Problem Set #4

Joe Michail
DATA SCI 423 – Machine Learning
Northwestern University

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Problem 7.2

See attached page for code. I was able to get down to 9 mis-classifications total.

Problem 7.3

See attached page for code. I was able to get no mis-classifications as stated in the problem.

Problem 7.4

Starting with equation (7.20):

$$g(\mathbf{w}_0, ..., \mathbf{w}_{C-1}) = \frac{1}{P}\sum_{p=1}^{P} \max_{j=0,...,C-1_{j\neq y_p}} (0, \mathring{\mathbf{x}}_p^T(\mathbf{w}_j - \mathbf{w}_{y_p})).$$

For $C = 2$:

$$g(\mathbf{w}_0, \mathbf{w}_1) = \frac{1}{P}\sum_{p=1}^{P} \max_{j\neq y_p}(0, \mathring{\mathbf{x}}_p^T(\mathbf{w}_j - \mathbf{w}_{y_p})).$$

From the argument in chapter 6: $\mathring{\mathbf{x}}_p^T\mathbf{w} > 0\ (y_p = 1); \mathring{\mathbf{x}}_p^T\mathbf{w} < 0\ (y_p = -1)$. Combining the two gives: $-y_p\mathring{\mathbf{x}}_p^T\mathbf{w} < 0$, so:

$$g(\mathbf{w}_0, \mathbf{w}_1) = \frac{1}{P}\sum_{p=1}^{P} \max_{j\neq y_p}(0, -y_p\mathring{\mathbf{x}}_p^T(\mathbf{w}_j - \mathbf{w}_{y_p})).$$

In the binary case: $y_p\mathring{\mathbf{x}}_p^T\mathbf{w}_{y_p} = 0$. In addition, in the binary case $\mathbf{w}_0 = \mathbf{w}_1 = \mathbf{w}$ (since there is only one boundary and set of weights), so:

$$g(\mathbf{w}) = \frac{1}{P}\sum_{p=1}^{P} \max(0, -y_p\mathring{\mathbf{x}}_p^T\mathbf{w}).$$

QED.

Problem 7.6

Starting with equation (7.24):

$$g(\mathbf{w}_0, ..., \mathbf{w}_{C-1}) = \frac{1}{P}\sum_{p=1}^{P}\log\left(1 + \sum_{j=0;j\neq y_p}^{C-1} e^{\mathring{\mathbf{x}}_p^T(\mathbf{w}_j - \mathbf{w}_{y_p})}\right).$$

Plugging in $C = 2$:

$$g(\mathbf{w}_0, \mathbf{w}_1) = \frac{1}{P}\sum_{p=1}^{P}\log\left(1 + \sum_{j=0;j\neq y_p}^{1} e^{\mathring{\mathbf{x}}_p^T(\mathbf{w}_j - \mathbf{w}_{y_p})}\right).$$

In the binary case $\mathbf{w}_0 = \mathbf{w}_1 = \mathbf{w}$ and $\mathring{\mathbf{x}}_p^T\mathbf{w}_{y_p} = 0$ so,

$$g(\mathbf{w}) = \frac{1}{P}\sum_{p=1}^{P}\log\left(e^0 + e^{\mathring{\mathbf{x}}_p^T\mathbf{w}}\right).$$

The softmax is defined as $\text{softmax}(s_0, s_1) = \log(e^{s_0} + e^{s_1})$, therefore it obviously follows that:

$$g(\mathbf{w}) = \frac{1}{P}\sum_{p=1}^{P}\log\left(e^0 + e^{\mathring{\mathbf{x}}_p^T\mathbf{w}}\right) = \frac{1}{P}\sum_{p=1}^{P}\log(e^0 + e^{\mathring{\mathbf{x}}_p^T\mathbf{w}}) = \frac{1}{P}\sum_{p=1}^{P}\text{softmax}(0, \mathring{\mathbf{x}}_p^T\mathbf{w}).$$

QED.

Problem 7.8

Start with softmax:

$$g(\mathbf{w}_0, ..., \mathbf{w}_{C-1}) = \frac{1}{P}\sum_{p=1}^{P}\log\left(\sum_{j=0}^{C-1} e^{\mathring{\mathbf{x}}_p^T\mathbf{w}_j}\right) - \mathring{\mathbf{x}}_p^T\mathbf{w}_{y_p}.$$

Taking the gradient with respect to $\mathbf{w}_c$:

$$\nabla_{\mathbf{w}_c} g = \frac{1}{P}\sum_{p}\nabla_{\mathbf{w}_c}\log\left(\sum_{j} e^{\mathring{\mathbf{x}}_p^T\mathbf{w}_j}\right) - \nabla_{\mathbf{w}_c}\mathring{\mathbf{x}}_p^T\mathbf{w}_{y_p}.$$

The second term is a constant and applying the differentiation to the first term yields:

$$\nabla_{\mathbf{w}_c} g = \frac{1}{P}\sum_{p}\frac{e^{\mathring{\mathbf{x}}_p^T\mathbf{w}_c}}{\sum_d e^{\mathring{\mathbf{x}}_p^T\mathbf{w}_d}}\mathring{\mathbf{x}}_p^T.$$

Taking the gradient again with respect to $\mathbf{w}_c$ (to get the diagonal):

$$\nabla^2 g = \frac{1}{P}\sum_p \left[\frac{e^{\mathring{\mathbf{x}}_p^T\mathbf{w}_c}}{\sum_d e^{\mathring{\mathbf{x}}_p^T\mathbf{w}_d}} - \left(\frac{e^{\mathring{\mathbf{x}}_p^T\mathbf{w}_c}}{\sum_d e^{\mathring{\mathbf{x}}_p^T\mathbf{w}_d}}\right)^2\right]\mathring{\mathbf{x}}_p\mathring{\mathbf{x}}_p^T$$
$$= \frac{1}{P}\sum_p \left[\frac{e^{\mathring{\mathbf{x}}_p^T\mathbf{w}_c}}{\sum_d e^{\mathring{\mathbf{x}}_p^T\mathbf{w}_d}}\left(1 - \frac{e^{\mathring{\mathbf{x}}_p^T\mathbf{w}_c}}{\sum_d e^{\mathring{\mathbf{x}}_p^T\mathbf{w}_d}}\right)\right]\mathring{\mathbf{x}}_p\mathring{\mathbf{x}}_p^T$$

Since all terms are positive, this means the sum of the eigenvalues (and the eigenvalues themselves) are positive, so the softmax is always convex.
Now the perceptron:

$$g(\mathbf{w}_0, ..., \mathbf{w}_{C-1}) = \frac{1}{P}\sum_p \max_{j=0,...,C-1}(0, \mathring{\mathbf{x}}_p^T(\mathbf{w}_j - \mathbf{w}_{y_p})).$$

Taking the gradient with respect to $\mathbf{w}_c$:

$$\nabla_{\mathbf{w}_c} g = \frac{1}{P}\sum_p \max_{j=0,...,C-1}(0, \nabla_{\mathbf{w}_c}\mathring{\mathbf{x}}_p^T(\mathbf{w}_j - \mathbf{w}_{y_p}))$$
$$= \frac{1}{P}\sum_p \max_{j=0,...,C-1}(\mathbf{0}, \mathring{\mathbf{x}}_p).$$

Taking the gradient again with respect to $\mathbf{w}_c$:

$$\nabla^2 g = \frac{1}{P}\sum_p \max_{j=0,...,C-1}(\mathbf{0}, \mathbf{0})$$
$$= \mathbf{0}.$$

Since the eigenvalues are all non-negative, this implies the perceptron cost function is always convex.

Problem 9.2

See attached page for code. In general, I was able to get the same results as in the textbook. The edge-based method classified about 2,000-3,000 more letters correctly than the pixel-based one after 20 iterations. Therefore, the edge-based detector reigns supreme here.

HW 4

May 17, 2020

[1]:
```
import numpy as np
import matplotlib.pyplot as plt
from sklearn.multiclass import *
from sklearn.linear_model import *
import sklearn.metrics as metrics
from sklearn.datasets import fetch_openml
from sklearn.model_selection import *
from scipy import ndimage

import warnings
warnings.simplefilter("ignore")
```

1 Problem 7.2

[2]:
```
data = np.loadtxt("4class_data.csv", delimiter=',')
x, y = data[:-1, :], data[-1:, :].flatten()

print(x.shape)
print(y.shape)
```

```
(2, 40)
(40,)
```

[3]:
```
OvA = OneVsRestClassifier(Perceptron(max_iter=100000, early_stopping=False,\
                                    fit_intercept=True, warm_start=False,␣
 ↪penalty='l1',\
                                    alpha=0.0005))
OvA.fit(x.T, y)
```

[3]:
```
OneVsRestClassifier(estimator=Perceptron(alpha=0.0005, class_weight=None,
                                         early_stopping=False, eta0=1.0,
                                         fit_intercept=True, max_iter=100000,
                                         n_iter_no_change=5, n_jobs=None,
                                         penalty='l1', random_state=0,
                                         shuffle=True, tol=0.001,
                                         validation_fraction=0.1, verbose=0,
                                         warm_start=False),
```

```
                    n_jobs=None)
```

[4]:
```
OvA_pred = OvA.predict(x.T)
```

[5]:
```
mat = metrics.confusion_matrix(y.flatten(), OvA_pred)
```

[6]:
```
mat
```

[6]:
```
array([[9, 1, 0, 0],
       [1, 8, 0, 1],
       [1, 1, 5, 3],
       [0, 1, 0, 9]], dtype=int64)
```

[7]:
```
print("Number of classifications: %s" % str(np.sum(mat) - np.sum(np.diag(mat))))
```

```
Number of classifications: 9
```

2 Problem 7.3

[8]:
```
data = np.loadtxt("3class_data.csv", delimiter=',')
x, y = data[:-1, :], data[-1:, :].flatten()

print(x.shape)
print(y.shape)
```

```
(2, 30)
(30,)
```

[9]:
```
MC = Perceptron(max_iter=100000, early_stopping=False,\
                                    fit_intercept=True, warm_start=False,␣
 ↪penalty='l1',\
                                    alpha=0.0005)
```

[10]:
```
MC.fit(x.T, y)
```

[10]:
```
Perceptron(alpha=0.0005, class_weight=None, early_stopping=False, eta0=1.0,
           fit_intercept=True, max_iter=100000, n_iter_no_change=5, n_jobs=None,
           penalty='l1', random_state=0, shuffle=True, tol=0.001,
           validation_fraction=0.1, verbose=0, warm_start=False)
```

[11]:
```
pred = MC.predict(x.T)
```

[12]:
```
mat = metrics.confusion_matrix(y, pred)
```

[13]:
```
mat
```

[13]: array([[10, 0, 0],
       [0, 10, 0],
       [0, 0, 10]], dtype=int64)

[14]:
```
print("Number of classifications: %s" % str(np.sum(mat) - np.sum(np.diag(mat))))
```

```
Number of classifications: 0
```

3 Problem 9.2

[15]:
```
x, y = fetch_openml('mnist_784', version=1, return_X_y = True)
y = y.astype(int)
```

[16]:
```
X_train, X_test, y_train, y_test = train_test_split(x, y, train_size=50000)
```

3.1 Pixel-based training

[17]:
```
#Arrays to store test and training histograms
pixel_hists_train = np.zeros((X_train.shape[0], 256))
pixel_hists_test  = np.zeros((X_test.shape[0], 256))

#Training
for i in range(pixel_hists_train.shape[0]):
    count = np.histogram(X_train[i], bins=256, range=[0, 255])[0]
    pixel_hists_train[i, :] = 1.0*count

#Test
for i in range(pixel_hists_test.shape[0]):
    count = np.histogram(X_test[i], bins=256, range=[0, 255])[0]
    pixel_hists_test[i, :] = 1.0*count
```

[18]:
```
#Arrays to store score and cost functions
pix_score = np.array([])
pix_cost  = np.array([])

#Run through 1 to 20 steps
for j in range(1, 21):
    softmax = SGDClassifier(loss = 'log', penalty='l1', alpha=0.01, max_iter=j,␣
 ↪\
                            early_stopping=False, warm_start=True)

    #Fit and predict
    softmax.fit(pixel_hists_train, y_train)
    y_pred = softmax.predict_proba(pixel_hists_test)

    #Calculate a score (% right) and cost function; save
```

```
    pix_score = np.append(pix_score, softmax.score(pixel_hists_test, y_test))
    pix_cost  = np.append(pix_cost, metrics.log_loss(y_test, y_pred,␣
 ↪normalize=True))
```

3.2 Edge training

[19]:
```
#Arrays to store test and training histograms
edge_hists_train = np.zeros((X_train.shape[0], 8))
edge_hists_test = np.zeros((X_test.shape[0], 8))

#Training
for i in range(edge_hists_train.shape[0]):
    reshape = X_train[i].reshape((28, 28))

    #Calculate Gaussian derivatives to find edges in x and y directions
    dIdx = ndimage.filters.gaussian_filter(reshape, [1, 1], order=[0,1],␣
 ↪mode='nearest')
    dIdy = ndimage.filters.gaussian_filter(reshape, [1, 1], order=[1,0],␣
 ↪mode='nearest')

    #Determine the angle of the edge
    angle = np.arctan(dIdy / dIdx)

    #Make histogram
    count = np.histogram(angle, bins=8, range=[-np.pi/2, np.pi/2])[0]
    edge_hists_train[i, :] = 1.0 * count

#Test
for i in range(edge_hists_test.shape[0]):
    reshape = X_test[i].reshape((28, 28))

    dIdx = ndimage.filters.gaussian_filter(reshape, [1, 1], order=[0,1],␣
 ↪mode='nearest')
    dIdy = ndimage.filters.gaussian_filter(reshape, [1, 1], order=[1,0],␣
 ↪mode='nearest')

    angle = np.arctan(dIdy / dIdx)

    count = np.histogram(angle, bins=8, range=[-np.pi/2, np.pi/2])[0]
    edge_hists_test[i, :] = 1.0 * count
```

[20]:
```
edge_score = np.array([])
edge_cost  = np.array([])

for j in range(1, 21):
```

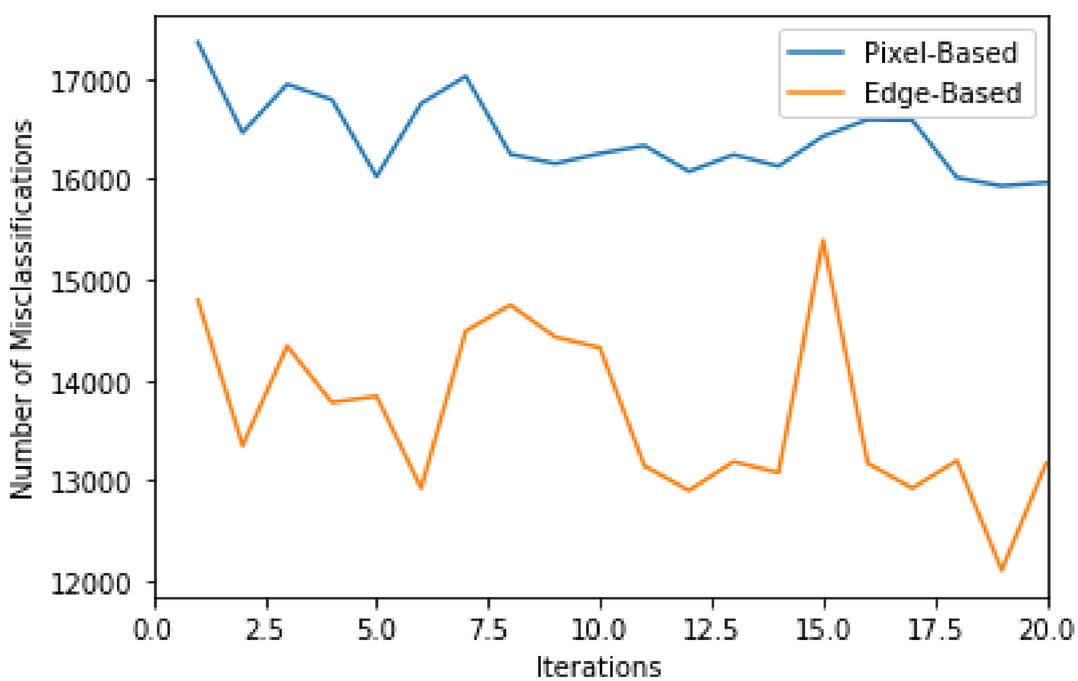
```
    softmax = SGDClassifier(loss = 'log', penalty='l1', alpha=0.01, max_iter=j,␣
 ↪\
                             early_stopping=False, warm_start=True)

    softmax.fit(edge_hists_train, y_train)
    y_pred = softmax.predict_proba(edge_hists_test)

    edge_score = np.append(edge_score, softmax.score(edge_hists_test, y_test))
    edge_cost  = np.append(edge_cost,  metrics.log_loss(y_test, y_pred,␣
 ↪normalize=True))
```

[21]:
```
plt.plot(list(range(1, 21)), (1.0 - pix_score) * X_test.shape[0],␣
 ↪label="Pixel-Based")
plt.plot(list(range(1, 21)), (1.0 - edge_score) * X_test.shape[0],␣
 ↪label="Edge-Based")
plt.xlabel("Iterations")
plt.ylabel("Number of Misclassifications")
plt.legend()
plt.xlim(0, 20)
```

[21]: (0, 20)



[22]:
```
plt.plot(list(range(1, 21)), pix_cost, label="Pixel-Based")
plt.plot(list(range(1, 21)), edge_cost, label="Edge-Based")
```

```python
plt.xlabel("Iterations")
plt.ylabel("Cost")
plt.legend()
```

[22]: <matplotlib.legend.Legend at 0x25b6ae96408>