Postulates of QM, Part I

- Up til now we have studied the Schr eqn, but we haven't discussed the fundamentals of QM
- We have seen that the following associations

Schr w.f. $\Longleftrightarrow$ config. or location of particle

eigenvalues of energy $\Longleftrightarrow$ definite energies of system

$|\psi(x)|^2$ $\Longleftrightarrow$ probability of finding particle @ x

Provide sensible physical interp & correct predictions

- But we have been following a semi-historical approach, discovering as we go.
- Really QM is a completely dif foundation from CM, & ~~sketch 2 stat~~ modern thinking has entire QM framework built up from a few postulates

Postulate 1 : Location & motion of a particle at any ~~#~~ given time are described completely ~~by~~ as a vector in a Hilbert space ~~[illegible]~~ $\mathcal{H}$, ~~[illegible]~~ a complex ~~vector~~ vector space with positive norm.

Unpack terms:

Vector space V : Set of objects v_i on which ① multiplication by a number & ② addition are well defined.

Multiplication $\alpha \in F$, where F is either $\underset{\text{set of real #'s}}{\mathbb{R}}$ or $\underset{\text{" " complex "}}{\mathbb{C}}$

$v \in V \longrightarrow$

$$\alpha v \in V$$

- If $F = \mathbb{R}$ only, then V is a "real vector space"
- But in QM we need to work in complex vector space, so we have $F = \mathbb{C}$

Addition :

$$v, w \in V \Rightarrow v + w \in V$$

Examples of complex vector spaces :

① n-tuple

$$a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

$$\Rightarrow \quad \alpha a = \begin{pmatrix} \alpha a_1 \\ \alpha a_2 \\ \vdots \\ \alpha a_n \end{pmatrix} \qquad a + b = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix}$$

(Mult criterion obviously met if mult works this way)

(Similar for addition criterion)

- We can define linear independence :

b is linearly independent of $\{a_k\}$ $k = 1 \ldots m$, if there is no choice of α_k s.t. $b = \sum_{k=1}^{m} \alpha_k a_k$

(Impossible to construct b from linear combos of a_k)

- In this representation of the n-dimensional vector space, each

$$e_k = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow \text{kth entry}$$

e_k is linearly independent from set of all the others.

- Also every vector in space can be written

$$a = \sum_{k=1}^{n} a_k e_k$$

~~set of vectors~~

- $\Rightarrow$ e_k form a "basis" for V.
  - Also it's a "minimal" basis. n-dimensional vector space requires a basis of at least n elements — with n-element ones being "minimal"
  - Many possible choices of minimal basis for V, all containing n elements

Example 2 of Complex vector space:

Space of complex-valued functions $\varphi(x)$

- $\alpha\, \varphi(x)$ + $\varphi_1(x) + \varphi_2(x)$ still fall within space, so criteria are met ✓
- Can think of $\varphi(x)$ as ∞-length n-tuple:

$$\begin{pmatrix} \varphi(x_1) \\ \varphi(x_2) \\ \vdots \\ \vdots \end{pmatrix}$$

- For space of functions with same B.C.s, Sturm-Liouville tells us that eigenfunctions of any positive, self-adjoint operator form complete basis for that space of fns.
- E.g. $\psi \to 0$ @ $\pm\infty$. $\{\varphi_i(x)\}$ eigenfunctions of H form complete basis

* - New step here is to consider that we are talking about a vector space, complex infinite-dimensional ∇ to be more precise.

Next def: <u>Positive Norm</u>:

– First: <u>Inner product</u>:

Takes vectors v_1, v_2 to a complex #:

$$N(v_1, v_2) = N(v_1, v_2) = c_{12} \in \mathbb{C}$$

such that N is linear in 2nd arg:

$$N(v_1, \alpha v_2 + \beta v_3) = \alpha N(v_1, v_2) + \beta N(v_1, v_3)$$

+ "antilinear" in 1st arg:

$$N(\alpha v_1 + \beta v_2, v_3) = \alpha^* N(v_1, v_3) + \beta^* N(v_2, v_3)$$

where α^* is complex conj. of α.

– We say that the inner product is a positive norm if

$N(v, v)$ is <u>real</u> + <u>positive</u> for any $v \in \bar{V}$ s.t. $v \neq 0$

– Both of our vector space examples have natural inner products with positive norm:

- For ~~vec~~ vector space of ~~n~~ n-tuples:

$$N(a,b) = \sum_{i=1}^{n} a_i^* b_i$$

$$\Rightarrow N(a,a) = \sum_{i=1}^{n} |a_i|^2 > 0 \quad \text{if} \quad a \neq 0 \quad \checkmark$$

- For vector space of functions:

$$N(\varphi_1; \varphi_2) = \int dx\, \varphi_1^*(x)\, \varphi_2(x)$$

$$\Rightarrow N(\varphi_1, \varphi_1) = \int dx\, |\varphi_1(x)|^2 > 0 \quad \text{if} \quad \varphi_1 \neq 0$$

* - $\psi(t)$ describes motion of vector throughout $\mathcal{H}$

- We have now defined all terms in Postulate 1.

- A few more concepts are helpful.

- Vector is normalized if

$N(v, v) = 1$

a) n-tuple: ~~(n-tuple vector model)~~

$$\sum_i |a_i|^2 = 1$$

b) function ~~(function space model)~~

$$\int dx\, |\psi(x)|^2 = 1$$

- In general v can be "normalized" as follows; assuming $v \neq 0$

$$v_1 = \frac{1}{\sqrt{N(v,v)}}\, v$$

- So we have generalized the concept of unit vectors. Don't usually think of $f(x)$ as unit vector, but if normalized it really can be a unit vector for some corresponding Hilbert space

- Normalized $\varphi_i(x)$ are unit vectors in Hilbert space of fns $\to 0$ at ∞

- Now that we have generalized concept of unit vectors, let's also generalize <u>orthogonality</u>

- Orthogonal if $N(v_1, v_2) = 0$
  - For n-tuple $\sum_{i=1}^{n} a_i^* b_i = 0$
  - For functions
$$\int dx\, \varphi_1^*(x)\, \varphi_2(x) = 0$$

- It is convenient to choose basis for $\mathcal{H}$ in which basis vectors are normalized & ~~or~~ mutually orthogonal

- For basis $\{v_k\}$ of $\mathcal{H}$, $N(v_i, v_j) = \delta_{ij}$

  δ_{ij} is Kronecker delta,

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

- Such a basis is "orthonormal"

- In 3D real space

$$\hat{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \hat{y} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \hat{z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

  form an orthonormal basis.

- Alternatively,

$$\hat{r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$
$$\hat{\theta} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$$
$$\hat{\phi} = (-\sin\phi, \cos\phi, 0)$$

  is also an orthonormal basis

- For our $\mathcal{H}$, it is possible states of the particle which form the "axes". Just as for real 3D space, many choices exist for which axes to use.

- Once we have orthonormal basis $\{v_k\}$, we can get very explicit representation of inner product:

Let a + b be any vectors in $\mathcal{H}$. Then

$$a = \sum \alpha_k v_k \qquad b = \sum \beta_k v_k$$

$$\Rightarrow N(a,b) = N\left(\sum_k \alpha_k v_k, \sum_\ell \beta_\ell v_\ell\right)$$

$$= \sum_{k\ell} \alpha_k^* \beta_\ell \overbrace{N(v_k, v_\ell)}^{\delta_{k\ell}} = \sum_k \alpha_k^* \beta_k$$

- Recall before we had for n-tuple
  $N(a,b) = \sum a_i^* b_i$ which looks same b/c unit vectors

$$\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$ are one ~~also~~ good choice for orthonormal basis

- But for vector space of functions, we had

$\cancel{N(\varphi_1, \varphi_2) = \int dx\, \varphi_1^*(x) \varphi_2(x)}$

$$N(\psi_1, \psi_2) = \int dx\, \psi_1^*(x) \psi_2(x)$$

- That looks at first pretty different. But using orthonormal basis to represent any $\psi(x)$, it really does behave same way as for n-tuple.

  $\Rightarrow$ "Vector space" description seems more reasonable ~~bean~~

- Finally "Dual Space"

- For fixed v_1,

  $N(v_1, v_2)$ is a mapping from $\mathcal{H}$ to $\mathbb{C}$:

  $$N(v_1, \underbrace{\cdot}_{\text{Denotes entire space}}) : \mathcal{H} \to \mathbb{C}$$

- (A complex # sets assigned to each vector in Hilbert space. I.e. mapped. Mapping changes with v_1.)

- Set of such maps (for dif v_1) is called the "dual space" $\mathcal{H}^*$

- We can think of $N(v_1, v_2)$ as describing action of vector v_1 onto vector v_2.

- Dirac expressed it in elegant notation:

  $$N(v_1, v_2) = \langle v_1 | v_2 \rangle \qquad \text{"bracket of } v_1 \text{ \& } v_2\text{"}$$

- So we have 2 spaces. A space of vectors & a space of maps. Latter is dual space.

- Notation suggests we can pull bracket apart into pieces:

$$\langle v_1 | v_2 \rangle$$

$\langle v_1 |$ — bra vector $\in \mathcal{H}^*$

$| v_2 \rangle$ — ket vector $\in \mathcal{H}$

If $|v_1\rangle$ is $\psi_1(x)$, $\langle v_1|$ is not just $\psi_1^*(x)$. it is that & $\int \rightarrow$ to make a ~~s...~~ map.

- Next: Results of Measurements ... about to get less abstract.