① consider a system with

$$[H, A] = i\hbar\omega B$$
$$[H, B] = -i\hbar\omega A$$

recall Ehrenfest's theorem (see previous discussion!)

$$\frac{d\langle \Theta\rangle}{dt} = \frac{1}{i\hbar}\langle [\Theta, H]\rangle = \frac{i}{\hbar}\langle [H, \Theta]\rangle$$

plug these commutators into this result to get a coupled pair of differential equations,

$$\frac{d\langle A\rangle}{dt} = \frac{i}{\hbar}\langle [H, A]\rangle = \frac{i}{\hbar}\langle i\hbar\omega B\rangle = -\omega\langle B\rangle$$

$$\frac{d\langle B\rangle}{dt} = \frac{i}{\hbar}\langle [H, B]\rangle = \frac{i}{\hbar}\langle -i\hbar\omega A\rangle = +\omega\langle A\rangle$$

so:

$$\langle \dot{A}\rangle = -\omega\langle B\rangle \qquad \langle \dot{B}\rangle = +\omega\langle A\rangle$$

apply another time derivative to both sides:

$$\langle \ddot{A}\rangle = -\omega\langle \dot{B}\rangle = -\omega(\omega\langle A\rangle) \quad \text{so} \quad \langle \ddot{A}\rangle = -\omega^2\langle A\rangle$$
$$\langle \ddot{B}\rangle = +\omega\langle \dot{A}\rangle = +\omega(-\omega\langle B\rangle) \quad \text{so} \quad \langle \ddot{B}\rangle = -\omega^2\langle B\rangle$$

solve:

$$\langle A\rangle(t) = \alpha\cos(\omega t) + \beta\sin(\omega t)$$
$$\langle B\rangle(t) = \gamma\cos(\omega t) + \rho\sin(\omega t)$$

fix $\alpha, \beta, \gamma, \rho$. $\langle A\rangle(t=0) \equiv A_0$ $\quad\langle B\rangle(t=0) \equiv B_0$

so $\quad \alpha = A_0 \qquad \gamma = B_0$

$$\langle \dot{A}\rangle = -\omega A_0\sin(\omega t) + \omega\beta\cos(\omega t)$$
$$= -\omega\langle B\rangle$$

so $$\langle B\rangle = -\beta\cos(\omega t) + A_0\sin(\omega t)$$
$$= B_0\cos(\omega t) + \rho\sin(\omega t)$$

$$\rho = A_0 \qquad \beta = -B_0$$

$$\boxed{\begin{aligned}\langle A\rangle(t) &= A_0\cos(\omega t) - B_0\sin(\omega t)\\ \langle B\rangle(t) &= B_0\cos(\omega t) + A_0\sin(\omega t)\end{aligned}}$$

② $H|\psi_1\rangle = E_1|\psi_1\rangle \qquad H|\psi_2\rangle = E_2|\psi_2\rangle \qquad H = H^\dagger$ so:

$\langle\psi_1|H = \langle\psi_1|\cdot E_1 \qquad \langle\psi_2|H = E_2\langle\psi_2| \qquad E_1 = E_1^* \quad E_2 = E_2^*$

a. construct:

$$\langle\psi_1|H|\psi_2\rangle = \langle\psi_1|\psi_2\rangle E_2 \quad \text{or} \qquad \text{(H acts to right)}$$
$$= E_1\langle\psi_1|\psi_2\rangle \qquad \text{(H acts to left)}$$

$(E_2 - E_1)\langle\psi_1|\psi_2\rangle = 0$

$E_1 \neq E_2$ so $\boxed{\langle\psi_1|\psi_2\rangle = 0}$

b. $A|\psi_1\rangle = |\psi_2\rangle \qquad A|\psi_2\rangle = |\psi_1\rangle$

write as a ~~[illegible]~~ matrix:

$$A = \begin{pmatrix} \langle\psi_1|A|\psi_1\rangle & \langle\psi_1|A|\psi_2\rangle \\ \langle\psi_2|A|\psi_1\rangle & \langle\psi_2|A|\psi_2\rangle \end{pmatrix} = \begin{pmatrix} \langle\psi_1|\psi_2\rangle & \langle\psi_1|\psi_1\rangle \\ \langle\psi_2|\psi_2\rangle & \langle\psi_2|\psi_1\rangle \end{pmatrix}$$

$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

diagonalize to find eigenvalues & eigenvectors:

$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \qquad \boxed{\lambda = \pm 1}$

$\lambda = +1, \quad \vec{v}_1 = \begin{pmatrix} a \\ b \end{pmatrix}$

$A\vec{v}_1 = +\vec{v}_1 \quad$ so $\quad (A - I)\vec{v}_1 = 0$

$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} b-a \\ a-b \end{pmatrix} \qquad a = b \qquad \boxed{\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

$\lambda = -1, \quad \vec{v}_{-1} = \begin{pmatrix} c \\ d \end{pmatrix}$

$A\vec{v}_{-1} = -\vec{v}_{-1} \quad$ so $\quad (A + I)\vec{v}_{-1} = 0$

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} c+d \\ c+d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad c = -d \qquad \boxed{\vec{v}_{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

c. set up system at $t=0$ as:

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\left(|\psi_1\rangle - |\psi_2\rangle\right)$$ eigenstate of A with $\lambda=-1$

time evolution is given by the Schrödinger equation:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-iE_1t/\hbar}|\psi_1\rangle - e^{-iE_2t/\hbar}|\psi_2\rangle\right)$$

probability to return to initial state is:

$$P(t) = |\langle\psi(t)|\psi(0)\rangle|^2$$

$$\langle\psi(0)|\psi(t)\rangle = \frac{1}{2}\left(\langle\psi_1| - \langle\psi_2|\right)\left(e^{-iE_1t/\hbar}|\psi_1\rangle - e^{-iE_2t/\hbar}|\psi_2\rangle\right)$$

$$= \frac{1}{2}\left(e^{-iE_1t/\hbar} + e^{-iE_2t/\hbar}\right)$$ since $\langle\psi_1|\psi_2\rangle = 0$, $\langle\psi_1|\psi_1\rangle = \langle\psi_2|\psi_2\rangle = 1$

factor out a common multiple:

$$e^{-iE_1t/\hbar} = e^{-i(E_1+E_2)t/2\hbar}\cdot e^{-iE_1t/2\hbar}\, e^{+iE_2t/2\hbar}$$

$$e^{-iE_2t/\hbar} = e^{-i(E_1+E_2)t/2\hbar}\cdot e^{-iE_2t/2\hbar}\, e^{+iE_1t/2\hbar}$$

$$\langle\psi(0)|\psi(t)\rangle = e^{-i(E_1+E_2)t/2\hbar}\,\underbrace{\frac{1}{2}\left(e^{i(E_2-E_1)t/2\hbar} + e^{-i(E_2-E_1)t/2\hbar}\right)}_{\cos\left(\frac{(E_2-E_1)t}{2\hbar}\right)}$$

$$\langle\psi(0)|\psi(t)\rangle = e^{-i(E_1+E_2)t/2\hbar}\cos\left[\frac{1}{2\hbar}(E_2-E_1)t\right]$$

$$P(t) = |\langle\psi(0)|\psi(t)\rangle|^2$$

$$\boxed{P(t) = \cos^2\left[\frac{(E_2-E_1)t}{2\hbar}\right]}$$

(3) a. $\langle P\rangle = \int_{-\infty}^{+\infty} \psi^*(x)\left[-i\hbar\frac{\partial}{\partial x}\right]\psi(x)\,dx \qquad \psi^*(x) = \psi(x)$

$$\langle P\rangle = -i\hbar\int_{-\infty}^{+\infty} \underbrace{\psi(x)\frac{\partial\psi}{\partial x}}_{=\frac{1}{2}\frac{\partial}{\partial x}(\psi^2)}\,dx$$

$$\langle P\rangle = -\frac{i\hbar}{2}\int_{-\infty}^{+\infty}\frac{\partial}{\partial x}\left(\psi^2(x)\right)dx$$

$$= -\frac{i\hbar}{2}\left[\psi^2(x)\right]_{-\infty}^{+\infty}$$ must vanish $\psi(\pm\infty) = 0$ if normalizable

$$\boxed{\langle P\rangle = 0}$$

(can also show this neatly via integration by parts)

b. suppose $\int_{-\infty}^{+\infty}\psi^*(x)\left[-i\hbar\frac{\partial}{\partial x}\right]\psi(x)\,dx = \langle P\rangle$

find, for $\phi(x) \equiv e^{ip_0x/\hbar}\psi(x)$, $\langle\phi|P|\phi\rangle$

$$\langle\phi|P|\phi\rangle = \int_{-\infty}^{+\infty}\phi^*(x)\left[-i\hbar\frac{\partial}{\partial x}\right]\phi(x)\,dx$$

$$-i\hbar\frac{\partial}{\partial x}\phi(x) = -i\hbar\frac{\partial}{\partial x}\left[e^{ip_0x/\hbar}\psi(x)\right]$$

$$= p_0 e^{ip_0x/\hbar}\psi(x) + e^{ip_0x/\hbar}\left[-i\hbar\frac{\partial}{\partial x}\right]\psi(x)$$

$$\langle\phi|P|\phi\rangle = p_0\int_{-\infty}^{+\infty}\psi^* e^{-ip_0x/\hbar}e^{ip_0x/\hbar}\psi(x)\,dx + \int_{-\infty}^{+\infty}\psi^*\underbrace{e^{-ip_0x/\hbar}e^{ip_0x/\hbar}}_{1}\left[-i\hbar\frac{\partial}{\partial x}\right]\psi(x)\,dx$$

$$= p_0\underbrace{\int_{-\infty}^{+\infty}\psi^*\psi\,dx}_{\text{normalized to }1} + \underbrace{\int_{-\infty}^{+\infty}\psi^*\left[-i\hbar\frac{\partial}{\partial x}\right]\psi\,dx}_{\langle P\rangle}$$

$$\boxed{\langle\phi|P|\phi\rangle = p_0 + \langle P\rangle}$$

④ $Q = a\left(|1\rangle\langle 1| + |1\rangle\langle 2| - |2\rangle\langle 2|\right)$

$a \in \mathbb{R}$

$$Q = \begin{pmatrix} \langle 1|Q|1\rangle & \langle 1|Q|2\rangle \\ \langle 2|Q|1\rangle & \langle 2|Q|2\rangle \end{pmatrix} = \begin{pmatrix} a & +a \\ 0 & -a \end{pmatrix}$$

diagonalize to find the eigenvalues & vectors of Q

$$\det(Q - \lambda I) = \begin{vmatrix} a-\lambda & a \\ 0 & -a-\lambda \end{vmatrix} = -(a+\lambda)(a-\lambda) = 0$$

$\Rightarrow \boxed{\lambda = \pm a}$

$\lambda = +a \quad \vec{V}_+ = \begin{pmatrix} A \\ B \end{pmatrix}$

$Q - \lambda I = \begin{pmatrix} 0 & a \\ 0 & -2a \end{pmatrix}$

$\begin{pmatrix} 0 & a \\ 0 & -2a \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} aB \\ -2aB \end{pmatrix} \qquad B = 0 \quad$ so $\boxed{\vec{V}_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$

$\lambda = -a \quad \vec{V}_- = \begin{pmatrix} C \\ D \end{pmatrix}$

$Q - \lambda I = \begin{pmatrix} 2a & a \\ 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 2a & a \\ 0 & 0 \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} (2C+D)a \\ 0 \end{pmatrix} \qquad 2C + D = 0$

$D = -2C \qquad \boxed{\vec{V}_- = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}}$

distinct eigenvalues, but eigenvectors not orthogonal!

$$\langle V_+|V_-\rangle = 1/\sqrt{5} \neq 0$$

why not? $\quad Q$ is not Hermitian! (which we usually assume)

$$Q^+ = \begin{pmatrix} a & 0 \\ a & -a \end{pmatrix} \neq Q$$

so diff e'vectors not orthogonal