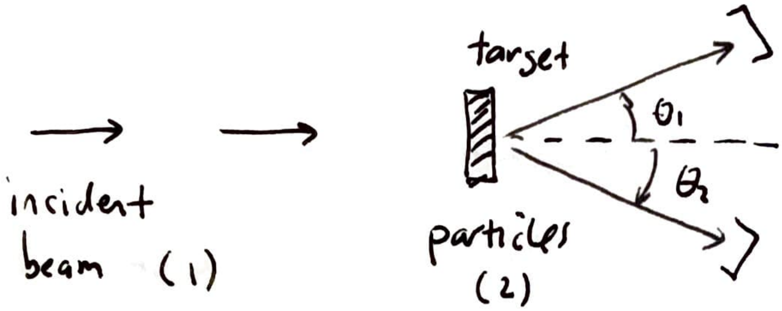
Scattering Theory



$$\underset{\text{incident}}{(1)} + \underset{\text{target}}{(2)} \rightarrow \underbrace{(3) + (4)}_{\text{other particles}}$$

At high energy, we can have new particles that appear.

For now, study scattering where initial and final state are composed of particles of same type [(1) + (2)].

Elastic scattering - none of the particles internal states change during collision.

Inelastic scattering - internal states can change.

Study at first <u>elastic</u> scattering between incident and target particles.

If we could use classical mechanics, we could determine particle trajectories.

In QM we must study evolution of wave-functions

Assumptions

i) Assume (1) & (2) have no spin

ii) Assume elastic scattering (ignoring internal structure)

iii) Assume "thin target", no multiple scatterings

iv) Neglect any possible coherence between waves scattered by the different particles which make up target.

(Note: this is a good approximation if wave packet spread is small compared to average distance between particles).

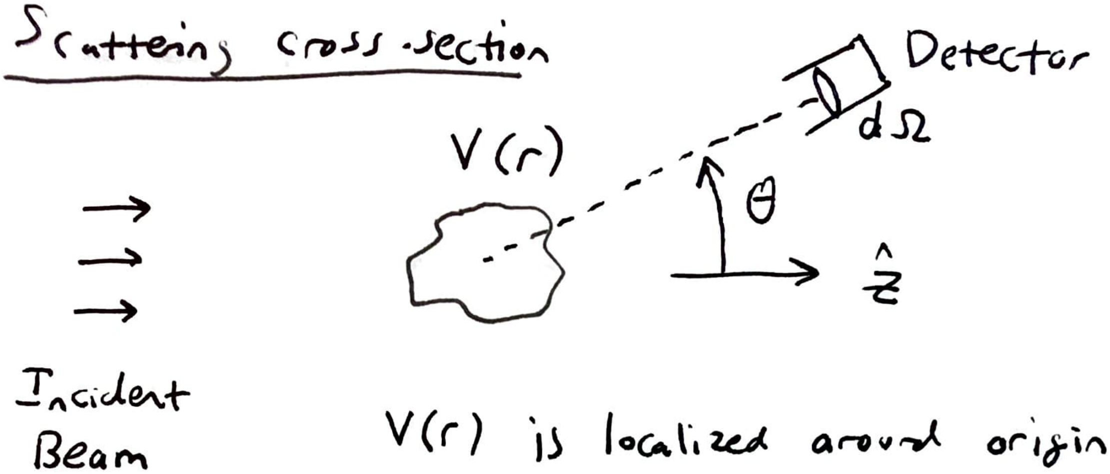
This excludes phenomena e.g. Bragg diffraction, scattering of neutrons by phonons of a solid..

In this limit flux of particles detected is N× flux scattered by any particle.

v) Assume potential energy $V(\vec{r}_1 - \vec{r}_2)$ describes the interaction between particles (1) & (2) $\vec{r} = (\vec{r}_1 - \vec{r}_2)$ relative coordinate.

In center of mass (COM) frame, like scattering of a single "relative" particle of mass μ by a potential $V(\vec{r})$, $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$.

Scattering cross-section



$V(r)$ is localized around origin.

Let F_i be the flux of incident particles $\left(\#/\text{Area}\cdot\text{time}\right)$.

Let $dn \equiv$ # of particles scattered per unit time into the solid angle $d\Omega$ about the direction (θ, φ)

$$dn \equiv F_i \underbrace{\sigma(\theta, \varphi)} d\Omega$$

this is differential scattering cross-section.

$$\sigma \equiv \int \sigma(\theta, \varphi)\, d\Omega$$

total scattering cross-section

measured in barns : $10^{-24}\ \text{cm}^2$

Note : we have ignored incident particles hitting detector. To measure at $\theta = 0$, we can extrapolate $\sigma(\theta, \varphi)$ for small θ.

Note : concept of cross-section is general. Not only applies to elastic scattering.

Stationary scattering states

In the remote past, we know the state of a particle.
it is not yet affected by $V(r)$

Define a Hamiltonian $H = H_0 + V(r)$

$$H_0 = \frac{p^2}{2\mu}$$

rather than using wave-packets look at stationary states first.

$$\psi(r,t) = \varphi(r)\, e^{-iEt/\hbar}$$

$\varphi(r)$ satisfies

$$\left[-\frac{\hbar^2}{2\mu}\nabla^2 + V(r)\right]\varphi(r) = E\,\varphi(r)$$

Assume $V(r)$ decreases faster than $\frac{1}{r}$ as $r \to \infty$.

$$\lim_{r\to\infty} r\,V(r) = 0$$

(excludes coloumb potential - will return to this later...)

Incident particle has energy

$E = \frac{\hbar^2 k^2}{2\mu}$. Define $U(r)$ where $V(r) = \frac{\hbar^2}{2\mu} U(r)$

$$\Rightarrow \left[\nabla^2 + k^2 - U(r)\right]\varphi(r) = 0$$

want to find a physical solution that corresponds to description of scattering process.

Stationary scattering states: $V_k^{(diff)}$ are the wave-functions.

In remote past, we have plane waves.

$$\sim e^{ikz}$$

During scattering, have something complicated.

After scattering $t \to \infty$

$$e^{ikz} + \text{scattered wave packet.}$$

$\Rightarrow$

$V_k^{(diff)}$ will be a superposition of scattered wave and incident wave.

In a given direction (θ, φ), the radial dependence of scattered wave must be $\frac{e^{ikr}}{r}$.

$$(\nabla^2 + k^2) \frac{e^{ikr}}{r} = 0 \quad \text{if} \quad r \gtrsim r_0 \text{ for some positive } r_0.$$

(the $\frac{1}{r}$ ensures that flux of energy through a sphere of radius r is r-independent. probability flux in Q.M.)

$$V_k^{(diff)}(r) \underset{r \to \infty}{\sim} e^{ikz} + \underbrace{f_k(\theta, \varphi)}_{\substack{\text{scattering amplitude} \\ \text{depends on } V(r).}} \frac{e^{ikr}}{r}$$

<u>Remark</u>: we can expand a wave-packet in terms of plane waves:

$$\psi(r,t) = \int_0^\infty dk\, g(k)\, v_k^{(diff)}\, e^{-iE_k t/\hbar} \qquad \text{etc.}$$

There is a "probability current" associated with a wave-function.

$$\vec{J}(\vec{r}) = \frac{1}{\mu} \mathrm{Re}\left[\varphi^*(\vec{r}) \frac{\hbar}{i} \vec{\nabla} \varphi(\vec{r})\right]$$

$$|J_i| = \frac{\hbar k}{\mu}$$

$$(J_d)_r = \frac{\hbar k}{\mu} \frac{1}{r^2} |f_k(\theta,\varphi)|^2$$

$$(J_d)_\theta = \frac{\hbar}{\mu} \frac{1}{r^3} \mathrm{Re}\left[\frac{1}{i} f_k^*(\theta,\varphi) \frac{\partial}{\partial\theta} f_k(\theta,\varphi)\right]$$

$$(J_d)_\varphi = \frac{\hbar}{\mu} \frac{1}{r^3 \sin\theta} \mathrm{Re}\left[\frac{1}{i} f_k^*(\theta,\varphi) \frac{\partial}{\partial\varphi} f_k(\theta,\varphi)\right]$$

$$F_i = C|J_i| = C\frac{\hbar k}{\mu}$$

$$dn = C\,\vec{J}_d \cdot d\vec{s} = C(J_d)_r r^2 d\Omega$$

$$= C\frac{\hbar k}{\mu} |f_k(\theta,\varphi)|^2 d\Omega$$

$$\Rightarrow \quad \sigma(\theta,\varphi) = |f_k(\theta,\varphi)|^2$$

scattering amplitude.

Remarks: Interference between incident and scattered waves: e^{ikz} can interfere w/ scattered wave.

these terms only appear when discussing the forward scattering $\theta = 0$.

usually we measure w/ a detector outside of region hit by incident beam, so we can neglect this.

Comment:

we need destructive interference between forward scattered wave-packets and the incident plane wave to ensure global conservation of particle #.

- particles scattered into other directions must have left the beam! there must be a deficiency in forward direction for conservation of particle #.

Integral Scattering equation

Eigenvalue equation of H: $[\nabla^2 + k^2 - \mathcal{U}(\vec{r})]\varphi(\vec{r}) = 0$

rewrite $(\nabla^2 + k^2)\varphi(\vec{r}) = \mathcal{U}(\vec{r})\varphi(\vec{r})$

Suppose $\exists$ a Green's function $G(r)$ satisfying $(\nabla^2 + k^2) G(r) = \delta(\vec{r})$.

It satisfies:

$$\varphi(\vec{r}) = \varphi_0(\vec{r}) + \int d^3r' \, G(\vec{r}-\vec{r}\,') \, u(\vec{r}\,') \, \varphi(\vec{r}\,')$$

where $\varphi_0(\vec{r})$ satisfies $(\nabla^2 + k^2)\varphi_0(\vec{r}) = 0$

Consider

$$(\nabla^2 + k^2)\varphi(r) = (\nabla^2 + k^2)\int d^3r' \, G(\vec{r}-\vec{r}\,') \, u(\vec{r}\,') \, \varphi(\vec{r}\,')$$

If we move operator into integral, it acts on $\vec{r}$ not $\vec{r}\,'$

$$(\nabla^2 + k^2)\varphi(\vec{r}) = \int d^3r' \, \delta(r-r') \, u(r') \, \varphi(r')$$

$$= u(r)\,\varphi(\vec{r}).$$

Let's solve for Green's function:

$$(\nabla^2 + k^2) G(r) = \delta(r)$$

Away from origin, $(\nabla^2 + k^2) G(r) = 0$.

we saw that $(\nabla^2 + k^2)\frac{e^{ikr}}{r} = 0$ for $r \geq r_0$ for positive r_0.

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

$$\nabla^2\left(\frac{e^{ikr}}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\frac{e^{ikr}}{r}\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left[\frac{ik}{r}e^{ikr} - \frac{1}{r^2}e^{ikr}\right]\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left[ikre^{ikr} - e^{ikr}\right]$$

$$= \frac{1}{r^2}\left[ike^{ikr} - k^2re^{ikr} - ike^{ikr}\right]$$

$$= -\frac{k^2}{r}e^{ikr}$$

$$(\nabla^2 + k^2)\frac{e^{ikr}}{r} = 0$$

so $G(r) \propto \dfrac{e^{\pm ikr}}{r}$

As $r \to 0$ $G(r)$ must behave as $-\dfrac{1}{4\pi r}$ since

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta(r)$$

$$(\nabla^2 + k^2)\frac{e^{\pm ikr}}{r} = -4\pi\delta(r)$$

e.g. $$\nabla^2\left[\frac{e^{ikr}}{r}\right] = \frac{1}{r}\nabla^2 e^{ikr} + e^{ikr}\nabla^2\left(\frac{1}{r}\right) + 2\nabla\left(\frac{1}{r}\right)\cdot\nabla\left(e^{ikr}\right)$$

$$= \frac{1}{r}\left(-k^2e^{ikr} + \frac{2ik}{r}e^{ikr} - e^{ikr}4\pi\delta(r) - \frac{2}{r^2}ike^{ikr}\right)$$

So we have

$$G_\pm(\vec{r}) = -\frac{1}{4\pi}\frac{e^{\pm ikr}}{r}$$

we can write

$$\nabla^2 G_\pm(\vec{r}) = -k^2 G_\pm(\vec{r}) + \delta(\vec{r})$$

we call these outgoing and incoming Green's functions

For our scattering problem we seek a solution involving

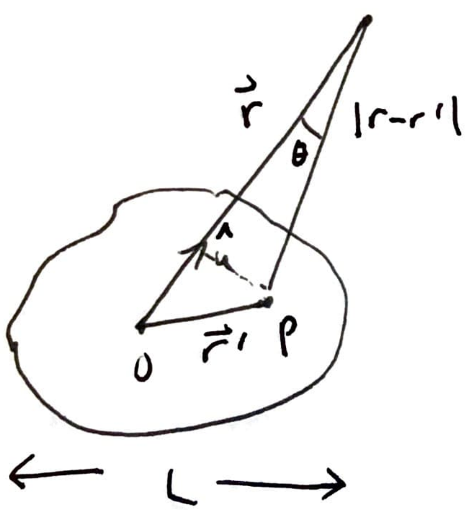
$$\varphi_0 \sim e^{ikz}$$

and outgoing Green's function $G_+(\vec{r})$.

<u>Claim</u>: Solutions to the following equation present the required asymptotic behavior:

$$\star \quad \psi_k^{(\text{diff})} = e^{ikz} + \int d^3r' \, G_+(\vec{r}-\vec{r}\,') \, \mathcal{U}(\vec{r}\,') \, \psi_k^{\text{diff}}(\vec{r}\,')$$

Assume we have potential inside a region with effective linear dimension L



Assume $r \gg L$

$r' \lesssim L$

If $r \gg L$, angle θ is small, we can approximate

$$|r - r'| \simeq r - \vec{u}\cdot\vec{r}\,'$$

$$\vec{u} \text{ is } \frac{\vec{r}}{|\vec{r}|}.$$

If r is large

$$G_+(\vec{r}-\vec{r}\,') = -\frac{1}{4\pi}\frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} \underset{r\to\infty}{\sim} -\frac{1}{4\pi}\frac{e^{ikr}}{r} e^{-ik\vec{u}\cdot\vec{r}\,'}$$

substitute into integral equation, we find

$$\psi_k^{\text{diff}} \underset{r\to\infty}{\sim} e^{ikz} - \frac{1}{4\pi}\frac{e^{ikr}}{r}\int d^3r' \, e^{-ik\vec{u}\cdot\vec{r}\,'} \, \mathcal{U}(\vec{r}\,') \, \psi_k^{\text{diff}}(\vec{r}\,')$$

Note that this looks like

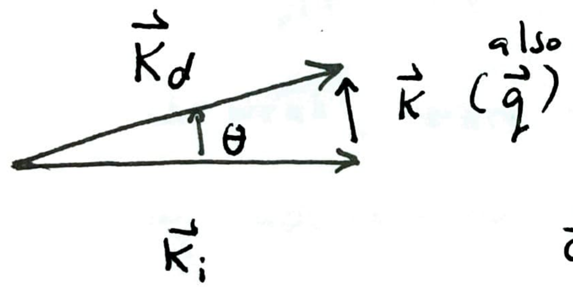
$$V_k^{diff} \underset{r\to\infty}{\sim} e^{ikz} + f_k(\theta,\varphi)\,\frac{e^{ikr}}{r}$$

since the integral depends on the angles θ, φ through $\vec{u}$ not r.

therefore we can get

$$f_k(\theta,\varphi) = -\frac{1}{4\pi}\int d^3r'\, e^{ik\vec{u}\cdot\vec{r}\,'}\,\mathcal{U}(\vec{r}\,')\,V_k^{diff}(\vec{r}\,')$$

thus solutions of equation ★ are the stationary scattering states.



$e^{ikz} = e^{i\vec{K}_i\cdot\vec{r}}$ incident

$K_d = k\vec{u}$ scattered wave vector

$\vec{q} = \vec{K}_d - \vec{K}_i$ transfered wave vector also "scattering" wave vector.

Born Approximation

Start from integral scattering equation

$$V_k^{diff}(\vec{r}) = e^{i\vec{K}_i\cdot\vec{r}} + \int d^3r'\, G_+(\vec{r}-\vec{r}\,')\,\mathcal{U}(\vec{r}\,')\,\underbrace{V_k^{diff}(\vec{r}\,')}$$

Let's construct an expansion

write $V_k^{diff}(\vec{r}\,') = e^{i\vec{K}_i\cdot\vec{r}\,'} + \int d^3r''\, G_+(\vec{r}\,'-\vec{r}\,'')\,\mathcal{U}(\vec{r}\,'')\,V_k^{diff}(\vec{r}\,'')$

Insert this into above

$$V_k^{diff}(\vec{r}) = e^{i\vec{k}_i\cdot\vec{r}} + \int d^3r'\, G_+(\vec{r}-\vec{r}\,')\, u(\vec{r}\,')\, e^{i\vec{k}_i\cdot\vec{r}\,'}$$

$$+ \int d^3r' \int d^3r''\, G_+(\vec{r}-\vec{r}\,')\, u(\vec{r}\,')\, G_+(\vec{r}\,'-\vec{r}\,'')\, u(\vec{r}\,'')\, V_k^{diff}(\vec{r}\,'')$$

we can repeat this. The first 2 terms are containing known quantities. Only 3rd term contains unknown function $V_k^{diff}(\vec{r}\,'')$.

This is called <u>Born expansion</u> of scattering wave function.

Each term gives one more factor of potential V.

If V is weak, each term gets smaller we can get an approximation for $V_k^{diff}(\vec{r})$ in terms of known quantities, up to some order.

substitute into $f_k(\theta,\varphi)$ we get Born Approximation for scattering amplitude.

At first order in u, replace $V_k^{diff}(\vec{r}\,')$ by $e^{i\vec{k}_i\cdot\vec{r}\,'}$

<u>Born Approximation</u>

$$f_k(\theta,\varphi) = -\frac{1}{4\pi}\int d^3r'\, e^{ik\vec{u}\cdot\vec{r}\,'}\, u(r')\, e^{i\vec{k}_i\cdot\vec{r}\,'}$$

$$= -\frac{1}{4\pi}\int d^3r'\, e^{-i(\vec{k}_d-\vec{k}_i)\cdot\vec{r}\,'}\, u(\vec{r}\,')$$

$$= -\frac{1}{4\pi}\int d^3r'\, e^{-i\vec{q}\cdot\vec{r}\,'}\, u(\vec{r}\,')$$

$\vec{q}$ is the scattering wave-vector

Mathematically just the fourier transform of the potential.

Differential cross-section -

$$\sigma_K^{(B)}(\theta,\varphi) = \frac{\mu^2}{4\pi^2\hbar^4}\left|\int d^3r\, e^{-i\vec{q}\cdot\vec{r}}\, V(\vec{r})\right|^2$$

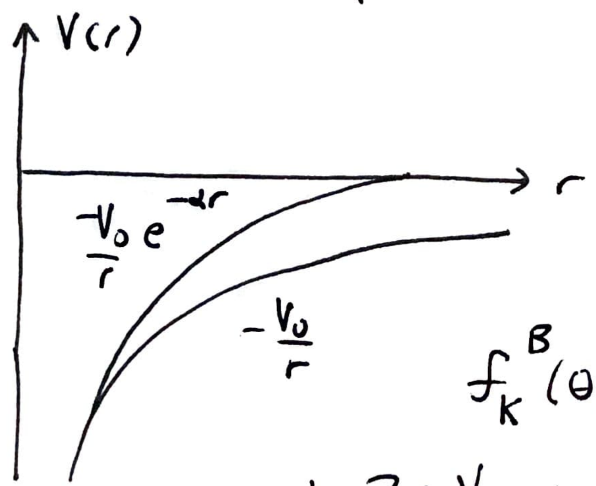
Studying variation of differential cross-section in terms of scattering direction and incident energy, gives information about potential $V(\vec{r})$.

- tends to get better at <u>higher energies</u>, V has a smaller effect.

<u>Example</u>

Yukawa potential $V(r) = V_0 \frac{1}{r} e^{-\alpha r}$

Let $r_0 = \frac{1}{\alpha}$



Use Born Approximation to determine scattering amplitude

$$f_K^B(\theta,\varphi) = -\frac{1}{4\pi}\frac{2\mu V_0}{\hbar^2}\int d^3r\, e^{-i\vec{q}\cdot\vec{r}}\,\frac{e^{-\alpha r}}{r}$$

$$= -\frac{1}{4\pi}\frac{2\mu V_0}{\hbar^2}\frac{4\pi}{|q|}\int_0^\infty r\,dr\,\sin(|q|r)\,\frac{e^{-\alpha r}}{r}$$

$$f_K^B(\theta,\varphi) = \frac{-2\mu V_0}{\hbar^2} \frac{1}{\alpha^2 + |q|^2}$$

$$|q| = 2K \sin\left(\frac{\theta}{2}\right)$$

$$\sigma_K^B(\theta) = \frac{2\mu^2 V_0^2}{\hbar^4} \frac{1}{\alpha^2 + 4K^2 \sin^2\left(\frac{\theta}{2}\right)}$$

$$\sigma_{TOT} = \frac{4\mu^2 V_0^2}{\hbar^4} \frac{4\pi}{\alpha^2(\alpha^2 + 4K^2)}$$

Remark: We can get Coulomb potential if we take limit $\alpha \to 0$. $V_0 = Z_1 Z_2 e^2$ $\quad e^2 = \frac{q^2}{4\pi\epsilon_0}$

$$\sigma^B_{\text{coloumb}}(\theta) = \frac{4\mu^2}{\hbar^2} \frac{Z_1^2 Z_2^2 e^4}{16 K^4 \sin^4\left(\frac{\theta}{2}\right)}$$

$$= \frac{Z_1^2 Z_2^2 e^4}{16 E^2 \sin^4\left(\frac{\theta}{2}\right)}$$

matches Rutherford Scattering!

this is not a rigorous proof but interesting to show there is agreement.

In reality you never get infinite range potential (charges screen each other) so divergence at $\theta = 0$ is not really physical.