QUANTUM MECHANICS

VOLUME II

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1. Angular Momentum. Notations and Conventions

The following notations and conventions concerning the angular momentum will be adopted throughout the Appendix.

Units: $\hbar = 1$.

Angular momentum components: $\boldsymbol{J}$, angular momentum operator, having cartesian coordinates J_x, J_y and J_z.

$$J_\pm = J_x \pm iJ_y. \tag{C.1}$$

Commutation relations

$$[J_x, J_y] = iJ_z \qquad [J_y, J_z] = iJ_x \qquad [J_z, J_x] = iJ_y \tag{C.2}$$

$$[J_z, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = 2J_z. \tag{C.3}$$

Basis vectors of a standard representation $\{\boldsymbol{J}^2 J_z\}$: $|\tau JM\rangle$

$$\boldsymbol{J}^2|\tau JM\rangle = J(J+1)\,|\tau JM\rangle \tag{C.4}$$

$$J_z|\tau JM\rangle = M|\tau JM\rangle \tag{C.5}$$

$$J_\pm|\tau JM\rangle = \sqrt{J(J+1)-M(M\pm 1)}\,|\tau J\, M\pm 1\rangle \tag{C.6}$$

$$\langle\tau JM|\tau' J' M'\rangle = \delta_{\tau\tau'}\,\delta_{JJ'}\,\delta_{MM'} \tag{C.7}$$

(J integral or half-integral ≥ 0, $M = -J, -J+1, \ldots, +J$).

τ denotes the quantum numbers that must be added to J and M to form a complete set; in the rest of the Appendix it will be omitted when not needed.

I. CLEBSCH–GORDON (C.–G.) COEFFICIENTS AND "3j" SYMBOLS

2. Definition and Notations

$\boldsymbol{j}_1$, $\boldsymbol{j}_2$ angular momenta of quantum systems 1 and 2 respectively.
$\boldsymbol{J}$, angular momentum of the total system of 1 and 2 taken together:

$$\boldsymbol{J} = \boldsymbol{j}_1 + \boldsymbol{j}_2. \tag{C.8}$$

The tensor product of the $(2j_1+1)$ vectors of system 1

$$|j_1 m_1\rangle \qquad (j_1 \text{ fixed}, m_1 = -j_1, \ldots, +j_1)$$

by the $(2j_2+1)$ vectors of system 2

$$|j_2 m_2\rangle \qquad (j_2 \text{ fixed}, m_2 = -j_2, \ldots, +j_2)$$

gives the $(2j_1+1)(2j_2+1)$ simultaneous eigenvectors of $\boldsymbol{j}_1^2$, $\boldsymbol{j}_2^2$, j_{1z}, j_{2z}, the vectors

$$|j_1 j_2 m_1 m_2\rangle \equiv |j_1 m_1\rangle |j_2 m_2\rangle \tag{C.9}$$

from which we can obtain, by a unitary transformation, the $(2j_1+1)(2j_2+1)$ simultaneous eigenvectors of $\boldsymbol{j}_1^2$, $\boldsymbol{j}_2^2$, $\boldsymbol{J}^2$, J_z, the vectors

$$|j_1 j_2 JM\rangle \tag{C.10}$$

$$(J = |\,j_1 - j_2\,|, \ldots, j_1 + j_2;\ M = -J, \ldots, +J).$$

Definition

The Clebsch–Gordon [1]), or vector addition, coefficients

$$\langle j_1 j_2 m_1 m_2 | JM\rangle$$

are the coefficients of that unitary transformation:

$$|j_1 j_2 JM\rangle = \sum_{m_1, m_2} |j_1 j_2 m_1 m_2\rangle\, \langle j_1 j_2 m_1 m_2 | JM\rangle. \tag{C.11}$$

Phase convention [2])

We complete the definition of the vectors in (C.9) and (C.10) by fixing their relative phases as follows:

(*i*) the $|j_1 m_1\rangle$, the $|j_2 m_2\rangle$ and the $|j_1 j_2 JM\rangle$ obey relations (C.6);

(*ii*) $\langle j_1 j_2 j_1 (j_1 - J)\,|JJ\rangle$ real > 0.

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[1]) There are many symbols employed in the literature to denote the Clebsch–Gordon coefficients. We note in particular:

$\langle j_1 j_2 m_1 m_2 | j_1 j_2 JM\rangle$ [Condon and Shortly, *Theory of Atomic Spectra* (University Press, Cambridge, 4th ed., 1957)].

$C_{j_1 j_2}(JM;\, m_1 m_2)$ [Blatt and Weisskopf, *Theoretical Nuclear Physics* (Wiley, New York, 1952)].

$S^{j_1 j_2}_{Jm_1 m_2}$ [E. P. Wigner, *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra* (English translation; Academic Press, New York, 1958)].

[2]) This convention is adopted by most authors, in particular by Wigner, Condon and Shortly, Blatt and Weisskopf, *op. cit.* note 1 of this page, and by Racah, Phys. Rev. 62 (1942) 437.

"3j" *symbol* (Wigner)[1]:

$$\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -M \end{pmatrix} \equiv \frac{(-)^{j_1-j_2+M}}{\sqrt{2J+1}} \langle j_1 j_2 m_1 m_2 | J M \rangle. \qquad \text{(C.12)}$$

3. Principal Properties

Reality. They are all real:

$$\langle j_1 j_2 m_1 m_2 | J M \rangle^* = \langle j_1 j_2 m_1 m_2 | J M \rangle.$$

Selection rules

(*i*) $m_1 + m_2 = M$;

(*ii*) $|j_1 - j_2| < J < j_1 + j_2$ ("triangular inequalities").

If these two conditions are not met, $\langle j_1 j_2 m_1 m_2 | J M \rangle = 0$.

Symmetries:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \text{ is:}$$

(*i*) invariant in a circular permutation of the three columns;

(*ii*) multiplied by $(-)^{j_1+j_2+j_3}$ in a permutation of two columns;

(*iii*) multiplied by $(-)^{j_1+j_2+j_3}$ when we simultaneously change the signs of m_1, m_2, and m_3.

Consequences:

$$\langle j_1 j_2 m_1 m_2 | J M \rangle = (-)^{j_1+j_2-J} \langle j_2 j_1 m_2 m_1 | J M \rangle \qquad \text{(C.13a)}$$

$$= (-)^{j_1-J+m_2} \sqrt{\frac{2J+1}{2j_1+1}} \langle J j_2 M -m_2 | j_1 m_1 \rangle \qquad \text{(C.13b)}$$

$$= (-)^{j_2-J-m_1} \sqrt{\frac{2J+1}{2j_2+1}} \langle j_1 J -m_1 M | j_2 m_2 \rangle \qquad \text{(C.13c)}$$

$$= (-)^{j_1+j_2-J} \langle j_1 j_2 -m_1 -m_2 | J -M \rangle. \qquad \text{(C.13d)}$$

[1] Racah, *op. cit.*, note 2, p. 1055, employs the symbol:

$$V(abc, \alpha\beta\gamma) \equiv (-)^{a-b-c} \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} \equiv \frac{(-)^{c-\gamma}}{\sqrt{2c+1}} \langle a b \alpha \beta | c -\gamma \rangle.$$

Orthogonality relations

$$\sum_{m_1=-j_1}^{+j_1} \sum_{m_2=-j_2}^{+j_2} \langle j_1 j_2 m_1 m_2 | J M \rangle \langle j_1 j_2 m_1 m_2 | J' M' \rangle = \delta_{JJ'} \delta_{MM'} \qquad \text{(C.14a)}$$

$$(|j_1 - j_2| < J < j_1 + j_2; \quad -J < M < J)$$

$$\sum_{J=|j_1-j_2|}^{j_1+j_2} \sum_{M=-J}^{+J} \langle j_1 j_2 m_1 m_2 | J M \rangle \langle j_1 j_2 m_1' m_2' | J M \rangle = \delta_{m_1 m_1'} \delta_{m_2 m_2'} \qquad \text{(C.14b)}$$

$$(-j_1 < m_1 < +j_1; \quad -j_2 < m_2 < +j_2)$$

$$\sum_{m_1=-j_1}^{+j_1} \sum_{m_2=-j_2}^{+j_2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3' \\ m_1 & m_2 & m_3' \end{pmatrix} = \frac{1}{2j_3+1} \delta_{j_3 j_3'} \delta_{m_3 m_3'} \qquad \text{(C.15a)}$$

$$\sum_{j_3=|j_1-j_2|}^{j_1+j_2} \sum_{m_3=-j_3}^{+j_3} (2j_3+1) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1' & m_2' & m_3 \end{pmatrix} = \delta_{m_1 m_1'} \delta_{m_2 m_2'}. \qquad \text{(C.15b)}$$

Composition relation for the spherical harmonics

$$\int Y_{l_1}^{m_1}(\Omega) Y_{l_2}^{m_2}(\Omega) Y_{l_3}^{m_3}(\Omega) \mathrm{d}\Omega = \left[\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}\right]^{\frac12} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \qquad \text{(C.16)}$$

whence:

$$Y_{l_1}^{m_1}(\Omega) Y_{l_2}^{m_2}(\Omega)$$

$$= \sum_{L=|l_1-l_2|}^{l_1+l_2} \sum_{M=-L}^{L} \left[\frac{(2l_1+1)(2l_2+1)}{4\pi(2L+1)}\right]^{\frac12} \langle l_1 l_2 0 0 | L 0 \rangle \langle l_1 l_2 m_1 m_2 | L M \rangle Y_L^M(\Omega) \qquad \text{(C.17a)}$$

$$= \sum_{L=|l_1-l_2|}^{l_1+l_2} \sum_{M=-L}^{L} (-)^M \left[\frac{(2l_1+1)(2l_2+1)(2L+1)}{4\pi}\right]^{\frac12} \begin{pmatrix} l_1 & l_2 & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & L \\ m_1 & m_2 & M \end{pmatrix} Y_L^{-M}(\Omega). \qquad \text{(C.17b)}$$

4. Methods of Calculation

Recursion relations

Relating the C.-G. whose arguments differ at most by:

(*i*) $\Delta J = 0 \qquad \Delta M = +1$

$$\sqrt{J(J+1) - M(M+1)} \langle j_1 j_2 m_1 m_2 | J M \rangle$$

$$= \sqrt{j_1(j_1+1) - m_1(m_1+1)} \langle j_1 j_2\, m_1+1\; m_2 | J\; M+1 \rangle \qquad \text{(C.18)}$$

$$+ \sqrt{j_2(j_2+1) - m_2(m_2+1)} \langle j_1 j_2\, m_1\; m_2+1 | J\; M+1 \rangle$$

(*ii*) $\Delta J = 0 \qquad \Delta M = -1$

$$\sqrt{J(J+1) - M(M-1)} \langle j_1 j_2 m_1 m_2 | J M \rangle$$

$$= \sqrt{j_1(j_1+1) - m_1(m_1-1)} \langle j_1 j_2\, m_1-1\; m_2 | J\; M-1 \rangle \qquad \text{(C.19)}$$

$$+ \sqrt{j_2(j_2+1) - m_2(m_2-1)} \langle j_1 j_2\, m_1\; m_2-1 | J\; M-1 \rangle$$

(*iii*) $\Delta J = \pm 1 \qquad \Delta M = 0$

$$A_0\langle j_1 j_2 m_1 m_2|JM\rangle = A_+\langle j_1 j_2 m_1 m_2|J+1\ M\rangle + A_-\langle j_1 j_2 m_1 m_2|J-1\ M\rangle \tag{C.20}$$

with

$$A_0 = m_1 - m_2 + M\,\frac{j_2(j_2+1) - j_1(j_1+1)}{J(J+1)} \qquad (M = m_1 + m_2)$$

$$A_+ = f(J+1)$$

$$A_- = f(J)$$

$$f(x) = \sqrt{x^2 - M^2}\left[\frac{[(j_1+j_2+1)^2 - x^2][x^2 - (j_1-j_2)^2]}{4x^2(2x-1)(2x+1)}\right]^{\frac{1}{2}}$$

The Racah formula

$$\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} = (-)^{a-b-\gamma}\sqrt{\Delta(abc)}\sqrt{(a+\alpha)!\,(a-\alpha)!\,(b+\beta)!\,(b-\beta)!\,(c+\gamma)!\,(c-\gamma)!}$$
$$\times \sum (-)^t[t!(c-b+t+\alpha)!\,(c-a+t-\beta)!\,(a+b-c-t)!\,(a-t-\alpha)!\,(b-t+\beta)!]^{-1} \tag{C.21}$$

with $\qquad (\alpha+\beta+\gamma = 0, \qquad |a-b| \le c \le a+b)$

$$\Delta(abc) \equiv \frac{(a+b-c)!\,(b+c-a)!\,(c+a-b)!}{(a+b+c+1)!} \tag{C.22}$$

$\sum_t$ extends over all integral values of t for which the factorials have a meaning, i.e. for which the arguments of the factorials are positive or null ($0! = 1$). The number of terms in this sum is $\nu+1$, where ν is the smallest of the nine numbers:

$$\begin{matrix} a \pm \alpha & b \pm \beta & c \pm \gamma \\ a+b-c & b+c-a & c+a-b. \end{matrix}$$

5. Special Values and Tables

Special values

(*i*) J and M taking their maximum value:

$$\langle j_1 j_2 j_1 j_2 | j_1+j_2\ j_1+j_2\rangle = 1;$$

(*ii*) one of the j null: $\langle j0m0|jm\rangle = 1$ or

$$\begin{pmatrix} j & j & 0 \\ m & -m & 0 \end{pmatrix} = \frac{(-)^{j-m}}{\sqrt{2j+1}};$$

(*iii*) $m_1 = m_2 = m_3 = 0$:

if $l_1 + l_2 + l_3$ is odd,

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} = 0; \tag{C.23a}$$

if $2p \equiv l_1 + l_2 + l_3$ is even

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} = (-)^p\sqrt{\Delta(l_1\,l_2\,l_3)}\,\frac{p!}{(p-l_1)!\,(p-l_2)!\,(p-l_3)!} \tag{C.23b}$$

(l_1, l_2, l_3 integers ≥ 0 verifying the "triangular inequalities").

Special cases of the Racah formula

The formulae below, or those obtained from them by use of the symmetry relations, give the C.-G. in the following special cases:

(*i*) $m_1 = \pm j_1$ or $m_2 = \pm j_2$ or $M = \pm J$:

$$\langle j_1 j_2 m_1 m_2 | JJ\rangle = \langle j_2 j_1 - m_2 - m_1 | J - J\rangle$$
$$= (-)^{j_1 - m_1}\sqrt{\frac{(2J+1)!\,(j_1+j_2-J)!}{(j_1+j_2+J+1)!\,(J+j_1-j_2)!\,(J+j_2-j_1)!}}\sqrt{\frac{(j_1+m_1)!\,(j_2+m_2)!}{(j_1-m_1)!\,(j_2-m_2)!}} \tag{C.24}$$
$$(m_1 + m_2 = J);$$

(*ii*) one of the j is the sum of the two others:

if $J = j_1 + j_2$

$$\langle j_1 j_2 m_1 m_2 | JM\rangle = \sqrt{\frac{(2j_1)!\,(2j_2)!}{(2J)!}}\sqrt{\frac{(J+M)!\,(J-M)!}{(j_1+m_1)!\,(j_1-m_1)!\,(j_2+m_2)!\,(j_2-m_2)!}} \tag{C.25}$$

$$\langle J j_2 M - m_2 | j_1 m_1\rangle = (-)^{j_2 - m_2}\sqrt{\frac{2j_1+1}{2J+1}}\,\langle j_1 j_2 m_1 m_2 | JM\rangle. \tag{C.26}$$

Tables of the "3j" symbols

The following tables give expressions for the symbol

$$\begin{pmatrix} j & s & (j+e) \\ m & \mu & (-m-\mu) \end{pmatrix}$$

as a function of j and m for

$$s = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2$$
$$0 \le e \le s \qquad 0 \le \mu \le s.$$

With these, and with the aid of the symmetry relations, we can easily calculate any of the C.-G. for which one of the j is equal to 0, $\frac{1}{2}$, 1, $\frac{3}{2}$ or 2