MATH 420 - FALL 2019
ASSIGNMENT 1

Note: Most of these problems are taken from *Partial Differential Equations*, by L.C. Evans. In the following you may assume that U is a bounded open subset of $\mathbb{R}^n$ unless otherwise stated.

(1) We say $v \in C^2(\overline{U})$ is *subharmonic* if $\Delta v \geqslant 0$ in U.
  (a) Show that if v is subharmonic then the Mean Value Inequality
$$(*) \qquad v(x) \leqslant \fint_{B(x,r)} v dy, \quad \text{for all } B(x,r) \subset U$$
holds.
  (b) Show that $\max_{\overline{U}} v = \max_{\partial U} v$.
  (c) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. If u is harmonic show that $\phi(u)$ is subharmonic.
  (d) If u is harmonic show that $|Du|^2$ is subharmonic.
  (e) Show that if $v, w \in C^2(\overline{U})$ are subharmonic then $\max(v, w)$ satisfies the Mean Value Inequality $(*)$.

(2) Let $u \in C^2(\overline{U})$ solve
$$-\Delta u = f \text{ in } U, \quad u = g \text{ on } \partial U.$$
Show that
$$\max_{\overline{U}} |u| \leqslant C(\max_{\partial U} |g| + \max_{\overline{U}} |f|),$$
for C depending only on U.

(Hint: $-\Delta(u + \frac{|x|^2}{2n}\lambda) \leqslant 0$ for $\lambda := \max_{\overline{U}} |f|$.)

(3) (Reflection Principle) Let U^+ denote the open half-ball $\{x \in \mathbb{R}^n \mid |x| < 1,\ x_n > 0\}$. Assume $u \in C^2(U^+) \cap C(\overline{U^+})$ is harmonic in U^+ with $u = 0$ on $\partial U^+ \cap \{x_n = 0\}$. Define
$$v(x) = \begin{cases} u(x) & x_n \geqslant 0 \\ -u(x_1, \ldots, x_{n-1}, -x_n) & x_n < 0 \end{cases}$$
for x in the open ball $U = \{|x| < 1\}$. Use the Poisson formula for the ball to show that v is harmonic within U.

(4) (a) Consider $u, v \in C^2(U) \cap C^0(\overline{U})$ with u harmonic and v subharmonic. Show that if $v \leqslant u$ on ∂U then $v \leqslant u$ on U.

(b) Let U be the annulus $\{x \in \mathbb{R}^2 \mid R_1 < |x| < R_2\} \subset \mathbb{R}^2$ for constants $R_2 > R_1 > 0$. Let $u \in C^2(U) \cap C^0(\overline{U})$ be subharmonic on U. Show that $\max_{|x|=r} u(x)$ is convex as a function of $\log r$, when $R_1 < r < R_2$.

Hint: recall that $\log|x|$ *is a harmonic function on* $\mathbb{R}^2 \setminus \{0\}$.

(c) Let u be C^2 and subharmonic on $\{x \in \mathbb{R}^2 \mid |x| < R\}$ for some $R > 0$. Show that $\max_{|x|=r} u(x)$ is nondecreasing in r for $0 \leqslant r < R$.

(d) Show that if $u \in C^2(\mathbb{R}^2)$ is subharmonic and

$$\frac{u(x)}{\log|x|} \to 0, \quad \text{as } |x| \to \infty,$$

then u is constant. (In particular, a bounded subharmonic function on $\mathbb{R}^2$ is constant.)

(5) (Kelvin transform) The *Kelvin transform* $\mathcal{K}u$ of a function $u : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\mathcal{K}(u) = u(\overline{x})|\overline{x}|^{n-2},$$

where $\overline{x} = x/|x|^2$. Show that if u is harmonic then so is $\mathcal{K}u$.