Define $\begin{cases} n \equiv \#\text{ spins parallel to } B \text{ (up)} \\ N-n = \#\text{ spins antiparallel to } B \text{ (down)} \end{cases}$

$$E = n(-\mu B) + (N-n)(\mu B) = (N-2n)\mu B$$

$\Rightarrow$ given N, B, n specifies energy

$$\boxed{n = \frac{N}{2} - \frac{E}{2\mu B}}$$

$\Rightarrow$ # microstates with energy E is given by way n of N spins can be up

$$\boxed{\Omega(n, N) = \frac{N!}{n!(N-n)!}}$$

Harmonic oscillator

$$E = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

Classical: microstate specified by (x, p)

point in phase space

x, p continuous variables $\Rightarrow$ compute $\underbrace{g(E)\Delta E}_{\#\text{ microstates between } E,\, E+\Delta E}$

density of states

$\Rightarrow$ easier to compute $\Gamma(E)$

microstate with energy less than or equal to E

$$g(E)\Delta E = \Gamma(E+\Delta E) - \Gamma(E) \approx \frac{d\Gamma(E)}{dE}\Delta E$$

How do we count microstates?

→ (x,p) point in phase space

→ # microstates connected to area in phase space, specified by E, i.e. $\Gamma(E)$ bounds this area

Given trajectory of particle $x(t), p(t)$, part of phase space covered is given by E

area

$$E = \frac{(p(t))^2}{2m} + \frac{1}{2}k(x(t))^2$$

dividing both sides by E

$$\frac{(p(t))^2}{2mE} + \frac{(x(t))^2}{2E/m\omega^2} = 1 \qquad \omega^2 = k/m$$

part of phase space covered is <u>ellipse</u>

$$\frac{x^2}{a^2} + \frac{p^2}{b^2} = 1$$

$a = 2E/m\omega^2 \qquad b = 2mE$

$\Rightarrow$ area of phase space covered is <u>area</u> of this ellipse

$$\pi ab = \pi \sqrt{\frac{2E}{m\omega^2}} \sqrt{2mE} = \pi \sqrt{\frac{4E^2}{\omega^2}} = 2\pi E / \omega^2$$

$\Rightarrow$ x, p continous, infinite # microstates in this area

$\Rightarrow$ to count them, divide into sections of size $\Delta x \Delta p$

$$\Gamma_{cl}(E) = \frac{\text{area in phase space}}{\Delta x \Delta p} = \frac{2\pi E}{\omega \Delta x \Delta p}$$

Classical physics does not specify how to chose $\Delta x, \Delta p$

<u>Quantum</u>: energy levels are quantized (have discrete values)

$\hbar \equiv h/2\pi$

$$E = (n + \tfrac{1}{2})\hbar\omega \qquad (n = 0, 1, 2, \ldots)$$

Now, $\Gamma(E)$ is straightword to calculate since the states are discrete

$$\Gamma_{qn}(E) = n = \frac{E}{\hbar\omega} - \frac{1}{2} \rightarrow \frac{E}{\hbar\omega}$$

for $E \gg \hbar\omega$

Now there is no arbitrary "unit of phase space" $\Delta x \Delta p$

$\Rightarrow$ want $\Gamma_{cl}(E) = \Gamma_{qm}(E)$

$$\frac{2\pi E}{\omega \Delta x \Delta p} = \frac{E}{\frac{h}{2\pi}\omega}$$

$$\Rightarrow h = \Delta x \Delta p$$

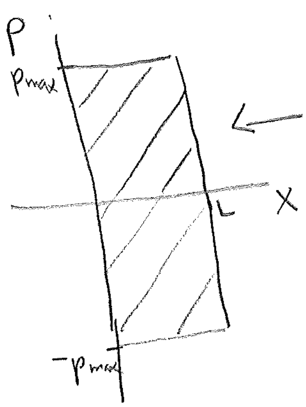
- we cannot specify microstate more precisely than this

$\Rightarrow$ consistent with Heisenberg uncertainity principle

Want to build to N particles in 3D box, but start simple

Particle in 1D box

classical : particle of mass m confined to 1D box of length L



area in phase space

$0 \le x \le L$

$0 \le p \le p_{max} = \sqrt{2mE}$

$\left(\frac{p^2}{2m} = E\right.$ no potential$\left.\right)$

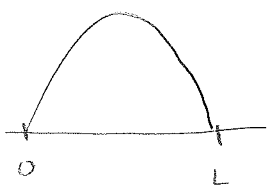
$$\Gamma_{cl} = \frac{\text{area in phase space}}{\Delta x \, \Delta p} = \frac{2L\, p_{max}}{\Delta x\, \Delta p} = \frac{2L\,(2mE)^{1/2}}{\Delta x\, \Delta p}$$

density of states : $g(E) \approx \dfrac{d\Gamma(E)}{dE}$

$$\approx \frac{2L \cdot \frac{1}{2}(2mE)^{-1/2}}{\Delta x\, \Delta p}$$

$$\approx \frac{L}{\sqrt{2mE}} \, \frac{1}{\Delta x\, \Delta p}$$

quantum : particle has wave properties



$\lambda = 2L$ (particle wavefunction zero at the walls)

wavelengths consistent with boundary conditions at $x = 0$, $x = L$

$$\lambda_n = \frac{2L}{n} \qquad n = 1, 2, 3, \ldots$$

de Broglie $p = h/\lambda$

$$E_n = \frac{p_n^2}{2m} = \frac{h^2}{2m\lambda_n^2} = \frac{n^2 h^2}{8mL^2}$$

$$\Rightarrow \quad n = \frac{2L}{h}(2mE)^{1/2}$$

microstates with energy $\leq E$

microstates correspond to values of n differing by 1

$$\Gamma_{qm}(E) = n = \frac{2L}{h}(2mE)^{1/2}$$

$$\Gamma_{cl}(E) = \Gamma_{qm}(E) \quad \Rightarrow \quad \Delta x\, \Delta p = h$$

(again!)

Particle in 2D box

- assume energy levels are quantized
- particle of mass m in box of side length L

$$E = \frac{1}{2m}\left(p_x^2 + p_y^2\right)$$

$$p_n = \frac{h}{\lambda_n} \leftarrow \lambda_n = \frac{2L}{n} \quad \text{each dimension}$$

$$p_x = \frac{hn_x}{2L}, \quad p_y = \frac{hn_y}{2L}$$

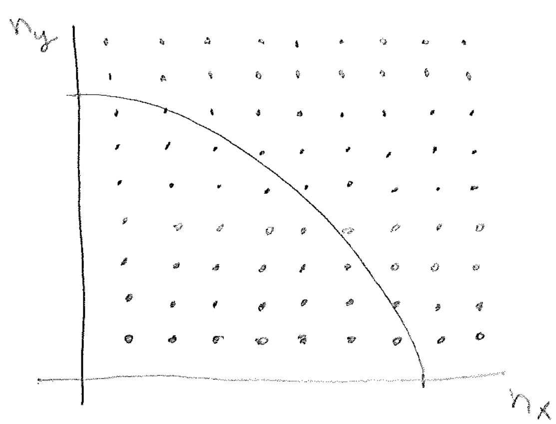
$$E = \frac{1}{2m}\left(\frac{h^2 n_x^2}{4L^2} + \frac{h^2 n_y^2}{4L^2}\right) = \frac{h^2}{8mL^2}\left(n_x^2 + n_y^2\right)$$

$\Rightarrow$ energy levels satisfy constraint

$$\left(n_x^2 + n_y^2\right) = \frac{8mEL^2}{h^2}$$

interpret this geometrically:

n_x, n_y integers ≥ 1



Define $R^2 \equiv n_x^2 + n_y^2$ so that

$$R^2 = \left(\frac{2L}{h}\right)^2 2mE$$

$\Rightarrow$ setting E sets R

$\rightarrow$ states inside positive quadrant of circle with radius R have energy $\leq$ E

Thus $\Gamma(E)$ = # microstates w/energy $\leq E$ is given by

$$\Gamma(E) = \frac{1}{4}\pi R^2 = \frac{1}{4}\pi\left(\frac{L}{h}\right)^2 8mE = \frac{2\pi m E L^2}{h^2}$$

$\Rightarrow$ note this is only true in the limit of large N

- our approximation of the circle overestimes the n_x, n_y inside, but approximation gets better and better as $N \to \infty$

Particle in 3D box

$$(n_x^2 + n_y^2 + n_z^2) = \frac{8mEL^2}{h^2}$$

Again, interpreting geometrically, we see

$$R^2 = n_x^2 + n_y^2 + n_z^2$$

and values of n_x, n_y, n_z that correspond to states w/energy $\leq E$ are given by positive octant of sphere of radius R

$$\Rightarrow \Gamma(E) = \frac{1}{8}\left(\frac{4}{3}\pi R^3\right) = \frac{\pi}{6}\left(\frac{8mEL^2}{h^2}\right)^{3/2} = \frac{2^{\cancel{3}^{2}} 2^{3/2}\pi m^{3/2} E^{3/2} L^3}{3\cancel{6}h^3}$$

$$\Gamma(E) = \frac{4\pi}{3}\frac{V}{h^3}(2mE)^{3/2}$$

$V \equiv L^3$

N particles in a 3D box

<u>trick</u>: count microstates assuming particles are distinguishable (easier) and then correct for our overcounting by dividing by N!

$\Rightarrow$ one particle in 3D box

microstates energy $\leq E$ = volume positive part of 3D sphere $R = \left(\frac{2L}{h}\right)(2mE)^{1/2}$

$\Rightarrow$ N particles in 3D box

microstates energy $\leq E$ = volume positive part of 3N-dimensional hypersphere $R = \left(\frac{2L}{h}\right)(2mE)^{1/2}$

will derive in discussion

$$V_n(R) = \frac{2\pi^{n/2}}{n\,\Gamma(n/2)} R^n$$

volume of n-dimensional hypersphere

gamma function

$\Gamma(n) = (n-1)!$ for integer n

(generalization of factorial)