By: Herminso Villarraga Gómez

Solución a los problemas Cap 1 Jackson 3ed

1.1 a) Por ley de Gauss $\int_S \vec{E}\cdot d\vec{a} = \frac{Q_{int}}{\epsilon_0} = \frac{1}{\epsilon_0}\int \rho_{int}\, dV$

donde S es una superficie gaussiana imaginaria. Supongamos que S es interior a la superficie del conductor; entonces si tuviesemos algun campo dentro de S, $E_{int} \neq 0$. luego las cargas libres ρ_{int} que existiesen en el interior se moverian en la direccion del campo; pero en el caso electrostático y en ausencia de campos externos las cargas no tienen por que moverse, luego $E_{int} = 0$ necesariamente, así:

$$\int_S \vec{E}_{int}\cdot d\vec{a} = 0 = \frac{1}{\epsilon_0}\int \rho_{int}\, dV$$

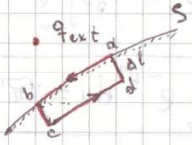
$\Rightarrow \int \rho_{int}\, dV = 0 \longrightarrow$ esto no implica que $\rho_{int} = 0$, pues puede estar la carga repartida de modo que la integral sume cero.

la ley de Gauss también se escribe como:

$\vec{\nabla}\cdot E_{int} = \rho_{int}/\epsilon_0 \Rightarrow E_{int} = 0$ implica $\rho_{int} = 0$

así en el interior del conductor no hay carga, luego debe estar en ~~el exterior~~. la superficie.

b) La segunda proposición es inmediata de la ley de Gauss, y para cargas exteriores al conductor hueco:



$$\oint \vec{E}_{int}\cdot d\vec{l} = 0 = \int_a^b \vec{E}_{int}\cdot d\vec{l} + \int_b^c \vec{E}_{int}\cdot d\vec{l} + \int_c^d \vec{E}_{int}\cdot d\vec{l} + \int_d^a \vec{E}_{int}\cdot d\vec{l}$$

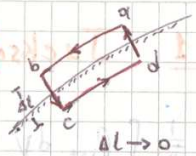
$\longrightarrow$ Cero en el limite $\Delta l \to 0$

$$= \underbrace{(V_b - V_a)}_{0} + \int_c^d \vec{E}_{int}\cdot d\vec{l}$$

S es equipotencial $\longrightarrow V_b = V_a$

$\Rightarrow 0 = \int_c^d \vec{E}_{int}\cdot d\vec{l} \longrightarrow \vec{E}_{int} = 0 \longrightarrow$ el interior del conductor no "ve" las cargas externas.

c)

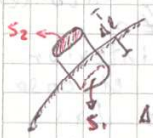


$$\oint \vec{E}\cdot d\vec{l} = 0 = \int_a^b \vec{E}_{ext_{\parallel}}\cdot d\vec{l} + \int_c^d \vec{E}_{int_{\parallel}}\cdot d\vec{l}$$

$\vec{E}_{int} = 0$ en el interior del conductor

$$\Rightarrow \quad 0 = \int_a^b \vec{E}_{ext_{\parallel}}\cdot d\vec{l} \longrightarrow \boxed{\vec{E}_{ext_{\parallel}} = 0}$$

luego el campo exterior debe ser normal



$$\int_S \vec{E}\cdot\hat{n}\,da = \frac{1}{\epsilon_0}\int_S \rho\, dv = \frac{1}{\epsilon_0}\int_S \sigma\, da$$

$$\Delta l \to 0 \qquad \int_{S_1} \underbrace{\vec{E}_{int}}_{0}\cdot\hat{n}\,da + \int_{S_2} \vec{E}_{ext}\cdot\hat{n}\,da = \frac{1}{\epsilon_0}\int_S \sigma\, da$$

$$S = S_2 = S_1 \longrightarrow \int_S \vec{E}_{ext}\cdot\hat{n}\,da = \int_S \frac{\sigma}{\epsilon_0}\,da \longrightarrow \underbrace{\vec{E}_{ext}\cdot\hat{n}}_{E_{normal}} = \frac{\sigma}{\epsilon_0}$$

pero $\vec{E}_{ext}$ es normal a la superficie

$$\Rightarrow \quad \boxed{\vec{E} = \frac{\sigma}{\epsilon_0}\hat{n}}$$

1.2 $$\delta(\vec{r}-\vec{r}\,') = \lim_{\alpha\to 0} D = \frac{1}{(2\pi)^{3/2}} \lim_{\alpha\to 0} \frac{1}{\alpha^3} e^{-\frac{1}{2\alpha^2}(x^2+y^2+z^2)}$$

como el sentido de la función delta se manifiesta al integrar sobre todo el espacio:

$$\int \delta(\vec{r}-\vec{r}\,')\,d\vec{r} = \frac{1}{(2\pi)^{3/2}} \lim_{\alpha\to 0} \frac{1}{\alpha^3} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2\alpha^2}(x^2+y^2+z^2)}\,dx\,dy\,dz$$

$$= \frac{1}{(2\pi)^{3/2}} \lim_{\alpha\to 0} \frac{1}{\alpha^3} \int_{-\infty}^{\infty} e^{-\frac{1}{2\alpha^2}x^2}dx \int_{-\infty}^{\infty} e^{-\frac{1}{2\alpha^2}y^2}dy \int_{-\infty}^{\infty} e^{-\frac{1}{2\alpha^2}z^2}dz$$

Sea $u = \frac{1}{\sqrt{2}\,\alpha}x_i \longrightarrow du = \frac{dx_i}{\sqrt{2}\,\alpha}$ y usando $\boxed{\int_{-\infty}^{\infty} e^{-u^2}du = \sqrt{\pi}}$

$$\int \delta(\vec{r}-\vec{r}\,')\,d\vec{r} = \frac{1}{(2\pi)^{3/2}} \lim_{\alpha\to 0} \frac{1}{\cancel{\alpha^3}} \left(\sqrt{2}\,\cancel{\alpha}\sqrt{\pi}\right)^3 = 1$$

$$= 1 \cdot 1 \cdot 1$$

$$= \int \delta(u-u')\,du \cdot \int \delta(v-v')\,dv \cdot \int \delta(w-w')\,dw$$

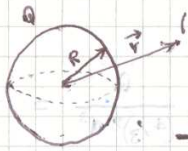
$$= \int \delta(u-u')\,\delta(v-v')\,\delta(w-w')\,uvw\,\frac{du}{u}\cdot\frac{dv}{v}\cdot\frac{dw}{w}$$

$$\int \delta(\vec{r}-\vec{r}\,')\,d\vec{r} = \int \delta(u-u')\,\delta(v-v')\,\delta(w-w')\,uvw\,d\vec{r}$$

$$\Rightarrow \quad \boxed{\delta(\vec{r}-\vec{r}\,') = \delta(u-u')\,\delta(v-v')\,\delta(w-w')\,uvw}$$

donde uvw es el Jacobiano de la transformación.

1.3 a)

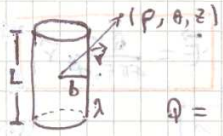


$$\rho \propto \delta(r-R) \longrightarrow \rho = K\delta(r-R)$$

$$Q = \int \rho\, dv = K\int \delta(r-R)\, r^2 d\Omega = KR^2(4\pi)$$

$$\longrightarrow K = Q/4\pi R^2 \longrightarrow \boxed{\rho(\vec{r}) = \frac{Q}{4\pi R^2}\delta(r-R)}$$

b)

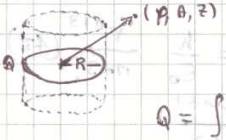


$$\rho \propto K\delta(\rho-b) \longrightarrow \rho = K\delta(\rho-b)$$

$$Q = \lambda L = \int \rho\, dv = \int_{-L/2}^{L/2}\int_0^{2\pi}\underbrace{\int_0^b K\delta(\rho-b)\,\rho\, d\rho}_{Kb}\, d\theta\, dz$$

$$\lambda \cancel{L} = (Kb)(2\pi)\cancel{L} \longrightarrow K = \frac{\lambda}{2\pi b} \longrightarrow \boxed{\rho(\vec{r}) = \frac{\lambda}{2\pi b}\delta(r-b)}$$

c)



$$\rho \propto \delta(z)\,u(R) \qquad u(R) = \begin{cases} 1 & \text{si } \rho < R \\ 0 & \text{si } R > \rho \end{cases}$$

$$Q = \int \rho\, dv = K\int \delta(z)\,u(R)\,\rho\, d\rho\, d\theta\, dz$$

$$Q = K\underbrace{\int_0^\infty u(R)\,\rho\, d\rho}_{\int_0^R 1\cdot\rho\, d\rho + \cancel{\int_R^\infty 0\cdot\rho\, d\rho}^{\,0} = \frac{R^2}{2}}\int_0^{2\pi} d\theta \int_{-\infty}^{\infty}\delta(z)\, dz = K\frac{R^2}{\cancel{2}}(\cancel{2}\pi)\cdot 1$$

$$\longrightarrow K = \frac{Q}{\pi R^2} \longrightarrow \boxed{\rho(\vec{r}) = \frac{Q}{\pi R^2}\delta(z)\,u(R)}$$

d)

$$\rho \propto \delta(\theta-\pi/2)\cdot u(R) \qquad u(R) = \begin{cases} 1 & \text{si } \rho < R \\ 0 & \text{si } \rho > R \end{cases}$$

$$Q = \int \rho\, dv = K\int \delta(\theta-\pi/2)\,u(R)\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= K\int_0^\infty u(R)\, r^2 dr \int_0^\pi \delta(\theta-\pi/2)\sin\theta\, d\theta \int_0^{2\pi} d\phi$$

$$= K\left\{\int_0^R 1\cdot r^2 dr + \cancel{\int_R^\infty 0\cdot r^2 dr}^{\,0}\right\}\sin\theta\Big|_{\theta=\pi/2}\cdot 2\pi$$

$$Q = K\frac{R^3}{3}\cdot\underbrace{(\sin\pi/2)}_{1}(2\pi)$$

$$\Rightarrow K = \frac{3Q}{2\pi R^3} \longrightarrow \boxed{\rho(\vec{r}) = \frac{3Q}{2\pi R^3}\delta(\theta-\pi/2)\,u(R)}$$

1.4 a) Esfera (cascarón) conductora de radio a y carga total Q

si $r < a \longrightarrow \oint \vec{E}\cdot\hat{n}\, da = \frac{1}{\epsilon_0}\int \rho_{int}\, dv \qquad \rho_{int} = 0 \longrightarrow \vec{E} = 0$

si $r > a \longrightarrow \oint \vec{E}\cdot\hat{n}\, da = \frac{1}{\epsilon_0}\int \rho_{int}\, dv$

$$EA = \frac{1}{\epsilon_0}Q \longrightarrow \boxed{\vec{E} = \frac{Q}{\epsilon_0(4\pi r^2)}} \qquad r > a$$

b) Esfera con densidad uniforme:

Si $r<a \rightarrow \int \vec{E}\cdot\hat{n}\,da = \frac{1}{\epsilon_0}\int \rho\, dv \qquad \rho = \frac{Q}{(4/3)\pi a^3} = \frac{3Q}{4\pi a^3}$

$$E\cdot 4\pi r^2 = \frac{1}{\epsilon_0}\int \frac{3Q}{4\pi a^3}\, r'^2\, dr'\, d\Omega$$

$$E\cdot 4\pi \cancel{r^2} = \frac{1}{\epsilon_0}\,\frac{3Q}{\cancel{4\pi} a^3}\,\frac{r^{\cancel{3}}}{3}\,\cancel{(4\pi)} \rightarrow \boxed{\vec{E} = \frac{Qr}{4\pi\epsilon_0 a^3}\,\hat{r}}$$

Si $r>a \rightarrow E\cdot 4\pi r^2 = \frac{Q}{\epsilon_0} \rightarrow \boxed{\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\,\hat{r}}$ $\quad r>a$

c) Si $\rho \propto r'^n$ para $r<a \qquad \rho = K r^n$

$$\Rightarrow \int \vec{E}\cdot\hat{n}\,da = \frac{1}{\epsilon}\int \rho\, dv = \frac{K}{\epsilon_0}\int_0^{4\pi}\int_0^{r} r'^n r'^2\, dr'\, d\Omega = \frac{K}{\epsilon_0}\,\frac{r^{n+3}}{n+3}(4\pi)$$

$$E(\cancel{4\pi} r^2) = \frac{K}{\epsilon_0}\,\frac{r^{n+3}}{n+3}\,\cancel{(4\pi)} \rightarrow E = \frac{K}{\epsilon_0}\,\frac{r^{n+1}}{n+3}$$

$$Q = \int_{\text{toda la esfera}} \rho\, dv = \int_0^{4\pi}\int_0^{a} K r'^n r'^2\, dr'\, d\Omega = K\,\frac{a^{n+3}}{n+3}(4\pi)$$

$$\rightarrow K = \frac{(n+3)Q}{4\pi a^{n+3}} \Rightarrow \boxed{\vec{E} = \frac{Q}{4\pi\epsilon_0}\,\frac{r^{n+1}}{a^{n+3}}\,\hat{r}} \quad \text{si } r<a$$

• Si $r>a \rightarrow E\cdot 4\pi r^2 = \frac{Q}{\epsilon_0} \Rightarrow \boxed{\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\,\hat{r}}$ si $r>a$

1.5 $\Phi = q\,\frac{e^{-\alpha r}}{r}\left(1+\frac{1}{2}\alpha r\right) = q\left(\frac{e^{-\alpha r}}{r} + \frac{1}{2}\alpha e^{-\alpha r}\right) + \frac{q}{r} - \frac{q}{r}$

$$= \underbrace{q\left(\frac{e^{-\alpha r}-1}{r}\right) + \frac{1}{2}\alpha q e^{-\alpha r}}_{\Phi'} + \frac{q}{r} = \Phi' + \frac{q}{r}$$

pot de una carga puntual

potencial efectivo de una distribución de carga con densidad

$$\rho' = -\epsilon_0 \nabla^2 \Phi' \quad \text{donde}$$

$$\nabla^2\Phi' = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi'}{dr}\right) + \cancel{\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi'}{\partial\theta}\right)}^{0} + \cancel{\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi'}{\partial\phi^2}}^{0}$$

$$= \frac{q}{r^2}\frac{d}{dr}\left[r^2\frac{d}{dr}\left(\frac{e^{-\alpha r}-1}{r}\right) + \frac{1}{2}\alpha r^2\frac{d}{dr}e^{-\alpha r}\right]$$

$$= \frac{q}{r^2}\frac{d}{dr}\left[r^2\left(\frac{-\alpha e^{-\alpha r}}{r} - \frac{e^{-\alpha r}-1}{r^2}\right) + \frac{1}{2}\alpha r^2\left(-\alpha e^{-\alpha r}\right)\right]$$

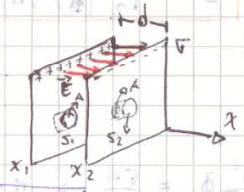
$$= \frac{q}{r^2}\left[\left(\alpha^2 e^{-\alpha r} r - \cancel{\alpha e^{-\alpha r}} + \cancel{\alpha e^{-\alpha r}}\right) - \frac{1}{2}\alpha\left(2r\alpha e^{-\alpha r} - \alpha^2 r^2 e^{-\alpha r}\right)\right]$$

$$\nabla^2 \Phi' = \cancel{\frac{q\,\alpha^2 e^{-\alpha r}}{r}} + \frac{q}{2}\alpha^3 e^{-\alpha r} - \cancel{\frac{q\,\alpha^2 e^{-\alpha r}}{r}}$$

$\Rightarrow \rho'(r) = -\epsilon_0 \frac{q}{2}\alpha^3 e^{-\alpha r}$ y con ello la distribución de carga total que produce el potencial Φ es

$\rho(r) = \rho'(r) + \text{Carga Puntual} \longrightarrow \boxed{\rho(r) = -\epsilon_0 \frac{q}{2}\alpha^3 e^{-\alpha r} + q\,\delta(r)}$

1.6 a)



$\int \vec{E}\cdot\hat{n}\,da = \frac{q}{\epsilon_0}$

Para una sola placa: $E(\cancel{2A}) = \frac{\sigma \cancel{A}}{\epsilon_0} \rightarrow E = \frac{\sigma}{2\epsilon_0}$

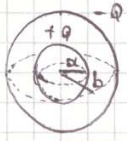
para las 2 placas: $E = \frac{\sigma}{2\epsilon_0} + \frac{\sigma}{2\epsilon_0} = \frac{\sigma}{\epsilon_0}$

$\boxed{\vec{E} = \frac{\sigma}{\epsilon_0}\hat{i}}$ Como $\vec{E} = -\frac{d\Phi}{dx} \longrightarrow \Phi = -\frac{\sigma}{\epsilon_0}x$

$\Phi_1 = -\frac{\sigma}{\epsilon_0}x_1$, $\Phi_2 = -\frac{\sigma}{\epsilon_0}x_2 \longrightarrow \Delta\Phi = \frac{\sigma}{\epsilon_0}(x_1 - x_2) = \frac{\sigma d}{\epsilon_0}$

$\Rightarrow C = \frac{Q}{\Delta\Phi} = \frac{\cancel{\sigma} A}{\cancel{\sigma} d/\epsilon_0} \longrightarrow \boxed{C = \frac{\epsilon_0 A}{d}}$

b)



Para: $a < r < b$: $\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$

$\Phi_{ab} = -\int_b^a \vec{E}\cdot d\vec{l} = -\int_b^a \frac{Q}{4\pi\epsilon_0 r^2}dr = \frac{Q}{4\pi\epsilon_0 r}\Big|_b^a$

$\Phi_{ab} = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right] = \frac{Q}{4\pi\epsilon_0}\left(\frac{b-a}{ab}\right) \longrightarrow \boxed{C = \frac{Q}{\Phi_{ab}} = \frac{4\pi\epsilon_0 ab}{b-a}}$

c)

$\int \vec{E}\cdot\hat{n}\,da = \frac{1}{\epsilon_0}\int \rho_l\,dl$ $\quad E(2\pi r\cancel{L}) = \frac{\rho_L \cancel{L}}{\epsilon_0}$

$\rightarrow \vec{E} = \frac{\rho_L}{2\pi\epsilon_0 r}\hat{r}$

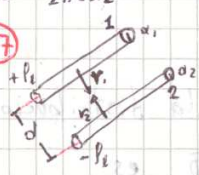
$\Phi_{ab} = -\int_b^a \vec{E}\cdot d\vec{l} = -\frac{\rho_L}{2\pi\epsilon_0}\int_b^a \frac{1}{r}dr = -\frac{\rho_L}{2\pi\epsilon_0}\ln\left(\frac{a}{b}\right)$

$C = \frac{Q}{\Phi_{ab}} = \frac{\rho_L L}{\Phi_{ab}} \longrightarrow \boxed{C = \frac{Q}{2\pi\epsilon_0}\ln(b/a)}$

$$\vec{E}_1 = \frac{\rho_1}{2\pi\epsilon_0 r_1}\hat{r}_1 \qquad \hat{r}_1 = -\hat{r}_2 \qquad \phi_{12} = -\int_2^1 (\vec{E}_1 + \vec{E}_2)\cdot d\hat{r}_1 = -\left[\int_2^1 \vec{E}_1 \cdot d\vec{r}_1 + \int_2^1 \vec{E}_2 \cdot (-d\vec{r}_2)\right]$$

$$\vec{E}_2 = \frac{\rho_1}{2\pi\epsilon_0 r_2}\hat{r}_2$$

1.7



$$\phi_{12} = -\left[\int_{d-a_2}^{a_1} \frac{\rho_l}{2\pi\epsilon_0 r_1} dr_1 + \int_{d-a_2}^{a_1} -\frac{\rho}{2\pi\epsilon_0 r_2}(-dr_2)\right]$$

$$\phi_{12} = -\int_2^1 \vec{E}\cdot d\vec{l} = -\frac{\rho_l}{2\pi\epsilon_0}\left[\int_{d-a_2}^{a_1} \frac{1}{r_1} dr_1 - \int_{a_2}^{d-a_1} \frac{1}{r_2} dr_2\right]$$

$$= -\frac{\rho_l}{2\pi\epsilon_0} \ln \frac{a_1 a_2}{(d-a_2)(d-a_1)} = \frac{\rho_l}{2\pi\epsilon_0} \ln \frac{(d-a_2)(d-a_1)}{a_1 a_2}$$

$$\frac{C}{L} = \frac{\rho_l}{\phi_{12}} = \frac{2\pi\epsilon_0}{\ln\left[\frac{(d-a_2)(d-a_1)}{a_1 a_2}\right]} \simeq \frac{\pi\epsilon_0}{\frac{1}{2}\ln\left(\frac{d^2}{a_1 a_2}\right)} \rightarrow \boxed{\frac{C}{L} = \frac{\pi\epsilon_0}{\ln\left(d/\sqrt{a_1 a_2}\right)}}$$

1.8 Usando $W = \frac{\epsilon_0}{2}\int E^2 dv$

A Condensador de placas paralelas:

$$W = \frac{\epsilon_0}{2}\int\left(\frac{\sigma}{\epsilon_0}\right)^2 dv = \frac{\sigma^2}{2\epsilon_0}\int dv = \boxed{\frac{\sigma^2}{2\epsilon_0} A d} \qquad \Phi_{12} = \frac{\sigma d}{\epsilon_0} \rightarrow \sigma = \frac{\epsilon_0 \Phi_{12}}{d}$$

$$\Rightarrow W = \frac{(\epsilon_0 \Phi_{12}/d)^2 A d}{2\epsilon_0} \rightarrow \boxed{W = \frac{1}{2}\frac{\epsilon_0 A}{d}\Phi_{12}^2}$$

Condensador esférico:

$$W = \frac{\epsilon_0}{2}\int\left(\frac{Q}{4\pi\epsilon_0 r^2}\right)^2 r^2 dr\, d\Omega = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0}\right)^2 (4\pi)\int_a^b \frac{1}{r^2} dr$$

$$= \frac{Q^2}{8\pi\epsilon_0}\left[-\frac{1}{b} + \frac{1}{a}\right] \rightarrow \boxed{W = \frac{Q^2}{8\pi\epsilon_0}\left(\frac{b-a}{ab}\right)}$$

$$\Phi_{ab} = \frac{Q}{4\pi\epsilon_0}\left(\frac{b-a}{ab}\right) \rightarrow Q = \frac{4\pi\epsilon_0 ab}{b-a}\Phi_{ab} \rightarrow$$

$$\Rightarrow W = \frac{(4\pi\epsilon_0 ab/b-a)^2}{8\pi\epsilon_0}\Phi_{ab}^2\left(\frac{b-a}{ab}\right) \rightarrow \boxed{W = 2\pi\epsilon_0 ab \frac{\Phi_{ab}^2}{b-a}}$$

Condensador cilíndrico:

$$\frac{W}{L} = \frac{\epsilon_0}{2}\int E^2 ds = \frac{\epsilon_0}{2}\int\left(\frac{\rho_L}{2\pi\epsilon_0 r}\right)^2 r\, dr\, d\theta = \frac{\rho_L^2}{8\pi\epsilon_0}\int_a^b \frac{1}{r} dr \cdot (2\pi)$$

$$\boxed{\frac{W}{L} = \frac{\rho_L^2}{4\pi\epsilon_0}\ln(b/a)} \qquad \Phi_{ab} = \frac{\rho_L}{2\pi\epsilon_0}\ln b/a$$

$$\rightarrow \frac{W}{L} = \frac{[2\pi\epsilon_0 \Phi_{ab}/\ln(b/a)]^2}{4\pi\epsilon_0}\ln b/a \rightarrow \boxed{\frac{W}{L} = \pi\epsilon_0 \frac{\Phi_{ab}^2}{\ln(b/a)}}$$

B Usando: $W = \frac{\epsilon_0}{2}|\vec{E}|^2$:

para placas paralelas: $W = \frac{\epsilon_0}{2}\left(\frac{\sigma}{\epsilon_0}\right)^2 = \frac{\epsilon_0}{2}\left(\frac{Q/A}{\epsilon_0}\right)^2 \Rightarrow \boxed{W = \frac{1}{2\epsilon_0}\frac{Q^2}{A^2}}$

$0 < r < d$

para el esférico: $W = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 r^2}\right)^2 \rightarrow \boxed{W = \frac{1}{32\pi^2\epsilon_0}\frac{Q^2}{r^4}}$ $\propto r^{-6}$

para el cilíndrico: $\frac{W}{L} = \frac{\epsilon_0}{2}\left(\frac{\rho_L}{2\pi\epsilon_0 r}\right)^2 \rightarrow \boxed{\frac{W}{L} = \frac{1}{8\pi^2\epsilon_0}\frac{\rho_L^2}{r^2}}$

1.9 a) Placas paralelas: $E = \frac{\sigma}{2\epsilon_0}$

$$F = QE = \frac{Q\sigma}{2\epsilon_0} = \frac{Q(Q/A)}{2\epsilon_0} \rightarrow \boxed{F = \frac{Q^2}{2\epsilon_0 A}}$$

Capacitor cilíndrico: $E = \frac{Q}{2\pi\epsilon_0 d}$

$$F = QE = Q\left(\frac{Q}{2\pi\epsilon_0 d}\right) \rightarrow \boxed{F = \frac{Q^2}{2\pi\epsilon_0 d}}$$

b) para las placas: $W = \frac{1}{2}CV^2 = \frac{1}{2}\frac{\epsilon_0 A}{d}V^2$ y $\boxed{F = \left(\frac{\partial W}{\partial d}\right)_V}$

$\Rightarrow \boxed{F = -\frac{\epsilon_0 A}{2d^2}V^2}$

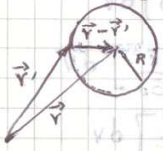
para los cilindros: $W = \frac{1}{2}\left(\frac{\pi\epsilon_0}{\ln(d/\sqrt{a_1 a_2})}\right)V^2$

$\rightarrow \boxed{F = \left(\frac{\partial W}{\partial d}\right)_V = \frac{1}{2}\frac{\pi\epsilon_0 V^2}{d\ln^2(d/\sqrt{a_1 a_2})}}$

Resultados similares en b) se obtienen con $Q = CV$

En a): Similarmente se hubiese podido usar para a): $\boxed{F = -\left(\frac{\partial W}{\partial d}\right)_V}$

1.10 Teorema del valor medio: para el espacio libre de cargas, el valor del potencial electrostático en algún punto es equivalente al promedio del potencial sobre la superficie de alguna esfera centrada en el punto.



$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}dv' + \frac{1}{4\pi}\oint_S\left[\frac{1}{|\vec{r}-\vec{r}'|}\frac{\partial\phi}{\partial n'} - \phi\frac{\partial}{\partial n'}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\right]da'$$

$$= \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(r')}{R}dv' + \frac{1}{4\pi}\oint_S\left[\frac{1}{R}\frac{\partial\phi}{\partial n'} - \phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\right]da'$$

0 $\rho(r') = 0$ en el int de la esfera.

$\vec{n}' = \vec{r}' - \vec{r} = R\hat{n}' \rightarrow \frac{\partial}{\partial n'}\left(\frac{1}{R}\right) = -\frac{1}{R^2}$

$$\Rightarrow \Phi(\vec{r}) = \frac{1}{4\pi R}\oint_S \vec{\nabla}\phi\cdot\hat{n}'da' - \frac{1}{4\pi}\oint_S \phi\left(-\frac{1}{R^2}\right)da'$$

$$= -\frac{1}{4\pi R}\oint_S \vec{E}\cdot\hat{n}'da' + \frac{1}{4\pi R^2}\oint_S \phi(\vec{r}')da'$$

$$= -\frac{1}{4\pi R}\int_V \vec{\nabla}'\cdot\vec{E}\,dv' + \frac{1}{4\pi R^2}\oint_S \phi(\vec{r}')da'$$

$$\Phi(\vec{r}) = -\frac{1}{4\pi R}\int_{V'}\frac{\rho(\vec{r}')}{\epsilon_0}dv' + \frac{1}{4\pi R^2}\oint_S \phi(\vec{r}')da'$$

0 No hay cargas en el volumen V'

$$\Rightarrow \boxed{\Phi(\vec{r}) = \frac{1}{4\pi R^2}\oint_S \phi(\vec{r}')ds}$$

By: Herminso Villarraga Gómez

$$\int_V \vec{\nabla}\psi\, dv = \int_S \psi\, \hat{n}\, da$$

1.11

en el int del conductor

$$\int \vec{\nabla}(\cancelto{0}{\rho/\epsilon_0})\, dv = \int_V \vec{\nabla}(\vec{\nabla}\cdot\vec{E})\, dv = \oint_S (\vec{\nabla}\cdot\vec{E})\,\hat{n}\, ds$$

$$= \oint_S (\vec{\nabla}\cdot\vec{E})\,\hat{n}\, ds = \oint_S (\vec{\nabla}\cdot\underbrace{(E\hat{n})}_{})\cdot\hat{n}\, ds$$

$\vec{E}$ es normal en la superficie de un conductor

$$= \oint_S \{(\vec{\nabla}E)\cdot\hat{n} + E\,\vec{\nabla}\cdot\hat{n}\}\cdot\hat{n}\, ds$$

$\boxed{\nabla\cdot\hat{n} = \frac{1}{R_1} + \frac{1}{R_2}}$ → Resultado de la geometría diferencial, donde R_1 y R_2 son los radios principales de la superficie.

$$\Rightarrow \quad = \oint_S \left[(\vec{\nabla}E)\cdot\hat{n} + E\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\right]\cdot\hat{n}\, ds$$

$$\Rightarrow \underbrace{(\vec{\nabla}E)\cdot\hat{n}}_{} = -E\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$$

$$\frac{\partial E}{\partial n} = -E\left(\frac{1}{R_1}+\frac{1}{R_2}\right) \longrightarrow \boxed{\frac{1}{E}\frac{\partial E}{\partial n} = -\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}$$

1.12 **Teorema de reciprocidad de Green:** Si Φ es el potencial debido a una densidad volumétrica de carga ρ dentro de V y a una densidad superficial de carga σ en S y Φ' es el potencial debido a otra distribución ρ', σ' tenemos:

S encierra a V y es conductora

$$\int_V \rho\,\Phi'\, dv + \int_S \sigma\,\Phi'\, da = \int_V \rho'\,\Phi\, dv + \int_S \sigma'\,\Phi\, da$$

D/: $\int_S (\phi\vec{\nabla}\psi - \psi\vec{\nabla}\phi)\cdot\hat{n}\, da = \int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, dv$ → Teo Green.

sea $\phi = \Phi$, $\psi = \Phi'$;

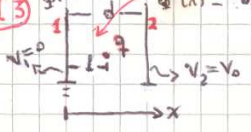
$$\int_S (\Phi\vec{\nabla}\Phi' - \Phi'\vec{\nabla}\Phi)\cdot\hat{n}\, da = \int_V (\Phi\nabla^2\Phi' - \Phi'\nabla^2\Phi)\, dv$$

$$\int_S \left[\Phi(-\vec{E}') - \Phi'(-\vec{E})\right]\cdot\hat{n}\, da = \int_V \left[\Phi\frac{\rho'}{\epsilon_0} - \Phi'\frac{\rho}{\epsilon_0}\right] dv$$

$$\int_S \left[-\Phi\frac{\sigma'}{\cancel{\epsilon_0}} + \Phi'\frac{\sigma}{\cancel{\epsilon_0}}\right] da = \int_V \left[\Phi\frac{\rho'}{\cancel{\epsilon_0}} - \Phi'\frac{\rho}{\cancel{\epsilon_0}}\right] dv$$

$$\Rightarrow \boxed{\int_V \rho\,\Phi'\, dv + \int_S \sigma\,\Phi'\, da = \int_V \Phi\,\rho'\, dv + \int_S \Phi\,\sigma'\, da}$$

1.13 $\Phi(0) = 0$ $\quad \Phi'(x) = V_0\frac{x}{d}$



$$\int \rho\,\Phi'\, dv + \int \sigma\,\Phi'\, da = \int \rho'\,\Phi\, dv + \int \sigma'\,\Phi\, da$$

$$\rho = q\,\delta(x-l)\,\delta(y)\,\delta(z)$$

$$\int_V q\,\delta(x-l)\,\delta(y)\,\delta(z)\, V_0\frac{x}{d}\, dv + \int_{S_1} \sigma_1 \cancelto{0}{V_1}\, da_1 + \int_{S_2} \sigma_2 \overbrace{V_2}^{V_0}\, da = \cancelto{0}{\int \rho'\,\Phi\, dv} + \cancelto{0}{\int_{S_1+S_2} \sigma'\,\Phi\, da}$$

$$\Rightarrow q\,V_0\frac{l}{d} + V_0\underbrace{\int_{S_2} \sigma_2\, da}_{q_2} = 0 \Rightarrow \boxed{q_2 = -q\frac{l}{d}}$$

por simetría para la otra placa $\boxed{q_1 = -\frac{q(d-l)}{d}}$

Nótese que $Q_{ind} = q_1 + q_2 = -q$ ✓

1.14 $$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\,dv \overset{\text{Teo Green}}{=} \oint_S \left[\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right] da$$

con $\phi = G(\vec{x},\vec{y})$ y $\psi = G(\vec{x}',\vec{y})$

$$\int_V \Big[G(\vec{x},\vec{y})\underbrace{\nabla_y^2 G(\vec{x}',\vec{y})}_{-4\pi\delta(\vec{x}'-\vec{y})} - G(\vec{x}',y)\overbrace{\nabla_y^2 G(\vec{x},\vec{y})}^{-4\pi\delta(\vec{x}-\vec{y})}\Big]\,dv = \oint_S \left[G(\vec{x},\vec{y})\frac{\partial G(\vec{x}',\vec{y})}{\partial n_y} - G(\vec{x}',\vec{y})\frac{\partial G(\vec{x},\vec{y})}{\partial n_y}\right] da_y$$

$$\Rightarrow -4\pi G(\vec{x},\vec{x}') + 4\pi G(\vec{x}',\vec{x}) = \oint_S \left[G(\vec{x},\vec{y})\frac{\partial G(\vec{x}',\vec{y})}{\partial n_y} - G(\vec{x}',\vec{y})\frac{\partial G(\vec{x},\vec{y})}{\partial n_y}\right] da_y$$

$$\rightarrow \boxed{G(\vec{x},\vec{x}') - G(\vec{x}',\vec{x}) = \frac{1}{4\pi}\oint_S \left[G(\vec{x}',\vec{y})\frac{\partial G(\vec{x},\vec{y})}{\partial n_y} - G(\vec{x},\vec{y})\frac{\partial G(\vec{x}',\vec{y})}{\partial n_y}\right] da_y}$$

a) Para C. Dirichlet: $G_D\big|_S = 0 \rightarrow \oint \left[\cancel{G_D(\,)}\frac{\partial G(\,)}{\partial n_y} - \cancel{G_D(\,)}\frac{\partial G(\,)}{\partial n_y}\right]$ ($\to 0$, $\to 0$)

$\Rightarrow G_D(\vec{x},\vec{x}') = G(\vec{x}',\vec{x}) \rightarrow$ Simetrica.

b) Para C. Neuman: $\partial G_N/\partial n_y\big|_S = -4\pi/S$

$$\Rightarrow \frac{1}{4\pi}\oint\left[G_N(\vec{x}',y)\frac{\partial G(\,)}{\partial n} - G_N(\vec{x},\vec{y})\frac{\partial G}{\partial n}\right] da_y = \frac{1}{S}\oint\left[G_N(\vec{x},y) - G_N(\vec{x}',y)\right] da$$

Si: $F(\vec{x}) = \frac{1}{S}\oint G_N(\vec{x},\vec{y})\,da_y \rightarrow \qquad = F(\vec{x}) - F(\vec{x}')$

$\rightarrow G_N(\vec{x},\vec{x}') - G(\vec{x}',\vec{x}) = F(\vec{x}) - F(\vec{x}') \rightarrow$ No simetrica

pero $\underbrace{G_N(\vec{x},\vec{x}') - F(\vec{x})}_{\rightarrow \text{ simetrica.}} = G(\vec{x}',\vec{x}) - F(\vec{x}')$

c) Para condiciones de Neuman:

$$\Phi(\vec{x}) = \langle\Phi\rangle_S + \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')G_N(\vec{x},\vec{x}')\,dv' + \frac{1}{4\pi}\oint_S \frac{\partial\Phi(\vec{x}')}{\partial n'}G_N(\vec{x},\vec{x}')\,da'$$

Si $G_N(\vec{x},\vec{x}') \rightarrow G_N(\vec{x},\vec{x}') - F(\vec{x})$:

$$\Phi_{Nuevo}(\vec{x}) = \langle\Phi\rangle_S + \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')G_N(\vec{x},\vec{x}')\,dv' - \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')F(\vec{x})\,dv'$$
$$+ \frac{1}{4\pi}\oint_S \frac{\partial\Phi(\vec{x}')}{\partial n'}G_N(\vec{x},\vec{x}')\,da' - \frac{1}{4\pi}\oint_S \frac{\partial\Phi(\vec{x}')}{\partial n'}F(\vec{x})\,da'$$

$$= \Phi(\vec{x}) - \frac{F(\vec{x})}{4\pi\epsilon_0}\left(\underbrace{\int_V \rho(\vec{x}')\,dv'}_{q_{enc}} + \epsilon_0\oint_S \frac{\partial\Phi(\vec{x}')}{\partial n'}\,da'\right)$$

$$\Phi_{Nuevo}(\vec{x}) = \Phi(\vec{x}) - \frac{F(\vec{x})}{4\pi\epsilon_0}\left(\cancel{q_{enc}} + \epsilon_0\underbrace{\cancel{\oint_S \vec{E}(\vec{x}')\cdot\vec{n}'\,da'}}_{q_{enc}/\epsilon_0}\right)$$

$$\boxed{\Phi_{Nuevo} = \Phi}$$

By: Herminso Villarraga Gómez

1.14, 2ed $W(\psi) = \frac{\epsilon_0}{2}\int_V |\vec{\nabla}\psi|^2 \, dv$

sea $\psi(\vec{r},\alpha) = \psi(\vec{r},0) + \alpha\, n(\vec{r})$ $\quad$ $\psi(\vec{r},0) \rightarrow$ mínimo valor de $\psi(\vec{r})$

$\alpha\, n(\vec{r}) \rightarrow$ variación alrededor de $\psi(\vec{r},0)$.

$$\Rightarrow W(\psi) = \frac{\epsilon_0}{2}\int_V \left[\vec{\nabla}\left(\psi(\vec{r},0) + \alpha\, n(\vec{r})\right)\right]^2 dv$$

$$= \frac{\epsilon_0}{2}\int_V \left[\vec{\nabla}\psi(\vec{r},0) + \alpha\vec{\nabla} n(\vec{r})\right]^2 dv \qquad \left.\frac{\partial W}{\partial \alpha}\right|_{\alpha=0} = 0 \rightarrow \text{Mínimo}$$

$$\frac{\partial W(\psi)}{\partial \alpha} = \frac{\epsilon_0}{\cancel{2}}\int_V \cancel{2}\left[\vec{\nabla}\psi(\vec{r},0) + \alpha\vec{\nabla} n(\vec{r})\right]\cdot\vec{\nabla} n(\vec{r})\, dv$$

$$\left.\frac{\partial W(\psi)}{\partial \alpha}\right|_{\alpha=0} = \epsilon_0\int_V \left[\vec{\nabla}\psi(\vec{r},0)\right]\cdot\vec{\nabla} n(\vec{r})\, dv \qquad \begin{array}{ll} u = \vec{\nabla}\psi & dv = \vec{\nabla} n\, dv \\ du = \vec{\nabla}^2\psi & v = n(\vec{r}) \end{array}$$

$$\delta W = \epsilon_0\left[\underbrace{\cancel{\vec{\nabla}\psi(\vec{r},0)\cdot n(\vec{r})\Big|_V}}_{0} - \int_V \nabla^2\psi(\vec{r},0)\, n(\vec{r})\, dv\right] = 0$$

si es evaluado en el límite del volumen (e.d en la superficie) donde por definición la variación es nula.

para que sea mínimo

$$\delta W = -\epsilon_0\int_V \nabla^2\psi(\vec{r},0)\, n(\vec{r})\, dv = 0 \Rightarrow \boxed{\nabla^2\psi(\vec{r},0) = 0}$$

en el interior de V

1.15 $W = \frac{1}{2}\oint_S \sigma\Phi\, da + \frac{\epsilon_0}{2}\int_V [\nabla\Phi]^2 dv$

$$\sigma = -\epsilon_0\left.\frac{\partial\Phi}{\partial n}\right|_S \rightarrow W = -\frac{\epsilon_0}{2}\left\{\oint_S \Phi\frac{\partial\Phi}{\partial n}\, da - \int_V [\nabla\Phi]^2 dv\right\}$$

$$\vec{E}\cdot\hat{n} = -\vec{\nabla}\Phi\cdot\hat{n} = -\frac{\partial\Phi}{\partial n} \rightarrow W = \frac{\epsilon_0}{2}\left\{\oint_S \Phi\vec{E}\cdot d\vec{a} + \int_V \vec{\nabla}\Phi\cdot\nabla\Phi\, dv\right\}$$

para superficies equipotenciales $\Phi|_S = cte \rightarrow \oint \Phi\vec{E}\cdot d\vec{a} = \Phi\int\vec{E}\cdot d\vec{a}$

$= \Phi\,(q_{int}/\epsilon_0)$

$$\Rightarrow W = \frac{\epsilon_0}{2}\left\{\Phi\frac{q_{int}}{\epsilon_0} + \int_V [\vec{\nabla}\Phi]^2 dv\right.$$

No hay q_{int} $\Rightarrow$ $\boxed{W = \frac{\epsilon_0}{2}\int_V [\vec{\nabla}\Phi]^2 dv}$ $\rightarrow$ la cual es mínimo según ejercicio 1.14-2ed

↳ Como no hay cargas en el interior $\Rightarrow \nabla^2\Phi = 0$ y ello implica que la expresión anterior sea mínimo según ejercicio 1.14-2ed

1.16 Supóngase que introducimos una superficie conductora S_0 no cargada. Sea $\vec{E}$ el campo antes de introducir S_0 al sistema y $\vec{E}'$ el campo después. El cambio de energía producido será:

$$U - U' = \frac{\epsilon_0}{2}\int_V [\vec{E}]^2 dv - \frac{\epsilon_0}{2}\int_V [\vec{E}']^2 dv$$

Los campos se anulan en el interior de las superficies conductoras S_0, S_i ($i = 1, 2, \ldots, n$; n = numero de conductores $\neq S_0$)
Sea V el volumen exterior a los n conductores S_i y V_0 el volumen de S_0

$$\Rightarrow U - U' = \frac{\epsilon_0}{2}\int_V [\vec{E}]^2 dv - \frac{\epsilon_0}{2}\int_{V-V_0} [\vec{E}']^2 dv$$

$$= \frac{\epsilon_0}{2}\int_{V-V_0} [\vec{E}]^2 dv + \frac{\epsilon_0}{2}\int_{V_0} [\vec{E}]^2 dv - \frac{\epsilon_0}{2}\int_{V-V_0} [\vec{E}']^2 dv$$

$$= \frac{\epsilon_0}{2}\int_{V_0} [\vec{E}]^2 dv + \frac{\epsilon_0}{2}\int_{V-V_0} (\vec{E}\cdot\vec{E} - \vec{E}'\cdot\vec{E}') dv$$

$$= \frac{\epsilon_0}{2}\int_{V_0} [\vec{E}]^2 dv + \frac{\epsilon_0}{2}\int_{V-V_0} (\vec{E}-\vec{E}')^2 dv + \epsilon_0\int_{V-V_0} \vec{E}'\cdot(\vec{E}-\vec{E}') dv$$

$$\vec{E}'\cdot(\vec{E}-\vec{E}') = -\nabla\phi'\cdot(\vec{E}-\vec{E}') = -\vec{\nabla}[\phi'(\vec{E}-\vec{E}')] + \phi'\underbrace{\vec{\nabla}(\vec{E}-\vec{E}')}_{\text{0 en } V-V_0 \text{ libre de cargas.}}$$

$$\Rightarrow U - U' = \frac{\epsilon_0}{2}\int_{V-V_0} (\vec{E}-\vec{E}')^2 dv + \epsilon_0\int_{V-V_0} \vec{\nabla}[\phi'(\vec{E}'-\vec{E})] + \frac{\epsilon_0}{2}\int_{V_0} (\vec{E})^2 dv$$

$$= \frac{\epsilon_0}{2}\int_{V-V_0} (\vec{E}-\vec{E}')^2 dv + \epsilon_0\oint_S \phi'(\vec{E}'-\vec{E})\cdot\hat{n}\, da + \frac{\epsilon_0}{2}\int_{V_0} (\vec{E})^2 dv$$

$$= \frac{\epsilon_0}{2}\int_{V-V_0} (\vec{E}-\vec{E}')^2 dv + \epsilon_0\sum_{i=1}^{n}\oint_{S_i} \phi_i'(\vec{E}'-\vec{E})\cdot\hat{n}\, da + \frac{\epsilon_0}{2}\int_{V_0} (\vec{E})^2 dv$$

$$= \frac{\epsilon_0}{2}\int_{V-V_0} (\vec{E}-\vec{E}')^2 dv + \epsilon_0\sum_{i=1}^{n}\phi_i'\oint_{S_i} (\vec{E}'-\vec{E})\cdot\hat{n}\, da + \frac{\epsilon_0}{2}\int_{V_0} (\vec{E})^2 dv$$

Como la carga sobre los conductores S_i no varía con la introducción de S_0 al sistema:

$$\int_{S_i} \vec{E}\cdot\hat{n}\, da = q_i/\epsilon_0 = \int_{S_i} \vec{E}'\cdot\hat{n}\, da \rightarrow \int_{S_i} (\vec{E}'-\vec{E})\cdot\hat{n}\, da = 0$$

$$\Rightarrow U - U' = \frac{\epsilon_0}{2}\int_{V-V_0} (\vec{E}-\vec{E}')^2 dv + \frac{\epsilon_0}{2}\int_{V_0} (\vec{E})^2 dv > 0 \rightarrow \boxed{U' < U}$$

1.17 a) $\epsilon_0 \int_V |\nabla\phi|^2 dv \overset{\text{teo Green}}{=} \epsilon_0 \left\{ \oint_S \phi \frac{\partial\phi}{\partial n} da - \int_V \phi \overbrace{\nabla^2\phi}^{0} dv \right\}$

$$= \epsilon_0 \sum_{i=1}^{n} V_i \int_{S_i} \frac{\partial\phi}{\partial n} da = \cancel{\epsilon_0} \sum_{i=1}^{n} V_i \int_{S_i} \frac{\sigma}{\cancel{\epsilon_0}} da$$

sea $V_k = 1$ el potencial del conductor k y $V_i = 0$ el conductor del i-ésimo $(i \neq k)$:

$$\Rightarrow \quad \epsilon_0 \int_V |\nabla\phi|^2 dv = V_k \int_{S_k} \sigma da = V_k Q_k = C_k \quad \text{ó} \quad \boxed{C = \epsilon_0 \int_V |\nabla\phi|^2 dv}$$

b) Supóngase $\psi = \Phi + \delta\psi \longrightarrow |\nabla\psi|^2 = |\nabla\Phi|^2 + |\nabla\delta\psi|^2 + 2\nabla\Phi\nabla\delta\psi$

$$\Rightarrow \quad \epsilon_0 \int_V |\nabla\psi|^2 dv = \underbrace{\epsilon_0 \int_V |\nabla\Phi|^2 dv}_{C} + \epsilon_0 \int_V |\nabla\delta\psi|^2 + 2\epsilon_0 \int_V \vec{\nabla}\Phi \cdot \vec{\nabla}\delta\psi dv$$

$$= C + \epsilon_0 \int_V |\nabla\delta\psi|^2 dv + 2\epsilon_0 \int_V \vec{\nabla}\Phi \cdot \vec{\nabla}\delta\psi dv$$

si $\psi = \Phi$ y $\phi = \delta\psi$ en $\int_V (\phi\nabla^2\psi + \vec{\nabla}\phi \cdot \vec{\nabla}\psi) dv = \oint_S \phi \frac{\partial\psi}{\partial n} da$ → identidad de Green.

$$\Rightarrow \quad \epsilon_0 \int_V |\nabla\psi|^2 dv = C + \epsilon_0 \int_V |\nabla\delta\psi|^2 dv - 2\epsilon_0 \left\{ \int_V \delta\psi \underbrace{\nabla^2\Phi}_{0} dv + \cancel{\oint_{\partial V} \delta\psi \frac{\partial\Phi}{\partial n} da} \right\}$$

$$= C + \epsilon_0 \int_V |\nabla\delta\psi|^2 dv \geq C$$

La aprox variacional requiere que las variaciones de la función de prueba tiendan a cero en la frontera.

así, para todo ψ que satisfaga las condiciones de frontera se tiene que

$$\boxed{C \leq \epsilon_0 \int_V |\nabla\psi|^2 dv}$$

1.18 Consideremos un volumen V libre de cargas, limitado por una superficie S compuesta por varias superficies separadas S_i (conductoras) tal que $\phi_i = 0$ si $i \neq 1$, $\phi_1 \neq 0$.

$$\Rightarrow \quad \Phi(\vec{r}) = \int_V \frac{\cancel{\rho(\vec{r}')}}{4\pi\epsilon_0} G(\vec{r},\vec{r}') dv' + \frac{1}{4\pi} \oint_S \left[\cancel{G(\vec{r},\vec{r}')} \frac{\partial\Phi}{\partial n'} - \Phi(\vec{r}') \frac{\partial G(\vec{r},\vec{r}')}{\partial n} \right] da'$$

V libre de cargas. C. Dirichlet.

$$\Phi(\vec{r}) = \frac{-1}{4\pi} \oint_S \Phi(\vec{r}') \frac{\partial G(\vec{r},\vec{r}')}{\partial n'} da = \frac{1}{4\pi} \sum_{i=1}^{n} \oint_{S_i} \Phi_i(\vec{r}') \frac{\partial G(\vec{r},\vec{r}')}{\partial n'} da'$$

↳ para cualquier $\vec{r}$ en V que no está sobre S_i.

Sobre la superficie $S_i \neq S_1$:

C. Dirichlet $\phi_i = 0$ $i \neq 1$.

$$\Phi(\vec{r}) = \int_{S_i} \frac{\sigma_i(\vec{r}') G(\vec{r},\vec{r}') dv}{4\pi\epsilon_0} + \frac{1}{4\pi} \oint_{S_i} \left[\cancel{G(\vec{r},\vec{r}')} \frac{\partial\phi_i}{\partial n'} - \cancel{\phi_i(\vec{r}')} \frac{\partial G(\vec{r},\vec{r}')}{\partial n'} \right] da'$$

$$\Rightarrow \quad \boxed{\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{S_1} \sigma_1(\vec{r}') G(\vec{r},\vec{r}') ds'}$$

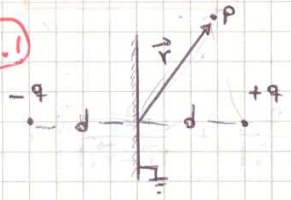
$$W = \frac{1}{2} \int \rho(\vec{r}) \Phi(\vec{r}) dv \qquad \begin{matrix} \rho \to \sigma \\ dv \to da \end{matrix} \Rightarrow W = \frac{1}{2} \int \sigma(\vec{r}) \Phi(\vec{r}) da$$

$$\Rightarrow \quad \boxed{W = \frac{1}{8\pi\epsilon_0} \int_{S_1} \int_{S_2} \sigma_1(\vec{r}) \sigma_1(\vec{r}') G(\vec{r},\vec{r}') da da'}$$

Solución a los problemas del Cap 2 Jackson 3ed.

2.1

a)



$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\vec{r}-d\hat{i}|} + \frac{-q}{|\vec{r}+d\hat{i}|}\right]$$

$$\sigma = -\epsilon_0 \frac{\partial \Phi}{\partial n}\Big|_S = -\epsilon_0 \frac{\partial \Phi}{\partial x}\Big|_{x=0}$$

$$\Phi(\vec{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{(x-d)^2+y^2+z^2}} - \frac{1}{\sqrt{(x+d)^2+y^2+z^2}}\right]$$

$$\frac{\partial \Phi}{\partial x}\Big|_{x=0} = \frac{q}{4\pi\epsilon_0}\left[\frac{(x-d)}{[(x-d)^2+y^2+z^2]^{3/2}} + \frac{(x+d)}{[(x+d)^2+y^2+z^2]^{3/2}}\right]_{x=0}$$

$$= \frac{q}{4\pi\epsilon_0}\frac{2d}{(d^2+y^2+z^2)^{3/2}}$$

$$\boxed{\sigma = -\epsilon_0 \frac{\partial \Phi}{\partial x}\Big|_{x=0} = -\frac{q}{2\pi}(d^2+y^2+z^2)^{-3/2}}$$

b) $\vec{F} = \frac{q(-q)}{4\pi\epsilon_0(2d)^2}\hat{i} \rightarrow \boxed{\vec{F} = -\frac{q^2}{16\pi\epsilon_0 d^2}}$

c) $$F = \int \frac{\sigma^2}{2\epsilon_0}da = \frac{1}{2\epsilon_0}\int_0^\infty\int_0^{2\pi}\frac{q^2d^2}{(2\pi)^2}\frac{\rho\, d\rho\, d\phi}{(d^2+\rho^2)^3}$$

$$= \frac{1}{2\epsilon_0}\frac{q^2d^2}{(2\pi)^{\cancel{2}}}\cancel{(2\pi)}\int_0^\infty \frac{2\rho\, d\rho}{2(d^2+\rho^2)^3} \qquad \begin{aligned} u &= d^2+\rho^2 \\ du &= 2\rho\, d\rho \end{aligned}$$

$$= \frac{q^2d^2}{8\pi\epsilon_0}\left[-\frac{1}{2}\frac{1}{(d^2+\rho^2)^2}\right]_0^\infty = \frac{q^2d^2}{8\pi\epsilon_0}\left[-\frac{1}{2d^4}\right]$$

$$\boxed{F = -\frac{q^2}{16\pi\epsilon_0 d^2}}$$

d) $$W = -\int \vec{F}\cdot d\vec{l} = \int_d^\infty \frac{q^2\, dx}{8\pi\epsilon\,\underbrace{(2d+2x)^2}} = \frac{q^2}{16\pi\epsilon_0}\frac{1}{(d+x)}\Big|_d^\infty$$

A medida que se mueve q hasta d+x, se mueve su imagen hasta -d-x.

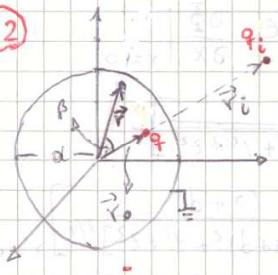
$$\boxed{W = -\frac{q^2}{16\pi\epsilon_0 d}}$$

e) $W = \frac{1}{8\pi\epsilon_0}\sum_{ij}\frac{q_i q_j}{|x_i - x_j|} = \frac{q^2}{8\pi\epsilon_0(2d)} \rightarrow \boxed{W = \frac{q^2}{16\pi\epsilon_0 d}}$

f) $\frac{q^2}{16\pi\epsilon_0 d} = 3.6 \text{ eV}$. ✓

2.2

a)



$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{|\vec{r}-\vec{r}_0|} + \frac{q_i}{|\vec{r}-\vec{r}_i|}\right\}$$

$$\Phi(r=a) = 0 \rightarrow \left.\frac{q}{|\vec{r}-\vec{r}_0|}\right|_{r=a} = -\left.\frac{q_i}{|\vec{r}-\vec{r}_i|}\right|_{r=a}$$

$$\Rightarrow \frac{q}{\sqrt{a^2+r_0^2-2ar_0\cos\beta}} = -\frac{q_i}{\sqrt{a^2+r_i^2-2ar_i\cos\beta}}$$

$$\frac{q}{a\sqrt{1+\frac{r_0^2}{a^2}-2\left(\frac{r_0}{a}\right)\cos\beta}} = \frac{-q_i}{r_i\sqrt{\left(\frac{a}{r_i}\right)^2+1-2\left(\frac{a}{r_i}\right)\cos\beta}}$$

Se satisface si $\frac{q}{a} = -\frac{q_i}{r_i}$ y $\frac{r_0}{a} = \frac{a}{r_i}$

$\Rightarrow \boxed{r_i = \frac{a^2}{r_0}}$ y $q_i = -\frac{q}{a}r_i = -q\frac{a}{r_0}$, $\boxed{q_i = -\frac{qa}{r_0}}$

$$\Rightarrow \Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{|\vec{r}-\vec{r}_0|} - \frac{q\,a/r_0}{\left|\vec{r}-\frac{a^2}{r_0}\hat{r}_0\right|}\right\}$$

$$\boxed{\Phi_{int}(\vec{r}) = \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{\sqrt{r^2+r_0^2-2rr_0\cos\beta}} - \frac{a}{\sqrt{r^2r_0^2+a^4-2a^2rr_0\cos\beta}}\right\}}$$

b) $$\sigma = \epsilon_0\left.\frac{\partial\Phi}{\partial r}\right|_{r=a} = \frac{q}{4\pi}\left\{-\frac{1}{\cancel{2}}\frac{\cancel{2}(r-r_0\cos\beta)}{(r^2+r_0^2-2rr_0\cos\beta)^{3/2}} + \frac{1}{\cancel{2}}\frac{\cancel{2}a(rr_0^2-a^2r_0\cos\beta)}{(r^2r_0^2+a^4-2a^2rr_0\cos\beta)^{3/2}}\right\}_{r=a}$$

$$= \frac{q}{4\pi}\left\{-\frac{(a-r_0\cos\beta)}{(a^2+r_0^2-2ar_0\cos\beta)^{3/2}} + \frac{a(ar_0^2-a^2r_0\cos\beta)}{(a^2r_0^2+a^4-2a^2ar_0\cos\beta)^{3/2}}\right\}$$

$$= \frac{q}{4\pi}\left\{-\frac{a-r_0\cos\beta}{r_0^3\left(\frac{a^2}{r_0^2}+1-2a\frac{1}{r_0}\cos\beta\right)^{3/2}} + \frac{\cancel{a^2}(r_0^2-ar_0\cos\beta)}{a^{\cancel{3}}r_0^3\left(1+\frac{a^2}{r_0^2}-2a\frac{1}{r_0}\cos\beta\right)^{3/2}}\right\}$$

$$= \frac{q}{4\pi}\left\{\frac{-(a-\cancel{r_0\cos\beta}) + r_0^2/a - \cancel{r_0\cos\beta}}{r_0^3\left(\frac{a^2}{r_0^2}+1-2a\frac{1}{r_0}\cos\beta\right)^{3/2}}\right\}$$

$$\boxed{\sigma = -\frac{q}{4\pi a r_0}\frac{(a^2/r_0^2)-1}{\left(\frac{a^2}{r_0^2}+1-2\frac{a}{r_0}\cos\beta\right)^{3/2}}}$$

tiene la misma forma que cuando la carga está afuera, pero note que ahora $a^2/r_0^2 > 1$.

Además tiene un menos (−) extra, de igual manera la carga inducida sobre la superficie interna de la esfera es:

$$q_{ind} = \int \sigma\, da = -\frac{q}{4\pi a r_0}\left(a^2/r_0^2 - 1\right)\int_0^{2\pi}\int_0^{\pi}\frac{a^2 \sin\theta\, d\theta\, d\phi}{\left(1+\frac{a^2}{r_0^2}-\frac{2a}{r_0}\cos\theta\right)^{3/2}}$$

Sea $u = 1 + a^2/r_0^2 - \frac{2a}{r_0}\cos\theta$

$du = (2a/r_0)\sin\theta\, d\theta$

$$q_{ind} = -\frac{q}{4\pi a r_0}\left(\frac{a^2}{r_0^2}-1\right)\int \frac{(r_0/2a)\,du}{u^{3/2}}\int_0^{2\pi} d\phi$$

$$= -\frac{q\,\cancel{a}}{2(\cancel{2\pi})\cancel{r_0}}\left(\frac{a^2}{r_0^2}-1\right)\left(\frac{\cancel{r_0}}{\cancel{2a}}\right)\left[\frac{u^{-1/2}}{-1/\cancel{2}}\right](\cancel{2\pi})$$

$$= \frac{q}{2}\left(\frac{a^2}{r_0^2}-1\right)\left[\frac{1}{\left(1+\frac{a^2}{r_0^2}-\frac{2a}{r_0}\cos\theta\right)^{1/2}}\right]_0^{\pi}$$

$$= \frac{q}{2}\left(\frac{a^2}{r_0^2}-1\right)\left[\frac{1}{\left(1+\frac{a^2}{r_0^2}+\frac{2a}{r_0}\right)^{1/2}} - \frac{1}{\left(1+\frac{a^2}{r_0^2}-\frac{2a}{r_0}\right)^{1/2}}\right]$$

$$= \frac{q}{2}\left(\frac{a^2}{r_0}-1\right)\left[\frac{1}{(1+a/r_0)} - \frac{1}{(a/r_0-1)}\right]$$

Note que la factorización debe ser tal que esta cantidad sea positiva por ser una magnitud

$$= -\frac{q}{2}\cancel{\left(1-\frac{a^2}{r_0}\right)}\left[\frac{(a/r_0-1)-(1+a/r_0)}{\cancel{a^2/r_0^2-1}}\right]$$

$$= +\frac{q}{\cancel{2}}\left[-\cancel{2}\right] \longrightarrow \boxed{q_{inducida} = -q}$$

Note que la carga inducida en la superficie interna de la esfera es $-q$ anulando el efecto de $+q$ en el interior de la cascara conductora

c) $$F = \frac{1}{4\pi\epsilon_0}\frac{q\cdot q_i}{|\vec{r}_0-\vec{r}_i|^2} = \frac{1}{4\pi\epsilon_0}\frac{q(-qa/r_0)}{(r_0-a^2/r_0)^2} = \frac{-q^2}{4\pi\epsilon_0}\frac{a}{r_0^3}\frac{1}{(1-a^2/r_0^2)^2}$$

$$\rightarrow \boxed{\vec{F}_q = \frac{q^2}{4\pi\epsilon_0}\frac{a}{r_0^3}\left(1-\frac{a^2}{r_0^2}\right)^{-2}\hat{r}_0}$$

Esto ocurre así para que se cumpla la ley de Gauss en la capa interna de la cascara de la esfera conductora. En el iterior de la cascara $\vec{E}=0$ como debe ser para un conductor.

d) Si se exije $\Phi(r=a)=V$

- Podemos suponer entonces que

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{|\vec{r}-\vec{r}_0|} - \frac{qa/r_0}{\left|\vec{r}-\frac{a^2}{r_0}\hat{r}_0\right|}\right\} + \frac{Va}{|\vec{r}|}$$

de modo que $\Phi(\vec{r}=a)=V$, pues la primera parte se anula tal como en la parte **a)** de la solución del problema. Ello consiste en superponer el problema de la parte a con el problema de una esfera sin carga q en su interior pero a potencial V cte sobre la superficie, sin embargo la solución anterior es inconsistente en el sentido de que

$$\vec{\nabla}\cdot\left(\frac{Va}{|\vec{r}|}\right) \neq 0$$

y ello significa que si $q=0$, $E=-\vec{\nabla}\Phi(\vec{r})\neq 0$ en el interior del conductor "$\rightarrow\leftarrow$". (Lo cual no debe ocurrir) Siempre en el espacio interno de un conductor en el que no hay cargas $\vec{E}_{int}=0$, ello implica que es mejor elejir

$$\boxed{\Phi_{int}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{|\vec{r}-\vec{r}_0|} - \frac{qa/r_0}{\left|\vec{r}-\frac{a^2}{r_0}\hat{r}_0\right|}\right\} + V} \quad \text{para } r<a$$

de modo que si $q_{int}=0 \rightarrow -\vec{\nabla}\Phi_{int}=\vec{E}_{int}=0$ ✓

- Cuando $q\rightarrow 0$, $\Phi(a)\rightarrow V$ y ello implica que la superficie de la esfera es una equipotencial y por lo tanto la ~~carga~~ densidad de carga que pueda existir en ella está uniformemente distribuida. Al poner q en el interior en $\vec{r}=\vec{r}_0$, se debe inducir la misma densidad de carga hallada en ~~de carga~~ **b)** pero en la superficie interior, de modo que la superficie exterior permanecerá con su distribución uniforme inicial sin nada adicional.

- La fuerza sobre q no cambia.

Si esfera tiene carga total Q: $\quad Q = \underbrace{Q_{int}}_{\text{Sobre } S_{int}} + \underbrace{Q_{out}}_{\text{Sobre } S_{out}}$

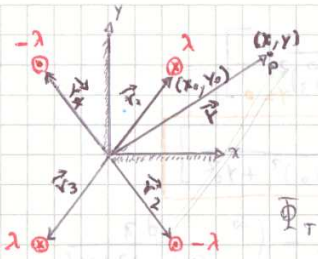
- Si imaginamos una cascara esférica, en el interior de la cáscara $\vec{E}=0$, $\Rightarrow Q_{int}=-q$ por ley de Gauss, $\rightarrow Q_{out}=Q-Q_{int}=Q+q$ y el potencial debido a Q_{out} es $V=Q_{out}/a=(Q+q)/a=$ cte. y aplicando el caso anterior

$$\boxed{\Phi_{int}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{|\vec{r}-\vec{r}_0|} - \frac{qa/r_0}{\left|\vec{r}-\frac{a^2}{r_0}\hat{r}_0\right|}\right\} + \frac{Q+q}{a}}$$

- La densidad de carga inducida en S_{int} no cambia y en S_{out} es $Q+q$ uniformemente distribuida. La fuerza sobre q no cambia.

2.3

a)



$$\Phi(x,y) = \frac{\lambda}{4\pi\epsilon_0} \ln \frac{R^2}{|\vec{r}-\vec{r}_1|^2}$$

$\downarrow$

Potencial de una línea de carga.

$$\Phi_T = \frac{\lambda}{4\pi\epsilon_0}\left[\ln \frac{R^2}{|\vec{r}-\vec{r}_1|^2} - \ln \frac{R^2}{|\vec{r}-\vec{r}_2|^2} + \ln \frac{R^2}{|\vec{r}-\vec{r}_3|^2} - \ln \frac{R^2}{|\vec{r}-\vec{r}_4|^2}\right]$$

$$\boxed{\Phi_T = \frac{\lambda}{4\pi\epsilon_0} \ln\left(\frac{|\vec{r}-\vec{r}_2|^2}{|\vec{r}-\vec{r}_1|^2}\cdot\frac{|\vec{r}-\vec{r}_4|^2}{|\vec{r}-\vec{r}_3|^2}\right)}$$

$$|\vec{r}-\vec{r}_i|^2 = (x-x_i)^2 + (y-y_i)^2$$

- Si $x=0 \longrightarrow |\vec{r}-\vec{r}_1| = |\vec{r}-\vec{r}_2|$ y $|\vec{r}-\vec{r}_3| = |\vec{r}-\vec{r}_4|$

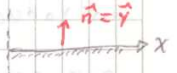
$$\Rightarrow \Phi(0,y) = \frac{\lambda}{4\pi\epsilon_0} \ln 1 = 0 \checkmark$$

- Si $y=0 \longrightarrow |\vec{r}-\vec{r}_1| = |\vec{r}-\vec{r}_4|$ y $|\vec{r}-\vec{r}_2| = |\vec{r}-\vec{r}_3|$

$$\Rightarrow \Phi(x,0) = \frac{\lambda}{4\pi\epsilon_0} \ln 1 = 0 \checkmark$$

b) $$\Phi_T = \frac{\lambda}{4\pi\epsilon_0}\left[\ln|\vec{r}-\vec{r}_2|^2 + \ln|\vec{r}-\vec{r}_4|^2 - \ln|\vec{r}-\vec{r}_1|^2 - \ln|\vec{r}-\vec{r}_3|^2\right]$$

$$\sigma = -\epsilon_0 \frac{\partial\Phi}{\partial n}\Big|_s \quad \text{si } y=0,\ x\geq 0 \longrightarrow \frac{\partial}{\partial n} = \frac{\partial}{\partial y}$$



$$\sigma = -\epsilon_0 \frac{\partial\Phi}{\partial y}\Big|_{y=0} \quad \text{como} \quad \frac{\partial}{\partial y}\left[\ln\left((x-x_i)^2+(y-y_i)^2\right)\right] = \frac{2(y-y_i)}{(x-x_i)^2+(y-y_i)^2} = \frac{2(y-y_i)}{|\vec{r}-\vec{r}_i|^2}$$

$$\Rightarrow \sigma = -\epsilon_0 \frac{\lambda}{4\pi\epsilon_0}\left[\frac{2(y-y_2)}{|\vec{r}-\vec{r}_2|^2} + \frac{2(y-y_4)}{|\vec{r}-\vec{r}_4|} - \frac{2(y-y_1)}{|\vec{r}-\vec{r}_1|^2} - \frac{2(y-y_3)}{|\vec{r}-\vec{r}_3|^2}\right]_{y=0}$$

$$= -\frac{\lambda}{2\pi}\left[-\frac{y_2}{|\vec{r}_1-\vec{r}_2|} - \frac{y_4}{|\vec{r}-\vec{r}_4|} + \frac{y_1}{|\vec{r}-\vec{r}_1|^2} + \frac{y_3}{|\vec{r}-\vec{r}_3|^2}\right] \qquad \begin{matrix} y_1 = y_0, & y_4 = y_0 \\ y_2 = -y_0, & y_3 = -y_0 \end{matrix}$$

$$= -\frac{\lambda}{2\pi}\left[\frac{y_0}{|\vec{r}-\vec{r}_2|^2} - \frac{y_0}{|\vec{r}-\vec{r}_4|^2} + \frac{y_0}{|\vec{r}-\vec{r}_1|^2} - \frac{y_0}{|\vec{r}-\vec{r}_3|^2}\right]$$

$$|\vec{r}-\vec{r}_2|^2 = (x-x_0)^2 + (\cancelto{0}{y}+y_0)^2 \qquad |\vec{r}-\vec{r}_4|^2 = (x+x_0)^2 + (\cancelto{0}{y}-y_0)^2$$

$$|\vec{r}-\vec{r}_1|^2 = (x-x_0)^2 + (\cancelto{0}{y}-y_0)^2 \qquad |\vec{r}-\vec{r}_3|^2 = (x+x_0)^2 + (\cancelto{0}{y}+y_0)^2$$

$\Rightarrow$ en $y=0$: $|\vec{r}-\vec{r}_2| = |\vec{r}-\vec{r}_1|$ y $|\vec{r}-\vec{r}_3| = |\vec{r}-\vec{r}_4| \Longrightarrow$

By: Herminso Villarraga Gómez

$$\Rightarrow \quad V = -\frac{y_0 \lambda}{\pi}\left[\frac{1}{|\vec{r}-\vec{r}_1|^2_{y=0}} - \frac{1}{|\vec{r}-\vec{r}_3|^2_{y=0}}\right]$$

$$\boxed{V = -\frac{\lambda y_0}{\pi}\left[\frac{1}{(x-x_0)^2+y_0^2} - \frac{1}{(x+x_0)^2+y_0^2}\right]}$$

c) $$Q_x = \int_0^\infty V\,dx = -\frac{\lambda y_0}{\pi}\left[\underbrace{\int_0^\infty \frac{dx}{(x-x_0)^2+y_0^2}}_{①} - \underbrace{\int_0^\infty \frac{dx}{(x+x_0)^2+y_0^2}}_{②}\right.$$

Para ①: $(x-x_0) = y_0 \tan u \qquad dx = y_0 \sec^2 u\, du$

si $x \to 0 \quad \tan u \to \tan^{-1}(-x_0/y_0)$

si $x \to \infty \quad \tan u \to \pi/2$.

$$① = \int \frac{y_0 \sec^2 u\, du}{y_0^2 \tan^2 u + y_0^2} = \int \frac{y_0 \sec^2 u}{y_0^2(\tan^2 u + 1)}\,du = \frac{1}{y_0} u \Big|_{tg^{-1}(-x_0/y_0)}^{\pi/2}$$

$$= \frac{1}{y_0}\left(\frac{\pi}{2} - tg^{-1}\left(-\frac{x_0}{y_0}\right)\right) = \frac{1}{y_0}\left(\frac{\pi}{2} + tg^{-1}\left(\frac{x_0}{y_0}\right)\right)$$

Para ②: $(x+x_0) = y_0 \tan u \qquad dx = y_0 \sec^2 u\, du$

$$② = \frac{1}{y_0} u \Big|_{tg^{-1}(x_0/y_0)}^{\pi/2} = \frac{1}{y_0}\left(\pi/2 - tg^{-1}\left(\frac{x_0}{y_0}\right)\right)$$

introduciendo ello en Q_x, $\rightarrow$ $\boxed{Q_x = -\frac{2\lambda}{\pi}\left[tg^{-1}\left(\frac{x_0}{y_0}\right)\right]}$

por analogía y dada la simetría: $\boxed{Q_y = -\frac{2\lambda}{\pi}\left[tg^{-1}\left(\frac{y_0}{x_0}\right)\right]}$

$$Q_T = Q_x + Q_y = -\frac{2\lambda}{\pi}\left[\underbrace{tg^{-1}\frac{x_0}{y_0} + tg^{-1}\frac{y_0}{x_0}}_{\pi/2}\right]$$

$$\boxed{tg^{-1} x + tg^{-1}\left(\frac{1}{x}\right) = \pi/2}$$

$$\Rightarrow \quad \boxed{Q_T = -\lambda}$$

d) $$\Phi = \frac{\lambda}{4\pi\epsilon_0} \ln\left(\frac{|\vec{r}-\vec{r}_2|^2\,|\vec{r}-\vec{r}_4|^2}{|\vec{r}-\vec{r}_1|^2\,|\vec{r}-\vec{r}_3|^2}\right)$$

$$|\vec{r}-\vec{r}_2|^2 = [(x-x_0)^2 + (y+y_0)^2] = x^2\left(1-\frac{x_0}{x}\right)^2 + y^2\left(1+\frac{y_0}{y}\right)^2$$

$$= x^2\left(1 - 2\frac{x_0}{x} + \frac{x_0^2}{x^2}\right) + y^2\left(1 + \frac{2y_0}{y} + \frac{y_0^2}{y^2}\right)$$

$$\simeq x^2 - 2x_0 x + y^2 + 2y_0 y \qquad \text{si: } x \gg x_0 \quad x_0/x \to 0$$

$$y \gg y_0 \quad y_0/y \to 0$$

$$\Rightarrow |\vec{r}-\vec{r}_2|^2 \simeq (x+y^2)\left[1-2\,\frac{xx_0-yy_0}{x^2+y^2}\right]$$

$$|\vec{r}-\vec{r}_4|^2 = (x+x_0)^2+(y-y_0)^2 = x^2+2x_0x+x_0^2+y^2-2yy_0+y_0^2$$

$$\simeq x^2+2x_0x+y^2-2y_0y \qquad y_0 << y\ ;\ x_0 << x$$

$$= (x^2+y^2)\left[1-2\,\frac{-xx_0+yy_0}{x^2+y^2}\right]$$

$$|\vec{r}-\vec{r}_1|^2 = (x-x_0)^2+(y-y_0)^2$$

$$\simeq (x^2+y^2)\left[1-2\,\frac{-xx_0-yy_0}{x^2+y^2}\right]$$

$$|\vec{r}-\vec{r}_3|^2 = (x+x_0)^2+(y+y_0)^2$$

$$\simeq (x^2+y^2)\left[1-2\,\frac{xx_0+yy_0}{x^2+y^2}\right]$$

$$\Rightarrow |\vec{r}-\vec{r}_2|^2|\vec{r}-\vec{r}_4|^2 = (x+y^2)^2\left[1-2\,\cancel{\frac{-xx_0+yy_0}{x^2+y^2}}-\cancel{\frac{2x_0x-yy_0}{x^2+y^2}}+4\,\frac{-(x_0x)^2-(yy_0)^2+2x_0y_0xx}{(x^2+y^2)^2}\right]$$

$$= (x^2+y^2)^2\left[1-4\,\frac{(xx_0-yy_0)^2}{(x^2+y^2)^2}\right]$$

$$|\vec{r}-\vec{r}_1|^2|\vec{r}-\vec{r}_3|^2 = (x^2+y^2)\left[1-4\,\frac{(xx_0+yy_0)^2}{(x^2+y^2)^2}\right]$$

$$\Rightarrow \Phi\,(x>>x_0,\ y>>y_0) = \frac{\lambda}{4\pi\epsilon_0}\left\{\ln\left[1-4\left(\frac{xx_0-yy_0}{x^2+y^2}\right)^2\right]-\ln\left[1-4\left(\frac{xx_0+yy_0}{x^2+y^2}\right)^2\right]\right\}$$

$(x^2+y^2) >> (x_0x \pm y_0y)$, si: $\boxed{\epsilon << 1 \rightarrow \ln(1+\epsilon)\approx\epsilon}$ con ello,

$$\Phi_{ASYMT} \approx \frac{\lambda}{4\pi\epsilon_0}\left\{-4\,\frac{(xx_0-yy_0)^2}{(x^2+y^2)^2}+4\,\frac{(x_0x+y_0y)^2}{(x^2+y^2)^2}\right\}$$

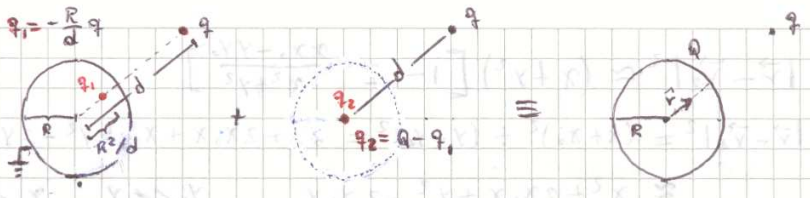
$$= \frac{\lambda}{4\pi\epsilon_0}\left[\frac{-4(\cancel{x^2x_0^2}-2xx_0yy_0+\cancel{y^2y_0^2})+4(\cancel{x^2x_0^2}+2xx_0yy_0+\cancel{y^2y_0^2})}{(x^2+y^2)^2}\right]$$

$$\boxed{\Phi_{asym} \simeq \frac{4\lambda}{\pi\epsilon_0}\,\frac{x_0x\,y_0y}{(x^2+y^2)^2}}$$

que corresponde a la contribución de cuadrupolo !

By: Herminso Villarraga Gómez

2.4



Problema 1 + Problema 2 es equivalente a

En el enunciado suponen $Q = q \rightarrow q_2 = q - q_1 = q + \frac{R}{d} q$

$q_2 = (d+R)q/d$

campo generado por las otras cargas.

$$\vec{F}_{\text{sobre } q} = q\vec{E} = \frac{q}{4\pi\epsilon_0}\left[\frac{q_1}{(d - R^2/d)^2} + \frac{q q_2}{d^2}\right]\hat{r} = \frac{q^2}{4\pi\epsilon_0}\left[\frac{-R/d}{(d - R^2/d)^2} + \frac{d+R}{d^3}\right]\hat{r}$$

$$\boxed{\vec{F}_q = \frac{q^2}{4\pi\epsilon_0}\left[\underbrace{\frac{-Rd}{(d^2-R^2)^2}}_{①} + \underbrace{\frac{d^2+Rd}{d^4}}_{②}\right]\hat{r}}$$

- Si $d \rightarrow R$ $|①| >> |②|$ y la fuerza es atractiva
- Si $d \rightarrow \infty$ $(d >> R) \rightarrow F \propto (-Rd + d^2 + Rd)/d^4$

  $F \sim 1/d^2$ y es repulsiva.

a) La fuerza se anula cuando $\frac{R\cancel{d}}{(d^2-R^2)^2} = \frac{d^2 + R\cancel{d}}{d^4}$

$\rightarrow Rd^4 = (d+R)(d^2-R^2)^2 = (d+R)(d^4 - 2d^2R^2 + R^4)$

$\cancel{R}d^4 = (d^5 - 2d^3R^2 + dR^4 + d^4R - 2d^2R^3 + \cancel{dR^4}) \rightarrow \boxed{d = 1.6178R}$

Numericamente

b) $F = \frac{q^2}{4\pi\epsilon_0}\left[\frac{-dR}{(d^2-R^2)^2} + \frac{d^2+dR}{d^4}\right]$ si $d = R + a$ tenemos:

$$F = \frac{q^2}{4\pi\epsilon_0}\left[\frac{-R(R+a)}{[(R+a)^2-R^2]^2} + \frac{(R+a)^2 + (R+a)R}{(R+a)^4}\right]$$

$$= \frac{q^2}{4\pi\epsilon_0}\left[\frac{-R^2 - aR}{[\cancel{R^2} + 2aR + a^2 - \cancel{R^2}]^2} + \frac{R^2 + 2aR + a^2 + R^2 + aR}{R^4 + 4aR^3 + 6R^2a^2 + 4Ra^3 + a^4}\right]$$

$$= \frac{q^2}{4\pi\epsilon_0}\left[\frac{-R^2 - aR}{4a^2R^2 + 4a^3R + a^4} + \frac{2R^2 + 3aR + a^2}{R^4 + 4aR^3 + 6R^2a^2 + 4Ra^3 + a^4}\right]$$

Si $R >> a \rightarrow a/R \rightarrow 0$ (q muy cerca a la esfera)

$\Rightarrow$

$$F = \frac{q^2}{4\pi\epsilon_0}\left[\frac{-1-a/R}{4a^2+4a^3/R+a^4/R} + \frac{2+3a/R+a^2/R^2}{R^2+4aR+6a^2+4a^3/R+a^4/R^2}\right]$$

$$\approx \frac{q^2}{4\pi\epsilon_0}\left[-\frac{1}{4a^2} + \frac{2}{R^2(1+4a/R+6a^2/R^2)}\right] \quad \text{si} \quad R \gg a$$

$$= \frac{q^2}{4\pi\epsilon_0}\left[-\frac{1}{4a^2} + \cancel{\frac{2}{R^2}}\right] \rightarrow 0 \qquad \frac{1}{R} \ll \frac{1}{a} \rightarrow \boxed{F \approx -\frac{q^2}{16\pi\epsilon_0 a^2}}$$

Es como si solo existiera la atracción de la carga imagen $q_1 = -\frac{R}{d}q$ cuando $d = R+a$ con $a \ll R$:

$$F_{qq_1} = -\frac{q^2}{4\pi\epsilon_0}\frac{R/(R+a)}{[R+a-R^2/(R+a)]} = -\frac{q^2}{4\pi\epsilon_0}\frac{R(R+a)}{[(R+a)^2-R^2]^2}$$

$$= -\frac{q^2}{4\pi\epsilon_0}\frac{R^2+aR}{(2aR+a^2)^2} = -\frac{q^2}{4\pi\epsilon_0}\frac{R^2+aR}{4a^2R^2+2a^3R+a^4}$$

$$= -\frac{q^2}{4\pi\epsilon_0}\frac{1+a/R}{4a^2+2a^3/R+a^4/R} \xrightarrow[R\gg a]{} \boxed{F \simeq -\frac{q^2}{4\pi\epsilon_0}\frac{1}{4a^2}}$$

Esto significa que la carga q solo siente el efecto de la fuerza de su imagen q_1 en la esfera cuando acercamos mucho la carga a la superficie de la esfera $d = R+a \approx R$ $(R \gg a)$, no importando en ese límite que la esfera este cargada (con Q).

c) Si la carga sobre la esfera es 2 veces la carga puntual q,

$$q_2 = 2q - q_1 = q(2+R/d)$$

$$\Rightarrow F = \frac{q^2}{4\pi\epsilon_0}\left[-\frac{dR}{(d^2-R^2)^2} + \frac{2d^2+dR}{d^4}\right] \quad \text{en este caso}$$

igualando a cero la fuerza se obtiene:

$$0 = 2d^5 - 4d^3R^2 - 2d^2R^3 + 2dR^4 + R^5$$

Solucionando analiticamente $\boxed{d = 1.43\,R}$ si $q \rightarrow 2q$

Si ahora, es la mitad:

$$F = \frac{q^2}{4\pi\epsilon_0}\left[-\frac{dR}{(d^2-R^2)^2} + \frac{d^2+2dR}{2d^4}\right]$$

$$F = 0 \rightarrow 0 = d^5 - 2d^3R^2 - 4d^2R^3 + dR^4 + 2R^5 \rightarrow \boxed{d = 1.88\,R}$$

si $q \rightarrow q/2$

By: Herminso Villarraga Gómez

2.5 $\vec{F} = -\dfrac{q^2 a}{4\pi\epsilon_0 r'^3}\dfrac{1}{(1-a^2/r'^2)^2}\hat{r}'$

a) $W = -\int_r^\infty \vec{F}\cdot d\vec{r} = \int_r^\infty |F(\vec{r}')|\,dr' = \dfrac{q^2 a}{4\pi\epsilon_0}\int_r^\infty \dfrac{dr'}{r'^3(1-a^2/r'^2)^2}$

$$= \frac{q^2 a}{4\pi\epsilon_0}\int_r^\infty \frac{r'\,dr'}{(r'^2-a^2)^2} \qquad u = r'^2 - a^2 \qquad du = 2r'\,dr'$$

$$= \frac{q^2 a}{4\pi\epsilon_0}\int \frac{du}{2u^2} = \frac{q^2 a}{8\pi\epsilon_0}\left(-\frac{1}{u}\right)_{r^2-a}^{\infty} \longrightarrow \boxed{W = \frac{q^2 a}{8\pi\epsilon_0}\frac{1}{(r^2-a^2)}}$$

$$E_{pot} = \frac{1}{8\pi\epsilon_0}\sum_{i\neq j}\frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|} = \frac{1}{8\pi\epsilon_0}\left\{\frac{q q_I}{|\vec{r}-\vec{r}_I|} + \frac{q_I q}{|\vec{r}_I - \vec{r}|}\right\} \qquad \begin{array}{l} q_I = -aq/|\vec{r}| \\ \vec{r}_I = a^2/|\vec{r}| \end{array}$$

$$= -\frac{q^2}{4\pi\epsilon_0}\frac{a/r}{|\vec{r}-(a^2/r)\hat{r}|} = -\frac{q^2 a}{4\pi\epsilon_0 r}\frac{1}{|(r-a^2/r)\hat{r}|}$$

$$\boxed{E_{pot} = -\frac{q^2 a}{4\pi\epsilon_0}\frac{1}{(r^2-a^2)}}$$

Nótese que $|W| > |E_p(r)|$

$E_p = W/2$.

$E_{pot} = W/2 \longrightarrow$ esto ocurre porque a medida que movemos la carga q desde el infinito, también se mueve su carga imagen q_i ejerciéndose el doble de esfuerzo que requeriría traer q cuando ya q_i se ha puesto en su posición final previamente (traer a q_i desde el infinito previamente en ausencia de otras cargas y por tanto de campo, no requiere trabajo)

b) $\vec{F} = \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{r^2}\left[Q - \dfrac{q a^3(2r^2-a^2)}{r(r^2-a^2)^2}\right]\hat{r}$

$$W = -\int_r^\infty \vec{F}\cdot d\vec{l} = -\int_r^\infty |\vec{F}|\,dr = \frac{-q}{4\pi\epsilon_0}\left[\int_r^\infty \frac{Q\,dr'}{r'^2} - \int_r^\infty \frac{q a^3(2r'^2-a^2)}{r'^3(r'^2-a^2)^2}\right]$$

$$= -\frac{q}{4\pi\epsilon_0}\left[\int_r^\infty \frac{Q\,dr'}{r'^2} - \int_r^\infty q\underbrace{\left\{\frac{a r'^4 - a r'^4 + 2a^3 r'^2 - a^5}{r'^3(r'^2-a^2)^2}\right\}}_{\frac{a r'^4 - a(r'^2-a^2)^2}{r'^3(r'^2-a^2)^2}}\right]$$

$$= -\frac{q}{4\pi\epsilon_0}\left[\int_r^\infty \frac{Q\,dr'}{r'^2} - q\int_r^\infty \underbrace{\frac{a r'\,dr'}{(r'^2-a^2)^2}} + q\int_r^\infty \frac{a\,dr'}{r'^3}\right]$$

$$= -\frac{q}{4\pi\epsilon_0}\left[-\frac{Q}{r'} + \frac{qa}{2}\frac{1}{(r^2-a^2)} - q\frac{a}{2r'^2}\right]_r^\infty$$

$$\boxed{W = \frac{q^2 a}{8\pi\epsilon_0(r^2-a^2)} - \frac{1}{4\pi\epsilon_0}\left[\frac{q^2 a}{2r^2} + \frac{qQ}{r}\right]}$$

$$W_i = \frac{q_i}{4\pi\epsilon_0}\sum_{j=1}^{n-1}\frac{q_j}{|\vec{r}_i-\vec{r}_j|} \qquad i\neq j \longrightarrow q_I = -\frac{a}{r}q \;.\; r_I = \frac{a^2}{r}$$

$$E_p \rightarrow W_q = \frac{q}{4\pi\epsilon_0}\left[\frac{q\,q_I}{|\vec{r}-\vec{r}_I|} + \frac{q(Q-q_I)}{|\vec{r}-\vec{0}|}\right] = \frac{q}{4\pi\epsilon_0}\left[\frac{-q^2a/r}{|(r-a^2/r)\hat{r}|} + \frac{qQ}{r} + \frac{aq^2/r}{r}\right]$$

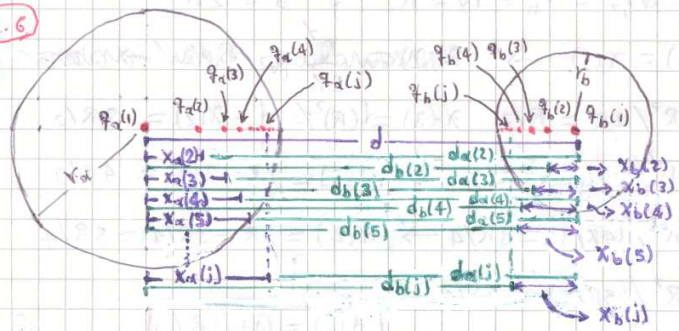
$$= \frac{q}{4\pi\epsilon_0}\left[-\frac{q^2a}{r(r-a^2/r)} + \frac{qQ}{r} + \frac{q^2a}{r^2}\right]$$

$$W_q = -\left[\frac{q^2a}{4\pi\epsilon_0(r^2-a^2)} - \frac{1}{4\pi\epsilon_0}\left(\frac{q^2a}{r^2} + \frac{qQ}{r}\right)\right]$$

Nótese que $|W_q| = |E_p| \neq W$ (trabajo) por las razones que se dieron en la parte a) para el primer y segundo término. Para el último no cambia pues Q es fijo en la esfera.

2.6

a)



$$x_a(2) = \frac{r_a^2}{d}$$

$$x_a(3) = r_a^2/d_b(2)$$

$$x_a(4) = r_a^2/d_b(3)$$

$$x_a(5) = r_a^2/d_b(4)$$

$$\vdots$$

$$\boxed{x_a(j) = r_a^2/d_b(j-1)}$$

donde $d_b(2) = d - x_b(2)$ con $x_b(2) = r_b^2/d$ con $d_a(2) = d - x_a(2)$

$d_b(3) = d - x_b(3)$ $\quad x_b(3) = r_b^2/d_a(2)$ $\quad d_a(3) = d - x_a(3)$

$d_b(4) = d - x_b(4)$ $\quad x_b(4) = r_b^2/d_a(3)$

$\vdots$

$$\boxed{d_b(j) = d - x_b(j)} \qquad \boxed{x_b(j) = r_b^2/d_a(j-1)} \qquad \boxed{d_a(j-1) = d - x(j-1)}$$

y $q_a(2) = -\frac{r_a}{d}q_b(1)$

$q_a(3) = -\frac{r_a}{d_b(2)}q_b(2)$ con $q_b(2) = -\frac{r_b}{d}q_a(1)$

$q_a(4) = -\frac{r_a}{d_b(3)}q_b(3)$ $\quad q_b(3) = -\frac{r_b}{d_a(2)}q_a(2)$

$\vdots$

$$\boxed{q_a(j) = -\frac{r_a}{d_b(j-1)}q_b(j-1)} \qquad \boxed{q_b(j-1) = -\frac{r_b}{d_a(j-2)}q_a(j-2)}$$

Si definimos $d_a(1) = d_b(1) = d$ y $x_a(1) = x_b(1) = 0$ se tienen las relaciones iterativas dadas por Jackson para $j = 2,3,4\ldots$

b) Nótese sin embargo, que las cargas $q_a(1)$ y $q_b(1)$ son aún desconocidas, pero pueden determinarse de:

$$\sum_{j=1}^{\infty} q_a(j) = Q_a \quad y \quad \sum_{j=2}^{\infty} q_b(j) = Q_b$$

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\sum_{j=1}^{n} \frac{q_a(j)}{|\vec{r} - x_a(j)\hat{x}|} + \sum_{j=1}^{n} \frac{q_b(j)}{|\vec{r} - d_b(j)\hat{x}|} \right) \rightarrow$$ Este potencial de satisfacer

Por construcción Φ es cte en las superficies $\Phi|_{S_1} = C_1$, $\Phi|_{S_2} = C_2$.

$$\boxed{F = \frac{1}{4\pi\epsilon_0} \sum_j \sum_k \frac{q_a(j)\, q_b(k)}{[d - x_a(j) - x_b(k)]^2}}$$

c) Si $Q_a = Q_b = Q$, $r_a = r_b = R$ y $d = 2R$

$\Rightarrow x_a(j) = x_b(j) = x(j)$

Con ello: $x(2) = R^2/2R = R/2 \longrightarrow d(2) = d - x(2) = 3R/2$

$x(3) = R^2/(3R/2) = 2R/3 \longrightarrow d(3) = 2R - x(3) = 4R/3$

$x(4) = R^2/(4R/3) = 3R/4 \longrightarrow d(4) = 2R - 3R/4 = 5R/4$

$x(5) = R^2/(5R/4) = 4R/5$

$$\boxed{d(j) = (j+1)R/j}$$

$$\boxed{x(j) = \frac{(j-1)}{j} R} \qquad j = 2, 3, 4, \ldots \quad y \quad x(1) = 0$$

Con ello $q(2) = -\frac{r_a}{d} q(1)$ si definimos $q(1) = q$

$q(2) = -\frac{R}{2R} q = -q/2$

$q(3) = -R\, q(2)/d(2) = -R(-q/2)/(3R/2) = q/3$

$q(4) = -R\, q(3)/d(3) = -R(q/3)/(4R/3) = -q/4$

$$\boxed{q(j) = (-1)^{j+1} \frac{q}{j}} \qquad j = 2, 3, 4, \ldots \qquad q(1) = q$$

de la condición: $\sum_{j=1}^{\infty} q(j) = q \underbrace{\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j}}_{\ln 2} = Q \longrightarrow \boxed{q = \frac{Q}{\ln 2}}$

$$F = \frac{1}{4\pi\epsilon_0} \sum_j \sum_k \frac{\left[(-1)^{j+1}\frac{q}{j}\right]\left[(-1)^{k+1}\frac{q}{k}\right]}{\left[2R - \frac{(j-1)}{j}R - \frac{(k-1)}{k}R\right]^2}$$

$$= \frac{1}{4\pi\epsilon_0} \sum_j \sum_k \frac{(-1)^{k+j}\, q^2/jk}{R^2\left[\frac{2jk-(j-1)k-(k-1)j}{jk}\right]^2} = \frac{q^2}{4\pi\epsilon_0 R^2} \sum_{j,k} \frac{(-1)^{j+k}/jk}{\left(\frac{k+j}{jk}\right)^2}$$

$$F = \frac{q^2}{4\pi\epsilon_0 R^2} \sum_j \sum_k \frac{(-1)^{j+k}\, jk}{(j+k)^2} = \frac{q^2}{4\pi\epsilon_0 R^2} \underbrace{(0.0739)}_{\text{Numericamente.}}$$

Como $q = Q/\ln 2$

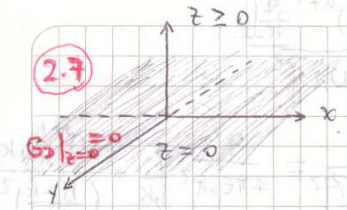
ó $$\boxed{F = \frac{Q^2}{4\pi\epsilon_0 R^2 (\ln 2)^2} \sum_j \sum_k \frac{(-1)^{j+k}\, jk}{(j+k)^2}} = \frac{Q^2}{4\pi\epsilon_0 R^2 (\ln 2)^2}(0.0739)$$

Si se compara este resultado con la fuerza que sentirían si las cargas Q estuviesen en los centros de las esferas:

$$F_c = \frac{Q^2}{4\pi\epsilon_0 (2R)^2} \Rightarrow F = \frac{4}{(\ln 2)^2}(0.0739)F_c = 0.615\, F_c$$

By: Herminso Villarraga Gómez

2.7



Recordemos que la funcion de Green es equivalente al potencial de una carga puntual de valor $4\pi\epsilon_0$ y su imagen elegidas de manera tal que se adecue a las condiciones de Dirichlet.

a) Para $\vec{r} = (x, y, z > 0)$, $\vec{r}' = (x', y', z' > 0)$ $\longrightarrow$ $\boxed{G(\vec{r},\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} - \frac{1}{|\vec{r}-\vec{r}_i|}}$

$\vec{r}_i$ debe ser la posición de la imagen $\rightarrow$ $\vec{r}_i = (x', y', -z')$

b) $$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \cancel{\rho(\vec{r}')}^{\,0}\, G_D(\vec{r},\vec{r}')\, dV - \frac{1}{4\pi}\oint \Phi(\vec{r}')\frac{\partial G_D}{\partial n'}\, da'$$

$\hat{n}' = -\hat{z}'$ pues queremos Φ en $z \geq 0$ y $\hat{n}'$ debe ser dirigido hacia afuera del volumen $z \geq 0$.

$$G(\vec{r},\vec{r}') = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}$$

$$\left.\frac{\partial G}{\partial n'}\right|_S = -\left.\frac{\partial G}{\partial z'}\right|_{z'=0} = -\left\{-\frac{1}{\cancel{2}}\frac{\cancel{2}(z-\cancel{z'})(-1)}{[(x-x')^2+(y-y')^2+(z-\cancel{z'})^2]^{3/2}} + \frac{1}{\cancel{2}}\frac{\cancel{2}(z+\cancel{z'})}{[\quad]^{3/2}}\right\}$$

$z' = 0$ $\quad z' = 0$ $\quad z' = 0$ $\quad z' = 0$

$$= -\left\{\frac{2z}{[(x-x')^2+(y-y')^2+z^2]^{3/2}}\right\}$$

$$\Rightarrow \Phi(\vec{r}) = -\frac{1}{4\pi}\oint_S \Phi(\vec{r}')\frac{-2z\, da'}{[(x-x')^2+(y-y')^2+z^2]^{3/2}}$$

$$= \frac{z}{2\pi}\oint_S \Phi(\vec{r}')\frac{\rho'\, d\rho'\, d\phi'}{[(\rho\cos\phi-\rho'\cos\phi')^2+(\rho\sin\phi-\rho'\sin\phi')^2+z^2]^{3/2}}$$

$$\Phi|_S = \Phi|_{z=0} = \begin{cases} V & \text{si} \quad 0 \leq \rho \leq a \\ 0 & \text{si} \quad \rho > a. \end{cases}$$

$$\Rightarrow \boxed{\Phi(\vec{r}) = \frac{zV}{2\pi}\int_0^{2\pi}\int_0^{a}\frac{\rho'\, d\rho'\, d\phi'}{[(\rho\cos\phi-\rho'\cos\phi')^2+(\rho\sin\phi-\rho'\sin\phi')^2+z^2]^{3/2}}}$$

c) si $\rho = 0$ $(x = 0, y = 0)$

$$\Phi(z) = \frac{zV}{2\pi}\int_0^{2\pi}\int_0^{a}\frac{\rho'\, d\rho'\, d\phi'}{[\underbrace{\rho'^2\cos^2\phi' + \rho'^2\sin^2\phi'}_{\rho'^2} + z^2]^{3/2}}$$

$u = \rho'^2 + z'^2$
$du = 2\rho' d\rho'$

$$\Phi(z) = \frac{zV}{\cancel{2\pi}}\,\cancel{(2\pi)}\int_0^a \frac{\rho' d\rho'}{(\rho'^2+z^2)^{3/2}} = zV\cdot\frac{1}{2}\left[\frac{1}{(-1/2)}(\rho'^2+z^2)^{-\frac{1}{2}}\right]_0^a$$

$$= -zV\left[\frac{1}{(a^2+z^2)^{1/2}} - \frac{1}{z}\right] \rightarrow \boxed{\Phi(z) = V\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right)}$$

d) Note que: $(\rho\cos\varphi - \rho'\cos\varphi')^2 + (\rho\sin\varphi - \rho'\sin\varphi')^2$

$$= \rho^2 + \rho'^2 - 2\rho\rho'\,(\underbrace{\cos\varphi\cos\varphi' + \sin\varphi\sin\varphi'}_{\cos(\varphi-\varphi')})$$

$$\Rightarrow \boxed{\Phi(\vec{r}) = \frac{zV}{2\pi}\int_0^{2\pi}\int_0^a \frac{\rho' d\rho' d\varphi'}{[\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')+z^2]^{3/2}}}$$

dada la simetría del problema respecto a φ, sea $\varphi = 0$

$$\Rightarrow \Phi(\vec{r}) = \frac{zV}{2\pi}\int_0^{2\pi}\int_0^a \frac{\rho' d\rho' d\varphi'}{[\rho^2+\rho'^2+z^2-2\rho\rho'\cos\varphi']^{3/2}} \qquad r^2 = \rho^2+z^2.$$

$$= \frac{zV}{2\pi r^3}\int_0^{2\pi}\int_0^a \frac{\rho' d\rho' d\varphi'}{\left[1+\frac{\rho'^2-2\rho\rho'\cos\varphi'}{r^2}\right]^{3/2}}$$

$$(1+x)^{-3/2} = 1 - \frac{3}{2}x + \frac{(-\frac{3}{2})(-\frac{5}{2})}{2!}x^2 + \cdots \qquad \text{si} \quad x \ll 1$$

$$= 1 - \frac{3}{2}x + \frac{15}{8}x^2 + \cdots$$

$$\left(1+\frac{\rho'^2-2\rho\rho'\cos\varphi'}{r^2}\right)^{-\frac{3}{2}} = \overset{①}{1} - \overset{②}{\frac{3}{2}\frac{\rho'^2}{r^2}} + \overset{③}{\frac{3\rho\rho'\cos\varphi'}{r^2}} + \overset{④}{\frac{15}{8}\frac{\rho'^4}{r^4}}$$
$$+ \underset{⑤}{\frac{15}{2}\frac{\rho^2\rho'^2\cos^2\varphi'}{r^4}} - \underset{⑥}{\frac{15}{2}\frac{\rho'^3\rho\cos\varphi'}{r^2}}$$

① $\displaystyle\int_0^{2\pi}\int_0^a \rho' d\rho' d\varphi' = \frac{a^2}{\cancel{2}}(\cancel{2}\pi) = \pi a^2$

② $\displaystyle\int_0^{2\pi}\int_0^a \frac{3}{2}\frac{\rho'^2}{r^2}(\rho' d\rho' d\varphi') = \frac{3}{\cancel{2}r^2}\frac{a^4}{4}(\cancel{2}\pi) = \frac{3}{4}\pi\frac{a^4}{r^2}$

③ $\displaystyle\int_0^{2\pi}\int_0^a \frac{3\rho}{r^2}\rho'\cos\varphi'(\rho' d\rho' d\varphi') = \frac{3\rho}{r^2}\cancel{\sin\varphi\Big|_0^{2\pi}}^{\,0}\frac{a^3}{3} = 0$

④ $\displaystyle\int_0^{2\pi}\int_0^a \frac{15}{8}\frac{\rho^2}{r^4}\rho'^4(\rho' d\rho' d\varphi') = \frac{15}{8r^4}(2\pi)\frac{a^6}{6} = \frac{5}{8}\frac{a^6}{r^4}\pi$

⑤ $$\int_0^{2\pi}\int_0^{a} \frac{15}{2}\frac{\rho^2}{r^4}\rho'^2 \overbrace{\frac{(1+\cos 2\varphi)}{2}}^{\cos^2\varphi} (\rho' d\rho' d\varphi') = \frac{15}{4}\frac{\rho^2}{r^4}\frac{a^2}{4}(2\pi)$$

⑥ $$\int_0^{2\pi} \cos\varphi' d\varphi' = 0 \rightarrow \int_0^{2\pi}\int_0^{a} \frac{15}{2}\frac{\rho}{r^2}\rho'^3 \cos\varphi' (\rho' d\rho' d\varphi') = 0$$

$$\Rightarrow \Phi(\vec{r}) = \frac{zV}{2\pi r^3}\left((2\pi)\frac{a^2}{2} - \frac{3}{8}(2\pi)\frac{a^4}{r^2} + \frac{5}{16}(2\pi)\frac{a^6}{r^4} + \frac{15}{16}(2\pi)\frac{\rho^2 a^4}{r^4} + \dots\right)$$

$$\boxed{\Phi(z) = \frac{zV}{2(\rho^2+z^2)^{3/2}}\left[1 - \frac{3}{4}\frac{a^2}{(\rho^2+z^2)} + \frac{5}{8}\frac{a^4+3\rho^2a^2}{(\rho^2+z^2)^2} + \dots\right]}$$

si: $\rho^2+z^2 >> a^2$

Si $\rho = 0 \rightarrow \Phi(z) = \frac{zVa^2}{2z^3}\left(1 - \frac{3}{4}\frac{a^2}{z^2} + \frac{5}{8}\frac{a^4}{z^4} + \dots\right)$

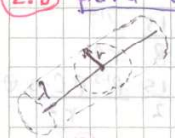
y usando en c) $\left(\frac{a^2}{z^2}+1\right)^{-1/2} = \left(1 - \frac{1}{2}\frac{a^2}{z^2} + \frac{3}{8}\frac{a^4}{z^4} - \frac{5}{16}\frac{a^6}{z^6} + \dots\right)$

tenemos $\Phi(z) = V\left(1 - \frac{z}{z(a^2/z^2-1)^{1/2}}\right) = V\left(\frac{1}{2}\frac{a^2}{z^2} + \frac{3}{8}\frac{a^4}{z^4} - \frac{5}{8}\frac{a^6}{z^6} + \dots\right)$

$$= \frac{Va^2}{2z^2}\left(1 - \frac{3}{4}\frac{a^2}{z^2} + \frac{5}{8}\frac{a^4}{z^4} + \dots\right)$$

Coincidiendo con $\Phi(z)$ hallado en d) para $z >> a$.

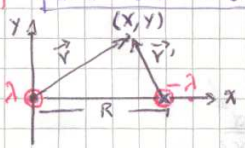
2.8 para una linea de carga:



$$\int \vec{E}\cdot\vec{n}\,da = \frac{Q_{int}}{\epsilon_0} \rightarrow E(2\pi r \cancel{l}) = \frac{\lambda \cancel{l}}{\epsilon_0} \rightarrow \vec{E} = \frac{\lambda}{2\pi\epsilon_0 r}\hat{r}$$

$$\varphi = -\int \vec{E}\cdot d\vec{r} = -\frac{\lambda}{2\pi\epsilon_0}\ln r$$

a) para dos lineas paralelas:



$$\Phi(\vec{r}) = -\frac{\lambda}{2\pi\epsilon_0}\ln r + \frac{\lambda}{2\pi\epsilon_0}\ln r' = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{r'}{r}$$

La superficie equipotencial es tal que:

$$\Phi = V = cte \rightarrow \ln\frac{r'}{r} = \frac{2\pi\epsilon_0 V}{\lambda} \rightarrow \frac{r'}{r} = e^{2\pi\epsilon_0 V/\lambda}$$

$$r'^2 = r^2 e^{4\pi\epsilon_0 V/\lambda}$$ donde $r'^2 = (x-R)^2 + y^2$, $r^2 = x^2+y^2$

$$\Rightarrow (x-R)^2 + y^2 = (x^2+y^2)e^{4\pi\epsilon_0 V/\lambda} = (x^2+y^2)K \quad \text{si: } K = e^{4\pi\epsilon_0 V/\lambda}$$

$$\rightarrow \quad x^2 + R^2 - 2Rx + y^2 - Kx^2 - Ky^2 = 0$$

$$R^2 - 2Rx + (1-K)x^2 + (1-K)y^2 = 0$$

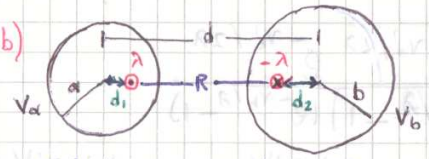
$$\frac{R^2}{(1-K)} - \frac{2Rx}{(1-K)} + x^2 + y^2 = \frac{(1-K)R^2}{(1-K)^2} - \frac{2Rx}{(1-K)} + x^2 + y^2 = 0$$

$$\frac{R^2}{(1-K)^2} - \frac{KR^2}{(1-K)^2} - \frac{2Rx}{(1-K)} + x^2 + y^2 = \left[x - \frac{R}{(1-K)}\right]^2 + y^2 - \frac{KR^2}{(1-K)^2} = 0$$

$$\Rightarrow \boxed{\left(x - \frac{R}{(1-K)}\right)^2 + y^2 = \frac{KR^2}{(1-K)^2}} \rightarrow$$ Circunferencia en el plano xy (cilindro circular en xyz)

Centro $= (h, k) = \left(\frac{R}{1-K}, 0\right)$ radio $= \frac{\sqrt{K}\,R}{|1-K|}$ con $K = e^{4\pi\epsilon_0 V/\lambda}$

b)



Cilindros equipotenciales.

Dos cilindros a potencial constante pueden pensarse como dos superficies equipotenciales diferentes del problema anterior (parte a), con

$$a = \frac{R\sqrt{K_1}}{(K_1 - 1)} \qquad b = \frac{R\sqrt{K_2}}{(K_2 - 1)}$$

con $K_1 = e^{4\pi\epsilon_0 V_a/\lambda}$
$K_2 = e^{4\pi\epsilon_0 V_b/(-\lambda)}$

De la geometría de arriba:

$$\begin{cases} d = R + d_1 + d_2 & \text{con} \\ d_1 = \dfrac{R}{K_1 - 1} \quad \text{y} \quad d_2 = \dfrac{R}{K_2 - 1} \end{cases}$$

Sea $V_a' = 4\pi\epsilon_0 V_a$
$V_b' = 4\pi\epsilon_0 V_b$ $\Rightarrow$ $d_1 = \dfrac{R}{e^{V_a'/\lambda} - 1}$ y $d_2 = \dfrac{R}{e^{-V_b'/\lambda} - 1}$

$$a = \frac{R e^{V_a'/2\lambda}}{e^{V_a'/\lambda} - 1} \quad \text{y} \quad b = \frac{R e^{-V_b'/2\lambda}}{e^{-V_b'/\lambda} - 1}$$

$$d^2 = (R + d_1 + d_2)^2 = R^2 + d_1^2 + d_2^2 + 2d_1 d_2 + 2Rd_1 + 2Rd_2$$

$$\Rightarrow d^2 - a^2 - b^2 = R^2 + \frac{R^2}{(e^{V_a'/\lambda} - 1)^2} + \frac{R^2}{(e^{-V_b'/\lambda} - 1)^2} + \frac{2R^2}{(e^{V_a'/\lambda} - 1)(e^{-V_b'/\lambda} - 1)} + \frac{2R^2}{(e^{V_a'/\lambda} - 1)}$$

$$+ \frac{2R^2}{(e^{-V_b'/\lambda} - 1)} - \frac{R^2 e^{V_a'/\lambda}}{(e^{V_a'/\lambda} - 1)} - \frac{R^2 e^{-V_b'/\lambda}}{(e^{-V_b'/\lambda} - 1)}$$

$$d^2-a^2-b^2 = R^2 - \frac{R^2(\cancel{e^{v_a'/\lambda}-1})}{(e^{v_a'/\lambda}-1)^2} - \frac{R^2(\cancel{e^{-v_b'/\lambda}-1})}{(e^{-v_b'/\lambda}-1)^2} + \frac{2R^2}{(e^{v_a'/\lambda}-1)(e^{v_b'/\lambda}-1)}$$

$$+ \frac{2R^2}{(e^{v_a'/\lambda}-1)} + \frac{2R^2}{(e^{-v_b'/\lambda}-1)}$$

$$= R^2 + \frac{R^2}{(e^{v_a'/\lambda}-1)} + \frac{R^2}{(e^{-v_b'/\lambda}-1)} + \frac{2R^2}{(e^{v_a'/\lambda}-1)(e^{-v_b'/\lambda}-1)}$$

$$= \frac{R^2(e^{v_a'/\lambda}e^{-v_b'/\lambda} - \cancel{e^{v_a'/\lambda}} - \cancel{e^{-v_b'/\lambda}} + 1)}{(e^{v_a'/\lambda}-1)(e^{-v_b'/\lambda}-1)} + \frac{R^2(\cancel{e^{-v_b'/\lambda}} \cancel{-2+e^{v_a'/\lambda}}) \cancel{+2R^2}}{(e^{v_a'/\lambda}-1)(e^{-v_b'/\lambda}-1)}$$

$$= \frac{R^2(e^{(v_a'-v_b')/\lambda}+1)}{(e^{v_a'/\lambda}-1)(e^{-v_b'/\lambda}-1)}$$

Note que: $2ab = 2\,\dfrac{R^2 e^{v_a'/2\lambda}\, e^{-v_b'/2\lambda}}{(e^{v_a'/\lambda}-1)(e^{-v_b'/\lambda}-1)}$

$$\Rightarrow \frac{d^2-a^2-b^2}{2ab} = \frac{e^{(v_a'-v_b')/\lambda}+1}{2e^{v_a'/2\lambda}e^{-v_b'/2\lambda}} = \frac{e^{(v_a'-v_b')/\lambda}+1}{2}\, e^{-(v_a'-v_b')/2\lambda}$$

$$= \frac{e^{(v_a'-v_b')/2\lambda}+e^{-(v_a'-v_b')/2\lambda}}{2} = \cosh\left(\frac{v_a'-v_b'}{2\lambda}\right)$$

$$\Rightarrow \frac{v_a'-v_b'}{2\lambda} = \cosh^{-1}\left(\frac{d^2-a^2-b^2}{2ab}\right) = \frac{4\pi\epsilon_0(v_a-v_b)}{2\lambda}$$

$$\Rightarrow (v_a-v_b) = \frac{\lambda}{2\pi\epsilon_0}\cosh^{-1}\left(\frac{d^2-a^2-b^2}{2ab}\right)$$

$$\frac{C}{L} = \frac{\lambda}{(v_a-v_b)} \longrightarrow \boxed{\frac{C}{L} = 2\pi\epsilon_0 \Big/ \cosh^{-1}\left(\frac{d^2-a^2-b^2}{2ab}\right)}$$

c) $$\frac{2\pi\epsilon_0 L}{C} = \cosh^{-1}\left[\frac{d^2[1-(a^2+b^2)/d^2]}{2ab}\right]$$

$$\Rightarrow \frac{d^2[1-(a^2+b^2)/d^2]}{2ab} = \cosh\frac{2\pi\epsilon_0 L}{C} = \frac{e^{\frac{2\pi\epsilon_0 L}{C}} + \cancel{e^{-\frac{2\pi\epsilon_0 L}{C}}}}{2}$$

~ 0 $\quad \frac{2\pi\epsilon_0 L}{C} \gg 1$

$$\Rightarrow \frac{d^2[1-(a^2+b^2)/d^2]}{ab} = \frac{2\pi\epsilon_0 L}{C} \Rightarrow$$

$$\frac{C}{L} = 2\pi\epsilon_0 \Big/ \ln\left[\frac{d^2[1-(a^2+b^2)/d^2]}{ab}\right] \approx 2\pi\epsilon_0 \Big/ \ln\frac{d^2}{ab}$$

$$\frac{C}{L} = \frac{\pi\epsilon_0}{\frac{1}{2}\ln\frac{d^2}{ab}} = \frac{\pi\epsilon_0}{\ln\left(\frac{d^2}{ab}\right)^{1/2}} \longrightarrow \boxed{\frac{C}{L} = \frac{\pi\epsilon_0}{\ln(d/\sqrt{ab})}}$$

tal como en el problema 1.7

d) Si: $a^2+b^2 > d^2 \rightarrow \frac{C}{L} \simeq 2\pi\epsilon_0 \Big/ \cosh^{-1}\left(\frac{a^2+b^2}{2ab}\right)$

Usemos $\boxed{\underbrace{\cosh^{-1}x}_{y} = \underbrace{\ln(x+\sqrt{x^2-1})}_{y}}$, demostremos esta relación:

$$\cosh y = x \qquad e^y = x+\sqrt{x^2-1} \rightarrow e^{-y} = \frac{1}{x+\sqrt{x^2-1}}$$

$$\frac{e^y-e^{-y}}{2} = x \Rightarrow \underbrace{e^y-e^{-y}}_{2x} = x+\sqrt{x^2-1} + \frac{1}{x+\sqrt{x^2-1}}$$

$$x = \frac{\sqrt{x^2-1}\,(x+\sqrt{x^2-1})+1}{x+\sqrt{x^2-1}}$$

$$= \frac{x\sqrt{x^2-1}+x^2-\cancel{1}+\cancel{1}}{x+\sqrt{x^2-1}}$$

$$\boxed{x=x} \longleftarrow x = \frac{x\,\cancel{(\sqrt{x^2-1}+x)}}{\cancel{x+\sqrt{x^2-1}}}$$

Esto es verdadero, luego la igualdad de arriba es cierta.

Usando ello: $\cos^{-1}\left(\frac{a^2+b^2}{2ab}\right) = \ln\left[\frac{a^2+b^2}{2ab} + \sqrt{\left(\frac{a^2+b^2}{2ab}\right)^2-1}\right]$

$$= \ln\left[\sqrt{\frac{a^4+2a^2b^2+b^4}{4a^2b^2} - \frac{4a^2b^2}{4a^2b^2}} + \frac{a^2+b^2}{2ab}\right]$$

$$= \ln\left[\frac{\sqrt{a^4+2a^2b^2+b^4-4a^2b^2}}{2ab} + \frac{a^2+b^2}{2ab}\right]$$

$$= \ln\left[\frac{(a^2-\cancel{b^2})}{2ab} + \frac{a^2+\cancel{b^2}}{2ab}\right] = \ln\left(\frac{a}{b}\right)$$

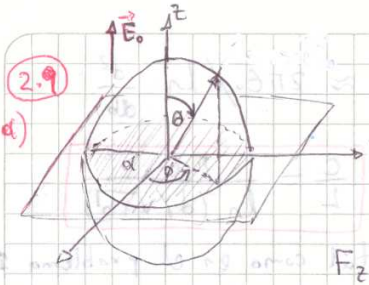
$$\Rightarrow \boxed{\frac{C}{L} = \frac{2\pi\epsilon_0}{\ln(a/b)}}$$

tal como se espera para dos cilindros concentricos.

By: Herminso Villarraga Gómez

2.9

a)



$$\sigma = 3\epsilon_0 E_0 \cos\theta \rightarrow \text{Ec. } (2.15)$$

$$dF = \frac{\sigma^2}{2\epsilon_0} da \rightarrow \text{Pg } 43 \text{ Jackson.}$$

$$dF_z = \frac{\sigma^2}{2\epsilon_0}\cos\theta\, da = \frac{9\epsilon_0^{\cancel{2}} E_0^2 \cos^3\theta\, da}{2\cancel{\epsilon_0}}$$

$$F_z = \int_0^{2\pi}\int_0^{\pi/2} \frac{9\epsilon_0 E_0^2}{2}\cos^3\theta \sin\theta\, d\theta \cdot a^2 d\phi$$

$$= \frac{9\epsilon_0 E_0^2}{\cancel{2}}(\cancel{2}\pi)\underbrace{\left(-\frac{\cos^4\theta}{4}\Big|_0^{\pi/2}\right)}_{1/4} a^2$$

$$\boxed{F_z = \frac{9}{4}\pi\epsilon_0 E_0^2 a^2}$$

b) $\sigma = 3\epsilon_0 E_0 \cos\theta + \dfrac{Q}{4\pi a^2}$

$$\frac{dF_z}{da} = \frac{\sigma^2}{2\epsilon_0}\cos\theta = \underbrace{\frac{9\epsilon_0^{\cancel{2}} E_0^2 \cos^3\theta}{2\cancel{\epsilon_0}}}_{①} + \underbrace{\frac{6\epsilon_0 E_0 \cos^2\theta}{2\epsilon_0 4\pi}\frac{Q}{a^2}}_{②} + \underbrace{\frac{Q^2\cos\theta}{16\pi^2 a^4 (2\epsilon_0)}}_{③}$$

① $\displaystyle\int_0^{2\pi}\int_0^{\pi/2}\cos^3\theta\sin\theta\, d\theta\, d\phi = (2\pi)\left(-\frac{\cos^4\theta}{4}\Big|_0^{\pi/2}\right) = \frac{\pi}{2}$

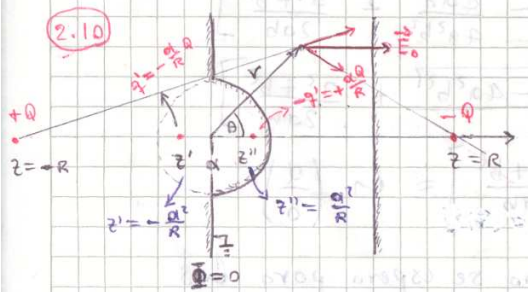
② $\displaystyle\int_0^{2\pi}\int_0^{\pi/2}\cos^2\theta\sin\theta\, d\theta\, d\phi = (2\pi)\left(-\frac{\cos^3\theta}{3}\Big|_0^{\pi/2}\right) = \frac{2\pi}{3}$

③ $\displaystyle\int_0^{2\pi}\int_0^{\pi/2}\cos\theta\sin\theta\, d\theta\, d\phi = (2\pi)\left(-\frac{\cos^2\theta}{2}\Big|_0^{\pi/2}\right) = \pi$

$$\Rightarrow F_z = \int_0^{2\pi}\int_0^{\pi/2}\frac{\sigma^2}{2\epsilon_0}\cos\theta\left(\sin\theta\, d\theta\, a^2 d\phi\right)$$

$$\boxed{F_z = \frac{9}{4}\pi\epsilon_0 E_0^2 a^2 + \frac{1}{2}QE_0 + \frac{1}{32}\frac{Q^2}{a^2\epsilon_0\pi}}$$

2.10



El campo uniforme $\vec{E}_0$ puede pensarse como generado por dos cargas puntuales en $z = \pm R$ con $R >> a$.

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{(r^2+R^2+2rR\cos\theta)^{1/2}} - \frac{Q}{(r^2+R^2-2rR\cos\theta)^{1/2}} - \frac{aQ/R}{\left(a^4/R^2+r^2+\frac{2a^2r}{R}\cos\theta\right)^{1/2}} + \frac{aQ/R}{\left(a^4/R^2+r^2-\frac{2a^2r}{R}\cos\theta\right)^{1/2}}\right]$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots \longrightarrow (1+x)^{-1/2} \simeq 1 - \frac{1}{2}x$$

$$\Rightarrow \Phi(\vec{r}) = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{R\left(\cancel{\frac{r^2}{R^2}} + 1 + \frac{2r}{R}\cos\theta\right)^{1/2}} - \frac{1}{R\left(\cancel{\frac{r^2}{R^2}} + 1 - 2\frac{r}{R}\cos\theta\right)^{1/2}}\right.$$

≈ 0 $\qquad$ ≈ 0

$R >> r, a.$

$$\left. - \frac{a}{rR}\frac{1}{\left(\cancel{\frac{a^4}{R^2r^2}} + 1 + \frac{2a^2}{Rr}\cos\theta\right)^{1/2}} + \frac{a}{Rr}\frac{1}{\left(\cancel{\frac{a^4}{R^2r^2}} + 1 - \frac{2a^2}{Rr}\cos\theta\right)^{1/2}}\right]$$

≈ 0 $\qquad$ ≈ 0

$$\Phi(\vec{r}) \simeq \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{R}\left\{\cancel{1} - \frac{r}{R}\cos\theta - \left(\cancel{1} + \frac{r}{R}\cos\theta\right)\right\} + \frac{a}{Rr}\left\{-\left(\cancel{1} - \frac{a^2}{Rr}\cos\theta\right) + \cancel{1} + \frac{a^2}{Rr}\cos\theta\right\}\right]$$

$$= \frac{Q}{4\pi\epsilon_0}\left[-\frac{2r}{R^2}\cos\theta + \frac{2a^3}{R^2r^2}\cos\theta\right] = -\frac{2Q}{4\pi\epsilon_0 R^2}\left[r - \frac{a^3}{r^2}\right]\cos\theta$$

Si $r >> a \longrightarrow \Phi(\vec{r}) \longrightarrow -\frac{2Q}{4\pi\epsilon_0 R^2}\underbrace{r\cos\theta}_{z} = -E_0 z \Rightarrow \boxed{E_0 = \frac{2Q}{4\pi\epsilon_0 R^2}}$

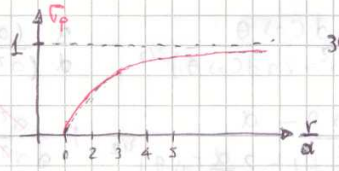
$$\Rightarrow \boxed{\Phi(\vec{r}) = -E_0\left(r - \frac{a^3}{r^2}\right)\cos\theta} \longrightarrow \text{potencial entre las placas}$$

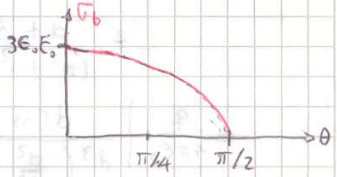
a) $\sigma_b = -\epsilon_0 \left.\frac{\partial\Phi}{\partial r}\right|_{r=a} = +\epsilon_0 E_0\left(1 + 2\frac{a^3}{r^3}\right)\cos\theta_{r=a} \longrightarrow \boxed{\sigma_b = 3\epsilon_0 E_0\cos\theta}$

$0 \le \theta \le \pi/2$

$$\sigma_p = -\epsilon_0 E_0\Big|_{\theta=\pi/2} = -\epsilon_0\frac{1}{r}\left.\frac{\partial\Phi}{\partial\theta}\right|_{\theta=\pi/2} = \epsilon_0 E_0\left(1 - \frac{a^3}{r^3}\right)\sin\theta\Big|_{\theta=\pi/2}$$

$$\boxed{\sigma_p = \epsilon_0 E\left(1 - \frac{a^3}{r^3}\right)}$$

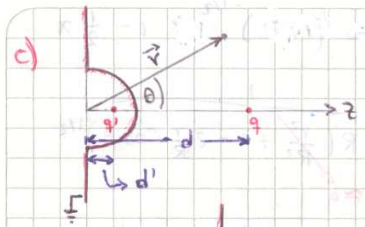




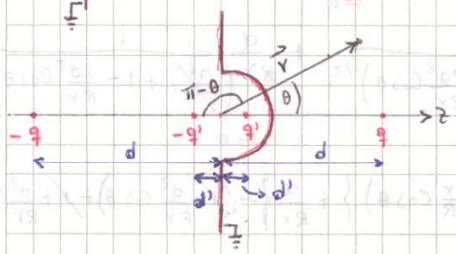
b) $Q_b = \int \sigma_b\, da = 3\epsilon_0 E_0 \int_0^{2\pi}\int_0^{\pi/2} \cos\theta\,(\sin\theta\, d\theta\, d\phi\, a^2)$

$$= 3\epsilon_0 E_0 (\cancel{2}\pi)\, a^2 \cancel{\frac{\sin^2\theta}{2}}^{\,1}\Big|_0^{\pi/2} \longrightarrow \boxed{Q_b = 3\pi\epsilon_0 E_0 a^2}$$

c)



Note que q' y q anulan el potencial sobre la protuberancia semi-esférica, pero no lo anulan sobre el plano $z=0$, para que ello ocurra ($\Phi(z=0)=0$) se necesitan un par de cargas opuestas $-q$ y $-q''$ en posiciones simétricas respecto a $z=0$.



$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\vec{r}-d\vec{k}|} + \frac{q'}{|\vec{r}-d'\vec{k}|} - \frac{q}{|\vec{r}+d\vec{k}|} - \frac{q'}{|\vec{r}+d'\vec{k}|}\right]$$

$$q' = -q\,a/d$$
$$d' = a^2/d$$

$\cos(\pi-\theta) = -\cos\theta$

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{\sqrt{r^2+d^2-2dr\cos\theta}} - \frac{q\,a/d}{\sqrt{r^2+(a^4/d^2)-(2a^2/d)\cos\theta}} - \frac{q}{\sqrt{r^2+d^2+2dr\cos\theta}} + \frac{q\,a/d}{\sqrt{r^2+(a^4/d^2)+(2a^2/d)\cos\theta}}\right\}$$

$$\frac{\partial\Phi}{\partial r} = \frac{q}{4\pi\epsilon_0}\left\{-\frac{1}{\cancel{2}}\frac{\cancel{2}r-\cancel{2}d\cos\theta}{(r^2+d^2-2dr\cos\theta)^{3/2}} + \frac{1}{\cancel{2}}\frac{a}{d}\frac{(\cancel{2}r-\cancel{2}(a^2/d)\cos\theta)}{(r^2+(a^4/d^2)-2(a^2/d)r\cos\theta)^{3/2}} + \frac{1}{\cancel{2}}\frac{\cancel{2}r+\cancel{2}d\cos\theta}{(r^2+d^2+2dr\cos\theta)^{3/2}} - \frac{1}{\cancel{2}}\frac{a}{d}\frac{(\cancel{2}r+\cancel{2}(a^2/d)\cos\theta)}{(r^2+(a^4/d^2)+2(a^2/d)r\cos\theta)^{3/2}}\right\}$$

$$\left.\frac{\partial\Phi}{\partial r}\right|_{r=a} = \frac{q}{4\pi\epsilon_0}\left\{-\frac{a-d\cos\theta}{(a^2+d^2-2ad\cos\theta)^{3/2}} + \frac{a}{d}\frac{(a-(a^2/d)\cos\theta)}{(a^2+(a^4/d^2)-2(a^3/d)\cos\theta)^{3/2}} + \frac{a+d\cos\theta}{(a^2+d^2+2ad\cos\theta)^{3/2}} - \frac{a}{d}\frac{(a+(a^2/d)\cos\theta)}{(a^2+(a^4/d^2)+2(a^3/d)\cos\theta)^{3/2}}\right\}$$

$$= \frac{q}{4\pi\epsilon_0}\left\{\frac{d\cos\theta-a}{d^3\left(\frac{a^2}{d^2}+1-2\frac{a}{d}\cos\theta\right)^{3/2}} + \frac{\cancel{a^2}}{d\cancel{a^3}}\frac{(1-(a/d)\cos\theta)}{\left(1+\frac{a^2}{d^2}-2\frac{a}{d}\cos\theta\right)^{3/2}} + \frac{d\cos\theta+a}{d^3\left(\frac{a^2}{d^2}+1+2\frac{a}{d}\cos\theta\right)^{3/2}} - \frac{\cancel{a^2}}{d\cancel{a^3}}\frac{(1+(a/d)\cos\theta)}{\left(1+\frac{a^2}{d^2}+2\frac{a}{d}\cos\theta\right)^{3/2}}\right\}$$

$$= \frac{q}{4\pi\epsilon_0}\left\{\frac{\cancel{(a/d)\cos\theta}-(a^2/d^2)}{ad\left(1+\frac{a^2}{d^2}-2\frac{a}{d}\cos\theta\right)^{3/2}} + \frac{1-\cancel{(a/d)\cos\theta}}{ad\left(1+\frac{a^2}{d^2}-2\frac{a}{d}\cos\theta\right)^{3/2}} + \frac{\cancel{(a/d)\cos\theta}+(a^2/d^2)}{ad\left(1+\frac{a^2}{d^2}+2\frac{a}{d}\cos\theta\right)^{3/2}} - \frac{1+\cancel{(a/d)\cos\theta}}{ad\left(1+\frac{a^2}{d^2}+2\frac{a}{d}\cos\theta\right)^{3/2}}\right\}$$

$$\Rightarrow \sigma = -\epsilon_0 \frac{\partial \Phi}{\partial r}\Big|_{r=a} = -\frac{q}{4\pi a d}\left(1-\frac{a^2}{d^2}\right)\left[\frac{1}{\left(1+\frac{a^2}{d^2}-2\frac{a}{d}\cos\theta\right)^{3/2}} - \frac{1}{\left(1+\frac{a^2}{d^2}+2\frac{a}{d}\cos\theta\right)^{3/2}}\right]$$

$$Q = \int_0^{2\pi}\int_0^{\pi/2} \sigma a^2 \sin\theta \, d\theta \, d\phi = (2\pi)a^2 \int_1^0 \sigma(-d\cos\theta)$$

$$= 2\pi a^2 \int_0^1 \sigma \, d\cos\theta$$

$$= -\frac{q}{4\pi a d}\left(1-\frac{a^2}{d^2}\right) 2\pi a^2 \int_0^1 \left[\frac{1}{\left(1+\frac{a^2}{d^2}-2\frac{a}{d}\cos\theta\right)^{3/2}} - \frac{1}{\left(1+\frac{a^2}{d^2}+2\frac{a}{d}\cos\theta\right)^{3/2}}\right] \cdot d\cos\theta$$

$$= \frac{q}{4}\left(1-\frac{a^2}{d^2}\right)\left[\int_{1+\frac{a^2}{d^2}}^{\left(1-\frac{a}{d}\right)^2} \frac{du}{u^{3/2}} + \int_{1+\frac{a^2}{d^2}}^{\left(1+\frac{a}{d}\right)^2} \frac{du}{u^{3/2}}\right]$$

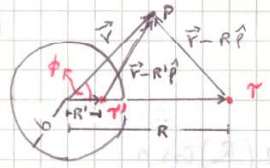
$$= -\frac{q}{2}\left(1-\frac{a^2}{d^2}\right)\left[\frac{1}{u^{1/2}}\Big|_{1+a^2/d^2}^{(1-a/d)^2} + \frac{1}{u^{1/2}}\Big|_{1+a^2/d^2}^{(1+a/d)^2}\right]$$

$$= -\frac{q}{2}\left(1+\frac{a}{d}\right)\left(1-\frac{a}{d}\right)\left[\frac{1}{1-a/d} + \frac{1}{1+a/d} - \frac{1}{(1+a^2/d^2)^{1/2}} - \frac{1}{(1+a^2/d^2)^{1/2}}\right]$$

$$= -\frac{q}{2}\left(1+\cancel{\frac{a}{d}}+1-\cancel{\frac{a}{d}} - 2\frac{(1-a^2/d^2)}{\sqrt{1-a^2/d^2}}\right)$$

$$\boxed{Q = -q\left(1-\frac{d^2-a^2}{d\sqrt{d^2-a^2}}\right)}$$

2.11



Vista desde arriba

potencial de una línea de carga.

$$\Phi(\vec{r}) = -\frac{\tau}{2\pi\epsilon_0}\ln|\vec{r}-R\hat{\rho}| - \frac{\tau'}{2\pi\epsilon_0}\ln|\vec{r}-R'\hat{\rho}|$$

$$= -\frac{\tau}{4\pi\epsilon_0}\ln|\vec{r}-R\hat{\rho}|^2 - \frac{\tau'}{4\pi\epsilon_0}\ln|\vec{r}-R'\hat{\rho}|^2$$

$$\Phi(\vec{r}) = -\frac{1}{4\pi\epsilon_0}\left[\tau\ln(r^2+R^2-2rR\cos\phi) + \tau'\ln(r^2+R'^2-2rR'\cos\phi)\right]$$

$$\Phi(r=b) = -\frac{1}{4\pi\epsilon_0}\left\{\tau\ln(b^2+R^2-2Rb\cos\phi) + \tau'\ln(b^2+R'^2-2R'b\cos\phi)\right\} = cte$$

Suponga que: $\cancel{\tau'/\tau = \gamma} \rightarrow \cancel{\dfrac{b^2+R^2-2Rb\cos\phi}{b^2+R'^2-2R'b\cos\phi} = K}$

$\cancel{\rightarrow b^2+R^2-2Rb\cos\phi = Kb^2+KR'^2-2KR'b\cos\phi}$

Si se hace: $\cancel{b^2+R^2 = Kb^2+KR'^2}$

$\cancel{Rb = KR'b}$

$$\Phi(r=b) = -\frac{\tau}{4\pi\epsilon_0}\left\{ \ln\left(b^2+R^2-2Rb\cos\phi\right) + \frac{\tau'}{\tau}\ln\left(b^2+R'^2-2R'b\cos\phi\right)\right\} = cte$$

para que $\Phi(r=b) = cte$ independiente de ϕ la mejor elección es:

$$\boxed{\tau' = -\tau} \quad y \quad \frac{b^2+R^2-2Rb\cos\phi}{b^2+R'^2-2R'b\cos\phi} = K = Kte.$$

$$\Rightarrow (b^2+R^2) - 2Rb\cos\phi = K(b^2+R'^2) - 2KR'b\cos\phi$$

$$\Rightarrow \left.\begin{aligned} b^2+R^2 &= K(b^2+R'^2) \\ R\cancel{b} &= KR'\cancel{b} \end{aligned}\right\} \Rightarrow \begin{aligned} b^2+R^2 &= \frac{R}{R'}(b^2+R'^2) \\ b^2+R^2 &= \frac{Rb^2}{R'} + RR' \end{aligned}$$

La igualdad anterior se cumple si $R' = R$ ó $R' = b^2/R$, La primera posibilidad no sirve pues $R' < b$, luego solo queda la segunda

$$\boxed{R' = b^2/R}$$

b) $$\boxed{\tau(r,\phi) = \frac{\tau}{4\pi\epsilon_0}\ln\left[\frac{r^2+(b^2/R)^2-2(b^2/R)r\cos\phi}{r^2+R^2-2Rr\cos\phi}\right]}$$

Si $r\to\infty \Rightarrow \Phi(r,\phi) = \frac{\tau}{4\pi\epsilon_0}\ln\left[\frac{1+(b^2/Rr)^2-2(b^2/Rr)\cos\phi}{1+\frac{R^2}{r^2}-2\frac{rR}{r^2}\cos\phi}\right]$

Si $r\to\infty$, $r \gg R$ $\quad \approx \frac{\tau}{4\pi\epsilon_0}\ln\left[\frac{1-2(b^2/Rr)\cos\phi}{1-2(R/r)\cos\phi}\right]$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$$

$$\ln(1-x) = -x + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \cdots$$

Si $r\to\infty \Leftrightarrow \Phi \to \frac{\tau}{4\pi\epsilon_0}\Big[-2\left(\frac{b^2}{Rr}\right)\cos\phi + 2\left(\frac{R}{r}\right)\cos\phi$

$$\underbrace{+\frac{1}{2}\left(2\frac{b^2}{Rr}\cos\phi\right)^2 - \frac{1}{2}\left(2\frac{R}{r}\cos\phi\right)^2 + \cdots}_{\approx 0 \text{ si } r\to\infty}\Big]$$

$$\Rightarrow \boxed{\Phi \approx \frac{\tau}{2\pi\epsilon_0}\frac{R(1-b^2/R^2)\cos\phi}{r}}$$

c) $\sigma = -\epsilon_0 \left.\frac{\partial\Phi}{\partial r}\right|_{r=b}$ $\quad \Phi(r,\phi) = \frac{\tau}{4\pi\epsilon_0}\Big[\ln\left(r^2+(b^2/R)^2-2(b^2/R)r\cos\phi\right) - \ln\left(r^2+R^2-2Rr\cos\phi\right)\Big]$

$$\sigma = -\cancel{\epsilon_0}\frac{\tau}{4\pi\cancel{\epsilon_0}}\left[\frac{2r-2(b^2/R)\cos\phi}{r^2+(b^2/R)^2-2(b^2/R)r\cos\phi} - \frac{2r-2R\cos\phi}{r^2+R^2-2Rr\cos\phi}\right]_{r=b}$$

$$\sigma = -\frac{\tau}{4\pi}\left[\frac{2b - 2(b^2/R)\cos\phi}{b^2 + (b^2/R)^2 - 2(b^2/R)b\cos\phi} - \frac{2b - 2R\cos\phi}{b^2 + R^2 - 2Rb\cos\phi}\right]$$

$$= -\frac{\tau}{4\pi}\left[\frac{2(R^2/b) - 2R\cos\phi}{R^2 + b^2 - 2Rb\cos\phi} - \frac{2b - 2R\cos\phi}{b^2 + R^2 - 2Rb\cos\phi}\right]$$

$$\boxed{\sigma = -\frac{\tau}{4\pi}\left[\frac{R^2 - b^2}{R^2 + b^2 - 2bR\cos\phi}\right]}$$

d) El potencial producido por la carga imagen es:

$$\Phi_{image} = \frac{\tau}{4\pi\epsilon_0}\ln\left(r^2 + (b^2/R)^2 - 2(b^2/R)r\cos\phi\right)$$

$$= \frac{\tau}{4\pi\epsilon_0}\ln\left|\vec{r} - (b^2/R)\hat{\rho}\right|^2 \qquad \nabla\Phi = \frac{\partial\Phi}{\partial\hat{\rho}}\hat{\rho} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\hat{\phi} + \frac{\partial\Phi}{\partial z}\hat{k}$$

$$\vec{E}_{image} = -\nabla\Phi_{image} = -\frac{\tau}{4\pi\epsilon_0}\frac{2r - 2(b^2/R)\cos\phi}{\left(r^2 + (b^2/R)^2 - 2(b^2/R)r\cos\phi\right)}\hat{\rho}$$

$$\vec{E}_{image}\,|\,r = R, \phi = 0) = -\frac{\tau}{2\pi\epsilon_0}\frac{\cancel{R - (b^2/R)}}{(R - (b^2/R))^{\cancel{2}}}\hat{\rho} = -\frac{\tau}{2\pi\epsilon_0}\frac{R}{R^2 - b^2}\hat{\rho}$$

$$\vec{f} = \frac{\vec{F}}{l} = \tau\vec{E}_{image} \Rightarrow \boxed{\vec{f} = -\frac{\tau^2}{2\pi\epsilon_0}\frac{R}{R^2 - b^2}\hat{\rho}} \rightarrow$$ Fuerza por unidad de Longit

2.12 $$\Phi(\rho,\phi) = a_0 + b_0\ln\rho + \sum_{n=1}^{\infty} a_n\rho^n\sin(n\phi + \beta_n) + \sum_{n=1}^{\infty} b_n\rho^{-n}\sin(n\phi + \delta_n)$$

$$\sin(n\phi + \beta_n) = \underbrace{\cos\beta_n}_{cte}\sin n\phi + \underbrace{\sin\beta_n}_{cte}\cos n\phi$$

evita singularidades en el origen.

$$\Rightarrow \Phi(\rho,\phi) = a_0 + b_0\ln\rho + \sum_{n=1}^{\infty}\rho^n\left\{\underbrace{a_n\cos\beta_n}\sin n\phi + \underbrace{a_n\sin\beta_n}\cos n\phi\right\}$$

$$= a_0 + b_0\ln\rho + \sum_{n=1}^{\infty}\rho^n\left\{\alpha_n\sin n\phi + \gamma_n\cos n\phi\right\}$$

evita singularidades en $\rho = 0$

$$\Phi(\rho,\phi) = A_0 + \sum_{n=1}^{\infty}\rho^n\left\{A_n\sin n\phi + B_n\cos n\phi\right\}$$

$$\Phi(b,\phi) = A_0 + \sum_{n=1}^{\infty} b^n\left\{A_n\sin n\phi + B_n\cos n\phi\right\}$$

$$\Rightarrow \int_0^{2\pi} \Phi(b,\phi)\cos(m\phi)\,d\phi = A_0 \cancelto{0}{\int_0^{2\pi}\cos(m\phi)\,d\phi}$$

$$+ \sum_{n=1}^{\infty} b^n \left\{ A_n \underbrace{\cancelto{0}{\int_0^{2\pi}\sin n\phi \cos m\phi\, d\phi}} + B_n \int_0^{2\pi} \cos n\phi \cos m\phi\, d\phi \right\}$$

$$= \sum_{n=1} b^n \left\{ A_n(0) + B_n(\pi \delta_{nm}) \right\} = B_m \pi b^m$$

$$\Rightarrow \boxed{B_n = \frac{1}{\pi b^n}\int_0^{2\pi} \Phi(b,\phi)\cos(n\phi)\,d\phi}$$

$$\boxed{A_n = \frac{1}{\pi b^n}\int_0^{2\pi} \Phi(b,\phi)\sin(n\phi)\,d\phi} \longrightarrow \text{de modo similar!}$$

Por otro lado:

$$\int_0^{2\pi} \Phi(b,\phi)\,d\phi = \int_0^{2\pi} A_0\,d\phi + \sum_{n=1}^{\infty} b^n \left\{ A_n \cancelto{0}{\int_0^{2\pi}\sin n\phi\, d\phi} + B_n \cancelto{0}{\int_0^{2\pi}\cos n\phi\, d\phi} \right\}$$

$$= 2\pi A_0$$

$$\rightarrow \boxed{A_0 = \frac{1}{2\pi}\int_0^{2\pi} \Phi(b,\phi)\,d\phi}$$

$$\Rightarrow \Phi(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(b,\phi)\,d\phi + \sum_{n=1}^{\infty} \rho^n \left\{ \frac{1}{\pi b^n}\left(\int_0^{2\pi} \Phi(b,\phi)\sin n\phi\, d\phi\right)\sin n\phi + \frac{1}{\pi b^n}\left(\int_0^{2\pi} \Phi(b,\phi)\cos(n\phi)\,d\phi\right)\cos n\phi \right\}$$

$$= \frac{1}{\pi}\int_0^{2\pi} \Phi(b,\phi') \left\{ \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{\rho}{b}\right)^n \underbrace{\left[\sin(n\phi')\sin n\phi + \cos n\phi' \cos n\phi\right]}_{\cos n(\phi'-\phi)} \right\} d\phi'$$

$$\sum_{n=1}^{\infty} \left(\frac{\rho}{b}\right)^n \cos n(\phi'-\phi) = \sum_{n=1}^{\infty} x^n \cos(n\alpha) \quad \text{si} \quad x = \frac{\rho}{b} \quad \alpha = \phi'-\phi$$

$$= \sum_{n=1}^{\infty} x^n \left(\frac{e^{in\alpha} + e^{-in\alpha}}{2}\right)$$

$$= \frac{1}{2}\left[\sum_{n=1}^{\infty} (x e^{i\alpha})^n + \sum_{n=1}^{\infty} (x e^{-i\alpha})^n - 2\right]$$

donde $0 < x = \frac{\rho}{b} < 1$ dado que ρ es interior

By: Herminso Villarraga Gómez

Usando ... $\boxed{\sum_{n=0}^{\infty} \alpha x^n = \frac{\alpha}{1-x}}$ Serie geometrica $0 < |x| < 1$

$$\sum_{n=1}^{\infty} x^n \cos(n\alpha) = \frac{1}{2}\left[\frac{1}{1-xe^{i\alpha}} + \frac{1}{1-e^{-i\alpha}} - 2\right]$$

$$= \frac{1}{2}\left[\frac{1-xe^{-i\alpha} + 1-xe^{-i\alpha}}{1-xe^{i\alpha} - xe^{-i\alpha} + x^2} - 2\right]$$

$$= \frac{1}{2}\left[\frac{2 + x(-e^{i\alpha} - e^{-i\alpha})}{1+x^2 - x(e^{i\alpha}+e^{-i\alpha})} - 2\right]$$

$$= \frac{1}{2}\left[\frac{2 - 2x\cos\alpha}{1+x^2-2x\cos\alpha} - 2\right]$$

$$= \left[\frac{1 - x\cos\alpha}{1+x^2-2x\cos\alpha} - 1\right]$$

$$= \left[\frac{1 - x\cos\alpha - 1 - x^2 + 2x\cos\alpha}{1+x^2-2x\cos\alpha}\right]$$

$$= \left[\frac{x\cos\alpha - x^2}{1+x^2-2x\cos\alpha}\right]$$

$$\Rightarrow \frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{\rho}{b}\right)^n \cos n(\phi'-\phi) = \frac{1}{2} + \underbrace{\sum_{n=1}^{\infty} x^n \cos(n\alpha)}_{} \qquad x = \frac{\rho}{b} \quad \alpha = \phi'-\phi$$

$$= \frac{1}{2} + \left[\frac{x\cos\alpha - x^2}{1+x^2-2x\cos\alpha}\right]$$

$$= \frac{1+x^2 - \cancel{2x\cos\alpha} + \cancel{2x\cos\alpha} - 2x^2}{2(1+x^2-2x\cos\alpha)}$$

$$= \frac{1}{2}\left[\frac{1-x^2}{1+x^2-2x\cos\alpha}\right]$$

$$= \frac{1}{2}\,\frac{1-(\rho/b)^2}{1+\left(\frac{\rho}{b}\right)^2 - 2\left(\frac{\rho}{b}\right)\cos(\phi'-\phi)}$$

$$\Rightarrow \Phi(\rho,b) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(b,\phi')\left[\frac{1-(\rho^2/b^2)}{1+\frac{\rho^2}{b^2} - 2\frac{\rho}{b}\cos(\phi'-\phi)}\right] d\phi'$$

multiplicando por b^2/b^2 :

$$\boxed{\Phi(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(b,\phi')\,\frac{b^2-\rho^2}{b^2+\rho^2-2\rho b\cos(\phi'-\phi)}\,d\phi'}$$

Integral de Poisson para $0<\rho<b$

Para el caso en que $b<\rho<\infty$, en (2.71) $a_n=0$, $b_0=0$ de modo que $\Phi(\rho\to\infty,\phi)\to 0$

$$\Rightarrow \Phi(\rho,\phi) = A_0 + \sum_{n=1}^{\infty} b_n \rho^{-n}\sin(n\phi+\beta_n)$$

$$= A_0 + \sum_{n=1}^{\infty} \rho^{-n} b_n \left\{\cos\beta_n \sin(n\phi) + \sin\beta_n\cos(n\phi)\right\}$$

$$= A_0 + \sum_{n=1}^{\infty} \rho^{-n} \left\{A_n \sin(n\phi) + B_n\cos(n\phi)\right\}$$

Aquí $A_0 = \frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi)\,d\phi$ $\quad A_n = \frac{b^n}{\pi}\int_0^{2\pi}\Phi(\rho,\phi)\sin(n\phi)\,d\phi$

$$B_n = \frac{b^n}{\pi}\int_0^{2\pi}\Phi(\rho,\phi)\cos(n\phi)\,d\phi$$

$$\Rightarrow \Phi(\rho,\phi) = \frac{1}{\pi}\int_0^{2\pi}\Phi(b,\phi')\left\{\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{b}{\rho}\right)^n\left[\sin(n\phi')\sin(n\phi)+\cos(n\phi')\cos(n\phi)\right]\right\}\cdot d\phi'$$

$$= \frac{1}{\pi}\int_0^{2\pi}\Phi(b,\phi')\left\{\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{b}{\rho}\right)^n\cos n(\phi'-\phi)\right\}d\phi'$$

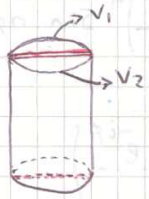
$$= \frac{1}{\pi}\int_0^{2\pi}\Phi(b,\phi')\left\{\frac{1}{2}\,\frac{1-(b/\rho)^2}{1+\left(\frac{b}{\rho}\right)^2-2\left(\frac{b}{\rho}\right)\cos(\phi'-\phi)}\right\}d\phi'\cdot\frac{\rho^2}{\rho^2}$$

$$\boxed{\Phi(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi')\,\frac{\rho^2-b^2}{\rho^2+b^2-2b\rho\cos(\phi'-\phi)}\,d\phi'}$$

Integral de Poisson para $b<\rho<\infty$.

Note que solo cambia el signo para el problema exterior.

2.13



$$\Phi(\rho,\phi) = a_0 + \cancel{b_0 \ln \rho}^{\,0} + \sum_{n=1}^{\infty} a_n \rho^n \sin(n\phi + \alpha_n) + \sum_{n=1}^{\infty} \cancel{b_n \rho^{-n}}^{\,0} \sin(n\phi + \beta_n)$$

Evita singularidades en $\rho = 0$

$$\Phi(\rho,\phi) = a_0 + \sum_{n=1}^{\infty} \rho^n \left(\underbrace{a_n \cos \alpha_n}_{A_n} \sin n\phi + \underbrace{a_n \sin \alpha_n}_{B_n} \cos n\phi\right)$$

$$= A_0 + \sum_{n=1}^{\infty} \rho^n \left(A_n \sin n\phi + B_n \cos n\phi\right)$$

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} \Phi(b,\phi)\, d\phi = \frac{1}{2\pi}\left[\int_0^{\pi} V_1\, d\phi + \int_{\pi}^{2\pi} V_2\, d\phi\right] = \frac{V_1 + V_2}{2}$$

$$A_n = \frac{1}{\pi b^n}\int_0^{2\pi} \Phi(b,\phi) \sin \phi\, d\phi =$$

$$= \frac{1}{\pi b^n}\left[V_1 \int_0^{\pi} \sin n\phi\, d\phi + V_2 \int_{\pi}^{2\pi} \sin n\phi\, d\phi\right]$$

$$= \frac{1}{\pi b^n}\left[-\frac{V_1}{n}\cos n\phi \Big|_0^{\pi} - \frac{V_2}{n}\cos n\phi \Big|_{\pi}^{2\pi}\right]$$

$$= -\frac{1}{n\pi b^n}\left[V_1(\cos n\pi - 1) + V_2(1 - \cos n\pi)\right] =$$

$$= \begin{cases} \dfrac{2}{n\pi b^n}\left[V_1 + V_2\right] & \text{si } n \text{ es impar} \\ 0 & \text{, en } n \text{ es par} \end{cases}$$

$$B_n = \frac{1}{\pi b^n}\int_0^{2\pi} \Phi(b,\phi) \cos n\phi\, d\phi$$

$$= \frac{1}{\pi b^n}\left[\int_0^{2\pi} V_1 \cos n\phi\, d\phi + V_2 \int_{\pi}^{2\pi} \cos n\phi\, d\phi\right]$$

$$= \frac{1}{\pi b^n}\left[\cancel{\frac{V_1}{n}\sin n\phi \Big|_0^{\pi}}^{\,0} + \cancel{\frac{V_2}{n}\sin n\phi \Big|_{\pi}^{2\pi}}^{\,0}\right]$$

$$\Phi(\rho,\phi) = \frac{V_1 + V_2}{2} + \sum_{n \text{ impar}} \frac{2\rho^n}{n\pi b^n}(V_1 - V_2)\sin n\phi$$

$$\Phi(\rho,\phi) = \frac{V_1+V_2}{2} + \frac{2(V_1-V_2)}{\pi}\sum_{n\ impar} \frac{1}{n}\left(\frac{\rho}{b}\right)^n \sin n\phi$$

Si $x = iy$:

$$\sum_{n\ impar} \frac{1}{n} x^n \sin(n\phi) = \sum_{n\ impar} \frac{1}{n}(iy)^n \left(\frac{e^{i\phi}-e^{-i\phi}}{2i}\right)$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} \frac{(iy)^{2n+1}}{2n+1}\,\frac{e^{i(2n+1)\phi}-e^{-i(2n+1)\phi}}{i}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} \frac{i^{2n}}{2n+1}\left[(ye^{i\phi})^{2n+1}-(ye^{-i\phi})^{2n+1}\right]$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left[(ye^{i\phi})^{2n+1}-(ye^{-i\phi})^{2n+1}\right]$$

Use $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = tg^{-1}x$

$$= \frac{1}{2}\left[tg^{-1}(ye^{i\phi}) - tg^{-1}(ye^{-i\phi})\right]$$

$$= \frac{1}{2}\, tg^{-1}\left(\frac{ye^{i\phi}-ye^{-i\phi}}{1+y^2}\right)$$

Use $tg^{-1}a - tg^{-1}b = tg^{-1}\left(\frac{a-b}{1+ab}\right)$

$$= \frac{1}{2}\, tg^{-1}\left(\frac{2iy\sin\phi}{1+y^2}\right) = \frac{1}{2}\, tg^{-1}\left(\frac{2x\sin\phi}{1-x^2}\right)$$

Así:

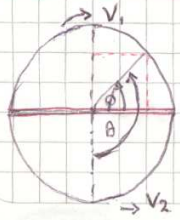
$$\boxed{\sum_{n\ impar} \frac{1}{n} x^n \sin(n\phi) = \frac{1}{2}\, tg^{-1}\left(\frac{2x\sin\phi}{1-x^2}\right)}$$

Usando esto último en el potencial:

$$\Phi(\rho,\phi) = \frac{V_1+V_2}{2} + \frac{V_1-V_2}{\pi}\, tg^{-1}\,\frac{2(\rho/b)\sin\phi}{1-(\rho/b)^2} \times \frac{b^2}{b^2}$$

$$\boxed{\Phi(\rho,\phi) = \frac{V_1+V_2}{2} + \frac{V_1-V_2}{\pi}\, tg^{-1}\left(\frac{2\rho b\sin\phi}{b^2-\rho^2}\right)} \longrightarrow$$

donde ϕ es el ángulo medido desde la ranura.



$\theta = \phi + 90^\circ$
$\phi = \theta - 90^\circ$ $\longrightarrow$ $\sin\phi = \sin(\theta - 90^\circ)$
$\sin\phi = \cos\theta$

b) $\sigma = -\epsilon_0 \left.\frac{\partial \Phi}{\partial \rho}\right|_{\rho=b}$ $\qquad \frac{d\tan^{-1}u}{d\rho} = \frac{1}{1+u^2}\frac{du}{d\rho}$

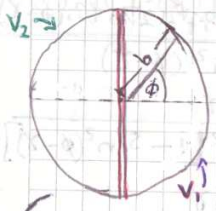
$$= -\epsilon_0 \frac{V_1 - V_2}{\pi}\left[\frac{1}{1+\left(\frac{2\rho b \sin\phi}{b^2-\rho^2}\right)^2}\left\{\frac{2b\sin\phi}{b^2-\rho^2} + \frac{2b\rho\sin\phi}{(b^2-\rho^2)^2}\cdot 2\rho\right\}\right]_{\rho=b}$$

$$= -\epsilon_0 \frac{V_1 - V_2}{\pi}\left[\frac{\cancel{(b^2-\rho^2)^2}}{(b^2-\rho^2)^2 + 4\rho^2 b^2 \sin^2\phi}\right\}\frac{2b\sin\phi(b^2-\rho^2) + 2b\rho\sin\phi\cdot 2\rho}{\cancel{(b^2-\rho^2)^2}}\Bigg\}\Bigg]_{\rho=b}$$

$$= -\epsilon_0 \frac{V_1 - V_2}{\pi}\;\frac{2\cancel{b^2}\cancel{\sin\phi}\cdot 2\cancel{b}}{4\cancel{b^2}b^{\cancel{2}}\sin^{\cancel{2}}\phi} \longrightarrow \boxed{\sigma = \epsilon_0 \frac{V_2 - V_1}{\pi b \sin\phi}}$$

$\cos\phi = \sin\theta$

Otro método de solución



$$\Phi(b,\phi) = \frac{V_1+V_2}{2} + \frac{V_1-V_2}{2}\,\mathrm{Sgn}(\cos\phi)$$

donde $\mathrm{Sgn}\,\phi = \begin{cases} 1 & \text{si } \cos\phi > 0 \\ -1 & \text{si } \cos\phi < 0 \end{cases}$

~~Usando~~ la integral de Poisson es:

$$\boxed{\Phi(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(b,\phi')\,\frac{b^2-\rho^2}{b^2+\rho^2-2\rho b\cos(\phi-\phi')}\,d\phi'}$$

El término $\frac{V_1+V_2}{2}$ es una cte que puede asumirse como ~~cambio~~ una referencia para el potencial que hace que se satisfagan las condiciones de frontera. De modo que podemos tomar otra referencia y definir un potencial Φ' que no incluya este término tal que

$$\Phi'(b,\phi) = \frac{V_1-V_2}{2}\,\mathrm{Sgn}(\cos\phi) \quad \text{y} \quad \Phi(\rho,\phi) = \Phi'(\rho,\phi) + \frac{V_1+V_2}{2}$$

y así podemos usar la integral de Poisson para hallar $\Phi'(\rho,\phi)$:

$$\Phi'(\rho,\phi) = \frac{V_1-V_2}{4\pi}\int_0^{2\pi}\frac{(b^2-\rho^2)\,\mathrm{Sgn}(\cos\phi)}{b^2+\rho^2-2\rho b\cos(\phi-\phi')}\,d\phi'$$

$$\Rightarrow \Phi'(\rho,\phi) = \frac{V_1-V_2}{4\pi}\left[\int_{-\pi/2}^{\pi/2}\frac{(b^2-\rho^2)\,d\phi'}{b^2+\rho^2-2\rho b\cos(\phi-\phi')} - \int_{\pi/2}^{3\pi/2}\frac{b^2-\rho^2}{b^2+\rho^2-2\rho b\cos(\,)}\,d\phi'\right]$$

Para la segunda integral sea: $\gamma = \phi'-\pi \rightarrow \phi' = \gamma+\pi$

y $\cos(\phi-\phi') = \cos(\phi-\gamma-\pi) = -\cos(\phi-\gamma)$

si $\gamma = \phi'-\pi \quad d\phi' = d\gamma$ y si $\phi' = \pi/2 \rightarrow \gamma = -\pi/2$

si $\phi' = 3\pi/2 \rightarrow \gamma = \pi/2$

y para la primera sea $\gamma = \phi'$

$$\Rightarrow \Phi'(\rho,\phi) = \frac{V_1-V_2}{4\pi}\left[\int_{-\pi/2}^{\pi/2}\frac{(b^2-\rho^2)\,d\gamma}{b^2+\rho^2-2\rho b\cos(\phi-\gamma)} - \int_{-\pi/2}^{\pi/2}\frac{(b^2-\rho^2)\,d\gamma}{b^2+\rho^2+2\rho b\cos(\phi-\gamma)}\right]$$

$$= \frac{V_1-V_2}{4\pi}(b^2-\rho^2)\int_{-\pi/2}^{\pi/2}\frac{[\cancel{b^2+\rho^2}+2b\rho\cos(\phi-\gamma)]-[\cancel{b^2+\rho^2}-2\rho b\cos(\phi-\gamma)]}{(b^2+\rho^2)^2-4\rho^2b^2\cos^2(\phi-\gamma)}\,d\gamma$$

$$= \frac{V_1-V_2}{\cancel{4}\pi}(b^2-\rho^2)\int_{-\pi/2}^{\pi/2}\frac{\cancel{4}b\rho\cos(\phi-\gamma)\,d\gamma}{b^4+\rho^4+2b^2\rho^2-4\rho^2b^2[1-\sin^2(\phi-\gamma)]}$$

$$= \frac{V_1-V_2}{\pi}(b^2-\rho^2)b\rho\int_{-\pi/2}^{\pi/2}\frac{\cos(\phi-\gamma)\,d\gamma}{(b^2-\rho^2)^2+4b^2\rho^2\sin^2(\phi-\gamma)}$$

Sea $u = \sin(\phi-\gamma) \rightarrow du = -\cos(\phi-\gamma)\,d\gamma$

si $\gamma = -\pi/2 \rightarrow u = \sin(\phi+\pi/2) = \cos\phi$

$\gamma = \pi/2 \rightarrow u = \sin(\phi-\pi/2) = -\cos\phi$

$$\Rightarrow \Phi'(\rho,\phi) = \frac{V_1-V_2}{\pi}(b^2-\rho^2)b\rho\int_{-\cos\phi}^{\cos\phi}\frac{du}{(b^2-\rho^2)^2+4b^2\rho^2u^2}$$

$$= \frac{(V_1-V_2)}{\pi}\frac{b\rho}{(b^2-\rho^2)}\int_{-\cos\phi}^{\cos\phi}\frac{du}{1+\left(\frac{2b\rho}{b^2-\rho^2}\right)^2u^2}$$

$$= \frac{V_1-V_2}{\pi}\frac{\cancel{b\rho}}{\cancel{(b^2-\rho^2)}}\cdot\left(\frac{\cancel{b^2-\rho^2}}{\cancel{2b\rho}}\right)tg^{-1}\left(\frac{2b\rho}{b^2-\rho^2}\right)\Bigg|_{-\cos\phi}^{\cos\phi}$$

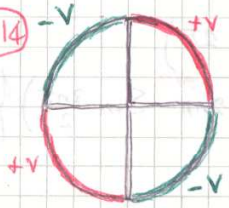
$$= \frac{V_1-V_2}{2\pi}\left[tg^{-1}\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right) - \underbrace{tg^{-1}\left(\frac{-2b\rho}{b^2-\rho^2}\cos\phi\right)}_{\tan^{-1}(-x) = -tg^{-1}x}\right]$$

$$= \frac{V_1-V_2}{\cancel{2}\pi}\cancel{2}\,tg^{-1}\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right)$$

$$\Rightarrow \Phi(\rho,\phi) = \Phi'(\rho,\phi) + \frac{V_1+V_2}{2}$$

$$\boxed{\Phi(\rho,\phi) = \frac{V_1-V_2}{\pi}\tan^{-1}\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right) + \frac{V_1+V_2}{2}}$$

2.14



$$\Phi(\rho,\phi) = a_0 + \cancel{b_0 \ln\rho}^{\,0} + \sum_{n=1}^{\infty} a_n \rho^n \sin(n\phi+\alpha_n) + \sum_{n=1}^{\infty} \cancel{b_n \rho^{-n}}^{\,0} \sin(n\phi+\beta_n)$$

Evita Singularidades en $\rho=0$.

$$\Phi(\rho,\phi) = A_0 + \sum_{n=1}^{\infty} \rho^n \left\{ A_n \sin n\phi + B_n \cos n\phi \right\}$$

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} \Phi(b,\phi)\,d\phi = \frac{V}{2\pi}\left\{\frac{\pi}{2} - \frac{\pi}{2} + \frac{\pi}{2} - \frac{\pi}{2}\right\} = 0$$

$$A_n = \frac{1}{\pi b^n}\int_0^{2\pi} \Phi(b,\phi)\sin(n\phi)\,d\phi$$

$$= \frac{1}{\pi b^n}\left\{ V\int_0^{\pi/2} \sin n\phi\,d\phi - V\int_{\pi/2}^{\pi} \sin n\phi\,d\phi + V\int_{\pi}^{3\pi/2} \sin n\phi\,d\phi - V\int_{3\pi/2}^{2\pi} \sin n\phi\,d\phi \right\}$$

$$= \frac{V}{\pi b^n}\left\{ -\frac{1}{n}\cos n\phi\Big|_0^{\pi/2} + \frac{1}{n}\cos n\phi\Big|_{\pi/2}^{\pi} - \frac{1}{n}\cos n\phi\Big|_{\pi}^{3\pi/2} + \frac{1}{n}\cos n\phi\Big|_{\frac{3\pi}{2}}^{2\pi} \right\}$$

$$\cos(n\pi/2) = \begin{cases} (-1)^{n/2} & \text{si } n \text{ es par} \\ 0 & \text{si } n \text{ es impar} \end{cases} \qquad \cos\left(n\frac{3\pi}{2}\right) = \begin{cases} (-1)^{n/2} & \text{si } n \text{ par} \\ 0 & \text{si } n \text{ impar} \end{cases}$$

$$\cos(n\pi) = \begin{cases} 1 & \text{si } n \text{ es par} \\ -1 & \text{si } n \text{ es impar} \end{cases} \qquad \cos n(0) = \cos n(2\pi) = 1 \text{ para cualquier } n \text{ entero}$$

$$\Rightarrow A_n = \frac{V}{n\pi b^n}\begin{cases} -[(-1)^{n/2}-1] + [1-(-1)^{n/2}] - [(-1)^{n/2}-1] + [1-(-1)^{n/2}] & \text{si } n \text{ par} \\ -[0-1] + [-1-0] - [0+1] + [1-0] & \text{si } n \text{ impar} \end{cases}$$

$$= \frac{V}{n\pi b^n}\begin{cases} 4\left[1-(-1)^{n/2}\right] & \text{si } n \text{ es par} \\ 0 & \text{si } n \text{ es impar} \end{cases}$$

$$A_n = \frac{4V}{n\pi b^n}\begin{cases} 2 & \text{si } n = 2, 6, 10, 14, \ldots \\ 0 & \text{si } n = 4, 8, 12, 16, \ldots \\ 0 & \text{si } n \text{ es impar} \end{cases}$$

$$B_n = \frac{1}{\pi b^n}\int_0^{2\pi} \Phi(b,\phi)\cos(n\phi)\,d\phi$$

$$= \frac{V}{\pi b^n}\left\{\int_0^{\pi/2} - \int_{\pi/2}^{\pi} + \int_{\pi}^{3\pi/2} - \int_{3\pi/2}^{2\pi}\right\}\cos n\phi\, d\phi$$

$$= \frac{V}{n\pi b^n}\left\{\left(\sin\frac{n\pi}{2} - \cancel{\sin 0}^{\,0}\right) - \left(\cancel{\sin n\pi}^{\,0} - \sin\frac{n\pi}{2}\right) + \left(\sin\frac{3n\pi}{2} - \cancel{\sin n\pi}^{\,0}\right) - \left(\cancel{\sin 2n\pi}^{\,0} - \sin\frac{3n\pi}{2}\right)\right\}$$

$$= \frac{2V}{n\pi b^n}\left\{\sin\frac{n\pi}{2} + \sin\frac{3n\pi}{2}\right\}$$

$$\sin\frac{n\pi}{2} = \begin{cases} 0 & \text{si } n \text{ par} \\ (-1)^{\frac{n-1}{2}} & \text{si } n \text{ impar}\end{cases} \qquad \sin\frac{3n\pi}{2} = \begin{cases} 0 & \text{si } n \text{ par} \\ -(-1)^{\frac{n-1}{2}} & \text{si } n \text{ impar}\end{cases}$$

$\Rightarrow B_n = 0$ en cualquier caso.

$$\Rightarrow \Phi(\rho,\phi) = \sum_{n}^{\infty} \frac{8V}{n\pi b^n}\rho^n \sin n\phi \quad \text{con } n = 2, 6, 10, 14, \ldots$$

$$\Phi(\rho,\phi) = \frac{8V}{\pi}\sum_{n=0}^{\infty}\left(\frac{\rho}{b}\right)^{4n+2}\frac{\sin[(4n+2)\phi]}{4n+2}$$

$$\boxed{\Phi(\rho,\phi) = \frac{4V}{\pi}\sum_{n=0}^{\infty}\left(\frac{\rho}{b}\right)^{4n+2}\frac{\sin[(4n+2)\phi]}{2n+1}}$$

b) $\displaystyle\sum_{n=0}^{\infty}\frac{x^{4n+2}}{2n+1}\sin[(4n+2)\phi]$ haciendo $x = iy$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(iy)^{4n+2}}{2n+1}\,\frac{e^{i(4n+2)\phi} - e^{-i(4n+2)\phi}}{i}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(i)^{4n+1}}{2n+1}\left[(ye^{i\phi})^{4n+2} - (ye^{-i\phi})^{4n+2}\right]$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{i^{2n}\,i^{2n+1}}{2n+1}\left[(y^2e^{i2\phi})^{2n+1} - (y^2e^{-2i\phi})^{2n+1}\right]$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left[(iy^2e^{i2\phi})^{2n+1} - (iy^2e^{-2i\phi})^{2n+1}\right]$$

Usando $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = \tan^{-1} x$

$$\sum_{n=0}^{\infty} \frac{x^{4n+2}}{2n+1} \sin[(4n+2)\phi] = \frac{1}{2}\left[\tan^{-1}(iy^2 e^{i2\phi}) - \tan^{-1}(iy^2 e^{-2i\phi})\right]$$

$x = iy$

$$\boxed{\tan^{-1} a - \tan^{-1} b = \tan^{-1}\left(\frac{a-b}{1+ab}\right)}$$

$$= \frac{1}{2}\tan^{-1}\left[\frac{iy^2(e^{i2\phi} - e^{-2i\phi})}{1+i^2y^4}\right]$$

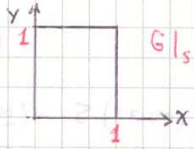
$$= \frac{1}{2}\tan^{-1}\left[\frac{i\, i^2x^2(2i\sin 2\phi)}{1 - x^4/i^4}\right]$$

$$= \frac{1}{2}\tan^{-1}\left[\frac{2(\rho/b)^2 \sin 2\phi}{1-(\rho/b)^4}\right]$$

$$= \frac{1}{2}\tan^{-1}\left[\frac{2\rho^2 b^2 \sin 2\phi}{b^2-\rho^2}\right]$$

$$\Rightarrow \boxed{\Phi(\rho,\phi) = \frac{2V}{\pi}\tan^{-1}\left[\frac{2\rho^2 b^2 \sin(2\phi)}{b^2-\rho^2}\right]}$$

2.15



$G|_s = 0$

$$\nabla'^2 G = -4\pi\delta(\vec{r}-\vec{r}') = -4\pi\delta(x-x')\delta(y-y')$$

$$G|_s = 0 \longrightarrow G(x'=0) = G(x'=1) = G(y'=0) = G(y'=1) = 0$$

Sea $G = g_x(x,x')\, g_y(y,y')$

$$\Rightarrow \nabla'^2 G = g_y \frac{\partial^2 g_x}{\partial x'^2} + g_x \frac{\partial^2 g_y}{\partial y'^2} = -4\pi\delta(x-x')\delta(y-y')$$

Si $x \neq x' \longrightarrow \underbrace{\frac{1}{g_x}\frac{\partial^2 g_x}{\partial x'^2}}_{-\alpha^2} + \underbrace{\frac{1}{g_y}\frac{\partial^2 g_y}{\partial y'^2}}_{\alpha^2} = 0$

① $\dfrac{\partial^2 g_x}{\partial x'^2} + \alpha^2 g_x = 0 \longrightarrow g_x = A\sin\alpha_{n_x} x' + B\cos\alpha_{n_x} x'$

$g_x(x'=0) = 0 \longrightarrow B = 0$, $g_x(x'=1) = 0 \longrightarrow \alpha = n\pi \quad n = 1, 2, \ldots$

$\Rightarrow g_x(x,x') = A_n(x)\sin(n\pi x)$

Por simetría de G para las condiciones de Dirichlet

$g_x(x,x') = g(x',x) \Rightarrow A_n(x)\sin(n\pi x') = A_n(x')\sin(n\pi x) \longrightarrow$

$\rightarrow A_n(x) = C_n \sin(n\pi x) \longrightarrow \boxed{g_x(x,x') = \sum_n C_n \sin(n\pi x)\sin(n\pi x')}$

$$\frac{\partial^2 g_x}{\partial x'^2} = -\sum_n n^2\pi^2 C_n \sin(n\pi x)\sin(n\pi x')$$

$$\Rightarrow g_y \frac{\partial^2 g_x}{\partial x'^2} + g_y \frac{\partial^2 g_y}{\partial y'^2} = \sum_n \left(-n^2\pi^2 g_y + \frac{\partial^2 g_y}{\partial y'^2}\right) C_n \sin(n\pi x)\sin(n\pi x')$$

$$-4\pi\,\delta(x-x')\,\delta(y-y') = \sum_n \left(-n^2\pi^2 g_y + \frac{\partial^2 g_y}{\partial y'^2}\right) C_n \sin(n\pi x)\sin(n\pi x')$$

podemos expandir $\delta(x-x')$ en funciones senoidales.

$\delta(x-x') = \sum_n D_n(x)\sin(n\pi x)$ y por simetría de la función delta de Dirac $\delta(x-x') = \delta(x'-x)$

$$\Rightarrow \sum_n D_n(x)\sin(n\pi x) = \sum_n D_n(x')\sin(n\pi x')$$

$\rightarrow D_n(x) = d_n \sin(n\pi x')$ es la condición que hace que la expansión sea simétrica.

$$\Rightarrow \delta(x-x') = \sum_n d_n \sin(n\pi x')\sin(n\pi x)$$

$$\int_0^1 \delta(x-x')\sin(m\pi x')\,dx' = \sum_n d_n \sin n\pi x \underbrace{\int_0^1 \sin(n\pi x')\sin(m\pi x')\,dx'}_{\frac{1}{2}\delta_{nm}}$$

$$\sin(m\pi x) = \frac{d_m}{2}\sin(m\pi x)$$

$$\Rightarrow d_n = 2 \rightarrow \boxed{\delta(x-x') = 2\sum_{n=1}^{\infty} \sin(n\pi x)\sin(n\pi x')}$$

Relación de Completez para las series senoidales.

$$\Rightarrow -4\pi\left(2\sum_{n=1}^{\infty}\sin(n\pi x)\sin(n\pi x')\right)\delta(y-y')$$
$$= \sum_n \left(-n^2\pi^2 g_y + \frac{\partial^2 g_y}{\partial y'^2}\right) C_n \sin(n\pi x)\sin(n\pi x')$$

lo más simple es suponer $\boxed{\left(-n^2\pi^2 g_y + \frac{\partial^2 g_y}{\partial y'^2}\right) = -4\pi\,\delta(y-y')}$

y $\boxed{C_n = 2}$ de modo que $g_x(x,x')$ coincida con $\delta(x-x')$ además estas condiciones son las que mejor se ajustan a una expansión de la forma: $G = 2\sum_{n=1}^{\infty} g_n(y,y')\sin(n\pi x)\sin(n\pi x')$

By: Herminso Villarraga Gómez

Si $y \neq y'$: $\dfrac{\partial^2 g_y}{\partial y'^2} - n^2\pi^2 g_y = 0 \longrightarrow g_y = A_{n1}\operatorname{Sinh}(n\pi y') + B_{n1}\operatorname{Cosh}(n\pi y')$

① Si $y'=0 \Rightarrow y'<y$, $0<y<1$ y $g_y(y'=0)=0 \longrightarrow B_{n1}=0$

② Si $y'=1 \Rightarrow y'>y$, $0<y<1$ y $g_y(y'=1)=0 = A_{n2}\operatorname{Sinh}(n\pi) + B_{n2}\operatorname{Cosh}(n\pi)$

$$\Rightarrow A_{n2}\left(\frac{e^{n\pi}-e^{-n\pi}}{2}\right) + B_{n2}\left(\frac{e^{n\pi}+e^{-n\pi}}{2}\right) = 0$$

$$(A_{n2}+B_{n2})e^{n\pi} + (B_{n2}-A_{n2})e^{-n\pi} = 0 \longrightarrow \begin{cases} A_{n2}+B_{n2} = -e^{-n\pi} \\ B_{n2}-A_{n2} = +e^{+n\pi} \end{cases}$$

$$\begin{aligned} A_{n2}+B_{n2} &= -e^{-n\pi} \\ -A_{n2}+B_{n2} &= e^{n\pi} \\ \hline 2B_{n2} &= e^{n\pi}-e^{-n\pi} \longrightarrow B_{n2} = \operatorname{Sinh}(n\pi) \end{aligned}$$

$$2A_{n2} = -e^{-n\pi} - e^{n\pi} \longrightarrow A_{n2} = \operatorname{Cosh}(n\pi)$$

$$\Rightarrow g_y = -\operatorname{Cosh}(n\pi)\operatorname{Sinh}(n\pi y') + \operatorname{Sinh}(n\pi)\operatorname{Cos}(n\pi y') \quad \text{si } y'>y$$

$$= \operatorname{Sinh}(n\pi - n\pi y')$$

$$\Rightarrow g_y = \begin{cases} A_{n1}\operatorname{Sinh}(n\pi y') & \text{si } y'<y \\ \gamma_n \operatorname{Sinh}[n\pi(1-y)] & \text{si } y'>y \end{cases}$$

③ Si $y=y'$ $\quad g_y(y'<y) = g_y(y'>y)$

$$A_{n1}\operatorname{Sinh}(n\pi y) = \gamma_n \operatorname{Sinh}[n\pi(1-y)]$$

$$A_{n1} = \underbrace{\frac{\gamma_n}{\operatorname{Sinh}(n\pi y)}}_{\beta_n}\operatorname{Sinh}[n\pi(1-y)]$$

$$\Rightarrow g_y = \begin{cases} \beta_n \operatorname{Sinh}[n\pi(1-y)]\operatorname{Sinh}(n\pi y') & \text{si } y'<y \\ \beta_n \operatorname{Sinh}[n\pi(1-y')]\operatorname{Sinh}(n\pi y) & \text{si } y'>y \end{cases}$$

Note que existe simetría respecto a y, y'

$$\Rightarrow g_y(y,y') = \beta_n \operatorname{Sinh}[n\pi(1-y_>)]\operatorname{Sinh}(n\pi y_<) \qquad \begin{aligned} y_> &= \text{Máx}\{y',y\} \\ y_< &= \text{Mín}\{y',y\} \end{aligned}$$

Para evaluar β_m debemos recurrir nuevamente a

$\dfrac{\partial^2}{\partial y'^2} g_y - n^2\pi^2 g_y = -4\pi\delta(y-y')$ e integrar en una vecindad de

en integrar respecto a y' en una vecindad muy pequeña alrededor de y:

$$\int_{y-\epsilon}^{y+\epsilon} \frac{\partial^2}{\partial y'^2} g_y \, dy' - n^2\pi^2 \cancel{\int_{y-\epsilon}^{y+\epsilon} g_y \, dy'} \to 0 \text{ si } \epsilon \to 0 = -4\pi \int_{y-\epsilon}^{y+\epsilon} \delta(y-y')\, dy'$$

$$\left.\frac{\partial g_y}{\partial y'}\right|_{y+\epsilon} - \left.\frac{\partial g_y}{\partial y'}\right|_{y-\epsilon} = -4\pi$$

Si $y' = y+\epsilon \rightarrow y' > y \Rightarrow \left.\frac{\partial g_y}{\partial y'}\right|_{y+\epsilon} = \beta_n(-n\pi)\cosh[n\pi(1-y')]\sinh(n\pi y)\Big|_{y+\epsilon}$

si $y' = y-\epsilon \rightarrow y' < y \Rightarrow \left.\frac{\partial g_y}{\partial y'}\right|_{y-\epsilon} = \beta_n n\pi \sinh[n\pi(1-y)]\cosh(n\pi y')\Big|_{y-\epsilon}$

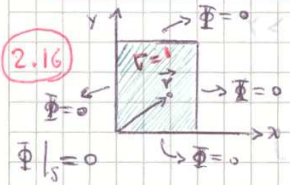
$$\lim_{\epsilon\to 0}\left\{\left.\frac{\partial g_y}{\partial y'}\right|_{y+\epsilon} - \left.\frac{\partial g_y}{\partial y'}\right|_{y-\epsilon}\right\} = \beta_n(n\pi)\Big[-\cosh[n\pi(1-y)]\sin n\pi y - \sinh[n\pi(1-y)]\cosh(n\pi y)\Big]$$

$$-4\pi = \beta_n(n\pi)\Big[-\sinh\big(n\pi(1-\cancel{y}) + \cancel{n\pi y}\big)\Big]$$

$$\frac{4}{n\sinh(n\pi)} = \beta_n \rightarrow \boxed{g_y = \frac{4\sinh[n\pi(1-y_>)]\sinh(n\pi y_<)}{n\sinh(n\pi)}}$$

$$\Rightarrow \boxed{G = 8\sum_{n=1}^{\infty} \frac{\sin(n\pi x)\sin(n\pi x')\sinh(n\pi x')\sinh(n\pi y_<)\sinh(n\pi(1-y_>))}{n\sinh(n\pi)}}$$

2.16



$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int G(\vec{r},\vec{r}\,')\rho(\vec{r}\,')\,dv' + \oint_S \Big[\cancel{\Phi(\vec{r}\,')\frac{\partial G}{\partial n}}\Big|_{r'} - \cancel{G_S}\frac{\partial \Phi}{\partial n}\Big|_{r'}\Big]\,da'$$

$\Phi|_S = 0$ $\quad 0$ C.D.

$$\Rightarrow \Phi(\vec{r}) = \frac{8}{4\pi\epsilon_0}\sum_{n=1}\frac{1}{n\sinh n\pi}\int_0^1\int_0^1 \sin(n\pi x)\sin(n\pi x')\sinh(n\pi y_<)\cdot\sinh(n\pi(1-y_>))\,dx'dy'$$

① $\int_0^1 \sin(n\pi x')\,dx' = -\frac{1}{n\pi}\cos n\pi x'\Big|_0^1 = -\frac{1}{n\pi}(\cos(n\pi)-1)$

$$= \begin{cases} 2/n\pi & \text{si } n \text{ es impar} \\ 0 & \text{si } n \text{ es par.} \end{cases}$$

② $= \int_0^1 \sinh(n\pi y_<)\sinh(n\pi(1-y_>))\,dy'$

$$= \left\{\sinh n\pi(1-y)\int_0^y \sinh(n\pi y')\,dy' + \sinh(n\pi y)\int_y^1 \sinh n\pi(1-y')\,dy'\right\}$$

$$\int_0^{y_0} \sinh(n\pi y')\,dy' = \frac{1}{n\pi}\cosh(n\pi y')\Big|_0^y = \frac{1}{n\pi}\big(\cosh(n\pi y)-1\big)$$

$$\int_y^1 \sinh(n\pi(1-y'))\,dy' = -\frac{1}{n\pi}\cos n\pi(1-y')\Big|_y^1$$

$$= -\frac{1}{n\pi}\left(\cancelto{1}{\cosh n\pi(1-1)} - \cosh n\pi(1-y)\right)$$

$$\Rightarrow ② = \sinh[n\pi(1-y)]\frac{1}{n\pi}(\cosh n\pi y - 1) - \sinh(n\pi y)\frac{1}{n\pi}(1-\cosh n\pi(1-y))$$

$$= \frac{1}{n\pi}\Big\{\sinh n\pi(1-y)\cosh n\pi y + \sinh n\pi y\cosh n\pi(1-y) - \sinh n\pi y - \sinh n\pi(1-y)\Big\}$$

$$= \frac{1}{n\pi}\Big\{\sinh(n\pi(1-y)+n\pi y) - \sinh n\pi y - \sinh n\pi(1-y)\Big\}$$

$$\Phi(\vec{r}) = \frac{8}{4\pi\epsilon_0}\sum_{n\ \text{impar}}\frac{1}{n\sinh n\pi}\left\{\frac{2}{n\pi}\sin n\pi x\left[\frac{1}{n\pi}(\sinh n\pi - \sinh n\pi y - \sinh n\pi(1-y))\right]\right\}$$

$$\boxed{\Phi(\vec{r}) = \frac{4}{\pi^3\epsilon_0}\sum_{n\ \text{impar}}\frac{\sin n\pi x}{n^3}\left[1 - \frac{\sinh n\pi y + \sinh n\pi(1-y)}{\sinh n\pi}\right]}$$

$$= \frac{4}{\pi^3\epsilon_0}\sum_{m=0}\frac{\sin(2m+1)\pi x}{(2m+1)^3}\left[\frac{\sinh(2m+1)\pi y_0 + \sinh[(2m+1)\pi(1-y)]}{\sinh(2m+1)\pi}\right]$$

$$\frac{\sinh n\pi y + \sinh n\pi(1-y)}{\sinh n\pi} = \frac{2\sinh\frac{1}{2}(n\pi y + n\pi(1-y))\cosh\frac{1}{2}(2n\pi y - n\pi)}{2\sinh\frac{n\pi}{2}\cosh\frac{n\pi}{2}}$$

$$= \frac{2\sinh\frac{1}{2}(n\pi y + n\pi(1-y))\cosh\frac{1}{2}(n\pi y - n\pi(1-y))}{2\sinh\frac{n\pi}{2}\cosh\frac{n\pi}{2}}$$

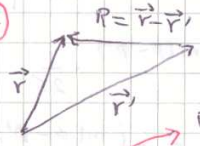
$$= \frac{\cosh n\pi(y-1/2)}{\cosh n\pi/2}$$

$$\Rightarrow \Phi(\vec{r}) = \frac{4}{\pi^3\epsilon_0}\sum_{n\ \text{impar}}\frac{\sin n\pi x}{n^3}\left[1 - \frac{\cosh[n\pi(y-1/2)]}{\cosh n\pi/2}\right]$$

$$\boxed{\Phi(\vec{r}) = \frac{4}{\pi^3\epsilon_0}\sum_{n=0}\frac{\sin(2n+1)\pi x}{(2n+1)^3}\left[1 - \frac{\cosh[(2n+1)\pi(y-1/2)]}{\cosh(2n+1)\pi/2}\right]}$$

2.17

a)



$$R = \vec{r} - \vec{r}' = [\underbrace{(x-x')^2 + (y-y')^2}_{a^2} + \underbrace{(z-z')^2}_{u^2}]^{1/2} = (a^2+u^2)^{1/2}$$

$$G = \int_{-z}^{z}\frac{du}{(a^2+u^2)^{1/2}} = \ln[(a^2+u^2)^{1/2} + u]_{-z}^{z}$$

Potencial de una carga puntual de valor $4\pi\epsilon_0$

$$= \ln\frac{(z^2+a^2)^{1/2}+z}{(z^2+a^2)^{1/2}-z} = \ln\frac{(1+a^2/z^2)^{1/2}+1}{(1+a^2/z)^{1/2}-1}$$

$$G = \ln\left[\frac{(1+\frac{1}{2}(a^2/z^2)+\cdots)+1}{(1+\frac{1}{2}(a^2/z^2)+\cdots)-1}\right] \approx \left(\frac{2+\frac{1}{2}a^2/z^2}{\frac{1}{2}a^2/z^2}\right)$$

$$= \ln\frac{4z^2+a^2}{a^2} = \ln(4z^2+a^2) - \ln a^2$$

Si $z \gg a \Rightarrow I \approx \underbrace{\ln 4z^2}_{cte} - \ln a^2 = G(x, x'; y, y')$

$\Rightarrow$ Podemos definir $G = -\ln a^2 = -\ln[(x-x')^2+(y-y')^2]$ ya que la cte $\ln 4z^2$ no es importante.

$\Rightarrow G(x,x';y,y') = -\ln[(x-x')^2+(y-y')^2]$

Si $x = \rho\cos\phi$, $y = \rho\sin\phi$ $\Rightarrow$ $(x-x')^2 = (\rho\cos\phi - \rho'\cos\phi')^2$

$$= \rho^2\cos^2\phi + \rho'^2\cos^2\phi - 2\rho\rho'\cos\phi\cos\phi'$$

$$(y-y') = (\rho\sin\phi - \rho'\sin\phi')^2 = \rho^2\sin^2\phi + \rho'^2\sin^2\phi - 2\rho\rho'\sin\phi\sin\phi'$$

$$(x-x')^2+(y-y')^2 = \rho^2+\rho'^2 - 2\rho\rho'(\cos\phi\cos\phi' + \sin\phi\sin\phi')$$

$$= \rho^2+\rho'^2 - 2\rho\rho'\cos(\phi-\phi')$$

$$G(x,x';y,y') = -\ln[\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')] = -2\log|\vec{\rho}-\vec{\rho}'|$$

$\nabla^2 G(r,r') = -4\pi\,\delta(r-r')$

b) $\int \nabla^2 G(\rho,\phi;\rho',\phi')\,dv' = -4\pi\int\delta(r-r')$

$$\int \nabla^2 G\cdot\rho'\,d\rho'\,d\phi' = -4\pi \longrightarrow \nabla^2 G = -\frac{4\pi}{\rho'}\delta(\rho-\rho')\delta(\phi-\phi')$$

$$\nabla^2 = \frac{1}{\rho'}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial}{\partial\rho'}\right) - \frac{1}{\rho'^2}\frac{\partial}{\partial\phi'^2}$$

Si $G = \frac{1}{2\pi}\sum_{-\infty}^{\infty} e^{im(\phi-\phi')}g_m(\rho,\rho')$ Por expansión de autofunciones de la parte angular

$$\rightarrow \nabla^2 G = \frac{1}{2\pi}\sum_{-\infty}^{\infty}\left\{\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial g_m}{\partial\rho'}\right)e^{im(\phi-\phi')} - \frac{m^2}{\rho'^2}e^{im(\phi-\phi')}\cdot g_m\right\}$$

$$-\frac{4\pi}{\rho'}\delta(\rho-\rho')\delta(\phi-\phi') = \sum_{-\infty}^{\infty}\left\{\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial g_m}{\partial\rho'}\right) - \frac{m^2}{\rho'^2}g_m\right\}e^{im(\phi-\phi')}\cdot\frac{1}{2\pi}$$

Sea $\delta(\phi) = \sum_{-\infty}^{\infty} C_n e^{in\phi} \rightarrow \delta(\phi-\phi') = \sum_{-\infty}^{\infty} C_n e^{in(\phi-\phi')}$

Expansión en series exponenciales de $\delta(\phi)$

1) Pues $g_m(\rho,\rho')$: θ, θ'

$$\Rightarrow \int_{-\pi}^{\pi} \delta(\phi-\phi')\, e^{in\phi'}\, d\phi' = \sum_{-\infty}^{\infty} C_n e^{im\phi} \underbrace{\int_{-\pi}^{\pi} e^{i(n-m)\phi'}\, d\phi'}_{2\pi\,\delta_{nm}}$$

$$e^{in\phi} = \sum_{-\infty}^{\infty} C_n e^{im\phi} \frac{\delta_{nm}}{2\pi} = \frac{C_n}{2\pi} e^{in\phi}$$

$$\Rightarrow C_n = 1/2\pi \qquad \Rightarrow \boxed{\delta(\phi-\phi') = \frac{1}{2\pi}\sum_{-\infty}^{\infty} e^{im(\phi-\phi')}} \rightarrow$$

Relación de completez para las exponenciales

$$\Rightarrow -\frac{4\pi}{\rho'}\delta(\rho-\rho')\cancel{\frac{1}{2\pi}}\sum_{-\infty}^{\infty} e^{im(\phi-\phi')} = \cancel{\frac{1}{2\pi}}\sum_{-\infty}^{\infty}\left\{\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial g_m}{\partial\rho'}\right) - \frac{m^2}{\rho'^2}g_m\right\} \times e^{im(\phi-\phi')}$$

Aquí la suposición más simple será:

$$\boxed{\frac{1}{\cancel{\rho'}}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial g_m}{\partial\rho'}\right) - \frac{m^2 g_m}{\rho'^{\cancel{2}}} = \frac{-4\pi}{\cancel{\rho'}}\delta(\rho-\rho')}$$

c) Si $\rho' \neq \rho$, $m > 0$

$$\left\{\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial}{\partial\rho'}\right) - \frac{m^2}{\rho'^2}\right\} g(\rho') = 0 \rightarrow g(\rho') = A_m\rho'^{\,m} + B_m\rho'^{\,-m}$$

- Si $\rho' < \rho \rightarrow g_m(\rho') = A_{1m}\rho'^{\,m} + B_{1m}\rho'^{\,-m}$
- Si $\rho' > \rho \rightarrow g_m(\rho') = A_{2m}\rho'^{\,m} + B_{2m}\rho'^{\,-m}$

Si $\rho' \rightarrow 0$ G es finito $\Rightarrow B_{1m} = 0$
Si $\rho' \rightarrow \infty$ G es finito $\rightarrow 0 \Rightarrow A_{2m} = 0$

$$\Rightarrow g(\rho') = \begin{cases} A_{1m}\rho'^{\,m} & \text{si } \rho' < \rho \\ B_{2m}\rho'^{\,-m} & \text{si } \rho' > \rho \end{cases}$$

Por continuidad

- Si $\rho' = \rho \rightarrow A_{1m}\rho^m = B_{2m}\rho^{-m}$

$\Rightarrow A_{1m} = \gamma_m\rho^{-m}$ y $B_{2m} = \gamma_m\rho^m \longrightarrow$ Elección de A_{1m} y B_{2m} que conserva la simetría de G.

$$\Rightarrow g_m(\rho') = \begin{cases} \gamma_m\rho^{-m}\rho'^{\,m} & \text{si } \rho' < \rho \\ \gamma_m\rho^m\rho'^{\,-m} & \text{si } \rho' > \rho \end{cases}$$

ó $g_m(\rho') = \gamma_m \rho_>^{-m}\rho_<^m$ $\qquad \rho_< = \text{mín}(\rho,\rho')$, $\rho_> = \text{máx}(\rho,\rho')$

Para evaluar γ_m recurrimos nuevamente a:

$$\frac{\partial}{\partial \rho'}\left(\rho' \frac{\partial g_m}{\partial \rho'}\right) - \frac{m^2 g_m}{\rho} = -4\pi\,\delta(\rho-\rho')$$

e integrar respecto a ρ' en una vecindad muy pequeña alrededor de ρ:

$$\int_{\rho-\epsilon}^{\rho+\epsilon} \frac{\partial}{\partial \rho'}\left(\rho' \frac{\partial g_m}{\partial \rho'}\right) d\rho' - m^2 \cancel{\int_{\rho-\epsilon}^{\rho+\epsilon} \frac{g(\rho')}{\rho'} d\rho'} = -4\pi \int_{\rho-\epsilon}^{\rho+\epsilon} \delta(\rho-\rho')\, d\rho'$$

0 si $\epsilon \to 0$

$$\rho' \frac{\partial g}{\partial \rho'}\Big|_{\rho+\epsilon} - \rho' \frac{\partial g}{\partial \rho'}\Big|_{\rho-\epsilon} = -4\pi$$

Si $\rho' = \rho+\epsilon \to \rho' > \rho \Rightarrow \rho' \frac{\partial g}{\partial \rho'}\Big|_{\rho+\epsilon} = \rho' \gamma_m (-m) \rho'^{-m-1} \rho^m \Big|_{\rho+\epsilon}$

Si $\rho' = \rho-\epsilon \to \rho' < \rho \Rightarrow \rho' \frac{\partial g}{\partial \rho'}\Big|_{\rho-\epsilon} = \rho' \gamma_m (+m) \rho^{-m} \rho'^{m-1} \Big|_{\rho-\epsilon}$

$$\Rightarrow \lim_{\epsilon \to 0} \left\{ \rho' \frac{\partial g}{\partial \rho'}\Big|_{\rho+\epsilon} - \rho' \frac{\partial g}{\partial \rho'}\Big|_{\rho-\epsilon} \right\} = -\rho' \gamma_m m \left(\rho^{-\cancel{m}-1} \rho^{\cancel{m}} + \rho^{-\cancel{m}} \rho^{\cancel{m}-1}\right)$$

$$-4\pi = -\cancel{\rho'} \gamma_m m \left(2 \cancel{\rho^{-1}}\right)$$

$$\Rightarrow \gamma_m = \frac{4\pi}{2m} \longrightarrow \boxed{g_m(\rho) = \frac{2\pi}{m} \rho_>^{-m} \rho_<^{m}} \quad \text{Si } m > 0$$

- Si $\rho' \neq \rho$, $m < 0$

$$\frac{\partial}{\partial \rho'}\left(\rho' \frac{\partial g}{\partial \rho'}\right) - \frac{m^2}{\rho'^2} g(\rho') = 0 \longrightarrow g(\rho') = A_m \rho'^{m} + B_m \rho'^{-m}$$

- Si $\rho' < \rho \to g = A_{1m} \rho'^{m} + B_{1m} \rho'^{-m}$
- Si $\rho' > \rho \to g = A_{2m} \rho'^{m} + B_{2m} \rho'^{-m}$ $\quad m < 0$

Si $\rho' \to 0$ g es finito $\to A_{1m} = 0$
Si $\rho' \to \infty$ $g \to 0$ $\to B_{2m} = 0$

Elección de A_{2m}, B_{1m} que conserva simetría.

- Si $\rho' = \rho \to B_{1m} \rho^{-m} = A_{2m} \rho^{m} \to B_{1m} = \gamma_m \rho^{m}$ y $A_{2m} = \gamma_m \rho^{-m}$

$$\Rightarrow g_m(\rho') = \begin{cases} \gamma_m \rho^{m} \rho'^{-m} & \text{si } \rho' < \rho \\ \gamma_m \rho^{-m} \rho'^{m} & \text{si } \rho' > \rho \end{cases} \quad \text{o} \quad g_m = \gamma_m \rho_>^{m} \rho_<^{-m}$$

$m < 0$

de nuevo integramos $\frac{\partial}{\partial \rho'}\left(\rho' \frac{\partial g}{\partial \rho'}\right) - \frac{m^2 g}{\rho} = -4\pi\,\delta(\rho-\rho')$

alrededor de ρ:

$$\int_{\rho-\epsilon}^{\rho+\epsilon} \frac{\partial}{\partial \rho'}\left(\rho' \frac{\partial g}{\partial \rho'}\right) d\rho' - m^2 \cancel{\int_{\rho-\epsilon}^{\rho+\epsilon} \frac{g(\rho')}{\rho'} d\rho'} \to 0 \text{ si } \epsilon \to 0 = -4\pi \int_{\rho-\epsilon}^{\rho+\epsilon} \delta(\rho-\rho')\, d\rho'$$

$$\rho' \frac{\partial g}{\partial \rho'}\Big|_{\rho+\epsilon} - \rho' \frac{\partial g}{\partial \rho'}\Big|_{\rho-\epsilon} = -4\pi$$

si $\rho' = \rho+\epsilon \to \rho' > \rho \Rightarrow \rho' \frac{\partial g}{\partial \rho'}\Big|_{\rho+\epsilon} = \rho' \gamma_m m \rho'^{\,m-1} \rho^{-m}\Big|_{\rho+\epsilon}$

Si $\rho' = \rho-\epsilon \to \rho' < \rho \Rightarrow \rho' \frac{\partial g}{\partial \rho'}\Big|_{\rho-\epsilon} = \rho' \gamma_m (-m) \rho^{m} \rho'^{\,-m-1}\Big|_{\rho-\epsilon}$

$$\Rightarrow \lim_{\epsilon \to 0} \left\{ \frac{\partial g}{\partial \rho'}\Big|_{\rho+\epsilon} - \frac{\partial g}{\partial \rho'}\Big|_{\rho-\epsilon} \right\} = \rho' m \gamma_m \left(\rho^{\cancel{m}-1} \cancel{\rho^{-m}} + \cancel{\rho^{m}} \rho^{-\cancel{m}-1} \right)$$

$$-4\pi = \cancel{\rho'} m \gamma_m \left(2 \cancel{\rho^{-1}}\right)$$

$$\Rightarrow \gamma_m = -\frac{4\pi}{2m} \longrightarrow \boxed{g_m = -\frac{2\pi}{m} \rho_>^{m} \rho_<^{-m}} \quad \text{si } m < 0$$

• Si $\rho' \neq \rho$, $m = 0$

$$\frac{\partial}{\partial \rho'}\left(\rho' \frac{\partial g}{\partial \rho'}\right) = 0 \longrightarrow \rho' \frac{\partial g}{\partial \rho'} = A_0 \Rightarrow g = A_0 \ln \rho' + B_0$$

• Si $\rho' < \rho \to g = A_{10} \ln \rho' + B_{10}$
• Si $\rho' > \rho \to g = A_{20} \ln \rho' + B_{20}$

Si $\rho' \to 0$ g es finito $\to A_{10} = 0$
Si $\rho' \to \infty$ $g \to 0$ $\to B_{20} = 0$

• Si $\rho' = \rho \to B_{10} = A_{20} \ln \rho \Rightarrow g = \begin{cases} A_{20} \ln \rho & \text{si } \rho' < \rho \\ A_{20} \ln \rho' & \text{si } \rho' > \rho \end{cases}$

ó $g = A_{20} \ln \rho_>$

integramos $\frac{\partial}{\partial \rho'}\left(\rho' \frac{\partial g}{\partial \rho'}\right) - \cancel{\frac{m^2 g}{\rho'}} \to 0 \quad m=0. = -4\pi \delta(\rho-\rho')$ alrededor de ρ

$$\int_{\rho-\epsilon}^{\rho+\epsilon} \frac{\partial}{\partial \rho'}\left(\rho' \frac{\partial g}{\partial \rho'}\right) d\rho' = -4\pi \int_{\rho-\epsilon}^{\rho+\epsilon} \delta(\rho-\rho')\, d\rho'$$

$$\rho' \frac{\partial g}{\partial \rho'}\Big|_{\rho+\epsilon} - \rho' \frac{\partial g}{\partial \rho'}\Big|_{\rho-\epsilon} = -4\pi$$

Si $\rho' = \rho + \epsilon \rightarrow \rho' > \rho \Rightarrow \rho' \frac{\partial g}{\partial \rho'}\Big|_{\rho+\epsilon} = \cancel{\rho'} A_{20} \frac{1}{\cancel{\rho'}}\Big|_{\rho+\epsilon} = A_{20}$

Si $\rho' = \rho - \epsilon \rightarrow \rho' < \rho \Rightarrow \rho' \underbrace{\cancel{\frac{\partial g}{\partial \rho'}}}_{0}\Big|_{\rho-\epsilon} = 0$

$$\Rightarrow \lim_{\epsilon \to 0} \left\{ \rho' \frac{\partial g}{\partial \rho'}\Big|_{\rho+\epsilon} - \rho' \frac{\partial g}{\partial \rho'}\Big|_{\rho-\epsilon} \right\} = A_{20} = -4\pi \rightarrow A_{20} = -4\pi$$

$$\Rightarrow \boxed{g = -4\pi \ln \rho_>}$$ Si $m=0$.

Ahora, $G = \frac{1}{2\pi} \sum_{-\infty}^{\infty} e^{im(\phi-\phi')} g_m(\rho,\rho')$

$$= \frac{1}{2\pi} g_0 + \frac{1}{2\pi} \sum_{m=-\infty}^{-1} g_{m<0}\, e^{im(\phi-\phi')} + \frac{1}{2\pi} \sum_{m=1}^{\infty} g_{m>0}\, e^{im(\phi-\phi')}$$

$$= \frac{1}{\cancel{2\pi}} (-\overset{2}{\cancel{4}}\pi \ln \rho_>) + \frac{1}{\cancel{2\pi}} \sum_{m=-\infty}^{-1} \left(-\frac{\cancel{2\pi}}{m} \rho_>^{m} \rho_<^{-m}\right) e^{im(\phi-\phi')}$$
$$+ \frac{1}{\cancel{2\pi}} \sum_{m=1}^{\infty} \left(\frac{\cancel{2\pi}}{m} \rho_>^{-m} \rho_<^{m}\right) e^{im(\phi-\phi')}$$

$$= -2 \ln \rho_> + \sum_{m=1}^{\infty} \left(-\frac{1}{-m} \rho_>^{-m} \rho_<^{+m}\right) e^{-im(\phi-\phi')}$$
$$+ \sum_{m=1}^{\infty} \left(\frac{1}{m} \rho_>^{-m} \rho_<^{m}\right) e^{im(\phi-\phi')}$$

$$= -\ln \rho_>^2 + \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\rho_<}{\rho_>}\right)^m \underbrace{\left(e^{im(\phi-\phi')} + e^{-im(\phi-\phi')}\right)}_{2\cos m(\phi-\phi')}$$

$$\boxed{G(\rho,\rho') = -\ln \rho_>^2 + 2 \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\rho_<}{\rho_>}\right)^m \cos m(\phi-\phi')}$$

2.18 $G = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{im(\phi-\phi')} g_m(\rho,\rho')$ donde g_m satisface:

$$\frac{\partial}{\partial \rho'}\left(\rho' \frac{\partial g_m}{\partial \rho'}\right) - \frac{m^2 g_m}{\rho'} = -4\pi \delta(\rho-\rho')$$

Si $\overset{m\neq 0}{\rho \neq \rho'} \rightarrow \frac{\partial}{\partial \rho'}\left(\rho' \frac{\partial g}{\partial \rho'}\right) - m^2 \frac{g_m}{\rho'} = 0$

$\rightarrow g_m = A_m \rho'^m + B_m \rho'^{-m}$

- Si $\rho' < \rho \rightarrow g_m(\rho') = A_{1m} \rho'^m + B_{1m} \rho'^{-m} = g_{1m}$
- Si $\rho' > \rho \rightarrow g_m(\rho') = A_{2m} \rho'^m + B_{2m} \rho'^{-m} = g_{2m}$

① Si $\rho' \neq \rho$, $m > 0$:

Si $\rho' \to 0$ g_m es finito $\Rightarrow$ $B_{1m} = 0$

Si $\rho' = b \to g_m(\rho, \rho' = b) = 0 \to A_{2m} b^m + B_{2m} b^{-m} = 0$

$\Rightarrow A_{2m} = \gamma_m b^{-m}$ y $B_{2m} = -\gamma_m b^m$ $\to$ Elección de A_{2m} y B_{2m} que conservan la simetría de g_m respecto ρ, ρ'.

$$\Rightarrow g_{2m} = \gamma_m b^{-m} \rho'^m - \gamma_m b^m \rho'^{-m}$$

$$g_m = \gamma_m \left[\left(\frac{\rho'}{b}\right)^m - \left(\frac{b}{\rho'}\right)^m \right] \quad \text{para } \rho' > \rho$$

• Si $\rho' = \rho \Rightarrow g_{1m}(\rho' = \rho) = g_{2m}(\rho' = \rho)$

$$A_{1m} \rho^m = \gamma_m \left[\left(\frac{\rho}{b}\right)^m - \left(\frac{b}{\rho}\right)^m \right] \to A_{1m} = \frac{\gamma_m}{\rho^m} \left[\quad \right]$$

$$\Rightarrow g_{1m} = \frac{\gamma_m}{\rho^m} \left[\left(\frac{\rho}{b}\right)^m - \left(\frac{b}{\rho}\right)^m \right] \rho'^m \quad \text{para } \rho' < \rho$$

para hallar γ_m usamos $\frac{\partial}{\partial \rho'}\left(\rho' \frac{\partial g_m}{\partial \rho'}\right) - \frac{m^2 g_m}{\rho'} = -4\pi \delta(\rho - \rho')$

e integramos alrededor de $\rho' = \rho$:

$$\int_{\rho-\epsilon}^{\rho+\epsilon} \frac{\partial}{\partial \rho'}\left(\rho' \frac{\partial g}{\partial \rho'}\right) d\rho' - m^2 \cancel{\int_{\rho-\epsilon}^{\rho+\epsilon} \frac{g(\rho')}{\rho'} d\rho'}^{\,0} = -4\pi \int_{\rho-\epsilon}^{\rho+\epsilon} \delta(\rho - \rho')\, d\rho' \qquad \epsilon \to 0$$

$$\rho' \frac{\partial g}{\partial \rho'}\Big|_{\rho+\epsilon} - \rho' \frac{\partial g}{\partial \rho'}\Big|_{\rho-\epsilon} = -4\pi$$

Si $\rho' = \rho + \epsilon \to \rho' > \rho \Rightarrow \rho' \frac{\partial g}{\partial \rho'} = \rho' \gamma_m m \left[\frac{\rho'^{m-1}}{b^m} + \frac{b^m}{\rho'^{m+1}} \right]$

Si $\rho' = \rho - \epsilon \to \rho' < \rho \Rightarrow \rho' \frac{\partial g}{\partial \rho'} = \rho' \frac{\gamma_m}{\rho^m} \left[\left(\frac{\rho}{b}\right)^m - \left(\frac{b}{\rho}\right)^m \right] m \rho'^{m-1}$

$$\Rightarrow \lim_{\epsilon \to 0} \left\{ \rho' \frac{\partial g}{\partial \rho'}\Big|_{\rho+\epsilon} - \rho' \frac{\partial g}{\partial \rho'}\Big|_{\rho-\epsilon} \right\} = -4\pi$$

$$\rho \gamma_m m \left[\frac{\rho^{m-1}}{b^m} + \frac{b^m}{\rho^{m+1}} \right] - \rho \frac{\gamma_m}{\rho^m} \left[\left(\frac{\rho}{b}\right)^m - \left(\frac{b}{\rho}\right)^m \right] m \rho^{m-1} = -4\pi$$

$$m \gamma_m \left\{ \cancel{\frac{\rho^m}{b^m}} + \frac{b^m}{\rho^m} - \cancel{\frac{\rho^m}{b^m}} + \frac{b^m}{\rho^m} \right\} = -4\pi \to \gamma_m = -\frac{4\pi}{2m} \left(\frac{\rho}{b}\right)^m$$

$$\Rightarrow g_m(\rho, \rho') = \begin{cases} -\frac{2\pi}{m} \left(\frac{\rho}{b}\right)^m \left[\left(\frac{\rho'}{b}\right)^m - \left(\frac{b}{\rho'}\right)^m \right] & \text{si } \rho' > \rho \\ -\frac{\frac{2\pi}{m} \left(\frac{\rho}{b}\right)^m}{\rho^m} \left[\left(\frac{\rho}{b}\right)^m - \left(\frac{b}{\rho}\right)^m \right] \rho'^m & \text{si } \rho' < \rho \end{cases}$$

ó $\quad g_m = -\frac{2\pi}{m}\left(\frac{\rho_<}{b}\right)^m\left[\left(\frac{\rho_>}{b}\right)^m - \left(\frac{b}{\rho_>}\right)^m\right] = -\frac{2\pi}{m}\left[\left(\frac{\rho_<\rho_>}{b^2}\right)^m - \left(\frac{\rho_<}{\rho_>}\right)^m\right]$

$$\boxed{g_m(\rho,\rho') = -\frac{2\pi}{m}\left[\left(\frac{\rho\rho'}{b^2}\right)^m - \left(\frac{\rho_<}{\rho_>}\right)^m\right]}$$ para $m>0$

② Si $\rho' \neq \rho$, $m<0$: $\quad \frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial g}{\partial\rho'}\right) - m^2\frac{g_m}{\rho'} = 0$

$\Rightarrow \quad g_m = A_m\rho'^m + B_m\rho'^{-m}$ con $m<0$

- Si $\rho' < \rho \rightarrow g_m(\rho') = A_{1m}\rho'^m + B_{1m}\rho'^{-m} = g_{1m}$
- Si $\rho' > \rho \rightarrow g_m(\rho') = A_{2m}\rho'^m + B_{2m}\rho'^{-m} = g_{2m}$ $\quad m<0$

Si $\rho' \to 0$ $\quad g_m$ es finito $\Rightarrow A_{1m} = 0$

Si $\rho' = b \rightarrow g_m(\rho, \rho'=b) = 0 \rightarrow A_{2m}b^m + B_{2m}b^{-m} = 0$

$\Rightarrow A_{2m} = \gamma_m b^{-m}$ y $B_{2m} = -\gamma_m b^m \rightarrow g_{2m} = \gamma_m\left[\left(\frac{\rho'}{b}\right)^m - \left(\frac{b}{\rho'}\right)^m\right]$ $\quad \rho'>\rho$

- Si $\rho' = \rho \rightarrow g_{1m}(\rho'=\rho) = g_{2m}(\rho'=\rho)$

$$B_{1m}\rho^{-m} = \gamma_m\left[\left(\frac{\rho}{b}\right)^m - \left(\frac{b}{\rho}\right)^m\right] \rightarrow B_{1m} = \gamma_m\rho^m\left[\quad\right]$$

$\Rightarrow g_{1m} = \gamma_m\rho^m\left[\left(\frac{\rho}{b}\right)^m - \left(\frac{b}{\rho}\right)^m\right]\rho'^{-m}$ $\quad$ si $\rho'<\rho$

de nuevo para hallar γ_m integramos $\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial g}{\partial\rho'}\right) - m^2\frac{g_m}{\rho'} = -4\pi\delta(\rho-\rho')$

en una vecindad de ρ: $\quad \epsilon\to 0$

$$\int_{\rho-\epsilon}^{\rho+\epsilon}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial g}{\partial\rho'}\right)d\rho' - m^2\cancel{\int_{\rho-\epsilon}^{\rho+\epsilon}\frac{g_m}{\rho'}d\rho'}^{\,0} = -4\pi\int_{\rho-\epsilon}^{\rho+\epsilon}\delta(\rho-\rho')$$

$$\rho'\frac{\partial g}{\partial\rho'}\Big|_{\rho+\epsilon} - \rho'\frac{\partial g}{\partial\rho'}\Big|_{\rho-\epsilon} = -4\pi$$

Si $\rho' = \rho+\epsilon \rightarrow \rho'>\rho \Rightarrow \rho'\frac{\partial g}{\partial\rho'} = \rho' m\gamma_m\left[\frac{\rho'^{m-1}}{b^m} + \frac{b^m}{\rho'^{m+1}}\right]$

Si $\rho' = \rho-\epsilon \rightarrow \rho'<\rho \Rightarrow \rho'\frac{\partial g}{\partial\rho'} = -\rho' m\gamma_m\rho^m\left[\left(\frac{\rho}{b}\right)^m - \left(\frac{b}{\rho}\right)^m\right]\rho'^{-m-1}$

$$\Rightarrow \lim_{\epsilon\to 0}\left\{\rho'\frac{\partial g}{\partial\rho'}\Big|_{\rho+\epsilon} - \rho'\frac{\partial g}{\partial\rho'}\Big|_{\rho-\epsilon}\right\} = -4\pi$$

$$\rho m\gamma_m\left[\frac{\rho^{m-1}}{b^m} - \frac{b^m}{\rho^{m+1}}\right] + \rho m\gamma_m\rho^m\left[\left(\frac{\rho}{b}\right)^m - \left(\frac{b}{\rho}\right)^m\right]\rho^{-m-1} = -4\pi$$

$$\Rightarrow \rho\, m\, \gamma_m \left\{ \frac{\rho^{m-1}}{b^m} + \cancel{\frac{b^m}{\rho^{m+1}}} + \frac{\rho^{m-1}}{b^m} - \cancel{\frac{b^m}{\rho^{m+1}}} \right\} = -4\pi \longrightarrow \gamma_m = -\frac{4\pi}{2m}\left(\frac{b}{\rho}\right)^m$$

$$\Rightarrow g_m(\rho,\rho') = \begin{cases} -\dfrac{2\pi}{m}\left(\dfrac{b}{\rho}\right)^m\left[\left(\dfrac{\rho'}{b}\right)^m - \left(\dfrac{b}{\rho'}\right)^m\right] & \text{para } \rho' > \rho \\ -\dfrac{2\pi}{m}\left(\dfrac{b}{\cancel{\rho}}\right)^m \cancel{\rho^m}\left[\left(\dfrac{\rho}{b}\right)^m - \left(\dfrac{b}{\rho}\right)^m\right]\rho'^{-m} & \text{para } \rho' < \rho \end{cases} \qquad m<0$$

ó $$g_m = -\frac{2\pi}{m}\left(\frac{b}{\rho_<}\right)^m\left[\left(\frac{\rho_>}{b}\right)^m - \left(\frac{b}{\rho_>}\right)^m\right] = -\frac{2\pi}{m}\left[\left(\frac{\rho_>}{\rho_<}\right)^m - \left(\frac{b^2}{\rho_<\rho_>}\right)^m\right]$$

$$g_m(\rho,\rho') = -\frac{2\pi}{m}\left[\left(\frac{\rho_>}{\rho_<}\right)^m - \left(\frac{b^2}{\rho\rho'}\right)^m\right] = \boxed{\frac{2\pi}{m}\left[\left(\frac{\rho\rho'}{b^2}\right)^{-m} - \left(\frac{\rho_<}{\rho_>}\right)^{-m}\right]} \quad \text{si } m<0$$

③ Si $\rho \neq \rho$, $m=0$

$$\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial g}{\partial\rho'}\right) = 0 \longrightarrow \rho'\frac{\partial g}{\partial\rho'} = A_0 \longrightarrow g = A_0 \ln\rho' + B_0$$

- Si $\rho' < \rho \longrightarrow g = A_{10}\ln\rho' + B_{10} = g_{10}$
- Si $\rho' > \rho \longrightarrow g = A_{20}\ln\rho' + B_{20} = g_{20}$

si $\rho' \to 0$ g es finito $\longrightarrow A_{10} = 0$

si $\rho' = b \longrightarrow g(\rho,\rho'=b) = 0 \longrightarrow A_{20}\ln b + B_{20} = 0 \rightarrow B_{20} = -A_{20}\ln b$

$$\Rightarrow g_{20} = A_{20}\ln\rho' - A_{20}\ln b = A_{20}\ln(\rho'/b) \qquad \text{si } \rho'>\rho$$

- Si $\rho' = \rho \longrightarrow g_{10}(\rho'=\rho) = g_{20}(\rho'=\rho)$

$$B_{10} = A_{20}\ln(\rho/b) = g_{10} \qquad \text{si } \rho'<\rho$$

para hallar A_{20} integramos $\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial g}{\partial\rho'}\right) - \cancel{m^2\frac{g}{\rho'}} = -4\pi\delta(\rho-\rho')$ $\to 0$ $m=0$

alrededor de ρ:

$$\int_{\rho-\epsilon}^{\rho+\epsilon}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial g}{\partial\rho'}\right)d\rho' = -4\pi\int_{\rho-\epsilon}^{\rho+\epsilon}\delta(\rho-\rho')\,d\rho'$$

$$\rho'\frac{\partial g}{\partial\rho'}\Big|_{\rho+\epsilon} - \rho'\frac{\partial g}{\partial\rho'}\Big|_{\rho-\epsilon} = -4\pi$$

si $\rho' = \rho+\epsilon \longrightarrow \rho'>\rho \Rightarrow \rho'\frac{\partial g}{\partial\rho'} = \cancel{\rho'}A_{20}\frac{1}{\cancel{\rho'}} = A_{20}$

si $\rho' = \rho-\epsilon \longrightarrow \rho'<\rho \Rightarrow \rho'\frac{\partial g}{\partial\rho'} = 0$

$$\Rightarrow \lim_{\epsilon\to 0}\left\{\rho'\frac{\partial g}{\partial \rho'}\Big|_{\rho+\epsilon} - \rho'\frac{\partial g}{\partial \rho'}\Big|_{\rho-\epsilon}\right\} = A_{20} = -4\pi$$

$$\Rightarrow g_0 = \begin{cases} -4\pi \ln(\rho'/b) & \text{si } \rho' > \rho \\ -4\pi \ln(\rho/b) & \text{si } \rho' < \rho \end{cases} \Rightarrow \boxed{g_0(\rho,\rho') = -4\pi \ln\left(\frac{\rho_>}{b}\right)}$$

para $m=0$

Ahora, $G = \frac{1}{2\pi}\sum_{-\infty}^{\infty} g_m(\rho,\rho')\, e^{im(\phi-\phi')}$

$$= \frac{1}{2\pi} g_0 + \frac{1}{2\pi}\sum_{m=-\infty}^{-1} g_{m<0}\, e^{im(\phi-\phi')} + \frac{1}{2\pi}\sum_{m=1}^{\infty} g_{m>0}\, e^{im(\phi-\phi')}$$

$$= \frac{1}{\cancel{2\pi}}\left[-\cancel{4\pi}^{\,2} \ln\left(\frac{\rho_>}{b}\right)\right] + \frac{1}{\cancel{2\pi}}\sum_{m=-\infty}^{-1} \frac{\cancel{2\pi}}{m}\left[\left(\frac{\rho\rho'}{b^2}\right)^{-m} - \left(\frac{\rho_<}{\rho_>}\right)^{-m}\right] e^{im(\phi-\phi')}$$

$$+ \frac{1}{\cancel{2\pi}}\sum_{m=1}^{\infty} -\frac{\cancel{2\pi}}{m}\left[\left(\frac{\rho\rho'}{b^2}\right)^{m} - \left(\frac{\rho_<}{\rho_>}\right)^{m}\right] e^{im(\phi-\phi')}$$

$$= -2\ln\left(\frac{\rho_>}{b}\right) + \sum_{m=1}^{\infty} \frac{1}{-m}\left[\left(\frac{\rho\rho'}{b^2}\right)^{+m} - \left(\frac{\rho_<}{\rho_>}\right)^{+m}\right] e^{-im(\phi-\phi')}$$

$$+ \sum_{m=1}^{\infty} -\frac{1}{m}\left[\left(\frac{\rho\rho'}{b^2}\right)^{m} - \left(\frac{\rho_<}{\rho_>}\right)^{m}\right] e^{im(\phi-\phi')}$$

$$\boxed{G(\rho,\rho') = -\ln\left(\frac{\rho_>}{b}\right)^2 - \sum_{m=1}^{\infty} \frac{1}{m}\left[\left(\frac{\rho\rho'}{b^2}\right)^{m} - \left(\frac{\rho_<}{\rho_>}\right)^{m}\right]\left\{2\cos m(\phi-\phi')\right\}}$$

$$\sum_{n=1}^{\infty}\frac{1}{n} x^n \cos n(\phi-\phi') = \sum_{n=1}^{\infty}\left[\int_0^x u^{n-1}\,du\right]\cos n(\phi-\phi')$$

$$= \int_0^x \left\{\frac{1}{u}\sum_{n=1}^{\infty} u^n \cos n(\phi-\phi')\right\} du$$

$$= \int_0^x \left\{\frac{1}{u}\sum_{n=1}^{\infty} u^n \left(\frac{e^{in(\phi-\phi')} + e^{-in(\phi-\phi')}}{2}\right)\right\} du$$

$$= \int_0^x \left\{\frac{1}{2u}\sum_{n=1}^{\infty}\left[\left(ue^{i(\phi-\phi')}\right)^n + \left(ue^{-i(\phi-\phi')}\right)^n\right]\right\} du$$

$$\boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x^n}}$$

$$\sum_{n=1}^{\infty} x^n = \frac{1}{1-x^n} - 1$$

$$= \int_0^x \left\{\frac{1}{2u}\left[\frac{1}{1-ue^{i(\phi-\phi')}} - 1 + \frac{1}{1-ue^{-i(\phi-\phi')}} - 1\right]\right\} du$$

$$= \int_0^x \left\{\frac{1}{2u}\left[\frac{1-ue^{-i(\phi-\phi')} + 1 - ue^{i(\phi-\phi')}}{1-ue^{i(\phi-\phi')} - ue^{-i(\phi-\phi')} + u^2} - 2\right]\right\} du$$

$$= \int_0^x \left\{ \frac{1}{2u} \left[\frac{2 - 2u\cos(\phi-\phi')}{1+u^2-2u\cos(\phi-\phi')} - 2 \right] \right\} du$$

$$= \int_0^x \left\{ \frac{1}{u} \left[\frac{\cancel{1} - \cancel{u\cos(\phi-\phi')} - (\cancel{1}+u^2 - \cancel{2u\cos(\phi-\phi')})}{1+u^2-2u\cos(\phi-\phi')} \right] \right\} du$$

$$= \int_0^x \left\{ \frac{1}{\cancel{u}} \left[\frac{\cancel{u}(-u+\cos(\phi-\phi'))}{1+u^2-2u\cos(\phi-\phi')} \right] \right\} du$$

$$= \int_0^x \left[\frac{\cos(\phi-\phi') - u}{1+u^2-2u\cos(\phi-\phi')} \right] du \qquad \begin{aligned} v &= 1+u^2-2u\cos(\phi-\phi') \\ dv &= 2u - 2\cos(\phi-\phi')\,du \end{aligned}$$

$$= -\frac{1}{2}\ln\left(1+u^2-2u\cos(\phi-\phi')\right)\Big|_0^x = -\frac{1}{2}\ln\left(1+x^2-2x\cos(\phi-\phi')\right)$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{1}{n} x^n \cos n(\phi-\phi') = -\frac{1}{2}\ln\left[1+x^2-2x\cos(\phi-\phi')\right]}$$

Usando esto,

$$\sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\rho\rho'}{b}\right)^m \cos m(\phi-\phi') = -\frac{1}{2}\ln\left[1+\left(\frac{\rho\rho'}{b^2}\right)^2 - 2\frac{\rho\rho'}{b}\cos(\phi-\phi')\right]$$

$$-\sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m \cos m(\phi-\phi') = \frac{1}{2}\ln\left[1+\left(\frac{\rho_<}{\rho_>}\right)^2 - 2\frac{\rho_<}{\rho_>}\cos(\phi-\phi')\right]$$

$$\sum_{m=1}^{\infty} \frac{1}{m}\left[\left(\frac{\rho\rho'}{b^2}\right)^m - \left(\frac{\rho_<}{\rho_>}\right)^m\right]\cos m(\phi-\phi') = -\frac{1}{2}\ln\left[\frac{1+(\rho\rho'/b^2)^2 - 2(\rho\rho'/b^2)\cos(\phi-\phi')}{1+(\rho_</\rho_>)^2 - 2(\rho_</\rho_>)\cos(\phi-\phi')}\right]$$

$$\Rightarrow G(\rho,\rho') = -\ln\left(\frac{\rho_>}{b}\right)^2 + \ln\left[\frac{\rho_>^2}{b^4}\,\frac{(b^4+\rho^2\rho'^2-2\rho\rho' b^2\cos(\phi-\phi'))}{(\rho_>^2+\rho_<^2-2\rho_<\rho_>\cos(\phi-\phi'))}\right]$$

$$= \ln\left[\frac{\cancel{\rho_>^2}}{b^4}\,\frac{(b^4+\rho^2\rho'^2-2\rho\rho' b^2\cos(\phi-\phi')}{(\rho_>^2+\rho_<^2-2\rho_<\rho_>\cos(\phi-\phi')} \Big/ \frac{\cancel{\rho_>^2}}{b^2}\right]$$

$$\boxed{G(\rho,\rho') = \ln\left[\frac{b^4+\rho^2\rho'^2-2\rho\rho' b^2\cos(\phi-\phi')}{b^2(\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')}\right]}$$

→ Note que esto es justo el potencial del problema 2.11 cuando $\gamma = 4\pi\epsilon_0$

b) $$\Phi(\rho,\phi) = \frac{1}{4\pi\epsilon_0}\int_V \cancel{\rho(\vec{r}')}^{\,0}\, G(\vec{r},\vec{r}')\,dv' - \frac{1}{4\pi}\oint_S \Phi(\vec{r}')\frac{\partial G_D}{\partial n'}\,da'$$

$$= -\frac{1}{4\pi}\oint_S \Phi(b,\phi')\frac{\partial G}{\partial \rho'}\Big|_{\rho'=b} da' \qquad da' = b\,d\phi$$

$\hat{n}'$ es un vector unitario dirigido hacia afuera del volumen de interés, en este caso $\hat{n}' = \hat{\rho}'$ (problema interior)

$$G = \ln\left(\rho^2\rho'^2 + b^4 - 2\rho\rho' b^2 \cos(\phi-\phi')\right) - \ln\left(b^2(\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi'))\right)$$

$$\frac{\partial G}{\partial \rho'} = \frac{2\rho'\rho^2 - 2\rho b^2\cos(\phi-\phi')}{\rho^2\rho'^2+b^4-2\rho\rho' b^2\cos(\phi-\phi')} - \frac{\cancel{b^2}(2\rho' - 2\rho\cos(\phi-\phi'))}{\cancel{b^2}(\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi'))}$$

$$\left.\frac{\partial G}{\partial \rho'}\right|_{\rho'=b} = \frac{2b\rho^2 - 2\rho b^2\cos(\phi-\phi')}{\rho^2 b^2 + b^4 - 2\rho b^3\cos(\phi-\phi')} - \frac{2b - 2\rho\cos(\phi-\phi')}{\rho^2+b^2-2\rho b\cos(\phi-\phi')}$$

$$= \frac{2b\rho^2 - \cancel{2\rho b^2\cos(\phi-\phi')} - (2b^3 - \cancel{2b^2\rho\cos(\phi-\phi')})}{b^2(\rho^2+b^2-2\rho b\cos(\phi-\phi'))}$$

$$= \frac{2\cancel{b}(\rho^2-b^2)}{b^{\cancel{2}}(\rho^2+b^2-2\rho b\cos(\phi-\phi'))}$$

$$\Rightarrow \Phi(\rho,\phi) = -\frac{1}{4\pi}\oint_0^{2\pi}\Phi(b,\phi')\frac{2(\rho^2-b^2)}{\cancel{b}(\rho^2+b^2-2\rho b\cos(\phi-\phi'))}\overbrace{(\cancel{b}\,d\phi')}^{da'}$$

$$\boxed{\Phi(\rho,\phi) = \frac{1}{2\pi}\oint_0^{2\pi}\Phi(b,\phi')\frac{(b^2-\rho^2)}{(\rho^2+b^2-2\rho b\cos(\phi-\phi'))}d\phi'} \rightarrow \text{Integral de Poisson.}$$

c) problema exterior (si $\rho' \neq \rho$, $m>0$)

Similar al problema interior sólo que ahora $A_{2m}=0$ y $B_{2m}\neq 0$ para hacer que $g_m(\rho'\to\infty)\to 0$, ello modifica g_{2m} quedando

$$g_{2m} = B_{2m}\rho'^{-m} \quad \text{si } \rho'>\rho.$$

como seguimos teniendo $g_m(\rho,\rho'=b)=0$ ello implica que tal como antes

$$g_m = \gamma_m\left[\left(\frac{\rho'}{b}\right)^m - \left(\frac{b}{\rho'}\right)^m\right] \quad \text{pero ahora para } \rho'<\rho$$

Por continuidad en $\rho=\rho'$:

$$\gamma_m\left[\left(\frac{\rho}{b}\right)^m - \left(\frac{b}{\rho}\right)^m\right] = B_{2m}\rho^{-m} \rightarrow B_{2m} = \gamma_m\rho^m\left[\left(\frac{\rho}{b}\right)^m - \left(\frac{b}{\rho}\right)^m\right]$$

Si $\rho'=\rho+\epsilon \rightarrow \rho'>\rho \Rightarrow \rho'\frac{\partial g}{\partial\rho'} = \rho'(-m)B_{2m}\rho'^{-m-1}$

Si $\rho'=\rho-\epsilon \rightarrow \rho'<\rho \Rightarrow \rho'\frac{\partial g}{\partial\rho'} = \rho' m\gamma_m\left[\frac{\rho'^{m-1}}{b^m} + \frac{b^m}{\rho'^{m+1}}\right]$

$$\Rightarrow \underbrace{\lim_{\epsilon\to 0}\left\{\rho'\frac{\partial g}{\partial\rho'}\Big|_{\rho+\epsilon} - \rho'\frac{\partial g}{\partial\rho'}\Big|_{\rho-\epsilon}\right\}}_{-4\pi} = -\rho m\gamma_m\left\{\left(\frac{\rho}{b}\right)^m - \cancel{\left(\frac{b}{\rho}\right)^m} + \left(\frac{\rho}{b}\right)^m + \cancel{\left(\frac{b}{\rho}\right)^m}\right\}$$

$$\Rightarrow \gamma_m = \frac{2\pi}{m}\left(\frac{b}{\rho}\right)^m$$

de modo que:

$$g_m(\rho,\rho') = \begin{cases} \frac{2\pi}{m}\left(\frac{b}{\rho}\right)^m \cancel{\rho^m}\left[\left(\frac{\rho}{b}\right)^m - \left(\frac{b}{\rho}\right)^m\right]\rho'^{-m} & \text{si } \rho'>\rho \\ \frac{2\pi}{m}\left(\frac{b}{\rho}\right)^m \left[\left(\frac{\rho'}{b}\right)^m - \left(\frac{b}{\rho'}\right)^m\right] & \text{si } \rho'<\rho \end{cases}$$

ó $g_m = \frac{2\pi}{m}\left(\frac{b}{\rho_>}\right)^m\left[\left(\frac{\rho_<}{b}\right)^m - \left(\frac{b}{\rho_<}\right)^m\right] = \frac{2\pi}{m}\left[\left(\frac{\rho_<}{\rho_>}\right)^m - \left(\frac{b^2}{\rho_>\rho_<}\right)^m\right]$

$$\boxed{g_m(\rho,\rho') = -\frac{2\pi}{m}\left[\left(\frac{b^2}{\rho\rho'}\right)^m - \left(\frac{\rho_<}{\rho_>}\right)^m\right]} \quad \text{con } m>0$$

problema exterior (si $\rho'\neq\rho$, $m<0$)

El procedimiento es el mismo esbozado en los casos anteriores. Aquí $A_{2m}\neq 0$ y $B_{2m}=0$ para hacer $g(\rho'\to\infty)=0$ ya que ahora $m<0$. El resultado como es de esperarse será:

$$\boxed{g_m(\rho,\rho') = -\frac{2\pi}{m}\left[\left(\frac{b^2}{\rho\rho'}\right)^{-m} - \left(\frac{\rho_<}{\rho_>}\right)^{-m}\right]} \quad \text{con } m<0$$

de modo que:

$$\frac{1}{2\pi}\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty} g_m e^{im(\phi-\phi')} = -2\sum_{m=1}^{\infty}\frac{1}{m}\left[\left(\frac{b^2}{\rho\rho'}\right)^m - \left(\frac{\rho_<}{\rho_>}\right)^m\right]\cos m(\phi-\phi')$$

$$= \ln\left[\frac{1+(b^2/\rho\rho')^2 - 2(b^2/\rho\rho')\cos(\phi-\phi')}{1+(\rho_</\rho_>)^2 - 2(\rho_<\rho_>)\cos(\phi-\phi')}\right]$$

Caso $m=0$, $\rho'\neq\rho$ para el problema exterior:

Este es similar que el del problema interior, solo que aquí $A_{20}=0$ y se hace para que $g(\rho'\to\infty)$ ...

$\Rightarrow g_{20} = B_{20}$ si $\rho'>\rho$

Como $g_0(\rho,\rho'=b)=0 \to A_{10}\ln b + B_{10} = 0 \to B_{10} = -A_{10}\ln b$

$\Rightarrow g_{10} = A_{10}\ln(\rho'/b)$ si $\rho'<\rho$

En $\rho=\rho' \to g_{10}(\rho'=\rho) = g_{20}(\rho'=\rho) \Rightarrow A_{10}\ln(\rho/b) = B_{20} = g_{20}$

si $\rho'=\rho+\epsilon \to \rho'>\rho \Rightarrow \rho'\frac{\partial g}{\partial\rho'} = 0$

si $\rho'=\rho-\epsilon \to \rho'<\rho \Rightarrow \rho'\frac{\partial g}{\partial\rho'} = \cancel{\rho'}A_{10}\frac{1}{\cancel{\rho'}}$ con lo que.

$$\lim_{\epsilon\to 0}\left\{\rho'\frac{\partial g}{\partial \rho'}\Big|_{\rho+\epsilon} - \rho'\frac{\partial g}{\partial \rho'}\Big|_{\rho-\epsilon}\right\} = -A_{10} = -4\pi \rightarrow A_{10} = 4\pi$$

con ello $g_0 = \begin{cases} 4\pi \ln(\rho'/b) & \rho' < \rho \\ 4\pi \ln(\rho/b) & \rho' > \rho \end{cases} \longrightarrow g_0 = -4\pi \ln\left(\frac{b}{\rho_<}\right)$

con esto se completa la solución para G:

$$\boxed{G(\rho,\rho') = -\ln\left(\frac{b}{\rho_<}\right)^2 - 2\sum_{m=1}^{\infty}\frac{1}{m}\left[\left(\frac{b^2}{\rho\rho'}\right)^m - \left(\frac{\rho_<}{\rho_>}\right)^m\right]\cos m(\phi-\phi')}$$

$$= \ln\left(\frac{\rho_<}{b}\right)^2 + \ln\left[\frac{1+(b^2/\rho\rho')^2 - 2(b^2/\rho\rho')\cos(\phi-\phi')}{1+(\rho_</\rho_>)^2 - 2(\rho_</\rho_>)\cos(\phi-\phi')}\right]$$

$$= \ln\left[\cancel{\frac{\rho_<^2}{b^2}\cdot\frac{\rho_>^2}{(\rho\rho')^2}}\,\frac{\rho^2\rho'^2 + b^4 - 2b^2\rho\rho'\cos(\phi-\phi')}{\rho_>^2 + \rho_<^2 - 2\rho_<\rho_>\cos(\phi-\phi')}\right]$$

$$\boxed{G(\rho,\rho') = \ln\left[\frac{b^4 + \rho^2\rho'^2 - 2b^2\rho\rho'\cos(\phi-\phi')}{b^2(\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')}\right]}$$

note que la expresión es idéntica a la del problema interior, de modo que al calcular el potencial mediante

$$\Phi(\rho,\phi) = -\frac{1}{4\pi}\oint_S \Phi(b,\phi)\frac{\partial G}{\partial n'}\Big|_S da' \qquad \hat{n}' = -\hat{\rho}'$$

lo único que cambia es un signo dado que $\partial G/\partial \hat{n}'$ ahora va en sentido contrario. Así lo que se obtiene es la integral de Poisson para el problema exterior:

$$\boxed{\Phi(\rho,\phi) = \frac{1}{2\pi}\oint_0^{2\pi}\Phi(b,\phi')\frac{b^2-\rho^2}{(\rho^2+b^2-2b\rho\cos(\phi-\phi'))}d\phi'}$$

Note que esta serie no se anula para $\rho' \to \infty$ ó $\rho \to \infty$ ya que en cualquiera de los casos sobrevive el término $-\ln(b/\rho_<)^2$ con $\rho_< = \text{máx}\{\rho,\rho'\}$

El problema puede salvarse si en la solución para G en el caso $m=0$ tenemos en cuenta que $G(\rho'\to\infty) = 0$ lo que hace que $g_0(\rho,\rho') = 0$ y no exista contribución para el caso $m=0$ a la serie de Fourier de modo que no exista el término antes mencionado y así la serie converja a cero para $\rho'\to\infty$ ó $\rho\to\infty$

2.19 $G = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{im(\phi-\phi')}$ donde g_m debe satisfacer

$$\frac{\partial}{\partial \rho'}\left(\rho' \frac{\partial g_m}{\partial \rho'}\right) - \frac{m^2 g_m}{\rho'} = -4\pi\,\delta(\rho-\rho') \quad \text{según problema (2.17).}$$

Si $\rho \neq \rho'$ $(m \neq 0)$ $\rightarrow$ $\frac{\partial}{\partial \rho'}\left(\rho' \frac{\partial g_m}{\partial \rho'}\right) - \frac{m^2 g_m}{\rho'} = 0$ para lo cual las soluciones son:

$$g_m = A_m \rho'^m + B_m \rho'^{-m}$$

Sea
$$g_{1m} = A_{1m}\rho'^m + B_{1m}\rho'^{-m} \quad \text{si } \rho' < \rho$$
$$g_{2m} = A_{2m}\rho'^m + B_{2m}\rho'^{-m} \quad \text{si } \rho' > \rho$$

<u>Si $\rho' \neq \rho$, $m \neq 0$:</u>

- Si $\rho' = b \rightarrow \rho' < \rho \Rightarrow A_{1m} b^m + B_{1m} b^{-m} = 0$

  $\Rightarrow A_{1m} = \gamma_m b^{-m}$ y $B_{1m} = -\gamma_m b^{+m}$ $\rightarrow$ Elección de Coeficientes que conserva la simetría de g_m

- Si $\rho' = c \rightarrow \rho' > \rho \Rightarrow A_{2m}\, {}^m + B_{2m}\, {}^{-m} = 0$

  $\Rightarrow A_{2m} = \beta_m c^{-m}$ y $B_{2m} = -\beta_m c^m$

- Si $\rho' = \rho \Rightarrow g_{1m}(\rho'=\rho) = g_{2m}(\rho'=\rho)$

  $\Rightarrow (\gamma_m b^{-m})\rho^m + (-\gamma_m b^m)\rho^{-m} = (\beta_m c^{-m})\rho^m + (-\beta_m c^m)\rho^{-m}$

$$\gamma_m\left[\left(\frac{\rho}{b}\right)^m - \left(\frac{b}{\rho}\right)^m\right] = \beta_m\left[\left(\frac{\rho}{c}\right)^m - \left(\frac{c}{\rho}\right)^m\right]$$

$$\Rightarrow \gamma_m = C_m\left[\left(\frac{\rho}{c}\right)^m - \left(\frac{c}{\rho}\right)^m\right] \quad \text{y} \quad \beta_m = \left[\left(\frac{\rho}{b}\right)^m - \left(\frac{b}{\rho}\right)^m\right]$$

$$\Rightarrow g_m = \begin{cases} C_m\left[\left(\frac{\rho}{c}\right)^m - \left(\frac{c}{\rho}\right)^m\right]\left[\left(\frac{\rho'}{b}\right)^m - \left(\frac{b}{\rho'}\right)^m\right] & \text{si } \rho' < \rho \\ C_m\left[\left(\frac{\rho}{b}\right)^m - \left(\frac{b}{\rho}\right)^m\right]\left[\left(\frac{\rho'}{c}\right)^m - \left(\frac{c}{\rho'}\right)^m\right] & \text{si } \rho' > \rho \end{cases}$$

$$\text{ó } g_m = C_m\left[\left(\frac{\rho_<}{b}\right)^m - \left(\frac{b}{\rho_<}\right)^m\right]\left[\left(\frac{\rho_>}{c}\right)^m - \left(\frac{c}{\rho_>}\right)^m\right]$$

$$= C_m\left[\left(\frac{\rho_<\rho_>}{bc}\right)^m + \left(\frac{bc}{\rho_<\rho_>}\right)^m - \left(\frac{b\rho_>}{c\rho_<}\right)^m - \left(\frac{c\rho_<}{b\rho_>}\right)^m\right]$$

$$\varrho_m = C_m \left[\left(\frac{\rho\rho'}{bc}\right)^m + \left(\frac{bc}{\rho\rho'}\right)^m - \left(\frac{b\rho_>}{c\rho_<}\right)^m - \left(\frac{c\rho_<}{b\rho_>}\right)^m \right]$$

para hallar C_m usamos $\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial\varrho_m}{\partial\rho'}\right) - \frac{m^2\varrho_m}{\rho'} = -4\pi\delta(\rho-\rho')$

e integramos alrededor de $\rho' = \rho$

$$\int_{\rho-\epsilon}^{\rho+\epsilon} \frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial\varrho}{\partial\rho'}\right)d\rho' - m^2\cancel{\int_{\rho-\epsilon}^{\rho+\epsilon}\frac{\varrho_m d\rho'}{\rho'}}^{\,0\ \epsilon\to 0} = -4\pi\int_{\rho-\epsilon}^{\rho+\epsilon}\delta(\rho-\rho')d\rho'$$

$$\rho'\frac{\partial\varrho_m}{\partial\rho'}\Big|_{\rho+\epsilon} - \rho'\frac{\partial\varrho_m}{\partial\rho'}\Big|_{\rho-\epsilon} = -4\pi$$

- Si: $\rho' = \rho+\epsilon \to \rho' > \rho$

$$\Rightarrow \rho'\frac{\partial\varrho}{\partial\rho'}\Big|_{\rho+\epsilon} = \rho' m C_m\left[\frac{\rho^m\rho'^{m-1}}{(bc)^m} - \frac{(bc)^m}{\rho^m\rho'^{m+1}} - \frac{b^m\rho'^{m-1}}{c^m\rho^m} + \frac{(c\rho)^m}{b^m\rho'^{m+1}}\right]$$

- Si $\rho' = \rho-\epsilon \to \rho' < \rho$

$$\Rightarrow \rho'\frac{\partial\varrho}{\partial\rho'}\Big|_{\rho-\epsilon} = \rho' m C_m\left[\frac{\rho^m\rho'^{m-1}}{(bc)^m} - \frac{(bc)^m}{\rho^m\rho'^{m+1}} + \frac{b^m\rho^m}{c^m\rho'^{m+1}} - \frac{c^m\rho'^{m-1}}{(b\rho)^m}\right]$$

$$\underbrace{\lim_{\epsilon\to 0}\left\{\rho'\frac{\partial\varrho_m}{\partial\rho'}\Big|_{\rho+\epsilon} - \rho'\frac{\partial\varrho_m}{\partial\rho'}\Big|_{\rho-\epsilon}\right\}}_{-4\pi} = mC_m\left\{\cancel{\frac{\rho^{2m}}{(bc)^m}} - \cancel{\frac{(bc)^m}{\rho^{2m}}} - \frac{b^m}{c^m} + \frac{c^m}{b^m} - \cancel{\frac{\rho^{2m}}{(bc)^m}} + \cancel{\frac{(bc)^m}{\rho^{2m}}} - \frac{b^m}{c^m} + \frac{c^m}{b^m}\right\}$$

$$\Rightarrow C_m = \frac{2\pi}{m}\left[\left(\frac{b}{c}\right)^m - \left(\frac{c}{b}\right)^m\right]^{-1}$$ con ello:

$$\boxed{\varrho_m(\rho,\rho') = \frac{2\pi}{m}\left[\left(\frac{b}{c}\right)^m - \left(\frac{c}{b}\right)^m\right]^{-1}\left[\left(\frac{\rho\rho'}{bc}\right)^m + \left(\frac{bc}{\rho\rho'}\right)^m - \left(\frac{b\rho_>}{c\rho_<}\right)^m - \left(\frac{c\rho_<}{b\rho_>}\right)^m\right]}$$ Si $m \neq 0$

Si $\rho' \neq \rho$, $m = 0$:

$$\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial\varrho}{\partial\rho'}\right) = 0 \to \rho'\frac{\partial\varrho}{\partial\rho'} = A_0 \to \varrho_0 = A_0\ln\rho' + B_0$$

Sea $\varrho_{10} = A_{10}\ln\rho' + B_{10}$ si $\rho' < \rho$

$\varrho_{20} = A_{20}\ln\rho' + B_{20}$ si $\rho' > \rho$

- Si $\rho' = b \to \rho' < \rho \Rightarrow A_{10}\ln b + B_{10} = 0 \to B_{10} = -A_{10}\ln b$
- Si $\rho' = c \to \rho' > \rho \Rightarrow A_{20}\ln c + B_{20} = 0 \to B_{20} = -A_{20}\ln c$

$$\Rightarrow g = \begin{cases} A_{10} \ln(\rho'/b) & \text{si } \rho' < \rho \\ A_{20} \ln(\rho'/c) & \text{si } \rho' > \rho \end{cases}$$

• Si $\rho' = \rho \Rightarrow A_{10} \ln(\rho/b) = A_{20} \ln(\rho/c)$

$\Rightarrow A_{10} = \gamma_0 \ln(\rho/c)$ y $A_{20} = \gamma_0 \ln(\rho/b) \rightarrow$ Elección que hace $g(\rho,\rho') = g(\rho',\rho)$

con ello $g_0 = \begin{cases} \gamma_0 \ln(\rho/c) \ln(\rho'/b) & \rho' < \rho \\ \gamma_0 \ln(\rho/b) \ln(\rho'/c) & \rho' > \rho \end{cases}$

ó $g_0(\rho,\rho') = \gamma_0 \ln(\rho_>/c) \ln(\rho_</b)$ cuando $m=0$

para hallar γ_0 integramos alrededor de ρ: $\frac{\partial}{\partial \rho'}\left(\rho' \frac{\partial g}{\partial \rho'}\right) - \cancel{m^2 \frac{g}{\rho'}} = -4\pi\,\delta(\rho-\rho')$ $\rightarrow 0$ $m=0$

$$\int_{\rho-\epsilon}^{\rho+\epsilon} \frac{\partial}{\partial \rho'}\left(\rho' \frac{\partial g}{\partial \rho'}\right) d\rho' = -4\pi \int_{\rho-\epsilon}^{\rho+\epsilon} \delta(\rho-\rho')$$

$$\rho' \frac{\partial g}{\partial \rho'}\Big|_{\rho+\epsilon} - \rho' \frac{\partial g}{\partial \rho'}\Big|_{\rho-\epsilon} = -4\pi$$

Si $\rho' = \rho+\epsilon \rightarrow \rho' > \rho \rightarrow \rho' \frac{\partial g_0}{\partial \rho'} = \cancel{\rho'} \frac{\gamma_0}{\cancel{\rho'}} \ln(\rho/b)$

Si $\rho' = \rho-\epsilon \rightarrow \rho' < \rho \rightarrow \rho' \frac{\partial g_0}{\partial \rho'} = \cancel{\rho'} \gamma_0 \ln(\rho/c) \frac{1}{\cancel{\rho'}}$

$$\Rightarrow \underbrace{\lim_{\epsilon \to 0} \left\{ \rho' \frac{\partial g}{\partial \rho'}\Big|_{\rho+\epsilon} - \rho' \frac{\partial g}{\partial \rho'}\Big|_{\rho-\epsilon} \right\}}_{-4\pi} = \gamma_0 \left\{ \ln \frac{\rho}{b} - \ln \frac{\rho}{c} \right\} = \gamma_0 \ln\left(\frac{c}{b}\right)$$

$$\Rightarrow \gamma_0 = -4\pi / \ln(c/b)$$

$$\Rightarrow \boxed{\gamma_0 = -4\pi \frac{\ln(\rho_>/c) \ln(\rho_</b)}{\ln(c/b)}}$$ para $m=0$.

Ahora, $G = \frac{1}{2\pi} \sum_{-\infty}^{\infty} g_m(\rho,\rho') e^{im(\phi-\phi')}$

$$= \frac{1}{2\pi} g_0 + \frac{1}{2\pi} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} g_m e^{im(\phi-\phi')}$$

$$= \frac{1}{2\pi} g_0 + \frac{1}{2\pi} \sum_{m=1}^{\infty} \left(g_m e^{im(\phi-\phi')} + g_{-m} e^{-im(\phi-\phi')} \right)$$

$$g_{-m} = \frac{2\pi}{-m} \frac{\left[\left(\frac{\rho\rho'}{bc}\right)^{-m} + \left(\frac{bc}{\rho\rho'}\right)^{-m} - \left(\frac{b\rho_>}{c\rho_<}\right)^{-m} - \left(\frac{c\rho_<}{b\rho_>}\right)^{-m} \right]}{\left[\left(\frac{b}{c}\right)^{-m} - \left(\frac{c}{b}\right)^{-m} \right]} = g_m$$

$$\Rightarrow G = \frac{1}{2\pi} g_0 + \frac{1}{2\pi}\sum_{m=1}^{\infty} g_m \underbrace{\left(e^{im(\phi-\phi')} + e^{-im(\phi-\phi')}\right)}_{2\cos m(\phi-\phi')}$$

$$= \frac{1}{\cancel{2\pi}}\left(-4\cancel{\pi}\,\frac{\ln(\rho_>/c)\ln(\rho_<\!/b)}{\ln(c/b)}\right)$$

$$+ \frac{1}{\cancel{2\pi}}\sum_{m=1}^{\infty}\frac{\cancel{2\pi}}{m}\,\frac{\left[\left(\frac{\rho\rho'}{bc}\right)^m + \left(\frac{bc}{\rho\rho'}\right)^m - \left(\frac{b\rho_>}{c\rho_<}\right)^m - \left(\frac{c\rho_<}{b\rho_>}\right)^m\right]}{\left[\left(\frac{b}{c}\right)^m - \left(\frac{c}{b}\right)^m\right]}(2\cos m(\phi-\phi'))$$

$$= \frac{2\ln(c/\rho_>)\cdot 2\ln(\rho_<\!/b)}{2\ln(c/b)}$$

$$+ 2\sum_{m=1}^{\infty}\frac{\cos m(\phi-\phi')}{m\left[\left(\frac{b}{c}\right)^m - \left(\frac{c}{b}\right)^m\right]}\left[\left(\frac{\rho_<}{b}\right)^m - \left(\frac{b}{\rho_<}\right)^m\right]\left[\left(\frac{\rho_>}{c}\right)^m - \left(\frac{c}{\rho_>}\right)^m\right]$$

$$= \frac{\ln(c/\rho_>)^2\ln(\rho_<\!/b)^2}{\ln(c/b)^2}$$

$$+ 2\sum_{m=1}^{\infty}\frac{\cos m(\phi-\phi')}{m\cancel{\left(\frac{c}{b}\right)^m}\left[\left(\frac{b}{c}\right)^{2m} - 1\right]}\,\frac{1}{\cancel{b^m}}\left[\rho_<^m - \frac{b^{2m}}{\rho_<^m}\right]\cdot\left[\frac{\rho_>^m}{c^{2m}} - \frac{1}{\rho_>^m}\right]\cancel{c^m}$$

$$\boxed{G(\rho,\rho') = \frac{\ln(c^2/\rho_>^2)\ln(\rho_<^2/b^2)}{\ln(c^2/b^2)} + 2\sum_{m=1}^{\infty}\frac{\left[\rho_<^m - b^{2m}/\rho_<^m\right]\left[1/\rho_>^m - \rho_>^m/c^{2m}\right]\cos m(\phi-\phi')}{m\left[1-(b/c)^{2m}\right]}}$$

2.20 $\sigma(\rho,\phi) = \frac{\lambda}{a}\sum_{n=0}^{3}(-1)^n\delta(\rho-a)\,\delta(\phi-n\pi/2)$

$$G(\rho,\phi;\rho',\phi') = -\ln\rho_>^2 + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m\cos m(\phi-\phi')$$

a) $\Phi(\rho,\phi) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{r}')\,G_D(\vec{r},\vec{r}')\,dv' - \frac{1}{4\pi}\oint_S \Phi(\vec{r}')\frac{\partial G_D}{\partial n'}\,da'$

Superficie

Ley de Coulomb

0 si la integral se extiende al espacio infinito. $\Phi(\vec{r}'\to\infty) = 0$.

$$= \frac{1}{4\pi\epsilon_0}\int\frac{\lambda}{a}\left(\sum_{n=0}^{3}(-1)^n\delta(\rho'-a)\,\delta(\phi'-n\pi/2)\right)\left(2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m\cos m(\phi-\phi') - \ln\rho_>^2\right)\rho'\,d\rho'\,d\phi'$$

$$\Phi = \frac{2\lambda}{4\pi\epsilon_0 a}\sum_{m=1}^{\infty}\sum_{n=0}^{3}\frac{(-1)^n}{m}\left\{\int \delta(\rho'-a)\delta(\phi'-\tfrac{n\pi}{2})\left(\frac{\rho_<}{\rho_>}\right)^m \cos m(\phi-\phi')\,\rho'd\rho'd\phi' - \int \delta(\rho'-a)\delta(\phi'-\tfrac{n\pi}{2})\ln \rho_>^2\cdot\rho'd\rho'd\phi'\right\}$$

$$= \frac{2\lambda}{4\pi\epsilon_0 a}\sum_{m=1}^{\infty}\sum_{n=0}^{3}\frac{(-1)^n}{m}\left\{\int \delta(\rho'-a)\left(\frac{\rho_<}{\rho_>}\right)^m \rho'd\rho'\cdot\overbrace{\int \delta(\phi'-\tfrac{n\pi}{2})\cos m(\phi-\phi')\,d\phi'}^{\cos m(\phi-n\pi/2)} - \int \delta(\rho'-a)\ln \rho_>^2\cdot\rho'd\rho'\underbrace{\int \delta(\phi'-\tfrac{n\pi}{2})\,d\phi'}_{1}\right\}$$

Si $\rho' < \rho$:

$$\int \delta(\rho'-a)\left(\frac{\rho_<}{\rho_>}\right)^m \rho'd\rho' = \int \delta(\rho'-a)\left(\frac{\rho'}{\rho}\right)^m \rho'd\rho' = \left(\frac{a}{\rho}\right)^m a$$

$$\int \delta(\rho'-a)\ln \rho_>^2\,\rho'd\rho' = \int \delta(\rho'-a)\ln \rho^2\,\rho'd\rho' = (\ln \rho^2)\,a$$

Si $\rho' > \rho$:

$$\int \delta(\rho'-a)\left(\frac{\rho_<}{\rho_>}\right)^m \rho'd\rho' = \int \delta(\rho'-a)\left(\frac{\rho}{\rho'}\right)^m \rho'd\rho' = \left(\frac{\rho}{a}\right)^m a$$

$$\int \delta(\rho'-a)\ln \rho_>^2\,\rho'd\rho' = \int \delta(\rho'-a)\ln \rho'^2\,\rho'd\rho' = (\ln a^2)\,a.$$

Note que si $\rho' < \rho \rightarrow a < \rho$ dado que justamente las líneas de carga se encuentran en $\rho' = a$. Similarmente si $\rho' > \rho \rightarrow a > \rho$, de modo que si definimos a partir de ahora:

$$\rho_< = \min\{\rho, a\} \quad \text{y} \quad \rho_> = \max\{\rho, a\}$$

$$\Rightarrow \int \delta(\rho'-a)\left(\frac{\rho_<}{\rho_>}\right)^m \rho'd\rho' = \left(\frac{\rho_<}{\rho_>}\right)^m a$$

$$\int \delta(\rho'-a)\ln \rho_>^2\,\rho'd\rho' = (\ln \rho_>^2)\,a.$$

$$\Rightarrow \Phi = \frac{2\lambda}{4\pi\epsilon_0 \cancel{a}}\sum_{m=1}^{\infty}\sum_{n=0}^{3}\frac{(-1)^n}{m}\left\{\left(\frac{\rho_<}{\rho_>}\right)^m \cancel{a}\cos m(\phi-\tfrac{n\pi}{2}) - (\ln \rho_>^2)\cancel{a}\right\}$$

$$= \frac{2\lambda}{4\pi\epsilon_0}\sum_{m=1}^{\infty}\frac{1}{m}\left\{\left(\frac{\rho_<}{\rho_>}\right)^m \sum_{n=0}^{3}(-1)^n\cos m(\phi-\tfrac{n\pi}{2}) - \ln \rho_>^2 \underbrace{\cancel{\sum_{n=0}^{3}(-1)^n}}_{0}\right\}$$

$$(-1)^n \cos\left[m\left(\phi - \frac{n\pi}{2}\right)\right] = (-1)^n \left\{ \cos m\phi \cos \frac{mn\pi}{2} - \sin m\phi \sin \frac{mn\pi}{2} \right\}$$

Si $n = 0$ $\quad = \cos m\phi$

Si $n = 1$ $\quad = -\left\{ \cos m\phi \cos \frac{m\pi}{2} - \sin m\phi \sin \frac{m\pi}{2} \right\}$

Si $n = 2$ $\quad = \cos m\phi \cos m\pi - \sin m\phi \underbrace{\sin m\pi}_{0}$

Si $n = 3$ $\quad = -\left\{ \cos m\phi \cos \frac{3m\pi}{2} - \sin m\phi \sin \frac{3m\pi}{2} \right\}$

Si $n = 0$ $\quad = \cos m\phi$

Si $n = 1$ $\quad = -\left\{ \cos m\phi \begin{bmatrix} 0 & \text{si } m \text{ impar} \\ -1 & \text{si } m = 2, 6, 10 \ldots \\ 1 & \text{si } m = 4, 8, 12 \ldots \end{bmatrix} - \sin m\phi \begin{bmatrix} 0 & \text{si } m \text{ par} \\ 1 & \text{si } m = 1, 5, 9 \ldots \\ -1 & \text{si } m = 3, 7, 11 \ldots \end{bmatrix} \right.$

Si $n = 2$ $\quad = \cos m\phi \begin{bmatrix} 1 & \text{si } m \text{ par} \\ -1 & \text{si } m \text{ impar} \end{bmatrix}$

Si $n = 3$ $\quad = -\left\{ \cos m\phi \begin{bmatrix} 0 & \text{si } m \text{ impar} \\ -1 & \text{si } m = 2, 6, 10 \ldots \\ 1 & \text{si } m = 4, 8, 12 \ldots \end{bmatrix} - \sin m\phi \begin{bmatrix} 0 & \text{si } m \text{ par} \\ -1 & \text{si } m = 1, 5, 9 \ldots \\ 1 & \text{si } m = 3, 7, 11 \ldots \end{bmatrix} \right.$

$$\sum_{n=0}^{3} (-1)^n \cos m\left(\phi - \frac{n\pi}{2}\right) = \begin{cases} 4 \cos(m\phi) & \text{si } m = 2, 6, 10, \ldots \\ 0 & \text{en otro caso} \end{cases}$$

$$\Rightarrow \Phi = \frac{2\lambda}{\cancel{4}\pi\epsilon_0} \sum_{m=2,6,10\ldots}^{\infty} \frac{1}{m} \left(\frac{\rho_<}{\rho_>}\right)^m \cdot \cancel{4} \cos(m\phi)$$

$$\boxed{\Phi = \frac{2\lambda}{\pi\epsilon_0} \sum_{k=0}^{\infty} \frac{1}{(4k+2)} \left(\frac{\rho_<}{\rho_>}\right)^{4k+2} \cos\left[(4k+2)\phi\right]}$$

b) $$\frac{(z - ia)(z + ia)}{(z - a)(z + a)} = \frac{z^2 + a^2}{z^2 - a^2} = \frac{1 + (a/z)^2}{1 - (a/z)^2}$$

donde $x = a/z$

$$\Rightarrow \ln \frac{(z - ia)(z + ia)}{(z - a)(z + a)} = \ln\left(\frac{1 + x^2}{1 - x^2}\right) = \ln(1 + x^2) - \ln(1 - x^2)$$

Usemos: $$\boxed{\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots}$$

$$\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots$$

$$\ln(1 + x) - \ln(1 - x) = 2x + 2\frac{x^3}{3} + 2\frac{x^5}{5} + \cdots$$

$$\ln(1 + x^2) - \ln(1 - x^2) = 2x^2 + 2\frac{x^6}{3} + 2\frac{x^{10}}{5} + \cdots$$

$$\Rightarrow \ln\left(\frac{1+x^2}{1-x^2}\right) = 2\left[x^2 + \frac{x^6}{3} + \frac{x^{10}}{5} + \dots\right] = 2\sum_{k=0}^{\infty} \frac{x^{4k+2}}{2k+1}$$

$$\Rightarrow \ln \frac{(z-ia)(z+ia)}{(z-a)(z+a)} = 2\sum_{k=0}^{\infty} \frac{1}{2k+1}\left(\frac{a}{z}\right)^{4k+2}$$

Note entonces que si evaluamos el potencial para $\rho > a$ nos aparece el término anterior:

$$\Phi(\rho>a) = \frac{2\lambda}{\pi\epsilon_0}\sum_{k=0}^{\infty} \frac{1}{(4k+2)}\left(\frac{a}{\rho}\right)^{4k+2} \cos\left[(4k+2)\phi\right]$$

$$= \frac{2\lambda}{\pi\epsilon_0}\sum_{k=0}^{\infty} \frac{1}{2(2k+1)}\left(\frac{a}{\rho}\right)^{4k+2} \mathrm{Re}\left\{e^{-i(4k+2)\phi}\right\}$$

Note que
$$\left.\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos(-\theta) + i\sin(-\theta) \\ &= \cos\theta - i\sin\theta \end{aligned}\right\} \Rightarrow \mathrm{Re}\, e^{\pm i\theta} = \cos\theta$$

hemos usado esto arriba escogiendo $e^{-i(4k+2)\phi}$ como la forma compleja cuya parte real es $\cos(4k+2)\phi$.

$$\Rightarrow \Phi(\rho>a) = \frac{\lambda}{\pi\epsilon_0}\sum_{k=0}^{\infty} \frac{1}{(2k+1)} \mathrm{Re}\left(\frac{a}{\rho e^{i\phi}}\right)^{4k+2}$$

$$= \frac{\lambda}{\pi\epsilon_0}\sum_{k=0}^{\infty} \frac{1}{2k+1} \mathrm{Re}\left(\frac{a}{z}\right)^{4k+2} \quad \text{si } z = \rho e^{i\phi}$$

$$= \frac{\lambda}{2\pi\epsilon_0} \mathrm{Re}\left\{2\sum_{k=0}^{\infty} \frac{1}{2k+1}\left(\frac{a}{z}\right)^{4k+2}\right\}$$

$$\boxed{\Phi(\rho>a) = \frac{\lambda}{2\pi\epsilon_0} \mathrm{Re}\left\{\ln\left[\frac{(z-ia)(z+ia)}{(z-a)(z+a)}\right]\right\}}$$

$\ln x^n = n \ln x$

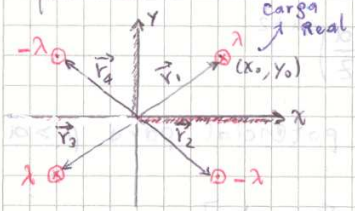
La solución del problema (2.3) era:

$$\Phi = \frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{|\vec{r}-\vec{r}_2|^2}{|\vec{r}-\vec{r}_1|^2}\cdot\frac{|\vec{r}-\vec{r}_4|^2}{|\vec{r}-\vec{r}_3|^2}\right] = \frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{|\vec{r}-\vec{r}_2|\,|\vec{r}-\vec{r}_4|}{|\vec{r}-\vec{r}_1|\,|\vec{r}-\vec{r}_3|}\right]^2$$

$$\Phi = \frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{|\vec{r}-\vec{r}_2|\,|\vec{r}-\vec{r}_4|}{|\vec{r}-\vec{r}_1|\,|\vec{r}-\vec{r}_3|}\right]$$

La cual coincide con la solución de arriba si hacemos una rotacion de ejes para la configuración del problema

(2.3) tal que la línea que conecta el origen del sistema de coordenadas con la carga real coincida con el eje x positivo.



En el sistema original:

$$\Phi = \frac{\lambda}{2\pi\epsilon_0} \ln\left[\frac{|\vec{r}-\vec{r}_2|\,|\vec{r}-\vec{r}_4|}{|\vec{r}-\vec{r}_1|\,|\vec{r}-\vec{r}_3|}\right]$$

En el sistema rotado:

$$\Phi' = \frac{\lambda}{2\pi\epsilon_0} \ln\left[\frac{|\vec{r}+y_0\hat{y}'|\,|\vec{r}-y_0\hat{y}'|}{|\vec{r}-x_0\hat{x}'|\,|\vec{r}+x_0\hat{x}'|}\right]$$

donde $\hat{x}'$, $\hat{y}'$ son los vectores unitarios del sistema rotado

Las soluciones coinciden en el caso en que $y_0 = a$, $x_0 = a$

Caso en el que la línea de carga real esta a 45° de los planos conductores.

$$\Phi' = \frac{\lambda}{2\pi\epsilon_0} \ln\left[\frac{|\vec{r}+a\hat{y}'|\,|\vec{r}-a\hat{y}'|}{|\vec{r}-a\hat{x}'|\,|\vec{r}+a\hat{y}'|}\right]$$

Es suficiente con asociar el eje $\hat{y}'$ con el eje imaginario y $\hat{x}'$ con el eje real, de modo que $\vec{r} = x\hat{x}' + y\hat{y}'$ se pueda escribir como el complejo $z = x + iy$, asociando de esta manera a $\vec{r}$ con un complejo z en el plano complejo. Con las asociaciones anteriores vemos que las soluciones coinciden cuando $x_0 = y_0 = a$.

c) Cerca del origen $\rho < a \Rightarrow \Phi = \frac{\lambda}{\pi\epsilon_0} \sum_{k=0}^{\infty} \frac{1}{2k+1}\left(\frac{\rho}{a}\right)^{4k+2} \cos(4k+2)\phi$

en cartesianas:

$$\rho = \sqrt{x^2+y^2} \longrightarrow \Phi = \frac{\lambda}{\pi\epsilon_0} \sum_{k=0}^{\infty} \frac{1}{2k+1} \frac{(x^2+y^2)^{2k+1}}{a^{4k+2}} \cos(4k+2)\phi$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$E_x = -\frac{\partial\Phi}{\partial x} = \frac{-\lambda}{\pi\epsilon_0} \sum_{k=0}^{\infty} \frac{1}{(2k+1)a^{4k+2}} \left\{ (2k+1)(x^2+y^2)^{2k}(2x)\cos(4k+2)\phi - (4k+2)(x^2+y^2)^{2k+1}\sin(4k+2)\phi \cdot \frac{\partial\phi}{\partial x} \right\}$$

$$E_y = -\frac{\partial\Phi}{\partial y} = -\frac{\lambda}{\pi\epsilon_0} \sum_{k=0}^{\infty} \frac{1}{(2k+1)a^{4k+2}} \left\{ (2k+1)(x^2+y^2)^{2k}(2y)\cos(4k+2)\phi - (4k+2)(x^2+y^2)^{2k+1}\sin(4k+2)\phi \frac{\partial\phi}{\partial x} \right\}$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) \rightarrow \frac{\partial\phi}{\partial x} = \frac{(-y/x^2)}{1+(y/x)^2} = -\frac{y}{x^2+y^2}$$

$$\frac{\partial\phi}{\partial y} = \frac{1/x}{1+(y/x)^2} = \frac{x}{x^2+y^2} \longrightarrow$$

$$\boxed{\begin{aligned} E_x &= -\frac{\lambda}{\pi\epsilon_0}\sum_{k=0}^{\infty}\frac{2(x^2+y^2)^{2k}}{a^{4k+2}}\left\{x\cos(4k+2)\phi + y\sin(4k+2)\phi\right\} \\ E_y &= -\frac{\lambda}{\pi\epsilon_0}\sum_{k=0}^{\infty}\frac{2(x^2+y^2)^{2k}}{a^{4k+2}}\left\{y\cos(4k+2)\phi - x\sin(4k+2)\phi\right\} \end{aligned}}$$

Conservando únicamente términos con $K=0$ y $K=1$

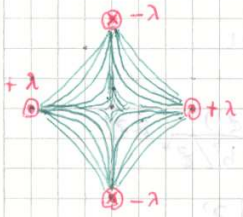
$$E_x \simeq -\frac{2\lambda}{\pi\epsilon_0}\left\{\frac{1}{a^2}\left[x\cos 2\phi + y\sin 2\phi\right] + \frac{(x^2+y^2)^2}{a^6}\left[x\cos 6\phi + y\sin 6\phi\right]\right\}$$

$$E_y \simeq -\frac{2\lambda}{\pi\epsilon_0}\left\{\frac{1}{a^2}\left[y\cos 2\phi - x\sin 2\phi\right] + \frac{(x^2+y^2)^2}{a^6}\left[y\cos 6\phi - x\sin 6\phi\right]\right\}$$

Si $y=0 \rightarrow \phi = 0°$

$$\Rightarrow E_x = -\frac{2\lambda}{\pi\epsilon_0}\left\{\frac{x}{a^2} + \frac{x^5}{a^6}\right\} \quad y \quad E_y = -\frac{2\lambda}{\pi\epsilon_0}\left\{-\frac{x}{a^2} - \frac{x^5}{a^6}\right\}$$

Lo cual coincide con las líneas de campo mostradas en el enunciado del problema. Note que cerca al origen se tiene que: E_x es aproximadamente igual a E_y en magnitud y sus direcciones debe ser según las ec. de arriba que:



- Si $x>0 \rightarrow E_x<0$ y si $y>0 \rightarrow E_y>0$
- Si $x<0 \rightarrow E_x>0$ si $y<0 \rightarrow E_y<0$

En $y=0$ $\quad E_x^{(0)} = -\frac{2\lambda}{\pi\epsilon_0}\frac{x}{a^2}$ si $K=0$ (cuadrupolo - 2^2 polo)

$E_x^{(1)} = -\frac{2\lambda}{\pi\epsilon_0}\frac{x^5}{a^6}$ si $K=1$ (2^6 - polo)

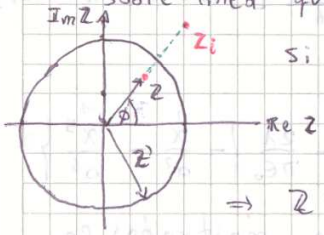
$$\Rightarrow \boxed{\frac{E_x^{(0)}}{E_x^{(1)}} = \left(\frac{a}{x}\right)^4} \rightarrow$$ razón de contribución de 2^2-polo respecto a 2^6 polo en E_x.

By: Herminso Villarraga Gómez

2.21 El teorema de Cauchy expresa que si z es interior a la curva C y $F(z) = \Phi(z)$ entonces

$$\Phi(z) = \frac{1}{2\pi i} \oint \frac{\Phi(z')}{z'-z} dz' \quad \text{donde} \quad z = \rho e^{i\phi}$$

Usando la sugerencia de Jackson, sea z_i el punto imagen de z y escojamos C como una circunferencia de radio b centrada en el origen. La imagen de z bajo la curva C será un punto en el plano complejo ubicado a una distancia $|z_i| = b^2/|z|$ del origen y sobre línea que une el origen y z.



si $|z| = \rho \rightarrow z_i = \frac{b^2}{\rho}$

ϕ expresa el ángulo entre el eje real y z, este permanece invariable para z_i.

$$\Rightarrow z = \rho e^{i\phi} \rightarrow z_i = \frac{b^2}{\rho} e^{i\phi} = \frac{b^2}{z^*}, \quad z^* = \rho e^{-i\phi}$$

Como $z_i = b^2/z^*$ está fuera de la curva C entonces:

$$\frac{1}{2\pi i} \oint \frac{\Phi(z')}{z' - b^2/z^*} dz' = 0 \quad \text{de modo que}$$

$$\Phi(z) - 0 = \frac{1}{2\pi i} \oint \frac{\Phi(z')}{z'-z} dz' - \frac{1}{2\pi i} \oint \frac{\Phi(z')}{z' - b^2/z^*} dz'$$

$$\Phi(z) = \frac{1}{2\pi i} \oint \Phi(z') \left\{ \frac{1}{z'-z} - \frac{1}{z' - b^2/z^*} \right\} dz'$$

dado que z' es un punto sobre la curva C, $z'z'^* = b^2$

$$\Rightarrow z' - \frac{b^2}{z^*} = z' - \frac{z'z'^*}{z^*} = \frac{z'}{z^*}(z^* - z'^*)$$

$$\Rightarrow \frac{1}{z'-z} - \frac{1}{z' - b^2/z^*} = \frac{1}{z'-z} + \frac{z^*/z'}{z'^* - z^*} = \frac{z'^* - z^* + z^* - zz^*/z'}{|z'-z|^2}$$

$$= \frac{1}{z'} \frac{z'z^* - zz^*}{|z'-z|^2}$$

$$\Rightarrow \Phi(z) = \frac{1}{2\pi i} \oint \Phi(z') \frac{z'z^* - zz^*}{|z'-z|^2} \frac{dz'}{z'}$$

$z = \rho e^{i\phi}$ $\quad z' = b e^{i\phi'} \rightarrow zz^* = \rho^2, \quad z'z'^* = b^2 \quad dz'/z' = i d\phi'$

$$\frac{dz'}{z'} = i\,d\phi' \quad y \quad |z'-z|^2 = (z'-z)(z'^*-z^*)$$

$$|z'-z|^2 = z'z'^* + zz^* - z'z^* - zz'^*$$
$$= b^2 + \rho^2 - b e^{i\phi'}\rho e^{-i\phi} - \rho e^{i\phi} b e^{-i\phi'}$$
$$= b^2 + \rho^2 - b\rho\left(e^{i(\phi'-\phi)} + e^{-i(\phi'-\phi)}\right)$$
$$= b^2 + \rho^2 - b\rho\left(2\cos(\phi-\phi')\right)$$

$\rho' = b$

$$\Rightarrow \Phi(\rho,\phi) = \frac{1}{2\pi \cancel{i}}\oint \Phi(\rho',\phi')\,\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi-\phi')}(\cancel{i}\,d\phi')$$

$$\boxed{\Phi(\rho,\phi) = \frac{1}{2\pi}\oint \Phi(b,\phi')\,\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi-\phi')}\,d\phi'}$$

2.22 Recordando que la funcion de Green para la esfera es:

Ec (2.17) $\rightarrow$ $$G(\vec{x},\vec{x}') = \frac{1}{(x^2+x'^2-2xx'\cos\gamma)^{1/2}} - \frac{1}{\left(\frac{x^2x'^2}{a^2}+a^2-2xx'\cos\gamma\right)^{1/2}}$$

y que para el problema interior $\hat{n}' = \hat{r}'$ ya que $\hat{n}'$ es un vector unitario dirigido hacia afuera del volumen de interés (en este caso el volumen de interés está adentro de la esfera).

$$\Rightarrow \left.\frac{\partial G}{\partial n'}\right|_S = +\left.\frac{\partial G}{\partial r'}\right|_{r'=a} = +\frac{x^2-a^2}{a(x^2+a^2-2ax\cos\gamma)^{3/2}} \qquad \text{Ec. (2.18)}$$

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\cancel{\int_V \rho(\vec{r}')\,G_D\,dv'}^{\,0} - \frac{1}{4\pi}\oint_S \Phi(\vec{r}')\,\frac{\partial G(\vec{r},\vec{r}')}{\partial n'}\,\underbrace{da'}_{a^2 d\Omega'}$$

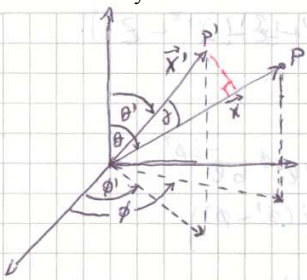
No hay cargas

$$\Phi(\vec{r}) = \frac{1}{4\pi}\oint \Phi(a,\theta',\phi')\,\frac{(a^2-x^2)}{\cancel{a}(x^2+a^2-2ax\cos\gamma)^{3/2}}(\cancel{a^2}\,d\Omega')$$

$$\Phi(r=a) = \begin{cases} V & \text{si } 0<\theta<\pi/2 \\ -V & \text{si } \frac{\pi}{2}<\theta<\pi \end{cases}$$

$$\Rightarrow \Phi(\vec{r}) = \frac{V}{4\pi}\int_0^{2\pi} d\phi' \left\{-\int_{-1}^{0} + \int_0^1\right\} d(\cos\theta')\,\frac{a(a^2-x^2)}{(x^2+a^2-2ax\cos\gamma)^{3/2}}$$

By: Herminso Villarraga Gómez



$$\vec{x} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$
$$\vec{x}' = \sin\theta'\cos\phi'\,\hat{i} + \sin\theta'\sin\phi'\,\hat{j} + \cos\theta'\,\hat{k}$$

$$\vec{x}\cdot\vec{x}' = \sin\theta\sin\theta'\cos\phi\cos\phi' + \sin\theta\sin\theta'\sin\phi\sin\phi' + \cos\theta\cos\theta'$$

$$\downarrow$$

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\underbrace{(\cos\phi\cos\phi' + \sin\phi\sin\phi')}_{\cos(\phi-\phi')}$$

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$$

$$\left.\begin{aligned}\cos\theta' &= -\cos(\pi-\theta')\\ \cos(\phi-\phi') &= -\cos(\phi-\phi'-\pi)\end{aligned}\right\}\ \text{sea entonces}\quad \begin{aligned}\theta' &= \pi-\theta'\\ \phi' &= \phi'+\pi\end{aligned}$$

$$\Rightarrow \cos\gamma \to -\cos\gamma = -\cos\theta\cos\theta' - \sin\theta\sin\theta'\cos(\phi-\phi')$$

de modo que $\left\{\begin{aligned}\theta' &\to \pi-\theta'\\ \phi' &\to \phi'+\pi\end{aligned}\right. \to -\int_{-1}^{0} d(\cos\theta') \to \int_0^1 d(\cos\theta')$

y con este cambio de variable que es en últimas una rotación de ejes:

$$\Phi = \frac{V\alpha(a^2-x^2)}{4\pi}\int_0^{2\pi} d\phi' \int_0^1 d(\cos\theta')\left[\frac{1}{(a^2+x^2-2ax\cos\gamma)^{3/2}} - \frac{1}{(a^2+x^2+2ax\cos\gamma)^{3/2}}\right]$$

podemos calcular Φ en el eje z para el cual $\theta=0 \to \cos\theta = 1$, $\sin\theta = 0$ y $\cos\gamma = \cos\theta'$:

$$\Phi(z) = \frac{V\alpha(a^2-x^2)}{2(\cancel{2\pi})}(\cancel{2\pi})\int_0^1\left[\frac{d(\cos\theta')}{(a^2+x^2-2ax\cos\theta')^{3/2}} - \frac{d(\cos\theta')}{(a^2+x^2+2ax\cos\theta')^{3/2}}\right]$$

$$= \frac{V\alpha(a^2-x^2)}{2}\left[\frac{1}{-\cancel{2}ax}\frac{-\cancel{2}}{(a^2+x^2-2ax\cos\theta')^{1/2}} - \frac{1}{\cancel{2}ax}\frac{-\cancel{2}}{(a^2+x^2+2ax\cos\theta')^{1/2}}\right]_0^1$$

Como $x = |\vec{x}| = z$ y evaluando:

$$\Phi(z) = \frac{\alpha V(a^2-z^2)}{2}\left[\frac{1}{az(a^2+z^2-2az)^{1/2}} + \frac{1}{az(a^2+z^2+2az)^{1/2}} - \frac{2}{az(a^2+z^2)^{1/2}}\right]$$

$$= \frac{\cancel{\alpha} V(a^2-z^2)}{2\cancel{\alpha} z}\left[\frac{1}{(a-z)} + \frac{1}{(a+z)} - \frac{2}{(a^2+z^2)^{1/2}}\right]$$

$$= \frac{V(a^2-z^2)}{2z}\left[\frac{a+\cancel{z}+a-\cancel{z}}{a^2-z^2} - \frac{2}{(a^2+z^2)^{1/2}}\right]$$

$$\Phi(z) = \frac{V(a^2-z^2)}{2z}\left[\frac{2a}{(a^2-z^2)} - \frac{2}{(a^2+z^2)^{1/2}}\right]$$

$$= V\left[\frac{a}{z} - \frac{(a^2-z^2)}{z(a^2+z^2)^{1/2}}\right] \rightarrow \boxed{\Phi(z) = \frac{Va}{z}\left[1 - \frac{(a^2-z^2)}{a\sqrt{a^2+z^2}}\right]}$$

Usando: $\boxed{(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots}$ $\quad x<1$

$$\left[1+\left(\frac{z}{a}\right)^2\right]^{-1/2} = 1 - \frac{1}{2}\left(\frac{z}{a}\right)^2 + \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{2}\frac{z^4}{a^4} + \dots$$

$$= 1 - \frac{1}{2}\frac{z^2}{a^2} + \frac{3}{8}\frac{z^4}{a^4}$$

$$\frac{a^2-z^2}{a^2\left(1+\frac{z^2}{a^2}\right)^{1/2}} = \frac{a^2-z^2}{a^2}\left(1+\frac{z^2}{a^2}\right)^{-1/2} = \left(1-\frac{z^2}{a^2}\right)\left(1+\frac{z^2}{a^2}\right)^{-1/2}$$

$$= 1 - \frac{1}{2}\frac{z^2}{a^2} + \frac{3}{8}\frac{z^4}{a^4} + \dots$$

$$- \frac{z^2}{a^2} + \frac{1}{2}\frac{z^4}{a^4} - \frac{3}{8}\frac{z^6}{a^6} + \dots$$

$$1 - \frac{a^2-z^2}{a^2(1+(z/a)^2)^{1/2}} = \frac{3}{2}\frac{z^2}{a^2} - \frac{7}{8}\frac{z^4}{a^4} + \dots$$

$$\Rightarrow \boxed{\Phi(z) = \frac{Va}{z}\left(\frac{3}{2}\frac{z^2}{a^2} - \frac{7}{8}\frac{z^4}{a^4} + \dots\right)} \qquad (z<a).$$

b) De la ec. (2.22) $\quad \Phi(z) = V\left(1 - \frac{(z^2-a^2)}{z\sqrt{a^2+z^2}}\right) \quad$ si $z>a$

$\Rightarrow$ para $z>a$:

$$E_z(z) = -\frac{\partial}{\partial z}\Phi(z) = V\left\{\frac{2z^2\sqrt{a^2+z^2} - (z^2-a^2)\left[\sqrt{a^2+z^2} + \frac{1}{2}z(a^2+z^2)^{-1/2}2z\right]}{z^2(\sqrt{a^2+z^2})^2}\right\}$$

$$= V\left\{\frac{2}{\sqrt{}} - \left(\frac{z^2}{z^2}+\frac{a^2}{z^2}\right)\frac{1}{\sqrt{}} - \frac{z^2-a^2}{(\sqrt{})^3}\right\}$$

$$= V\left\{\frac{2(a^2+z^2) - (a^2+z^2) + \frac{a^2}{z^2}(a^2+z^2) - (z^2-a^2)}{(\sqrt{})^3}\right\}$$

$$= V\left\{\frac{2a^2 + a^4/z^2 + a^2}{(\sqrt{})^3}\right\} = \frac{Va^2}{(\sqrt{})^3}\left(3 + \frac{a^2}{z^2}\right)$$

$$\Rightarrow \boxed{E_z(z) = \frac{V a^2}{(a^2+z^2)^{3/2}}\left(3 + \frac{a^2}{z^2}\right)} \quad \text{si } z > a$$

Si $z < a$: $\Phi(z) = V\left(\frac{a}{z} - \frac{a^2 - z^2}{z\sqrt{a^2+z^2}}\right) = V\left(\frac{a}{z} + \frac{z^2-a^2}{z\sqrt{a^2+z^2}}\right)$

$$\Rightarrow E_z(z) = -\frac{\partial}{\partial z} V\left[\frac{a}{z}\right] - V\frac{\partial}{\partial z}\left(\frac{z^2-a^2}{z\sqrt{a^2+z^2}}\right)$$

$$= \frac{Va}{z^2} - \frac{Va^2}{(a^2+z^2)^{3/2}}\left(3 + \frac{a^2}{z^2}\right) = -\frac{V}{a}\left[-\frac{a^2}{z^2} + \frac{3a^3 + a^5/z^2}{(a^2+z^2)^{3/2}}\right]$$

$$= -\frac{V}{a}\left[-\frac{a^2}{z^2} + \frac{3 + a^2/z^2}{(1+z^2/a^2)^{3/2}}\right]$$

$$\boxed{E_z(z) = -\frac{V}{a}\left[\frac{3 + (a/z)^2}{(1+(z/a)^2)^{3/2}}\right]} \quad \text{Si } |z| < a$$

$$[1+(z/a)^2]^{-3/2} = 1 - \frac{3}{2}\frac{z^2}{a^2} + \dots$$

$$\Rightarrow E_z(z) \simeq -\frac{V}{a}\left[3 + \cancel{\frac{a^2}{z^2}} - \frac{9}{2}\frac{z^2}{a^2} - \frac{3}{2}\cancel{\frac{a^2}{z^2}}\cancel{\frac{z^2}{a^2}} + \dots - \cancel{\frac{a^2}{z^2}}\right]$$

$$E_z(z) \simeq -\frac{V}{a}\left[\frac{3}{2} - \frac{9}{2}\frac{z^2}{a^2} + \dots\right] \Rightarrow \boxed{E_z(0) \simeq -\frac{3V}{2a}}$$

Si $|z| < a$: $E_z(a) = -\frac{V}{a}\left[\frac{3+1}{2^{3/2}} - 1\right] = -\frac{V}{a}\left[\frac{4}{\sqrt{2^2 \cdot 2}} - 1\right]$

$$= -\frac{V}{a}\left[\frac{2}{\sqrt{2}} - 1\right] = -\frac{V}{a}\left[\sqrt{2} - 1\right]$$

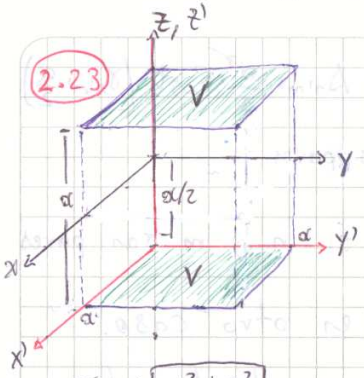
$$\boxed{E_z(a) = -\frac{V}{a}(-1+\sqrt{2})} \quad \text{para } z \to a_-$$

Si $z > a$ $\Rightarrow E_z(a) = \frac{Va^2}{a(2a^2)^{3/2}}(3+1) = \frac{Va^2}{2\sqrt{2}\,a^3} \cdot 4 = \frac{V\sqrt{2}}{a}$

$$\Rightarrow \boxed{E_z(a) = \frac{\sqrt{2}\,V}{a}} \quad \text{para } z \to a_+$$

$z = z' - a/2$, así si $z' = 0 \rightarrow z = -a/2$
$z' = a \rightarrow z = a/2$.

2.23



Podemos trabajar el problema con los planos a potencial V en $z = \pm a/2$ y luego hacer una transformación de coordenadas $z \rightarrow z - a/2$.

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \qquad \Phi = X(x) Y(y) Z(z)$$

$$\underbrace{\frac{1}{X(x)} \frac{d^2 X}{dx^2}}_{-\alpha^2} + \underbrace{\frac{1}{Y(y)} \frac{d^2 Y}{dy^2}}_{-\beta^2} + \underbrace{\frac{1}{Z(z)} \frac{d^2 Z}{dz^2}}_{\gamma^2} = 0$$

$$X \sim e^{\pm i\alpha x} \qquad Y \sim e^{\pm i\beta y} \qquad Z \sim e^{\pm \gamma z}$$

$$\gamma = \sqrt{\alpha^2 + \beta^2}$$

$X(x) = A \sin \alpha x + B \cos \alpha x$, $\quad X(0) = 0 \rightarrow B = 0$, $\quad X(a) = 0 \rightarrow \alpha = n\pi/a$

$Y(y) = C \sin \beta y + D \cos \beta y$, $\quad Y(0) = 0 \rightarrow D = 0$, $\quad Y(a) = 0 \rightarrow \beta = m\pi/a$

$Z(z) = E \sinh \gamma z + F \cosh \gamma z$

$$Z(a/2) = Z(-a/2)$$

$$E \sinh(\gamma a/2) + \cancel{F \cosh(\gamma a/2)} = E \sinh(-\gamma a/2) + F \cosh(-\gamma a/2)$$
$$= -E \sinh(\gamma a/2) + \cancel{F \cosh(\gamma a/2)}$$

$$E \cancel{\sinh(\gamma a/2)} = -E \cancel{\sinh(\gamma a/2)} \rightarrow E = -E \rightarrow E = 0$$

$$\Rightarrow \Phi = X(x) Y(y) Z(z) = (A \sin \alpha_n x)(C \sin \beta_m y)(F \cosh \gamma_{nm} z)$$
$$= A_{nm} \sin \alpha_n x \sin \beta_m y \cosh \gamma_{nm} z$$

$$\alpha_n = n\pi/a \qquad \beta_m = m\pi/a \qquad \gamma_n = \sqrt{\alpha_n^2 + \beta_m^2}$$

$$\Phi = \sum_{n,m=1}^{\infty} A_{nm} \sin \alpha_n x \sin \beta_m y \cosh \gamma_{nm} z$$

$$\Phi(z = a/2) = V = \sum_{n,m=1}^{\infty} A_{nm} \sin \alpha_n x \sin \beta_m y \cosh(\gamma_{nm} a/2)$$

$$\int_0^a \int_0^a V \sin\left(\frac{n'\pi x}{a}\right) \sin\left(\frac{m'\pi y}{a}\right) dx\, dy$$
$$= \sum_{n,m=1}^{\infty} A_{nm} \underbrace{\int_0^a \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n'\pi x}{a}\right) dx}_{(a/2)\delta_{nn'}} \overbrace{\int_0^a \sin\left(\frac{m\pi y}{a}\right) \sin\left(\frac{m'\pi y}{a}\right) dy}^{(a/2)\delta_{mm'}} \times \cosh(\gamma_{nm} a/2)$$

$$V\left(\frac{a}{n'\pi}\right)\left(\frac{a}{m'\pi}\right)\left[\cos \frac{n'\pi x}{a}\right]_0^a \left[\cos \frac{m'\pi x}{a}\right]_0^a = \sum_{n,m=1}^{\infty} A_{nm} \frac{a}{2}\delta_{nn'} \frac{a}{2}\delta_{mm'} \times \cosh(\gamma_{nm} a/2)$$

$$\Rightarrow V \frac{\cancel{a^2}}{n'm'\pi^2}\underbrace{\left[(-1)^{n'}-1\right]\left[(-1)^{m'}-1\right]}_{\substack{4 \text{ si } n' \text{ y } m' \text{ son ambos pares.} \\ 0 \text{ en otro caso.}}} = A_{n'm'}\frac{\cancel{a^2}}{4}\cosh\left(\gamma_{n'm'}\frac{a}{2}\right)$$

$$\Rightarrow A_{nm} = \begin{cases} \dfrac{16V}{\pi^2 nm \cosh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)} & \text{si } n \text{ y } m \text{ son pares} \\ 0 & \text{en otro caso.} \end{cases}$$

Sustituyendo A_{nm} en $\Phi(x,y,z)$ con la transformación $z \to z - a/2$.

$$\boxed{\Phi = 16V \sum_{n,m \text{ pares}}^{\infty} \frac{\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)}{\pi^2 nm \cosh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)} \cosh\left(\frac{\pi}{a}\sqrt{n^2+m^2}\left(z-\frac{a}{2}\right)\right)}$$

b) $$\Phi\left(\frac{a}{2},\frac{a}{2},\frac{a}{2}\right) = 16V \sum_{n,m \text{ pares}}^{\infty} \frac{\sin(n\pi/2)\sin(m\pi/2)\cancelto{1}{\cosh 0}}{\pi^2 nm \cosh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)}$$

$$= 16V \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}}{\pi^2 (2i+1)(2j+1)\cosh \frac{\pi}{2}\sqrt{(2i+1)^2+(2j+1)^2}}$$

El resultado exacto debe ser el promedio del potencial en las 6 caras (problema 2.28).

Así $\boxed{\Phi\left(\frac{a}{2},\frac{a}{2},\frac{a}{2}\right) = \frac{V}{3}}$

$\hat{n} = -\hat{z}$

c) $$\sigma = \epsilon_0 E_n = -\epsilon_0 \frac{\partial \Phi}{\partial n}\Big|_s$$

$\hat{n}$ es un vector normal unitario dirigido hacia la región de interés (hacia el interior del cubo en este caso).

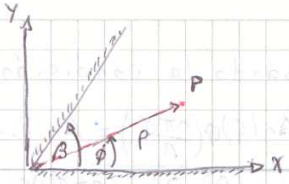
$$= \epsilon_0 \frac{\partial \Phi}{\partial z}\Big|_{z=a}$$

$$= 16\epsilon_0 V \sum_{n,m \text{ pares}}^{\infty} \left\{ \frac{\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)\sinh\left(\frac{\pi}{a}\sqrt{n^2+m^2}\left(z-\frac{a}{2}\right)\right)}{\pi^2 nm \cosh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)} \Bigg|_0^a \right\} * \frac{\pi}{a}\sqrt{n^2+m^2}$$

$$= \frac{16\epsilon_0 V}{\pi^2}\frac{\cancel{\pi}}{a}\sum_{n,m \text{ pares}}^{\infty} \frac{\sqrt{n^2+m^2}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)\sinh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)}{nm\cosh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)}$$

$$\boxed{\sigma = \frac{16\epsilon_0 V}{\pi a}\sum_{n,m \text{ pares}}^{\infty} \frac{\sqrt{n^2+m^2}}{nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)\tanh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)}$$

2.24



$$f(\phi) = \sum_m A_m \sin(m\pi\phi/\beta)$$

multiplicando por $\sin(n\pi\phi/\beta)$ e integrando.

$$\int_0^\beta \sum_m A_m \sin\left(\frac{m\pi\phi}{\beta}\right)\sin\left(\frac{n\pi\phi}{\beta}\right)d\phi = \int_0^\beta f(\phi)\sin\left(\frac{n\pi\phi}{\beta}\right)d\phi$$

$$\sum_m A_m \underbrace{\int_0^\beta \sin\left(\frac{m\pi\phi}{\beta}\right)\sin\left(\frac{n\pi\phi}{\beta}\right)d\phi}_{\frac{\beta}{2}\delta_{mn}} = \int_0^\beta f(\phi)\sin\left(\frac{n\pi\phi}{\beta}\right)d\phi$$

$$\sum_m A_m \frac{\beta}{2}\delta_{mn} = \frac{A_n}{2}\beta \qquad = \int_0^\beta f(\phi)\sin\left(\frac{n\pi\phi}{\beta}\right)d\phi$$

$$\rightarrow A_m = \frac{2}{\beta}\int_0^\beta f(\phi)\sin\left(\frac{n\pi\phi}{\beta}\right)d\phi$$

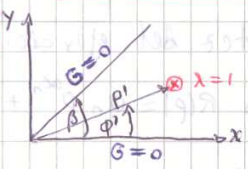
$$\rightarrow f(\phi) = \sum_m A_m \sin\left(\frac{m\pi\phi}{\beta}\right) = \sum_m \left(\frac{2}{\beta}\int_0^\beta f(\phi')\sin\left(\frac{m\pi\phi'}{\beta}\right)\right)\sin\left(\frac{m\pi\phi}{\beta}\right)$$

$$= \int_0^\beta \frac{2}{\beta} f(\phi') \sum_m^\infty \sin\left(\frac{m\pi\phi'}{\beta}\right)\sin\left(\frac{m\pi\phi}{\beta}\right)$$

$$\int_0^\beta f(\phi')\delta(\phi-\phi')d\phi' = \int_0^\beta f(\phi')\left(\frac{2}{\beta}\sum_{m=1}^\infty \sin\left(\frac{m\pi\phi'}{\beta}\right)\sin\left(\frac{m\pi\phi}{\beta}\right)\right)$$

$$\rightarrow \boxed{\delta(\phi-\phi') = \frac{2}{\beta}\sum_{m=1}^\infty \sin\left(\frac{m\pi\phi'}{\beta}\right)\sin\left(\frac{m\pi\phi}{\beta}\right)}$$

2.25



$$\frac{1}{\cancel{\rho}}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial G}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 G}{\partial\varphi^2} = -\frac{4\pi}{\cancel{\rho}}\delta(\rho-\rho')\delta(\varphi-\varphi')$$

Si $\varphi' \neq \varphi$ $\qquad \frac{\partial}{\partial\rho}\left(\rho\frac{\partial G}{\partial\rho}\right) + \frac{1}{\rho}\frac{\partial^2 G}{\partial\varphi^2} = 0$

Sea $G = R(\rho)\psi(\varphi) \rightarrow \underbrace{\frac{\rho}{R(\rho)}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R(\rho)}{\partial\rho}\right)}_{+k^2} + \underbrace{\frac{1}{\Phi(\varphi)}\frac{\partial^2\psi(\varphi)}{\partial^2\varphi^2}}_{-k^2} = 0$

$$\rightarrow \frac{d^2\psi}{d\varphi^2} + k^2\phi = 0 \rightarrow \psi = A\sin\alpha\varphi + B\cos\alpha\varphi$$

$$\psi(0) = 0 \rightarrow B = 0$$
$$\psi(\beta) = 0 \rightarrow \alpha = n\pi/\beta$$

$\Rightarrow \quad \psi = A_n \sin\left(\frac{n\pi}{\beta}\varphi\right)$. y usando la condición de simetría $\psi(\varphi,\varphi') = \psi(\varphi',\varphi) \rightarrow A_n \sin\left(\frac{n\pi}{\beta}\varphi\right) = A_n \sin\left(\frac{n\pi}{\beta}\varphi'\right)$

$$A_n(\varphi,\varphi') = C_n \sin\left(\frac{n\pi}{\beta}\varphi'\right) \rightarrow \boxed{\psi_n = C_n \sin\left(\frac{n\pi\varphi'}{\beta}\right)\sin\left(\frac{n\pi}{\beta}\varphi\right)}$$

$$\psi = \sum_n C_n \sin\alpha_n\varphi \sin\alpha_n\varphi' \qquad \alpha_n = n\pi/\beta$$

$$\frac{\partial^2\psi}{\partial\varphi^2} = -\sum_n \alpha_n^2 C_n \sin\alpha_n\varphi \sin\alpha_n\varphi' \qquad G(\rho,\varphi) = R(\rho)\psi(\varphi)$$

$$\frac{\partial}{\partial\rho}\left(\rho\frac{\partial G}{\partial\rho}\right) + \frac{1}{\rho}\frac{\partial^2 G}{\partial\varphi^2} = -4\pi\,\delta(\rho-\rho')\delta(\varphi-\varphi')$$

$$\psi(\varphi)\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) + \frac{R(\rho)}{\rho}\frac{\partial^2\psi}{\partial\varphi^2} = -4\pi\,\delta(\rho-\rho')\delta(\varphi-\varphi')$$

$$\sum_n\left\{\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) - \alpha_n^2\frac{R(\rho)}{\rho}\right\} C_n \sin\alpha_n\varphi \sin\alpha_n\varphi' = -4\pi\,\delta(\rho-\rho')\delta(\varphi-\varphi')$$

La elección más simple es $\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) - \alpha_n^2\frac{R(\rho)}{\rho} = -4\pi\,\delta(\rho-\rho')$ y $\delta(\varphi-\varphi') = \sum_n C_n \sin\alpha_n\varphi \sin\alpha_n\varphi'$, de hecho para está última tenemos $\int_0^\beta \delta(\varphi-\varphi')\sin(\alpha_m\varphi')\,d\varphi' = \sum_n C_n \sin\alpha_n\varphi \underbrace{\int_0^\beta \sin\alpha_n\varphi' \cdot \sin\alpha_m\varphi'\,d\varphi'}_{\frac{\beta}{2}\delta_{nm}}$

$$\sin\alpha_m\varphi = \sum_n C_n \sin\alpha_n\varphi \cdot \frac{\beta}{2}\delta_{nm}$$

$$\cancel{\sin\alpha_m\varphi} = C_m\frac{\beta}{2}\cancel{\sin\alpha_m\varphi} \rightarrow \boxed{C_m = \frac{2}{\beta}}$$

lo cual da justo la relación de completez del ejercicio (2.24).

<u>Si $\rho \neq \rho'$</u> $\quad \frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) - \alpha_n\frac{R}{\rho} = 0 \rightarrow R(\rho) = a_n\rho^{\alpha_n} + b_n\rho^{-\alpha_n}$

- Si $\rho < \rho' \rightarrow R(\rho) = a_{1n}\rho^{\alpha_n} + b_{1n}\rho^{-\alpha_n} = R_1(\rho)$
- Si $\rho > \rho' \rightarrow R(\rho) = a_{2n}\rho^{\alpha_n} + b_{2n}\rho^{-\alpha_n} = R_2(\rho)$

Si $\rho \rightarrow 0 \quad \rho < \rho', \quad R(\rho\rightarrow 0) \rightarrow 0 \Rightarrow b_{1n} = 0$

Si $\rho \rightarrow \infty \quad \rho > \rho', \quad R(\rho\rightarrow\infty) \rightarrow 0 \Rightarrow a_{2n} = 0$

Si $\rho = \rho' \rightarrow a_{1n}\rho'^{\alpha_n} = b_{2n}\rho'^{-\alpha_n}$

$\Rightarrow a_{1n} = \gamma_n\rho'^{-\alpha_n}$ y $b_{2n} = \gamma_n\rho'^{\alpha_n}$ Elección de coeficientes que conserva la simetría de G: $G(\rho,\rho') = G(\rho',\rho)$.

$$\Rightarrow R(\rho,\rho') = \begin{cases} \gamma_n \rho'^{-\alpha_n} \rho^{\alpha_n} & \text{si } \rho<\rho' \\ \gamma_n \rho'^{\alpha_n} \rho^{-\alpha_n} & \text{si } \rho>\rho' \end{cases}$$

ó $R(\rho,\rho') = \gamma_n \rho_<^{\alpha_n} \rho_>^{-\alpha_n}$ con $\rho_> = \text{máx}\{\rho,\rho'\}$, $\rho_< = \text{mín}\{\rho,\rho'\}$

Para hallar γ_n integramos $\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) - \alpha_n^2 \frac{R}{\rho} = -4\pi\,\delta(\rho-\rho')$ en una vecindad de ρ':

$$\int_{\rho'-\epsilon}^{\rho'+\epsilon} \frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right)d\rho - \alpha_n^2 \cancel{\int_{\rho'-\epsilon}^{\rho'+\epsilon} \frac{R(\rho)}{\rho}d\rho}^{\,0\ \ \epsilon\to 0} = -4\pi\int_{\rho'-\epsilon}^{\rho'+\epsilon}\delta(\rho-\rho')\,d\rho$$

$$\rho\frac{\partial R}{\partial\rho}\Big|_{\rho'+\epsilon} - \rho\frac{\partial R}{\partial\rho}\Big|_{\rho'-\epsilon} = -4\pi$$

Si $\rho=\rho'+\epsilon \rightarrow \rho>\rho' \rightarrow \rho\frac{\partial R}{\partial\rho} = \rho(-\alpha_n)\gamma_n \rho'^{\alpha_n}\rho^{-\alpha_n-1}$

Si $\rho=\rho'-\epsilon \rightarrow \rho<\rho' \rightarrow \rho\frac{\partial R}{\partial\rho} = \rho(\alpha_n)\gamma_n \rho^{\alpha_n-1}\rho'^{-\alpha_n}$

$$\Rightarrow \underbrace{\lim_{\epsilon\to 0}\left\{\rho\frac{\partial R}{\partial\rho}\Big|_{\rho'+\epsilon} - \rho\frac{\partial R}{\partial\rho}\Big|_{\rho'-\epsilon}\right\}}_{-4\pi} = -\rho'\alpha_n\gamma_n\left\{\rho'^{\alpha_n}\rho'^{-\alpha_n-1} + \rho'^{\alpha_n-1}\rho'^{-\alpha_n}\right\}$$

$$= -\cancel{\rho'}\alpha_n\gamma_n\left\{2\cancel{\rho'^{-1}}\right\}$$

$$\Rightarrow \gamma_n = \frac{4\pi}{2\alpha_n} = \frac{\cancel{4\pi}}{2(n\cancel{\pi}/\beta)} = \frac{2\beta}{n} \rightarrow \boxed{R(\rho) = \frac{2\beta}{n}\left(\frac{\rho_<}{\rho_>}\right)^{\frac{n\pi}{\beta}}}$$

$$\Rightarrow G = R(\rho)\psi(\varphi) = \frac{2\cancel{\beta}}{n}\left(\frac{\rho_<}{\rho_>}\right)^{n\pi/\beta}\cdot\frac{2}{\cancel{\beta}}\sin\left(\frac{n\pi}{\beta}\varphi\right)\sin\left(\frac{n\pi}{\beta}\varphi'\right)$$

$$\boxed{G(\rho,\varphi;\rho',\varphi') = 4\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\rho_<}{\rho_>}\right)^{n\pi/\beta}\sin\left(\frac{n\pi}{\beta}\varphi\right)\sin\left(\frac{n\pi}{\beta}\varphi'\right)}$$

b) En el desarrollo del problema (2.18) usamos y demostramos que: $\boxed{\sum_{n=1}^{\infty}\frac{1}{n}x^n\cos n\alpha = -\frac{1}{2}\ln\left(1-2x\cos\alpha+x^2\right)}$

Como $2\sin\alpha_1\varphi\sin\alpha_1\varphi' = \cos\alpha_1(\varphi-\varphi') - \cos\alpha_1(\varphi+\varphi')$

$$G = 2\sum_{n=1}^{\infty}\frac{1}{n}x^n\cos n\alpha_1 - 2\sum_{n=1}^{\infty}\frac{1}{n}x^n\cos n\alpha_2$$

con $x = \left(\frac{\rho_<}{\rho_>}\right)^{\pi/\beta}$, $\alpha_1 = \frac{\pi}{\beta}(\varphi-\varphi')$ y $\alpha_2 = \frac{\pi}{\beta}(\varphi_1+\varphi_2) \Rightarrow$

$$G = -\ln(1+x^2-2x\cos\alpha_1) + \ln(1+x^2-2x\cos\alpha_2)$$

$$= \ln\left[\frac{1+x^2-2x\cos\alpha_2}{1+x^2-2x\cos\alpha_1}\right]$$

$$= \ln\left[\frac{1+(\rho_</\rho_>)^{2\pi/\beta}-2(\rho_<\rho_>)^{\pi/\beta}\cos\frac{\pi}{\beta}(\varphi+\varphi')}{1-(\rho_<\rho_>)^{2\pi/\beta}-2(\rho_<\rho_>)^{\pi/\beta}\cos\frac{\pi}{\beta}(\varphi-\varphi')}\right]$$

$$= \ln\left[\frac{(\rho_>)^{2\pi/\beta}+\rho_<^{2\pi/\beta}-2(\rho_<\cdot\rho_>)^{\pi/\beta}\cos\frac{\pi}{\beta}(\varphi+\varphi')}{\rho_>^{2\pi/\beta}+\rho_<^{2\pi/\beta}-2(\rho_<\rho_>)^{\pi/\beta}\cos\frac{\pi}{\beta}(\varphi-\varphi')}\right]$$

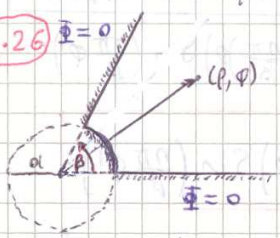
$$\boxed{G(\rho,\varphi;\rho',\varphi') = \ln\left[\frac{\rho^{2\pi/\beta}+\rho'^{2\pi/\beta}-2(\rho\rho')^{\pi/\beta}\cos(\pi(\varphi+\varphi')/\beta)}{\rho^{2\pi/\beta}+\rho'^{2\pi/\beta}-2(\rho\rho')^{\pi/\beta}\cos(\pi(\varphi-\varphi')/\beta)}\right]}$$

c) Si $\beta=\pi \Rightarrow G = \ln\left[\frac{\rho^2+\rho'^2-2\rho\rho'\cos(\varphi+\varphi')}{\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')}\right]$

Si $\beta=\pi/2 \Rightarrow G = \ln\left[\frac{\rho^4+\rho'^4-2(\rho\rho')^2\cos(\varphi+\varphi')}{\rho^4+\rho'^4-2(\rho\rho')^2\cos(\varphi-\varphi')}\right]$

Estos son justamente los resultados que se obtienen para el semi-espacio infinito ($\beta=\pi$) y para dos planos conductores a tierra intersectados perpendicularmente ($\beta=\pi/2$). Compárese también estos resultados con los del problema 2.17 y 2.18.

2.26



$$\Phi = a_0 + b_0\ln\rho + \sum_{n=1}^{\infty} a_n\rho^n\sin(n\phi+\alpha_n) + \sum_{n=1}^{\infty} b_n\rho^{-n}\sin(n\phi+\alpha_n)$$

$$\sin(n\phi+\alpha_n) = \sin n\phi\,\underbrace{\cos\alpha_n}_{A} + \cos n\phi\,\underbrace{\sin\alpha_n}_{B}$$

$$\Phi = A_0 + B_0\ln\rho + \sum_{n=1}^{\infty}\rho^n(A_n\sin n\phi + B_n\cos n\phi) + \sum_{n=1}^{\infty}\rho^{-n}(A_n'\sin n\phi + B_n'\cos n\phi)$$

Si $\varphi=0 \rightarrow \Phi(\varphi=0)=0 \rightarrow A_0=B_0=B_n=B_n'=0$

si $\varphi=\beta \rightarrow \Phi(\varphi=\beta)=0 \rightarrow \sum_{n=1}^{\infty}(A_n\rho^n+A_n'\rho^{-n})\sin n\beta = 0$

$\Rightarrow n\beta = m\pi \rightarrow n = \frac{m\pi}{\beta} \Rightarrow \Phi = \sum_{n=1}^{\infty}(A_n\rho^n+A_n'\rho^{-n})\sin\frac{m\pi}{\beta}$

Si $\rho \to a \to \Phi(\rho,\phi)=0 \to \sum_{n=1}^{\infty}(A_n a^n + A_n' a^{-n})\sin\frac{m\pi}{\beta}=0$

$\to A_n a^n + A_n' a^{-n} = 0 \to A_n' = -A_n a^{2n}$

con ello: $\boxed{\Phi(\rho,\phi)=\sum_{m=1}^{\infty} A_m\left[\rho^{m\pi/\beta}-\left(\frac{a^2}{\rho}\right)^{m\pi/\beta}\right]\sin\left(\frac{m\pi}{\beta}\phi\right)}$

b) $E_\rho = -\frac{\partial\Phi}{\partial\rho} = -\sum_{m=1}^{\infty} A_m\left[\frac{m\pi}{\beta}\rho^{\frac{m\pi}{\beta}-1}+\frac{m\pi}{\beta}\frac{a^{2m\pi/\beta}}{\rho^{\frac{m\pi}{\beta}+1}}\right]\sin\left(\frac{m\pi}{\beta}\phi\right)$

$= -\sum_{m=1}^{\infty}\frac{m\pi A_m}{\beta\rho}\left[\rho^{m\pi/\beta}+\left(\frac{a^2}{\rho}\right)^{m\pi/\beta}\right]\sin\left(\frac{m\pi}{\beta}\phi\right)$

$E_\phi = -\frac{1}{\rho}\frac{\partial\Phi}{\partial\phi} = -\frac{1}{\rho}\sum_{m=1}^{\infty}A_m\left[\rho^{m\pi/\beta}-\left(\frac{a^2}{\rho}\right)^{m\pi/\beta}\right]\left(\frac{m\pi}{\beta}\right)\cos\left(\frac{m\pi}{\beta}\phi\right)$

$= -\sum_{m=1}^{\infty}\frac{m\pi A_m}{\beta\rho}\left[\rho^{m\pi/\beta}-\left(\frac{a^2}{\rho}\right)^{m\pi/\beta}\right]\cos\left(\frac{m\pi}{\beta}\phi\right)$

Conservando solo el término con $m=1$ el campo eléctrico es:

$$\boxed{\vec{E}\approx -\frac{\pi A_1}{\beta\rho}\left\{\left[\rho^{\pi/\beta}+\left(\frac{a^2}{\rho}\right)^{\pi/\beta}\right]\sin\left(\frac{\pi\phi}{\beta}\right)\vec{\rho}+\left[\rho^{\pi/\beta}-\left(\frac{a^2}{\rho}\right)^{\pi/\beta}\right]\cos\left(\frac{\pi\phi}{\beta}\right)\vec{\phi}\right\}}$$

$\sigma(a,\phi)=\epsilon_0 E_\rho\big|_{\rho=a} = -\frac{\epsilon_0\pi A_1}{\beta a}\left[2a^{\pi/\beta}\right]\sin\left(\frac{\pi\phi}{\beta}\right)$

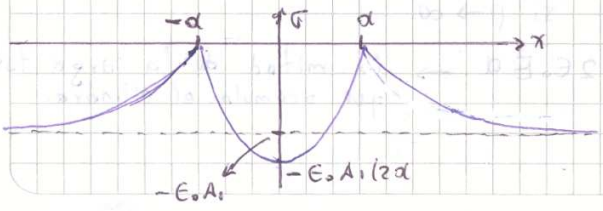
$\sigma(\rho,0)=\epsilon_0 E_\phi\big|_{\phi=0} = -\frac{\epsilon_0\pi A_1}{\beta\rho}\left[\rho^{\pi/\beta}-\left(\frac{a^2}{\rho}\right)^{\pi/\beta}\right]$

$\sigma(\rho,\beta)=\sigma(\rho,0)$

c) Si $\beta=\pi$ $\quad \sigma(a,\phi) = -\epsilon_0 A_1(2a)\sin\phi$

$\sigma(\rho,0) = -\frac{\epsilon_0 A_1}{\rho}\left[\rho-\frac{a^2}{\rho}\right] = -\epsilon_0 A_1\left[1-\frac{a^2}{\rho^2}\right]$

$\sigma(\rho,\pi)=\sigma(\rho,0)$



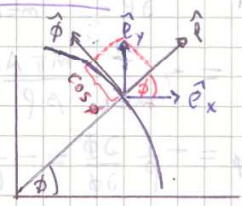
- Note que si $\rho\to\infty$. $\sigma(\rho,0)\to -\epsilon_0 A_1$
- Si $\phi=\frac{\pi}{2}$, $\sigma(a,\frac{\pi}{2}) = -\epsilon_0 A_1(2a)$
- Si $\phi=0,\pi \to \sigma(a,0)=0$, $\sigma(a,\pi)=0$

Si $\beta = \pi$

$$\vec{E} \simeq -\frac{A_1}{\rho}\left\{\left[\rho + \frac{a^2}{\rho}\right]\sin\phi\,\hat{\rho} + \left[\rho - \frac{a^2}{\rho}\right]\cos\phi\,\hat{\phi}\right\}$$

$$= -A_1\left\{\left[1 + \frac{a^2}{\rho^2}\right]\sin\phi\,\hat{\rho} + \left[1 - \frac{a^2}{\rho^2}\right]\cos\phi\,\hat{\phi}\right\}$$

Si $\rho >> a \rightarrow \vec{E} \simeq -A_1\underbrace{\left\{\sin\phi\,\hat{\rho} + \cos\phi\,\hat{\phi}\right\}}_{\hat{e}_y}$



$\Rightarrow$ Si $\rho >> a \rightarrow \boxed{\vec{E} \simeq -A_1\hat{e}_y}$

El campo eléctrico lejos del cilindro es perpendicular al plano.

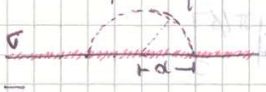
de aquí $A_1 = -E$ siendo E la magnitud del campo lejos del cilindro.

$\Rightarrow \sigma(a,\phi) = -2\epsilon_0 a A_1 \sin\phi = 2\epsilon_0 a E \sin\phi$

$Q_1 = \int_0^{\pi} \sigma\, d\phi = \int_0^{\pi} 2\epsilon_0 a E \sin\phi = -4\epsilon_0 a E \rightarrow$ Carga total sobre el semi-cilindro

$\hookrightarrow$ Esto es realmente carga por unidad de longitud del semi-cilindro.

Note que para un plano infinito:



$\therefore E = \frac{\sigma}{2\epsilon_0} \rightarrow \sigma = 2\epsilon_0 E \rightarrow$ Carga por unidad de area.

Así Q_1 es dos veces la carga que acumula el plano ∞.

La carga que tendría la lamina plana ~~infinita~~ en el espacio ocupado por el cilindro si la densidad del plano allí fuese la misma que tiene lejos es:

$Q_2 = 2\int_0^a (-\epsilon_0 A_1)\, d\rho$ pues recuerde que $\sigma(\rho, 0) \rightarrow -\epsilon_0 A_1$ si $\rho \rightarrow \infty$.

$= -2\epsilon_0 A_1 a = +2\epsilon_0 E a \rightarrow$ la mitad de la carga total que acumula el cilindro.

2.27 Del problema (2.26), conservando solo los términos de menor orden:

$$\vec{E} \approx -\frac{\pi A_1}{\beta\rho}\left\{\left[\rho^{\pi/\beta}+\left(\frac{a^2}{\rho}\right)^{\pi/\beta}\right]\sin\left(\frac{\pi\phi}{\beta}\right)\hat{\rho}+\left[\rho^{\pi/\beta}-\left(\frac{a^2}{\rho}\right)^{\pi/\beta}\right]\cos\left(\frac{\pi\phi}{\beta}\right)\hat{\phi}\right\}$$

Si $\beta = 2\pi$:

$$\vec{E} \approx -\frac{A_1}{2\rho}\left\{\left[\rho^{\frac{1}{2}}+\left(\frac{a^2}{\rho}\right)^{1/2}\right]\sin\frac{\phi}{2}\,\hat{\rho}+\left[\rho^{1/2}-\left(\frac{a^2}{\rho}\right)^{1/2}\right]\cos\frac{\phi}{2}\,\hat{\phi}\right\}$$

$$\Rightarrow \sigma(a,\phi) = \epsilon_0 E_\rho\big|_{\rho=a} = -\frac{\epsilon_0 A_1}{\cancel{2}a}\left[\cancel{2}a^{\frac{1}{2}}\right]\sin\frac{\phi}{2} = -\frac{\epsilon_0 A_1}{2}\frac{1}{a^{1/2}}\sin\frac{\phi}{2}$$

$$\sigma(\rho,0) = \epsilon_0 E_\phi\big|_{\phi=0} = -\frac{\epsilon_0 A_1}{2\rho}\left[\rho^{1/2}-\left(\frac{a^2}{\rho}\right)^{1/2}\right] = -\frac{\epsilon_0 A_1}{2}\left[\frac{1}{\rho^{1/2}}-\frac{a}{\rho^{3/2}}\right]$$

$$\sigma(\rho,2\pi) = \sigma(\rho,0)$$

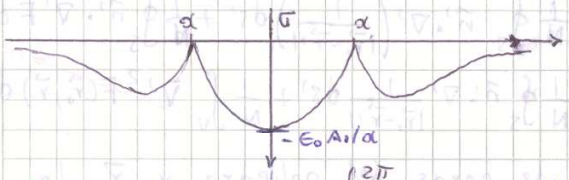
Note que si $\rho \to \infty \Rightarrow \sigma(\rho,0) = \sigma(\rho,2\pi) \to 0$

si $\rho = a \Rightarrow \sigma(a,0) = \sigma(a,2\pi) = 0$

si $\phi = 0 \to \sigma(a,0) = 0$

si $\phi = \pi \to \sigma(a,\pi) = -\epsilon_0 A_1/a$

si $\phi = 2\pi \to \sigma(a,2\pi) = 0$



b) $$Q_1 = -\epsilon_0 A_1 a^{-1/2}\int_0^{2\pi}\sin(\phi/2)\,d\phi = \epsilon_0 A_1 a^{-1/2}\,\frac{1}{2}\cos\phi/2\Big|_0^{2\pi}$$

$$Q_1 = \frac{\epsilon_0 A_1 a^{-1/2}}{\cancel{2}}\,\cancel{(2)} \to$$ carga total sobre el cilindro (carga por unidad de longit)

Si $\rho \gg a \Rightarrow \sigma(\rho,0) = -\frac{\epsilon_0 A_1}{2}\frac{1}{\rho^{1/2}} \to$ densidad de carga lejos del cilindro.

La carga que tendría la lámina plana en el espacio ocupado por el cilindro si la densidad del plano fuese la misma que tiene lejos es:

$$Q_2 = \cancel{2}\int_0^a\left[-\frac{\epsilon_0 A_1}{\cancel{2}}\frac{1}{\rho^{1/2}}\right]d\rho = 2\epsilon_0 A_1\sqrt{a} \to$$ la cual es el doble de la carga acumulada por el cilindro

2.28 En el interior del poliedro:

$$\Phi(\vec{r}) = \frac{1}{4\pi}\oint_S \left[G(\vec{r},\vec{r}\,')\frac{\partial \Phi(\vec{r}\,')}{\partial n'} - \Phi(\vec{r}\,')\frac{\partial G(\vec{r},\vec{r}\,')}{\partial n'}\right] ds'$$

Como $\Phi(\vec{r}\,') = V_i$ donde V_i es una cte para cada i, aún cuando puedan ser diferentes, implica que tenemos el problema de Dirichlet ($\Phi|_S$ dado) $\Rightarrow G(\vec{r},\vec{r}\,')|_S = 0$

$$\Rightarrow \Phi(\vec{r}) = -\frac{1}{4\pi}\oint_S \Phi(\vec{r}\,')\frac{\partial G(\vec{r},\vec{r}\,')}{\partial n'} ds'$$

$$= -\frac{1}{4\pi}\sum_{i=1}^{N}\left[V_i \oint_{S_i} \frac{\partial G(\vec{r},\vec{r}\,')}{\partial n'} ds'\right]$$

$S_i \rightarrow$ Superficie de una cara del cilindro

En el centro del poliedro, dada la distribución simétrica y regular de sus caras y dado que todas las caras son iguales y con la misma área:

$$\oint_{S_i} \frac{\partial G(\vec{r}_0,\vec{r}\,')}{\partial n'} ds' = \frac{1}{N}\oint_S \frac{\partial G(\vec{r}_0,\vec{r}\,')}{\partial n'} ds' = \frac{1}{N}\oint_S \nabla' G\cdot\hat{n}'\, ds'$$

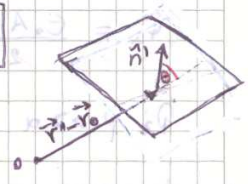
$$= \frac{1}{N}\oint_S \nabla'\left(\frac{1}{|\vec{r}_0-\vec{r}\,'|} + F(\vec{r}_0,\vec{r}\,')\right)\cdot\hat{n}'\, ds'$$

$$= \frac{1}{N}\oint_S \hat{n}'\cdot\nabla'\left(\frac{1}{|\vec{r}_0-\vec{r}\,'|}\right) ds' + \frac{1}{N}\oint_S \hat{n}'\cdot\nabla' F\, ds'$$

$$= \frac{1}{N}\oint_S \hat{n}'\cdot\nabla'\frac{1}{|\vec{r}_0-\vec{r}\,'|} ds' + \frac{1}{N}\int_V \cancelto{0}{\nabla'^2 F(\vec{r}_0,\vec{r}\,')}\, dV'$$

donde N es el número de caras del poliedro y $\vec{r}_0$ la posición de su centro.

$$\Rightarrow \Phi(\vec{r}_0) = -\frac{1}{4\pi N}\sum_{i=1}^{N}\left[V_i \oint_S \hat{n}'\cdot\nabla'\left(\frac{1}{|\vec{r}_0-\vec{r}\,'|}\right) ds'\right]$$



$$\nabla'\left(\frac{1}{|\vec{r}_0-\vec{r}\,'|}\right) = \frac{\vec{r}_0-\vec{r}\,'}{|\vec{r}_0-\vec{r}\,'|^3} = -\frac{\vec{r}\,'-\vec{r}_0}{|\vec{r}_0-\vec{r}\,'|^3}$$

$\hat{n}'$ es un vector dirigido hacia afuera del volumen de interés y $\vec{r}\,'-\vec{r}_0$ tambien no necesariamente paralelo a $\hat{n}'$ a menos que $\vec{r}\,'$ este en el centro de de las caras.

$$\hat{n}'\cdot\nabla'\left(\frac{1}{|\vec{r}_0-\vec{r}\,'|}\right) = -\hat{n}'\cdot\frac{(\vec{r}\,'-\vec{r}_0)}{|\vec{r}_0-\vec{r}\,'|^3} = \frac{-|\hat{n}'|\,|\vec{r}\,'-\vec{r}_0|\cos\theta}{|\vec{r}_0-\vec{r}\,'|^3}$$

$$\Rightarrow \hat{n}'\cdot\nabla'\left(\frac{1}{|\vec{r}_0-\vec{r}'|}\right) = \frac{-\cos\theta}{|\vec{r}_0-\vec{r}'|^2}$$ con esto

$$\Phi(\vec{r}_0) = -\frac{1}{4\pi N}\sum_{i=1}^{N}\left[V_i \underbrace{\oint_S -\frac{\cos\theta}{|\vec{r}_0-\vec{r}'|^2}\,ds'}\right]$$

$$= \frac{1}{4\pi N}\sum_{i=1}^{N}\left[V_i \oint_S d\Omega'\right] = \frac{1}{\cancel{4\pi} N}\sum_{i=1}^{N} V_i\,(\cancel{4\pi})$$

$$\Rightarrow \boxed{\Phi(\vec{r}_0) = \frac{1}{N}\sum_{i=1}^{N} V_i}$$ $\rightarrow$ donde N es el número de caras de un poliedro regular.

Otro método:

al llegar a:

$$\oint_{S_i} \frac{\partial G(\vec{r}_0,\vec{r}')}{\partial n'}\,ds' = \frac{1}{N}\oint_S \hat{n}'\cdot\nabla'\frac{1}{|\vec{r}-\vec{r}'|}\,ds'$$ por teo. divergencia

$$= \frac{1}{N}\int_V \nabla'^2\frac{1}{|\vec{r}-\vec{r}'|}\,dv'$$

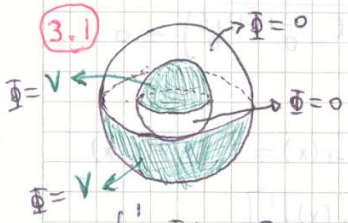
$$= \frac{1}{N}\int_V -4\pi\,\delta(\vec{r}-\vec{r}')\,dv'$$

$$= -\frac{4\pi}{N}$$

$$\Rightarrow \Phi(\vec{r}_0) = -\frac{1}{\cancel{4\pi}}\sum_{i=1}^{N} V_i\left(-\frac{\cancel{4\pi}}{N}\right) \rightarrow \boxed{\Phi(\vec{r}_0) = \frac{1}{N}\sum_{i=1}^{N} V_i}$$

Solución a los problemas del capitulo 3 Jackson 3ed

3.1



El potencial sobre S no depende de ϕ, solo de θ $\Rightarrow$ Simetría Acimutal:

$$\Phi(\vec{r}) = \sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right] P_l(\cos\theta)$$

$$\int_{-1}^{1} \Phi(r,\theta) P_{l'}(\cos\theta)\, d(\cos\theta) = \sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right] \underbrace{\int_{-1}^{1} P_{l'}(\cos\theta) P_l(\cos\theta)\, d(\cos\theta)}$$

$$= \sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right] \frac{2}{2l+1}\delta_{ll'}$$

$$= \left[A_{l'} r^{l'} + B_{l'} r^{-(l'+1)}\right] \cdot \frac{2}{2l'+1}$$

- Si $r = b$: $\Phi(b,\theta) = \begin{cases} 0 & \text{si } 0 \le \theta \le \pi/2 \rightarrow 1 > \cos\theta \ge 1 \\ V & \text{si } \frac{\pi}{2} \le \theta \le \pi \rightarrow 0 > \cos\theta \ge -1 \end{cases}$

$$\Rightarrow \int_{-1}^{0} V P_l(\cos\theta)\, d(\cos\theta) = \left[A_l b^l + B_l b^{-(l+1)}\right] \frac{2}{2l+1} \qquad (1)$$

- Si $r = a$: $\Phi(a,\theta) = \begin{cases} V & \text{si } 0 \le \theta \le \pi/2 \rightarrow 1 \ge \cos\theta > 0 \\ 0 & \text{si } \frac{\pi}{2} < \theta \le \pi \rightarrow 0 > \cos\theta \ge -1 \end{cases}$

$$\Rightarrow \int_{0}^{1} V P_l(\cos\theta)\, d(\cos\theta) = \left[A_l a^l + B_l a^{-(l+1)}\right] \frac{2}{2l+1} \qquad (2)$$

* Si l es par: $P_l(x)$ es par $\rightarrow \int P_l(x)\,dx$ es impar

$$\Rightarrow \int_0^1 P_l(x)\,dx = -\int_0^1 P_l(x)\,dx = \int_{-1}^{0} P_l(x)\,dx \quad \text{y con esto}$$

$$2\int_0^1 P_l(x)\,dx = \int_{-1}^{0} P_l(x)\,dx + \int_0^1 P_l(x)\,dx = \int_{-1}^{1} P_l(x)\,dx$$

$$\Rightarrow \int_0^1 P_l(x)\,dx = \frac{1}{2}\int_{-1}^{1} P_l(x)\,dx = \frac{1}{2}\int_{-1}^{1} P_l(x) P_0(x)\,dx = \frac{1}{\cancel{2}}\frac{\cancel{2}}{2l+1}\delta_{l0}$$

Con esto en (1) y (2) tenemos que:

Si $l = 0$ $\rightarrow$ $\left.\begin{aligned} \frac{V}{2} &= A_0 + B_0 b^{-1} \\ \frac{V}{2} &= A_0 + B_0 a^{-1} \end{aligned}\right\} \rightarrow A_0 = \frac{V}{2}, \quad B_0 = 0$

Si $l>0$ es par $\Rightarrow$ $0 = A_l b^l + B_l b^{-(l+1)}$

$$-\{0 = A_l a^l + B_l a^{-(l+1)}\}$$

$$B_l\left(b^{-(l+1)}a^l - a^{-(l+1)}b^l\right) = 0$$

$\Rightarrow$ $B_l = 0$ y $A_l = 0$.

* Si l es impar: Usemos $P'_{l+1}(x) - P'_{l-1}(x) = (2l+1)P_l(x)$

$$\Rightarrow \int_0^1 P_l(x)\,dx = \frac{1}{2l+1}\left[P_{l+1}(x)\Big|_0^1 - P_{l-1}(x)\Big|_0^1\right]$$

$$= \frac{1}{2l+1}\left[\underbrace{P_{l+1}(1) - P_{l-1}(1)}_{0} + P_{l-1}(0) - P_{l+1}(0)\right]$$

$P_n(1) = 1 \quad \forall n$

Si l es impar $\Rightarrow$ $P_l(x)$ es impar $\rightarrow$ $\int P_l(x)$ es par

$$\Rightarrow \int_0^1 P_l(x)\,dx = \int_0^{-1} P_l(x)\,dx = -\int_{-1}^0 P_l(x)\,dx$$

$$\Rightarrow \int_0^1 P_l(x)\,dx = -\int_{-1}^0 P_l(x)\,dx = \frac{1}{2l+1}\underbrace{\left[P_{l-1}(0) - P_{l+1}(0)\right]}_{C_l}$$

con esto en ① y ② tenemos:

$$\left\{-\frac{V}{2}C_l = A_l b^l + B_l b^{-(l+1)}\right\} \times(-a^l)$$

$$\left\{\frac{V}{2}C_l = A_l a^l + B_l a^{-(l+1)}\right\} \times b^l$$

$$\frac{V}{2}C_l\left(a^l + b^l\right) = B_l\left(-b^{-(l+1)}a^l + a^{-(l+1)}b^l\right)$$

$$\rightarrow B_l = \frac{V}{2}C_l\,\frac{a^l + b^l}{a^{-(l+1)}b^l - b^{-(l+1)}a^l} = \frac{V}{2}C_l\,\frac{a^l + b^l}{a^{-(l+1)}b^{-(l+1)}\left(b^{2l+1} - a^{2l+1}\right)}$$

$$\boxed{B_l = \frac{V}{2}C_l\,(ab)^{l+1}\,\frac{a^l + b^l}{b^{2l+1} - a^{2l+1}}}$$

Usando las mismas relaciones obtenemos A_l:

$$\left\{-\frac{V}{2}C_l = A_l b^l + B_l b^{-(l+1)}\right\} \times(-b^{l+1})$$

$$\left\{\frac{V}{2}C_l = A_l a^l + B_l a^{-(l+1)}\right\} \times(a^{l+1})$$

$$\frac{V}{2}C_l\left(a^{l+1} + b^{l+1}\right) = A_l\left(-b^{2l+1} + a^{2l+1}\right) \longrightarrow \boxed{A_l = \frac{V}{2}C_l\,\frac{a^{l+1} + b^{l+1}}{a^{2l+1} - b^{2l+1}}}$$

donde debemos recordar que $C_\ell = P_{\ell-1}(0) - P_{\ell+1}(0)$

$$\Rightarrow \boxed{\Phi = \frac{V}{2} + \frac{V}{2}\sum_{\ell\ \text{impar}}^{\infty}\left[\left(\frac{a^{\ell+1}+b^{\ell+1}}{a^{2\ell+1}-b^{2\ell+1}}\right)r^{\ell} + (ab)^{\ell+1}\left(\frac{a^{\ell}+b^{\ell}}{b^{2\ell+1}-a^{2\ell+1}}\right)r^{-(\ell+1)}\right] \times \left(P_{\ell-1}(0)-P_{\ell+1}(0)\right)P_\ell(\cos\theta)}$$

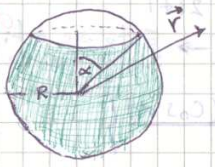
$$\Phi(\vec{r}) = \frac{V}{2} + \frac{V}{2}\left\{\left[\frac{a^2+b^2}{a^3-b^3}r + (ab)^2\frac{a+b}{b^3-a^3}r^{-2}\right]\left(1-(-1/2)\right)P_1(\cos\theta) + \left[\frac{a^4+b^4}{a^7-b^7}r^3 + (ab)^4\frac{a^3+b^3}{b^7-a^7}r^{-4}\right]\left((-1/2)-\frac{3}{8}\right)P_3(\cos\theta)+\cdots\right\}$$

$$= V\left\{\frac{1}{2} + \frac{3}{4}\left[\frac{a^2+b^2}{a^3-b^3}r + \left(\frac{ab}{r}\right)^2\frac{a+b}{b^3-a^3}\right]P_1(\cos\theta) - \frac{7}{16}\left[\frac{a^4+b^4}{a^7-b^7}r^3 + \left(\frac{ab}{r}\right)^4\frac{a^3+b^3}{b^7-a^7}\right]P_3(\cos\theta)+\cdots\right\}$$

Si $b\to\infty \Rightarrow \Phi(r) = V\left\{\frac{1}{2} + \frac{3}{4}\left(\frac{a}{r}\right)^2 P_1(\cos\theta) - \frac{7}{16}\left(\frac{a}{r}\right)^4 P_3(\cos\theta)+\cdots\right\}$

Si $a\to 0 \Rightarrow \Phi(r) = V\left\{\frac{1}{2} - \frac{3}{4}\left(\frac{r}{b}\right)P_1(\cos\theta) + \frac{7}{16}\left(\frac{r}{b}\right)^3 P_3(\cos\theta)+\cdots\right\}$

3.2



$\rho(\vec{r}) = \sigma(\theta)\,\delta(r-R)$ con

$$\sigma(\theta) = \begin{cases} \dfrac{Q}{4\pi R^2} & \text{si } \cos\alpha > \cos\theta > -1 \\ 0 & \text{si } 1 > \cos\theta > \cos\alpha \end{cases}$$

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{r}\,')G(\vec{r},\vec{r}\,')\,dv' - \frac{1}{4\pi}\oint_{S_\infty}\Phi(\vec{r}\,')\frac{\partial G}{\partial n'}\,da'$$

$\to 0$ cuando $S\to\infty$

$$G(\vec{r},\vec{r}\,') = \frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{\ell=0}^{\infty}\frac{r_<^{\ell}}{r_>^{\ell+1}}P_\ell(\cos\gamma)$$ $\to$ Función de Green para el espacio infinito

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\int\frac{r_<^{\ell}}{r_>^{\ell+1}}P_\ell(\cos\gamma)\,\sigma(\theta)\,\delta(r'-R)\,r'^2\,dr'\,d(\cos\theta')\,d\varphi' \quad (1)$$

Si $r < r'$: $\rightarrow$ $r_< = r$ y $r_> = r'$ (2)

$$\Rightarrow \Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\int \frac{r^l}{r'^{l+1}} P_l(\cos\gamma)\,\sigma(\theta)\,\delta(r'-R)\,r'^2 dr'\, d(\cos\theta')\, d\phi'$$

$$= \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r^l}{R^{l+1}}(2\pi)R^2\int P_l(\cos\gamma)\,\sigma(\theta)\, d(\cos\theta')$$

si: $\vec{r} = r\hat{z} \Rightarrow \gamma = \theta'$ pues recuérdese que γ es el ángulo entre $\vec{r}$ y $\vec{r}'$

$$\Rightarrow \Phi(z) = \frac{2\pi R^2}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r^l}{R^{l+1}}\int_0^{\pi} P_l(\cos\theta')\,\sigma(\theta)\, d(\cos\theta')$$

$$= \frac{R^2}{2\epsilon_0}\sum_{l=0}^{\infty}\frac{r^l}{R^{l+1}}\int_{-1}^{1} P_l(x)\,\sigma(x)\, dx$$

$$= \frac{R^2}{2\epsilon_0}\sum_{l=0}^{\infty}\frac{r^l}{R^{l+1}}\left\{\int_{-1}^{\cos\alpha} P_l(x)\frac{Q}{4\pi R^2}dx + \cancel{\int_{\cos\alpha}^{1} P_l(x)\cdot 0\, dx}\right\} \rightarrow 0$$

si: $l \neq 0 \rightarrow (2l+1)P_l(x) = P'_{l+1}(x) - P'_{l-1}(x)$ usando esto

$$\Phi(z) = \frac{Q}{8\pi\epsilon_0}\sum_{l=1}\frac{r^l}{R^{l+1}}\left[\frac{P_{l+1}(x) - P_{l-1}(x)}{2l+1}\right]_{-1}^{\cos\alpha} + \frac{Q}{8\pi\epsilon_0}\frac{r^0}{R^1}\int_{-1}^{\cos\alpha}\underbrace{P_0(x)}_{=1}dx$$

$$= \frac{Q}{8\pi\epsilon_0}\left\{\frac{r^0}{R^1}(\cos\alpha + 1) + \sum_{l=1}^{\infty}\frac{r^l}{R^{l+1}}\left\{\frac{P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha) + \underbrace{P_{l-1}(-1) - P_{l-1}(-1)}_{0}}{2l+1}\right\}\right\}$$

$P_1(\cos\alpha) = \cos\alpha$, $P_{-1}(\cos\alpha) = -1 \rightarrow$ Esto se define así por sugerencia de Jackson.

$$\Rightarrow \Phi(z) = \frac{Q}{8\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r^l}{R^{l+1}}\left\{\frac{P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)}{2l+1}\right\}$$

$$\Rightarrow \boxed{\Phi(\vec{r}) = \frac{Q}{8\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r^l}{R^{l+1}}\left\{\frac{P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)}{2l+1}\right\}P_l(\cos\theta)}$$

si $r < r'$.

Si $r > r'$: $\rightarrow$ $r_< = r'$ y $r_> = r$ de modo que la integral en r' en (1) queda: $\int \frac{r'^l}{r^{l+1}}\delta(r'-R)\,r'^2 dr' = \frac{R^l}{r^{l+1}}R^2$

y las demás integrales son como en el caso anterior quedando:

$$\Phi(\vec{r}) = \frac{Q}{8\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{R^l}{r^{l+1}}\left\{\frac{P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)}{2l+1}\right\}P_l(\cos\theta) \quad \text{si } r > r'$$

ESTILO

de modo que podemos escribir:

$$\boxed{\Phi(\vec{r}) = \frac{Q}{8\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r_<^{\,l}}{r_>^{\,l+1}}\left\{\frac{P_{l+1}(\cos\alpha)-P_{l-1}(\cos\alpha)}{2l+1}\right\}P_l(\cos\theta)}$$

donde $r_< = \min\{r, R\}$ y $r_> = \max\{r, R\}$

y se definió arbitrariamente $\underline{P_{-1}(\cos\alpha) \equiv -1}$

b) Si $r<r'$: $\Phi(r) = \frac{Q}{8\pi\epsilon_0}\left\{\frac{P_1(\cos\alpha)-P_{-1}(\cos\alpha)}{R}P_0(\cos\theta)\right.$

$$\left. + \frac{P_2(\cos\alpha)-P_0(\cos\alpha)}{3R^2}\,r\,P_1(\cos\theta) + \frac{P_3(\cos\alpha)-P_1(\cos\alpha)}{5R^3}\,r^2P_2(\cos\theta) + \dots\right\}$$

$$\vec{E} = -\vec{\nabla}\Phi = -\frac{\partial\Phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\cancel{\frac{\partial\Phi}{\partial\phi}}\hat{\phi}$$

$\to 0$ No depende de ϕ.

$$= -\frac{Q}{8\pi\epsilon_0}\left\{\hat{r}\left[\frac{P_2(\cos\alpha)-P_0(\cos\alpha)}{3R^2}P_1(\cos\theta) + \frac{P_3(\cos\alpha)-P_1(\cos\alpha)}{5R^3}(2r)P_2(\cos\theta+\dots\right]\right.$$

$$+\hat{\theta}\left[\frac{P_1(\cos\alpha)-P_{-1}(\cos\alpha)}{rR}\cancelto{0}{\frac{d}{d\theta}P_0(\cos\theta)} + \frac{P_2(\cos\alpha)-P_0(\cos\alpha)}{\cancel{r}(3R^2)}\cancel{r}\frac{dP_1(\cos\theta)}{d\theta}\right.$$

$$\left.\left. + \frac{P_3(\cos\alpha)-P_1(\cos\alpha)}{\cancel{r}(5R^3)}r^{\cancel{2}}\frac{dP_2(\cos\theta)}{d\theta}+\dots\right]\right\}$$

$$\vec{E}(r=0) = -\frac{Q}{8\pi\epsilon_0}\left\{\hat{r}\left[\frac{P_2(\cos\alpha)-P_0(\cos\alpha)}{3R^2}P_1(\cos\theta)\right]\right.$$

$$\left. + \hat{\theta}\left[\frac{P_2(\cos\alpha)-P_0(\cos\alpha)}{3R^2}\right]\frac{dP_1(\cos\theta)}{d\theta}\right\}$$

$P_0(\cos\alpha) = 1$

$P_1(\cos\alpha) = \cos\alpha$

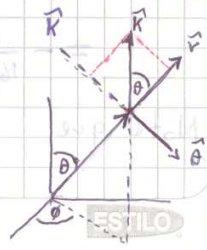
$P_1(\cos\theta) = \cos\theta \longrightarrow \frac{d}{d\theta}P_1(\cos\theta) = -\sin\theta$

$P_2(\cos\alpha) = \frac{1}{2}(3\cos^2\alpha - 1)$

$$\Rightarrow \vec{E}(r=0) = -\frac{Q}{8\pi\epsilon_0(3R^2)}\left[\frac{1}{2}(3\cos^2\alpha-1)-1\right]\underbrace{\left\{\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right\}}$$

$$= -\frac{Q}{8\pi\epsilon_0(\cancel{3}R^2)}\left[\frac{\cancel{3}}{2}\underbrace{(\cos^2\alpha-1)}_{-\sin^2\alpha}\right]\hat{k}$$

$$\boxed{\vec{E}(r=0) = \frac{Q}{16\pi\epsilon_0R^2}\sin^2\alpha\,\hat{k}}$$



Note que va unicamente en $\hat{k}$, pues dada la simetría es de esperarse que no tenga componentes en x ni en y.

c) Podemos expandir $P_\ell(x)$ alrededor de $x=1$ en series de Laurent:

$$P_\ell(x) = \sum_n \frac{1}{n!}\frac{d^n}{dx^n} P_\ell(x)\Big|_{x=1} (x-1)^n$$

$$\approx \underbrace{P_\ell(1)}_{1} + \frac{d}{dx} P_\ell(x)\Big|_{x=1} (x-1)$$

Si $x = \cos\alpha \rightarrow (x-1) = \cos\alpha - 1 = \sqrt{1-\sin^2\alpha} - 1$

$$= \left(1 - \frac{1}{2}\sin^2\alpha + \cdots\right) - 1 \approx -\frac{1}{2}\sin^2\alpha$$

$$\Rightarrow P_\ell(\cos\alpha) \approx 1 - \frac{1}{2}\frac{d}{dx} P_\ell(1)\sin^2\alpha$$

$$P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha) \approx 1 - \frac{1}{2}\frac{d}{dx}P_{\ell+1}(x)\Big|_{x=1}\sin^2\alpha - \left(1 - \frac{1}{2}\frac{d}{dx}P_{\ell-1}(x)\Big|_{x=1}\sin^2\alpha\right)$$

$$= -\frac{1}{2}\sin^2\alpha\left\{\frac{d}{dx}P_{\ell+1}(x)\Big|_{x=1} - \frac{d}{dx}P_{\ell-1}(x)\Big|_{x=1}\right\}$$

Usando que $(2\ell+1)P_\ell(x) = P'_{\ell+1}(x) - P'_{\ell-1}(x)$ si $\ell \neq 0$ tenemos:

$$P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha) \approx -\frac{1}{2}\sin^2\alpha\left\{(2\ell+1)P_\ell(1)\right\}$$

Si α es pequeño $\sin\alpha \approx \alpha$ $\Rightarrow$ $\approx -\frac{1}{2}\alpha^2(2\ell+1)P_\ell(1)$ si $\ell \neq 0$

$$\Rightarrow \Phi(\vec{r}) = \frac{Q}{8\pi\epsilon_0}\sum_{\ell=0}^{\infty}\frac{r_<^\ell}{r_>^{\ell+1}}\left\{\frac{P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha)}{(2\ell+1)}\right\}P_\ell(\cos\theta)$$

$$\approx \frac{Q}{8\pi\epsilon_0}\sum_{\ell=1}^{\infty}\frac{r_<^\ell}{r_>^{\ell+1}}\left\{-\frac{1}{2}\alpha^2 \underbrace{P_\ell(1)}_{1}\right\}P_\ell(\cos\theta) + \frac{Q}{8\pi\epsilon_0 r_>}(\cos\alpha + 1)$$

$$= -\frac{Q\alpha^2}{16\pi\epsilon_0}\left[-\frac{r_<^0}{r_>^1}\overbrace{P_0(\cos\theta)}^{1} + \sum_{\ell=0}^{\infty}\frac{r_<^\ell}{r_>^{\ell+1}}P_\ell(\cos\theta)\right] + \frac{Q(\cos\alpha+1)}{8\pi\epsilon_0 r_>}$$

$$= -\frac{Q\alpha^2}{16\pi\epsilon_0}\underbrace{\sum_{\ell=0}^{\infty}\frac{r_<^\ell}{r_>^{\ell+1}}P_\ell(\cos\theta)}_{\frac{1}{|\vec{r}-\vec{r}'|}} + \frac{Q}{8\pi\epsilon_0 r_>}(\underbrace{\cos\alpha}_{\approx 1} + 1) + \frac{Q}{16\pi\epsilon_0}\frac{\overbrace{\alpha^2}^{0}}{r_>}$$

$\alpha << 1$ rad

$r_> \geq R$

$\rightarrow \vec{r}' = R\hat{z}$

Antes que cero es despreciable frente a los demás términos

así para $\alpha << 1$ rad tenemos que:

$$\boxed{\Phi(\vec{r}) \simeq -\frac{Q\alpha^2}{16\pi\epsilon_0|\vec{r}-R\hat{z}|} + \frac{Q}{4\pi\epsilon_0 r_>}} \qquad r_> = \text{máx}\{r, R\}$$

Potencial de una carga Negativa puntual ubicada en $R\hat{z}$.

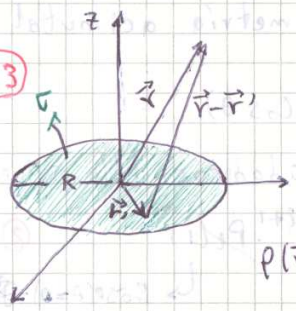
Potencial del cascarón esférico.

Por analogía del caso anterior si $\alpha \to \pi - \beta$ con $\beta << 1$ rad (para que las aproximaciones funcionen)

$$\boxed{\Phi(\vec{r}) \simeq \frac{Q\beta^2}{16\pi\epsilon_0|\vec{r}+R\hat{z}|}} \longrightarrow$$ potencial de una carga puntual positiva ubicada en $-R\hat{z}$.

~~Sin embargo escribir la expansión lo que se debe hacer es algo como:~~

3.3



$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}dv' + \frac{1}{4\pi}\int_S \left[\frac{1}{|\vec{r}-\vec{r}'|}\frac{\partial\Phi}{\partial n'} - \Phi\frac{\partial}{\partial n'}\frac{1}{|\vec{r}-\vec{r}'|}\right]da'$$

$\to 0$ si $S \to \infty$

$$\rho(\vec{r}') = \frac{c\,\delta(z')}{\sqrt{R^2-\rho'^2}}$$

Si $\vec{r} = z\hat{z} \longrightarrow |\vec{r}-\vec{r}'| = \sqrt{z^2+\rho'^2}$ $\qquad \vec{r}' = (\rho', \varphi')$ cilíndricas

$$\Rightarrow \Phi(z) = \frac{1}{4\pi\epsilon_0}\int_V \frac{c}{\sqrt{R^2-\rho'^2}}\cdot\frac{\delta(z')}{\sqrt{z^2+\rho'^2}}(\rho'\,d\rho'\,d\varphi'\,dz')$$

$$= \frac{c}{4\pi\epsilon_0}(2\pi)\int_0^R \frac{\rho'\,d\rho'}{\sqrt{(R^2-\rho'^2)(z^2+\rho'^2)}}$$

Usemos: $$\boxed{\int \frac{dx}{\sqrt{(ax+b)(px+q)}} = \frac{2}{\sqrt{-ap}}\tan^{-1}\sqrt{\frac{-p(ax+b)}{a(px+q)}}}$$

pero transformemos primero la integral mediante $u = \rho'^2$

$$\Rightarrow \int_0^R \frac{\rho'\,d\rho'}{\sqrt{(R^2-\rho'^2)(z^2+\rho'^2)}} = \frac{1}{2}\int_0^{R^2}\frac{du}{\sqrt{(R^2-u)(z^2+u)}} = \frac{1}{2}\frac{2}{\sqrt{1}}\tan^{-1}\sqrt{\frac{R^2-u}{z^2+u}}\Bigg|_0^{R^2}$$

$$= \cancel{\tan^{-1} 0}^{\,0} - \tan^{-1}(R/z)$$

$$\Rightarrow \Phi(z) = -\frac{C}{2\epsilon_0}\tan^{-1}\left(\frac{R}{z}\right)$$

• Si $\Phi(z=0) = V$ $\rightarrow$ $V = -\frac{C}{2\epsilon_0}\tan^{-1}\infty = -\frac{C}{2\epsilon_0}\frac{\pi}{2}$

$$\rightarrow C = -\frac{4\epsilon_0 V}{\pi} \Rightarrow \Phi(z) = \frac{2V}{\pi}\tan^{-1}\left(\frac{R}{z}\right)$$

$$\boxed{\tan^{-1} x = x - \frac{x^3}{2} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\Rightarrow \Phi(z) = \frac{2V}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{R}{z}\right)^{2n+1} \quad (1)$$

Por otro lado el problema tiene simetría acimutal de modo que:

$$\Phi(\vec{r}) = \sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right]P_l(\cos\theta)$$

a)

• Si $r > R$: $\Phi(r\to\infty) = 0$ $\rightarrow$ $A_l = 0$ con ello,

note que $\Phi(\vec{r} = z\hat{k}) = \sum_{l=0}^{\infty} B_l\left(\frac{1}{z}\right)^{l+1}\cancel{P_l(1)}^{\,1}$ (2)

$\hookrightarrow \theta = 0$ en $\vec{r} = z\hat{k}$

(1) y (2) deben ser equivalentes, luego

$l = 2n$ y $B_l = \frac{2V}{\pi}\frac{(-1)^n}{2n+1}R^{2n+1}$ con $n = l/2$

Así: $\Phi(\vec{r}) = \sum_{l=0}^{\infty}\left[\frac{2V}{\pi}\frac{(-1)^n}{2n+1}R^{2n+1}\right]r^{-(l+1)}P_l(\cos\theta)$ $\quad l = 2n$

$$\boxed{\Phi(\vec{r}) = \frac{2V}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{R}{r}\right)^{2n+1}P_{2n}(\cos\theta)} \quad (3) \quad \text{Si } r > R$$

b)

• Si $r < R$: $\Phi(r\to 0)$ debe ser finito $\rightarrow$ $B_l = 0$, con ello

si $r < R$ $\quad \Phi(\vec{r}) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$ (4)

por continuidad de las soluciones (3) y (4) deben ser equivalentes en $r = R$ $\Rightarrow$

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \frac{2V}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} P_{2n}(\cos\theta)$$

$$\Rightarrow A_l = \frac{2V}{\pi} \frac{(-1)^n}{(2n+1)R^l} \quad y \quad n = 2l$$

$$\Rightarrow \Phi(\vec{r}) = \sum_{l=0}^{\infty} \left[\frac{2V}{\pi} \frac{(-1)^n}{(2n+1)R^l} \right] r^l P_l(\cos\theta) \quad \text{con } l = 2n$$

ó $\boxed{\Phi(\vec{r}) = \frac{2V}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{r}{R}\right)^{2n} P_{2n}(\cos\theta)}$ 5

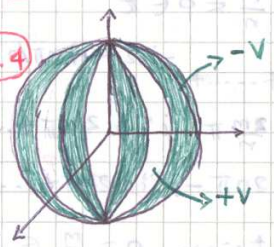
Si $r < R$.

c) $$C = \frac{Q}{V} = \frac{C}{V} \int_0^R \frac{\rho\, d\rho\, d\varphi}{\sqrt{R^2-\rho^2}} = \frac{(4\epsilon_0 \cancel{V}/\cancel{\pi})}{\cancel{V}} (2\cancel{\pi}) \int_0^R \frac{\rho\, d\rho}{\sqrt{R^2-\rho^2}}$$

$$= 8\epsilon_0 \int_{R^2}^{0} \frac{\left(-\frac{1}{2} du\right)}{u^{1/2}} = -8\epsilon_0 \frac{1}{2} 2u^{1/2} \Big|_{R^2}^{0}$$

$\boxed{C = 8\epsilon_0 R}$ → capacitancia.

3.4



$$\Phi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[A_{lm} r^l + B_{lm} r^{-(l+1)} \right] Y_{lm}(\theta,\phi)$$

• Si $r < a$: $\Phi(r \to a)$ es finito → $B_{lm} = 0$

$$\Rightarrow \Phi(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} a^l Y_{lm}(\theta,\phi)$$

$$\int \Phi(a,\theta,\phi) Y^*_{l'm'}(\theta,\phi)\, d\Omega = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} a^l \underbrace{\int Y_{lm}(\theta,\phi) Y^*_{l'm'}(\theta,\phi)\, d\Omega}_{\delta_{ll'}\delta_{mm'}}$$

$$= A_{l'm'} a^{l'}$$

$$\Rightarrow A_{lm} = \frac{1}{a^l} \int_0^{\pi} \int_0^{2\pi} \Phi(a,\theta,\phi) Y^*_{lm}(\theta,\phi)\, d\Omega$$

Si tenemos $2n$ segmentos a potenciales fijos $\pm V$ alternadamente:

$$\Phi(a,\theta,\phi) = (-1)^K V \quad \text{si} \quad K \frac{\cancel{2}\pi}{2n} \le \phi \le (K+1)\frac{\cancel{2}\pi}{2n}$$

con $K = 0, 1, 2, \ldots, 2n-1$ de modo que:

$$A_{lm} = \frac{1}{a^l}\int_0^{\pi}\sum_{k=0}^{2n-1}(-1)^k V\int_{k\pi/n}^{(k+1)\pi/n}\overbrace{\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\,P_l^m(\cos\theta)\,e^{-im\phi}}^{Y^*_{lm}(\theta,\phi)}\underbrace{(\sin\theta\, d\theta\, d\phi)}_{d\Omega}$$

$$= \frac{V}{a^l}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\int_{-1}^{1}P_l^m(x)\,dx\cdot\sum_{k=0}^{2n-1}(-1)^k\int_{k\pi/n}^{(k+1)\pi/n}e^{-im\phi}\,d\phi$$

$$\int_{k\pi/n}^{(k+1)\pi/n}e^{-im\phi}\,d\phi = -\frac{1}{im}e^{-im\phi}\Big|_{k\pi/n}^{(k+1)\pi/n} = \frac{1}{im}\left\{e^{-i\frac{km\pi}{n}} - e^{-i\frac{(k+1)m\pi}{n}}\right\}$$

$$= -\frac{1}{im}\left\{e^{-i\frac{m}{n}\pi} - 1\right\}e^{-i\frac{km\pi}{n}}$$

$$\Rightarrow \sum_{k=0}^{2n-1}(-1)^k\int_{k\pi/n}^{(k+1)\pi/n}e^{-im\phi}\,d\phi = \frac{i}{m}\left(e^{-i\frac{m\pi}{n}} - 1\right)\sum_{k=0}^{2n-1}\left(-e^{-i\frac{m\pi}{n}}\right)^k \quad (1)$$

$\Phi \propto e^{im\phi}$ y $\Phi(\phi) = \Phi(\phi + 2\pi/n) \Rightarrow e^{im\phi} = e^{im(\phi+2\pi/n)}$

$\rightarrow e^{im\phi} = e^{im\phi}e^{i2m\pi/n} \Rightarrow e^{i2m\pi/n} = 1$

$e^{i2m\pi/n} = \cos\left(\frac{2m\pi}{n}\right) + i\sin\left(\frac{2m\pi}{n}\right) = 1 \rightarrow \frac{m}{n} = p \in \mathbb{Z}$

$$\sum_{k=0}^{2n-1}\left(-e^{-i\frac{m\pi}{n}}\right)^k = 1 - e^{-im\pi/n} + e^{-i2m\pi/n} - e^{-i3m\pi/n} + \cdots + e^{-i(2n-1)m\pi/2}$$

$$= 1 - \cos\frac{m}{n}\pi + i\sin\frac{m}{n}\pi + \cos\frac{2m}{n}\pi - i\sin\frac{2m}{n}\pi + \cdots$$

$$= 1 - \cos p\pi + \cancel{i\sin p\pi} + \cos 2p\pi - \cancel{i\sin 2p\pi} + \cdots$$

$$= 1 - \cos p\pi + \cos 2p\pi - \cos 3p\pi + \cdots \quad p = \frac{m}{n} \in \mathbb{Z}$$

- Si $p = m/n$ es par: $\cos kp\pi = 1$ ya que kp es par $\forall k$

$$\Rightarrow \sum_{k=0}^{2n-1}\left(-e^{-ip\pi}\right)^k = \underbrace{1 - \cos p\pi}_{0} + \underbrace{\cos 2p\pi - \cos 3p\pi}_{0} + \cdots$$

$$+ \cdots + \underbrace{\cos(2n-2)p\pi - \cos(2n-1)p\pi}_{0}$$

- Si $p = m/n$ es impar: $\cos kp\pi = \begin{cases} 1 & \text{si } k \text{ es par} \\ -1 & \text{si } k \text{ es impar}\end{cases}$

$$\Rightarrow \sum_{k=0}^{2n-1}\left(-e^{-ip\pi}\right)^k = 1 - (-1) + 1 - (-1) + \cdots + 1 - (-1) = 2n$$

$$\Rightarrow \sum_{k=0}^{2n-1}\left(-e^{-i\frac{m}{n}\pi}\right)^k = \begin{cases} 0 & \text{si } m/n \text{ es par} \\ 2n & \text{si } m/n \text{ es impar}\end{cases} \quad m/n = p \in \mathbb{Z}$$

Con ello en (1):

$$\sum_{k=0}^{2n-1}(-1)^k\int_{k\pi/n}^{(k+1)\pi/n} e^{-im\phi}\,d\phi = \begin{cases} \frac{i}{m}\left(e^{-im\pi/n}-1\right)\cdot 2n & \text{si } \frac{m}{n} \text{ impar} \\ 0 & \text{si } m/n \text{ es par}\end{cases}$$

pero $e^{-i\frac{m}{n}\pi} = \underbrace{\cos\frac{m}{n}\pi}_{-1 \text{ si } m/n \text{ es impar}} - i\underbrace{\sin\frac{m}{n}\pi}_{0 \text{ si } m/n\in\mathbb{Z}}$

$$\Rightarrow \sum_{k=0}^{2n-1}(-1)^k\int_{k\pi/n}^{(k+1)\pi/n} e^{-im\phi}\,d\phi = \begin{cases} \frac{i}{m}(-1-1)\cdot 2n & \text{si } \frac{m}{n} \text{ es impar} \\ 0 & \text{si } m/n \text{ es par}\end{cases}$$

Con todo esto:

$$A_{lm} = \begin{cases} \frac{V}{a^l}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\int_{-1}^{1}P_l^m(x)\,dx\cdot\left(-\frac{4in}{m}\right) & \text{con } \frac{m}{n} \text{ impar} \\ 0 \quad \text{si } m/n \text{ es par} & (m/n\in\mathbb{Z}).\end{cases}$$

Sea $\frac{m}{n} = (2j+1)$ con $j = 0, \pm1, \pm2, \ldots$

ello obliga a que m/n sea impar y $m = (2j+1)n$

$$\Rightarrow A_{l,(2j+1)n} = \frac{V}{a^l}\sqrt{\frac{2l+1}{4\pi}\frac{[l-(2j+1)n]!}{[l+(2j+1)n]!}}\int_{-1}^{1}P_l^{(2j+1)n}(x)\,dx\cdot\left(\frac{-4in}{(2j+1)n}\right)$$

$$j = 0, \pm1, \pm2, \ldots$$

Además:

- Si $l+m$ es impar $\rightarrow P_l^m$ es impar $\rightarrow \int P_l^m(x)\,dx$ es par

  con ello $\int_{-1}^{1}P_l^m(x)\,dx = 0$

- Si $l+m$ es par $\rightarrow P_l^m$ es par $\rightarrow \int P_l^m(x)\,dx$ es impar

  con ello $\int_{-1}^{1}P_l^m(x)\,dx \neq 0$

$$\Rightarrow \Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{-l}^{l}A_{lm}\,r^l\,Y_{lm}(\theta,\phi)$$

con $A_{lm} = \begin{cases} -\frac{i4nV}{ma^l}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\int_{-1}^{1}P_l^m(x)\,dx & \text{con } m/n \text{ impar} \\ 0 \quad \text{si } m/n \text{ es par}\end{cases}$

Además $A_{l,m}\neq 0$ solo si $l+m$ es par $\rightarrow \begin{cases} l \text{ y } m \text{ pares} \\ l \text{ y } m \text{ impares}\end{cases}$

b) Si $n=1$ $\rightarrow$ $A_{11} = -\frac{4iV}{\alpha}\sqrt{\frac{3}{4\pi}\cdot\frac{1}{2!}}\int_{-1}^{1} P_1^1(x)\,dx$

$$A_{1,-1} = \frac{i4V}{\alpha}\sqrt{\frac{3}{4\pi}\;2!}\int_{-1}^{1} P_1^{-1}(x)\,dx$$

Si usamos $\boxed{P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x)}$

$$A_{1,-1} = \frac{4iV}{\alpha}(-1)\int_{-1}^{1} P_1^1(x)\,dx$$

En general $A_{l,-m} = \frac{i4Vn}{m\alpha^l}\sqrt{\frac{2l+1}{4\pi}\frac{(l+m)!}{(l-m)!}}(-1)^m\frac{(l-m)!}{(l+m)!}\int_{-1}^{1} P_l^m(x)\,dx$

$$= (-1)^m \frac{i4Vn}{m\alpha^l}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\int_{-1}^{1} P_l^m(x)\,dx$$

$$A_{l,-m} = (-1)^m(-A_{lm}) = (-1)^{m+1}A_{lm}$$

m impar

Como $A_{lm}\neq 0$ para m impar $(-1)^{m+1}=1 \rightarrow \boxed{A_{l,-m} = A_{lm}}$

$$A_{11} = -\frac{4iV}{\alpha}\sqrt{\frac{3}{8\pi}}\int_{-1}^{1} P_1^1(x)\,dx = -\frac{4iV}{\alpha}\sqrt{\frac{3}{8\pi}}\int_{-1}^{1}(1-x^2)^{1/2}\,dx$$

$$= -\frac{4iV}{\alpha}\sqrt{\frac{3}{8\pi}}\,\frac{1}{2}\,\mathrm{Sin}^{-1}x\Big|_{-1}^{1} = -\frac{4iV}{\alpha}\sqrt{\frac{3}{8\pi}}\left(\frac{\pi}{2}-\frac{3\pi}{2}\right)$$

$$= \frac{iV}{\alpha}\sqrt{\frac{3\pi}{2}} = A_{1,-1}$$

$1+2$ es impar $\rightarrow \begin{cases} A_{2,-1} = 0 \\ A_{21} = 0 \end{cases}$ $m=2$ es par $\rightarrow \begin{cases} A_{2,-2} = 0 \\ A_{22} = 0 \end{cases}$

$2+3$ es impar $\rightarrow \begin{cases} A_{3,-2} = 0 \\ A_{32} = 0 \end{cases}$

$$A_{31} = -\frac{4iV}{\alpha^3}\sqrt{\frac{7}{48\pi}}\int_{-1}^{1} P_3^1(x)\,dx = -\frac{4iV}{\alpha^3}\sqrt{\frac{7}{48\pi}}\,\frac{3}{2}\int_{-1}^{1}(5x^2-1)(1-x^2)^{1/2}\,dx$$

$$= -\frac{6iV}{\alpha^3}\sqrt{\frac{7}{48\pi}}\int_{-1}^{1}\left[5x^2(1-x^2)^{1/2} - (1-x^2)^{1/2}\right]dx$$

$$\int_{-1}^{1} x^2(1-x^2)^{1/2}\,dx = \left(-\frac{x(1-x^2)^{3/2}}{4} + \frac{x\sqrt{1-x^2}}{8} + \frac{1}{8}\mathrm{Sin}^{-1}x\right)\Big|_{-1}^{1}$$

$$\int_{-1}^{1}(1-x^2)^{1/2}\,dx = \left(\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\mathrm{Sin}^{-1}x\right)\Big|_{-1}^{1} = \frac{1}{2}\left(\frac{\pi}{2}-\frac{3\pi}{2}\right)$$

By: Herminso Villarraga Gómez

$$\Rightarrow A_{31} = -\frac{6iV}{a^3}\sqrt{\frac{21}{144\pi}}\left[-\frac{5}{8}\pi + \frac{1}{2}\pi\right] = -\frac{6iV}{12a^3}\sqrt{21\pi}\left(\frac{-10+8}{16}\right)$$

$$A_{31} = \frac{iV}{16a^3}\sqrt{21\pi} = A_{3,-1}$$

$$A_{33} = -\frac{4iV}{3a^3}\sqrt{\frac{7}{4\pi}\frac{1}{6!}}\int_{-1}^{1} P_3^3(x)\,dx = -\frac{4iV}{3a^3}\sqrt{\frac{7}{2880\pi}}\int_{-1}^{1} 15(1-x^2)^{3/2}\,dx$$

$$\int_{-1}^{1}(1-x^2)^{3/2}\,dx = \left(\frac{x(1-x^2)^{3/2}}{4} + \frac{3x\sqrt{1-x^2}}{8} + \frac{3}{8}\sin^{-1}x\right)\Big|_{-1}^{1}$$

$$= \frac{3}{8}\left(\frac{\pi}{2} - \frac{3\pi}{2}\right)$$

$$A_{33} = -\frac{4iV}{3a^3}\sqrt{\frac{35}{14400\pi}}\,15\,\frac{3}{8}(-\pi) = \frac{15iV}{240a^3}\sqrt{35\pi}$$

$$A_{33} = \frac{iV}{16a^3}\sqrt{35\pi} = A_{3,-3}$$

$$\Phi(r,\theta,\phi) = r(A_{11}Y_{11} + A_{1,-1}Y_{1,-1}) - r^3(A_{31}Y_{31} + A_{3,-1}Y_{3,-1} + A_{33}Y_{33} + A_{3,-3}Y_{3,-3}) + \cdots$$

$$Y_{l,-m} = (-1)^m Y_{lm}^* \qquad A_{lm} = A_{l,-m} \quad m \text{ impar.}$$

$$\Rightarrow \Phi = rA_{11}(Y_{11} - Y_{11}^*) + r^3A_{31}(Y_{31} - Y_{31}^*) + r^3A_{33}(Y_{33} - Y_{33}^*) + \cdots$$

$$Y_{lm} = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\,P_l^m(\cos\theta)\,e^{im\phi}$$

$$rA_{11}(Y_{11} - Y_{11}^*) = \frac{riV}{a}\sqrt{\frac{3\pi}{2}}\sqrt{\frac{3}{8\pi}}(1-\cos^2\theta)^{1/2}(e^{i\phi} - e^{-i\phi})$$

$$= \frac{3}{4}\frac{iVr}{a}\sin\theta\left(\frac{e^{i\phi} - e^{-i\phi}}{2i}\right)\cdot 2i = -\frac{3V}{2}\frac{r}{a}\sin\theta\sin\phi$$

$$r^3A_{31}(Y_{31} - Y_{31}^*) = \frac{iVr^3}{16a^3}\sqrt{21\pi}\sqrt{\frac{7}{4\pi}\frac{2!}{4!}}\frac{3}{2}\left(5(\cos^2\theta)-1\right)(1-\cos^2\theta)^{1/2}(e^{i\phi} - e^{-i\phi})$$

$$= \frac{iV}{16}\frac{r^3}{a^3}\sqrt{21\pi}\sqrt{\frac{7}{48}\frac{3}{3\pi}}\frac{3}{\cancel{2}}(5\cos^2\theta - 1)\sin\theta\sin\phi\,\cancel{2}i$$

$$= -\frac{V}{16}\frac{r^3}{a^3}\frac{21}{12}\cdot 3(5\cos^2\theta - 1)\sin\theta\sin\phi$$

$$= -\frac{21V}{64}\frac{r^3}{a^3}\sin\theta(5\cos^2\theta - 1)\sin\phi$$

$$r^3 A_{33}\left(Y_{33}-Y_{33}^*\right) = \frac{iVr^3}{16a^3}\sqrt{35\pi}\sqrt{\frac{7}{4\pi}\frac{1}{6!}}\,15\left(1-\cos^2\theta\right)^{3/2}\left(\frac{e^{i3\phi}-e^{-i3\phi}}{2i}\right)$$

$$= -\frac{V}{8}\frac{r^3}{a^3}\frac{\sqrt{35\pi}}{120}\sqrt{35/\pi}\;15\sin^3\theta\sin 3\phi$$

$$= -\frac{35}{64}V\frac{r^3}{a^3}\sin^3\theta\sin(3\phi)$$

$$\Rightarrow \Phi = V\left\{\frac{3}{2}\frac{r}{a}\sin\theta\sin\phi + \frac{21}{64}\frac{r^3}{a^3}\sin\theta\left(5\cos^2\theta-1\right)\sin\phi + \frac{35}{64}\frac{r^3}{a^3}\sin^3\theta\sin 3\phi\right\}+\cdots$$

$$= V\left\{\frac{3}{2}\frac{r}{a}\sin\theta\sin\phi + \frac{7}{64}\left(\frac{r}{a}\right)^3\left[3\sin\theta\left(5\cos^2\theta-1\right)\sin\phi + \frac{5}{64}\sin^3\theta\sin 3\phi\right]\right\}+\cdots$$

$$\boxed{\sin 3\phi = 3\sin\phi - 4\sin^3\phi}$$

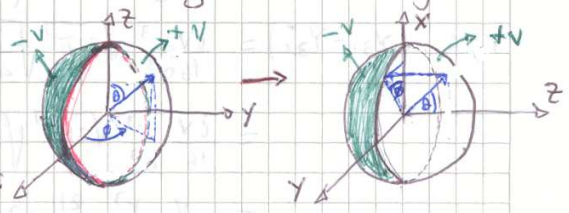
$$\Phi = V\left\{\frac{3}{2}\frac{r}{a}\sin\theta\sin\phi + \frac{7}{64}\left(\frac{r}{a}\right)^3\left[3\sin\theta\left(5\cos^2\theta-1\right)\sin\phi + \frac{15}{64}\sin^3\theta\sin\phi - \frac{20}{64}\sin^3\theta\sin^3\phi\right]\right\}+\cdots$$

pasando a cartesianas $x = r\sin\theta\cos\phi$ $\quad z = r\cos\theta$
$y = r\sin\theta\sin\phi$

$$\Rightarrow \Phi = V\left\{\frac{3}{2}\frac{y}{a} + \frac{7}{64}\left[3\frac{z}{a}\left(5\frac{z^2}{a^2}-\frac{r^2}{a^2}\right) - 20\frac{y^3}{a^3} + 15\frac{y}{a}\left(\frac{r^2}{a^2}-\frac{z^2}{a^2}\right)\right]\right\}+\cdots$$

Rotando las coordenadas para conseguir la configuración que produce (3.36):

$x \to y$, $y \to z$, $z \to x$



$$\Phi = V\left\{\frac{3}{2}\frac{z}{a} + \frac{7}{64}\left[3\frac{z}{a}\left(5\frac{x^2}{a^2}-\frac{r^2}{a^2}\right) - 20\frac{z^3}{a^3} + 15\frac{z}{a}\left(\frac{r^2}{a^2}-\frac{x^2}{a^2}\right)\right]\right\}+\cdots$$

Volviendo a coordenadas esféricas $x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

$$\Phi = V\left\{\frac{3}{2}\frac{r}{a}\cos\theta + \frac{7}{64}\frac{r^3}{a^3}\left[3\cos\theta\left(5\sin^2\theta\cos^2\phi - 1\right) - 20\cos^3\theta + 15\cos\theta\left(1-\sin^2\theta\sin^2\phi\right)\right]\right\} + \cdots$$

$$\Phi = V\left\{\frac{3}{2}\frac{r}{a}\cos\theta - \frac{7}{64}\frac{r^3}{a^3}\left[-12\cos\theta + 20\cos^3\theta\right] + \cdots\right\}$$

$$\Phi = V\left\{\frac{3}{2}\frac{r}{a}\underbrace{\cos\theta}_{P_1(\cos\theta)} - \frac{7}{8}\frac{r^3}{a^3}\underbrace{\left[-\frac{3}{2}\cos\theta + \frac{5}{2}\cos^3\theta\right]}_{P_3(\cos\theta)} + \cdots\right.$$

$$\boxed{\Phi(r,\theta,\phi) = V\left\{\frac{3}{2}\frac{r}{a}P_1(\cos\theta) - \frac{7}{8}\frac{r^3}{a^3}P_3(\cos\theta) + \cdots\right\}} \quad \text{(Ec. 3.36)}$$

3.5 El cálculo para la función de Green en el interior de la esfera es similar al de la seccion 2.6 de Jackson produciendo una ecuación idéntica a la ec. (2.19) de Jackson pero con un signo invertido.
La función de Green para el espacio interior es equivalente al potencial de una carga de valor $4\pi\epsilon_0$ en el interior de la esfera, el calculo de tal potencial y de $\partial\Phi/\partial n|_S$ se hace en el problema 2.2. Lo que se obtiene es:

$$\left.\frac{\partial G}{\partial n'}\right|_{r=a} = -\frac{(a^2-r^2)}{a\left(r^2+a^2-2ar\cos\gamma\right)^{3/2}} \qquad \text{Ec. similar a (2.19)}$$

con lo que para el interior de la esfera con condiciones de Dirichlet es:

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \underbrace{\rho(\vec{r}\,')}_{0}\, G(\vec{r},\vec{r}\,')\,dv' - \frac{1}{4\pi}\oint_S \Phi(\vec{r}\,')\frac{\partial G(\vec{r},\vec{r}\,')}{\partial n'}\,da'$$

"no hay cargas en el interior de la esfera."

$$\Phi(\vec{r}) = \frac{1}{4\pi}\int \Phi(a,\theta',\phi')\frac{(a^2-r^2)}{\cancel{a}\left(r^2+a^2-2ar\cos\gamma\right)^{3/2}}\left(a^{\cancel{2}}\,d\Omega\right)$$

$$= \frac{a}{4\pi}\int \frac{V(\theta',\phi')\,(a^2-r^2)\,d\Omega'}{\left(r^2+a^2-2ar\cos\gamma\right)^{3/2}}\cdot\frac{1/a^3}{\left(1/a^2\right)^{3/2}}$$

$$\Phi(\vec{r}) = \frac{1}{4\pi}\int \frac{V(\theta',\phi')\,(1-r^2/a^2)\,d\Omega'}{\left(1-2\frac{r}{a}\cos\gamma + r^2/a^2\right)^{3/2}}.$$

Recuérdese que $\boxed{g(t,x) = (1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\,t^n}$

$$\frac{\partial g(t,x)}{\partial t} = \frac{-1/2\,(2t-2x)}{(1-2xt+t^2)^{3/2}} = \frac{x-t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} n P_n(x)\,t^{n-1}$$

$$2t\,\frac{\partial g(t,x)}{\partial t} + g(x,t) = 2t\,\frac{x-t}{(1-2xt+t^2)^{3/2}} + \frac{1}{(1-2xt+t^2)^{1/2}}$$

$$2t\sum_{n=0}^{\infty} n P_n(x)\,t^{n-1} + \sum_{n=0}^{\infty} P_n(x)\,t^n = \frac{2xt-2t^2+(1-2xt+t^2)}{(1-2xt+t^2)^{3/2}}$$

$$\sum_{n=0}^{\infty}(2n+1)\,P_n(x)\,t^n = \frac{1-t^2}{(1-2xt+t^2)^{3/2}}$$

Si hacemos $t = r/a$ y $x = \cos\gamma$

$$\Phi(\vec{r}) = \frac{1}{4\pi}\int \frac{V(\theta',\phi')\left[1-r^2/a^2\right]d\Omega'}{\left(1-2\frac{r}{a}\cos\gamma+\frac{r^2}{a^2}\right)^{3/2}}$$

$$= \frac{1}{4\pi}\int V(\theta',\phi')\sum_{n=0}^{\infty}(2n+1)\,P_n(\cos\gamma)\left(\frac{r}{a}\right)^n d\Omega'$$

Usando ahora $\boxed{P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y^*_{lm}(\theta',\phi')\,Y_{lm}(\theta,\phi)}$

$$\Phi(\vec{r}) = \int V(\theta',\phi')\sum_{n=0}^{\infty}\sum_{m=-n}^{n} Y^*_{nm}(\theta',\phi')\,Y_{nm}(\theta,\phi)\left(\frac{r}{a}\right)^n d\Omega'$$

$$\boxed{\Phi(\vec{r}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\underbrace{\left(\int V(\theta',\phi')\,Y^*_{lm}(\theta',\phi')\,d\Omega'\right)}_{A_{lm}}\left(\frac{r}{a}\right)^l Y_{lm}(\theta,\phi)}$$

Otro método:

Podemos partir directamente de la función de Green para la esfera

$$G(\vec{r},\vec{r}\,') = \frac{1}{|\vec{r}-\vec{r}\,'|} - \frac{a}{r'\left|\vec{r}-\frac{a^2}{r'^2}\vec{r}\,'\right|} \qquad \text{(Ec. 2.16)}$$

y usar:

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{r_<^{\,l}}{r_>^{\,l+1}} \frac{1}{2l+1} Y_{lm}(\theta,\phi)\, Y^*_{lm}(\theta',\phi')$$

teniendo en cuenta que $r<r'$ y $r<\frac{a^2}{r'^2}|\vec{r}\,'|$

$$G(\vec{r},\vec{r}\,') = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1} Y^*_{lm}(\theta',\phi')\, Y_{lm}(\theta,\phi) \left(\frac{r^l}{r'^{l+1}} - \frac{a}{r'} \frac{r^l}{\left(\frac{a^2}{r'^2}r'\right)^{l+1}} \right)$$

$$= 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1} Y^*_{lm}(\theta',\phi')\, Y_{lm}(\theta,\phi) \left(\frac{r^l}{r'^{l+1}} - \frac{r'^l r^l}{a^{2l+1}} \right)$$

$$\frac{\partial G}{\partial r'} = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1} Y^*_{lm}(\theta',\phi')\, Y_{lm}(\theta,\phi) \left(-(l+1)\frac{r^l}{r'^{l+2}} - l\,\frac{r'^{l-1} r^l}{a^{2l+1}} \right)$$

$$\left.\frac{\partial G}{\partial r'}\right|_{r'=a} = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1} Y^*_{lm}(\theta',\phi')\, Y_{lm}(\theta,\phi) \left(-(l+1)\frac{r^l}{a^{2l+1}} - l\,\frac{r^l}{a^{2l+2}} \right)$$

$$= -4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{\cancel{2l+1}} Y^*_{lm}(\theta',\phi')\, Y_{lm}(\theta,\phi)\, \cancel{(2l+1)} \left(\frac{r}{a}\right)^l \frac{1}{a^2}$$

$$\Rightarrow \Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\underbrace{\cancel{\int \rho(\vec{r}\,')\, G(\vec{r},\vec{r}\,')\, dv'}}_{0} - \frac{1}{4\pi}\oint_S \Phi(\vec{r}\,')\, \frac{\partial G(\vec{r},\vec{r}\,')}{\partial n'}\, da'$$

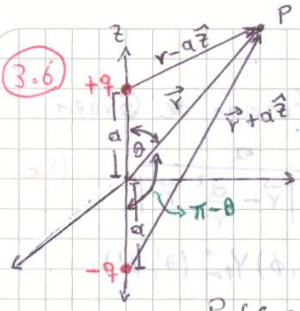
no hay cargas en el interior.

$$= \cancel{-}\frac{1}{\cancel{4\pi}} \int \Phi(\theta',\phi') \left(\cancel{-4\pi} \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{lm}(\theta',\phi')\, Y_{lm}(\theta,\phi) \left(\frac{r}{a}\right)^l \frac{1}{\cancel{a^2}} \left(\cancel{a^2}\, d\Omega'\right)\right)$$

$$\boxed{\Phi(\vec{r}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \underbrace{\left(\int \Phi(\theta',\phi')\, Y^*_{lm}(\theta',\phi')\, d\Omega' \right)}_{A_{lm}} \left(\frac{r}{a}\right)^l Y_{lm}(\theta,\phi)}$$

By: Herminso Villarraga Gómez

3.6



$$\Phi(\vec{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\vec{r}-a\hat{z}|} - \frac{1}{|\vec{r}+a\hat{z}|}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta) - \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l[\cos(\pi-\theta)]\right]$$

$$= \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}\left[P_l(\cos\theta) - P_l(\cos(\pi-\theta))\right]$$

$$P_l(\cos(\pi-\theta)) = P_l(-\cos\theta) = P_l(-x)$$

Si l es par $\rightarrow$ P_l es par $\rightarrow$ $P_l(-x) = P_l(x)$
Si l es impar $\rightarrow$ P_l es impar $\rightarrow$ $P_l(-x) = -P_l(x)$ $\Big\}$ $\boxed{P_l(-x) = (-1)^l P_l(x)}$

$$\Rightarrow \Phi(\vec{r}) = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}\left[P_l(\cos\theta) + (-1)^{l+1}P_l(\cos\theta)\right]$$

$$= \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}\left[1 + (-1)^{l+1}\right]P_l(\cos\theta)$$

Usando ahora que $P_l(\cos\theta) = \sqrt{\frac{4\pi}{2l+1}}\,Y_l^0(\theta,\phi)$

$$\boxed{\Phi(\vec{r}) = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}\left[1 + (-1)^{l+1}\right]\sqrt{\frac{4\pi}{2l+1}}\,Y_l^0(\theta,\phi)}$$

$r_< = a$, $r_> = r$

$r \gg a$

b) Conservando únicamente los dos primeros términos:

$$\Phi(\vec{r}) \approx \frac{q}{4\pi\epsilon_0}\left[\frac{a^0}{r^1}\underbrace{[1 + (-1)]}_{0}P_0(\cos\theta) + \frac{a^1}{r^2}[1 + (-1)^2]\underbrace{P_1(\cos\theta)}_{\cos\theta}\right]$$

$$\boxed{\Phi(\vec{r}) \approx \frac{q(2a)}{4\pi\epsilon_0 r^2}\cos\theta} \rightarrow \text{potencial de dipolo.}$$

c) $\Phi(\vec{r}) = \Phi_{dipolo} + \Phi_{esfera}$ donde Φ_{esfera} puede construirse como $\Phi_{esfera} = \sum_l [A_l r^l + B_l r^{-(l+1)}]P_l(\cos\theta)$ dada la simetría acimutal, y $B_l = 0$ para evitar singularidades en $r \rightarrow 0$. en Φ_{esfera}

$$\Rightarrow \Phi(\vec{r}) = \frac{q(2a)}{4\pi\epsilon_0 r^2}\cos\theta + \sum_l A_l r^l P_l(\cos\theta)$$

$$\Phi(r=b) = \frac{q(2a)}{4\pi\epsilon_0 b^2}\cos\theta + \sum_l A_l b^l P_l(\cos\theta) = 0$$

$$\rightarrow -\frac{q(2a)}{4\pi\epsilon_0 b^2}\cos\theta = A_0 + A_1 b\underbrace{P_1(\cos\theta)}_{\cos\theta} + A_2 b^2 P_2(\cos\theta) + \cdots$$

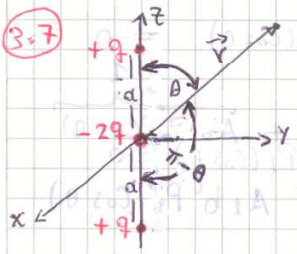
igualando coeficientes: $A_1 = -\dfrac{q(2a)}{4\pi\epsilon_0 b^3}$

$A_\ell = 0$ si $\ell \neq 1$

$$\rightarrow \Phi(\vec{r}) = \frac{q(2a)}{4\pi\epsilon_0 r^2}\cos\theta - \frac{q(2a)}{4\pi\epsilon_0 b^3} r\cos\theta$$

$$\boxed{\Phi(\vec{r}) = \frac{q(2a)}{4\pi\epsilon_0}\left[\frac{1}{r^2} - \frac{r}{b^3}\right]\cos\theta}$$

3.7



$$\Phi(\vec{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\vec{r}-a\hat{z}|} - \frac{2}{|\vec{r}|} + \frac{1}{|\vec{r}+a\hat{z}|}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\sum_{\ell=0}^{\infty}\frac{r_<^\ell}{r_>^{\ell+1}}P_\ell(\cos\theta) - \frac{2}{r} + \sum_{\ell=0}^{\infty}\frac{r_<^\ell}{r_>^{\ell+1}}P_\ell(\cos(\pi-\theta))\right]$$

$$P_\ell(\cos(\pi-\theta)) = P_\ell(-\cos\theta) = (-1)^\ell P_\ell(\cos\theta)$$

$$\Rightarrow \Phi(\vec{r}) = \frac{q}{4\pi\epsilon_0}\left[\sum_{\ell=0}^{\infty}\frac{r_<^\ell}{r_>^{\ell+1}}\underbrace{\left(1+(-1)^\ell\right)}_{0 \rightarrow \text{si } \ell \text{ es impar},\ 2 \text{ si } \ell \text{ es par.}}P_\ell(\cos\theta) - \frac{2}{r}\right]$$

incluyendo $\ell = 0$

$$\boxed{\Phi(\vec{r}) = \frac{2q}{4\pi\epsilon_0}\left[\sum_{\ell\ \text{par}}\frac{r_<^\ell}{r_>^{\ell+1}}P_\ell(\cos\theta) - \frac{1}{r}\right]}$$

Si $a \rightarrow 0$, $r_< = a$ y $r_> = r$ así

$$\Rightarrow \Phi(\vec{r}) = \frac{2q}{4\pi\epsilon_0}\left[\sum_{\ell\ \text{par}}\frac{a^\ell}{r^{\ell+1}}P_\ell(\cos\theta) - \frac{1}{r}\right]$$ y si $r \gg a$

$$r \gg a \rightarrow \approx \frac{2q}{4\pi\epsilon_0}\left[\cancel{\frac{1}{r}\underbrace{P_0(\cos\theta)}_{1}} + \frac{a^2}{r^3}P_2(\cos\theta) - \cancel{\frac{1}{r}}\right]$$

$$\boxed{\Phi(\vec{r}) = \frac{2(qa^2)}{4\pi\epsilon_0 r^3}P_2(\cos\theta)} \rightarrow \text{potencial cuadrupolar}$$

b) $\Phi(\vec{r}) = \Phi_a + \Phi_{esfera}$ donde b es el radio de la esfera.

$$\Phi_{esfera} = \sum_\ell\left[A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right]P_\ell(\cos\theta)$$

- Si $r < b$, Φ_{esfera} es finito si $r \rightarrow 0 \rightarrow B_\ell = 0$

$$\Rightarrow \Phi(\vec{r}) = \frac{2q}{4\pi\epsilon_0}\left[\sum_{\substack{\ell=0\\ \ell\ \text{par}}}^{\infty}\frac{r_<^\ell}{r_>^{\ell+1}}P_\ell(\cos\theta) - \frac{1}{r}\right] + \sum_{\ell=0}^{\infty}A_\ell r^\ell P_\ell(\cos\theta)$$

$r_< = \min\{a, r\}$, $r_> = \max\{a, r\}$

como la esfera esta a tierra: $\Phi(r=b)=0$, si $r=b>a$

$\Rightarrow r_< = a$ y $r_> = r = b$:

$$\Phi(r=b) = \frac{2q}{4\pi\epsilon_0}\left[\sum_{l\ par}\frac{a^l}{b^{l+1}}P_l(\cos\theta) - \frac{1}{b}\right] + \sum_{l=0}^{\infty} A_l b^l P_l(\cos\theta) = 0$$

$$0 = \frac{2q}{4\pi\epsilon_0}\left[\cancel{\frac{1}{b}}\underbrace{P_0(\cos\theta)}_{1} + \sum_{\substack{l=2\\ l\ par}}^{\infty}\frac{a^l}{b^{l+1}}P_l(\cos\theta) - \cancel{\frac{1}{b}}\right] + \sum_{l=0}^{\infty} A_l b^l P_l(\cos\theta)$$

$$0 = \sum_{\substack{l=2\\ l\ par}}^{\infty}\left(\frac{2q}{4\pi\epsilon_0}\frac{a^l}{b^{l+1}} + A_l b^l\right)P_l(\cos\theta) + A_0 \overbrace{P_0(\cos\theta)}^{1} + \sum_{\substack{l=1\\ l\ impar}} A_l b^l P_l(\cos\theta)$$

$$\Rightarrow \begin{cases} A_l = -\dfrac{2q}{4\pi\epsilon_0}\dfrac{a^l}{b^{2l+1}} & \text{si } l \text{ es par diferente de cero.} \\ A_l = 0 & \text{si } l \text{ es impar ó cero} \end{cases}$$

$$\Rightarrow \Phi(\vec{r}) = \frac{2q}{4\pi\epsilon_0}\left[\sum_{\substack{l=0\\ l\ par}}\frac{a^l}{r^{l+1}}P_l(\cos\theta) - \frac{1}{r}\right] + \sum_{\substack{l\ par\\ l\neq 0}}\frac{-2q}{4\pi\epsilon_0}\frac{a^l}{b^{2l+1}}r^l P_l(\cos\theta)$$

$$\boxed{\Phi(\vec{r}) = \frac{2q}{4\pi\epsilon_0}\left[\sum_{l\ par}\frac{a^l}{r^{l+1}}P_l(\cos\theta) - \sum_{\substack{l\ par\\ l\neq 0}}\frac{a^l}{b^{2l+1}}r^l P_l(\cos\theta) - \frac{1}{r}\right]}$$

si $b>r>a$

- Si $r<b$ pero a la vez $r<a<b$

El potencial de la esfera sigue siendo el mismo ya encontrado con $\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$ pero ahora

$r_< = r$ y $r_> = a$ con lo que el potencial debido a unicamente las cargas puntuales es ahora:

$$\frac{2q}{4\pi\epsilon_0}\left[\sum_{l\ par}\frac{r^l}{a^{l+1}}P_l(\cos\theta) - \frac{1}{r}\right]$$

$$\Rightarrow \Phi(\vec{r}) = \frac{2q}{4\pi\epsilon_0}\left[\sum_{\substack{l=0\\ l\ par}}\frac{r^l}{a^{l+1}}P_l(\cos\theta) - \frac{1}{r}\right] + \sum_{\substack{l\ par\\ l\neq 0}}\frac{-2q}{4\pi\epsilon_0}\frac{a^l}{b^{2l+1}}r^l P_l(\cos\theta)$$

$$\Rightarrow \boxed{\Phi(\vec{r}) = \frac{2q}{4\pi\epsilon_0}\left[\sum_{l\ \text{par}} \frac{r^l}{a^{l+1}} P_l(\cos\theta) - \sum_{\substack{l\ \text{par} \\ l\neq 0}} \frac{a^l}{b^{2l+1}} r^l P_l(\cos\theta) - \frac{1}{r}\right]}$$

si $a > r > 0$ $(a<b)$

Las dos soluciones se pueden juntar para tener

$$\boxed{\Phi(\vec{r}) = \frac{2q}{4\pi\epsilon_0}\left[\sum_{l\ \text{par}} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta) - \sum_{l\ \text{par}>0} \frac{a^l}{b^{2l+1}} r^l P_l(\cos\theta) - \frac{1}{r}\right]}$$

$r_> = \text{máx}\{a, r\}$ y $r_< = \text{mín}\{a, r\}$.

- Si $a \to 0$ $(r >> a)$: $r_< = a$ y $r_> = r$

$$\Phi(\vec{r}) = \frac{q}{2\pi\epsilon_0}\left[\sum_{l\ \text{par}} \frac{a^l}{r^{l+1}} P_l(\cos\theta) - \sum_{l\ \text{par}>0} \frac{a^l}{b^{2l+1}} r^l P_l(\cos\theta) - \frac{1}{r}\right]$$

$$= \frac{q}{2\pi\epsilon_0}\left[\left(\cancel{\frac{1}{r}} + \frac{a^2}{r^3} P_2(\cos\theta) + \cdots\right) - \left(\frac{a^2}{b^5} P_2(\cos\theta) + \cdots\right) - \cancel{\frac{1}{r}}\right]$$

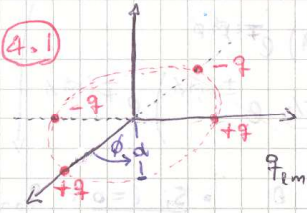
$$\approx \frac{q}{2\pi\epsilon_0}\left(\frac{a^2}{r^3} - \frac{a^2}{b^5} r^2\right) P_2(\cos\theta)$$ si $r >> a$

$$\boxed{\Phi(\vec{r}) \approx \frac{2(qa^2)}{4\pi\epsilon_0 r^3}\left(1 - \frac{r^5}{b^5}\right) P_2(\cos\theta)}$$ $r >> a$

By: Herminso Villarraga Gómez

Solución a los problemas del capitulo 4 Jackson 3ed

4.1



$$\rho(\vec{r}) = \frac{q}{r^2}\,\delta(r-a)\,\delta(\cos\theta)\left\{\delta(\phi)+\delta(\phi-\pi/2)-\delta(\phi-\pi)-\delta(\phi-3\pi/2)\right\}$$

$$q_{lm} = \int Y^*_{lm}(\theta',\phi')\,r'^l\,\rho(r')\,dv'$$

$$\int = q_1\delta$$

$$q_{lm} = \int\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\,P_l^m(\cos\theta)\,e^{-im\phi}\,r'^l\,\frac{q}{\cancel{r^2}}\,\delta(r-a)\,\delta(\cos\theta)$$

$$\cdot\left\{\delta(\phi)+\delta(\phi-\pi/2)-\delta(\phi-\pi)-\delta(\phi-\tfrac{3\pi}{2})\right\}\cancel{r^2}dr\,d(\cos\theta)\,d\phi$$

$$= \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\,P_l^m(0)\,a^l\,q\left\{1+e^{-im\pi/2}-e^{-im\pi}-e^{-i3m\pi/2}\right\}$$

$$e^{-im\pi} = \cos m\pi - i\,\cancelto{0}{\sin m\pi} = (-1)^m$$

$$e^{-i3m\pi/2} = e^{-im\pi/2}\,e^{-im\pi} = e^{-im\pi/2}\,(-1)^m$$

$$\Rightarrow q_{lm} = \sqrt{}\,P_l^m(0)\,a^l\,q\left\{1+e^{-im\pi/2}-(1)^m-(-1)^m e^{-im\pi/2}\right\}$$

$$= \sqrt{}\,\underbrace{P_l^m(0)}_{0 \text{ si } l+m \text{ impar}}\,a^l\,q\,\underbrace{\left(1-(-1)^m\right)}_{\substack{0 \text{ si } m \text{ es par} \\ 2 \text{ si } m \text{ es impar}}}\left(1+e^{-im\pi/2}\right)$$

$l+m$ es par si l es par y m es par (m par $\rightarrow q_{lm}=0$)
si l es impar y m es impar

→ Solo queda que l y m sean impares juntos

Si m es impar $\rightarrow$ $\left(1-(-1)^m\right)\left(1+e^{-im\pi/2}\right) = 2\left(1+\cancelto{0}{\cos\frac{m\pi}{2}}-i\sin\frac{m\pi}{2}\right)$ m impar

$$\Rightarrow \boxed{q_{lm} = 2\left(1-i\sin\frac{m\pi}{2}\right)q\,a^l\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\,P_l^m(0)}$$

siempre que l y m sean impares y $q_{lm}=0$ en otro caso.

$$q_{1,\pm1} = 2\left(1-i\sin\pm\pi/2\right)q\,a\sqrt{\frac{3}{4\pi}\frac{(1\mp1)!}{(1\pm1)!}}\,P_1^{\pm1}(0)$$

$$P_l^m(\cos\theta) = \sqrt{\frac{4\pi}{2l+1}\frac{(l+m)!}{(l-m)!}}\,Y_{lm}(\theta,\phi)\,e^{-im\phi} \Rightarrow$$

$$Y_{l,-m}(\theta,\phi) = (-1)^m Y^*_{lm}(\theta,\phi)$$

$$\boxed{q_{lm} = 2\left(1 - i\sin\frac{m\pi}{2}\right) q a^l Y_{lm}(\pi/2,\phi)\, e^{-im\phi}}$$

l y m impares

$\underline{l=1}$: $q_{1,\pm1} = 2(1 - i\sin\pm\pi/2)\, q a\, Y_{1,\pm1}(\pi/2,\phi)\, e^{\mp i\phi}$

$$= 2(1\mp i)\, q a\, (\pm1)\left(-\sqrt{\frac{3}{8\pi}}\, \underbrace{\sin\pi/2}_{1}\, \cancel{e^{\pm i\phi}}\right) e^{\mp i\phi}$$

$$\boxed{q_{1,\pm1} = \mp(1\mp i)\sqrt{\frac{3}{2\pi}}\, q a}$$

$q_{10} = 0$ • $\underline{\text{Si } l=0}$: $q_{00} = 0$

$\underline{l=2} \longrightarrow q_{20} = 0,\ q_{2,\pm1} = 0,\ q_{2,\pm2} = 0$

$\underline{l=3}$ $q_{3,\pm1} = 2(1 - i\sin\pm\pi/2)\, q a^3 Y_{3,\pm1}(\pi/2,\phi)\, e^{\mp i\phi}$

$$= 2(1\mp i)\, q a^3 (\pm1)\left(\frac{-1}{4}\sqrt{\frac{21}{4\pi}}\, \underbrace{\sin\pi/2}_{1}\, \left(5\cancel{\cos^2\frac{\pi}{2}}^{\,0} - 1\right)\cancel{e^{\pm i\phi}}\right)\cancel{e^{\mp i\phi}}$$

$$\boxed{q_{3,\pm1} = \pm(1\mp i)\sqrt{\frac{21}{16\pi}}\, q a^3}$$

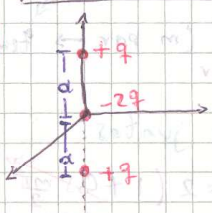
$q_{30} = 0$, $q_{3,\pm2} = 0$

$$q_{3,\pm3} = 2(1 - i\sin\pm3\pi/2)\, q a^3 Y_{3,\pm3}(\pi/2,\phi)\, e^{\mp i3\phi}$$

$$= 2(1\pm i)\, q a^3 (\pm1)\left(-\frac{1}{4}\sqrt{\frac{35}{4\pi}}\, \underbrace{\sin^3\frac{\pi}{2}}_{1}\, \cancel{e^{\pm 3i\phi}}\right) e^{\mp i3\phi}$$

$$\boxed{q_{3,\pm3} = \mp(1\pm i)\sqrt{\frac{35}{16\pi}}\, q a^3}$$

b)



$$\rho(\vec{r}) = \frac{q}{2\pi r^2}\left\{\delta(r-a)\,\delta(\cos\theta - 1) + \delta(r-a)\,\delta(\cos\theta + 1) - \delta(r)\right\}$$

Note que $\displaystyle\int_{r<a} \frac{q\,\delta(r)}{2\pi \cancel{r^2}}\, \cancel{r^2}\, dr'\, d\Omega' = 2q$

$$q_{lm} = \int Y^*_{lm}(\theta',\phi')\, r'^l\, \rho(r')\, dv'$$

$$q_{lm} = \int \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta')\, e^{-im\phi'}\, r'^l\, \frac{q}{2\pi \cancel{r'^2}}\Bigg\} \qquad \Big\{\cancel{r'^2}\, dr'\, d(\cos\theta')\, d\phi'$$

$$= \frac{q}{2\pi}\sqrt{}\left\{\int_0^{2\pi} e^{-im\phi'}\, d\phi'\left(P_l^m(1) + P_l^m(-1)\right) a^l - \int_0^{2\pi} e^{-im\phi'}\, d\phi' \underbrace{\int r'^l\, \delta(r')\, dr'}\int_{-1}^{1} P_l^m(\cos\theta')\, d(\cos\theta')\right\}$$

$$\int_0^{2\pi} e^{-im\phi'}\, d\phi' = 2\pi\delta_{m0} \qquad \begin{cases} 0 & \text{si } l \neq 0 \\ 1 & \text{si } l = 0 \end{cases}$$

By: Herminso Villarraga Gómez

$P_0 = 1$
$P_1 = x$
$P_2 =$

$$\text{Si } l=0 \rightarrow \int_{-1}^{1} \overbrace{P_l^m(\cos\theta')}^{1 \text{ si } l=0} d(\cos\theta') = 2$$

$$\Rightarrow q_{lm} = \frac{q}{\cancel{2\pi}} \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, \cancel{2\pi}\delta_{m0} \left\{ \left(P_l^m(1) + P_l^m(-1)\right)\alpha^l - 2\delta_{l0} \right\}$$

$$= q\sqrt{\frac{2l+1}{4\pi}} \left\{ \left(\underbrace{P_l(1)}_{1} + \underbrace{P_l(-1)}_{(-1)^l}\right)\alpha^l - 2\delta_{l0} \right\} \qquad \text{si } m=0$$

$$q_{lm} = q\sqrt{\frac{2l+1}{4\pi}} \left\{ \underbrace{\left(1+(-1)^l\right)}_{\substack{2 \text{ si } l \text{ es par} \\ 0 \text{ si } l \text{ es impar.}}} \alpha^l - 2\delta_{l0} \right\} \qquad \text{si } m=0$$

$$\boxed{q_{lm} = \begin{cases} 2q\sqrt{\frac{2l+1}{4\pi}}\,\alpha^l & \text{si } m=0,\ l \text{ par} \neq 0 \\ 0 & \text{en otro caso} \end{cases}}$$

Si $l=1$: $q_{10}=0$ $\quad q_{1,\pm1}=0$ $\quad$ • Si $l=0$: $q_{00}=0$

Si $l=2$: $\boxed{q_{20} = q\sqrt{\frac{5}{\pi}}\,\alpha^2}$ $\quad q_{2,\pm1}=0$ $\quad q_{2,\pm2}=0$

Si $l=3$: $q_{3,m}=0$ $\quad m=0,\pm1,\pm2,\pm3$

Si $l=4$: $\boxed{q_{4,0} = q\sqrt{\frac{9}{\pi}}\,\alpha^4}$, $\quad q_{4,m}=0$ $\quad m=\pm1,\pm2,\pm3,\pm4$

c) $$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{2l+1}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}q_{lm}$$

$$= \frac{1}{4\pi\epsilon_0}\sum_{\substack{l \text{ par} \\ l\neq0}}^{\infty} \frac{4\pi}{2l+1}\frac{Y_{l0}(\theta,\phi)}{r^{l+1}}q_{l0} = \frac{1}{4\pi\epsilon_0}\sum_{\substack{l \text{ par} \\ l\neq0}}^{\infty}\frac{4\pi}{2l+1}\sqrt{\frac{2l+1}{4\pi}}$$

$$= \frac{1}{4\pi\epsilon_0}\sum_{\substack{l \text{ par} \\ l\neq0}}^{\infty}\frac{\cancel{4\pi}}{\cancel{2l+1}}\frac{\cancel{\sqrt{\frac{2l+1}{4\pi}}}\,P_l(\cos\theta)\cancelto{1}{e^{i0}}}{r^{l+1}}\left(2q\cancel{\sqrt{\frac{2l+1}{4\pi}}}\,\alpha^l\right)$$

$$\boxed{\Phi(\vec{r}) = \frac{q}{2\pi\epsilon_0}\sum_{k=1}^{\infty}\frac{\alpha^{2k}}{r^{2k+1}}P_{2k}(\cos\theta)} = \frac{q}{2\pi\epsilon_0 r}\left\{\frac{1}{2}(3\cos^2\theta-1)\frac{\alpha^2}{r^2}+\dots\right\}$$

En el plano x-y, $\cos\theta=0 \rightarrow \Phi(r,\theta=\pi/2) = -\frac{q}{4\pi\epsilon_0}\frac{\alpha^2}{r^3}+\dots$

d) $$\Phi_{exact}(r,\pi/2) = \frac{1}{4\pi\epsilon_0}\left\{\frac{2q}{\sqrt{r^2+a^2}} - \frac{2q}{r}\right\} = \frac{q}{2\pi\epsilon_0 r}\left\{(1+a^2/r^2)^{-1/2}-1\right\}$$

$$= \frac{q}{2\pi\epsilon_0 r}\left\{\cancel{1} - \frac{1}{2}\left(\frac{a}{r}\right)^2 + \frac{3}{8}\left(\frac{a}{r}\right)^4 + \dots - \cancel{1}\right\}$$

→ Coincide con la expansión multipolar !!

4.2 $$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\vec{p}\cdot\frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3} = \frac{1}{4\pi\epsilon_0}\int \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\cdot\vec{p}\,\delta^3(\vec{r}-\vec{r}_0)\,dv'$$

$$= \frac{1}{4\pi\epsilon_0}\int \vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\cdot\vec{p}\,\delta^3(\vec{r}'-\vec{r}_0)\,dv' \qquad \nabla\left(\frac{a}{b}\right) = \frac{(\nabla a)}{b} + a\nabla\left(\frac{1}{b}\right)$$

$$= \frac{1}{4\pi\epsilon_0}\int\left\{\vec{\nabla}'\left[\frac{\vec{p}\cdot\delta^3(\vec{r}'-\vec{r}_0)}{|\vec{r}-\vec{r}'|}\right] - \frac{1}{|\vec{r}-\vec{r}'|}\left(\vec{p}\cdot\vec{\nabla}'\delta^3(\vec{r}'-\vec{r}_0)\right)\right\}dv'$$

$$= \frac{1}{4\pi\epsilon_0}\oint_S \frac{\vec{p}\cdot\hat{n}'}{|\vec{r}-\vec{r}'|}\delta^3(\vec{r}'-\vec{r}_0)\,da' + \frac{1}{4\pi\epsilon_0}\int\frac{1}{|\vec{r}-\vec{r}'|}\underbrace{\left(-\vec{p}\cdot\vec{\nabla}'\delta^3(\vec{r}'-\vec{r}_0)\right)}_{\rho(\vec{r}')}dv'$$

0 pues $\vec{r}_0$ es interior a V y no contenido en S.

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}dv' \longrightarrow \boxed{\rho(\vec{r}) = -\vec{p}\cdot\vec{\nabla}'\delta^3(\vec{r}-\vec{r}_0)}$$

Por energía: $$W = -\vec{p}\cdot\vec{E}(\vec{r}_0) = \int\delta^3(\vec{r}'-\vec{r}_0)\left(-\vec{p}\cdot\vec{E}(\vec{r}_0)\right)dv'$$

$$= \int\delta^3(\vec{r}-\vec{r}_0)\,\vec{p}\cdot\vec{\nabla}'\Phi(\vec{r}')\,dv'$$

$$W = \int\left\{\vec{\nabla}'\left[\vec{p}\,\Phi(\vec{r}')\,\delta^3(\vec{r}'-\vec{r}_0)\right] - \Phi(\vec{r}')\,\vec{\nabla}'\left(\vec{p}\,\delta^3(\vec{r}'-\vec{r}_0)\right)\right\}dv'$$

$$= \oint_S(\vec{p}\cdot\hat{n}')\,\Phi(\vec{r}')\,\delta^3(\vec{r}'-\vec{r}_0)\,da' + \int\underbrace{\left[-\vec{p}\cdot\vec{\nabla}'\delta^3(\vec{r}'-\vec{r}_0)\right]}_{\rho(\vec{r}')}\Phi(\vec{r}')\,dv'$$

0 $\vec{r}_0$ es interior a V y no $\in$ S.

$$W = \int\rho(\vec{r}')\,\Phi(\vec{r}')\,dv' \longrightarrow \boxed{\rho(\vec{r}) = -\vec{p}\cdot\vec{\nabla}'\delta^3(\vec{r}-\vec{r}_0)}$$

4.3

4.5 $\vec{F} = \int \rho(\vec{r})\, \vec{E}^{(0)}(\vec{r})\, dv \quad \rightarrow \quad F_i = \int \rho(\vec{r})\, E_i^{(0)}(\vec{r})\, dv$

$$E_i^{(0)}(\vec{r}) = E_i^{(0)}(0) + \sum_j x_j \left(\frac{\partial E_i^{(0)}(y)}{\partial y_j}\right)_{y=0} + \frac{1}{2}\sum_{j,k} x_j x_k \left(\frac{\partial^2 E_i^{(0)}(y)}{\partial y_j \partial y_k}\right)_{y=0} + \cdots$$

$$\Rightarrow F_i = E_i^{(0)}(0) \underbrace{\int \rho(\vec{r})\, dv}_{q} + \sum_j \left(\frac{\partial E_i^{(0)}(y)}{\partial y_j}\right)_{y=0} \underbrace{\int x_j \rho(\vec{r})\, dv}_{p_j}$$

$$+ \frac{1}{2}\sum_{j,k} \left(\frac{\partial^2 E_i^{(0)}(y)}{\partial y_j \partial y_k}\right)_{y=0} \int x_j x_k \rho(\vec{r})\, dv + \cdots$$

$$= q E_i^{(0)}(0) + \left[\vec{\nabla}_y E_i^{(0)}(y) \cdot \vec{p}\right]_{y=0} + \frac{1}{2}\sum_{j,k} \left(\frac{\partial^2 E_i^{(0)}(y)}{\partial y_j \partial y_k}\right)_{y=0} \int x_j x_k \rho(\vec{r})\, dv + \cdots$$

$$\vec{\nabla}_y E_i^{(0)}(y) \cdot \vec{p} = \frac{\partial E_i^{(0)}}{\partial y_1} p_1 + \frac{\partial E_i^{(0)}}{\partial y_2} p_2 + \frac{\partial E_i^{(0)}}{\partial y_3} p_3$$

Como $\boxed{\vec{\nabla} \times \vec{E} = 0} = \left(\frac{\partial E_3}{\partial y_2} - \frac{\partial E_2}{\partial y_3}\right)\hat{\imath} - \left(\frac{\partial E_3}{\partial y_1} - \frac{\partial E_1}{\partial y_3}\right)\hat{\jmath} + \left(\frac{\partial E_2}{\partial y_1} - \frac{\partial E_1}{\partial y_2}\right)\hat{k} = \vec{0}$

$$\Rightarrow \frac{\partial E_1}{\partial y_2} = \frac{\partial E_2}{\partial y_1}, \quad \frac{\partial E_1}{\partial y_3} = \frac{\partial E_3}{\partial y_1} \quad y \quad \frac{\partial E_2}{\partial y_3} = \frac{\partial E_3}{\partial y_2} \rightarrow \boxed{\frac{\partial E_i}{\partial y_j} = \frac{\partial E_j}{\partial y_i}}$$

$$\vec{\nabla}_y E_1(y) \cdot \vec{p} = \frac{\partial E_1}{\partial y_1} p_1 + \frac{\partial E_1}{\partial y_2} p_2 + \frac{\partial E_1}{\partial y_3} p_3 = \frac{\partial E_1}{\partial y_1} p_1 + \frac{\partial E_2}{\partial y_1} p_2 + \frac{\partial E_3}{\partial y_1} p_3$$

$$\vec{\nabla}_y E_2(y) \cdot \vec{p} = \frac{\partial E_2}{\partial y_1} p_1 + \frac{\partial E_2}{\partial y_2} p_2 + \frac{\partial E_2}{\partial y_3} p_3 = \frac{\partial E_1}{\partial y_2} p_1 + \frac{\partial E_2}{\partial y_2} p_2 + \frac{\partial E_3}{\partial y_2} p_3$$

$$\vec{\nabla}_y E_3(y) \cdot \vec{p} = \frac{\partial E_3}{\partial y_1} p_1 + \frac{\partial E_3}{\partial y_2} p_2 + \frac{\partial E_3}{\partial y_3} p_3 = \frac{\partial E_1}{\partial y_3} p_1 + \frac{\partial E_2}{\partial y_3} p_2 + \frac{\partial E_3}{\partial y_3} p_3$$

En general $\left[\vec{\nabla}_y E_i^{(0)}(y) \cdot \vec{p}\right]_{y=0} = \left[\vec{p} \cdot \frac{\partial \vec{E}}{\partial y_i}\right]_{y=0}$

con ello $\vec{\nabla}_x E_1(x) \cdot \vec{p}\, \hat{\imath} + \nabla_x E_2(x) \cdot \vec{p}\, \hat{\jmath} + \nabla_x E_2(x) \cdot \vec{p}\, \hat{k}$

$$= \vec{p} \cdot \frac{\partial}{\partial x_1} \vec{E}\, \hat{\imath} + \vec{p} \cdot \frac{\partial}{\partial x_2} \vec{E}\, \hat{\jmath} + \vec{p} \frac{\partial}{\partial x_3} \vec{E}\, \hat{k} \Big|_{x=0}$$

$$= \frac{\partial}{\partial x_1}(\vec{p} \cdot \vec{E})\hat{\imath} + \frac{\partial}{\partial x_2}(\vec{p} \cdot \vec{E})\hat{\jmath} + \frac{\partial}{\partial x_3}(\vec{p} \cdot \vec{E})\hat{k} \Big|_{x=0}$$

$$= \left[\vec{\nabla}(\vec{p} \cdot \vec{E})\right]_{x=0}$$

Como $\vec{p}$ es un número y no una función de $\vec{x}$ diferenciable (Ec 4.24)

$$\Rightarrow \vec{F} = q\vec{E}^{(0)}(0) + \left[\vec{\nabla}(\vec{p} \cdot \vec{E}^{(0)}(x))\right]_{x=0} + \cdots$$

$\nabla \times \vec{E} = 0$

para el 3er término usamos también $\frac{\partial E_i}{\partial y_j} = \frac{\partial E_j}{\partial y_i}$

$$\frac{1}{2}\sum_{j,k}\left(\frac{\partial^2 E_i^{(0)}(y)}{\partial y_i\,\partial y_k}\right)_{y=0}\int x_j x_k \rho(\vec{r})\,dv = \frac{\partial}{\partial y_i}\frac{1}{2}\sum_{j,k}\left(\frac{\partial E_j^{(0)}(y)}{\partial y_k}\right)_{y=0}\int x_j x_k \rho\,dv$$

$$= \frac{\partial}{\partial y_i}\frac{1}{2}\int\left\{\sum_{j,k}\left(\frac{\partial E_j^{(0)}(y)}{\partial y_k}\right)_{y=0} x_j x_k - \frac{1}{3}r^2 \underbrace{\sum_{j,k}\left(\frac{\partial E_j^{(0)}(y)}{\partial y_k}\right)_{y=0}\delta_{jk}}_{0 \text{ si } \vec{\nabla}\cdot\vec{E}^{(0)}=0}\right\}\rho\,dv$$

$$= \frac{\partial}{\partial y_i}\frac{1}{6}\sum_{j,k}\left(\frac{\partial E_j^{(0)}(y)}{\partial y_k}\right)_{y=0}\underbrace{\int\left(3x_i x_k - r^2\delta_{jk}\right)\rho\,dv}_{Q_{jk}}$$

juntando los términos con $i = 1, 2, 3$ en un vector obtenemos:

$$\nabla_y\left[\frac{1}{6}\sum_{j,k} Q_{jk}\left(\frac{\partial E_j^{(0)}(y)}{\partial y_k}\right)_{y=0}\right]$$

$$\Rightarrow \boxed{\vec{F} = q\vec{E}^{(0)} + \left[\vec{\nabla}(\vec{p}\cdot\vec{E}^{(0)}(x))\right]_{x=0} + \left\{\nabla\left[\frac{1}{6}\sum_{j,k} Q_{jk}\frac{\partial E_j^{(0)}(x)}{\partial x_k}\right]\right\}_{x=0} + \cdots}$$

Note que esto es $\vec{F} = -\vec{\nabla}W(x)$ si W es dado por (4.24):

$$W = q\Phi(x) - \vec{p}\cdot\vec{E}^{(0)}(x) - \frac{1}{6}\sum_{j,k} Q_{jk}\frac{\partial E_j^{(0)}(x)}{\partial x_k} + \cdots$$

b) $\vec{N} = \int \vec{r}\times\left[\rho(\vec{r})\vec{E}^{(0)}(\vec{r})\right]dv$

4.6 $W = -\frac{1}{6}\sum_{i,j=1}^{3} Q_{ij}\frac{\partial E_j(0)}{\partial x_i}$ → interacción eléctrica de cuadrupolo

a) $\vec{\nabla}\cdot\vec{E}(0) = \cancelto{0}{\rho(0)}/\epsilon_0 \longrightarrow \frac{\partial E_x(0)}{\partial x} + \frac{\partial E_y(0)}{\partial y} + \frac{\partial E_z(0)}{\partial z} = 0$

Por la simetría cilíndrica del campo $\frac{\partial E_x}{\partial x} = \frac{\partial E_y}{\partial y}$

$\Rightarrow 2\frac{\partial E_x(0)}{\partial x} + \frac{\partial E_z(0)}{\partial z} = 0 \longrightarrow \frac{\partial E_x(0)}{\partial x} = \frac{\partial E_y(0)}{\partial y} = -\frac{1}{2}\frac{\partial E_z(0)}{\partial z}$

También por simetría cilíndrica $Q_{11} = Q_{22}$ y como

$Tr(Q) = Q_{11} + Q_{22} + Q_{33} = 0 = 2Q_{11} + Q_{33} \longrightarrow Q_{11} = Q_{22} = -\frac{1}{2}Q_{33}$

además $Q_{ij} = 0$ si $i \neq j$ según Jackson Pg 151.

$\Rightarrow W = -\frac{1}{6}Q_{33}\frac{\partial E}{\partial z}(0)\left[\frac{1}{4} + \frac{1}{4} + 1\right] = -\frac{1}{4}Q_{33}\frac{\partial E_z(0)}{\partial z}$

$Q = \frac{1}{e}Q_{33}$ → momento cuadrupolar del estado nuclear (página 151 Jackson)

$\Rightarrow \boxed{W = -\frac{1}{4}eQ\frac{\partial E_z(0)}{\partial z}}$

b) $\frac{\partial E_z(0)}{\partial z} = -\frac{4W}{eQ} = -\frac{4W}{eQ\left(\frac{e}{4\pi\epsilon_0 a_0^3}\right)}\left(\frac{e}{4\pi\epsilon_0 a_0^3}\right)$

$\frac{e^2}{4\pi\epsilon_0} = \alpha\hbar c = \frac{\alpha h c}{2\pi}$ con $\alpha = 1/137$

$\Rightarrow \frac{4W}{eQ\left(\frac{e}{4\pi\epsilon_0 a_0^3}\right)} = \frac{4(W/h)2\pi a_0^3}{Q\alpha c} = \frac{4\cdot 10^7 s^{-1}\, 2\pi\,(0.529\times10^{-10})^3 m^3}{2\times10^{-28} m^2 (1/137)\times 3\times10^8 m/s}$

$\Rightarrow \boxed{\frac{\partial E_z}{\partial z}(0) = -0.085\left(\frac{e}{4\pi\epsilon_0 a_0^3}\right)}$

c) La ec. de un esferoide simetrico respecto a z es

$\frac{x^2+y^2}{b^2} + \frac{z^2}{a^2} = 1$ ó $\frac{\rho^2}{b^2} + \frac{z^2}{a^2} = 1 \longrightarrow z = \pm\sqrt{1 - \frac{\rho^2}{b^2}}\,a$

$$\Rightarrow \quad V = \int_0^{2\pi}\int_0^b \int_{-a\sqrt{1-\rho^2/b^2}}^{a\sqrt{1-\rho^2/b^2}} \rho\, dz\, d\rho\, d\phi =$$

$$= 2\pi \int_0^b 2a\sqrt{1-\frac{\rho^2}{b^2}}\,\rho\, d\rho \qquad u = 1-\frac{\rho^2}{b^2} \qquad du = -\frac{2\rho\, d\rho}{b^2}$$

$$= 4\pi a\left[\frac{\cancel{2}}{3}\left(1-\frac{\rho^2}{b^2}\right)^{3/2}\left(-\frac{b^2}{\cancel{2}}\right)\right]_0^b = \frac{4\pi}{3}ab^2$$

$$\Rightarrow \quad \rho_c = eZ/V = 3Ze/(4\pi a b^2)$$

$$Q_{ij} = \int \rho(\vec{r}\,')\left[3x_i' x_j' - \vec{r}\,'^2\delta_{ij}\right] dv' \qquad r'^2 = \rho^2 + z^2$$

$$Q_{33} = \rho_c \int_0^{2\pi}\int_0^b \int_{-a\sqrt{1-\rho^2/b^2}}^{a\sqrt{1-\rho^2/b^2}} \underbrace{\left[3z^2 - (\rho^2+z^2)\right]}_{2z^2-\rho^2} dz(\rho\, d\rho)\, d\phi$$

$$= \rho_c(2\pi)\int_0^b \left[\frac{2z^3}{3} - \rho^2 z\right]_{-a\sqrt{1-\rho^2/b^2}}^{a\sqrt{1-\rho^2/b^2}} \rho\, d\rho$$

$$= \rho_c(2\pi)\int_0^b 2\left[\frac{2}{3}a^3\left(1-\frac{\rho^2}{b^2}\right)^{3/2} - \rho^2 a\sqrt{1-\frac{\rho^2}{b^2}}\right]\rho\, d\rho$$

$$= \rho_c(2\pi)\,\frac{4}{15}ab^2(a^2-b^2) = \frac{3Ze}{\cancel{4\pi}\,\cancel{ab^2}}(2\pi)\frac{\cancel{4}}{15}\cancel{ab^2}(a^2-b^2)$$

$$Q_{33} = \frac{2}{5}Ze(a^2-b^2)$$

$$Q = \frac{1}{e}Q_3 = \frac{2}{5}Z(a^2-b^2) = \frac{2}{5}Z(a-b)\underbrace{\frac{(a+b)}{2}}_{R = 7\times10^{-15}\,m}2 = \frac{4}{5}ZR(a-b)$$

$$\Rightarrow \quad \frac{a-b}{R} = \frac{5Q}{4ZR^2} = \frac{5\cdot 2.5\times10^{-28}\,m^2}{4\cdot 63\cdot(7\times10^{-15})^2\,m^2} \longrightarrow \boxed{\frac{a-b}{R} = 0.101}$$

4.7 $$q_{lm} = \int \rho(\vec{r}\,')\,Y_{lm}^*(\theta',\phi')\,r'^l\, dv' \qquad \rho(\vec{r}\,') = \frac{1}{64\pi}r^2e^{-r}\sin^2\theta$$

$$= \frac{1}{64\pi}\int\left(r'^2e^{-r'}\sin^2\theta'\right)\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\,P_l^m(\cos\theta')\,e^{-im\phi'}\,r'^2dr' \times d(\cos\theta')\,d\phi'$$

$$\int_0^{2\pi} e^{-im\phi'}d\phi' = \int_0^{2\pi} e^{i(0-m)\phi'}d\phi' = 2\pi\delta_{m0} \longrightarrow q_{lm} = 0 \text{ si } m\neq 0$$

$$q_{l0} = \frac{1}{64\cancel{\pi}}(2\cancel{\pi})\int r'^{l+4}e^{-r'}\sin^2\theta'\sqrt{\frac{2l+1}{4\pi}}\,P_l(\cos\theta')\,dr'\,d(\cos\theta')$$

$$q_{l0} = \frac{1}{32}\sqrt{\frac{2l+1}{4\pi}}\int_0^\infty r'^{\,l+4} e^{-r'} dr' \underbrace{\int (1-\cos^2\theta')\, P_l(\cos\theta')\, d(\cos\theta')}_{\int_{-1}^{1}(1-x^2)P_l(x)\,dx}$$

$$\int_{-1}^{1} \overset{P_0(x)}{(1-x^2)}\, P_l(x)\, dx = \overbrace{\frac{2}{2l+1}}^{2}\delta_{0l} - \int_{-1}^{1} x^2 P_l(x)\, dx$$

$$P_2(x) = \frac{1}{2}(3x^2-1) \rightarrow \quad x^2 = \frac{2P_2(x)+1}{3} = \frac{2P_2(x)+P_0(x)}{3}$$

$$\Rightarrow \int_{-1}^{1} x^2 P_l(x)\, dx = \frac{2}{3}\int_{-1}^{1} P_2(x) P_l(x)\, dx + \frac{1}{3}\int_{-1}^{1} P_0(x) P_l(x)\, dx$$

$$= \frac{2}{3}\cdot\frac{2}{2l+1}\delta_{2l} + \frac{1}{3}\,\frac{2}{2l+1}\delta_{0l}$$

$$= \frac{2}{3}\cdot\frac{2}{5}\delta_{2l} + \frac{1}{3}\cdot\frac{2}{1}\delta_{0l} = \frac{4}{15}\delta_{2l} + \frac{2}{3}\delta_{0l}$$

$$\Rightarrow \int_{-1}^{1}(1-x^2)P_l(x)\,dx = 2\delta_{0l} - \left(\frac{4}{15}\delta_{2l} + \frac{2}{3}\delta_{0l}\right) = \frac{4}{3}\delta_{0l} - \frac{4}{15}\delta_{2l}$$

$$\Rightarrow q_{l0} = \frac{1}{32}\sqrt{\frac{2l+1}{4\pi}}\int_0^\infty r'^{\,l+4} e^{-r'} dr' \left(\frac{4}{3}\delta_{l0} - \frac{4}{15}\delta_{l2}\right)$$

$$q_{00} = \frac{1}{32}\sqrt{\frac{1}{4\pi}}\int_0^\infty r'^{\,4} e^{-r'} dr' \cdot \left(\frac{4}{3}\delta_{00} - \cancel{\frac{4}{15}\delta_{00}}^{\,0}\right)$$

$$q_{20} = \frac{1}{32}\sqrt{\frac{5}{4\pi}}\int_0^\infty r'^{\,6} e^{-r'} dr' \left(\cancel{\frac{4}{3}\delta_{20}}^{\,0} - \frac{4}{15}\delta_{20}\right)$$

$$\boxed{q_{l0} = 0 \quad \text{si} \quad l \neq 0, 2}$$

$$\boxed{\int_0^\infty r^n e^{-r}\, dr = n!} \Rightarrow \quad q_{00} = \cancel{\frac{1}{32}}\sqrt{\frac{1}{4\pi}}\cdot 4!\,\cancel{\frac{4}{3}} \qquad \boxed{q_{00} = \frac{1}{\sqrt{4\pi}}}$$

$$q_{20} = \frac{1}{32}\sqrt{\frac{5}{4\pi}}\cdot 6!\left(-\frac{4}{15}\right) \qquad \boxed{q_{20} = -6\sqrt{\frac{5}{4\pi}}}$$

$$\Rightarrow \Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1} q_{lm}\, r^{-(l+1)}\, Y_l^m(\theta,\phi)$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{4\pi}{1} q_{00}\, r^{-1}\, Y_0^0(\theta,\phi) + \frac{4\pi}{5} q_{20}\, r^{-3}\, Y_2^0(\theta,\phi)\right]$$

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\cancel{4\pi}\sqrt{\frac{1}{\cancel{4\pi}}}\,\frac{1}{r}\sqrt{\frac{1}{\cancel{4\pi}}} - \frac{\cancel{4\pi}}{5}\cdot 6\sqrt{\frac{5}{\cancel{4\pi}}}\,\frac{1}{r^3}\sqrt{\frac{5}{\cancel{4\pi}}}\,P_2(\cos\theta)\right]$$

$\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$

$$\boxed{\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0 r}\left[1 - \frac{6}{r^2}P_2(\cos\theta)\right]} \longrightarrow$$ Potencial lejos de la distribución (Exp multipolar)

b) $$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,dv'$$

$$\rho(\vec{r}) = \frac{1}{64\pi}\,r^2 e^{-r}\sin^2\theta$$

$$= \frac{1}{\cancel{4\pi}\epsilon_0}\int \rho(\vec{r}\,')\left(\cancel{4\pi}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y^*_{lm}(\theta',\phi')\,Y_{lm}(\theta,\phi)}{2l+1}\,\frac{r_<^{\,l}}{r_>^{\,l+1}}\right)dv'$$

$$= \frac{1}{64\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y_{lm}(\theta,\phi)}{2l+1}\int_0^{\infty}\frac{r_<^{\,l}}{r_>^{\,l+1}}\,r'^2 e^{-r'}(r'^2dr')$$

$$\times \underbrace{\int_0^{2\pi}\int_0^{\pi} Y^*_{lm}(\theta',\phi')\sin^2\theta'\,(\sin\theta' d\theta')\,d\phi'}_{A}$$

$$A = \int_0^{2\pi}\int_0^{\pi}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\,P_l^m(\cos\theta')\,e^{-im\phi'}\sin^2\theta'\,d(\cos\theta')\,d\phi'$$

$$\int_0^{2\pi} e^{-im\phi'}d\phi' = \int_0^{2\pi} e^{i(0-m)\phi'}d\phi' = 2\pi\,\delta_{m0}$$

$$\Rightarrow A = \int_0^{\pi}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\,P_l^m(\cos\theta')\sin^2\theta'\,d(\cos\theta')\cdot 2\pi\,\delta_{m0}$$

$A = 0$ si $m \neq 0$ y $A = 2\pi\int_0^{\pi}\sqrt{\frac{2l+1}{4\pi}}\,P_l(\cos\theta')\sin^2\theta'\,d(\cos\theta')$ si $m=0$

$$A = 2\pi\sqrt{\frac{2l+1}{4\pi}}\underbrace{\int_{-1}^{1}P_l(\cos\theta')(1-x^2)\,dx}$$ con $x = \cos\theta$

Esto se calculó atras

$$A = \begin{cases} 2\pi\sqrt{\frac{2l+1}{4\pi}}\left(\frac{4}{3}\delta_{0l} - \frac{4}{15}\delta_{2l}\right) & \text{si } m=0 \\ 0 & \text{en otro caso. } (m\neq 0)\end{cases}$$

$$\Rightarrow \Phi(\vec{r}) = \frac{1}{64\cancel{\pi}\epsilon_0}\sum_{l=0}^{\infty}\frac{Y_{l0}(\theta,\phi)}{2l+1}\int_0^{\infty}\frac{r_<^{\,l}}{r_>^{\,l+1}}\,r'^4 e^{-r'}dr'\cdot 2\cancel{\pi}\sqrt{\frac{2l+1}{4\pi}}\left(\frac{4}{3}\delta_{0l} - \frac{4}{15}\delta_{2l}\right)$$

$$= \frac{1}{32\epsilon_0}\sum_{l=0}^{\infty}\frac{\sqrt{\frac{\cancel{2l+1}}{4\pi}}\,P_l(\cos\theta)}{\cancel{2l+1}}\int_0^{\infty}\frac{r_<^{\,l}}{r_>^{\,l+1}}\,r'^4 e^{-r'}dr'\sqrt{\frac{\cancel{2l+1}}{4\pi}}\left(\frac{4}{3}\delta_{0l} - \frac{4}{15}\delta_{2l}\right)$$

$$\Phi(\vec{r}) = \frac{1}{128\pi\epsilon_0}\sum_{l=0}^{\infty} P_l(\cos\theta)\int_0^{\infty}\frac{r_<^{\,l}}{r_>^{\,l+1}}\, r'^4 e^{-r'}dr'\left(\frac{4}{3}\delta_{0l} - \frac{4}{15}\delta_{2l}\right)$$

$$= \frac{1}{128\pi\epsilon_0}\left\{ P_0(\cos\theta)\underbrace{\int_0^{\infty}\frac{1}{r_>}\, r'^4 e^{-r'}dr'}_{I_0}\cdot\frac{4}{3} + P_2(\cos\theta)\underbrace{\int_0^{\infty}\frac{r_<^{\,2}}{r_>^{\,3}}\, r'^4 e^{-r'}dr'}_{I_2}\left(-\frac{4}{15}\right)\right\}$$

$0 < r < \infty$, de modo que:

$$I_0 = \underbrace{\int_0^{r}\frac{1}{r}\, r'^4 e^{-r'}dr'}_{①} + \underbrace{\int_r^{\infty}\frac{1}{r'}\, r'^4 e^{-r'}dr'}_{②}$$

$$I_2 = \underbrace{\int_0^{r}\frac{r'^2}{r^3}\, r'^4 e^{-r'}dr'}_{③} + \underbrace{\int_r^{\infty}\frac{r^2}{r'^3}\, r'^4 e^{-r'}dr'}_{④}$$

Para hacer las anteriores integrales úsese:

$$\boxed{\int x^n e^{ax}dx = \frac{e^{ax}}{a}\left(x^n - \frac{n\,x^{n-1}}{a} + \frac{n(n-1)x^{n-2}}{a^2} - \cdots + \frac{(-1)^n n!}{a^n}\right)}$$

siempre que n sea un entero positivo.

$$① = \frac{1}{r}\frac{e^{-r'}}{(-1)}\left(r'^4 - \frac{4r'^3}{(-1)} + \frac{4(3)r'^2}{(-1)^2} - \frac{4(3)(2)r'}{(-1)^3} + \frac{4!}{(-1)^4}\right)\Bigg|_0^{r}$$

$$② = \frac{e^{-r}}{(-1)}\left(r'^3 - \frac{3r'^2}{(-1)} + \frac{3(2)r'}{(-1)^2} - \frac{3!}{(-1)^3}\right)\Bigg|_r^{\infty}$$

$$① = -\frac{e^{-r}}{r}\left(r^4 + 4r^3 + 12r^2 + 24r + 24\right) + \frac{24}{r}$$

$$② = e^{-r}\left(r^3 + 3r^2 + 6r + 6\right)$$

$$I_0 = ① + ② = -\frac{e^{-r}}{r}\left(0 + r^3 + 6r^2 + 18r + 24\right) + \frac{24}{r} \longrightarrow I_0$$

$$③ = \frac{e^{-r'}}{(-1)}\left(r'^6 - \frac{6r^5}{(-1)} + \frac{6(5)r'^4}{(-1)^2} - \frac{6(5)(4)r'^3}{(-1)^3} + \frac{6(5)(4)(3)r'^2}{(-1)^4} - \frac{6(5)(4)(3)(2)r'}{(-1)^5} + \frac{6!}{(-1)^6}\right)\Bigg|_0^{r}$$

$$④ = r^2\frac{e^{-r'}}{(-1)}\left(r' - \frac{1!}{(-1)}\right)\Bigg|_r^{\infty}$$

$$③ = -\frac{e^{-r}}{r^3}\left(r^6+6r^5+30r^4+120r^3+360r^2+720r+720\right)+\frac{720}{r^3}$$

$$④ = r^2e^{-r}(r+1)$$

$$I_2 = ③+④ = -\frac{e^{-r}}{r^3}\left(0+5r^5+30r^4+120r^3+360r^2+720r+720\right)+\frac{720}{r^3}$$

$$I_2 = -\frac{5e^{-r}}{r^3}\left(r^5+6r^4+24r^3+72r^2+144r+144\right)+\frac{5(144)}{r^3}$$

$$\Rightarrow \Phi(\vec{r}) = \frac{1}{128\pi\epsilon_0}\left\{\overbrace{P_0(\cos\theta)}^{1}\, I_0\cdot\frac{4}{3} + P_2(\cos\theta)\, I_2\cdot\left(-\frac{4}{15}\right)\right\}$$

$$= \frac{1}{96\pi\epsilon_0}\left\{\underbrace{\frac{1}{r}\left(24-e^{-r}(24+18r+6r^2+r^3)\right)}_{(*)}\right.$$

$$\left. - \underbrace{P_2(\cos\theta)\,\frac{1}{r^3}\left(144-e^{-r}(144+144r+72r^2+24r^3+6r^4+r^5)\right)}_{(**)}\right\}$$

- Si $r\to\infty$, $e^{-r}\to 0$ $\Rightarrow$ $\boxed{\Phi(\vec{r})\to\frac{1}{4\pi\epsilon_0 r}\left\{1-\frac{6}{r^2}P_2(\cos\theta)\right\}}$

reproduce los resultados de a)

- Si $r\to 0$ usemos la expansion de Taylor alrededor de cero:

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots$$

Usando esto en (*) y (**)

$$(*) = \frac{1}{r}\left[24-\left(1-r+\frac{r^2}{2}-\cdots\right)(24+18r+6r^2+r^3)\right]$$

$$= \frac{1}{r}\left[\cancel{24}-\left(\cancel{24}-24r+\frac{24}{2}r^2-\cdots\right)-\left(18r-18r^2+\frac{18}{2}r^3-\cdots\right)+\cdots\right]$$

$$= \frac{1}{r}\left[6r+\left(-\frac{24}{2}+18+\cdots\right)r^2+\left(-18+\cdots\right)r^3+\cdots\right]$$

$$= 6+d_1$$

$$(**) = \frac{1}{r^3}\left[144-\left(1-r+\frac{r^2}{2}-\frac{r^3}{3!}+\cdots\right)(144+144r+72r^2+24r^3+6r^4+r^5)\right]$$

$$(**) = \frac{1}{r^3}\Big[\cancel{144} - \Big(\cancel{144} - \cancel{144r} + \frac{144}{2}r^2 - \frac{144}{3!}r^3 + \frac{144}{4!}r^4 - \frac{144}{5!}r^5 + \dots\Big)$$

$$-\Big(\cancel{144r} - \cancel{144r^2} + \frac{144}{2}r^3 - \frac{144}{3!}r^4 + \frac{144}{4!}r^5 - \frac{144}{5!}r^6\Big)$$

$$-\Big(\cancel{72r^2} - \cancel{72r^3} + \frac{72}{2}r^4 - \frac{72}{3!}r^5 + \dots\Big)$$

$$-\Big(\cancel{24r^3} - \cancel{24r^4} + \frac{24}{2}r^5 - \frac{24}{3!}r^6 + \dots\Big)$$

$$-\Big(\cancel{6r^4} - \cancel{6r^5} + \frac{6}{2}r^6 - \frac{6}{3!}r^7 + \dots\Big)$$

$$-\Big(\cancel{r^5} - \cancel{r^6} + \frac{r^7}{2} - \frac{r^8}{3!} + \dots\Big)\Big]$$

$$= \frac{1}{r^3}\left[\left(\frac{144}{5!} - 1\right)r^5 + \dots\right] = \frac{24}{120}r^2 + \mathcal{O}_2$$

$$\Rightarrow \Phi(\vec{r}) = \frac{1}{96\pi\epsilon_0}\left\{6 + \mathcal{O}_1 + \left(\frac{24}{120}r^2 + \mathcal{O}_2\right)P_2(\cos\theta)\right\}$$

$$\Phi(\vec{r}) \approx \frac{1}{96\pi\epsilon_0}\left[6 + \frac{24}{120}r^2 P_2(\cos\theta)\right] \qquad \text{si } r \to 0$$

$$\boxed{\Phi(\vec{r}) \approx \frac{1}{4\pi\epsilon_0}\left[\frac{1}{4} - \frac{r^2}{120}P_2(\cos\theta)\right]} \qquad \text{si } r \to 0.$$

c) La energía de interacción cuadrupolar es dada por el tercer término de la ec. (4.24) de Jackson:

$$W = -\frac{1}{6}\sum_{ij} Q_{ij}\frac{\partial E_j}{\partial x_i}\Big|_{x=0}$$

donde $\vec{E}$ es el campo "externo" producido por la distribución de carga electrónica $\rho(\vec{r})$. Por la simetría cilíndrica de la distribución nuclear (Jackson Pg 151) $\frac{\partial E_x}{\partial x} = \frac{\partial E_y}{\partial y}$ y como $\nabla\cdot\vec{E} = 0$

$$\Rightarrow 2\frac{\partial E_x(0)}{\partial x} + \frac{\partial E_z}{\partial z} = 0 \longrightarrow \frac{\partial E_x}{\partial x}(0) = \frac{\partial E_y}{\partial y}(0) = -\frac{1}{2}\frac{\partial E_z}{\partial z}(0)$$

además por simetría cilíndrica $Q_{11} = Q_{22}$ y como

$$Tr(Q) = Q_{11} + Q_{22} + Q_{33} = 0 = 2Q_{11} + Q_{33} \longrightarrow Q_{11} = Q_{22} = -\frac{1}{2}Q_{33}$$

también $Q_{ij} = 0$ si $i \neq j$ segun jackson Pg. 151

$$\Rightarrow W = -\frac{1}{6} Q_{33} \left[\frac{1}{4} + \frac{1}{4} + 1 \right] \frac{\partial E}{\partial z}(0) = -\frac{1}{4} Q_{33} \frac{\partial E_z}{\partial z}(0)$$

$Q = \frac{1}{e} Q_{33} \longrightarrow$ momento cuadrupolar del estado nuclear (pg 151 Jackson)

$$\Rightarrow W = -\frac{1}{4} e Q \frac{\partial E_z}{\partial z}(0)$$ $\longrightarrow$ Esto es la parte a) del problema 4.6.

donde $E_z = \frac{\partial}{\partial z} \Phi(\vec{r}) \Big|_{r=0}$, con $\Phi(\vec{r})$ el potencial producido por la distribución electrónica. En las cercanías del núcleo ese potencial es dado en la parte b) de este problema:

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left[\frac{1}{4} - \frac{r^2}{120} P_2(\cos\theta) \right] \qquad P_2 = \frac{1}{2}(3\cos^2\theta - 1)$$

$$= \frac{1}{4\pi\epsilon_0} \left[\frac{1}{4} - \frac{1}{240} (3\underbrace{r^2\cos^2\theta}_{z^2} - r^2) \right]$$

$$= \frac{1}{4\pi\epsilon_0} \left[\frac{1}{4} - \frac{1}{240} (3z^2 - (x^2+y^2-z^2)) \right]$$

$$= \frac{1}{4\pi\epsilon_0} \left[\frac{1}{4} - \frac{1}{240} (2z^2 - x^2 - y^2) \right]$$

$$\frac{\partial E_z}{\partial z}(0) = \frac{\partial}{\partial z}\left(-\frac{\partial}{\partial z} \Phi(\vec{r}) \right) = \frac{1}{240\pi\epsilon_0} \Rightarrow |W| = \frac{1}{240} \frac{eQ}{4\pi\epsilon_0}$$

para que $|W|$ sea dimensionalmente correcto, debemos multiplicar por e y dividir por a_0^3 lo cual es legal ya que el enunciado del problema dice que e es la unidad de carga y a_0 la unidad de longitud

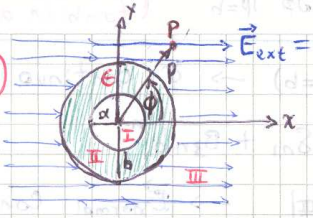
$$\Rightarrow \boxed{|W| = \frac{1}{240} \frac{e^2}{4\pi\epsilon_0 a_0} \frac{Q}{a_0^2}} \longrightarrow \frac{|W|}{\hbar} = \frac{1}{240} \frac{e^2}{4\pi\epsilon_0 \hbar} \frac{Q}{a_0^3}$$

$$\frac{e^2}{4\pi\epsilon_0} = \alpha \hbar c = \frac{\alpha h c}{2\pi} \quad \text{con} \quad \alpha = \frac{1}{137}$$

$$\Rightarrow \boxed{\frac{|W|}{h} = \frac{1}{240} \frac{\alpha c Q}{2\pi a_0^3}} = \frac{1}{240} \frac{1}{137} \frac{(3\times10^8\, m.s^{-1})(10^{-28}\, m^2)}{2\pi(0.529\times10^{-10}\, m)^3}$$

$$\boxed{\frac{|W|}{h} = 0.982\ MHz}$$

4.8



si $\rho \to \infty \Rightarrow \Phi = -E_0 x$

para que $\vec{E} = -\nabla\Phi = E_0\hat{\imath}$ en $\rho\to\infty$

$$\Rightarrow \Phi = -E_0 \overbrace{\rho\cos\phi}^{x} \quad \text{si } \rho\to\infty$$

$$\Phi(\rho,\phi) = \cancel{a_0} + \cancel{b_0 \ln\rho} + \sum_{n=1}^{\infty}\left[a_n\rho^n \sin(n\phi+\alpha_n) + b_n\rho^{-n}\sin(n\phi+\beta_n)\right]$$

$\Phi \to 0$ en $\rho\to\infty$ $\quad$ evita singularidades en el origen.

$$\Phi(\rho,\phi) = \sum_{n=1}^{\infty}\Big\{ a_n\rho^n\left[\overbrace{\cos\alpha_n}^{cte}\sin n\phi + \overbrace{\sin\alpha_n}^{cte}\cos n\phi\right] + b_n\rho^{-n}\left[\underbrace{\cos\beta_n}_{cte}\sin n\phi + \underbrace{\sin\beta_n}_{cte}\cos n\phi\right]\Big\}$$

Note que $\Phi(\rho,\phi) = \Phi(\rho,-\phi)$ por lo que Φ no debe tener dependencia de $\sin n\phi$, solo de funciones cosenos que son pares

$$\Rightarrow \Phi(\rho,\phi) = \sum_{n=1}^{\infty}\left(A_n\rho^n + B_n\rho^{-n}\right)\cos n\phi \quad \text{con esto:}$$

Φ finito en $\rho\to 0$

$$\Phi_I = \sum_{n=1}^{\infty}\left(A_{1n}\rho^n + \cancel{B_{1n}\rho^{-n}}\right)\cos n\phi \qquad \text{si } \rho<a$$

$$\Phi_{II} = \sum_{n=1}^{\infty}\left(A_{2n}\rho^n + B_{2n}\rho^{-n}\right)\cos n\phi \qquad \text{si } a<\rho<b$$

$$\Phi_{III} = \sum_{n=1}^{\infty}\left(A_{3n}\rho^n + B_{3n}\rho^{-n}\right)\cos n\phi \qquad \text{si } \rho>b$$

Si $\rho\to 0$, Φ debe ser finito $\Rightarrow B_{1n} = 0$

Si $\rho\to\infty$, $\Phi = -E_0\rho\cos\phi = \sum_{n=1}^{\infty}\left(A_{3n}\rho^n + \cancel{B_{3n}\rho^{-n}}\right)\cos n\phi$ $\quad \rho\to\infty$

$$\Rightarrow A_{31} = -E_0, \quad A_{3n} = 0 \quad \text{si } n\neq 1$$

Si $\rho = a$, $\Phi_I(\rho=a) = \Phi_{II}(\rho=a)$ $\quad$ Continuidad del potencial

$$\Rightarrow A_{1n}a^n = A_{2n}a^n + B_{2n}a^{-n} \qquad (1)$$

y $\quad \epsilon_0\dfrac{\partial\Phi_I}{\partial\rho}\Big|_{\rho=a} = \epsilon\dfrac{\partial\Phi_{II}}{\partial\rho}\Big|_{\rho=a}$ $\quad$ Componentes normales de $\vec{D}$ continuas

$$\Rightarrow \epsilon_0 A_{1n}\cancel{n}a^{n-1} = \epsilon\left(A_{2n}\cancel{n}a^{n-1} - B_{2n}\cancel{n}a^{-n-1}\right) \qquad (2)$$

también $\quad \dfrac{1}{\rho}\dfrac{\partial\Phi_I}{\partial\phi}\Big|_{\rho=a} = \dfrac{1}{\rho}\dfrac{\partial\Phi_{II}}{\partial\phi}\Big|_{\rho=a}$ $\quad$ Componentes tangenciales de $\vec{E}$ continuas (Esto conduce de nuevo a (1)

también $\frac{1}{\rho}\frac{\partial \Phi_{II}}{\partial \phi}\Big|_{\rho=b} = \frac{1}{\rho}\frac{\partial \Phi_{III}}{\partial \phi}\Big|_{\rho=b}$ $\rightarrow$ $\vec{E}$ normal continua (pero ello conduce también a ③)

Si $\rho = b$,

$$\Phi_{II}(\rho=b) = \Phi_{III}(\rho=b) \rightarrow \Phi \text{ continuo}$$

$$\Rightarrow A_{2n} b^n + B_{2n} b^{-n} = -E_0 b \delta_{n1} + B_{3n} b^{-n} \quad ③$$

$$\epsilon \frac{\partial \Phi_{II}}{\partial \rho}\Big|_{\rho=b} = \epsilon_0 \frac{\partial \Phi_{III}}{\partial \rho}\Big|_{\rho=b} \rightarrow \vec{D}_{normal} \text{ continuo.}$$

$$\Rightarrow \epsilon\left(A_{2n}\cancel{n} b^{n-1} - \frac{\cancel{n} B_{2n}}{b^{n+1}}\right) = \epsilon_0\left(-E_0 \delta_{n1} - \frac{n B_{3n}}{b^{n+1}}\right) \quad ④$$

$$= \epsilon_0\left(-E_0 \cancel{n} \delta_{n1} - \frac{\cancel{n} B_{3n}}{b^{n+1}}\right)$$

Note que δ_{n1} contribuye solo cuando $n=1$ de modo que podemos incluir el factor n sin afectar la expresión

multiplicando ① $\times a^n$, ② $\times \frac{a^{n+1}}{\epsilon_0}$, ③ $\times b^n$ y ④ $\times \frac{b^{n+1}}{\epsilon_0}$ y reorganizando tenemos:

$$A_{1n} a^{2n} - A_{2n} a^{2n} - B_{2n} = 0$$
$$A_{1n} a^{2n} - \epsilon_r A_{2n} a^{2n} + \epsilon_r B_{2n} = 0$$
$$A_{2n} b^{2n} + B_{2n} - B_{3n} = -E_0 b^{n+1} \delta_{n1}$$
$$\epsilon_r A_{2n} b^{2n} - \epsilon_r B_{2n} + B_{3n} = -E_0 b^{n+1} \delta_{n1}$$

donde $\epsilon_r = \epsilon/\epsilon_0$. Tenemos el sistema:

$$\begin{pmatrix} a^{2n} & -a^{2n} & -1 & 0 \\ a^{2n} & -\epsilon_r a^{2n} & \epsilon_r & 0 \\ 0 & b^{2n} & 1 & -1 \\ 0 & \epsilon_r b^{2n} & -\epsilon_r & 1 \end{pmatrix} \begin{pmatrix} A_{1n} \\ A_{2n} \\ B_{2n} \\ B_{3n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -E_0 b^{n+1} \delta_{n1} \\ -E_0 b^{n+1} \delta_{n1} \end{pmatrix}$$

Si $n > 1$ el sistema es homogéneo, y tiene solución no trivial si el Determinante de la matriz de coeficientes es igual a cero.

$$\text{Det} = a^{2n}\begin{vmatrix} -\epsilon_r a^{2n} & \epsilon_r & 0 \\ b^{2n} & 1 & -1 \\ \epsilon_r b^{2n} & -\epsilon_r & 1 \end{vmatrix} + a^{2n}\begin{vmatrix} a^{2n} & \epsilon_r & 0 \\ 0 & 1 & -1 \\ 0 & -\epsilon_r & 1 \end{vmatrix} - \begin{vmatrix} a^{2n} & -\epsilon_r a^{2n} & 0 \\ 0 & b^{2n} & -1 \\ 0 & \epsilon_r b^{2n} & 1 \end{vmatrix}$$

$$= a^{2n}\left[-\epsilon_r a^{2n}(1-\epsilon_r) - \epsilon_r b^{2n}(1+\epsilon_r)\right] + a^{2n}\left[a^{2n}(1-\epsilon_r)\right] - \left[a^{2n} b^{2n}(1+\epsilon_r)\right]$$

$$= a^{2n}\left[a^{2n}(-\epsilon_r + \epsilon_r^2 + 1 - \epsilon_r) - b^{2n}(\epsilon_r + \epsilon_r^2 + 1 + \epsilon_r)\right]$$

$$\text{Det} = a^{2n}\left[a^{2n}(\epsilon_r - 1)^2 - b^{2n}(\epsilon_r + 1)^2\right] \neq 0 \rightarrow A_{1n} = A_{2n} = B_{2n} = B_{3n} = 0$$

si $n > 1$

$\text{Det} \neq 0 \rightarrow$ Solución trivial

Si $n=1$:

$$(A_{11} - A_{21})a^2 - B_{21} = 0 \quad (5)$$

$$(A_{11} - \epsilon_r A_{21})a^2 + \epsilon_r B_{21} = 0 \quad (6)$$

$$A_{21} b^2 + B_{21} - B_{31} = -E_0 b^2 \quad (7)$$

$$\epsilon_r A_{21} b^2 - \epsilon_r B_{21} + B_{31} = -E_0 b^2 \quad (8)$$

$$(7)+(8) = (\epsilon_r+1)A_{21} b^2 - (\epsilon_r-1)B_{21} = -2E_0 b^2 \quad (9)$$

$$(6)-(5) = -(\epsilon_r-1)A_{21} a^2 + (\epsilon_r+1)B_{21} = 0 \quad (10)$$

$$(\epsilon_r-1)(10) + (\epsilon_r+1)(9) = \left[-(\epsilon_r-1)^2 a^2 + (\epsilon_r+1)^2 b^2\right] A_{21} = -2E_0(\epsilon_r+1)b^2$$

$$\Rightarrow \boxed{A_{21} = \frac{2E_0(\epsilon_r+1)b^2}{(\epsilon_r-1)^2 a^2 - (\epsilon_r+1)^2 b^2}}$$

de (9):

$$B_{21} = \frac{-2E_0 b^2 - (\epsilon_r+1)b^2 A_{21}}{-(\epsilon_r-1)}$$

$$= \frac{2E_0 b^2 + (\epsilon_r+1)b^2\left[\dfrac{2E_0(\epsilon_r+1)b^2}{(\epsilon_r-1)^2 a^2 - (\epsilon_r+1)^2 b^2}\right]}{(\epsilon_r-1)}$$

$$= \frac{2E_0 b^2\left[(\epsilon_r-1)^2 a^2 - \cancel{(\epsilon_r+1)^2 b^2}\right] + \cancel{2E_0(\epsilon_r+1)^2 b^4}}{\cancel{(\epsilon_r-1)}\left[(\epsilon_r-1)^2 a^2 - (\epsilon_r+1)^2 b^2\right]}$$

$$\boxed{B_{21} = \frac{2E_0 b^2 (\epsilon_r-1) a^2}{(\epsilon_r-1)^2 a^2 - (\epsilon_r+1)^2 b^2}}$$

de (5):

$$A_{11} = \frac{B_{21}}{a^2} + A_{21}$$

$$= \frac{2E_0 b^2 \cancel{(\epsilon_r-1)}}{(\epsilon_r-1)^2 a^2 - (\epsilon_r+1)^2 b^2} + \frac{2E_0 \cancel{(\epsilon_r+1)} b^2}{(\epsilon_r-1) a^2 - (\epsilon_r+1)^2 b^2}$$

$$\boxed{A_{11} = \frac{4E_0 \epsilon_r b^2}{(\epsilon_r-1)^2 a^2 - (\epsilon_r+1)^2 b^2}}$$

de (7):

$$B_{31} = A_{21} b^2 + B_{21} + E_0 b^2$$

$$= \frac{2E_0(\epsilon_r+1)b^4}{(\epsilon_r-1)^2 a^2 - (\epsilon_r+1)^2 b^2} + \frac{2E_0 b^2 a^2 (\epsilon_r-1)}{(\epsilon_r-1)^2 a^2 - (\epsilon_r+1)^2 b^2} + E_0 b^2$$

$$= \frac{2E_0 b^2\left[(\epsilon_r+1)b^2 + (\epsilon_r-1)a^2\right]}{(\epsilon_r-1)^2 a^2 - (\epsilon_r+1)^2 b^2} + \frac{E_0 b^2\left[(\epsilon_r-1)^2 a^2 - (\epsilon_r+1)^2 b^2\right]}{(\epsilon_r-1)^2 a^2 - (\epsilon_r+1)^2 b^2}$$

$$B_{31} = \frac{E_0 b^2 \left\{ 2(\epsilon_r+1)b^2 + 2(\epsilon_r-1)a^2 + (\epsilon_r-1)^2 a^2 - (\epsilon_r+1)^2 b^2 \right\}}{(\epsilon_r-1)^2 a^2 - (\epsilon_r+1)^2 b^2}$$

$$= \frac{E_0 b^2 \left\{ 2(\cancel{\epsilon_r}+1)b^2 + 2(\cancel{\epsilon_r}-1)a^2 + (\epsilon_r^2 - \cancel{2\epsilon_r}+1)a^2 - (\epsilon_r^2 + \cancel{2\epsilon_r}+1)b^2 \right\}}{(\epsilon_r-1)a^2 - (\epsilon_r-1)b^2}$$

$$\boxed{B_{31} = \frac{E_0 b^2 (\epsilon_r^2-1)(a^2-b^2)}{(\epsilon_r-1)^2 a^2 - (\epsilon_r+1)^2 b^2}}$$

$$\Rightarrow \Phi(\rho,\phi) = \underbrace{\frac{E_0}{(\epsilon_r^2+1)^2 b^2 - (\epsilon_r^2-1)^2 a^2}}_{c} \times \begin{cases} (-4\epsilon_r b^2)\,\rho\cos\phi & \text{si } r<a \quad \rho<a \\ (-2b^2)\left((\epsilon_r+1)\rho + (\epsilon_r-1)\frac{a^2}{\rho}\right)\cos\phi & a<\rho<b \\ (b^2-a^2)(\epsilon_r^2-1)\frac{b^2}{\rho}\cos\phi - E_0 r\cos\phi & \rho>b \end{cases}$$

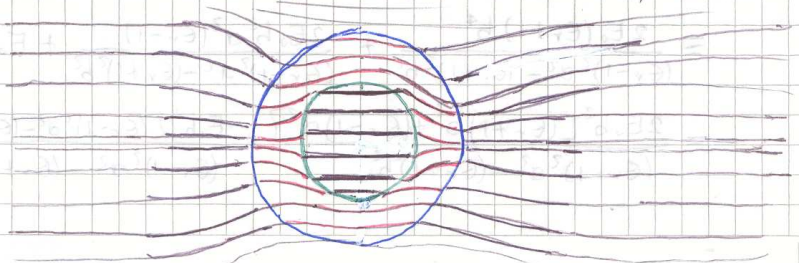
$$\boxed{\Phi(\rho,\phi) = \begin{cases} c(-4\epsilon_r b^2)\,\rho\cos\phi & \rho<a \\ c(-2b^2)\left((\epsilon_r+1)\rho + (\epsilon_r-1)\frac{1}{\rho}\right)\cos\phi & a<\rho<b \\ c(b^2-a^2)(\epsilon_r^2-1)\frac{b^2}{\rho}\cos\phi - E_0\rho\cos\phi & \rho>b \end{cases}}$$

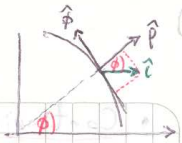
Note que si $\epsilon_r \to 1 \Rightarrow \Phi \to -E_0 r\cos\phi$ en todas las regiones.

b) $\vec{E} = -\nabla\Phi = -\left(\frac{\partial\Phi}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\hat{\phi}\right)$

$$\vec{E} = \begin{cases} -c(-4\epsilon_r b^2)\overbrace{\left[\cos\phi\,\hat{\rho} - \sin\phi\,\hat{\phi}\right]}^{\hat{c}} & \rho<a \\ -c(-2b^2)\left\{\left[(\epsilon_r+1) - (\epsilon_r-1)\frac{a^2}{\rho^2}\right]\cos\phi\,\hat{\rho} - \left[(\epsilon_r+1) + (\epsilon_r-1)\frac{a^2}{\rho^2}\right]\sin\phi\,\hat{\phi}\right\} & a<\rho<b \\ +c(b^2-a^2)(\epsilon_r^2-1)\frac{b^2}{\rho^2}\left[\cos\phi\,\hat{\rho} + \sin\phi\,\hat{\phi}\right] + E_0\underbrace{\left[\cos\phi\,\hat{\rho} - \sin\phi\,\hat{\phi}\right]}_{\hat{c}} & \rho>b \end{cases}$$

$\vec{F} = q\vec{E}$ de modo que es suficiente con graficar $\vec{E}$ para conocer las líneas de fuerza. si $b = 2a$, $\epsilon = 5\epsilon_0$





c) Si $a \to 0$, $\quad C \to \dfrac{E_0}{(\epsilon_r+1)^2 b^2}$

$$\Rightarrow \vec{E}(\vec{r}) = \begin{cases} -\dfrac{E_0}{(\epsilon_r+1)^2 \cancel{b^2}}(-2\cancel{b^2})\,\cancel{(\epsilon_r+1)}\,\overbrace{(\cos\phi\,\hat{\rho} - \sin\phi\,\hat{\phi})}^{\hat{i}} & \text{si } r<b \\ +\dfrac{E_0}{(\epsilon_r+1)^2 \cancel{b^2}}\,\cancel{b^2}(\epsilon_r^2-1)\dfrac{b^2}{\rho^2}\left[\cos\phi\,\hat{\rho} + \sin\phi\,\hat{\phi}\right] + E_0\,\hat{i} & r>b \end{cases}$$

$$\boxed{\vec{E}(\vec{r}) = \begin{cases} \dfrac{2E_0}{(\epsilon_r+1)}\,\hat{i} & \text{si } \rho<b \\ E_0\,\hat{i} + E_0\dfrac{(\epsilon_r^2-1)}{(\epsilon_r+1)^2}\dfrac{b^2}{\rho^2}\left[\cos\phi\,\hat{\rho} + \sin\phi\,\hat{\phi}\right] & \text{si } \rho>b \end{cases}}$$

$\hookrightarrow$ Campo para un cilindro dielectrico de radio b en un campo externo $\vec{E} = E_0\,\hat{i}$

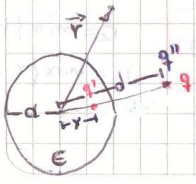
Si $b \to \infty$ $\qquad b >> a \Rightarrow C \to \dfrac{E_0}{(\epsilon_r+1)^2 b^2}$

$$\Rightarrow \vec{E}(\vec{r}) = \begin{cases} -\dfrac{E_0}{(\epsilon_r+1)\cancel{b^2}}(-4\epsilon_r\cancel{b^2})\,\hat{i} & \text{si } \rho<a \\ -\dfrac{E_0}{(\epsilon_r+1)^2\cancel{b^2}}(-2\cancel{b^2})\left\{(\epsilon_r+1)\underbrace{(\cos\phi\,\hat{\rho} - \sin\phi\,\hat{\phi})}_{\hat{i}} - (\epsilon_r-1)\dfrac{a^2}{\rho^2}\cdot(\cos\phi\,\hat{\rho} + \sin\phi\,\hat{\phi})\right\} & \rho>a \end{cases}$$

$$\Rightarrow \boxed{\vec{E}(\vec{r}) = \begin{cases} \dfrac{4\epsilon_r E_0}{(\epsilon_r+1)}\,\hat{i} & \text{si } \rho<a \\ \dfrac{2E_0}{(\epsilon_r+1)}\,\hat{i} - \dfrac{2E_0(\epsilon_r-1)}{(\epsilon_r+1)^2}\dfrac{a^2}{\rho^2}(\cos\phi\,\hat{\rho} + \sin\phi\,\hat{\phi}) & \text{si } \rho>a \end{cases}}$$

$\hookrightarrow$ Campo para un cilindro hueco de radio a (E_0 en su interior) sumergido en un dielectrico con campo externo en $\rho \to \infty$, $\vec{E} = \dfrac{2E_0}{\epsilon_r+1}\,\hat{i}$ (Note la similaridad entre $b \to \infty$ y $a \to 0$ si intercambiamos E_0 con $2E_0/(\epsilon_r+1)$ y a por b)

4.9 Por método de Imagenes:



$$\Phi_{ext} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|\vec{r}-d\hat{k}|} + \frac{q'}{|\vec{r}-r\hat{k}|}\right) \qquad \text{si } r>a$$

$$= \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\left[q\frac{r_<^l}{r_>^{l+1}} + q'\frac{r^l}{r^{l+1}}\right]P_l(\cos\theta) \qquad \text{si } r>a$$

$$\Phi_{int} = \frac{1}{4\pi\epsilon}\frac{q'''}{|\vec{r}-d\hat{k}|} = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty} q''\frac{r^l}{d^{l+1}}P_l(\cos\theta) \qquad \text{si } r<a$$

• Continuidad de $\vec{E}_{\parallel}$ $\quad \frac{1}{r}\frac{\partial \Phi_{ext}}{\partial \theta}\Big|_{r=a} = \frac{1}{r}\frac{\partial \Phi_{int}}{\partial \theta}\Big|_{r=a}$

$$\Rightarrow \quad \frac{q}{\epsilon_0}\frac{a^l}{d^{l+1}} + \frac{q'}{\epsilon_0}\frac{y^l}{a^{l+1}} = \frac{q''}{\epsilon}\frac{a^l}{d^{l+1}} \qquad (1)$$

• Continuidad de $\vec{D}_{normal}$ $\quad \epsilon_0 \frac{\partial \Phi_{ext}}{\partial r}\Big|_{r=a} = \epsilon \frac{\partial \Phi_{int}}{\partial r}\Big|_{r=a}$.

$$\Rightarrow \quad q\, l\, \frac{a^{l-1}}{d^{l+1}} - \frac{q'(l+1)y^l}{a^{l+2}} = q''\, l\, \frac{a^{l-1}}{d^{l+1}} \qquad (2)$$

Sea $y = a^2/d$ $\longrightarrow$ del método de imágenes para una esfera conductora.

$$\Rightarrow \quad (1) \rightarrow \frac{q}{\epsilon_0}\frac{a^l}{d^{l+1}} + \frac{q'}{\epsilon_0}\frac{a^{l-1}}{d^l} = \frac{q''}{\epsilon}\frac{a^l}{d^{l+1}} \qquad (2) \rightarrow q\, l\, \frac{a^{l-1}}{d^{l+1}} - \frac{q'(l+1)a^{l-2}}{d^l} = q''\, l\, \frac{a^{l-1}}{d^{l+1}}$$

$$\frac{a}{\epsilon l}(2) \longrightarrow \quad \frac{q}{\epsilon}\frac{a^l}{d^{l+1}} - \frac{q'(l+1)a^{l-1}}{l\epsilon d^l} = \frac{q''}{\epsilon}\frac{a^l}{d^{l+1}} \qquad (2')$$

sumando:

$$\left(\frac{1}{\epsilon_0} - \frac{1}{\epsilon}\right)\frac{\cancel{a^l}}{d^{l+1}}q + \left(\frac{1}{\epsilon_0} + \frac{(l+1)}{l\epsilon}\right)\frac{q'\cancel{a^{l-1}}}{\cancel{d^l}} = 0$$

$$\Rightarrow \quad -\frac{(\epsilon-\epsilon_0)}{\epsilon\epsilon_0}\frac{q}{d} = \frac{l\epsilon + (l+1)\epsilon_0}{l\epsilon\epsilon_0}\frac{q'}{a}$$

$$(1-\epsilon/\epsilon_0)\frac{q}{d} = \frac{l(1+\epsilon/\epsilon_0)+1}{l}\frac{q'}{a} \Rightarrow \boxed{q' = q\frac{a}{d}\frac{l(1-\epsilon/\epsilon_0)}{1+l(1+\epsilon/\epsilon_0)}}$$

Sustituyendo esto en (2'):

$$\frac{q''}{\epsilon}\frac{\cancel{a^l}}{\cancel{d^{l+1}}} = \frac{q}{\epsilon_0}\frac{\cancel{a^l}}{\cancel{d^{l+1}}} + \frac{q}{\epsilon_0}\frac{\cancel{a^l}}{\cancel{d^{l+1}}}\frac{l(1-\epsilon/\epsilon_0)}{1+l(1+\epsilon/\epsilon_0)} = \frac{1+l(1+\cancel{\epsilon/\epsilon_0})+l(1-\cancel{\epsilon/\epsilon_0})}{[1+l(1+\epsilon/\epsilon_0)]\epsilon_0}q$$

$$= \frac{1+2l}{[1+l(1+\epsilon/\epsilon_0)]\epsilon_0}q \longrightarrow \boxed{q'' = q\frac{\epsilon}{\epsilon_0}\frac{2l+1}{1+l(1+\epsilon/\epsilon_0)}}$$

$$\Rightarrow \Phi_{ext} = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\left[\frac{r_<^l}{r_>^{l+1}} + \frac{(a^2/d)^l}{r^{l+1}}\frac{a}{d}\frac{l(1-\epsilon/\epsilon_0)}{1+l(1+\epsilon/\epsilon_0)}\right]P_l(\cos\theta)$$

$$\boxed{\Phi_{ext} = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\left[\frac{r_<^l}{r_>^{l+1}} + \frac{a^{2l+1}}{(rd)^{l+1}}\frac{l(1-\epsilon/\epsilon_0)}{1+l(1+\epsilon/\epsilon_0)}\right]P_l(\cos\theta)}$$

$r_< = \min\{r, d\}$
$r_> = \max\{r, d\}$

$$\boxed{\Phi_{int} = \frac{q}{4\pi\cancel{\epsilon}}\sum_{l=0}^{\infty}\frac{\cancel{\epsilon}}{\epsilon_0}\frac{2l+1}{1+l(1+\epsilon/\epsilon_0)}\frac{r^l}{d^{l+1}}P_l(\cos\theta)}$$

Por expansión en $P_l(\cos\theta)$ (por simetría acimutal)

$$\Phi_{int} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \qquad \text{si } r<a.$$

$$\Phi_{ext} = \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta) + \frac{q}{4\pi\epsilon_0}\frac{1}{|\vec{r}-d\hat{k}|} \qquad \text{si } r>a.$$

$$= \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta) + \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta)$$

- Continuidad de $\vec{D}_{normal}$ $\quad \epsilon \left.\frac{\partial \Phi_{int}}{\partial r}\right|_{r=a} = \epsilon_0 \left.\frac{\partial \Phi_{ext}}{\partial r}\right|_{r=a}$

$$\Rightarrow \frac{\epsilon}{\epsilon_0} l A_l a^{l-1} = -(l+1) B_l a^{-(l+2)} + \frac{l q}{4\pi\epsilon_0}\frac{a^{l-1}}{d^{l+1}}$$

$$A_l = \frac{\epsilon_0}{\epsilon}\left[-\frac{(l+1)}{l} B_l a^{-(2l+1)} + \frac{q}{4\pi\epsilon_0 d^{l+1}}\right]$$

Aquí tambien sirve $\Phi_1(r=a) = \Phi_2(r=a)$

- Continuidad de $\vec{E}_{\parallel}$: $\longrightarrow \left.\frac{\partial \Phi_{int}}{\partial \theta}\right|_{r=a} = \left.\frac{\partial \Phi_{ext}}{\partial \theta}\right|_{r=a}$

$$\rightarrow A_l a^l = B_l a^{-(l+1)} + \frac{q}{4\pi\epsilon_0}\frac{a^l}{d^{l+1}} \rightarrow B_l = A_l a^{2l+1} - \frac{q}{4\pi\epsilon_0}\frac{a^{2l+1}}{d^{l+1}}$$

con ello $\quad A_l = \frac{\epsilon_0}{\epsilon}\left[-\frac{(l+1)}{l} A_l + \frac{(l+1)q}{4\pi\epsilon_0 d^{l+1}} + \frac{q}{4\pi\epsilon_0 d^{l+1}}\right]$

$$\left[\frac{\epsilon}{\epsilon_0} + \frac{(l+1)}{l}\right] A_l = \frac{(2l+1)q}{4\pi\epsilon_0 l d^{l+1}} \longrightarrow \boxed{A_l = \frac{2l+1}{\left[\frac{\epsilon}{\epsilon_0} + \frac{l+1}{l}\right]}\frac{q}{4\pi\epsilon_0 l d^{l+1}}}$$

$$\Rightarrow B_l = \frac{2l+1}{\left[\frac{\epsilon}{\epsilon_0} + (l+1)/l\right]}\frac{q a^{2l+1}}{4\pi\epsilon_0 l d^{l+1}} - \frac{q l}{4\pi\epsilon_0 l}\frac{a^{2l+1}}{d^{l+1}}$$

$$= \frac{(2l+1) - l[\epsilon/\epsilon_0 + (l+1)/l]}{[\epsilon/\epsilon_0 + (l+1)/l]\, 4\pi\epsilon_0 l}\frac{q a^{2l+1}}{d^{l+1}}$$

$$\boxed{B_l = \frac{1-\epsilon/\epsilon_0}{[\epsilon/\epsilon_0 + (l+1)/l]}\frac{q a^{2l+1}}{4\pi\epsilon_0 d^{l+1}}}$$

$r_< = \min\{r, d\}$

$r_> = \max\{r, d\}$

$$\Rightarrow \boxed{\Phi_{int} = \sum_{l=0}^{\infty}\frac{2l+1}{\epsilon/\epsilon_0 + (l+1)/l}\frac{q}{4\pi\epsilon_0 l}\frac{r^l}{d^{l+1}} P_l(\cos\theta)}$$

$$\boxed{\Phi_{ext} = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\left[\frac{1-\epsilon/\epsilon_0}{\epsilon/\epsilon_0 + (l+1)/l}\frac{a^{2l+1}}{(rd)^{l+1}} + \frac{r_<^l}{r_>^{l+1}}\right] P_l(\cos\theta)}$$

Si se quiere la respuesta en términos de armónicos esféricos usese

$$P_l(\cos\theta) = \sqrt{\frac{4\pi}{2l+1}}\, Y_l^0(\theta,\phi)$$

b) En las proximidades del centro de la esfera:

$$\Phi = \Phi_{int} = A_0 P_0(\cos\theta) + A_1 r P_1(\cos\theta) + A_2 r^2 P_2(\cos\theta) + \cdots$$

$$= \frac{q}{4\pi\epsilon_0 d}\left[1 + \frac{3r\cos\theta}{(\epsilon/\epsilon_0+2)d} + \frac{5r^2(\frac{1}{2})(3\cos^2\theta-1)}{(2\epsilon/\epsilon_0+3)d^2} + \cdots\right]$$

$$= \frac{q}{4\pi\epsilon_0 d}\left[1 + \frac{3z}{(\epsilon/\epsilon_0+2)d} + \frac{5(3z^2-(x^2+y^2+z^2))}{(2\epsilon/\epsilon_0+3)d^2} + \cdots\right]$$

$$\Rightarrow \vec{E} = -\nabla\Phi = -\left(\frac{\partial\Phi}{\partial x}\hat{i} + \frac{\partial\Phi}{\partial y}\hat{j} + \frac{\partial\Phi}{\partial z}\hat{k}\right)$$

$$\vec{E} \simeq \frac{q}{4\pi\epsilon_0 d^2}\left[\frac{5x}{(2\epsilon/\epsilon_0+3)d}\hat{i} + \frac{5y}{(2\epsilon/\epsilon_0+3)d}\hat{j} + \left(\frac{3}{\epsilon/\epsilon_0+2} + \frac{10z}{(2\epsilon/\epsilon_0+3)d}\right)\hat{k}\right]$$

c) Si $\epsilon/\epsilon_0 \longrightarrow \infty$, $\rightarrow A_l = 0$ si $l \neq 0$ y $A_0 = 1$

$$\Rightarrow \Phi_{int} = \frac{q}{4\pi\epsilon_0 d} \quad y \quad E_{int} = 0$$

(Campo y potencial en el interior de una esfera conductora con carga total inducida q)

$$B_l \rightarrow -\frac{q a^{2l+1}}{4\pi\epsilon_0 d^{l+1}} \quad \text{si } l \neq 0 \quad y \quad B_0 = 0$$

$$\Rightarrow \Phi_{ext} = \frac{q}{4\pi\epsilon_0}\left\{\sum_{l=1}^{\infty}\left(\frac{-a^{2l+1}}{d^{l+1}}\right) r^{-(l+1)} P_l(\cos\theta) + \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta)\right\}$$

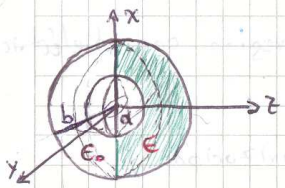
$$= \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\left[\frac{r_<^l}{r_>^{l+1}} - \frac{a^{2l+1}}{(rd)^{l+1}}\right] P_l(\cos\theta) + \frac{q}{4\pi\epsilon_0}\frac{a}{(rd)}$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\vec{r}_> - \vec{r}_<|} - \frac{a}{|\vec{r}d - a^2\hat{n}|}\right] + \frac{qa/d}{4\pi\epsilon_0 r}$$

$r_< = \min\{r, d\}$ y $r_> = \max\{r, d\}$

$$\Rightarrow \Phi_{ext} = \underbrace{\frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r} - d\hat{n}|}}_{\text{Carga real}} - \underbrace{\frac{qa/d}{|\vec{r} - (a^2/d)\hat{n}|}}_{\text{Carga Imagen}} + \underbrace{\frac{qa/d}{4\pi\epsilon_0 r}}_{\text{Carga imagen en el origen que hace que } Q_{Total}=0 \text{ sobre la esfera.}}$$

4.10 $\Phi(r,\theta) = \sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right] P_l(\cos\theta)$



$$\vec{D} = \epsilon(\vec{r})\vec{E} = -\epsilon(\vec{r})\vec{\nabla}\Phi \qquad \epsilon(\vec{r}) = \begin{cases} \epsilon & 0 \le \theta < \frac{\pi}{2} \\ \epsilon_0 & \frac{\pi}{2} < \theta \le \pi \end{cases}$$

$$D_r\hat{r} + D_\theta\hat{\theta} + D_\phi = -\epsilon(\vec{r})\left(\frac{\partial\Phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\theta} + 0\hat{\phi}\right)$$

$$\Rightarrow D_r = -\epsilon(\vec{r})\sum_{l=0}^{\infty}\left[lA_l r^l - (l+1)B_l r^{-(l+2)}\right]P_l(\cos\theta)$$

$\int_S \vec{D}\cdot\hat{n}\,da = Q$ siendo S la superficie de una esfera gaussiana entre a y b. ($\hat{n} = \hat{r}$)

$$\int D_r\left(r^2\sin\theta\, d\theta\, d\phi\right) = Q \longrightarrow 2\pi\int_{-1}^{1} D_r r^2\, d(\cos\theta) = Q$$

Note que el resultado de la integral es independiente de r, de modo que $D_r r^2$ debe ser independiente de r:

$$D_r r^2 = -\epsilon(\vec{r})\sum_{l=0}^{\infty}\left[lA_l r^{l+2} - (l+1)B_l r^{-l}\right]P_l(\cos\theta)$$

Como debe ser independiente de r $\Rightarrow$ $A_l = 0 \quad B_l = 0$ si $l \neq 0$

$$\Rightarrow D_r r^2 = +\epsilon(\vec{r})B_0\underbrace{P_0(\cos\theta)}_{1}$$

$$2\pi\int_{-1}^{1} D_r r^2\, d(\cos\theta) = 2\pi\int_{-1}^{1}\epsilon(\vec{r})B_0\, d(\cos\theta) = 2\pi B_0\left[\int_{-1}^{0}\epsilon_0\, d(\cos\theta) + \int_{0}^{1}\epsilon\, d(\cos\theta)\right]$$

$$Q = 2\pi B_0\left[\epsilon_0 + \epsilon\right] \longrightarrow \boxed{B_0 = \frac{Q}{2\pi\epsilon_0(1+\epsilon/\epsilon_0)}}$$

$$\Rightarrow \Phi = \frac{Q}{2\pi\epsilon_0(1+\epsilon/\epsilon_0)}r^{-1}\underbrace{P_0(\cos\theta)}_{1} \longrightarrow$$ el potencial no depende de θ, solo de r.

$$\Rightarrow \vec{E} = -\nabla\Phi \longrightarrow \boxed{\vec{E} = \frac{Q}{2\pi\epsilon_0(1+\epsilon/\epsilon_0)r^2}\hat{r}}$$

b) $\int_S \vec{D}\cdot\hat{n}\,da = Q = \int_S \sigma_{int}\,da \longrightarrow \sigma_{int} = D_r\Big| = \epsilon(\vec{r})E_r\Big|_{r=a} = -\epsilon(\vec{r})\frac{\partial\Phi}{\partial n}\Big|_S$ ($\hat{n} = \hat{r}$)

$$\Rightarrow \boxed{\sigma_{int} = \frac{Q}{2\pi\epsilon_0(1+\epsilon/\epsilon_0)a^2}\times\begin{cases}\epsilon & \text{si } 0 \le \theta < \pi/2 \\ \epsilon_0 & \text{si } \frac{\pi}{2} < \theta \le \pi\end{cases}}$$

c) $\vec{P} = (\epsilon - \epsilon_0)\vec{E} = \frac{Q}{2\pi r^2}\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}\hat{r}$ para la región con dieléctrico

$\sigma_{pol} = \vec{P}\cdot\underset{-\hat{r}}{\hat{n}}\Big|_S = -\frac{Q}{2\pi a^2}\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}$ en la región con dieléctrico

En la región sin dielectrico no hay polarización y por tanto no hay σ_{pol}.

Otra forma de ver esto es: $\vec{\nabla}\cdot\vec{E} = 0$

y $\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}(\rho_{pol} - \nabla\cdot\vec{P}) = 0 \rightarrow \boxed{\rho_{pol} = -\vec{\nabla}\cdot\vec{P}}$ → Ec. 4.33 de Jackson

$$\rightarrow \int \rho_{pol}\, dv = -\int \vec{\nabla}\cdot\vec{P}\, dv = -\int\left(\frac{\partial}{\partial r}P_r + \cancelto{0}{\frac{\partial}{\partial\theta}P_\theta} + \cancelto{0}{\frac{\partial}{\partial\phi}P_\phi}\right) dv$$

como lo que realmente tenemos es densidad superficial

$$\Rightarrow \int \sigma_{pol}\, da = -\int\left(\frac{\partial}{\partial r}P_r\right) da \rightarrow \sigma_{pol} = -P = -\epsilon_0\chi_e E\big|_S$$

$$\Rightarrow \sigma_{pol} = -\epsilon_0(\epsilon(\vec{r}) - \epsilon_0)E\big|_S \qquad \chi_e = \epsilon - \epsilon_0.$$

$$\boxed{\sigma_{pol} = -(\epsilon(\vec{r})/\epsilon_0 - 1)\frac{Q}{2\pi(1+\epsilon/\epsilon_0)a^2}}$$ con $\epsilon(\vec{r}) = \begin{cases}\epsilon & 0 \le \theta < \frac{\pi}{2} \\ \epsilon_0 & \frac{\pi}{2} < \theta < \pi\end{cases}$

$$\sigma_{pol} = \begin{cases} -\frac{Q(\epsilon/\epsilon_0 - 1)}{2\pi a^2(1+\epsilon/\epsilon_0)} & \text{si } 0 \le \theta < \pi/2 \\ 0 & \text{si } \frac{\pi}{2} < \theta \le 0 \end{cases}$$

Note que $\sigma_{pol} + \sigma_{Lib\,int} = -\frac{Q(\cancel{\epsilon} - \epsilon_0)}{2\pi a^2(1+\epsilon/\epsilon_0)\epsilon_0} + \cancel{\frac{\epsilon Q}{2\pi\epsilon_0(1+\epsilon/\epsilon_0)a^2}}$

Para cualquier $\theta \rightarrow \sigma_{tot} = \frac{Q}{2\pi a^2(1+\epsilon/\epsilon_0)} = \epsilon_0 E$ como es de esperarse.

4.11

By: Herminso Villarraga Gómez

Solución a los problemas del cap 5 Jackson 3ed

5.1 $d\vec{B} = \frac{\mu_0}{4\pi} I\, d\vec{l}' \times \left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right) \rightarrow \vec{B} = \frac{\mu_0}{4\pi} I \oint d\vec{l}' \times \left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right)$

$\oint_c \vec{A}\cdot d\vec{l} = \int_s (\vec{\nabla}\times\vec{A})\cdot\hat{n}\, da \rightarrow$ teo Stokes

Realmente usaremos $\boxed{\oint_c d\vec{l}\times\vec{P} = \int_s (\hat{n}\, da \times \vec{\nabla})\times\vec{P}} \rightarrow$

Forma Alterna del teorema de Stokes.

que se obtiene de hacer $\vec{A} = \vec{a}\times\vec{P}$ en el teo. de Stokes con $\vec{a}$ un vector cte en magnitud y dirección pero de elección arbitraria: Como

$$\oint_c (\vec{a}\times\vec{P})\cdot d\vec{l} = \int_s [\vec{\nabla}\times(\vec{a}\times\vec{P})]\cdot d\vec{l}$$

Use 1 Use 2

$(\vec{a}\times\vec{b})\cdot\vec{c} = (\vec{b}\times\vec{c})\cdot\vec{a} = -(\vec{c}\times\vec{b})\cdot\vec{a}$ 1

$(\vec{A}\times\vec{B})\cdot\vec{C} = (\vec{C}\times\vec{A})\cdot\vec{B}$ 2

$$-\oint_c (d\vec{l}\times\vec{P})\cdot\vec{a} = \int_s (\underbrace{\hat{n}\, da\times\vec{\nabla}}_{c})\cdot(\underbrace{\vec{a}}_{a}\times\underbrace{\vec{P}}_{b})$$

usando 1

$$= -\int_s [(\hat{n}\, da\times\vec{\nabla})\times\vec{P}]\cdot\vec{a}$$

$$\left(\oint_c (d\vec{l}\times\vec{P}) - \int_s (\hat{n}\, da\times\vec{\nabla})\times\vec{P}\right)\cdot\vec{a} = 0$$

aunque la dirección de $\vec{a}$ es cte, es arbitraria

$$\Rightarrow \boxed{\oint_c d\vec{l}\times\vec{P} = \int_s (\hat{n}\, da\times\vec{\nabla})\times\vec{P}} \rightarrow$$

forma alterna del teo de Stokes.

Usando esto:

$$\vec{B} = \frac{\mu_0 I}{4\pi}\oint d\vec{l}'\times\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right) = \frac{\mu_0 I}{4\pi}\int (\hat{n}\, da'\times\vec{\nabla}')\times\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right)$$

$$= -\frac{\mu_0 I}{4\pi}\int (\hat{n}\, da'\times\nabla)\times\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right)$$

$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$

$$= \frac{\mu_0 I}{4\pi}\int \left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right)\times(\hat{n}'\, da'\times\nabla) = \frac{\mu_0 I}{4\pi}\left[\int \vec{\nabla}\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right)\hat{n}\, da' - \vec{\nabla}\cdot\left(\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right)\cdot\hat{n}\, da'\right)\right]$$

$$\Rightarrow \vec{B} = \frac{\mu_0 I}{4\pi}\left[\underbrace{\int_S \vec{\nabla}\cdot\overbrace{\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)}^{\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}}\hat{n}\,da'}_{-4\pi\int_S \delta(\vec{r}-\vec{r}')\,\hat{n}\,da' \;\to\; 0} - \vec{\nabla}\int \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\cdot\hat{n}\,da'\right]$$

Ya que $\vec{r} \neq \vec{r}'$ sobre S
($\vec{r}'$ está sobre la espira
$\vec{r} \in S$ sobre el pto de observación)

$$\Rightarrow \vec{B} = \frac{\mu_0 I}{4\pi}\vec{\nabla}\int\left[\left(\frac{\vec{r}'-\vec{r}}{|\vec{r}-\vec{r}'|^3}\right)\cdot ds'\right]$$

Note que $\Omega = \int \frac{\vec{R}\cdot d\vec{A}}{R^3}$ luego si $\vec{R} = \vec{r}' - \vec{r}$

$$\boxed{\vec{B} = \frac{\mu_0 I}{4\pi}\vec{\nabla}\Omega}$$

5.2 El problema 5.1 define un potencial escalar magnético $\Phi_M = -\frac{\mu_0 I}{4\pi}\Omega$