Physics 412-2 (Spring 2020) **Problem Set 3** Professor Gerald Gabrielse
Posted: 26 Jan. 2020 Due: 9 am on Mon. 3 Feb. 2020

The problem sets are a serious part of the learning experience in this class. The problem sets will sometimes deliberately range away from what can be covered in the lectures in some cases. The goal to expose you to basic concepts and important examples of quantum mechanics. Problems will often be divided into many small parts to guide your through a solution. Corrections to the assignments, if needed, will be posted to the class web page. Your solutions should be placed in the box outside my office.

1. **Spherical Tensors**

   (a) Start from the definition of a spherical tensor T_{kq},

   $$DT_{kq}D^{\dagger} = \sum_{q'} D^{(k)*}_{q'q}.$$

   Prove the the first part of the alternative definition,

   $$[J_z, T_{kq}] = \hbar q T_{kq}.$$

   (b) From the same starting point, prove the second part of the alternative definition,

   $$[J_{\pm}, T_{kq}] = \hbar\sqrt{(k \mp q)(k \pm q + 1)}T_{k,q\pm 1}.$$

   (c) Write down the spherical tensor components in terms of the Cartesian components of a vector $\vec{W}$. Show that these spherical vector components satify the alternative definition for spherical tensors.

   (d) A dot product makes a scalar from two vectors. Starting from the dot product in Cartesian coordinates, derive the dot product of two vectors in terms of their spherical tensor components.

   (e) Show that you get the same result as in the previous section using the general formula for doing this using Clebsch-Gordon coefficients.

   (f) A cross product makes a vector from two vectors. Starting from the dot product in Cartesian coordinates, derive the dot product of two vectors in terms of their spherical tensor components.

   (g) Show that you get the same result as in the previous section using the general formula for doing this using Clebsch-Gordon coefficients.

2. **Coupling of Angular Momenta.** For the following important cases, use the standard table of Clebsch-Gordan coefficients you were introduced to in class, and which is posted on the website. Use Mathematica to verify the CG coefficients you extracted from the table. Hand in the code that you used with your solution.

   (a) For the coupling of the electron spin with the 2p states of the hydrogen atom, express the coupled states in terms of uncoupled states.

(b) For the coupling of the electron spin with the 2p states of the hydrogen atom, express the uncoupled states in terms of the coupled states.

(c) For the hyperfine states of the hydrogen ground state, express the coupled states in terms of uncoupled states.

(d) For the hyperfine states of the hydrogen ground state, express the uncoupled states in terms of the coupled states.

(e) For the coupling of the two electrons in the ground state of the helium atom, express the coupled states in terms of uncoupled states.

(f) For the coupling of the two electrons in the ground state of the helium atom, express the uncoupled states in terms of the coupled states.

3. **Hydrogen matrix elements:** Using the Wigner-Eckhart Theorem and Mathematica (as much as possible) to minimize your labor, compute first a reduced matrix element and then all matrix elements of the operators x, y and z for each of the following sets of states.

   (a) Between the coupled $n = 1$ states of hydrogen formed by coupling orbital angular momentum to electron spin, assuming no nuclear spin.

   (b) Between the coupled $n = 2$ states of hydrogen formed by coupling orbital angular momentum to electron spin, assuming no nuclear spin.

   (c) Between the coupled $n = 1$ and the coupled $n = 2$ states of hydrogen formed by coupling orbital angular momentum to electron spin, assuming no nuclear spin.

4. **Couple two spherical tensors to form other spherical tensors:** Form the spherical tensors that you can from the following:

   (a) From the position vector and the momentum operator.

   (b) From the position vector and the orbital angular momentum operator.

   (c) From the orbital angular momentum operator and itself.

   (d) What is the parity of each of these operators?