# Physics 414-2 Problem Set 7

May 14, 2022

**Due: Friday, May 20 at 4 pm**

**1. Warm up problem on heat capacities.** Consider an ideal gas that satisfies the equation $PV = Nk_BT$, where $P$ is the presure, $V$ is the volume, $N$ is the number of particles, $k_B$ is Boltzmann's constant, and $T$ is the temperature. Show that the following relation holds between the heat capactity $C_P$ at constant pressure and the heat capacity $C_V$ at constant volume: $C_P - C_V = Nk_B$. This means that it requires more heat to increase the temperature by a given amount at constant pressure than at constant volume.

**2. Heat capacity of a dielectric.** In lecture, we considered a set of fixed, closed, conducting surfaces with the space between these surfaces filled by a uniform dielectric medium. We defined various thermodynamic potentials that had either $\vec{E}$ or $\vec{D}$ as their natural variables. We found that changes $\delta\vec{E}$ in the electric field correspond to changes $\delta\phi_j$ in the potentials on the surfaces of the conductors, and that changes $\delta\vec{D}$ in the displacement field correspond to changes $\delta q_j$ in the excess charges on the surfaces of the conductors. Therefore, holding $\vec{E}$ fixed corresponds to holding the potentials on the surfaces of the conductors fixed, and holding $\vec{D}$ fixed corresponds to holding the excess charges on the surfaces of the conductors fixed.

In this problem, we will investigate the heat capacity of a linear, isotropic dielectric (so that $\vec{D} = \epsilon\vec{E}$) in two different cases: 1. we hold $\vec{D}$ (and therefore the charges on the surfaces of the conductors) fixed and 2. we hold $\vec{E}$ (and therefore the potentials on the surfaces of the conductors) fixed. We note that $\vec{D}$ and $\vec{E}$ will in general have spatial dependence $\vec{D}(\vec{r})$ and $\vec{E}(\vec{r})$. For notational convenience, throughout this problem we will not explicitly write out the argument $\vec{r}$, and we will just write $\vec{D}$ and $\vec{E}$. It will be important to note that $\epsilon$ can depend on temperature.

(a) Since $\vec{D} = \epsilon\vec{E}$, changes in $\delta\vec{D}$ are related to changes in $\delta\vec{E}$ (at fixed temperature, so that $\epsilon$ is fixed) by the relation $\delta\vec{D} = \epsilon\delta\vec{E}$. Recalling the definition of the thermodynamic potential $F$ from lecture, show that at temperature $T$ and displacement field $\vec{D}$,

$$F(T, \vec{D}) = F(T, 0) + \frac{1}{4\pi} \int_V dV \frac{D^2}{2\epsilon}. \tag{1}$$

Here, $F(T, 0)$ is the value of $F$ at temperature $T$ and zero displacement field.

(b) Find the entropy $S(T, \vec{D})$ at temperature $T$ and displacement field $\vec{D}$ in terms of $S(T, 0)$, the entropy at temperature $T$ and zero displacement field. Note that the dielectric constant $\epsilon$ can depend on temperature.

(c) Show that the head capacity $C_D$ at fixed displacement field and temperature $T$ is given by

$$C_D = C_0 - \frac{1}{4\pi} \int_V dV D^2 T \frac{d^2}{dT^2} \left( \frac{1}{2\epsilon} \right), \tag{2}$$

where $C_0$ is the heat capacity at zero field ($\vec{D} = \vec{E} = 0$). Note that the heat capacity is affected by how the dielectric constant changes with temperature.

(d) Now, we consider the case in which we hold $\vec{E}$ (and therefore the potentials of the conductors) fixed. Show that the difference between the heat capacity at fixed $\vec{E}$, which we denote as $C_E$, and the heat capacity at fixed $\vec{D}$ is

$$C_E - C_D = \frac{T}{4\pi} \int_V dV \frac{E^2}{\epsilon} \left( \frac{d\epsilon}{dT} \right)^2. \tag{3}$$

**3. Electrostriction.** For some dielectric materials, it is important to take the pressure and volume into account when considering thermodynamics. If the dielectric constant $\epsilon$ depends on pressure, then the application of a uniform electric field of magnitude $E$ can cause the volume of the dielectric to change by an amount $\Delta V = V - V_0$, where $V$ is the volume with the electric field present and $V_0$ is the volume without the electric field. This phenomenon is called electrostriction. Show that

$$\frac{\Delta V}{V} = -\frac{E^2}{8\pi} \left( \frac{\partial \epsilon}{\partial P} \right)_{T,E}, \tag{4}$$

where $\left( \frac{\partial \epsilon}{\partial P} \right)_{T,E}$ is the partial derivative of the dielelectric constant with respect to pressure at fixed temperature and electric field.

**4. Nonisotropic dielectrics.** As we discussed in lecture, in dielectric materials that are not isotropic, the simple relation $\vec{D} = \epsilon \vec{E}$ will not generally hold. Consider, for example, a dielectric material with $\vec{D}$ and $\vec{E}$ related by $D_i = \epsilon_i E_i$ for $i = x, y, z$. Assuming that this dielectric material fills all space, find the potential $\phi(\vec{r})$ from a point charge $q$ located at the origin.

**5. Nonlinear response of materials to electric fields.** In class, we studied materials in which the polarization (i.e., the density of electric dipoles) of the material depended linearly on the electric field. In general, the induced polarization can have nonlinear dependence on the electric field. As a model for such behavior, in this problem we will consider an anharmonic modification to the Drude-Lorentz model. Specifically, we consider an electric field in the x-direction, and we add a restoring force $-max^2$ to the usual linear restoring force. Here, $a$ is a coefficient that determines the strength of the anharmonicity.

(a) Consider the case of a sufficiently small electric field so that the equations of motion for the anharmonic Drude-Lorentz model described above can be perturbatively solved as a sum of terms of increasing order in the electric field. For an electric field that oscillates with frequency $\omega$, solve for the effect of the anharmonicity on the induced polarization, keeping terms up to second order in the electric field amplitude.

(b) Based on your solution to part (a), explain why the nonlinearity of the material causes electromagnetic waves to be generated at frequency $2\omega$. This process is called second harmonic generation.

**6. The Faraday Effect: rotation of polarization in a dielectric medium with an applied magnetic field.** Consider light with frequency $\omega$ (time dependence $e^{-i\omega t}$) that propagates in a transparent, dielectric, non-magnetic medium. The light propagates parallel to an applied uniform magnetic field $\vec{B} = B_0\hat{z}$ (that is, the light propagates in the $z$ direction). Treat the material as being made up of bound electrons with very weak damping, so that you can take the limit of the damping time $\tau \to \infty$. The electrons have number density $n_0$ and resonance frequency $\omega_0$. Because of the magnetic field, it will be important to consider the Lorentz force $-e\frac{\vec{v}}{c}\times\vec{B}$, where $\vec{v}$ is the velocity of an electron.

Recall from class that light with positive circular polarization has an electric field proportional to the vector $\hat{x} + i\hat{y}$, and that light with negative circular polarization has an electric field proportional to the vector $\hat{x} - i\hat{y}$.

(a) Show that an electromagnetic wave with positive (negative) circular polarization has an index of refraction $n_+$ ($n_-$), defined so that

$$n_+^2 = 1 + \frac{4\pi n_0 e^2/m}{\omega_0^2 - \omega^2 + \omega\omega_c} \tag{5}$$

$$n_-^2 = 1 + \frac{4\pi n_0 e^2/m}{\omega_0^2 - \omega^2 - \omega\omega_c}, \tag{6}$$

where $\omega_c \equiv \frac{eB_0}{mc}$ and $m$ is the electron mass.

(b) A plane wave propagating in the $z$ direction and linearly polarized in the $x$

direction is incident at $z = 0$ on a slab of such material that extends from $z = 0$ to $z = d$. For sufficiently small $B_0$, and assuming that we $\omega$ is not too close to $\omega_0$, we can approximate $\omega_c$ as being small and Taylor expand our expressions to first order in $\omega_c$. In this limit, show that after passing through the material, the wave has its angle of polarization rotated by an amount approximately equal to

$$\Delta\theta \approx \frac{\omega_c d}{c} \frac{\omega^2}{\omega_0^2 - \omega^2} \frac{n^2 - 1}{2n}, \tag{7}$$

where $n \approx \frac{1}{2}(n_+ + n_-)$.

(c) If we place a mirror at $z = d$ that reflects the wave back through the material, will the polarization keep rotating in the same direction, or will it rotate in the opposite direction? Explain your reasoning.