**Applied Nonlinear Dynamics 322**

Spring 2022

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**Problem Set 2** **Saturday, April 9, 2022** **due Thursday, April 14, 2022**

1. For
$$\frac{dx}{dt} = f(x,\mu) \qquad -\infty < x < \infty \tag{1}$$
complete the decision tree shown in Fig.1 to include the decisions that identify saddle-node, transcritical, supercritical pitchfork, and subcritical pitchfork bifurcations.

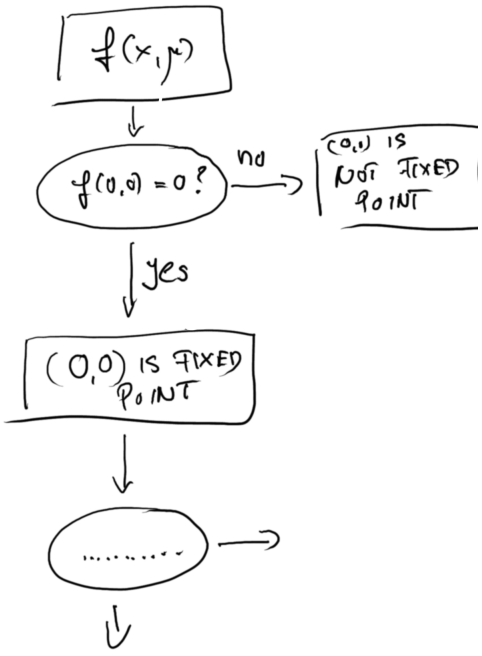


Figure 1: Beginning of a decision tree that organizes the different bifurcations (saddle-node, transcritical, supercritical pitchfork, subcritical pitchfork) that can occur in (1).

2. **Ferromagnetism as a Bifurcation Problem** (cf. Strogatz 3.6.7)
Magnets loose their magnetization when they are heated up above a certain temperature, the Curie temperature, after Pierre Curie, who studied not only radioactivity, but also magentism). Only below the Curie temperature the interaction between the microcscopic magnetic moments (e.g. electron spins), which tries to lign them up with each other leading to a non-zero macroscopic magnetization $M$, is strong enough to overcome the disordering thermal motion. Using statistical mechanics one can show

that within a certain approximation ('mean-field theory') the magnetization of a piece of iron in the presence of a magnetic field is described by the implicit equation

$$F(m,T,h) \equiv \tanh\left(\frac{1}{kT}\left(H+MJ\right)\right) - M = 0, \tag{2}$$

where $M$ is the macroscopic magnetization of the material, $T$ the absolute temperatue, $H$ the external magnetic field, $J$ the coupling strength between the spins, and $k$ is the Boltzmann constant.

(a) For $H = 0$ show that the system undergoes a bifurcation as $T$ is varied. What is the Curie temperature $T_c$ in terms of the parameters of the simple model (2)? Identify the type of the bifurcation.

(b) For $H \neq 0$ obtain the qualitative behavior of the solutions to (2) graphically. Sketch the bifurcation diagram with $T$ as the bifurcation parameter for $H > 0$ and for $H < 0$. Describe how the bifurcation diagram in part (a) is perturbed by the magnetic field $H$ and how the bifurcation diagram changes as $H$ is varied from negative to positive values.

(c) In the $T - H$-plane sketch qualitatively the regions in which (2) has 1 and 3 solutions, respectively.

3. **A Biochemical Switch**
Do problem 3.7.5 in Strogatz. The cited paper by Lewis et al is on Canvas.