Spin $\frac{1}{2}$

- Added ad hoc in undergrad physics

- Arises naturally and unavoidably from rotations approach

- Cannot get spin by starting with $[x,p_x]=i\hbar$

no such internal degrees of freedom with spin

General angular momentum – so far

① Generator of rotations

$$\underbrace{\mathcal{D}_{\hat{n}}(\epsilon)}_{\text{unitary}} = \overleftrightarrow{I} - \frac{i\epsilon}{\hbar}\underbrace{\vec{J}\cdot\hat{n}}_{\text{Hermitian}}$$

$\vec{J} = J_x\hat{x} + J_y\hat{y} + J_z\hat{z}$

set of 3 operators

we consider such "vector operators" in more detail later

② $[J_x, J_y] = i\hbar J_z$, etc.

Represent states as N-dimensional vectors

$\therefore J_i$ is an $N \times N$ matrix

$\underline{N=1}$: Can we satisfy ② with numbers?

$\Rightarrow$ clearly not since numbers commute

$\underline{N \geq 3}$: Consider later

$\underline{N=2}$: Can we satisfy ② with 2x2 matrices?

Consider Pauli matrices

$$\boxed{\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.}$$

spans 2x2 space with

$$\overset{\leftrightarrow}{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

ie. $\mathcal{O} = a\overset{\leftrightarrow}{1} + b\sigma_x + c\sigma_y + d\sigma_z$

$$\sigma_x^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_y^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore \boxed{\sigma_i^2 = \overset{\leftrightarrow}{1}} \qquad \vec{\sigma}^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2$$

$$\boxed{\vec{\sigma}^2 = 3\overset{\leftrightarrow}{1}}$$

$\underline{\text{Commutator:}}$ $\underbrace{[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k}_{\text{not quite ang. mom.}}$ $\Leftarrow$ verify by doing explicitly

Consider: $\left[\frac{1}{2}\hbar\sigma_i, \frac{1}{2}\hbar\sigma_j\right] = \frac{1}{\cancel{2}}\hbar\,\frac{1}{2}\hbar\left(\cancel{2}i\,\epsilon_{ijk}\,\sigma_k\right)$

$$= i\hbar\,\epsilon_{ijk}\left(\frac{1}{2}\hbar\sigma_k\right)$$

$\therefore$ $\boxed{S_i \equiv \frac{1}{2}\hbar\sigma_i}$ is an angular momentum

What does this S_i mean?

$N=2 \rightsquigarrow$ 2 base states $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$

$$S_z\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}\hbar\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}\hbar\begin{pmatrix}1\\0\end{pmatrix}$$

$$S_z\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{2}\hbar\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{2}\hbar\begin{pmatrix}0\\-1\end{pmatrix} = -\frac{1}{2}\hbar\begin{pmatrix}0\\1\end{pmatrix}$$

$\therefore$ Eigenstate of S_z $\begin{pmatrix}1\\0\end{pmatrix}$ has eigenvalue $\frac{1}{2}\hbar$

Eigenstate $\begin{pmatrix}0\\1\end{pmatrix}$ has eigenvalue $-\frac{1}{2}\hbar$

Spin $\frac{1}{2}$ $\leftarrow$ angular momentum

up $\rightarrow \pm\frac{1}{2}\hbar \leftarrow$ down

Operator $\vec{S}^2$:

$$\vec{S}^2 = \left(\frac{1}{2}\hbar\right)^2\underbrace{\left(\sigma_x^2+\sigma_y^2+\sigma_z^2\right)}_{3\hat{1}} = \frac{3}{4}\hbar^2\,\hat{1}$$

$\therefore$ $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$ have the same eigenvalue for $\vec{S}^2 \longrightarrow \underline{\frac{3}{4}\hbar^2}$

Operator $S_\pm \equiv S_x \pm i S_y = \frac{1}{2}\hbar\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pm i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right] = \frac{\hbar}{2}\begin{pmatrix} 0 & 1\pm 1 \\ 1\mp 1 & 0 \end{pmatrix}$

$= \frac{1}{2}\hbar\begin{pmatrix} 0 & 1\pm 1 \\ 1\mp 1 & 0 \end{pmatrix}$

$S_+ = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. $\quad S_+\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$

$S_+\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hbar\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ⇐ raising operator

$S_- = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. $\quad S_-\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hbar\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ← lowering operator

Rotation Operator for Spin $\frac{1}{2}$ System

$$D_{\hat{n}}(\alpha) = e^{-\frac{i\vec{S}\cdot\hat{n}}{\hbar}\alpha} = e^{-i\frac{\alpha}{2}\vec{\sigma}\cdot\hat{n}}$$

$$= \sum_{\substack{k=0\\ \text{even}}}^{\infty} \frac{\left(\frac{-i\alpha}{2}\right)^k (\vec{\sigma}\cdot\hat{n})^k}{k!} + \sum_{\substack{k=1\\ \text{odd}}}^{\infty} \frac{\left(\frac{-i\alpha}{2}\right)^k (\vec{\sigma}\cdot\hat{n})^k}{k!}$$

$\hat{n}$ = some axis

$$\vec{\sigma}\cdot\hat{n} = n_x\sigma_x + n_y\sigma_y + n_z\hat{\sigma}_z = \begin{pmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{pmatrix}$$

$$(\vec{\sigma}\cdot\vec{n})^2 = \begin{pmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{pmatrix}\begin{pmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{pmatrix}$$

$$= \begin{pmatrix} n_z^2 + n_x^2 + n_y^2 & 0 \\ 0 & n_x^2 + n_y^2 + n_z^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hat{1}$$

$$(\vec{\sigma}\cdot\hat{n})^3 = (\vec{\sigma}\cdot\hat{n})\underbrace{(\vec{\sigma}\cdot\hat{n})^2}_{1} = \vec{\sigma}\cdot\hat{n}$$

<u>nifty:</u> $(\vec{\sigma}\cdot\vec{n})^k = \begin{cases} \vec{\sigma}\cdot\hat{n}, & k \text{ odd} \\ \hat{1}, & k \text{ even} \end{cases}$

$i^0 = 1$
$i^2 = -1$
$i^4 = 1$

$$D_{\hat{n}}(\alpha) = \sum_{\substack{k=0\\ \text{even}}}^{\infty} \frac{\left(\frac{-i\alpha}{2}\right)^k (\vec{\sigma}\cdot\hat{n})^k}{k!} + \sum_{\substack{k=1\\ \text{odd}}}^{\infty} \frac{\left(\frac{-i\alpha}{2}\right)^k (\vec{\sigma}\cdot\hat{n})^k}{k!}$$

$\hat{1}$ $\qquad$ $\vec{\sigma}\cdot\hat{n}$

$$\boxed{D_{\hat{n}}(\alpha) = \overleftrightarrow{1}\cos\left(\frac{\alpha}{2}\right) - i\,\vec{\sigma}\cdot\hat{n}\,\sin\frac{\alpha}{2}}$$

big simplification for special case $N=2$

In matrix form

$$\boxed{\mathcal{D}_{\hat{n}}(\alpha) = \begin{pmatrix} \cos\frac{\alpha}{2} & 0 \\ 0 & \cos\frac{\alpha}{2} \end{pmatrix} - i \begin{pmatrix} n_z & n_x - i n_y \\ n_x + i n_y & -n_z \end{pmatrix} \sin\left(\frac{\alpha}{2}\right) = \begin{pmatrix} \cos\frac{\alpha}{2} - i n_z \sin\frac{\alpha}{2} & -i(n_x - i n_y)\sin\frac{\alpha}{2} \\ -i(n_x + i n_y)\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} + i n_z \sin\frac{\alpha}{2} \end{pmatrix}}$$

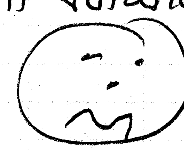
Rotation of wavefunc

Rotation of a statevector by 2π ← for spin $\frac{1}{2}$

$$D_{\hat{n}}(2\pi) = \overleftrightarrow{1}\cos\left(\tfrac{2\pi}{2}\right) - i(\vec{\sigma}\cdot\hat{n})\cancel{\sin}\left(\tfrac{2\pi}{2}\right)$$
$$= -\overleftrightarrow{1}$$

$$|\alpha\rangle \rightarrow \underline{\underline{D_{\hat{n}}(2\pi)|\alpha\rangle = -|\alpha\rangle}}$$

← state vector changes sign for 2π rotation



What about operators?

$$|\alpha\rangle \rightarrow |\tilde{\alpha}\rangle = D|\alpha\rangle$$
$$|\beta\rangle \rightarrow |\tilde{\beta}\rangle = D|\beta\rangle$$

Need $\langle\tilde{\alpha}|\tilde{\theta}|\tilde{\beta}\rangle = \langle\alpha|\theta|\beta\rangle$

$$\langle\alpha|D^{\dagger}\tilde{\theta}D|\beta\rangle = \langle\alpha|\theta|\beta\rangle$$

$$\boxed{\tilde{\theta} = D\theta D^{\dagger}}$$

For 2π rotation: $\tilde{\theta} = (-\overleftrightarrow{1})\,\theta\,(-\overleftrightarrow{1}) = \theta$

$\therefore$ observables do not change under 2π rotations