Wavepackets

Gaussian $\psi(x)$

$$\psi(x) = \frac{1}{(\pi\Delta^2)^{1/4}} e^{-x^2/2\Delta^2} \longleftrightarrow \tilde{\psi}(k) = (4\pi\Delta^2)^{1/4} e^{-\frac{\Delta^2}{2}k^2}$$

Take case of $V(x)=0$. How does it evolve in time? Need to write in expansion of eigenfunctions of H.

$$\psi(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{\psi}(k)$$

(expression above)

coefficients describing amount & phase of this $\vec{k}$ component

eigenfunctions of H with $E = \frac{\hbar^2 k^2}{2m}$

$$\psi(x,t) = \int \frac{dk}{2\pi} e^{ikx} e^{-\frac{i\hbar k^2}{2m}t} (4\pi\Delta^2)^{1/4} e^{-\frac{\Delta^2}{2}k^2}$$

$$\psi(x,t) = \int \frac{dk}{2\pi} (4\pi\Delta^2)^{1/4} e^{-\frac{1}{2}\left(\Delta^2 + \frac{i\hbar t}{m}\right)k^2 + ikx}$$ ← Need to complete square in k

Define $\boldsymbol{\Delta}^2 = \Delta^2 + \frac{i\hbar t}{m}$

$$\psi(x,t) = \int \frac{dk}{2\pi} (4\pi\Delta^2)^{1/4} e^{-\frac{1}{2}\boldsymbol{\Delta}^2\left(k - \frac{ix}{\boldsymbol{\Delta}^2}\right)^2} e^{-\frac{1}{2}\frac{x^2}{\boldsymbol{\Delta}^2}}$$

$$= \frac{(4\pi\Delta^2)^{1/4}}{2\pi} \sqrt{\pi \cdot \frac{2}{\boldsymbol{\Delta}^2}}\; e^{-\frac{1}{2}\frac{x^2}{\boldsymbol{\Delta}^2}}$$

$$= \frac{\Delta^{1/2}}{\pi^{1/4}\boldsymbol{\Delta}} e^{-\frac{1}{2}\frac{x^2}{\boldsymbol{\Delta}^2}}$$

Let's consider $|\psi(x,t)|^2$

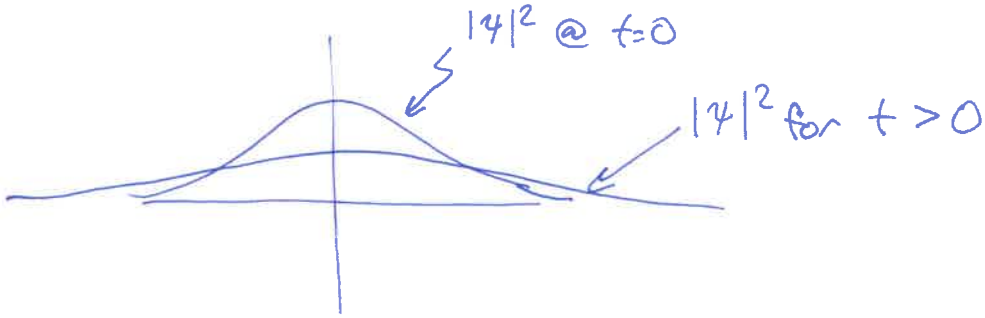
$$\left|\frac{1}{\boldsymbol{\Delta}^2}\right| = \frac{1}{\Delta^4 + \frac{\hbar^2 t^2}{m^2}}$$

$\left(c = x^2,\; a+ib = \boldsymbol{\Delta}^2 = \Delta^2 + \frac{i\hbar t}{m}\right)$

And consider $\left|e^{-\frac{c}{\alpha}}\right|$, where $\alpha = a+ib$

$$\left|e^{-\frac{c}{\alpha}}\right| = \left|e^{-\frac{c}{a+ib}}\right| = \left|e^{-\frac{(c)(a-ib)}{a^2+b^2}}\right| = e^{-\frac{ca}{a^2+b^2}}$$

$$\Rightarrow |\psi(x,t)|^2 = \left(\frac{\Delta^{1/2}}{\pi^{1/4}}\right)^2 \frac{1}{\Delta^4 + \frac{\hbar^2t^2}{m^2}} \times e^{-\left(\frac{\Delta^2}{\Delta^4 + \frac{\hbar^2t^2}{m^2}}\right)x^2}$$



- Probability distribution gets broader and shorter, but stays gaussian (Recall we proved that prob. conserved by Schr. $\Rightarrow$ shorter must go along w/ broader)
- What happens if we put particle in $V(x) =$ square well or harmonic oscillator?
- Notice that for $V(x)=0$, it spreads out quickly if Δ small + slower if Δ large. Consistent w/ expectations of momentum distributions for tightly vs loosely confined particles

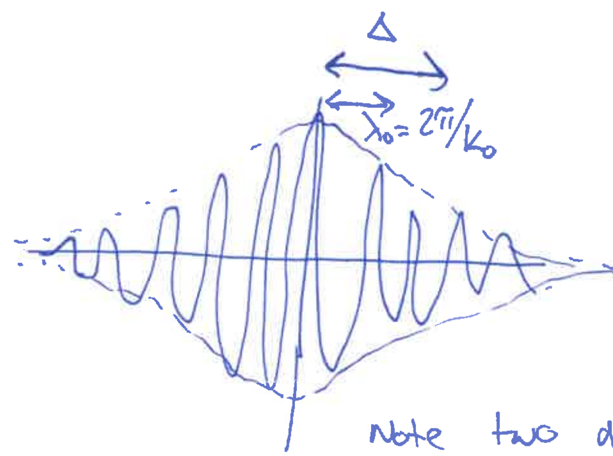
In some sense, the different momentum components of the original w.f.s run out ward at different speeds.

Now, let's give our localized particle some (net) momentum.

$$\psi(x) = \frac{1}{(\pi\Delta^2)^{1/4}} e^{-x^2/2\Delta^2} e^{ik_0 x}$$

@ $t=0$

(Just looking @ $\text{Re}(\psi)$ we don't know direction)



Note two different length scales.

Would need to plot $\text{Im}\,\psi$ to see ~~that~~ whether k is ~~bsome~~ positive or negative. Phase of $\text{Im}\,\psi$ vs $\text{Re}\,\psi$ will tell.

- Note, there is not a well-defined momentum. This is not a plane wave! We expect delocalization over time.
- Let's see if center moves as expected.

$$\boxed{\psi(x,t=0) = \frac{1}{(\pi\Delta^2)^{1/4}} e^{-x^2/2\Delta^2} e^{ik_0x}}$$

$$\tilde{\psi}(k) = \int dx\, e^{-ikx}\psi(x,t=0) = \int dx \frac{1}{(\pi\Delta^2)^{1/4}} e^{-x^2/2\Delta^2} e^{-i(k-k_0)x}$$

(coefficients of energy eigenstates, found by F.T. of $t=0$ wavefunction)

This is exactly the same integral we had before (F.T. of simple gaussian) for the case $k_0=0$. We can just take $k \to k-k_0$ in ~~that~~ the result from that integral.

$$\Rightarrow \boxed{\tilde{\psi}(k) = (4\pi\Delta^2)^{1/4} e^{-\frac{\Delta^2}{2}(k-k_0)^2}}$$

This makes good sense. The momentum distribution looks exactly the same as before except it is now centered around $k=k_0$ rather than $k=0$.

Now let's find $\psi(x,t)$

$$\psi(x,t) = \int \frac{dk}{2\pi} e^{+ikx}\tilde{\psi}(k) e^{-\frac{i\hbar k^2}{2m}t}$$

This is an integral over all k, so we can replace $k \to k+k_0$

Then we have:

~~This is a sum over eigenfunctions of H, so putting in ~~energy~~ time dependence is easy.~~

$$\cancel{\psi(x,t) = \int \frac{dk}{2\pi} \left(\frac{4\pi}{\Delta^2}\right)^{1/4} e^{ikx} e^{-\frac{\Delta^2}{2}(k-k_0)^2} e^{-\frac{i\hbar k^2}{2m}t}}$$

~~Shift back $k \to k+k_0$~~

$$\psi(x,t) = \left(\frac{\Delta^2}{4\pi^3}\right)^{1/4} \int dk \; e^{i(k+k_0)x} e^{-\frac{\Delta^2}{2}k^2} e^{-\frac{i\hbar(k+k_0)^2}{2m}t}$$

$$= \left(\frac{\Delta^2}{4\pi^3}\right)^{1/4} \int dk \; e^{-\frac{k^2}{2}\overbrace{\left(\Delta^2 + \frac{i\hbar}{m}t\right)}^{\tilde{\Delta}^2}} e^{ik\underbrace{\left(x - \overbrace{\frac{\hbar k_0 t}{m}}^{v_0 t}\right)}_{\alpha}} e^{ik_0 x} e^{-i\overbrace{\frac{\hbar k_0^2}{2m}t}^{\omega_0 t}}$$

Complete square in exponent

$$= \left(\frac{\Delta^2}{4\pi^3}\right)^{1/4} \int dk \; e^{-\frac{\tilde{\Delta}^2}{2}\left(k - \frac{i\alpha}{\tilde{\Delta}^2}\right)^2} e^{-\frac{\alpha^2}{2\tilde{\Delta}^2}} e^{ik_0 x} e^{-i\omega_0 t}$$

$$= \left(\frac{\Delta^2}{4\pi^3}\right)^{1/4} \left(\frac{2\pi}{\tilde{\Delta}^2}\right)^{1/2} e^{-(x-v_0 t)^2/2\tilde{\Delta}^2} e^{ik_0 x} e^{-i\omega_0 t}$$

Gaussian envelope

Same characteristic wavelength

Time-dep phase

① Moves with speed $\frac{p_0}{m}$, as expected

② Spreads out in time, as expected