Exercise 1 : Invent your own gauge

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(helped by Aaron King)

Eqns. of motion:
$$\nabla^2 \phi + \frac{\partial (\vec{\nabla} \cdot \vec{A})}{\partial t} = -\frac{\rho}{\varepsilon_0}$$
$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} + \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t}\right)$$

Choose : $\nabla^2 \vec{A} = 0$

$$\left.\begin{aligned} \phi' &= \phi - \frac{\partial \chi}{\partial t} \\ \vec{A}' &= \vec{A} + \vec{\nabla} \chi \end{aligned}\right\} \longrightarrow \nabla^2 \vec{A}' = \nabla^2 \vec{A} + \nabla^2 \vec{\nabla} \chi = 0$$
$$\nabla^2 \vec{A} = -\nabla^2 \nabla \chi = f(\vec{x}, t) = 0$$

Eqns of motion become:
$$\nabla^2 \phi + \frac{\partial (\vec{\nabla} \cdot \vec{A})}{\partial t} = -\frac{\rho}{\varepsilon_0}$$
$$-\frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} + \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t}\right)$$

Conditions for invented gauge:
$$\phi' = \phi - \frac{\partial \chi}{\partial t}$$
$$\vec{A}' = \vec{A} + \vec{\nabla} \chi$$
$$-\nabla^2 \nabla \chi = 0$$

Exercise 2

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$\vec{L} = \frac{1}{c}\int d^3r\, \vec{r} \times (\vec{E} \times \vec{B})$

show: $\vec{L} = \frac{1}{c}\int d^3r \left[\vec{E} \times \vec{A} + \sum_{\ell=1}^{3} E_\ell (\vec{r} \times \vec{\nabla}) A_\ell\right]$

$B = \vec{\nabla} \times \vec{A}$

$$\vec{L} = \frac{1}{c}\int d^3r\, \vec{r} \times (\vec{E} \times \vec{\nabla} \times \vec{A})$$

$$= \frac{1}{c}\int d^3r\, \vec{r} \times \left([\vec{E} \cdot \vec{A}]\vec{\nabla} - [\vec{E} \cdot \vec{\nabla}]\vec{A}\right)$$

$$= \frac{1}{c}\int d^3r \left(-\vec{r} \times [\vec{E} \cdot \vec{\nabla}]\vec{A} + \vec{r} \times [\vec{E} \cdot \vec{A}]\vec{\nabla}\right)$$

$$= \frac{1}{c}\int d^3r \left(\underbrace{-\vec{r} \times [\vec{E} \cdot \vec{\nabla}]\vec{A}}_{[\vec{E} \cdot \vec{\nabla}]\cancel{\vec{r}} \times \vec{A}} + \underbrace{(E_1 A_1 + E_2 A_2 + E_3 A_3)}_{\sum_{\ell=1}^{3} E_\ell A_\ell}(\vec{r} \times \vec{\nabla}) \right)$$

$$= \frac{1}{c}\int d^3r \left(\vec{E} \times \vec{A} + \sum_{\ell=1}^{3} E_\ell (\vec{r} \times \vec{\nabla}) A_\ell\right)$$

Exercise 3: Calculate $N = \varepsilon_0 \int d^3r \, [\vec r \times (\vec E \times \vec B)]$

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$$\left.\begin{aligned} \vec E &= \frac{-e\vec r}{4\pi\varepsilon_0 r^3} \\ \vec B &= \frac{\mu_0}{4\pi}\left[3\frac{(\vec m\cdot\vec r)\vec r}{r^5} - \frac{\vec m}{r^3}\right] \end{aligned}\right\} \text{ for } r > r_0$$

$$N = \varepsilon_0 \int d^3r \left[\vec r \times \left(\frac{-e\vec r}{4\pi\varepsilon_0 r^3} \times \frac{\mu_0}{4\pi}\left[3\frac{(\vec m\cdot\vec r)\vec r}{r^5} - \frac{\vec m}{r^3}\right]\right)\right]$$

$$= \cancel{\varepsilon_0} \int d^3r \left(\frac{-e}{4\pi\cancel{\varepsilon_0} r^3}\right)\left(\frac{\mu_0}{4\pi}\right)\left[\vec r \times \left(\vec r \times \left[3\frac{(\vec m\cdot\vec r)\vec r}{r^5} - \frac{\vec m}{r^3}\right]\right)\right]$$

$$= \frac{-e\mu_0}{16\pi^2}\int d^3r \frac{1}{r^3}\left(\vec r \times \left[\vec r \times \frac{3(\vec m\cdot\vec r)\vec r}{r^5} - \vec r \times \frac{\vec m}{r^3}\right]\right)$$

$$= \frac{-e\mu_0}{16\pi^2}\int d^3r \frac{1}{r^3}\left[\underbrace{\left(\vec r \times \left[\vec r \times \frac{3(\vec m\cdot\vec r)\vec r}{r^5}\right]\right)}_{= \frac{3}{r^5}(\vec m\cdot\vec r)(\vec r \times \cancel{\vec r \times \vec r}) = 0} - \underbrace{\left(\vec r \times \left[\vec r \times \frac{\vec m}{r^3}\right]\right)}_{= \frac{-1}{r^3}(\vec r \times [\vec r \times \vec m])}\right]$$

$$= \frac{-1}{r^3}\left([\vec r\cdot\vec m]\vec r - [\vec r\cdot\vec r]\vec m\right)$$

$$= \frac{-1}{r^3}\left([\vec r\cdot\vec m]\vec r - r^2\vec m\right)$$

$$= \frac{+e\mu_0}{16\pi^2}\int d^3r \frac{1}{r^3}\frac{1}{r^3}\left([\vec r\cdot\vec m]\vec r - r^2\vec m\right)$$

$$= \frac{e\mu_0}{16\pi^2}\int d^3r \frac{1}{r^6}\left(\underbrace{\left[r\hat r \cdot \frac{e\hbar}{2mc}(\cos\theta\,\hat r - \sin\theta\,\hat\theta)\right] r\hat r}_{\frac{e\hbar}{2mc}[r\cos\theta]\, r\hat r} - \underbrace{r^2\frac{e\hbar}{2mc}[\cos\theta\,\hat r - \sin\theta\,\hat\theta]}_{+\frac{e\hbar}{2mc}[-r^2\cos\theta\,\hat r + r^2\sin\theta\,\hat\theta]}\right)$$

$$= \frac{e\mu_0}{16\pi^2}\left(\frac{e\hbar}{2mc}\right)\int d^3r \frac{1}{r^{\cancel{6}4}}\left(\cancel{r^2\cos\theta\,\hat r} - \cancel{r^2\cos\theta\,\hat r} + \cancel{r^2}\sin\theta\,\hat\theta\right)$$

$$= \frac{e^2\mu_0\hbar}{32\pi^2 mc}\int d^3r \frac{1}{r^4}\sin\theta\,\hat\theta$$

$$= \frac{e^2\mu_0\hbar}{32\pi^2 mc}\int_{r_0}^{\infty}\frac{1}{r^{\cancel{4}2}}\cancel{r^2}dr \int_0^{\pi}\sin\theta\,\hat\theta\, d\theta \int_0^{2\pi} d\phi$$

$$= \frac{e^2\mu_0\hbar}{16\,\cancel{32}\pi^{\cancel{2}} mc}\left[\frac{-1}{r}\right]_{r_0}^{\infty}\left[-\cos\theta\right]_0^{\pi}(\cancel{2\pi})\,\hat\theta$$

$$= \frac{e^2\mu_0\hbar}{16\pi mc}\left(\cancelto{0}{\frac{-1}{\infty}} + \frac{1}{r_0}\right)\left(-\cancel{\cos\pi} + \cancelto{1}{\cos 0}\right)\hat\theta$$

$$= \boxed{\frac{e^2\mu_0\hbar}{16\pi mc\, r_0}\,\hat\theta}$$