$\boxed{\text{Spherical Tensors}}$ — operators that rotate like spherical harmonics

Rotation of a spherical harmonic

θ, ϕ

$$Y_{lm}(\hat{r}) = \langle \hat{r} | lm \rangle \quad \Leftarrow \text{a number}$$

For a rotation that rotates a vector $\vec{V} \to R\vec{V}$
state vectors transform $|\alpha\rangle \to \mathcal{D}|\alpha\rangle$

$$|\hat{r}\rangle \to |\hat{r}'\rangle = \mathcal{D}|\hat{r}\rangle$$

$$Y_{lm}(\hat{r}') = \langle \hat{r}' | lm \rangle$$

$$= \langle \hat{r} | \mathcal{D}^\dagger | lm \rangle$$

$$= \sum_{m'} \underbrace{\langle \hat{r} | lm' \rangle}_{Y_{lm'}(\hat{r})} \underbrace{\langle lm' | \mathcal{D}^\dagger | lm \rangle}_{\underbrace{\langle lm | \mathcal{D} | lm' \rangle^*}_{\mathcal{D}^{(l)*}_{mm'}}}$$

$$= \sum_{m'} Y_{lm'}(\hat{r}) \, \mathcal{D}^{(l)*}_{mm'}$$

Spherical harmonics regarded operators

$Y_{lm}(\vec{V})$ is an operator

$\vec{V}$ ← vector operator

eg. $Y_{10}(\hat{r}) \sim \dfrac{z}{r}$

$Y_{10}(\vec{x}) \sim z$

$Y_{10}(\vec{p}) \sim p_z$

$Y_{10}(\vec{L}) \sim L_z$

etc.

For a vector

$$V'_i = \sum_j R_{ij} V_j$$

For a vector operator

$$\langle\tilde{\alpha}|V_i|\tilde{\alpha}\rangle = \sum_j R_{ij} \langle\alpha|V_j|\alpha\rangle$$

For spherical harmonics

$$Y_{lm}(\hat{r}') = \sum_{m'} Y_{lm'}(\hat{r})\, D^{(l)*}_{mm'}$$

For a spherical harmonic operator

$$\langle\tilde{\alpha}|Y_{kq}(\vec{V})|\tilde{\alpha}\rangle = \sum_{q'} \langle\alpha|Y_{kq'}(\vec{V})|\alpha\rangle\, D^{(k)*}_{qq'}$$

$$\boxed{D^\dagger Y_{kq} D = \sum_{q'} Y_{kq'}\, D^{(k)*}_{qq'}}$$

Use unitarity of D

$$\sum_q D^{(k)}_{qq''}\, D^\dagger Y_{kq} D = \sum_q D^{(k)}_{qq''} \sum_{q'} Y_{kq'}\, D^{(k)*}_{qq'}$$

$$= \sum_{q'} Y_{kq'} \sum_q \langle kq|D|kq''\rangle \langle kq'|D^\dagger|kq\rangle$$

complete set of states (rot. do not change k)

$$= \sum_{q'} Y_{kq'} \langle kq'|D^\dagger D|kq''\rangle$$

$$= Y_{kq''}$$

$q'' \to q$
$q \to q'$

$$Y_{kq} = \sum_{q'} D^{(k)}_{q'q}\, D^\dagger Y_{kq'} D$$

$$\boxed{D Y_{kq} D^\dagger = \sum_{q'} Y_{kq'}\, D^{(k)}_{q'q}}$$

equiv.

like: $$\boxed{D|kq\rangle = \sum_{q'} |kq'\rangle\, D^{(k)}_{q'q}}$$

similar form for op. & states

Spherical Tensors — operators that rotate like angular momentum eigenstates i.e. like spherical harmonics

$$\boxed{\mathcal{D}\, T_{kq}\, \mathcal{D}^{\dagger} = \sum_{q'} T_{kq'}\, \mathcal{D}^{(k)}_{q'q}}$$

$$\boxed{\mathcal{D}^{\dagger}\, T_{kq}\, \mathcal{D} = \sum_{q'} T_{kq'}\, \mathcal{D}^{(k)*}_{qq'}}$$

we showed equivalence for spherical harmonics e.g. <u>Position (Vector Operator)</u>

$$rY_{11} = -r\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi} = -\sqrt{\frac{3}{4\pi}}\frac{1}{\sqrt{2}}(x+iy)$$

$$rY_{10} = r\sqrt{\frac{3}{4\pi}}\cos\theta = \sqrt{\frac{3}{4\pi}}\, z$$

$$rY_{1-1} = r\sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi} = \sqrt{\frac{3}{4\pi}}\frac{1}{\sqrt{2}}(x-iy)$$

<u>Since $\mathcal{D}$ does not affect r, get spherical tensor</u>

$$\boxed{\begin{aligned} r_{11} &\equiv -\frac{1}{\sqrt{2}}(x+iy) \\ r_{10} &\equiv z \\ r_{1-1} &= \frac{1}{\sqrt{2}}(x-iy) \end{aligned}}$$

rotate like state vector $|k_q\rangle$

<u>Since all vector operators rotate the same</u> – make for any vector operator

$$\boxed{\begin{aligned} V_{11} &= -\frac{1}{\sqrt{2}}(V_x + iV_y) \\ V_{10} &= V_z \\ V_{1-1} &= \frac{1}{\sqrt{2}}(V_x - iV_y) \end{aligned}}
\qquad
\boxed{\begin{aligned} V_x &= -\frac{1}{\sqrt{2}}(V_{11} - V_{1-1}) \\ V_y &= +\frac{i}{\sqrt{2}}(V_{11} + V_{1-1}) \\ V_z &= V_{10} \end{aligned}}$$

show later that V_{1q} are sph. tensor comp.

$\boxed{\text{Equivalent Definition}}$ — for spherical tensors

For infinitesmal rotations: $\mathcal{D} = 1 - \frac{i\epsilon}{\hbar}\vec{J}\cdot\hat{n}$

$$\mathcal{D}\, T_{kq}\, \mathcal{D}^{\dagger} = \sum_{q'} T_{kq'}\, \mathcal{D}^{(k)}_{q'q}$$

$$\left[1 - \frac{i\epsilon}{\hbar}\vec{J}\cdot\hat{n}\right] T_{kq} \left[1 + \frac{i\epsilon}{\hbar}\vec{J}\cdot\hat{n}\right] = \sum_{q'} T_{kq'} \langle kq' | 1 - \frac{i\epsilon}{\hbar}\vec{J}\cdot\hat{n} | kq \rangle$$

$$\cancel{T_{kq}} \cancel{-} \cancel{\frac{i\epsilon}{\hbar}}\left[\vec{J}\cdot\hat{n}, T_{kq}\right] + \cancel{O(\epsilon^2)} = \sum_{q'} \cancel{T_{kq'}\delta_{q'q}} \cancel{-} \cancel{\frac{i\epsilon}{\hbar}} T_{kq'} \langle kq' | \vec{J}\cdot\hat{n} | kq \rangle$$

$$\left[\vec{J}\cdot\hat{n}, T_{kq}\right] = \sum_{q'} T_{kq'} \langle kq' | \vec{J}\cdot\hat{n} | kq \rangle$$

$$\Rightarrow \boxed{\left[J_z, T_{kq}\right] = \hbar q\, T_{kq}}$$

$$\Rightarrow \left[J_{\pm}, T_{kq}\right] = \sum_{q'} T_{kq'} \langle kq' | J_{\pm} | kq \rangle$$

$$= \sum_{q'} T_{kq'}\, \hbar\sqrt{(k \mp q)(k \pm q + 1)}\; \delta_{q', q\pm 1}$$

$$\boxed{\left[J_{\pm}, T_{kq}\right] = \hbar\sqrt{(k \mp q)(k \pm q + 1)}\; T_{k\, q\pm 1}}$$

problem ?

VECTOR OPERATORS AS SPHERICAL TENSORS

$\rightarrow [V_i, J_j] = i\hbar \epsilon_{ijk} V_k$

Form $\boxed{\begin{aligned} V_{1\pm1} &= \mp \tfrac{1}{\sqrt{2}}(V_x \pm i V_y) \\ V_{10} &= V_z \end{aligned}}$ ← just like r_{1q} (expect all vector op. rotate the same)

Show $[J_z, V_{1q}] = \hbar q V_{1q}$

$[J_z, V_{10}] = -[V_z, J_z] = 0$ ✓

$[J_z, V_{1\pm1}] = -[\mp\tfrac{1}{\sqrt{2}}(V_x \pm iV_y), J_z] = \pm\tfrac{1}{\sqrt{2}}\{\underbrace{[V_x, J_z]}_{-i\hbar V_y} \pm i \underbrace{[V_y, J_z]}_{i\hbar V_x}\}$

$= \pm \hbar \tfrac{1}{\sqrt{2}}(\mp V_x - iV_y)$

$= \hbar(-\tfrac{1}{\sqrt{2}})(V_x \pm iV_y) = \pm\hbar V_{1\pm1}$ ✓

Show $[J_\pm, V_{1q}] = \hbar\sqrt{(k\mp q)(k\pm q+1)}\, V_{1q\pm1}$ $\xrightarrow{\text{ie.}}$ $[J_+, V_{11}] = 0$, $[J_-, V_{11}] = \hbar\sqrt{2}V_{10}$, $[J_\pm, V_{10}] = \hbar\sqrt{2}\, V_{1,\pm1}$, $[J_+, V_{1-1}] = \hbar\sqrt{2}V_{10}$, $[J_-, V_{1-1}] = 0$ } show these

①,⑤ $[J_\pm, V_{1\pm1}] = -[\mp\tfrac{1}{\sqrt{2}}(V_x \pm iV_y), J_x \pm iJ_y]$

$= \pm\tfrac{1}{\sqrt{2}}\{\underbrace{[V_x, J_x]}_{0} \pm i\underbrace{[V_x, J_y]}_{i\hbar V_z} \pm i\underbrace{[V_y, J_x]}_{-i\hbar V_z} - \underbrace{[V_y, J_y]}_{0}\}$

$= 0$ ✓

②,④ $[J_\mp, V_{1\pm1}] = -[\mp\tfrac{1}{\sqrt{2}}(V_x \pm iV_y), J_x \mp iJ_y] = \pm\tfrac{1}{\sqrt{2}}\{\underbrace{[V_x, J_x]}_{=0} \mp i\underbrace{[V_x, J_y]}_{i\hbar V_z} \pm i\underbrace{[V_y, J_x]}_{-i\hbar V_z} - \underbrace{[V_y, J_y]}_{0}\}$

$= \pm\tfrac{1}{\sqrt{2}}\hbar 2(\pm)V_z = \hbar\sqrt{2}V_{10}$ ✓

$[J_\pm, V_{10}] = -[V_z, J_x \pm iJ_y] = -\underbrace{[V_z, J_x]}_{i\hbar V_y} \mp i\underbrace{[V_z, J_y]}_{-i\hbar V_x} = -i\hbar V_y \mp \hbar V_x$

$= \mp\hbar(V_x \pm iV_y) = \hbar\sqrt{2}(\mp)\tfrac{1}{\sqrt{2}}(V_x \pm iV_y)$

$= \hbar\sqrt{2}V_{1\pm1}$ ✓

∴ $\boxed{V_{1q} \text{ are spherical tensor components when } [V_i, J_j] = i\hbar\epsilon_{ijk}V_k}$