MATH 420 - FALL 2019
ASSIGNMENT 3

Note: Most of these problems are taken from *Partial Differential Equations*, by L.C. Evans. Assume that U is a bounded domain with smooth boundary ∂U, unless otherwise stated. Recall that $\Delta u = \sum_{i=1}^n u_{x_i x_i}$.

(1) The idea of this problem is to get a proof of Harnack's inequality for harmonic functions using the maximum principle.

(a) Show that if $A = (A_{ij})$ is a symmetric $n \times n$ matrix then
$$\sum_{i,j=1}^n A_{ij}^2 \geqslant \frac{1}{n}(\mathrm{trace}(A))^2.$$

(b) Let $u > 0$ be a smooth harmonic function on U, and define $v = \log u$. Show that $\Delta v = -|Dv|^2$.

(c) Let B be an open ball in U and let ζ be a smooth function compactly supported in U such that $0 \leqslant \zeta \leqslant 1$ on U with $\zeta = 1$ on B. Define $Q = \zeta^2 |Dv|^2$. Show that if Q achieves its maximum at the point $x_0 \in U$ then at this point we have
$$0 \geqslant \Delta Q \geqslant -C|Dv|^2 - C\zeta|Dv|^3 + 2\zeta^2|D^2 v|^2,$$
for C independent of v. Hint: use (b) and the fact that $DQ = 0$ at x_0.

(d) Conclude, using (a), (b), (c) above that Q is bounded from above at x_0, and hence on U.

(e) Use (d) to prove that for any open connected set $V \subset\subset U$ we have the Harnack inequality for smooth harmonic functions $u > 0$ in U:
$$\sup_V u \leqslant C \inf_V u,$$
for a constant C depending only on V and U.

(f) Show that in (e) the assumption $u > 0$ can be weakened to $u \geqslant 0$.

(2) Let $u \in C^\infty(\overline{U})$ be a solution of
$$Lu = -\sum_{i,j=1}^n a^{ij} u_{x_i x_j} = 0, \qquad \text{in } U.$$
Set $v := |Du|^2 + \lambda u^2$, for $\lambda > 0$ a large constant.

(i) Show that $Lv \leqslant 0$ in U if λ is sufficiently large.

(ii) Deduce that
$$\|Du\|_{L^\infty(U)} \leqslant C(\|Du\|_{L^\infty(\partial U)} + \|u\|_{L^\infty(\partial U)}).$$

(3) Assume $u \in C^\infty(\overline{U})$ solves $Lu = -\sum_{i,j=1}^n a^{ij} u_{x_i x_j} = f$ in U, with $u = 0$ on ∂U where $|f| \leqslant K$ for a constant K. Fix $x_0 \in \partial U$. A *barrier* at x_0 is a C^2 function w such that

$$Lw \geqslant 1 \text{ in } U, \ w(x_0) = 0, \ w \geqslant 0 \text{ on } \partial U.$$

Show that if w is a barrier at x_0 there exists a constant C such that

$$|Du(x_0)| \leqslant K \left| \frac{\partial w}{\partial \nu}(x_0) \right|,$$

where $\partial/\partial\nu$ denotes the derivative in the outward normal direction at a point on the boundary of U.

(4) Let $V \subset\subset U$. Show by example that if $u \in L^1(U)$ satisfies

$$\|D^h u\|_{L^1(V)} \leqslant C$$

for all $0 < |h| < \frac{1}{2}\mathrm{dist}(V, \partial U)$, it does not necessarily follow that $u \in W^{1,1}(V)$.

(5) Let $u \in H^1(\mathbb{R}^n)$ have compact support and be a weak solution of

$$-\Delta u + c(u) = f, \qquad \text{in } \mathbb{R}^n,$$

where $f \in L^2(\mathbb{R}^n)$ and $c : \mathbb{R} \to \mathbb{R}$ is smooth with $c(0) = 0$ and $c' \geqslant 0$. Prove that $u \in H^2(\mathbb{R}^n)$.

Hint: follow the standard proof for interior estimates as in class, but without the cutoff function