Postulates of QM, Part II

Postulate 2

(physicists use "self-adjoint" "Hermitian" interchageably, even though mathematicians would distinguish)

Every measurable quantity corresponds to a linear, self-adjoint operator Θ acting on states in $\mathcal{H}$. If $|\psi\rangle$ is an eigenvector of Θ:

$$\Theta|\psi\rangle = \lambda|\psi\rangle$$

then the measurement of Θ yields a definite value λ if the system is prepared in state $|\psi\rangle$.

- Case where system is not in an eigenstate is covered in Postulates 3-5
- Self-adjoint: Θ is s.-a. if

$$\langle v_1 | \Theta v_2 \rangle = \langle \Theta v_1 | v_2 \rangle$$

- "Self-adjoint" ≈ "Hermitian". Mathematically these are different, related to questions of boundedness but for our QM purposes, they are the same.

- More generally the "Hermitian adjoint" of operator Φ is defined by eqn.

$$\langle v_1 | \Phi v_2 \rangle = \langle \Phi^\dagger v_1 | v_2 \rangle \text{ for all } v_1, v_2 \text{ in } \mathcal{H}$$

Hermitian adjoint.

Φ is self-adjoint if $\Phi^\dagger = \Phi$

- For vector space of functions, self-adjoint means

$$\int dx \, \varphi_1^*(x) \, \Theta \, \varphi_2(x) = \int dx \left(\Theta \varphi_1(x)\right)^* \varphi_2(x),$$

which is def. we used for Sturm-Liouville

- What does s-a mean for n-tuple model of $\mathcal{H}$?
- For n-tuple model of Hilbert space:

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

Most general linear Mapping from a to b given by $\quad m = \begin{pmatrix} m_{11} & m_{12} & \cdots \\ m_{21} & & \\ \vdots & & \end{pmatrix}$

$$b_i = \sum_{j=1}^{n} m_{ij} a_j \, , \quad m_{ij} \in \mathbb{C}$$

- If we think of m_{ij} as a $n \times n$ matrix,

$$b = \underset{\sim}{m} \cdot a$$

- Condition of self-adjointness is

$$\langle a | \sigma b \rangle = \langle \sigma a | b \rangle$$

(From def. of inner product)

$$\Rightarrow \sum_{ij} a_i^* (m_{ij} b_j) = \sum_{ij} (m_{ij} a_j)^* b_i$$

- Interchanging i,j

$$= \sum_{ij} a_i^* m_{ji}^* b_j$$

- Since this must hold (for s-a operator) for any pair of vectors $|a\rangle, |b\rangle$, we must have

$$m_{ij} = m_{ji}^*$$

- In terms of matrices

$$\underset{\sim}{m} = (\underset{\sim}{m}^T)^*$$

$$\begin{pmatrix} m_{11} & m_{12} & \cdots \\ \underbrace{m_{21}}_{= m_{12}^*} & & \\ \vdots & & \end{pmatrix}$$

Only true if m is Hermitian

- We can define Hermitian adjoint of a matrix $\underset{\sim}{n}$ s.t.

$$\underset{\sim}{n}^{+} \equiv (\underset{\sim}{n}^{T})^{*}$$

H.A.

$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ is Hermitian

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ are not

Observables never look like this

These can correspond to observables

We call $\underset{\sim}{n}$ "self-adjoint" or "Hermitian" if $\underset{\sim}{n} = \underset{\sim}{n}^{+}$

- We saw from Sturm-Liouville that if an operator $\mathcal{O}$ was self-adjoint, eigenfunctions of $\mathcal{O}$ satisfied certain properties. Now generalize to any Hilbert space.

- Theorem:

Let $\mathcal{O}$ be a self-adjoint operator on $\mathcal{H}$. Then the eigenvectors of $\mathcal{O}$ form a complete basis for $\mathcal{H}$.

- For space of functions, proof is S-L theorem.
- For n-tuples you prove by diagonalizing arbitrary matrix + using well-known results from lin. algebra

- From this result, some consequences follow:

(1) <u>Eigenvalues of O are real</u>

v_i is e.v. of O:

$$O v_i = \lambda_i v_i$$

$$\Rightarrow \langle v_i | O v_i \rangle = \langle v_i | \lambda_i v_i \rangle = \lambda_i \langle v_i | v_i \rangle$$

But also

$$\langle v_i | O v_i \rangle = \langle O v_i | v_i \rangle = \langle \lambda_i v_i | v_i \rangle = \lambda_i^* \langle v_i | v_i \rangle$$

$$\Rightarrow \lambda_i = \lambda_i^*$$

* comes from the def. of inner product,

(2) <u>If v_i, v_j are e.vects of O with dif e.vs</u>

$$\Rightarrow \underline{\langle v_i | v_j \rangle = 0}$$

$$\langle v_i | O v_j \rangle = \lambda_j \langle v_i | v_j \rangle$$

$$= \langle O v_i | v_j \rangle = \lambda_i \langle v_i | v_j \rangle \quad (\text{since } \lambda_i = \lambda_i^*)$$

- This can be true one of two ways: either $\lambda_i = \lambda_j$ or

$$\langle v_i | v_j \rangle = 0$$

3 Let $|v_i\rangle$ be an orthonormal basis of $\mathcal{H}$.

$$\Rightarrow \sum |v_i\rangle\langle v_i| = \mathbb{1}$$

Where $\mathbb{1}$ is unit operator $\mathbb{1}|v\rangle = |v\rangle$

* - This will be true if $|v_i\rangle$ are normalized e.v.s of $\hat{O}$

Proof:

If $\{|v_i\rangle\}$ is orthonormal basis, any $|v\rangle$ is

$$|v\rangle = \sum_j \alpha_j |v_j\rangle$$

$$\Rightarrow \left(\sum_i |v_i\rangle\langle v_i|\right)|v\rangle = \sum_{i,j} |v_i\rangle\underbrace{\langle v_i|v_j\rangle}_{\delta_{ij} \text{ since orthonormal}}\alpha_j$$

$$= \sum_j |v_j\rangle\alpha_j = |v\rangle$$

$$\Rightarrow \sum_i |v_i\rangle\langle v_i| = \mathbb{1} \quad ✓$$

Just projects onto basis & writes ~~[illegible]~~ as sum of basis vectors

- On Hilbert space of n-tuples,

a basis vector $|v_i\rangle$ is an n-tuple with

kth entry $(v_i)_k$

- Using linear algebra formalism, $|v_i\rangle$ is a column vector $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \end{pmatrix}$ and from def. of inner product $\langle v_i|$ is a row vector $(v_1^*, v_2^*, v_3^* \ldots)$

- Then above formula reads

(In component form)

$$\sum_i (v_i)_k (v_i)_l^* = \delta_{kl}$$

$\Longleftarrow$

(In matrix form)

$$\sum_i \underset{\sim}{v}_i (\underset{\sim}{v}_i^T)^* = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots\ 1 \end{pmatrix}$$

~~(easy to see for diagonalized basis, but here is an example from a different basis)~~

~~where I have represented v_i as a column vector, that order is necessary to get an $n\times n$ matrix~~

Example

$$v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$$= \cancel{\left(1/\sqrt{2}, 1/\sqrt{2}\right)\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} + \left(1/\sqrt{2}, -1/\sqrt{2}\right)\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}}$$

$$= \overbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix}}^{v_1} + \overbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \end{pmatrix}}^{v_2}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$$

- We usually like to think of the identity operator $\sum |v_i\rangle\langle v_i|$ as just projecting an arbitrary state onto some choice of basis.

- For instance we can use the eigenstates of H with either of these potentials to represent any complex-valued function

$V(x)$ $\qquad\qquad$ $V'(x)$

Eigenstates $|0\rangle, |1\rangle, \ldots$ $\qquad\qquad$ $|0'\rangle, |1'\rangle, \ldots$

Then we can represent for instance $|0\rangle$ as

$$
\begin{aligned}
|0\rangle &= c_0 |0'\rangle + c_1 |1'\rangle + \ldots \\
&= \langle 0'|0\rangle |0'\rangle + \langle 1'|0\rangle |1'\rangle + \ldots \\
&= |0'\rangle\langle 0'|0\rangle + |1'\rangle\langle 1'|0\rangle + \ldots \\
&= \underbrace{\sum |n'\rangle\langle n'|}_{\mathbb{1}} 0\rangle
\end{aligned}
$$

Unitary Operators - Different class from Hermitian

- Preserve inner product : $\langle v|w\rangle = \langle \mathcal{U}v|\mathcal{U}w\rangle$
- These are generalization of 3D rotation, which leaves lengths & dot products of vectors invariant.

- In general $\mathcal{O}^{-1}\mathcal{O} = \mathbb{1}$ for any operator
- $\mathcal{U}$ has special property that $\mathcal{U}^{+} = \mathcal{U}^{-1}$

- Using def of Hermition adjoint

$$\langle \mathcal{U}v|\mathcal{U}w\rangle = \langle v|\mathcal{U}^{+}\mathcal{U}w\rangle$$

- Since this equals $\langle v|w\rangle$ by def of unitary,

$$\Rightarrow \mathcal{U}^{+}\mathcal{U} = \mathbb{1} \quad \Rightarrow \quad ①\ \mathcal{U}^{+} = \mathcal{U}^{-1}, \quad ②\ \mathcal{U}\mathcal{U}^{+} = \mathbb{1}$$

$+ \ \mathcal{U}\mathcal{U}^{-1} = 1$

- Note that these are not self-adjoint :

$\mathcal{U}^{+} \neq \mathcal{U}$ except for special cases $(\mathcal{U} = \mathbb{1}, \mathcal{U} = \Pi)$ where $\mathcal{U}\mathcal{U} = \mathbb{1}$

~~On Hilbert space of n-tuples, this can be written~~ ~~is a matrix mult. which~~

$$\cancel{\sum_i (\mathcal{U}^{+})_{ki}(\mathcal{U})_{il} = \delta_{kl}}$$

~~or~~

$$\cancel{\sum_i (\mathcal{U}_{ik})^{*}\,\mathcal{U}_{il} = \delta_{kl}}$$

~~recall in general $\underset{\sim}{n}^{+} = (\underset{\sim}{n}^{T})^{*}$~~

~~Schr~~ ~~Max~~

~~$|\uparrow\rangle + |\downarrow\rangle$~~

~~① How could you tell?~~

- It turns out that the columns of $\underset{\sim}{U}$ are an orthonormal basis for n-tuples

Proof

- U preserves orthogonality + normalization, by def.
- Consider $\underset{\sim}{U}$ acting on the orthonormal basis

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \text{ etc}$$

$$\underbrace{\begin{pmatrix} \vdots & U_{12} & \vdots \\ \vdots & U_{22} & \vdots \\ \vdots & U_{32} & \vdots \end{pmatrix}}_{\underset{\sim}{U}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} U_{12} \\ U_{22} \\ U_{32} \end{pmatrix}$$

And having U act on a dif. basis vector gives a different column.

$\Rightarrow$ The columns of U are an orthonormal basis, since that (simplest) basis vector set is

Special Operator Summary

Any Operator A

- Hermitian adjoint $A^\dagger$:

  $\langle \psi | A \chi \rangle = \langle A^\dagger \psi | \chi \rangle$

- Inverse A^{-1}:

  $A A^{-1} = A^{-1} A = \mathbb{1}$

Hermitian Operator $\mathcal{O}$

- $\mathcal{O} = \mathcal{O}^\dagger$:

$\Rightarrow \langle \psi | \mathcal{O} \chi \rangle = \langle \mathcal{O} \psi | \chi \rangle$

$\Rightarrow$ Eigenvalues of $\mathcal{O}$ are real

$\Rightarrow$ Eigen vectors of $\mathcal{O}$ can be chosen such that they form an orthonormal basis for $\mathcal{H}$

- Requires normalizing
- Requires ~~m~~ taking right care in degenerate subspace

e.g. $\mathcal{O} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad |\varphi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \varphi_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

both are eigenvectors, but not orthogonal.

Infinite # of choices, with $\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

or $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ being popular choices

Unitary Operator $\mathcal{U}$

$\mathcal{U}^\dagger \mathcal{U} = \mathbb{1}$:

$\Rightarrow \langle \mathcal{U}\psi | \mathcal{U}\chi \rangle = \langle \psi | \chi \rangle$

$\Rightarrow \mathcal{U}^\dagger = \mathcal{U}^{-1}$

$\Rightarrow$ $\mathcal{U}$ rotates $|\psi\rangle$ in $\mathcal{H}$.

Equivalently, $\mathcal{U}$ transforms one orthonormal basis for $\mathcal{H}$ into another