Problem Set #2

Solutions

1 **a** It can be shown that eigenfunctions of a symmetric potential are either even or odd.

Odd: $\varphi(x) = -\varphi(-x)$

Continuity of w.f. means $\varphi(0_+) = \varphi(0_-)$

$\Rightarrow \varphi(0) = -\varphi(0) \Rightarrow \boxed{\varphi(0) = 0}$

Even: $\varphi(x) = \varphi(-x)$

$$\frac{d}{dx}\varphi(x) = \frac{d}{d(-x)}\varphi(-x) = -\frac{d}{dx}\varphi(-x)$$

(Just taking $x \to -x$)

$\Rightarrow \varphi'(x) = -\varphi'(-x)$

Smoothness of w.f. means $\varphi'(0_+) = \varphi'(0_-)$

$\Rightarrow \varphi'(0) = -\varphi'(0) \Rightarrow \boxed{\varphi'(0) = 0}$

b) $x>0$, φ odd

2

Take $y = cx+d \Rightarrow \frac{d}{dy} = \frac{1}{c}\frac{d}{dx}$

We want $\frac{d^2}{dy^2}\varphi - y\varphi = 0$

$$\frac{d^2}{dx^2}\varphi + \frac{2m}{\hbar^2}(E_n - ax)\varphi = 0$$

$$\Downarrow$$

$$\frac{d^2}{dy^2}\varphi + \underbrace{\frac{2m}{c^2\hbar^2}(E_n - ax)}_{-y = -cx-d}\varphi = 0$$

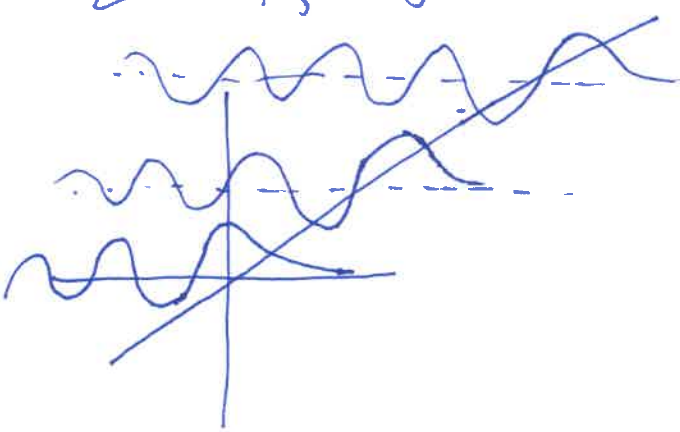
$$\Rightarrow \frac{-2ma}{c^2\hbar^2} = -c \Rightarrow c = \left(\frac{2ma}{\hbar^2}\right)^{1/3}$$

$$\Rightarrow \frac{2mE_n}{c^2\hbar^2} = -d \Rightarrow d = -\frac{2mE_n}{\hbar^2}\cdot\left(\frac{2ma}{\hbar^2}\right)^{-2/3}$$

$$\Rightarrow \boxed{\frac{d^2}{dy^2}\varphi(y) - y\,\varphi(y) = 0, \text{ where } y = \left(\frac{2ma}{\hbar^2}\right)^{1/3}\left(x - \frac{E_n}{a}\right)}$$

The $\varphi(y)$ which satisfy $\varphi \to 0$ as $x \to \infty$ are the $A_i(y)$.

If we just had $V(x) = ax$, what we have just worked out shows that every choice of E_n has a solution to the Schr. equation. Each E_n has a unique & ~~different~~ offset for the Airy function which solves the equation, e.g.:



However, we are solving instead for $V(x) = a|x|$, and this imposes the even/odd condition for $\varphi(x=0)$.
(For odd ~~function~~ φ)
We must choose the E_n such that $\varphi(y)$ has a zero at $x=0$. $\propto A_i(y)$

The Airy function has zeroes at $A_i(a_k) = 0$

$$\Rightarrow a_k = y\big|_{x=0} = \left(\frac{2ma}{\hbar^2}\right)^{1/3}\left(0 - \frac{E_n}{a}\right)$$

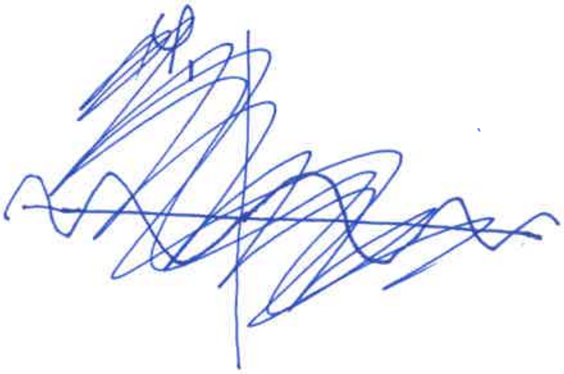
Then e.g. (using table)

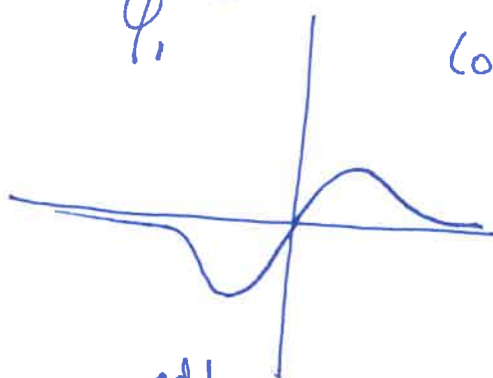
$$E_1 = (-2.34)\left(\frac{2ma}{\hbar^2}\right)^{-1/3}(-a)$$

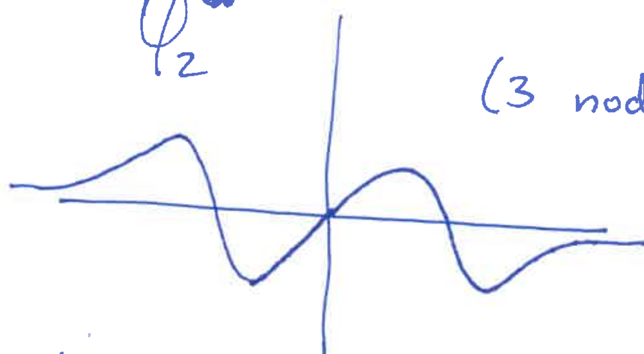
$$E_2^* = (+4.09)\left(\frac{2ma}{\hbar^2}\right)^{-1/3}(a), \text{ etc.}$$

Then for odd solutions, we have

$$\boxed{\varphi_k^{odd}(x) \propto \begin{cases} Ai\left[\left(\frac{2ma}{\hbar^2}\right)^{1/3}\left(x-\frac{E_k}{a}\right)\right], & x>0 \\ -Ai\left[\left(\frac{2ma}{\hbar^2}\right)^{1/3}\left(\underbrace{|x|}_{=-x}-\frac{E_k}{a}\right)\right], & x<0 \end{cases}}$$







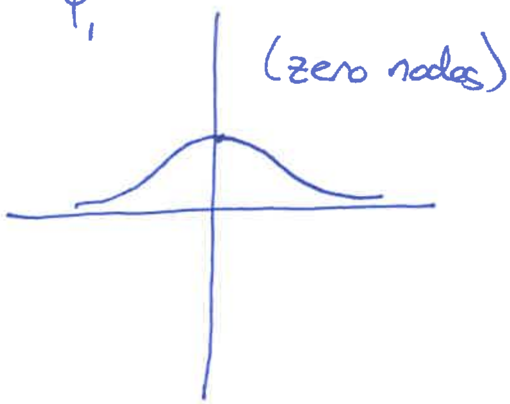
c For even parity solutions, we need a zero in $Ai'(y)$ at $x=0$.

$$\Rightarrow \underbrace{a_k'}_{k\text{th zero of } Ai'(y)} = \left(\frac{2ma}{\hbar^2}\right)^{1/3}\left(-\frac{E_k}{a}\right)$$

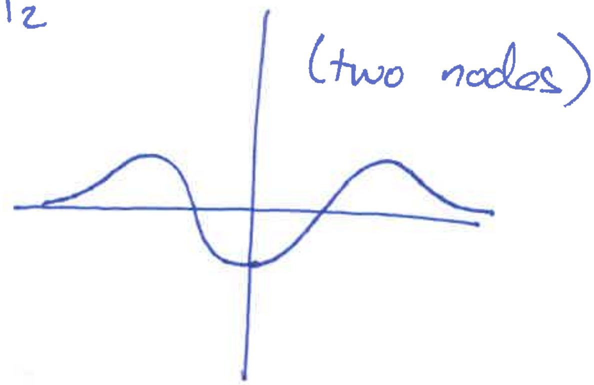
$$\Rightarrow \boxed{\varphi_k^{even} \propto \begin{cases} Ai\left[\left(\frac{2ma}{\hbar^2}\right)^{1/3}\left(x-\frac{E_k}{a}\right)\right], & x>0 \\ Ai\left[\left(\frac{2ma}{\hbar^2}\right)^{1/3}\left(\underbrace{|x|}_{=-x}-\frac{E_k}{a}\right)\right], & x<0 \end{cases}}$$

These look like

φ_1^{even}



φ_2^{even}



We see from the table that the a_k are interleaved between the a_k'.

$\Rightarrow$ Between every pair of even-function eigenenergies, there is an odd-function eigenenergy.

$\boxed{2}$

a) $$f(b)-f(a) = \int_a^b \frac{df(x)}{dx}\,dx = \int_a^b \frac{f(x)-1}{L}\,dx$$

b) $$f(b)-f(a) \approx \left(\left.\frac{df(x)}{dx}\right|_{x=a}\right)(\Delta x) = \frac{f(a)-1}{L}\,\Delta x$$

c) Here, I will take $L=1$ to demonstrate.

$$f(0.1) \approx f(0) + \frac{f(0)-1}{L}\,\Delta x = 2 + \frac{\Delta x}{L} = 2.1$$

$$f(0.2) \approx f(0.1) + \frac{f(0.1)-1}{L}\,\Delta x \approx 2.1 + 1.1\,\frac{\Delta x}{L} = 2.21$$