Reminder: a function $f(z)$ is continuous at $z = z_0$ if $f(z_0)$ exists and $f(z) \to f(z_0)$ as $z \to z_0$, which means $|f(z) - f(z_0)| \longrightarrow 0$ as $|z - z_0| \to 0$.

Note 1: $z = x + iy$, $z_0 = x_0 + iy_0$

Since $|z - z_0| = \sqrt{(x-x_0)^2 + (y-y_0)^2}$,

$$|z - z_0| \to 0 \iff x \to x_0 \ \& \ y \to y_0$$

Note 2: the same is true for $f(z)$: if

$$f(z) = u(x,y) + i\,v(x,y), \quad f(z_0) = u(x_0,y_0) + i\,v(x_0,y_0)$$

Then $f(z) \to f(z_0) \iff u(x,y) \to u(x_0,y_0)$ & $v(x,y) \to v(x_0,y_0)$

Thus, $f(z) \to f(z_0)$ as $z \to z_0 \iff$
$u(x,y) \to u(x_0,y_0)$ & $v(x,y) \to v(x_0,y_0)$ as $x \to x_0$, $y \to y_0$

$z = x + iy$
$z_0 = x_0 + iy_0$

|| $u(x,y)$, $v(x,y)$ - continuous $\Rightarrow$ $f(z)$ - continuous

$$\begin{matrix} u(x,y) \to u(x_0,y_0) \\ v(x,y) \to v(x_0,y_0) \end{matrix} \Rightarrow f(z) = u(x,y) + i\,v(x,y) \longrightarrow$$
$$\longrightarrow u(x_0,y_0) + i\,v(x_0,y_0) = f(z_0)$$

Ex. $\lim_{z \to 0} |z|^2 = ?$ $\quad |z|^2 = x^2 + y^2$

As $z \to 0$, we have $x \to 0, y \to 0 \Rightarrow \lim_{z \to 0} |z|^2 = 0 =$ the value of $|z|^2$ at $z = 0$

$\therefore$ ~~z~~ $f(z) = |z|^2$ is continuous at $z = 0$, and in fact at any z.

Ex. $\lim_{z\to 0} f(z)$ where $f(z) = \frac{z}{\bar{z}}$ does not exist.

Indeed, if $z_n = x_n + i0 \Rightarrow$

$$f(z_n) = \frac{z_n}{\bar{z}_n} = \frac{x_n}{x_n} = 1 \to 1 \text{ as } z_n \to 0$$

But if $z_n = 0 + iy_n \Rightarrow$

$$f(z_n) = \frac{z_n}{\bar{z}_n} = \frac{iy_n}{-iy_n} = -1 \to -1 \text{ as } z_n \to 0$$

so that we get different limits for different sequences approaching $z=0$ $\therefore$ no common limit exists $\Rightarrow$

$\lim_{z\to 0} f(z)$ does not exist

Def. The derivative $f'(z)$ of the function $f(z)$ is

$$f'(z) = \lim_{\Delta z\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

provided this limit exists.

Ex. $f(z) = z^2 \Rightarrow$

$$f'(z) = \lim_{\Delta z\to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z\to 0} \frac{2z\Delta z + (\Delta z)^2}{\Delta z} = \lim_{\Delta z\to 0} (2z + \Delta z) = 2z$$

Ex. $f(z) = |z|^2$

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{|z+\Delta z|^2 - |z|^2}{\Delta z} = \frac{(z+\Delta z)(\bar{z} + \overline{\Delta z}) - z\bar{z}}{\Delta z} =$$

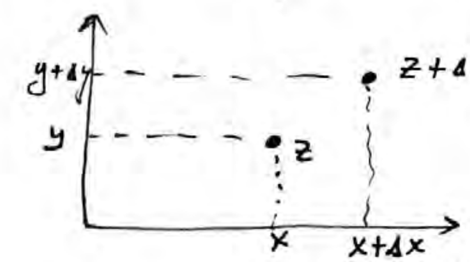
$$= \frac{z\cdot\overline{\Delta z} + \Delta z\cdot\bar{z} + \Delta z\cdot\overline{\Delta z}}{\Delta z} = z\,\underbrace{\frac{\overline{\Delta z}}{\Delta z}}_{\text{does not have a limit}} + \underbrace{\bar{z}}_{\to\,\bar{z}} + \underbrace{\overline{\Delta z}}_{\to\,0 \text{ as } \Delta z \to 0}$$

$\therefore$ thus derivative of $f(z) = |z|^2$ does not exist (unless $z=0$ when the 'bad' term disappears, and then $f'(0) = 0$)

Note that $f(z) = |z|^2 = x^2 + y^2$, so that as a function of x & y it is nice and smooth. Still, $f'(z)$ does not exist for $z \neq 0$. Thus, the notion of the derivative requires further discussion.

Consider again the definition of the derivation of $f(z)$ at the point z:

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$



$$f(z) = u(x,y) + i\, v(x,y)$$

$$f(z+\Delta z) = u(x+\Delta x, y+\Delta y) + i\, v(x+\Delta x, y+\Delta y)$$

$z = x + iy$

$z + \Delta z = x + \Delta x + i(y + \Delta y)$

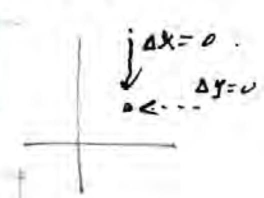
Then

$$f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\overbrace{[u(x+\Delta x, y+\Delta y) - u(x,y)]}^{\Delta u} + i \overbrace{[v(x+\Delta x, y+\Delta y) - v(x,y)]}^{\Delta v}}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y}$$

Suppose the derivative exists, i.e., the limit exists for any choice of $(\Delta x, \Delta y) \to 0$. For example, take $\Delta y = 0$ and consider the limit as $\Delta x \to 0$ $(\Delta z = \Delta x + i \cdot 0 \to 0)$

Then $$f'(z) = \lim_{\Delta x \to 0} \frac{\Delta u + i\Delta v}{\Delta x} = \boxed{\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}}$$



Now consider $\Delta z = 0 + i \cdot \Delta y \to 0$. Then

$$f'(z) = \lim_{\Delta y \to 0} \frac{\Delta u + i\Delta v}{i\Delta y} = \frac{\partial v}{\partial y} + \frac{1}{i}\frac{\partial u}{\partial y} = \boxed{\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}}$$

Thus,

if the derivative $f'(z)$ of $f(z) = u(x,y) + i v(x,y)$ exists at $z = x + iy$, then

$$\text{Re } f'(z) = \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \& \quad \text{Im } f'(z) = \boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

Thus, a necessary condition for $f'(z)$ to exist is

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

— Cauchy - Riemann Conditions
(CR conditions)

A quick way to derive the CR conditions:

$$f(z) = u(x,y) + i v(x,y) \Rightarrow \frac{d}{dx} \Rightarrow f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

$$f(z) = u(x,y) + i v(x,y) \Rightarrow \frac{d}{dy} \Rightarrow \underbrace{f'(z) \cdot \frac{\partial z}{\partial y}}_{\text{chain rule}} = \left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y}\right) \cancel{\xi}$$

i

Thus, $u_x + i v_x = \cancel{\xi}(u_y + i v_y)\frac{1}{i} \Rightarrow$

$u_x = \cancel{\ \ } V_y$, $\cancel{V_x}$ $u_y = -V_x$

Sufficient Condition

If $f(z) = u(x,y) + i v(x,y)$, $z = x + iy$ and

(1) CR Conditions for u & v are satisfied and

(2) u_x, $\cancel{\ \ }$ u_y, v_x, v_y are continuous (at a point of interest

then the derivative of $f(z)$ exists at this point and

$$f'(z) = u_x + i v_x \quad \left(= \frac{1}{i}(u_y + i v_y) = v_y - i u_y \right)$$

Ex. $f(z) = z^2 = (x+iy)^2 = \underbrace{x^2 - y^2}_{u} + i\underbrace{2xy}_{v}$

$u_x = 2x \quad \ominus \quad v_x = 2y$ || All the partial derivatives
$u_y = -2y \quad v_y = 2x$ || u_x, u_y, v_x, v_y – cont.

$\Rightarrow f'(z)$ exists and $f'(z) = u_x + iv_x = 2x + i2y = 2z$

Ex $f(z) = |z|^2 = x^2+y^2 \Rightarrow u(x,y) = x^2+y^2$
$v(x,y) \equiv 0$

$u_x = 2x \mid v_x = 0$
$u_y = 2y \mid v_y = 0$

$\Rightarrow$ CR conditions are satisfied only at $x=y=0$

No derivative unless $z=0$, where $f'(0)=0$.

Ex $f(z) = e^z = e^x(\cos y + i\sin y) \Rightarrow u(x,y) = e^x\cos y$
$v(x,y) = e^x \sin y$

$u_x = e^x\cos y \quad \ominus \quad v_x = e^x \sin y$
$u_y = -e^x\sin y \quad v_y = e^x\cos y$

$f'(z) = u_x + iv_x = e^x\cos y + ie^x\sin y = e^x(\cos y + i\sin y) = e^z$

Cauchy – Riemann Conditions in Polar Form

$f(z) = u(r,\theta) + iv(r,\theta)$, $\quad z = re^{i\theta}$

Using the chain rule:

$\frac{d}{dr}$: $\quad f'(z)\underbrace{\frac{dz}{dr}}_{=e^{i\theta}} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} \Rightarrow \boxed{f'(z) = e^{-i\theta}(u_r + iv_r)}$

$\frac{d}{d\theta}$: $\quad f'(z)\underbrace{\frac{dz}{d\theta}}_{=rie^{i\theta}} = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} \Rightarrow f'(z) = \frac{1}{ir}e^{-i\theta}(u_\theta + iv_\theta)$

$\Rightarrow e^{-i\theta}(u_r + iv_r) = \frac{1}{ir}e^{-i\theta}(u_\theta + iv_\theta) \Rightarrow \boxed{\begin{aligned} u_r &= \tfrac{1}{r}v_\theta \\ v_r &= -\tfrac{1}{r}u_\theta \end{aligned}}$

Ex. $f(z) = z^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n(\cos n\theta + i \sin n\theta)$

$$u(r,\theta) = r^n \cos n\theta, \quad v(r,\theta) = r^n \sin n\theta$$

$$u_r = n r^{n-1} \cos n\theta \qquad v_r = n r^{n-1} \sin n\theta$$

$$u_\theta = - n r^n \sin nx \qquad v_\theta = n r^n \cos n\theta$$

$$\boxed{u_r = \frac{1}{r} v_\theta}, \quad \boxed{v_r = -\frac{1}{r} u_\theta}$$

$$f'(z) = e^{-i\theta}(u_r + i v_r) = e^{-i\theta}\left(n r^{n-1}\cos n\theta + i\, n r^{n-1} \sin n\theta\right)$$

$$= n r^{n-1} e^{-i\theta}(\cos n\theta + i \sin n\theta) = n r^{n-1} e^{-i\theta} \cdot e^{in\theta} =$$

$$= n r^{n-1} e^{i(n-1)\theta} = n (re^{i\theta})^{n-1} = n z^{n-1}$$

We have the usual rules of differentiation, assuming $f(z)$ and $g(z)$ are differentiable:

$$\frac{d}{dz}(fg) = f'g + fg', \quad \frac{d}{dz}(f+g) = f' + g', \quad \frac{d}{dz}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}, \ldots$$

Thus, we don't have to verify that, say, $f(z) = (z^2+z)e^z$ is differentiable – it is differentiable because z, z^2, e^z are differentiable, and we can compute the derivative using the standard rules of differentiation

$$\frac{d}{dz}\left[(z^2+z)e^z\right] = (2z+1)e^z + (z^2+z)e^z = (z^2+3z+1)e^z$$