① derive Ehrenfest's theorem, for $\hat{A}$ not explicitly dependent on time, as:

$$\frac{d\langle A\rangle}{dt} = \frac{1}{i\hbar}\langle [A,H]\rangle$$

write $\langle A\rangle$ using Dirac notation for a generic state:

$$\langle A\rangle = \langle\psi|A|\psi\rangle$$

$$\underbrace{\frac{d\langle A\rangle}{dt}}_{} = \frac{d\langle\psi|}{dt}A|\psi\rangle + \langle\psi|\underbrace{\frac{dA}{dt}}_{}|\psi\rangle + \langle\psi|A\frac{d|\psi\rangle}{dt}$$

dependence of expectation value of operator A on time (not generically zero even if $dA/dt = 0$)

dependence of the operator itself on time $= 0$

use Schrödinger equation:

$$+i\hbar\frac{d|\psi\rangle}{dt} = H|\psi\rangle \iff -i\hbar\frac{d\langle\psi|}{dt} = \langle\psi|H \qquad (H = H^+)$$

$$\frac{d|\psi\rangle}{dt} = +\frac{1}{i\hbar}H|\psi\rangle \qquad \frac{d\langle\psi|}{dt} = -\frac{1}{i\hbar}\langle\psi|H$$

$$\frac{d\langle A\rangle}{dt} = \frac{-1}{i\hbar}\langle\psi|HA|\psi\rangle + 0 + \frac{1}{i\hbar}\langle\psi|AH|\psi\rangle$$

$$\frac{d\langle A\rangle}{dt} = \frac{1}{i\hbar}\langle\psi|AH - HA|\psi\rangle$$ recognize commutator

$$\frac{d\langle A\rangle}{dt} = \frac{1}{i\hbar}\langle\psi|[A,H]|\psi\rangle$$

$$\boxed{\frac{d\langle A\rangle}{dt} = \frac{1}{i\hbar}\langle [A,H]\rangle}$$

② $f(x) \equiv \begin{cases} (1-x/L) & 0 \le x \le L \\ 0 & \text{otherwise} \end{cases}$

$f(x)$ has Fourier decomposition: $f(x) = \begin{cases} \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) & 0 \le x \le L \\ 0 & \text{otherwise} \end{cases}$

(as it vanishes at $x=0$ and L)

find the coefficients by exploiting the orthogonality of the sine functions:

$$\int_0^L \sin(n\pi x/L)\sin(m\pi x/L)\,dx = \frac{L}{\pi}\int_0^\pi \sin(n\theta)\sin(m\theta)\,d\theta = \frac{L}{\pi}\cdot\frac{\pi}{2}\delta_{nm}$$

$$\int_0^L f(x)\sin(m\pi x/L)\,dx = \int_0^L \left(\sum_{n=1}^{\infty} C_n \sin(n\pi x/L)\right)\sin(m\pi x/L)\,dx$$

$$\int_0^L f(x)\sin(m\pi x/L)\,dx = \sum_{n=1}^{\infty} C_n \underbrace{\left[\int_0^L \sin(n\pi x/L)\sin(m\pi x/L)\,dx\right]}_{\frac{L}{2}\delta_{nm}}$$

$$\int_0^L f(x)\sin(m\pi x/L)\,dx = \sum_{n=1}^{\infty} \frac{L}{2} C_n \delta_{nm} = \frac{L}{2} C_m$$

so the coefficients are given by:

$$C_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad \theta: 0 \to \pi$$

$$C_n = \frac{2}{L}\int_0^L \left(1 - \frac{x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx \qquad \text{let } \theta = \frac{\pi x}{L} \quad d\theta\cdot\frac{L}{\pi} = dx$$

$$C_n = \frac{2}{\pi}\int_0^\pi \left(1 - \frac{\theta}{\pi}\right)\sin(n\theta)\,d\theta$$

$$C_n = \frac{2}{\pi}\underbrace{\int_0^\pi \sin(n\theta)\,d\theta}_{\equiv I_1} - \frac{2}{\pi^2}\underbrace{\int_0^\pi \theta\sin(n\theta)\,d\theta}_{\equiv I_2} \qquad (A)$$

first evaluate I_1:

$$I_1 = \int_0^\pi \sin(n\theta)\,d\theta = -\frac{1}{n}\cos(n\theta)\Big]_0^\pi$$

$$= -\frac{1}{n}\cos(n\pi) - \left(-\frac{1}{n}\right)$$

$$= \frac{1}{n}\left(1 - \cos(n\pi)\right)$$

now I_2 using integration by parts:

$$\begin{array}{ll} \theta & -\frac{1}{n}\cos(n\theta) \\ d\theta & \sin(n\theta)\,d\theta \end{array}$$

$$I_2 = \int_0^\pi \underbrace{\theta}_{u}\,\underbrace{\sin(n\theta)\,d\theta}_{dv} \qquad u = \theta \quad v = -\frac{1}{n}\cos(n\theta)$$

$$\int u\,dv = \int d(uv) - \int v\,du = uv\Big| - \int v\,du$$

$$I_2 = -\frac{\theta}{n}\cos(n\theta)\Big]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(n\theta)\,d\theta$$
$$= -\frac{\pi}{n}\cos(n\pi) - 0 + \underbrace{\frac{1}{n^2}\sin(n\theta)\Big]_0^{\pi}}_{0}$$
$$= -\frac{\pi}{n}\cos(n\pi)$$

$$C_n = \frac{2}{\pi} I_1 - \frac{2}{\pi^2} I_2$$
$$= \frac{2}{\pi}\cdot\frac{1}{n}\left(1-\cos(n\pi)\right) + \frac{2}{\pi^{\cancel{2}}}\cdot\frac{\cancel{\pi}}{n}\cos(n\pi)$$
$$= \frac{2}{\pi n}\left(1-\cos(n\pi)\right) + \frac{2}{\pi n}\cos(n\pi)$$
$$= \frac{2}{n\pi}$$

$$\boxed{C_n = \frac{2}{n\pi}}$$

problems 3, 4, and 5 see last week's solutions.