbf{u}$ at a given time t. Within a first-order perturbation theory, the volume of the deformed infinitesimal volume element, $\mathrm{d}V$, is related to its initial volume $\mathrm{d}V_x$ by the following relation

$$\mathrm{d}V = \big(1 + \boldsymbol{\nabla} \cdot \mathbf{u}(\mathbf{x}, t)\big)\, \mathrm{d}V_x \tag{2.9}$$

and, so, we can write

$$V(t) = \int_{\mathcal{V}(t)} \mathrm{d}V = V(0) + \int_{\mathcal{V}} \boldsymbol{\nabla} \cdot \mathbf{u}(\mathbf{x}, t)\, \mathrm{d}V_x \tag{2.10}$$

In light of this, the volume integral of the divergence of the displacement field yields the volume variation with respect to the initial volume at time, i.e., $V(t) - V(0)$. This quantity can be evaluated also by the following surface integral

$$V(t) - V(0) = \int_{\delta\mathcal{V}} h(\mathbf{x}, t)\, \mathrm{d}S_x \tag{2.11}$$

where $\delta\mathcal{V}$ is the surface of the region $\mathcal{V}$, $\mathrm{d}S_x$ is the area of an infinitesimal surface element and $h(\mathbf{x})$ being the (signed) height of the perturbed surface with respect to the initial one. In particular the latter can be quantified by the scalar product between the displacement and the (outward) normal to the surface

$$h(\mathbf{x}, t) = \hat{\mathbf{n}}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}, t) \tag{2.12}$$

and, so, by comparing eqs. (2.10) and (2.11), we obtain the divergence theorem

$$\int_{\delta\mathcal{V}} \hat{\mathbf{n}}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}, t)\, \mathrm{d}S_x = \int_{\mathcal{V}} \boldsymbol{\nabla} \cdot \mathbf{u}(\mathbf{x}, t)\, \mathrm{d}V_x \tag{2.13}$$

This proof, based on the displacement field, is mainly geometric. Nonetheless, it can be shown that the divergence theorem holds for any vector, $\mathbf{a}$, and tensor, $\mathbf{A}$, field

$$\int_{\delta\mathcal{V}} \hat{\mathbf{n}}(\mathbf{x}) \cdot \mathbf{a}(\mathbf{x},t)\,\mathrm{d}S_x = \int_{\mathcal{V}} \boldsymbol{\nabla} \cdot \mathbf{A}(\mathbf{x},t)\,\mathrm{d}V_x \tag{2.14a}$$

$$\int_{\delta\mathcal{V}} \hat{\mathbf{n}}(\mathbf{x}) \cdot \mathbf{A}(\mathbf{x},t)\,\mathrm{d}S_x = \int_{\mathcal{V}} \boldsymbol{\nabla} \cdot \mathbf{A}(\mathbf{x},t)\,\mathrm{d}V_x \tag{2.14b}$$

In the case of tensor field it is important to keep in mind that the scalar product between the normal and the tensor field is the left one.

2.3 Momentum equation

Let us start from the second Newton's law, which states that the total force acting of a body, $\mathbf{F}$, is equal to its mass m by the acceleration $\mathbf{A}$

$$\mathbf{F}(t) = m\,\mathbf{A}(t) \tag{2.15}$$

with t being the time.

Within the framework of continuous body, where each infinitesimal volume element may have a different acceleration (i.e., it may depend on the vector position $\mathbf{x}$), the right-hand side of eq. (2.15) can be rewritten as the following volume integral

$$m\,\mathbf{A}(t) = \int_{\mathcal{V}} \ddot{\mathbf{u}}(\mathbf{x},t)\,\mathrm{d}m = \int_{\mathcal{V}} \rho(\mathbf{x})\,\ddot{\mathbf{u}}(\mathbf{x},t)\,\mathrm{d}V_x \tag{2.16}$$

where the double dot $\cdot\cdot$ stands for the second order time derivative, $\mathcal{V}$ is the region occupied by the continuum and $\mathrm{d}m = \rho\,\mathrm{d}V_x$ is the mass of an infinitesimal volume element, with ρ and $\mathrm{d}V_x$ being its density and volume.

The total force acting of the body $\mathbf{F}$ can be decomposed into the contribution due to volume (as the gravity force) and surface forces

$$\mathbf{F}(t) = \mathbf{F}_{\mathrm{vol}}(t) + \mathbf{F}_{\mathrm{sur}}(t) = \int_{\mathcal{V}} \mathbf{f}(\mathbf{x}, t)\, \mathrm{d}V_x + \int_{\delta\mathcal{V}} \mathbf{T}(\mathbf{x}, t)\, \mathrm{d}S_x \tag{2.17}$$

where $\mathbf{f}$ is the body force per unit volume acting on an infinitesimal volume element and $\mathbf{T}$ is the traction (i.e., the surface force per unit area) acting on infinitesimal surface element of the surface $\delta\mathcal{V}$ with area $\mathrm{d}S_x$.

By considering that the traction can be expressed as the scalar product between the (symmetric) stress tensor $\boldsymbol{\sigma}$ and the (outward) normal $\hat{\mathbf{n}}$ to the infinitesimal surface element

$$\mathbf{T}(\mathbf{x}, t) = \boldsymbol{\sigma}(\mathbf{x}, t) \cdot \hat{\mathbf{n}}(\mathbf{x}) \tag{2.18}$$

and making use of the divergence theorem, eq. (2.14), the total force can be recast as the following volume integral

$$\mathbf{F}(t) = \int_{\mathcal{V}} \Big(\mathbf{f}(\mathbf{x}, t) + \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}(\mathbf{x}, t) \Big)\, \mathrm{d}V_x \tag{2.19}$$

The second Newton's law, eq. (2.15), together with eqs. (2.16) and (2.19), thus guarantees that the following volume integral equals to zero

$$\int_{\mathcal{V}} \Big(\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}(\mathbf{x}, t) + \mathbf{f}(\mathbf{x}, t) - \rho(\mathbf{x})\, \ddot{\mathbf{u}}(\mathbf{x}, t) \Big)\, \mathrm{d}V = \mathbf{0} \tag{2.20}$$

Also, in view of the arbitrariness of the volume $\mathcal{V}$, we can require that the integrand of eq. (2.20) have to be zero everywhere (i.e., for each infinitesimal volume element)

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}(\mathbf{x}, t) + \mathbf{f}(\mathbf{x}, t) = \rho(\mathbf{x})\, \ddot{\mathbf{u}}(\mathbf{x}, t) \tag{2.21}$$

that is the so called momentum equation

2.4 Linear elasticity

Let us introduce the strain tensor

$$\boldsymbol{\varepsilon}(\mathbf{x},t) = \frac{1}{2}\left[\boldsymbol{\nabla}\otimes\mathbf{u}(\mathbf{x},t) + \big(\boldsymbol{\nabla}\otimes\mathbf{u}(\mathbf{x},t)\big)^{\mathrm{T}}\right] \tag{2.22}$$

with $\mathbf{u}$ being the displacement field. Into Cartesian components, it reads

$$\varepsilon_{ij}(\mathbf{x},t) = \frac{1}{2}\left(\frac{\partial u_i(\mathbf{x},t)}{\partial x_j} + \frac{\partial u_j(\mathbf{x},t)}{\partial x_i}\right) \tag{2.23}$$

The modern generalization of the Hooke's law is that each component of the stress tensor is a linear combination of all components of the strain tensor

$$\boldsymbol{\sigma}(\mathbf{x},t) = \mathbf{c}(\mathbf{x}) : \boldsymbol{\varepsilon}(\mathbf{x},t) \tag{2.24}$$

where : stands for the double scalar product and $\mathbf{c}$ is the fourth-order elastic tensor. Into Cartesian components, it reads

$$\sigma_{ij}(\mathbf{x},t) = c_{ij\ell k}(\mathbf{x})\,\varepsilon_{\ell k}(\mathbf{x},t) \tag{2.25}$$

The Cartesian components of the fourth-order elastic tensor, are sometimes called elastic constants. Considering the symmetry of the stress and strain tensors, $\sigma_{ij} = \sigma_{ji}$ and $\varepsilon_{ij} = \varepsilon_{ji}$, and the existence of the strain-energy function (see section 2.5), the elastic constants satisfy

$$c_{ij\ell k} = c_{ji\ell k} = c_{ijk\ell} = c_{\ell kij} \tag{2.26}$$

These symmetry relations restrict to 21 the number of independent elastic constants and, in particular, allow us to rewrite eq. (2.25) as follows

$$\boldsymbol{\sigma}(\mathbf{x},t) = \mathbf{c}(\mathbf{x}) : \big(\boldsymbol{\nabla}\otimes\mathbf{u}(\mathbf{x},t)\big) = \big(\boldsymbol{\nabla}\otimes\mathbf{u}(\mathbf{x},t)\big) : \mathbf{c}(\mathbf{x}) \tag{2.27}$$

It can be shown that the most general isotropic fourth-order elastic tensor has the form

$$c_{ij\ell k}(\mathbf{x}) = \lambda(\mathbf{x})\, \delta_{ij}\, \delta_{\ell k} + \mu(\mathbf{x}) \left(\delta_{i\ell}\, \delta_{jk} + \delta_{ik}\, \delta_{j\ell}\right) \tag{2.28}$$

It involves only two independent constants, λ and μ, known as Lamè moduli. In this simple case, the Hooke's law reads

$$\boldsymbol{\sigma}(\mathbf{x}, t) = \lambda(\mathbf{x})\, \boldsymbol{\nabla} \cdot \mathbf{u}(\mathbf{x}, t)\, \mathbf{1} + 2\, \mu(\mathbf{x})\, \boldsymbol{\varepsilon}(\mathbf{x}, t) \tag{2.29}$$

with $\mathbf{1}$ being the identity tensor, or, equivalently, into Cartesian components

$$\sigma_{ij}(\mathbf{x}, t) = \lambda(\mathbf{x})\, \frac{\partial u_k(\mathbf{x}, t)}{\partial x_k}\, \delta_{ij} + 2\, \mu(\mathbf{x})\, \varepsilon_{ij}(\mathbf{x}, t) \tag{2.30}$$

2.5 Strain-energy function

Let us assume that the process are adiabatic and reversible (there is no time for heat diffusion during the seismic wave propagation, which occurs on much shorter times). We thus neglect the heating and we can compare the rate of doing mechanical work $\dot{W}$ with the rate of change of the kinetic $\dot{K}$ and internal energy $\dot{E}$

$$\dot{W}(t) = \dot{K}(t) + \dot{E}(t) \tag{2.31}$$

with the dot $\cdot$ stands for the time derivative.

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\sigma} \cdot \mathbf{u}') = (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}) \cdot \mathbf{u}' + (\boldsymbol{\nabla} \otimes \mathbf{u}') : \boldsymbol{\sigma} \tag{2.32}$$

The rate of doing mechanical work reads

$$\dot{W}(t) = \int_{\mathcal{V}} \mathbf{f}(\mathbf{x}, t) \cdot \dot{\mathbf{u}}(\mathbf{x}, t)\, \mathrm{d}V_x + \int_{\mathcal{S}} \mathbf{T}(\mathbf{x}, t) \cdot \dot{\mathbf{u}}(\mathbf{x}, t)\, \mathrm{d}S_x$$

$$
\begin{aligned}
&= \int_{\mathcal{V}} \Big[\mathbf{f}(\mathbf{x},t)\cdot\dot{\mathbf{u}}(\mathbf{x},t) + \boldsymbol{\nabla}\cdot\big(\boldsymbol{\sigma}(\mathbf{x},t)\cdot\dot{\mathbf{u}}(\mathbf{x},t)\big)\Big]\,\mathrm{d}V_x \\
&= \int_{\mathcal{V}} \Big[\big(\mathbf{f}(\mathbf{x},t) + \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}(\mathbf{x},t)\big)\cdot\dot{\mathbf{u}}(\mathbf{x},t) + \boldsymbol{\nabla}\otimes\dot{\mathbf{u}}(\mathbf{x},t) : \boldsymbol{\sigma}(\mathbf{x},t)\Big]\,\mathrm{d}V_x \\
&= \int_{\mathcal{V}} \Big[\rho(\mathbf{x})\,\ddot{\mathbf{u}}(\mathbf{x},t)\cdot\dot{\mathbf{u}}(\mathbf{x},t) + \dot{\boldsymbol{\varepsilon}}(\mathbf{x},t) : \boldsymbol{\sigma}(\mathbf{x},t)\Big]\,\mathrm{d}V_x
\end{aligned}
\tag{2.33}
$$

where we have used the divergence theorem, the momentum equation, eq. (2.21).

The kinetic energy, instead, can be expressed as follows

$$
K(t) = \frac{1}{2}\int_{\mathcal{V}} k(\mathbf{x},t)\,\mathrm{d}V \tag{2.34}
$$

with k being the kinetic energy per unit volume

$$
k(\mathbf{x},t) = \frac{1}{2}\,\rho(\mathbf{x})\,\dot{\mathbf{u}}(\mathbf{x},t)\cdot\dot{\mathbf{u}}(\mathbf{x},t) \tag{2.35}
$$

and note that its time derivative yields

$$
\dot{K}(t) = \int_{\mathcal{V}} \rho(\mathbf{x})\,\dot{\mathbf{u}}(\mathbf{x},t)\cdot\ddot{\mathbf{u}}(\mathbf{x},t)\,\mathrm{d}V \tag{2.36}
$$

From the comparison of eqs. (2.31), (2.33) and (2.36), we can understand that the rate of change of the internal energy is given by

$$
\dot{E}(t) = \int_{\mathcal{V}} \dot{e}(\mathbf{x},t)\,\mathrm{d}V \tag{2.37}
$$

with $\dot{e}$ being the rate of change of the internal energy per unit volume

$$
\dot{e}(\mathbf{x},t) = \dot{\boldsymbol{\varepsilon}}(\mathbf{x},t) : \boldsymbol{\sigma}(\mathbf{x},t) \tag{2.38}
$$

Within an infinitesimal time interval δt, the change of the internal energy per unit volume yields

$$\delta e(\mathbf{x}, t) = \delta \boldsymbol{\varepsilon}(\mathbf{x}, t) : \boldsymbol{\sigma}(\mathbf{x}, t) \tag{2.39}$$

which means that

$$\frac{\partial e}{\partial \varepsilon_{ij}} = \sigma_{ij} \tag{2.40}$$

once e is seen as function of the strain components. Moreover, within the assumption of linear elasticity, eq. (2.25), we also have

$$\frac{\partial \sigma_{ij}}{\partial \varepsilon_{\ell k}} = c_{ij\ell k} \tag{2.41}$$

and, so, the elastic constants can be regarded as the second order partial derivative of the internal energy per unit volume with respect to the strain components

$$\frac{\partial e}{\partial \varepsilon_{\ell k} \partial \varepsilon_{ij}} = c_{ij\ell k} \tag{2.42}$$

From this perspective and in view of the fact that we can change the order of the partial derivative, we obtain the last symmetry mentioned in eq. (2.26) of section 2.4

$$c_{ij\ell k} = c_{\ell k ij} \tag{2.43}$$

Having proved also this symmetry, we can then define the internal energy per unit volume as follows

$$e(\mathbf{x}, t) = \tfrac{1}{2}\, \boldsymbol{\varepsilon}(\mathbf{x}, t) : \boldsymbol{\sigma}(\mathbf{x}, t) \tag{2.44}$$

that is sometimes called also strain-energy function. Indeed, it is straightforward to show that its time derivative satisfies eq. (2.38). We note that also that it has been defined so that $e = 0$ when there is no deformation.

2.6 Seismic wave equation

Let su consider an isotropic and homogeneous elastic continuum. In this case, the only elastic constants characterizing the elastic tensor are the two Lamè parameters and we further assume that they are constant (they do not depend on space). We thus rewrite eq. (2.29) as follows

$$\boldsymbol{\sigma}(\mathbf{x},t) = \lambda\,\boldsymbol{\nabla}\cdot\mathbf{u}(\mathbf{x},t)\,\mathbf{1} + \mu\,\boldsymbol{\nabla}\otimes\mathbf{u}(\mathbf{x},t) + \mu\,\big(\boldsymbol{\nabla}\otimes\mathbf{u}(\mathbf{x},t)\big)^{\mathrm{T}} \tag{2.45}$$

Then, by making use of the following identities

$$\boldsymbol{\nabla}\cdot\big(\boldsymbol{\nabla}\cdot\mathbf{u}(\mathbf{x},t)\,\mathbf{1}\big) = \boldsymbol{\nabla}\cdot\big(\boldsymbol{\nabla}\otimes\mathbf{u}(\mathbf{x},t)\big)^{\mathrm{T}} = \boldsymbol{\nabla}\big(\boldsymbol{\nabla}\cdot\mathbf{u}(\mathbf{x},t)\big) \tag{2.46a}$$

$$\boldsymbol{\nabla}\cdot\big(\boldsymbol{\nabla}\otimes\mathbf{u}(\mathbf{x},t)\big) = \nabla^2\mathbf{u}(\mathbf{x},t) \tag{2.46b}$$

the momentum equation, eq. (2.21), becomes the so called seismic wave equation

$$(\lambda+\mu)\,\boldsymbol{\nabla}\big(\boldsymbol{\nabla}\cdot\mathbf{u}(\mathbf{x},t)\big) + \mu\,\nabla^2\mathbf{u}(\mathbf{x},t) + \mathbf{f}(\mathbf{x},t) = \rho\,\ddot{\mathbf{u}}(\mathbf{x},t) \tag{2.47}$$

Also, by making use of eq. (1.19), the seismic wave equation can be recast in the following form

$$c_p^2\,\boldsymbol{\nabla}\big(\boldsymbol{\nabla}\cdot\mathbf{u}(\mathbf{x},t)\big) - c_s^2\,\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{u}(\mathbf{x},t) + \rho^{-1}\,\mathbf{f}(\mathbf{x},t) = \ddot{\mathbf{u}}(\mathbf{x},t) \tag{2.48}$$

where, as we are going to show, c_p and c_s are the P and S wave velocities

$$\rho\,c_p^2 = \lambda + 2\,\mu \tag{2.49a}$$

$$\rho\,c_s^2 = \mu \tag{2.49b}$$

We recall again that the density and Lamè parameters are constant in space and, so, the seismic wave velocities do.

2.6.1 Helmholtz representation theorem

Let us now consider the Helmholtz representation theorem

$$\mathbf{u} = \boldsymbol{\nabla}\Phi + \boldsymbol{\nabla} \times \boldsymbol{\Psi} \tag{2.50}$$

where Φ and $\boldsymbol{\Psi}$ are the scalar and (solenoidal) vector potentials, with $\boldsymbol{\nabla}\cdot\boldsymbol{\Psi} = 0$, and substitute it into the momentum equation for the homogeneous space. After some straightforward algebra, we obtain

$$\boldsymbol{\nabla}\left[c_p^2\nabla^2\Phi - \ddot{\Phi}\right] + \boldsymbol{\nabla} \times \left[c_s^2\nabla^2\boldsymbol{\Psi} - \ddot{\boldsymbol{\Psi}}\right] + \rho^{-1}\,\mathbf{f} = \mathbf{0} \tag{2.51}$$

2.6.2 Plane waves

Let us consider the seismic wave equation in the form of eq. (2.51) expressed in terms of the scalar and vector potentials, eq. (2.50), in the case of zero body force and assuming that the vector potential is simply proportional to a constant unit vector

$$\mathbf{f}(\mathbf{x}, t) = \mathbf{0} \tag{2.52a}$$

$$\boldsymbol{\Psi}(\mathbf{x}, t) = \Psi(\mathbf{x}, t)\,\hat{\mathbf{a}} \tag{2.52b}$$

Within this assumption, eq. (2.50) becomes

$$\mathbf{u} = \boldsymbol{\nabla}\Phi + \boldsymbol{\nabla}\Psi \times \hat{\mathbf{a}} \tag{2.53}$$

and eq. (2.51) simplifies into

$$\boldsymbol{\nabla}\left[c_p^2\,\nabla^2\Phi - \ddot{\Phi}\right] + \boldsymbol{\nabla} \times \left\{\hat{\mathbf{a}}\left[c_s^2\nabla^2\Psi - \ddot{\Psi}\right]\right\} = \mathbf{0} \tag{2.54}$$

A specific set of solutions of eq. (2.54) can be found setting to zero both terms within the square brackets, that is to solve the following differential equation

$$c^2\,\nabla^2\phi - \ddot{\phi} = 0 \tag{2.55}$$

with $\phi = \Phi, \Psi$ and $c = c_p, c_s$, respectively.

Let us test the following trial solution

$$\phi(\mathbf{x}, t) = F(t - \mathbf{p}\cdot\mathbf{x}) \tag{2.56}$$

the gradient and Laplacian of which yield

$$\boldsymbol{\nabla}\phi(\mathbf{x}, t) = -\mathbf{p}\,f(t - \mathbf{p}\cdot\mathbf{x}) \tag{2.57a}$$

$$\nabla^2\phi(\mathbf{x}, t) = \mathbf{p}\cdot\mathbf{p}\,f(t - \mathbf{p}\cdot\mathbf{x}) \tag{2.57b}$$

with $f = \dot{F}$. Here F (or f) is an arbitrary function.

After substitution of eq. (2.56) into eq. (2.55), we obtain the constraint $\mathbf{p}\cdot\mathbf{p} = c^{-2}$ that we satisfy choosing $\mathbf{p}$ as

$$\mathbf{p} = \hat{\mathbf{p}}/c \tag{2.58}$$

with $\hat{\mathbf{p}}$ being a unit vector. This prove that eq. (2.56) with $\mathbf{p}$ given by eq. (2.58) solves eq. (2.55) and, so, it can be seen as the solution for the potentials Φ and Ψ from which we can obtain the displacement field. In particular, we can write

$$\mathbf{u}(\mathbf{x}, t) = \hat{\mathbf{p}}\,f_p(t - \hat{\mathbf{p}}\cdot\mathbf{x}/c_p) + \hat{\mathbf{a}}\times\hat{\mathbf{p}}\,f_s(t - \hat{\mathbf{p}}\cdot\mathbf{x}/c_s) \tag{2.59}$$

where we have chosen $f = -f_p\,c_p$ and $f = f_s\,c_s$ as arbitrary functions. We note also that the divergence of the vector potential yields

$$\boldsymbol{\nabla}\cdot\boldsymbol{\Psi}(\mathbf{x}, t) = -\hat{\mathbf{p}}\cdot\hat{\mathbf{a}}\,f_s(t - \hat{\mathbf{p}}\cdot\mathbf{x}/c_s) \tag{2.60}$$

and, so, in order that it yields zero, the two unit vectors $\hat{\mathbf{p}}$ and $\hat{\mathbf{a}}$ must be orthogonal. In this respect $\hat{\mathbf{a}} \times \mathbf{p}$ is orthogonal and it is also a unit vector. This allows us to rewrite eq. (2.59) as follows

$$\mathbf{u}(\mathbf{x}, t) = \hat{\mathbf{p}}\, f_p(t - \hat{\mathbf{p}} \cdot \mathbf{x}/c_p) + \hat{\mathbf{q}}\, f_s(t - \hat{\mathbf{p}} \cdot \mathbf{x}/c_s) \tag{2.61}$$

with $\hat{\mathbf{q}}$ being a unit vector perpendicular to $\hat{\mathbf{p}}$ and $\hat{\mathbf{a}}$.

We note that $\hat{\mathbf{p}}$ can be seen as the direction of propagation of the seismic waves and that the first term in eq. (2.61) represents P waves because the displacement occurs along the direction of propagation. The second term, instead, represents S waves because the displacement is perpendicular to the direction of propagation.

Fourier transform

Let us now consider the Fourier transform of an arbitrary function $f(t - t_0)$

$$\int_{\infty}^{\infty} f(t - t_0)\, e^{-i\,\omega\,t}\, \mathrm{d}t = e^{-i\,\omega\,t_0}\, \tilde{f}(\omega) \tag{2.62}$$

In this respect, the Fourier transform of the plane seismic wave, eq. (2.61), can be written as

$$\mathbf{u}(\mathbf{x}, \omega) = \hat{\mathbf{p}}\, \tilde{f}_p(\omega)\, e^{-i\,\omega\,\hat{\mathbf{p}}\cdot\mathbf{x}/c_p} + \hat{\mathbf{q}}\, \tilde{f}_s(\omega)\, e^{-i\,\omega\,\hat{\mathbf{p}}\cdot\mathbf{x}/c_s} \tag{2.63}$$

Also, the plane seismic wave in the time domain can be seen as

$$\mathbf{u}(\mathbf{x}, t) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} \left[\hat{\mathbf{p}}\, \tilde{f}_p(\omega)\, e^{i\,\omega\,(t-\hat{\mathbf{p}}\cdot\mathbf{x})/c_p} + \hat{\mathbf{q}}\, \tilde{f}_s(\omega)\, e^{i\,\omega\,(t-\hat{\mathbf{p}}\cdot\mathbf{x})/c_s}\right] \mathrm{d}\omega \tag{2.64}$$

It is noteworthy that the term of the form $\exp\big(i\,\omega\,(t - \hat{\mathbf{p}}\cdot\mathbf{x}/c)\big)$ is also solution of eq. (2.55) and represents a monochromatic wave with angular frequency ω. In this respect, eq. (2.64) means that the displacement field can be seen as the superimposition of monochromatic waves.

Equation (2.64) is often rewritten in terms of the wavenumber vector

$$\mathbf{k} = \omega\,\hat{\mathbf{p}}/c \tag{2.65}$$

as follows

$$\mathbf{u}(\mathbf{x},t) = \frac{1}{2\,\pi}\int_{-\infty}^{\infty}\Big[\hat{\mathbf{p}}\,\tilde{f}_p(\omega)\,e^{i\,(\omega\,t-\mathbf{k}_c\cdot\mathbf{x})} + \hat{\mathbf{q}}\,\tilde{f}_s(\omega)\,e^{i\,(\omega\,t-\mathbf{k}_s\cdot\mathbf{x})}\Big]\,\mathrm{d}\omega \tag{2.66}$$

We note that similar representations hold for the potentials Φ andf Ψ. In particular, eq. (2.56) can be recast as follows

$$\phi(\mathbf{x},t) = \frac{1}{2\,\pi}\int_{-\infty}^{\infty}\tilde{F}(\omega)\,e^{i\,(\omega\,t-\mathbf{k}\cdot\mathbf{x})}\,\mathrm{d}\omega \tag{2.67}$$

2.6.3 Monochromatic waves

When the spectrum in the frequency domain is concentrated at a specific frequency ω_0 like

$$\tilde{f}(\omega) = \pi\,\Big(A\,e^{i\,\phi}\,\delta(\omega-\omega_0) + A\,e^{-i\,\phi}\,\delta(\omega+\omega_0)\Big) \tag{2.68}$$

eq. (2.64) becomes

$$\begin{aligned}\mathbf{u}(\mathbf{x},t) = {} & A_p\,\hat{\mathbf{p}}\,\cos\big(\omega_0\,(t-\hat{\mathbf{p}}\cdot\mathbf{x}/c_p)+\phi_p\big)\\ & + A_s\,\hat{\mathbf{q}}\,\cos\big(\omega_0\,(t-\hat{\mathbf{p}}\cdot\mathbf{x}/c_s)+\phi_s\big)\end{aligned} \tag{2.69}$$

Note that eq. (2.68) satisfies the property $\tilde{f}(\omega) = \tilde{f}^*(-\omega)$, which guarantees that the inverse Fourier transform is a real function in the time domain.

Chapter 3

Representation of seismic sources

In this chapter, we will discuss the Betti's and representation theorems and we will use them to discuss the spatial reciprocity of the elastodynamics Green function and the equivalent body force to a displacement dislocation across a fault.

3.1 The Betti's theorem

Let us consider two displacements fields $\mathbf{u}$ and $\mathbf{u}'$ that solve the following momentum equations

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}(\mathbf{x}, \tau) + \mathbf{f}(\mathbf{x}, \tau) = \rho(\mathbf{x})\, \ddot{\mathbf{u}}(\mathbf{x}, \tau) \tag{3.1a}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}'(\mathbf{x}, \tau') + \mathbf{f}'(\mathbf{x}, \tau') = \rho(\mathbf{x})\, \ddot{\mathbf{u}}'(\mathbf{x}, \tau') \tag{3.1b}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ are the respective stress fields, while $\mathbf{f}$ and $\mathbf{f}'$ are the respective body forces. Note that we have evaluated the eqs. (3.1a) and (3.1b)) at the same spatial point $\mathbf{x}$ but at different times: τ and τ', respectively. In particular, in the following we shall choose τ' as

$$\tau' = t - \tau \tag{3.2}$$

in the perspective of performing the time convolution between the fields related to the two solutions $\mathbf{u}$ and $\mathbf{u}'$. In this respect, we recall that the time convolution between two time dependent functions f and h is defined as follows

$$f(t) \star h(t) = \int_{-\infty}^{\infty} f(t-\tau)\, h(\tau)\, \mathrm{d}t = \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, \mathrm{d}\tau \tag{3.3}$$

Let us now consider the scalar product of eq. (3.1a) with $\mathbf{u}'(\mathbf{x}, \tau')$ and of eq. (3.1b) with $\mathbf{u}(\mathbf{x}, \tau)$ and write them in Cartesian components

$$\big(\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}\big) \cdot \mathbf{u}' + \mathbf{f} \cdot \mathbf{u}' = \rho\, \ddot{\mathbf{u}} \cdot \mathbf{u}' \tag{3.4a}$$

$$\big(\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}'\big) \cdot \mathbf{u} + \mathbf{f}' \cdot \mathbf{u} = \rho\, \ddot{\mathbf{u}}' \cdot \mathbf{u} \tag{3.4b}$$

Here, for the sake of brevity, we have omitted the spatial and temporal dependences of the fields. We will explicit it later when it will be strictly necessary.

By making use of the following identity

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\sigma} \cdot \mathbf{u}') = (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}) \cdot \mathbf{u}' + (\boldsymbol{\nabla} \otimes \mathbf{u}') : \boldsymbol{\sigma} \tag{3.5}$$

eq. (3.4) becomes

$$\boldsymbol{\nabla} \cdot \big(\boldsymbol{\sigma} \cdot \mathbf{u}'\big) + \mathbf{f} \cdot \mathbf{u}' - \rho\, \ddot{\mathbf{u}} \cdot \mathbf{u}' = \big(\boldsymbol{\nabla} \otimes \mathbf{u}'\big) : \boldsymbol{\sigma} \tag{3.6a}$$

$$\boldsymbol{\nabla} \cdot \big(\boldsymbol{\sigma}' \cdot \mathbf{u}\big) + \mathbf{f}' \cdot \mathbf{u} - \rho\, \ddot{\mathbf{u}}' \cdot \mathbf{u} = \big(\boldsymbol{\nabla} \otimes \mathbf{u}\big) : \boldsymbol{\sigma}' \tag{3.6b}$$

Furthermore, in view of eq. (2.27) and considering that

$$\left(\boldsymbol{\nabla}\otimes\mathbf{u}'\right):\boldsymbol{\sigma}=\left(\boldsymbol{\nabla}\otimes\mathbf{u}'\right):\mathbf{c}:\left(\boldsymbol{\nabla}\otimes\mathbf{u}\right)=\boldsymbol{\sigma}':\left(\boldsymbol{\nabla}\otimes\mathbf{u}\right) \tag{3.7}$$

we note that the two right hand sides of eqs. (3.6a) and (3.6b) are equal and, so, we can equal the two left-hand sides and obtain

$$\boldsymbol{\nabla}\cdot\left(\boldsymbol{\sigma}\cdot\mathbf{u}'\right)-\boldsymbol{\nabla}\cdot\left(\boldsymbol{\sigma}'\cdot\mathbf{u}\right)+\mathbf{f}\cdot\mathbf{u}'-\mathbf{f}'\cdot\mathbf{u}=\rho\left(\ddot{\mathbf{u}}\cdot\mathbf{u}'-\ddot{\mathbf{u}}'\cdot\mathbf{u}\right) \tag{3.8}$$

Let us now make the choice for the time τ' at which we evaluate the auxiliary fields given by eq. (3.2) and define

$$f(\mathbf{x},\tau)=\dot{\mathbf{u}}(\mathbf{x},\tau)\cdot\mathbf{u}'(\mathbf{x},t-\tau)+\mathbf{u}(\mathbf{x},\tau)\cdot\dot{\mathbf{u}}'(\mathbf{x},t-\tau) \tag{3.9}$$

By requiring that the two displacement fields are quiescent in the past, which means that they are zero somewhen in the past

$$\mathbf{u}(\mathbf{x},\tau<t_0)=\mathbf{0} \tag{3.10a}$$

$$\mathbf{u}'(\mathbf{x},\tau<t_0')=\mathbf{0} \tag{3.10b}$$

for some times t_0 and t_0', respectively, and that they remain finite at later times, we note that

$$\lim_{\tau\to\pm\infty} f(\mathbf{x},\tau)=0 \tag{3.11}$$

The function f given by eq. (3.9) has been introduced because its time derivative yields

$$\dot{f}(\mathbf{x},\tau)=\ddot{\mathbf{u}}(\mathbf{x},\tau)\cdot\mathbf{u}'(\mathbf{x},t-\tau)-\ddot{\mathbf{u}}'(\mathbf{x},t-\tau)\cdot\mathbf{u}(\mathbf{x},\tau) \tag{3.12}$$

that is just the term within the round brackets of right hand side of eq. (3.8) with the choice of $\tau'=t-\tau$. In this respect, we can write

$$\begin{aligned}&\boldsymbol{\nabla}\cdot\big(\boldsymbol{\sigma}(\mathbf{x},\tau)\cdot\mathbf{u}'(\mathbf{x},t-\tau)\big)-\boldsymbol{\nabla}\cdot\big(\boldsymbol{\sigma}'(\mathbf{x},t-\tau)\cdot\mathbf{u}(\mathbf{x},\tau)\big)\\&+\mathbf{f}(\mathbf{x},\tau)\cdot\mathbf{u}'(\mathbf{x},t-\tau)-\mathbf{f}'(\mathbf{x},t-\tau)\cdot\mathbf{u}(\mathbf{x})=\rho\,\dot{f}(\mathbf{x},\tau)\end{aligned}\tag{3.13}$$

and, after time integration with respect to τ from $-\infty$ to ∞, we obtain

$$\begin{aligned}&\int_{-\infty}^{\infty}\Big[\boldsymbol{\nabla}\cdot\big(\boldsymbol{\sigma}(\mathbf{x},\tau)\cdot\mathbf{u}'(\mathbf{x},t-\tau)\big)-\boldsymbol{\nabla}\cdot\big(\boldsymbol{\sigma}'(\mathbf{x},t-\tau)\cdot\mathbf{u}(\mathbf{x},\tau)\big)\\&+\mathbf{f}(\mathbf{x},\tau)\cdot\mathbf{u}'(\mathbf{x},t-\tau)-\mathbf{f}'(\mathbf{x},t-\tau)\cdot\mathbf{u}(\mathbf{x},\tau)\Big]\,\mathrm{d}\tau=0\end{aligned}\tag{3.14}$$

In fact, in view of eq. (3.11), the time integration of $\dot{f}$ yields zero

$$\int_{-\infty}^{\infty}\dot{f}(\mathbf{x},\tau)\,\mathrm{d}\tau=f(\mathbf{x},\infty)-f(\mathbf{x},-\infty)=0\tag{3.15}$$

In the end, by taking the volume integral of eq. (3.14) and making use of the divergence theorem over a volume $\mathcal{V}$ with boundary $\delta\mathcal{V}$, we obtain the Betti's theorem for displacement fields with a quiescence past

$$\begin{aligned}&\int_{-\infty}^{\infty}\Big[\int_{\delta\mathcal{V}}\Big(\mathbf{u}'(\mathbf{x},t-\tau)\cdot\mathbf{T}(\mathbf{x},\tau)-\mathbf{u}(\mathbf{x},\tau)\cdot\mathbf{T}'(\mathbf{x},t-\tau)\Big)\,\mathrm{d}S_x\\&+\int_{\mathcal{V}}\Big(\mathbf{u}'(\mathbf{x},t-\tau)\cdot\mathbf{f}(\mathbf{x},\tau)-\mathbf{u}(\mathbf{x},\tau)\cdot\mathbf{f}'(\mathbf{x},t-\tau)\Big)\,\mathrm{d}V_x\Big]\,\mathrm{d}\tau=0\end{aligned}\tag{3.16}$$

where $\mathbf{T}$ and $\mathbf{T}'$ are the two tractions over the surface

$$\mathbf{T}(\mathbf{x},t)=\boldsymbol{\sigma}(\mathbf{x},t)\cdot\hat{\mathbf{n}}(\mathbf{x})\qquad\mathbf{x}\in\delta\mathcal{V}\tag{3.17a}$$

$$\mathbf{T}'(\mathbf{x},t)=\boldsymbol{\sigma}'(\mathbf{x},t)\cdot\hat{\mathbf{n}}(\mathbf{x})\qquad\mathbf{x}\in\delta\mathcal{V}\tag{3.17b}$$

3.2 The space reciprocity

Let us now specify the Betti's theorem in the case in which the two body forces $\mathbf{f}$ and $\mathbf{f}'$ are both impulsive and point-like

$$\mathbf{f}(\mathbf{x},t) = \mathbf{F}\,\delta(\mathbf{x}-\boldsymbol{\xi})\,\delta(t) \tag{3.18a}$$

$$\mathbf{f}'(\mathbf{x},t) = \mathbf{F}'\,\delta(\mathbf{x}-\boldsymbol{\xi}')\,\delta(t) \tag{3.18b}$$

where δ stands for the Dirac delta distribution, $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$ are the two vector positions where they are concentrated, while $\mathbf{F}$ and $\mathbf{F}'$ are two arbitrary constant vectors. Also, let us assume that the surface δV is traction free for both displacement fields

$$\mathbf{T}(\mathbf{x},t) = \mathbf{0} \qquad \mathbf{x} \in \delta\mathcal{V} \tag{3.19a}$$

$$\mathbf{T}'(\mathbf{x},t) = \mathbf{0} \qquad \mathbf{x} \in \delta\mathcal{V} \tag{3.19b}$$

Within these assumptions, the Betti's theorem, eq. (3.16), thus yields

$$\mathbf{u}(\boldsymbol{\xi}',t)\cdot\mathbf{F}' = \mathbf{u}'(\boldsymbol{\xi},t)\cdot\mathbf{F} \tag{3.20}$$

Let us now suppose that the displacement fields can be expressed in terms of the elastodynamics Green function $\mathbf{G}$ as follows

$$\mathbf{u}(\mathbf{x},t) = \mathbf{G}(\mathbf{x},t,\boldsymbol{\xi})\cdot\mathbf{F} \tag{3.21a}$$

$$\mathbf{u}'(\mathbf{x},t) = \mathbf{G}(\mathbf{x},t,\boldsymbol{\xi}')\cdot\mathbf{F}' \tag{3.21b}$$

as it is in the case of solution of linear differential system. Equation (3.20) thus yields

$$\mathbf{F}'\cdot\mathbf{G}(\boldsymbol{\xi}',t,\boldsymbol{\xi})\cdot\mathbf{F} = \mathbf{F}\cdot\mathbf{G}(\boldsymbol{\xi},t,\boldsymbol{\xi}')\cdot\mathbf{F}' \tag{3.22}$$

and, in view of the arbitrariness of $\mathbf{F}$ and $\mathbf{F}$, we can write

$$\mathbf{G}(\boldsymbol{\xi}', t, \boldsymbol{\xi}) = \mathbf{G}^{\mathrm{T}}(\boldsymbol{\xi}, t, \boldsymbol{\xi}') \tag{3.23}$$

which specify a purely spatial reciprocity.This result implies that, if we know the Green function for the displacement field due to a body force concentrated at $\boldsymbol{\xi}'$ and evaluated at $\boldsymbol{\xi}$, then we also know the Green function for the displacement field due to a body force concentrated at $\boldsymbol{\xi}$ and evaluated at $\boldsymbol{\xi}'$.

3.3 Representation theorem

Let us consider the Betti's theorem, eq. (3.16), in the case that the continuum contains a fault with surface $\mathcal{S}_f$ and that the volume $\mathcal{V}$ contains all the continuum but for the fault. As depicted in fig. 3.1, the surface $\delta\mathcal{V}$ thus composes of the outer surface $\mathcal{S}_0$ and the two surfaces just above, $\mathcal{S}_f^+$, and below, $\mathcal{S}_f^-$, the fault. We also requires that the outer surface $\mathcal{S}_0$ is traction free for both displacement fields

$$\mathbf{T}(\mathbf{x}, t) = \mathbf{0} \qquad \mathbf{x} \in \mathcal{S}_0 \tag{3.24a}$$

$$\mathbf{T}'(\mathbf{x}, t) = \mathbf{0} \qquad \mathbf{x} \in \mathcal{S}_0 \tag{3.24b}$$

and that the displacement field $\mathbf{u}'$ is due to an impulsive and point-like force (as in section 3.2, eq. (3.18b)) concentrated at $\boldsymbol{\xi}' \in \mathcal{V}$ which does not belong to the fault . No other assumptions, instead, are made for the displacement field $\mathbf{u}$, but for the fact that it has a quiescent past.

Within these assumption, the Betti's theorem, eq. (3.14), becomes

$$\mathbf{u}(\boldsymbol{\xi}', t) \cdot \mathbf{F}' = \int_{-\infty}^{\infty} \left[\int_{\mathcal{V}} \mathbf{f}(\mathbf{x}, \tau) \star \mathbf{u}'(\mathbf{x}, t - \tau) \; \mathrm{d}V_x \right.$$

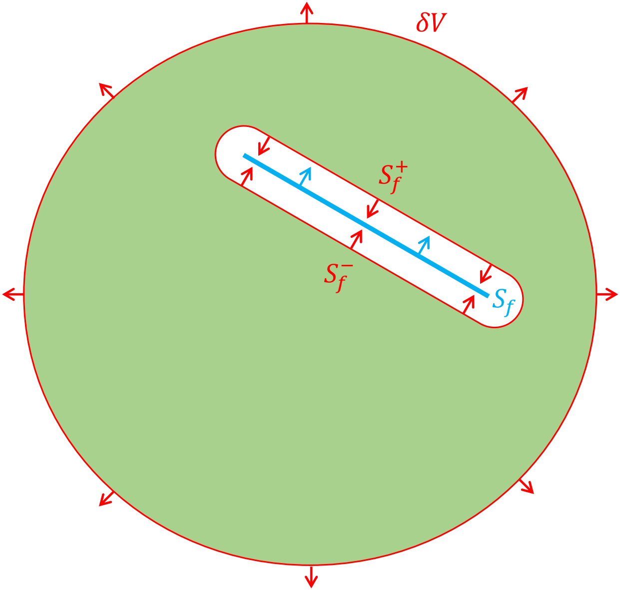

Figure 3.1: Cartoon depicting the volume $\mathcal{V}$ (green region) which excludes the fault surface $\mathcal{S}_f$ (blue line). The red contours represent the outer surface $\delta\mathcal{V}$ which composes of $\mathcal{S}_0$ and the two surfaces just above, $\mathcal{S}_f^+$, and below, $\mathcal{S}_f^-$, the fault. The red arrows represents the normal unit vectors $\hat{\mathbf{n}}$ pointing outwards to the volume δV and the blue arrows represent the normal unit vectors over the fault surface.

$$+ \int_{\mathcal{S}_f^+} \big(\mathbf{u}'(\mathbf{x}, t-\tau) \cdot \mathbf{T}(\mathbf{x}, \tau) - \mathbf{u}(\mathbf{x}, \tau) \cdot \mathbf{T}'(\mathbf{x}, t-\tau)\big)\, \mathrm{d}S_x$$

$$+ \int_{\mathcal{S}_f^-} \big(\mathbf{u}'(\mathbf{x}, t-\tau) \cdot \mathbf{T}(\mathbf{x}, \tau) - \mathbf{u}(\mathbf{x}, \tau) \cdot \mathbf{T}'(\mathbf{x}, t-\tau)\big)\, \mathrm{d}S_x\bigg]\, \mathrm{d}\tau \tag{3.25}$$

Furthermore, we note that the displacement field $\mathbf{u}'$ is continuous across the fault, as well as the stress field $\boldsymbol{\sigma}'$. In light of this, eq. (3.25) becomes

$$\mathbf{u}(\boldsymbol{\xi}', t) \cdot \mathbf{F}' = \int_{-\infty}^{\infty} \bigg[\int_{\mathcal{V}} \mathbf{f}(\mathbf{x}, \tau) \cdot \mathbf{u}'(\mathbf{x}, t-\tau)\ \mathrm{d}V_x$$

$$+ \int_{\mathcal{S}_f} \Big(- \mathbf{u}'(\mathbf{x}, t-\tau) \cdot \boldsymbol{\sigma}(\mathbf{x}^+, \tau) + \mathbf{u}(\mathbf{x}^+, \tau) \cdot \boldsymbol{\sigma}'(\mathbf{x}, t-\tau)$$

$$+ \mathbf{u}'(\mathbf{x}, t-\tau) \cdot \boldsymbol{\sigma}(\mathbf{x}^+, \tau) - \mathbf{u}(\mathbf{x}^+, \tau) \cdot \boldsymbol{\sigma}'(\mathbf{x}, t-\tau)\Big) \cdot \hat{\mathbf{n}}(\mathbf{x})\, \mathrm{d}S_x\bigg]\, \mathrm{d}\tau \tag{3.26}$$

with

$$\mathbf{x}^{\pm} = \mathbf{x} \pm \epsilon\, \hat{\mathbf{n}}(\mathbf{x}) \tag{3.27}$$

where ε is the distance between $\mathcal{S}_f$ and the two surfaces S_f^+ and Sf^-, while $\hat{\mathbf{n}}$ is the normal unit vector of the fault surface $\mathcal{S}_f$ pointing towards the top surface $\mathcal{S}_f^+$. As shown in fig. 3.1, we note that the normal unit vector over the above surface $\mathcal{S}_f^+$ (red arrows) is opposite to the normal unit vector over the fault surface $\mathcal{S}_f$ (blue arrows).

By defining the following discontinuities across the fault

$$\delta\mathbf{u}(\mathbf{x}, t) = \lim_{\varepsilon \to 0} \Big[\mathbf{u}(\mathbf{x}^+, t) - \mathbf{u}(\mathbf{x}^-, t)\Big] \tag{3.28a}$$

$$\delta \mathbf{T}(\mathbf{x},t) = \lim_{\varepsilon \to 0} \Big[\boldsymbol{\sigma}(\mathbf{x}^+,t) - \boldsymbol{\sigma}(\mathbf{x}^-,t) \Big] \cdot \hat{\mathbf{n}}(\mathbf{x}) \tag{3.28b}$$

eq. (3.26) becomes

$$\mathbf{u}(\boldsymbol{\xi}',t) \cdot \mathbf{F}' = \int_{-\infty}^{\infty} \Bigg[\int_{\mathcal{V}} \mathbf{f}(\mathbf{x},\tau) \star \mathbf{u}'(\mathbf{x},t-\tau)\,\mathrm{d}V_x \\ + \int_{\mathcal{S}_f} \Big(\delta \mathbf{u}(\mathbf{x},\tau) \otimes \hat{\mathbf{n}}(\mathbf{x}) : \boldsymbol{\sigma}'(\mathbf{x},t-\tau) + \delta \mathbf{T}(\mathbf{x},\tau) \cdot \mathbf{u}'(\mathbf{x},t-\tau) \Big)\,\mathrm{d}S_x \Bigg]\,\mathrm{d}\tau \tag{3.29}$$

In the following, we shall focus only on the case in which the displacement field $\mathbf{u}$ is discontinuous across the fault, while the traction is continuous as it is in the case of an earthquake. Within this assumption the traction discontinuity is zero, $\delta \mathbf{T} = 0$, and we can further simplify eq. (3.29) as follows

$$\mathbf{u}(\boldsymbol{\xi}',t) \cdot \mathbf{F}' = \int_{-\infty}^{\infty} \Bigg[\int_{\mathcal{V}} \mathbf{f}(\mathbf{x},\tau) \star \mathbf{u}'(\mathbf{x},t-\tau)\,\mathrm{d}V_x \\ + \int_{\mathcal{S}_f} \delta \mathbf{u}(\mathbf{x},\tau) \otimes \hat{\mathbf{n}}(\mathbf{x}) : \boldsymbol{\sigma}'(\mathbf{x},t-\tau)\,\mathrm{d}S_x \Bigg]\,\mathrm{d}\tau \tag{3.30}$$

Furthermore, by recalling that the displacement field $\mathbf{u}'$ is due to an impulsive and point-like force concentrated at $\boldsymbol{\xi}'$, we can use eq. (3.21a) and write

$$\boldsymbol{\sigma}'(\mathbf{x},t) = \mathbf{c}(\mathbf{x}) : \boldsymbol{\nabla}_x \otimes \mathbf{G}(\mathbf{x},t,\boldsymbol{\xi}') \cdot \mathbf{F}' \tag{3.31}$$

This allows to recast eq. (3.30) in terms of the only elastodynamics Green function

$$\mathbf{u}(\boldsymbol{\xi}',t) \cdot \mathbf{F}' = \Bigg\{ \int_{-\infty}^{\infty} \Bigg[\int_{\mathcal{V}} \mathbf{f}(\mathbf{x},\tau) \cdot \mathbf{G}(\mathbf{x},t-\tau,\boldsymbol{\xi}')\,\mathrm{d}V_x$$

$$+\int_{\mathcal{S}_f} \mathbf{m}(\mathbf{x},\tau) : \boldsymbol{\nabla}_x \otimes \mathbf{G}(\mathbf{x}, t-\tau, \boldsymbol{\xi}')\,\mathrm{d}S_x \Bigg] \,\mathrm{d}\tau \Bigg\} \cdot \mathbf{F}' \tag{3.32}$$

where $\mathbf{m}$ is the moment density tensor

$$\mathbf{m}(\mathbf{x},t) = \delta\mathbf{u}(\mathbf{x},t) \otimes \hat{\mathbf{n}}(\mathbf{x}) : \mathbf{c}(\mathbf{x}) \tag{3.33}$$

which describes the geometry of the fault and the time evolution of the dislocation over it.

In the end, in view of the arbitrariness of $\mathbf{F}'$, eq. (3.32) implies that

$$\mathbf{u}(\boldsymbol{\xi}',t) = \int_{-\infty}^{\infty} \Bigg[\int_{\mathcal{V}} \mathbf{f}(\mathbf{x},\tau) \cdot \mathbf{G}(\mathbf{x}, t-\tau, \boldsymbol{\xi}')\,\mathrm{d}V_x \\ + \int_{\mathcal{S}_f} \mathbf{m}(\mathbf{x},\tau) : \boldsymbol{\nabla}_x \otimes \mathbf{G}(\mathbf{x}, t-\tau, \boldsymbol{\xi}')\,\mathrm{d}S_x \Bigg] \,\mathrm{d}\tau \tag{3.34}$$

and, for the sake of clarity, we change the variables $\boldsymbol{\xi}' \to \mathbf{x}$ and $\mathbf{x} \to \boldsymbol{\xi}$, and write

$$\mathbf{u}(\mathbf{x},t) = \int_{-\infty}^{\infty} \Bigg[\int_{\mathcal{V}} \mathbf{f}(\boldsymbol{\xi},\tau) \cdot \mathbf{G}(\boldsymbol{\xi}, t-\tau, \mathbf{x})\,\mathrm{d}V_\xi \\ + \int_{\mathcal{S}_f} \mathbf{m}(\boldsymbol{\xi},\tau) : \boldsymbol{\nabla}_\xi \otimes \mathbf{G}(\boldsymbol{\xi}, t-\tau, \mathbf{x})\,\mathrm{d}S_\xi \Bigg] \,\mathrm{d}\tau \tag{3.35}$$

where the surface and volume integrals are performed with respect to $\boldsymbol{\xi}$.

Equation (3.35) allows us to calculate the displacement field $\mathbf{u}$ at the observation point $\mathbf{x}$ in terms of (i) the time convolution and volume integral of the body force $\mathbf{f}$ and the elastodynamics Green function for an impulsive and point-like force and of (ii) the time convolution and integral over the fault surface of the displacement discontinuity and a term involving the spatial derivative of the elastodynamics Green function. This is the representation theorem for the displacement field due to a body force $\mathbf{f}$ and a prescribed displacement discontinuity over the fault.

In view of the space reciprocity of the Green function, eq. (3.23), we recast eq. (3.35) as follows

$$\mathbf{u}(\mathbf{x},t) = \int_{-\infty}^{\infty} \left[\int_{\mathcal{V}} \mathbf{f}(\boldsymbol{\xi},\tau) \cdot \mathbf{G}^{\mathrm{T}}(\mathbf{x}, t-\tau, \boldsymbol{\xi})\, \mathrm{d}V_{\xi} \right. \\ \left. + \int_{\mathcal{S}_f} \mathbf{m}(\boldsymbol{\xi},\tau) : \boldsymbol{\nabla}_{\xi} \otimes \mathbf{G}^{\mathrm{T}}(\mathbf{x}, t-\tau, \boldsymbol{\xi})\, \mathrm{d}S_{\xi} \right] \mathrm{d}\tau \tag{3.36}$$

3.4 Equivalent body force

The first and second terms in the right-hand side of eq. (3.34) constitute two distinct contributions to the displacement field. It is interesting to determine which is the body force that yields the same displacement field due to a displacement discontinuity over the fault surface. We can do it by equaling them and rewriting the surface integral as follows

$$\int_{\mathcal{V}} \mathbf{f}(\boldsymbol{\xi},\tau) \cdot \mathbf{G}^{\mathrm{T}}(\mathbf{x}, t-\tau, \boldsymbol{\xi})\, \mathrm{d}V_{\xi} = \int_{\mathcal{S}_f} \mathbf{m}(\boldsymbol{\xi},\tau) : \boldsymbol{\nabla}_{\xi} \otimes \mathbf{G}^{\mathrm{T}}(\mathbf{x}, t-\tau, \boldsymbol{\xi})\, \mathrm{d}S_{\xi} \\ = -\int_{\mathcal{V}} \int_{\mathcal{S}_f} \mathbf{m}(\boldsymbol{\xi},\tau) : \left[\boldsymbol{\nabla}_{\eta} \delta(\boldsymbol{\eta}-\boldsymbol{\xi}) \otimes \mathbf{G}^{\mathrm{T}}(\mathbf{x}, t, \boldsymbol{\eta})\right] \mathrm{d}S_{\xi}\, \mathrm{d}V_{\eta} \\ = \int_{\mathcal{V}} \left[-\int_{\mathcal{S}_f} \mathbf{m}(\boldsymbol{\xi},\tau) \cdot \boldsymbol{\nabla}_{\eta} \delta(\boldsymbol{\eta}-\boldsymbol{\xi})\, \mathrm{d}S_{\xi} \right] \cdot \mathbf{G}^{\mathrm{T}}(\mathbf{x}, t, \boldsymbol{\eta})\, \mathrm{d}V_{\eta} \tag{3.37}$$

Here, we have made use of the elastodynamics Green function recast as the following volume integral

$$\mathbf{G}^{\mathrm{T}}(\mathbf{x}, t, \boldsymbol{\xi}) = \int_{\mathcal{V}} \delta(\boldsymbol{\eta} - \boldsymbol{\xi})\, \mathbf{G}^{\mathrm{T}}(\mathbf{x}, t, \boldsymbol{\eta})\, \mathrm{d}V_\eta \tag{3.38}$$

and its partial derivative

$$\boldsymbol{\nabla}_\xi \otimes \mathbf{G}^{\mathrm{T}}(\mathbf{x}, t, \boldsymbol{\xi}) = -\int_{\mathcal{V}} \boldsymbol{\nabla}_\eta \delta(\boldsymbol{\eta} - \boldsymbol{\xi}) \otimes \mathbf{G}^{\mathrm{T}}(\mathbf{x}, t, \boldsymbol{\eta})\, \mathrm{d}V_\eta \tag{3.39}$$

Then, by noting then both the left and right hand sides of eq. (3.37) involves a volume integral, we can understand that it can be satisfied once the body force is chosen as follows

$$\mathbf{f}(\boldsymbol{\eta}, \tau) = \int_{\mathcal{S}_f} \mathrm{d}\mathbf{f}(\boldsymbol{\eta}, \tau, \boldsymbol{\xi}) \tag{3.40}$$

with

$$\mathrm{d}\mathbf{f}(\boldsymbol{\eta}, \tau, \boldsymbol{\xi}) = -\mathbf{m}(\boldsymbol{\xi}, \tau) \cdot \boldsymbol{\nabla}\delta(\boldsymbol{\eta} - \boldsymbol{\xi})\, \mathrm{d}S_\xi \tag{3.41}$$

3.4.1 Centre of explosion, dipole and double couple

Let us now consider the case of an isotropic elastic medium. In this case the moment density tensor given by eq. (3.33) reads

$$\mathbf{m} = \lambda \left(\delta\mathbf{u} \cdot \hat{\mathbf{n}}\right) \mathbf{1} + \mu \left(\delta\mathbf{u} \otimes \hat{\mathbf{n}} + \hat{\mathbf{n}} \otimes \delta\mathbf{u}\right) \tag{3.42}$$

Also, let us distinguish between tensile and shear displacement dislocations

$$\delta\mathbf{u} = \delta u^{\perp}\, \hat{\mathbf{n}} + \delta u^{\parallel}\, \hat{\mathbf{s}} \tag{3.43}$$

with $\hat{\mathbf{s}} \cdot \hat{\mathbf{n}} = 0$. With this distinction eq. (3.42) becomes

$$\mathbf{m} = \left(\lambda\, \mathbf{1} + 2\,\mu\, \hat{\mathbf{n}} \otimes \hat{\mathbf{n}}\right) \delta u^{\perp} + \mu \left(\hat{\mathbf{s}} \otimes \hat{\mathbf{n}} + \hat{\mathbf{n}} \otimes \hat{\mathbf{s}}\right) \delta u^{\parallel} \tag{3.44}$$

Then, by making use of the following identity

$$\begin{aligned}\hat{\mathbf{a}}\cdot\boldsymbol{\nabla}\delta(\boldsymbol{\eta}-\boldsymbol{\xi}) &= \lim_{\varepsilon\to 0}\frac{\delta(\boldsymbol{\eta}+\varepsilon\,\hat{\mathbf{a}}-\boldsymbol{\xi})-\delta(\boldsymbol{\eta}-\varepsilon\,\hat{\mathbf{a}}-\boldsymbol{\xi})}{2\,\varepsilon}\\ &= -\lim_{\varepsilon\to 0}\frac{\delta(\boldsymbol{\eta}-(\boldsymbol{\xi}+\varepsilon\,\hat{\mathbf{a}}))-\delta(\boldsymbol{\eta}-(\boldsymbol{\xi}-\varepsilon\,\hat{\mathbf{a}}))}{2\,\varepsilon}\end{aligned} \tag{3.45}$$

eq. (3.41) can be recast as follows

$$\mathrm{d}\mathbf{f}(\boldsymbol{\eta},\tau,\boldsymbol{\xi}) = \mathrm{d}\mathbf{f}_{\mathrm{dcp}}(\boldsymbol{\eta},\tau,\boldsymbol{\xi}) + \mathrm{d}\mathbf{f}_{\mathrm{dip}}(\boldsymbol{\eta},\tau,\boldsymbol{\xi}) + \mathrm{d}\mathbf{f}_{\mathrm{exp}}(\boldsymbol{\eta},\tau,\boldsymbol{\xi}) \tag{3.46}$$

with

$$\begin{aligned}\mathrm{d}\mathbf{f}_{\mathrm{dcp}}(\boldsymbol{\eta},\tau,\boldsymbol{\xi}) = \lim_{\varepsilon\to 0}\frac{\mu\,\delta u^{\parallel}\,\mathrm{d}S_{\xi}}{2\,\varepsilon}\Big\{&\hat{\mathbf{s}}\left[\delta(\boldsymbol{\eta}-(\boldsymbol{\xi}+\varepsilon\,\hat{\mathbf{n}}))-\delta(\boldsymbol{\eta}-(\boldsymbol{\xi}-\varepsilon\,\hat{\mathbf{n}}))\right]\\ &+\hat{\mathbf{n}}\left[\delta(\boldsymbol{\eta}-(\boldsymbol{\xi}+\varepsilon\,\hat{\mathbf{s}}))-\delta(\boldsymbol{\eta}-(\boldsymbol{\xi}-\varepsilon\,\hat{\mathbf{s}}))\right]\Big\}\end{aligned} \tag{3.47a}$$

$$\mathrm{d}\mathbf{f}_{\mathrm{dip}}(\boldsymbol{\eta},\tau,\boldsymbol{\xi}) = \lim_{\varepsilon\to 0}\frac{2\,\mu\,\delta u^{\perp}\,\mathrm{d}S_{\xi}}{2\,\varepsilon}\hat{\mathbf{n}}\left[\delta(\boldsymbol{\eta}-(\boldsymbol{\xi}+\varepsilon\,\hat{\mathbf{n}}))-\delta(\boldsymbol{\eta}-(\boldsymbol{\xi}-\varepsilon\,\hat{\mathbf{n}}))\right] \tag{3.47b}$$

$$\mathrm{d}\mathbf{f}_{\mathrm{exp}}(\boldsymbol{\eta},\tau,\boldsymbol{\xi}) = \lim_{\varepsilon\to 0}\frac{2\,\mu\,\delta u^{\perp}\,\mathrm{d}S_{\xi}}{2\,\varepsilon}\hat{\mathbf{x}}_{x}\left[\delta(\boldsymbol{\eta}-(\boldsymbol{\xi}+\varepsilon\,\hat{\mathbf{x}}_{k}))-\delta(\boldsymbol{\eta}-(\boldsymbol{\xi}-\varepsilon\,\hat{\mathbf{x}}_{k}))\right] \tag{3.47c}$$

Here, the first term, $\mathrm{d}\mathbf{f}_{\mathrm{dcp}}$, represents a double couple of forces with zero net force and torque (fig. 3.2). The second term, $\mathrm{d}\mathbf{f}_{\mathrm{dip}}$, instead, represents one dipole of point-like forces (opposite to each other) along the normal to the infinitesimal surface element and concentrated at points shifted with respect to $\boldsymbol{\xi}$ in the same direction of the forces (fig. 3.3). In the end, the third term, $\mathrm{d}\mathbf{f}_{\mathrm{exp}}$, represents a centre of explosion characterized by three dipoles of point-like forces. The amplitude of the forces are equal in all directions and, so, it is isotropic.

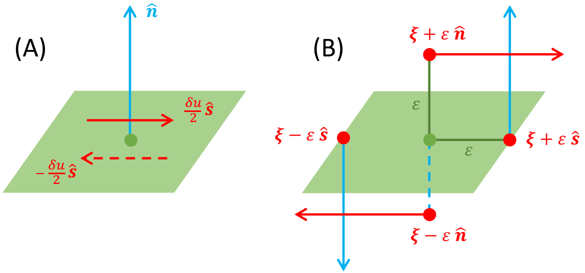

Figure 3.2: Cartoons depicting (A) the tangential (or shear) displacement dislocation $\delta u\,\hat{\mathbf{s}}$ of an infinitesimal surface element of the fault (green square) with normal $\hat{\mathbf{n}}$, (B) the equivalent double couple of forces with zero net force and torque.

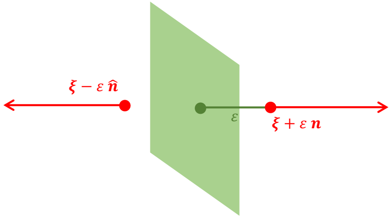

Figure 3.3: Cartoon depicting a dipole of forces along the normal to the infinitesimal surface element and shifted in the same direction of the forces.

Chapter 4

Seismic waves and sources

In order to calculate the displacement field due to a displacement discontinuity over a fault making use of the representation theorem, eq. (3.36), hereinafter we will derive first the analytical solution for the displacement field due to a point-like force concentrated at $\boldsymbol{\xi}$

$$\mathbf{f}(\mathbf{x}, t) = \delta(\mathbf{x} - \boldsymbol{\xi})\, \mathbf{P}(t) \tag{4.1}$$

with $\mathbf{P}$ being a vector constant in space. This is possible within the assumption of a homogeneous and isotropic continuum (i.e., when the elastic constants obeys eq. (2.28) with λ and μ being constant in space).

In this perspective, it is convenient to introduce the following spherical coordinate system for identifying the observation point $\mathbf{x}$ with respect to the point where the force is concentrated

$$\mathbf{x} = \boldsymbol{\xi} + r\, \hat{\mathbf{r}}(\theta, \phi) \tag{4.2}$$

where r is the distance between $\mathbf{x}$ and $\boldsymbol{\xi}$

$$r = \|\mathbf{x} - \boldsymbol{\xi}\| \tag{4.3}$$

while $\hat{\mathbf{r}}$ is the radial unit vector calculated from $\boldsymbol{\xi}$ which depends on the colatitude and longitude only.

We refer to the reader to 1.4 for further details about the spherical coordinate system.

4.1 Elastic waves due to point-like forces

For the homogeneous and isotropic continuum (i.e., when the elastic constants obeys eq. (2.28) with λ and μ being constant in space) and making use of the Helmholtz representation theorem, the seismic wave equation takes the form given by eq. (2.51), which we report hereinafter

$$\boldsymbol{\nabla}\left[c_p^2\nabla^2\Phi - \ddot{\Phi}\right] + \boldsymbol{\nabla}\times\left[c_s^2\nabla^2\boldsymbol{\Psi} - \ddot{\boldsymbol{\Psi}}\right] + \frac{\mathbf{f}}{\rho} = \mathbf{0} \tag{4.4}$$

Also, let us further assume that the scalar and vector potentials take the following form

$$\Phi = \boldsymbol{\nabla}\cdot\mathbf{A}_p \tag{4.5a}$$

$$\boldsymbol{\Psi} = -\boldsymbol{\nabla}\times\mathbf{A}_s \tag{4.5b}$$

with $\mathbf{A}_p$ and $\mathbf{A}_s$ being two vector fields. Within this assumption, the displacement field in eq. (2.50) becomes

$$\mathbf{u} = \boldsymbol{\nabla}\,\boldsymbol{\nabla}\cdot\mathbf{A}_p - \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{A}_p = \boldsymbol{\nabla}\,\boldsymbol{\nabla}\cdot\left(\mathbf{A}_p - \mathbf{A}_s\right) + \nabla^2\mathbf{A}_s \tag{4.6}$$

After substitution of eq. (4.5) into eq. (4.4), we obtain

$$\nabla\,\nabla\cdot\left[c_p^2\nabla^2\mathbf{A}_p-\ddot{\mathbf{A}}_p\right]-\nabla\times\nabla\times\left[c_s^2\nabla^2\mathbf{A}_s-\ddot{\mathbf{A}}_s\right]+\frac{\mathbf{f}}{\rho}=\mathbf{0} \tag{4.7}$$

This form of the seismic wave equation is useful for finding the solution for the displacement field in the case of a point-like body force like eq. (4.1). Indeed, by making use of the following identities

$$\delta(\mathbf{x}-\boldsymbol{\xi})=-\frac{1}{4\,\pi}\,\nabla^2 r^{-1} \tag{4.8}$$

we can recast eq. (4.1) as follows

$$\begin{aligned}\mathbf{f}(\mathbf{x},t)&=\mathbf{P}(t)\,\nabla^2\left(-\frac{1}{4\,\pi\,r}\right)=-\nabla^2\left(\frac{\mathbf{P}(t)}{4\,\pi\,r}\right)\\&=-\nabla\nabla\cdot\left(\frac{\mathbf{P}(t)}{4\,\pi\,r}\right)+\nabla\times\nabla\times\left(\frac{\mathbf{P}(t)}{4\,\pi\,r}\right)\end{aligned} \tag{4.9}$$

that, once substituted in eq. (4.7), yields

$$\begin{aligned}&\nabla\,\nabla\cdot\left[c_p^2\,\nabla^2\mathbf{A}_p(\mathbf{x},t)-\ddot{\mathbf{A}}_p(\mathbf{x},t)-\frac{\mathbf{P}(t)}{4\,\pi\,\rho\,r}\right]\\&-\nabla\times\nabla\times\left[c_s^2\nabla^2\mathbf{A}_s(\mathbf{x},t)-\ddot{\mathbf{A}}_s(\mathbf{x},t)-\frac{\mathbf{P}(t)}{4\,\pi\,\rho\,r}\right]=\mathbf{0}\end{aligned} \tag{4.10}$$

In this respect, the solution of the following differential equation

$$c^2\,\nabla^2\mathbf{A}(\mathbf{x},t)-\ddot{\mathbf{A}}(\mathbf{x},t)-\frac{\mathbf{P}(t)}{4\,\pi\,\rho\,r}=\mathbf{0} \tag{4.11}$$

allows us to obtain both $\mathbf{A}_p$ and $\mathbf{A}_s$, as well as the displacement field $\mathbf{u}$ using eq. (4.6).

4.1.1 The analytical solution

Let us first consider the solution of the homogeneous differential equation

$$c^2\,\nabla^2\mathbf{A}_{\mathrm{homo}} - \ddot{\mathbf{A}}_{\mathrm{homo}} = \mathbf{0} \tag{4.12}$$

Let us assume that $\mathbf{A}_{\mathrm{homo}}$ depends on the vector position $\mathbf{x}$ only trough the its module r (i.e., the distance from the origin where the body force is concentrated). In this case and for $r > 0$, the Laplacian yields

$$\nabla^2\mathbf{A}_{\mathrm{homo}} = \frac{1}{r}\,\frac{\partial^2}{\partial r^2}\big(r\,\mathbf{A}_{\mathrm{homo}}\big) \tag{4.13}$$

and eq. (4.12) becomes

$$c^2\,\frac{\partial^2}{\partial r^2}\big(r\,\mathbf{A}_{\mathrm{homo}}\big) - \frac{\partial^2}{\partial t^2}\big(r\,\mathbf{A}_{\mathrm{homo}}\big) = \mathbf{0} \tag{4.14}$$

that is solved by

$$\mathbf{A}_{\mathrm{homo}}(\mathbf{x},t) = \frac{\mathbf{Q}(g)}{r} \tag{4.15}$$

where $\mathbf{Q}$ is an arbitrary vector function of the auxiliary variable g relating the space and time through the velocity c

$$g = t - r/c \tag{4.16}$$

This homogeneous solution, however, is singular at $\boldsymbol{\xi}$, where the point-like force is concentrated (i.e., where $r = 0$). In order to understand the role of this singularity we need to properly take into account the spatial derivative at the origin. In this perspective, let us consider the following identity

$$\nabla^2(f\,\mathbf{a}) = \mathbf{a}\,\nabla^2 f + f\,\nabla^2\mathbf{a} + 2\,(\boldsymbol{\nabla} f)\cdot(\boldsymbol{\nabla}\mathbf{a}) \tag{4.17}$$

and specify it in the case in which the scalar, f, and vector, $\mathbf{a}$, fields depend on r only

$$\begin{aligned}\nabla^2 (f\,\mathbf{a}) &= \mathbf{a}\,\nabla^2 f + f\,\left(\frac{\partial^2 \mathbf{a}}{\partial r^2} + \frac{2}{r}\,\frac{\partial \mathbf{a}}{\partial r}\right) + 2\,\frac{\partial f}{\partial r}\,\frac{\partial \mathbf{a}}{\partial r} \\ &= \mathbf{a}\,\nabla^2 f + 2\,\left(\frac{f}{r} + \frac{\partial f}{\partial r}\right)\,\frac{\partial \mathbf{a}}{\partial r} + f\,\frac{\partial^2 \mathbf{a}}{\partial r^2}\end{aligned} \tag{4.18}$$

In light of this, the Laplacian of the homogeneous solution yields

$$\nabla^2 \mathbf{A}_{\mathrm{homo}}(\mathbf{x},t) = -4\,\pi\,\delta(\mathbf{x}-\boldsymbol{\xi})\,\mathbf{Q}(t) + \frac{1}{r\,c^2}\,\ddot{\mathbf{Q}}(g) \tag{4.19}$$

where we made use of $\delta(\mathbf{x}-\boldsymbol{\xi})\,\mathbf{Q}(t-r/c) = \delta(\mathbf{x}-\boldsymbol{\xi})\,\mathbf{Q}(t)$ due to the fact that the Dirac delta is concentrated at $r = 0$. Furthermore, substituting eq. (4.19) into the left-hand side of eq. (4.12), we obtain

$$c^2\,\frac{\partial^2 (r\,\mathbf{A}_{\mathrm{homo}}(\mathbf{x},t))}{\partial r^2} - \frac{\partial^2 (r\,\mathbf{A}_{\mathrm{homo}}(\mathbf{x},t))}{\partial t^2} = 4\,\pi\,c^2\,\delta(\mathbf{x}-\boldsymbol{\xi})\,\mathbf{Q}(t) \tag{4.20}$$

This means that eq. (4.15) can be regarded as solution of the homogeneous differential equation, eq. (4.12), everywhere but for an infinitesimal neighboorhood of the origin due to the presence of the Dirac delta distribution in the right-hand side.

Starting from this knowledge, let us try the following alternative solution for the dishomogeneous differential equation, eq. (4.11)

$$\mathbf{A}(\mathbf{x},t) = \frac{\mathbf{Q}(g) - \mathbf{Q}(t)}{r} \tag{4.21}$$

By considering that

$$\nabla^2 \mathbf{A}(\mathbf{x}, t) = \frac{\ddot{\mathbf{Q}}(g)}{r\, c^2} \tag{4.22a}$$

$$\ddot{\mathbf{A}}(\mathbf{x}, t) = \frac{\ddot{\mathbf{Q}}(g) - \ddot{\mathbf{Q}}(t)}{r} \tag{4.22b}$$

and after substitution into eq. (4.11), we thus obtain

$$\ddot{\mathbf{Q}}(t) = \frac{\mathbf{P}(t)}{4\,\pi\,\rho} \tag{4.23}$$

This result allows us to understand that the solution of the dishomogeneous differential equation, eq. (4.11), takes the form given by eq. (4.21), provided that the vector function $\mathbf{Q}$ is related to $\mathbf{P}$ through eq. (4.23).

According to eq. (4.6) and focusing our attention everywhere but the point where the point-like force is concentrated (i.e. $r > 0$), the displacement field is given by

$$\begin{aligned} \mathbf{u}(\mathbf{x}, t) &= \boldsymbol{\nabla}\, \boldsymbol{\nabla} \cdot \left(\frac{\mathbf{Q}(g_p) - \mathbf{Q}(g_s)}{r} \right) + \nabla^2 \left(\frac{\mathbf{Q}(g_s)}{r} \right) \\ &= \boldsymbol{\nabla}\, \boldsymbol{\nabla} \cdot \left(\frac{\mathbf{Q}(g_p) - \mathbf{Q}(g_s)}{r} \right) + \frac{\mathbf{P}(g_s)}{4\,\pi\,\rho\, c_s^2\, r} \end{aligned} \tag{4.24}$$

where we made use of eqs. (4.19) and (4.23) and

$$g_p = t - r/c_p \tag{4.25a}$$

$$g_s = t - r/c_s \tag{4.25b}$$

In order to obtain an explicit expression for the displacement field in terms of the vector $\mathbf{P}$ only, we first note that, thanks to integration by partes and within the assumption of quiescent past, we can write

$$\begin{aligned}\int_{-\infty}^{g} \ddot{\mathbf{Q}}(\tau)\,(t-\tau)\,\mathrm{d}\tau &= \dot{\mathbf{Q}}(\tau)\,(t-\tau)\Big|_{-\infty}^{g} + \int_{-\infty}^{g} \dot{\mathbf{Q}}(\tau)\,\mathrm{d}\tau \\ &= \mathbf{Q}(g) + (t-g)\,\dot{\mathbf{Q}}(g) = \mathbf{Q}(g) + \frac{r}{c}\,\dot{\mathbf{Q}}(g)\end{aligned} \tag{4.26}$$

This allows us to recast the following partial derivatives as

$$\frac{\partial}{\partial r}\left(\frac{\mathbf{Q}(g)}{r}\right) = -\int_{-\infty}^{g} \frac{\mathbf{P}(\tau)}{4\,\pi\,\rho\,r^2}\,(t-\tau)\,\mathrm{d}\tau \tag{4.27a}$$

$$\frac{\partial^2}{\partial r^2}\left(\frac{\mathbf{Q}(g)}{r}\right) = \frac{\mathbf{P}(g)}{4\,\pi\,\rho\,c^2\,r} + 2\int_{-\infty}^{g} \frac{\mathbf{P}(\tau)}{4\,\pi\,\rho\,r^3}\,(t-\tau)\,\mathrm{d}\tau \tag{4.27b}$$

Then, making use of eq. (1.29), we obtain

$$\begin{aligned}\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\frac{\mathbf{Q}(q)}{r}\right) &= \left(3\,\hat{\mathbf{r}}\otimes\hat{\mathbf{r}} - \mathbf{1}\right)\cdot\int_{-\infty}^{g} \frac{\mathbf{P}(\tau)}{4\,\pi\,\rho\,r^3}\,(t-\tau)\,\mathrm{d}\tau \\ &\quad + \hat{\mathbf{r}}\otimes\hat{\mathbf{r}}\cdot\left(\frac{\mathbf{P}(g)}{4\,\pi\,\rho\,c^2\,r}\right)\end{aligned} \tag{4.28}$$

and, once substituted into eq. (4.24), we obtain

$$\begin{aligned}\mathbf{u}(\mathbf{x},t) &= \left(3\,\hat{\mathbf{r}}\otimes\hat{\mathbf{r}} - \mathbf{1}\right)\cdot\int_{g_p}^{g_s} \frac{\mathbf{P}(\tau)}{4\,\pi\,\rho\,r^3}\,(t-\tau)\,\mathrm{d}\tau \\ &\quad + \hat{\mathbf{r}}\otimes\hat{\mathbf{r}}\cdot\left(\frac{\mathbf{P}(g_p)}{4\,\pi\,\rho\,c^2\,r}\right) + \left(\mathbf{1} - \hat{\mathbf{r}}\otimes\hat{\mathbf{r}}\right)\cdot\left(\frac{\mathbf{P}(g)}{4\,\pi\,\rho\,c^2\,r}\right)\end{aligned} \tag{4.29}$$

In the end, by noting that

$$\int_{g_p}^{g_s} \mathbf{P}(\tau)\,(t-\tau)\,\mathrm{d}\tau = \int_{r/c_p}^{r/c_s} \mathbf{P}(t-\tau)\,\tau\,\mathrm{d}\tau \tag{4.30}$$

we obtain

$$\mathbf{u}(\mathbf{x},t) = \left(3\,\hat{\mathbf{r}}\otimes\hat{\mathbf{r}} - \mathbf{1}\right)\cdot\int_{r/c_s}^{r/c_p}\frac{\mathbf{P}(t-\tau)}{4\,\pi\,\rho\,r^3}\,\tau\,\mathrm{d}\tau + \hat{\mathbf{r}}\otimes\hat{\mathbf{r}}\cdot\left(\frac{\mathbf{P}(g_p)}{4\,\pi\,\rho\,c^2\,r}\right) + \left(\mathbf{1} - \hat{\mathbf{r}}\otimes\hat{\mathbf{r}}\right)\cdot\left(\frac{\mathbf{P}(g)}{4\,\pi\,\rho\,c^2\,r}\right) \tag{4.31}$$

4.1.2 Elastodynamics Green function

In the case in which the vector $\mathbf{P}$ describes an impulsive force

$$\mathbf{P}(t) = \mathbf{F}\,\delta(t) \tag{4.32}$$

eq. (4.31) becomes

$$\mathbf{u}(\mathbf{x},t) = \hat{\mathbf{G}}(\mathbf{x}-\boldsymbol{\xi},t)\cdot\mathbf{F} \tag{4.33}$$

with

$$\hat{\mathbf{G}}(\mathbf{x}-\boldsymbol{\xi},t) = \frac{3\,\hat{\mathbf{r}}\otimes\hat{\mathbf{r}} - \mathbf{1}}{4\,\pi\,\rho\,r^3}\,t\left[H(t-r/c_p) - H(t-r/c_s)\right] + \frac{\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}}{4\,\pi\,\rho\,r}\,\frac{\delta(t-r/c_p)}{c_p^2} + \frac{\mathbf{1}-\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}}{4\,\pi\,\rho\,r}\,\frac{\delta(t-r/c_s)}{c_s^2} \tag{4.34}$$

Here the two Heaviside functions result from the fact that the integral in the first terms of the right-hand side of eq. (4.31) differs from zero only when $t \in [r/c_p, r/c_s]$.

After comparison of eqs. (3.21a) and (4.33), we thus understand that $\hat{\mathbf{G}}$ can be seen as the elastodynamics Green function

$$\mathbf{G}(\mathbf{x},t,\boldsymbol{\xi}) = \hat{\mathbf{G}}(\mathbf{x}-\boldsymbol{\xi},t) \tag{4.35}$$

4.2 Elastic waves due to displacement dislocations

The representation theorem as given by eq. (3.36) allows us to obtain the displacement field due to displacement dislocations in the homogeneous full space in terms of the elastodynamics Green function due to a point like seismic source, eq. (4.35). In particular, let us consider the following problem

$$\begin{cases} \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}(\mathbf{x}, t) = \rho\, \ddot{\mathbf{u}}(\mathbf{x}, t) \\ \Delta \mathbf{u}(\mathbf{x}, t) = \lim_{\varepsilon \to 0} \left[\mathbf{u}(\mathbf{x}^+, t) - \mathbf{u}(\mathbf{x}^-, t) \right] \end{cases} \tag{4.36}$$

with $\mathbf{x}^\pm$ given by eq. (3.27). This problem correspond to the one considered for obtaining eq. (3.36), but for the fact that the body force $\mathbf{f}$ entering eq. (3.1a) is now zero. In this respect, eq. (3.36) simplifies into

$$\mathbf{u}(\mathbf{x}, t) = \int_{\mathcal{S}_f} \mathrm{d}\mathbf{u}(\mathbf{x}, t, \boldsymbol{\xi}) \tag{4.37}$$

where $\mathrm{d}\mathbf{u}$ is the contribution to the displacement due to the infinitesimal surface element of the fault located at $\boldsymbol{\xi}$

$$\mathrm{d}\mathbf{u}(\mathbf{x}, t, \boldsymbol{\xi}) = \int_{-\infty}^{\infty} \mathrm{d}\mathbf{m}(\boldsymbol{\xi}, t - \tau) : \boldsymbol{\nabla}_\xi \otimes \hat{\mathbf{G}}(\mathbf{x} - \boldsymbol{\xi}, \tau)\, \mathrm{d}\tau\, \mathrm{d}S_\xi \tag{4.38}$$

Here we have made use of the elastodynamics Green function for the homogeneous and isotropic full space, eqs. (4.34) and (4.35).

4.2.1 The gradient of the elastodynamics Green function

Let us now evaluate the gradient of the elastodynamics Green function, eq. (4.35), with respect to the coordinates of the vector position $\boldsymbol{\xi}$ where the point-like force is concentrated. In this perspective, from eq. (4.3), we first obtain the following identities

$$\nabla_\xi\, r = -\hat{\mathbf{r}} \tag{4.39a}$$

$$\nabla_\xi \left(\frac{\mathbf{1}}{r^n}\right) = \frac{n\,\mathbf{K}^{(1)}}{r^{n+1}} \tag{4.39b}$$

$$\nabla_\xi \left(\frac{\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}}{r^n}\right) = \frac{(n+2)\,\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}\otimes\hat{\mathbf{r}} - \mathbf{K}^{(2)} - \mathbf{K}^{(3)}}{r^{n+1}} \tag{4.39c}$$

where ∇_ξ is the gradient operator with respect to $\boldsymbol{\xi}$, $n \in \mathbb{N}$ and

$$\mathbf{K}^{(1)} = \hat{\mathbf{r}}\otimes\hat{\mathbf{x}}_k\otimes\hat{\mathbf{x}}_k = \hat{\mathbf{r}}\otimes\mathbf{1} \tag{4.40a}$$

$$\mathbf{K}^{(2)} = \hat{\mathbf{x}}_k\otimes\hat{\mathbf{r}}\otimes\hat{\mathbf{x}}_k = (\hat{\mathbf{r}}\otimes\mathbf{1})^{(132)} \tag{4.40b}$$

$$\mathbf{K}^{(3)} = \hat{\mathbf{x}}_k\otimes\hat{\mathbf{x}}_k\otimes\hat{\mathbf{r}} = \mathbf{1}\otimes\hat{\mathbf{r}} \tag{4.40c}$$

In light of this, the gradient of the elastodynamics Green function can be recast as follows

$$\begin{aligned}
\nabla_\xi\otimes\mathbf{G}(\mathbf{x},t,\boldsymbol{\xi}) &= \nabla_\xi\otimes\frac{3\,\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}-\mathbf{1}}{4\,\pi\,\rho\,r^3}\,t\left[H(t-r/c_p)-H(t-r/c_s)\right] \\
&\quad+\left[\nabla_\xi\otimes\frac{\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}}{4\,\pi\,\rho\,r}+\frac{3\,\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}-\hat{\mathbf{r}}\otimes\mathbf{1}}{4\,\pi\,\rho\,r^2}\right]\frac{\delta(t-r/c_p)}{c_p^2} \\
&\quad-\left[\nabla_\xi\otimes\frac{\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}-\mathbf{1}}{4\,\pi\,\rho\,r}+\frac{3\,\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}-\hat{\mathbf{r}}\otimes\mathbf{1}}{4\,\pi\,\rho\,r^2}\right]\frac{\delta(t-r/c_s)}{c_s^2} \\
&\quad+\frac{\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}}{4\,\pi\,\rho\,r}\,\frac{\dot{\delta}(t-r/c_p)}{c_p^3}-\frac{\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}-\hat{\mathbf{r}}\otimes\mathbf{1}}{4\,\pi\,\rho\,r}\,\frac{\dot{\delta}(t-r/c_s)}{c_s^3} \\
&= \mathbf{A}^{(1)}\,\frac{t\left[H(t-r/c_p)-H(t-r/c_s)\right]}{4\,\pi\,\rho\,r^4}
\end{aligned}$$

$$+ \mathbf{A}^{(2)} \frac{\delta(t - r/c_p)}{4\,\pi\,\rho\,r^2\,c_p^2} + \mathbf{A}^{(3)} \frac{\delta(t - r/c_s)}{4\,\pi\,\rho\,r^2\,c_s^2}$$

$$+ \mathbf{A}^{(4)} \frac{\dot{\delta}(t - r/c_p)}{4\,\pi\,\rho\,r\,c_p^3} + \mathbf{A}^{(5)} \frac{\dot{\delta}(t - r/c_s)}{4\,\pi\,\rho\,r\,c_s^3} \tag{4.41}$$

with

$$A_{ijk}^{(1)} = 15\,\hat{\mathbf{r}} \otimes \hat{\mathbf{r}} \otimes \hat{\mathbf{r}} - 3\left(\mathbf{K}^{(1)} + \mathbf{K}^{(2)} + \mathbf{K}^{(3)}\right) \tag{4.42a}$$

$$A_{ijk}^{(2)} = 6\,\hat{\mathbf{r}} \otimes \hat{\mathbf{r}} \otimes \hat{\mathbf{r}} - \left(\mathbf{K}^{(1)} + \mathbf{K}^{(2)} + \mathbf{K}^{(3)}\right) \tag{4.42b}$$

$$A_{ijk}^{(3)} = -6\,\hat{\mathbf{r}} \otimes \hat{\mathbf{r}} \otimes \hat{\mathbf{r}} + 2\,\mathbf{K}^{(1)} + \mathbf{K}^{(2)} + \mathbf{K}^{(3)} \tag{4.42c}$$

$$A_{ijk}^{(4)} = \hat{\mathbf{r}} \otimes \hat{\mathbf{r}} \otimes \hat{\mathbf{r}} \tag{4.42d}$$

$$A_{ijk}^{(5)} = -\hat{\mathbf{r}} \otimes \hat{\mathbf{r}} \otimes \hat{\mathbf{r}} + \mathbf{K}^{(1)} \tag{4.42e}$$

Thanks to the analytical expression for the gradient of the elastodynamics Green function, eq. (4.41), we can calculate the contribution $\mathrm{d}\mathbf{u}$ to the displacement field due to an infinitesimal surface element of the fault plane, eq. (4.38) and, so, the displacement field $\mathbf{u}$, eq. (4.37).

4.2.2 The analytical solution

Let us consider a planar fault with normal $\hat{\mathbf{n}} = \mathbf{x}_3$ and, for simplicity, assume that the direction of the displacement dislocation is the same everywhere and parallel to $\mathbf{x}_1$

$$\delta\mathbf{u}(\boldsymbol{\xi}, t) = \delta u(\boldsymbol{\xi}, t)\,\hat{\mathbf{x}}_1 \tag{4.43}$$

Furthermore, within the assumption of isotropic elastic continuum, the moment density tensor reads

$$\mathbf{m}(\boldsymbol{\xi}, t) = \mu\, \delta u(\boldsymbol{\xi}, t) \left(\hat{\mathbf{x}}_1 \otimes \hat{\mathbf{x}}_3 + \hat{\mathbf{x}}_3 \otimes \hat{\mathbf{x}}_1\right) \tag{4.44}$$

After substitution of eq. (4.44) into eq. (4.38) and some straightforward algebra, we obtain

$$\begin{aligned} \mathrm{d}\mathbf{u}(\mathbf{x}, t, \boldsymbol{\xi}) = \frac{\mu\, \mathrm{d}S_\xi}{4\,\pi\,\rho} \Bigg[& \frac{\mathbf{B}^{(1)}(\theta, \phi)}{r^4} \int_{r/c_p}^{r/c_s} \delta u(\boldsymbol{\xi}, t - \tau)\, \tau\, \mathrm{d}\tau \\ & + \frac{\mathbf{B}^{(2)}(\theta, \phi)}{c_p^2\, r^2}\, \delta u(\boldsymbol{\xi}, g_p) + \frac{\mathbf{B}^{(3)}(\theta, \phi)}{c_s^2\, r^2}\, \delta u(\boldsymbol{\xi}, g_s) \\ & + \frac{\mathbf{B}^{(4)}(\theta, \phi)}{c_p^3\, r}\, \delta \dot{u}(\boldsymbol{\xi}, g_p) + \frac{\mathbf{B}^{(5)}(\theta, \phi)}{c_s^3\, r}\, \delta \dot{u}(\boldsymbol{\xi}, g_s) \Bigg] \end{aligned} \tag{4.45}$$

where

$$\mathbf{B}^{(1)} = 15\, \hat{\mathbf{r}}\, \sin(2\,\theta)\, \cos\phi - 6\, \mathbf{p} \tag{4.46a}$$

$$\mathbf{B}^{(2)} = 6\, \hat{\mathbf{r}}\, \sin(2\,\theta)\, \cos\phi - 2\, \mathbf{p} \tag{4.46b}$$

$$\mathbf{B}^{(3)} = -6\, \hat{\mathbf{r}}\, \sin(2\,\theta)\, \cos\phi + 3\, \mathbf{p} \tag{4.46c}$$

$$\mathbf{B}^{(4)} = \hat{\mathbf{r}}\, \sin(2\,\theta)\, \cos\phi \tag{4.46d}$$

$$\mathbf{B}^{(5)} = -\hat{\mathbf{r}}\, \sin(2\,\theta)\, \cos\phi + \mathbf{p} \tag{4.46e}$$

with

$$\mathbf{p} = \hat{\mathbf{x}}_1\, \cos\theta + \hat{\mathbf{x}}_3\, \sin\theta\, \cos\phi = p_r\, \hat{\mathbf{r}} + p_\theta\, \hat{\boldsymbol{\theta}} + p_\phi\, \hat{\boldsymbol{\phi}} \tag{4.47}$$

and

$$p_r = \mathbf{p} \cdot \hat{\mathbf{r}} = \sin(2\,\theta)\, \cos\phi \tag{4.48a}$$

$$p_\theta = \mathbf{p} \cdot \hat{\boldsymbol{\theta}} = \cos(2\,\theta)\, \cos\phi \tag{4.48b}$$

$$p_\phi = \mathbf{p} \cdot \hat{\boldsymbol{\phi}} = -\cos\theta \, \sin\phi \tag{4.48c}$$

After substitution of eq. (4.47) into eq. (4.46), we obtain

$$\mathbf{B}^{(1)} = 9\, p_r \, \hat{\mathbf{r}} - 6 \left(p_\theta \, \hat{\boldsymbol{\theta}} + p_\phi \, \hat{\boldsymbol{\phi}} \right) \tag{4.49a}$$

$$\mathbf{B}^{(2)} = 4\, p_r \, \hat{\mathbf{r}} - 2 \left(p_\theta \, \hat{\boldsymbol{\theta}} + p_\phi \, \hat{\boldsymbol{\phi}} \right) \tag{4.49b}$$

$$\mathbf{B}^{(3)} = -3\, p_r \, \hat{\mathbf{r}} + 3 \left(p_\theta \, \hat{\boldsymbol{\theta}} + p_\phi \, \hat{\boldsymbol{\phi}} \right) \tag{4.49c}$$

$$\mathbf{B}^{(4)} = p_r \, \hat{\mathbf{r}} \tag{4.49d}$$

$$\mathbf{B}^{(5)} = p_\theta \, \hat{\boldsymbol{\theta}} + p_\phi \, \hat{\boldsymbol{\phi}} \tag{4.49e}$$

From eq. (4.45), we can understant that the contribution to the displacement field from an infinitesimal surface element composes of five terms. The dependence on the angular coordinates is controlled by the vector $\mathbf{B}^{(k)}$ and they have different dependence on the radial distance r. In particular, the first three terms decays as the inverse of the square radial distance (r^{-2}) and they do faster than the last two terms do, which instead decays only as the inverse of the radial distance. In light of these, we will refer to the first three terms as near-field contributions and to the last two terms as far-field contributions.

4.2.3 Far-field contributions

The last two terms in eq. (4.45) depends on time through the auxiliary variable g_p and g_s and decays as the inverse of the radial distance. They thus describe spherical P and S waves that propagates at the velocities of c_p and c_s, respectively, and the energy of which distributes over the spherical surfaces of increasing radius.

As we already assumed, the displacement discontinuity as a quiescent past. This means that it is zero before a certain given time, say $t = 0$. We made this assumption in order to obtain the Betti's theorem and the following representation theorem in Chapter 3. Let us further assume that the displacement discontinuity evolves in time only within a finite time intervals, say from $t = 0$ to $t = T$, with T being the duration of the rupture, and that it reaches a finite constant value at later times

$$\delta u(\boldsymbol{\xi}, t > T) = \Delta u(\boldsymbol{\xi}) \tag{4.50}$$

that is the total displacement discontinuity cumulated during the rupture.

Within this framework, we can understand that the P and S spherical waves contribute to the displacement field only within the time interval $[r/c, r/c+T]$, with $c = c_p, c_s$. They are indeed proportional to the time derivative of δu evaluated at $g = t - r/c$, which is zero for $g < 0$ and $g > T$.

Figure 4.1 shows the radiation pattern of the radial component of the P waves in x_1-x_3 plane.

4.2.4 Near-field contributions

Within the same assumptions made in section 4.2.3 and after that the passage of S waves (i.e., at times $t \geq r/c_s + T$), we can make use of the following identity

$$\begin{aligned} \int_{r/c_p}^{r/c_s} \delta u(\boldsymbol{\xi}, t-\tau)\, \tau \, \mathrm{d}\tau &= \Delta u(\boldsymbol{\xi}) \int_{r/c_p}^{r/c_s} \tau \, \mathrm{d}\tau \\ = \Delta u(\boldsymbol{\xi}) \left[\frac{1}{2}\, \tau^2 \right]_{r/c_p}^{r/c_s} &= \frac{\Delta u(\boldsymbol{\xi})}{2} \left[\left(\frac{r}{c_s} \right)^2 - \left(\frac{r}{c_p} \right)^2 \right] \end{aligned} \tag{4.51}$$

and eq. (4.45) becomes

$$\mathrm{d}\mathbf{u}(\mathbf{x}, t, \boldsymbol{\xi}) = \frac{\mu\, \Delta u(\boldsymbol{\xi})\, \mathrm{d}S_\xi}{4\, \pi\, \rho\, r^2} \left[\frac{2\, \mathbf{B}^{(2)} - \mathbf{B}^{(1)}}{2\, c_p^2} + \frac{2\, \mathbf{B}^{(3)} + \mathbf{B}^{(1)}}{2\, c_s^2} \right]$$

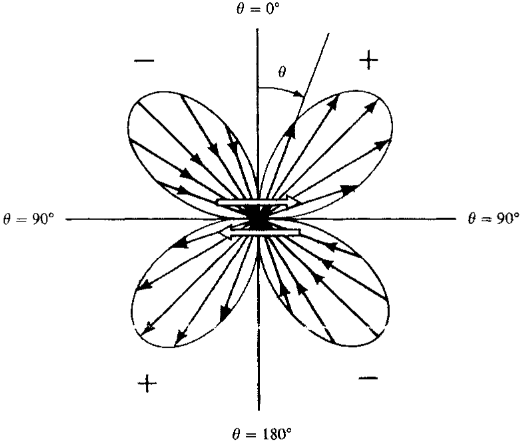

Figure 4.1: The radiation pattern of the radial component of the P waves in x_1-x_3 plane.

$$= \frac{\Delta u(\boldsymbol{\xi})\,\mathrm{d}S_\xi}{4\,\pi\,r^2}\left[\left(3-\frac{c_s^2}{c_p^2}\right)\frac{p_r\,\hat{\mathbf{r}}}{2}+\frac{c_s^2}{2\,c_p^2}\left(p_\theta\,\hat{\boldsymbol{\theta}}+p_\phi\,\hat{\boldsymbol{\phi}}\right)\right] \tag{4.52}$$

for $t > r/r_c + T$.

It is noteworthy that eq. (4.52) involves only the near-field contributions and that it describes a static contribution which only depends on the vector position (or the spherical coordinates r, θ and ϕ). In particular, it decays as the inverse of the square of the radial distance. In light of this, we can say that, after the passage of S waves, it is left a static displacement to which we will refer as the co-seismic displacement.

Chapter 5

Failure criteria and rupture dynamics

5.1 Shear deformation and elastic stress

Let us consider the following shear deformation loading the elastic isotropic continuum, with the displacement and strain fields given by

$$\mathbf{u} = 2\,\dot{\varepsilon}\,t\,x_1'\,\hat{\mathbf{x}}_2' \tag{5.1a}$$

$$\boldsymbol{\varepsilon} = \dot{\varepsilon}\,t\,(\hat{\mathbf{x}}_1' \otimes \hat{\mathbf{x}}_2' + \hat{\mathbf{x}}_2' \otimes \hat{\mathbf{x}}_1') \tag{5.1b}$$

Here, $\hat{\mathbf{x}}_1'$ and $\hat{\mathbf{x}}_2'$ two unit vectors (which are not the principal directions for the stress tensor) and $\dot{\varepsilon}$ is the strain rate which can be as high as $10^{-7}/\mathrm{yr}$ in active tectonic areas like subduction zones.

The resulting elastic stress is deviatoric and we consider it as superimposed on the initial lithostatic stress

$$\boldsymbol{\sigma} = -p\,\mathbf{1} + \sigma\,(\hat{\mathbf{x}}_1' \otimes \hat{\mathbf{x}}_2' + \hat{\mathbf{x}}_2' \otimes \hat{\mathbf{x}}_1') \tag{5.2}$$

where

$$\sigma = 2\,\mu\,\dot{\varepsilon}\,t = \dot{\sigma}\,t \tag{5.3}$$

with $\dot{\sigma} = 2\,\mu\,\dot{\varepsilon}$ being the (constant) stress rate. Assuming a shear modulus of $\mu = 30$ GPa and $\dot{\varepsilon} = 10^{-7}/\text{yr}$, we obtain $\dot{\sigma} = 6$ hPa/yr.

It is straightforward to show that the following unit vectors

$$\mathbf{x}_1 = \frac{\hat{\mathbf{x}}_1 + \hat{\mathbf{x}}_2}{\sqrt{2}} \tag{5.4a}$$

$$\mathbf{x}_2 = \frac{\hat{\mathbf{x}}_1 - \hat{\mathbf{x}}_2}{\sqrt{2}} \tag{5.4b}$$

correspond to the principal directions of both strain and stress fields

$$\boldsymbol{\varepsilon} = \dot{\varepsilon}\,t\,(\hat{\mathbf{x}}_1 \otimes \hat{\mathbf{x}}_1 - \hat{\mathbf{x}}_2 \otimes \hat{\mathbf{x}}_2) \tag{5.5a}$$

$$\boldsymbol{\sigma} = -p\,\mathbf{1} + \sigma\,(\hat{\mathbf{x}}_1 \otimes \hat{\mathbf{x}}_1 - \hat{\mathbf{x}}_2 \otimes \hat{\mathbf{x}}_2) \tag{5.5b}$$

We note that the two deviatoric principal stresses are opposite to each other and grow linearly with time, eq. (5.3). Also, we note that a positive deviatoric stress is associated to extension, while a negative one is associated to compression.

5.2 The Coulomb-Navier failure criterion

Brittle shear fracture under triaxial compressive stress is the most commonly occurring type of failure in the upper lithosphere. Although physical theories

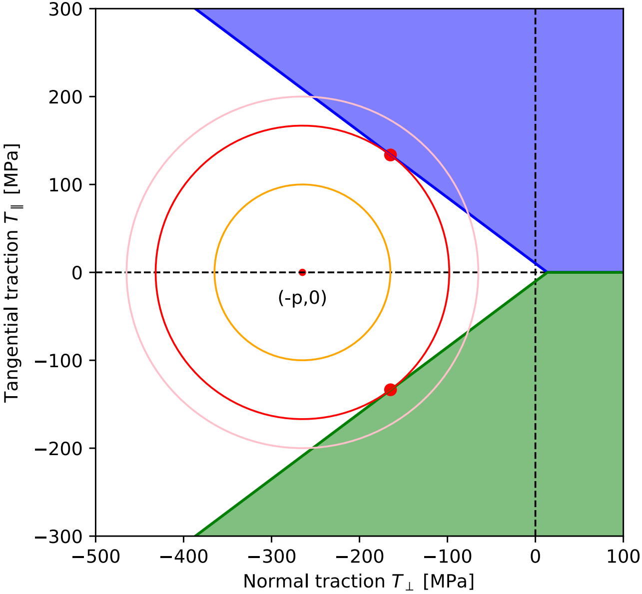

Figure 5.1: The representation of the Coulomb-Navier criterion in the stress space for $f_s = 3/4$, $S = 10$ MPa and $p = 265$ MPa (at 10 km depth for a crustal rock of density 2.7 kg/m^3). The green and blue regions are state of stress that breaks the rocks. The yellow, red and pink circles represent the stress states at increasing time. They are centered at $(-p, 0)$ and have a radius of $\sigma' = \dot{\sigma}'\, t$. The red circle (tangent to the green and blue regions) represents the first time at which the rock can breaks along two specific fault plane.

for brittle shear fracture exist, empirical failure criteria are usually sufficient to account for at least those characteristics of fracture which are of geodynamic significance. Among the empirical failure criteria, the Coulomb-Navier criterion is one of the most simple and commonly adopted in scientific literature. It states that the rocks breaks along a surface when the module of the shear stress, $|T_{\parallel}|$, exceeds the difference between the cohesive strength, S, and the normal stress, $T_{\perp}$, multiplied by the static friction coefficient, f_s,

$$|T_{\parallel}| \geq S - f_s \, T_{\perp} \tag{5.6}$$

Depending on the sign of the shear stress, the Coulomb-Navier criterion becomes

$$T_{\parallel} \geq S - f_s \, T_{\perp} \qquad T_{\parallel} > 0 \tag{5.7a}$$

$$T_{\parallel} \leq -S + f_s \, T_{\perp} \qquad T_{\parallel} < 0 \tag{5.7b}$$

which, in the stress space with the normal and shear stress along the abscissa and ordinate, identifies two straight lines, as shown in fig. 5.1, above and below which the rock breaks (blue and green regions).

The Mohr circle

Let us now consider the stress tensor given by eq. (5.5b) and a surface with unit normal in the x_1-x_2 plane

$$\hat{\mathbf{n}} = \cos\theta \, \hat{\mathbf{x}}_1 + \sin\theta \, \hat{\mathbf{x}}_2 \tag{5.8}$$

with θ being the angle that the plane surface forms with respect to the x_1 axis, as depicted in fig. 5.2. The unit vector tangential to the fault, instead, reads

$$\hat{\mathbf{v}} = \sin\theta \, \hat{\mathbf{x}}_1 - \cos\theta \, \hat{\mathbf{x}}_2 \tag{5.9}$$

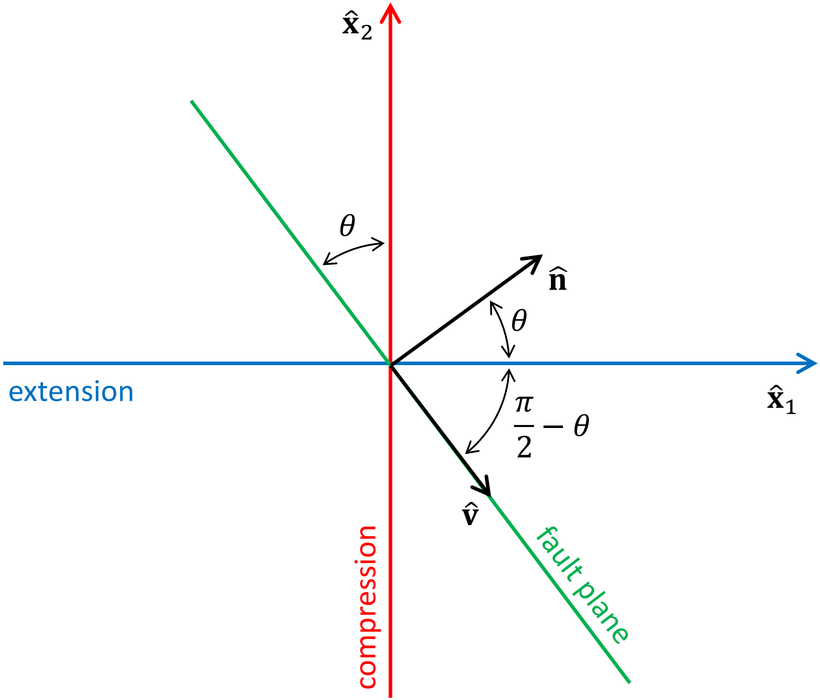

Figure 5.2: The cartoon of the extensional and compressional principal axes $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ and the cross section of the fault in the x_1-x_2 plane forming an angle θ with respect to the compressional axis.

Within this framework, the traction yields

$$\mathbf{T} = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} = \hat{\mathbf{x}}_1 \, (\sigma' - p) \, \cos\theta + \hat{\mathbf{x}}_2 \, (-\sigma' - p) \, \sin\theta \tag{5.10}$$

and we further decomposes it into the normal and tangential components

$$\mathbf{T} = T_\perp \, \hat{\mathbf{n}} + T_\| \, \hat{\mathbf{v}} \tag{5.11}$$

with $T_\perp$ and $T_\|$ being the normal and shear stresses

$$T_\perp = \mathbf{T} \cdot \hat{\mathbf{n}} = \sigma' \, \cos(2\,\theta) - p \tag{5.12a}$$

$$T_\| = \mathbf{T} \cdot \hat{\mathbf{v}} = \sigma' \, \sin(2\,\theta) \tag{5.12b}$$

Here, we have made use of the following trigonometric identities

$$\sin(2\,\theta) = 2\,\sin\theta\,\cos\theta \tag{5.13a}$$

$$\cos(2\,\theta) = \cos^2\theta - \sin^2\theta \tag{5.13b}$$

When plotted on the stress space, as shown in fig. 5.1, the above expressions for the normal and tangential tractions forms a circle centered in $(-p, 0)$ and of radius given by the principal deviatoric stress σ. The specific point of the circle referring to the state of stress depends on the angle θ which the normal forms with the x_1 axis, and it is identified by the angle $2\,\theta$ calculated from the abscissa, i.e. from the axis related to the normal traction.

5.3 Anderson's theory of faulting

Considering that the principal deviatoric stress grows linearly with time, eq. (5.3), we can immagine that the radius of the Mohr circle grows from zero, at the initial time $t = 0$, to a maximum radius where the circle becomes tangent to the

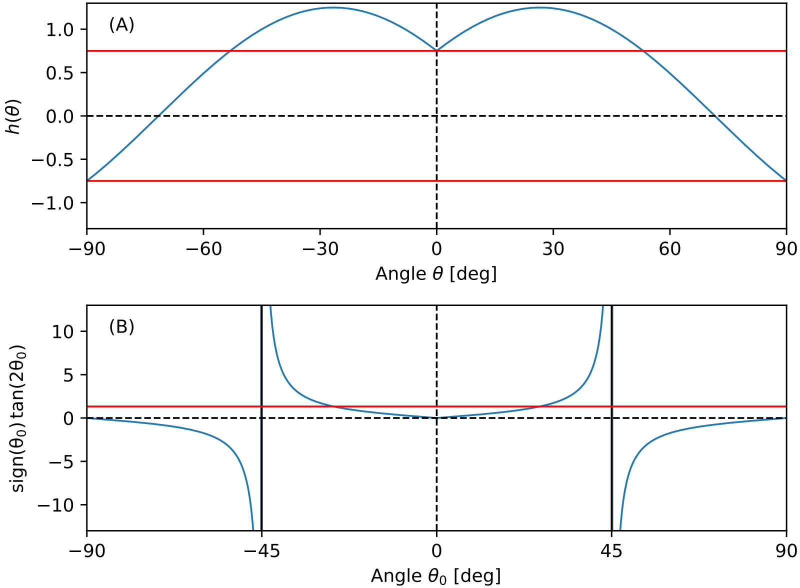

Figure 5.3: (A) The symmetric function $h(\theta)$ given by eq. (5.15) and (B) the function $\mathrm{sign}(\theta_0)\ \tan(2\,\theta_0)$. The two horizontal red lines in panel (A) correspond to the values $\pm f_s$, while the horizontal red line in panel (B) corresponds to the value $1/f_s$.

two straight lines resulting from the Coulomb-Navier criterion (the red circle in fig. 5.1). In this respect, there are two specific angles, which first satisfy the Coulomb-Navier criterion and, so, two fault planes that break before the others.

Restricting our attention to $\theta \in [-\pi/2, \pi/2]$, the Coulomb-Navier criterion becomes

$$\mathrm{sign}(\theta)\ \sin(2\,\theta)\,\sigma' \geq S - f_s\ \cos(2\,\theta)\,\sigma' + f_s\,p \tag{5.14}$$

and we recast it as follows

$$h(\theta)\,\sigma' \geq S + f_s\,p \tag{5.15}$$

with h being the following symmetric function

$$h(\theta) = f_s\ \cos(2\,\theta) + \mathrm{sign}(\theta)\,\sin(2\,\theta) \tag{5.16}$$

Figure 5.3(A) show h for the typical static friction of rocks, $f_s \approx 3/4$.

Considering that σ' grows with time, we have to find the maxima of h in order to find the two angles $\pm\theta_0$ for which the rock breaks early. We thus investigate its first-order derivative

$$\dot{h}(\theta) = 2\,\big(-f_s\ \sin(2\,\theta) + \mathrm{sign}(\theta)\ \cos(2\,\theta)\big) \tag{5.17}$$

and find its stationary points (i.e., when $\dot{h}(\theta_0) = 0$). As shown in fig. 5.3(B), this occurs when

$$\mathrm{sign}(\theta_0)\ \tan(2\,\theta_0) = \frac{1}{f_s} \tag{5.18}$$

In this respect, the two angles at which the rock breaks early are

$$\theta_0 = \pm\frac{1}{2}\ \arctan\left(\frac{1}{f_s}\right) \tag{5.19}$$

and, for $f_s \approx 3/4$, we obtain $\theta_0 \approx \pm 27^\circ$. This angle can be seen as the angle that the fault plane form with the compressional axis $\mathbf{x}_2$. On the contrary, the angle with respect to the extensional axis $\hat{\mathbf{x}}_2$ is $\pm\pi/2 \mp \theta_0 = \pm 63^\circ$.

Tectonic regimes

The cartoon depicted in fig. 5.2 can be regarded as an extensional regime with $\hat{\mathbf{x}}_1$ being parallel to the Earth surface and oriented along the maximum extension, while $\hat{\mathbf{x}}_2$ being perpendicular to the Earth surface. In this case, we have to immagine that the fault develops parallel to the x_3 axis. Within this framework, the dip angle of the fault is $\pi/2 - \theta_0$, that is a fault with a high angle of about 63° for $f_s \approx 3/4$.

Rotating the cartoon of 90°, instead, it can be considered as a compressional regimes with $\hat{\mathbf{x}}_2$ being parallel to the Earth surface and oriented along the maximum compression, while $\hat{\mathbf{x}}_1$ being perpendicular to the Earth surface. Within this framework, the dip angle of the fault is θ_0, that is a fault with a low angle of about 27° for $f_s \approx 3/4$.

In the end, supposing that both the extensional and compressional directions are parallel to the Earth surface, while $\hat{\mathbf{x}}_3$ being perpendicular to it, the cartoon depicts a trascurrent regime. In this case, we have to immagine that the fault develops parallel to the x_3 axis, which means that it is perpendicular to the Earth surface. The dip angle is thus 90°. The angle θ_0 now describes how the fault intersects the Earth surface and, so, it is related to the strike angle.

5.4 Rupture dynamics and seismic cycle

As depicted in fig. 5.4, let us consider a cube of side b and assume that the rock has already broken along a rectangular fault of area $A = b^2$ in the x_2-x_3 plane and perpendicular to the x_1 axis, at $x_1 = 0$, separating the left (green) and right (blue) blocks, and that the continuum is subjected to the following displacement field

$$\mathbf{u}(\mathbf{x}, t) = f(x_1, t)\,\mathbf{x}_2 \tag{5.20}$$

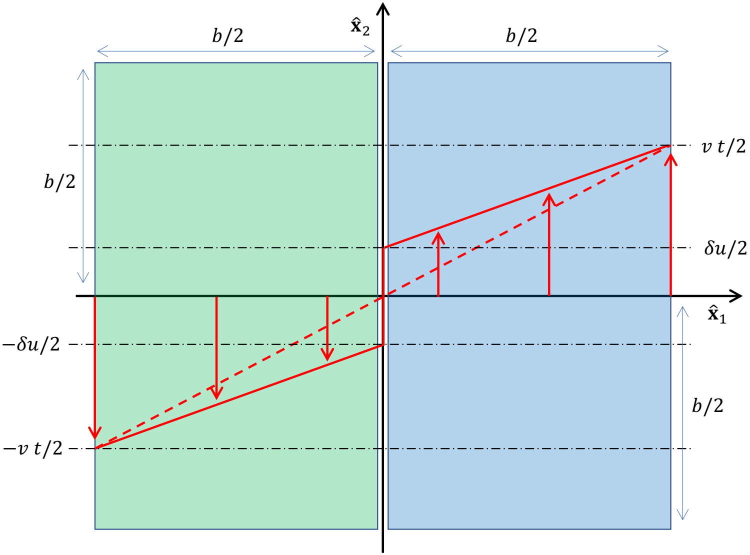

Figure 5.4: The left (green) and right (blue) blocks separated by the fault in the x_2-x_3 plane of area A. The dashed and solid lines represent the displacement field in the x_2 direction just before ($\delta u = 0$) and during the rupture ($\delta u \neq 0$) according to eqs. (5.20) and (5.21).

with

$$f(x_1) = \left[\left(v\, t - \delta u(t) \right) \frac{x_1}{b} + \frac{1}{2}\, \mathrm{sign}(x_1)\, \delta u(t) \right] \hat{\mathbf{x}}_2 \tag{5.21}$$

We note that the displacement field describes a relative velocity of $v\,\hat{\mathbf{x}}_2$ from the left and right sides (at $x_1 = \pm b/2$) and a displacement discontinuity of $\delta u\,\hat{\mathbf{x}}_2$ across the fault at $x_1 = 0$. Obviously the normal to the fault is $\hat{\mathbf{n}} = \hat{\mathbf{x}}_1$.

Within each block, the strain and stress fields are constants and read

$$\boldsymbol{\varepsilon} = \varepsilon(t) \left(\hat{\mathbf{x}}_1 \otimes \hat{\mathbf{x}}_2 + \hat{\mathbf{x}}_2 \otimes \hat{\mathbf{x}}_1 \right) \tag{5.22a}$$

$$\boldsymbol{\sigma} = -p\, \mathbf{1} + 2\, \mu\, \varepsilon(t) \left(\hat{\mathbf{x}}_1 \otimes \hat{\mathbf{x}}_2 + \hat{\mathbf{x}}_2 \otimes \hat{\mathbf{x}}_1 \right) \tag{5.22b}$$

with p being the lithostatic pressure and

$$\varepsilon(t) = \frac{v\, t - \delta u(t)}{2\, b} \tag{5.23}$$

Also, the traction on surfaces parallel to the fault, with normal $\hat{\mathbf{n}} = \hat{\mathbf{x}}_1$, is

$$\mathbf{T} = \boldsymbol{\sigma} \cdot \hat{\mathbf{x}}_1 = T_\perp\, \hat{\mathbf{x}}_1 + T_\parallel\, \hat{\mathbf{x}}_2 \tag{5.24}$$

with

$$T_\perp = -p \tag{5.25a}$$

$$T_\parallel(t) = 2\, \mu\, \varepsilon(t) = \frac{\mu\, (v\, t - \delta u(t))}{b} \tag{5.25b}$$

Let us assume that there is no dislocation on the fault until the Coulomb-Navier criterion is satisfied, say at the time $t = t_0$, i.e. $\delta u(t \leq t_0) = 0$. In this respect, the first time that the Coulomb-Navier criterion is satisfied is when

$$T_{\parallel}(t_0) = \frac{\mu\, v\, t_0}{b} = p\, f_s + S \tag{5.26}$$

that is

$$t_0 = \frac{b}{v}\, \frac{p\, f_s + S}{\mu} \tag{5.27}$$

As we are going to show, the slip velocity is much greater than the tectonic velocity v. During the rupture, we thus consider that shear stress varies only due to the slip and write

$$T_{\parallel}(t \geq t_0) = \frac{\mu\,(v\, t_0 - \delta u(t))}{b} = p\, f_s + S - \frac{\mu\, \delta u(t)}{b} \tag{5.28}$$

This traction hold on the two surfaces at $x_1 = \pm b/2$. On the contrary, during the rupture, the fault is no longer able to support static friction and, so, the shear stress at $x_1 = 0$ have to be quantified using a simple law of dynamic friction

$$T_{\parallel}(t \geq t_0) = f_d\, p \tag{5.29}$$

Within this framework, indeed, the total force acting on the right (blue) block is

$$\mathbf{F}(t) = H(t - t_0)\, A\, \hat{\mathbf{x}}_2 \left(p\,(f_s - f_d) + S - \frac{\mu\, \delta u(t)}{b} \right) \tag{5.30}$$

that we have obtained considering the two different shear stress acting on the right and left sides of the block, at $x_1 = b$ and $x_1 = 0$, eqs. (5.28) and (5.29). According to the second Newton's law, we can write also

$$\mathbf{F}(t) = \int_{\mathcal{V}} \rho\, \ddot{\mathbf{u}}(\mathbf{x}, t)\, \mathrm{d}V = \hat{\mathbf{x}}_2\, \rho\, A\, \delta\ddot{u}(t) \int_0^{b/2} \left(\frac{1}{2} - \frac{x_1}{b} \right) \mathrm{d}x_1$$

$$= \hat{\mathbf{x}}_2 \frac{\rho\, b\, A}{8} \delta\ddot{u}(t) \tag{5.31}$$

In this respect, we obtain the following second order differential equation

$$H(t - t_0)\, \frac{p\,(f_s - f_d) + S}{\mu}\, b = \delta u(t) + \frac{\rho\, b^2}{8\,\mu}\, \delta\ddot{u}(t) \tag{5.32}$$

that we can solve for obtaining the slip time evolution δu. As initial conditions, we can assume that the slip and the slip velocity are zero at the rupture time, that is

$$\begin{cases} \delta u(t_0) = 0 \\ \delta\dot{u}(t_0) = 0 \end{cases} \tag{5.33}$$

Then, defining

$$\Delta u = 2\, \frac{p\,(f_s - f_d) + S}{\mu}\, b \tag{5.34a}$$

$$T = \frac{\pi}{2\sqrt{2}} \sqrt{\frac{\rho}{\mu}}\, b = \frac{\pi}{2\sqrt{2}}\, \frac{b}{c_s} \tag{5.34b}$$

we can recast eq. (5.32) as follows

$$\frac{\Delta u}{2}\, H(t - t_0) = \delta u(t) + \left(\frac{T}{\pi}\right)^2 \ddot{u}(t) \tag{5.35}$$

It is solved by

$$\delta u(t + t_0) = \Delta u\, \frac{1 - \cos(\pi\, t/T)}{2}\, H(t) \tag{5.36}$$

and, so, we can understand that Δu is the maximum slip.

We note that at time $t = t_0 + T$, the slip reaches its maximum value, $\delta u(t_0 + T) = \Delta u$, and the slip velocity is zero, $\delta \dot{u}(t_0 + T) = 0$. At this time, the friction on the fault becomes static again and further motion is allowed only if the Coulomb-Navier criterion is still satisfied. On the other hand, the shear stress on the right side of the block, at $x_1 = b$, decreases by

$$\Delta\sigma = T_{\|}(t_0) - T_{\|}(t_0 + T) = \frac{\mu\,\Delta u}{b} = 2\left(p\,(f_s - f_d) + S\right) \tag{5.37}$$

and the Coulomb-Navier criterion is no longer satisfied. This means that the fault becomes locked and the dislocation at later times $t \geq t_0 + T$ remains fixed at its maximum value. According to this reasoning, we change eq. (5.36) as follows

$$\delta u(t + t_0) = \Delta u \left[\frac{1 - \cos(\pi\, t/T)}{2}\left(H(t) - H(t - T)\right) + H(t - T)\right] \tag{5.38}$$

Summary

The stress drop $\Delta\sigma$, the total (or maximum) slip Δu and the rise time T are given by

$$\Delta\sigma = 2\left(p\,(f_s - f_d) + S\right) \tag{5.39a}$$

$$\Delta u = \frac{\Delta\sigma}{\mu}\sqrt{A} \tag{5.39b}$$

$$T = \frac{\pi}{2\sqrt{2}}\,\frac{\sqrt{A}}{c_s} \tag{5.39c}$$

Also, we calculate the average slip velocity $\Delta u/T$ and the seismic moment M

$$\frac{\Delta u}{T} = \frac{2\sqrt{2}}{\pi}\,\frac{\Delta\sigma}{\mu}\,c_s \tag{5.40a}$$

$$M = \mu\,A\,\Delta u = \Delta\sigma\,A^{3/2} \tag{5.40b}$$

and write the moment magnitude as follows

$$\begin{aligned} M_W &= \frac{2}{3}\,\log_{10}\left(\frac{M}{\mathrm{N\,m}}\right) - 6 \\ &= \frac{2}{3}\,\log_{10}\left(\frac{\Delta\sigma}{\mathrm{MPa}}\right) + \log_{10}\left(\frac{A}{\mathrm{km}^2}\right) + 4 \end{aligned} \tag{5.41}$$

We note that the stress drop and the average slip velocity do not vary with the fault size b. In this respect, they are earthquake independent and depend on the material properties of the rocks (coefficients of friction, cohesive strengths, shear modulus and seismic velocity) and the lithostatic pressure.

Let us also assume that the fault is located at $h \approx 10$ km depth within a crust of $\rho \approx 3000$ kg/m^3 subjected to a gravity acceleration of $g \approx 10$ m/s^2. The lithostatic pressure $p \approx \rho\,g\,h = 300$ MPa. Also, let us assume that the shear modulus and S wave velocity are $\mu \approx 30$ GPa and $c_s \approx \sqrt{\mu/\rho} = 3.16$ km/s, and that the static and dynamic friction coefficients are $f_s \approx 1/20$ and $f_d \approx 9/10\,f_s$, and that the cohesive strength is zero, $S = 0$. With these choices, we obtain

$$\Delta\sigma \approx 3\,\mathrm{MPa} \tag{5.42a}$$

$$\frac{\Delta u}{T} \approx 28.5\,\mathrm{cm/s} \tag{5.42b}$$

$$\Delta u \approx 10^{-4}\,\sqrt{A} \tag{5.42c}$$

$$T = \frac{\sqrt{A}}{2.85\,\mathrm{km}}\,\mathrm{s} \tag{5.42d}$$

$$M_W \approx 4.3 + \log_{10}\left(\frac{A}{\mathrm{km}^2}\right) \tag{5.42e}$$

For a rupture area of $A = 10 \times 10\,\mathrm{km}^2$, we have $\sqrt{A} = 10\,\mathrm{km}$ and, so, $\Delta u = 1$ m, $T = 3.5$ s and $M_W = 6.3$ like the 2009 L'Aquila earthquake. For a rupture area of $500 \times 125\,\mathrm{km}^2$, we have $\sqrt{A} = 250\,\mathrm{km}$ and, so, $\Delta u = 25$ m, $T = 88$ s and $M_W = 9$ like a megathrust earthquake at subduction zones.

Chapter 6

Finiteness of the fault

In this Chapter we will investigate the effect of the finiteness of the fault on the seismic waves at distance much greater than the dimension of the fault. This goal allows us to focus only on the far-field contributions, thus neglecting the near-field ones. Furthermore, we will only discuss the main effect, thus neglecting second order contributions.

6.1 The main contribution

Let us consider a reference point over the fault and make it coincides with the origin of the Cartesian reference frame and denote with R, Θ and Φ the spherical coordinates and with $\hat{\mathbf{R}}$, $\hat{\mathbf{\Theta}}$ and $\hat{\mathbf{\Phi}}$ the spherical unit vector with respect to it. Within this reference system, the observation point $\mathbf{x}$ reads

$$\mathbf{x} = R\,\hat{\mathbf{R}} \tag{6.1}$$

where

$$\hat{\mathbf{R}} = \Gamma_j \, \hat{\mathbf{x}}_j \tag{6.2}$$

with Γ_i being the direction cosines

$$\Gamma_1 = \sin\Theta \, \cos\Phi \tag{6.3a}$$

$$\Gamma_2 = \sin\Theta \, \sin\Phi \tag{6.3b}$$

$$\Gamma_3 = \cos\Theta \tag{6.3c}$$

As far as the observation point $\mathbf{x}$ is far away from the fault (or its distance is much greater than the fault dimension), we can neglect the near-field terms as well as the difference between the spherical coordinates defined with respect to a fault point and those defined with respect to the origin. Within this approximation, eqs. (4.37) and (4.52) yield

$$\begin{aligned}\mathbf{u}(\mathbf{x}, t) = \frac{\mu}{4\,\pi\,\rho\,R} \Bigg[&\frac{\hat{\mathbf{\Theta}} \, \cos(2\,\Theta) \, \cos\Phi - \hat{\mathbf{\Phi}} \, \cos\Theta \, \sin\Phi}{c_s^3} \int_{\mathcal{S}_f} \delta\dot{u}(\boldsymbol{\xi}, g_s) \, \mathrm{d}S_\xi \\ &+ \frac{\hat{\mathbf{R}} \, \sin(2\,\Theta) \, \cos\Phi}{c_p^3} \int_{\mathcal{S}_f} \delta\dot{u}(\boldsymbol{\xi}, g_p) \, \mathrm{d}S_\xi \Bigg]\end{aligned} \tag{6.4}$$

6.2 The time dependence

The time dependence of the displacement field given by eq. (6.4) is completely characterized by the two terms of the following form

$$f(\mathbf{x}, t) = \mu \int_{\mathcal{S}_f} \delta\dot{u}(\boldsymbol{\xi}, t - r/c) \, \mathrm{d}S_\xi \tag{6.5}$$

with $c = c_p, c_s$. Within this term, the distance r between the surface element of the fault and the observation point cannot be replaced simply by R, as depicted in fig. 6.1. This can be understood by considering the following first-order expansion

$$r = |\mathbf{x} - \boldsymbol{\xi}| = R\sqrt{1 + \frac{\boldsymbol{\xi}\cdot\boldsymbol{\xi}}{R^2} - 2\,\frac{\hat{\mathbf{R}}\cdot\boldsymbol{\xi}}{R}} \approx R - \hat{\mathbf{R}}\cdot\boldsymbol{\xi} \tag{6.6}$$

and noting that the first order term $-\hat{\mathbf{R}}\cdot\boldsymbol{\xi}$ is of the order of the dimension of the fault. Let us say that the fault has dimension of the order of ten kilometers. Then, in this case, the terms r/c will differs from R/c by up to a few seconds, assuming a velocity c of the order of several kilometers per second. Such a difference is important in the seismic wave modeling and must be taken into account.

6.2.1 The Haskell model

For the sake of simplicity, let us assume that each point of the fault has the same time evolution U but for the fact that the rupture starts at different times, say at $t = t_0(\boldsymbol{\xi})$ as function of the position on the fault. This means that

$$\delta u(\boldsymbol{\xi}, t) = U(t - t_0(\boldsymbol{\xi})) \tag{6.7a}$$

$$\delta \dot{u}(\boldsymbol{\xi}, t) = V(t - t_0(\boldsymbol{\xi})) \tag{6.7b}$$

with $V = \dot{U}$. Equation (6.5) thus becomes

$$f(\mathbf{x}, t) = \mu \int_{\mathcal{S}_f} V(t - r/c - t_0(\boldsymbol{\xi}))\,\mathrm{d}S_\xi \tag{6.8}$$

Let us now investigate this function in the frequency domain. After Fourier transform, eq. (6.8) yields

$$\tilde{f}(\mathbf{x},\omega) = \mu \int_{\mathcal{S}_f} \int_{-\infty}^{\infty} V(t - r/c - t_0(\boldsymbol{\xi}))\, e^{-i\,\omega\,t}\, \mathrm{d}t\, \mathrm{d}S_\xi = \mu\, \tilde{V}(\omega)\, \tilde{F}(\mathbf{x},\omega) \tag{6.9}$$

with $\tilde{V}$ being the Fourier transform of V and F defined as follows

$$\tilde{V}(\omega) = \int_{-\infty}^{\infty} V(t)\, e^{-i\,\omega\,t}\, \mathrm{d}t \tag{6.10a}$$

$$\tilde{F}(\mathbf{x},\omega) = \int_{\mathcal{S}_f} e^{-i\,\omega\,(r/c+t_0(\boldsymbol{\xi}))}\, \mathrm{d}S_\xi \tag{6.10b}$$

Equation (6.9) shows how the frequency spectrum of the seismic waves can be seen as the product between one term related to the common time evolution of the dislocation velocity, $\tilde{V}$, and another term which takes into account the time at which the rupture of each surface element starts, t_0, and its distance from the observation point, r.

As depicted in fig. 6.1, let us consider a fault of length L and width W along the x_1 and x_2 axes, respectively, and denote with ψ the angle between the observation point and the x_1 axis. We note that

$$\Gamma_1 = \sin\Theta\,\cos\Phi = \cos\psi \tag{6.11}$$

Let us assume also that the rupture starts at $t = 0$ for all surface element located at $\xi_1 = 0$ and that it propagates along the x_1 direction at rupture velocity v

$$t_0(\boldsymbol{\xi}) = \xi_1/v \tag{6.12}$$

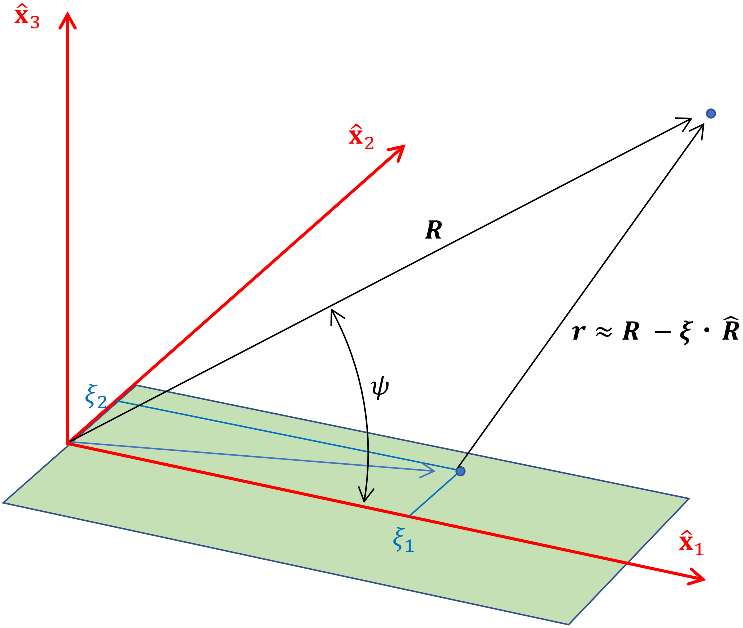

Figure 6.1: The Cartesian reference frame and the rectangular fault of length L and width W.

Theoretical works have shown that the rupture velocity is usually smaller than the seismic wave velocities and, so, $v < c_s < c_p$. A general rule of thumb is $v \approx 4/5\, c_s$.

Let us also consider the approximation given by eq. (6.6) and write eq. (6.10b) as follows

$$\begin{aligned} \tilde{F}(\mathbf{x},\omega) &= e^{-i\,\omega\,R} \int_0^L \int_{-W/2}^{W/2} e^{i\,\omega\,(\xi_1\ \cos\psi + \xi_2\,\Gamma_2)/c - \xi_1/v}\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2 \\ &= e^{-i\,\omega\,R}\, \tilde{\Xi}_1(\mathbf{x},\omega)\, \tilde{\Xi}_2(\mathbf{x},\omega) \end{aligned} \tag{6.13}$$

with

$$\tilde{\Xi}_1(\mathbf{x},\omega) = \int_0^L e^{i\,\omega\,\xi_1\,(\cos\psi/c - 1/v)}\, \mathrm{d}\xi_1 \tag{6.14a}$$

$$\tilde{\Xi}_2(\mathbf{x},\omega) = \int_{-W/2}^{W/2} e^{i\,\omega\,\xi_2\,\Gamma_2/c}\, \mathrm{d}\xi_2 \tag{6.14b}$$

The integral in eq. (6.14a) can be evaluated analytically by performing the following change of variable

$$t = T_L\, \frac{\xi_1}{L} \tag{6.15}$$

with T_L being the time defined as follows

$$T_L = L\,(1/v - \cos\psi/c) \tag{6.16}$$

We thus obtain

$$\tilde{\Xi}_1(\mathbf{x},\omega) = \frac{L}{T_L} \int_0^{T_L} e^{-i\,\omega\,t}\, \mathrm{d}t = L\, \frac{1 - e^{-i\,\omega\,T_L}}{i\,\omega\,T_L}$$

$$= L\, e^{-i\,\omega\, T_L/2}\, \frac{2\,\sin\big(\omega\, T_L/2\big)}{\omega\, T_L} \tag{6.17}$$

On the contrary, by assuming that $W \ll L$, we approximate the integral in eq. (6.14b) as follows

$$\tilde{\Xi}_2(\mathbf{x}, \omega) \approx W \tag{6.18}$$

It is noteworthy that the time T_L varies with the observation point and ranges from $L\,(1/v - 1/c)$ to $L\,(1/v + 1/c)$ for ψ ranging from 0 to π. Assuming $v = 4/5\, c_s$ and $c_p = \sqrt{3}\, c_s$, we obtain

$$\left(\frac{L}{4\, c_s}\right) \leq T_L^s \leq 9\, \left(\frac{L}{4\, c_s}\right) \tag{6.19}$$

for S waves, and

$$2.7\, \left(\frac{L}{4\, c_s}\right) \leq T_L^p \leq 7.3\, \left(\frac{L}{4\, c_s}\right) \tag{6.20}$$

for P waves.

It is important also to note that $\tilde{\Xi}_1$ is proportional to the Fourier transform of the box function in the time domain

$$\Xi_1(\mathbf{x}, t) = \frac{L}{T_L}\, \big(H(t) - H(t - T_L)\big) \tag{6.21}$$

In light of this, after inverse Fourier transform, eq. (6.9) becomes

$$f(\mathbf{x}, t) = \mu\, L\, W\, V(t - R/c) \star \frac{H(t) - H(t - T_L)}{T_L} \tag{6.22}$$

This results show how the rupture delay due to the finiteness of the fault, as well as the dimension of the fault itself, acts as moving average windows of the local time evolution of the slip velocity.

6.2.2 The ramp slip

Let us now make the assumption that the time evolution of the slip is the ramp function from 0 to T, with T being the rise time, and then it reaches its constant value Δu

$$U(t) = \Delta u \left[\frac{t}{T} \big(H(t) - H(t-T) \big) + H(t-T) \right] \tag{6.23}$$

$$V(t) = \frac{\Delta u}{T} \big(H(t) - H(t-T) \big) \tag{6.24}$$

Here, the slip velocity $\Delta u / T$ should be relatively earthquake independent and of the order of 1 m/s. With this choice, eq. (6.22) becomes

$$f(\mathbf{x}, t) = \frac{M_s}{T\,T_L} \int_{R/c}^{R/c+T} \big(H(t-\tau) - H(t - T_L - \tau) \big)\, \mathrm{d}\tau \tag{6.25}$$

with M_s being the seismic moment

$$M_s = \mu\, L\, W\, \Delta u \tag{6.26}$$

After some straightforward reasoning, eq. (6.25) becomes

$$\begin{aligned} f(\mathbf{x}, t + R/c) = \frac{M_s}{T\,T_L} \Big[&\big(t\, H(t) - (t-T)\, H(t-T) \big) \\ &- \big((t - T_L)\, H(t - T_L) - (t - T_L - T)\, H(t - T_L - T) \big) \Big] \end{aligned} \tag{6.27}$$

6.2.3 The Aki ω^2 model

The Fourier transform of the slip velocity given by eq. (6.24) yields

$$\tilde{V}_1(\omega) = \Delta u \, \frac{1 - e^{-i\,\omega\,T}}{i\,\omega\,T} = \Delta u \, e^{-i\,\omega\,T/2} \, \frac{2\,\sin\left(\omega\,T/2\right)}{\omega\,T} \tag{6.28}$$

and eq. (6.9) becomes

$$\tilde{f}(\mathbf{x},\omega) = M_s \, e^{-i\,\omega\,(T/2+T_L/2+R/c)} \, \frac{\sin\left(\omega\,T/2\right)}{\omega\,T/2} \, \frac{\sin\left(\omega\,T_L/2\right)}{\omega\,T_L/2} \tag{6.29}$$

We note that

$$\lim_{\omega \to 0} \tilde{f}(\mathbf{x},\omega) = M_s \tag{6.30}$$

and that we can consider the following envelop

$$\begin{aligned} \left|\tilde{f}(\mathbf{x},\omega)\right| &= M_s \, \frac{\left|\sin\left(\omega\,T/2\right)\,\sin\left(\omega\,T_L/2\right)\right|}{\omega^2\,T\,T_L/4} \\ &\leq U\,L\,W\,\frac{1}{(\omega/\omega_0)^2} \end{aligned} \tag{6.31}$$

with ω_0 being the corner angular frequency

$$\omega_0 = \frac{2}{\sqrt{T\,T_L}} \tag{6.32}$$

These properties are catched by the ω^2 model

$$\left|\tilde{f}(\mathbf{x},\omega)\right| = \frac{M_s}{1 + \left(\frac{\omega}{\omega_0}\right)^2} \tag{6.33}$$