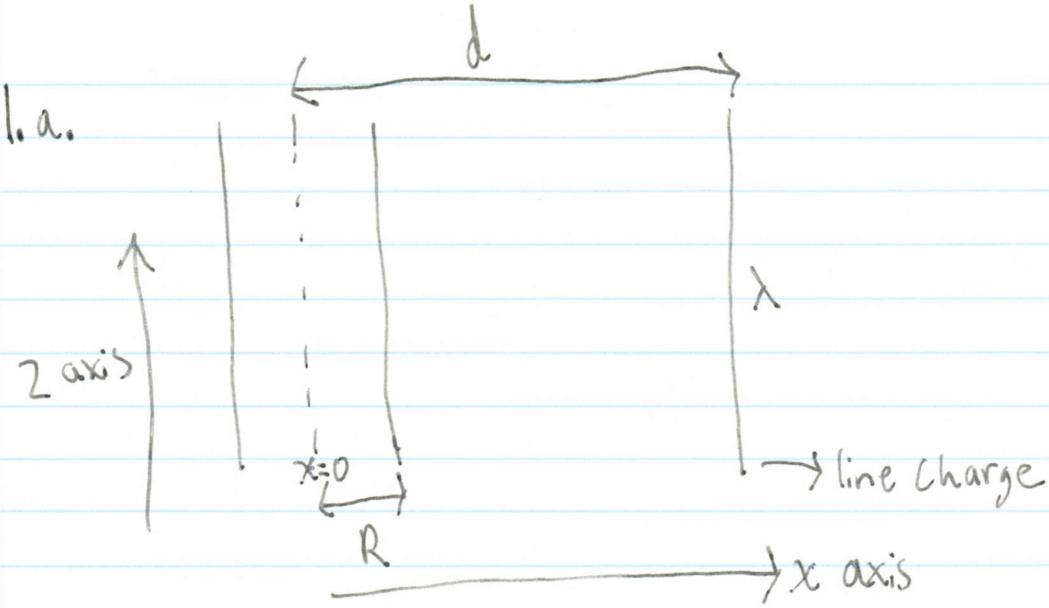
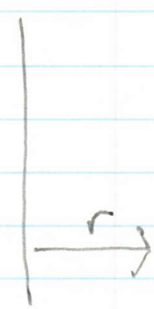
1.a.



By symmetry considerations, we would expect the image charge to be a line charge with charge density λ' located within the cylinder in the plane of the cylinder and the original line charge, and will be parallel to both (i.e., in the x-z plane, in the z direction).

By Gauss' law, the electric field a distance r away from a line charge λ is $\vec{E} = E_r \hat{r}$, with E_r such that

$$E_r \cdot 2\pi r = 4\pi\lambda \Rightarrow E_r = \frac{2\lambda}{r}$$



The corresponding potential is

$$\phi_\lambda(r) = -\int_{r_0}^{r} \frac{2\lambda}{r'} dr' = -2\lambda \ln\left(\frac{r}{r_0}\right)$$ where

the choice of r_0 is arbitrary.

For our two lines with charge densities λ and λ', with the λ line at $x=d$ and the λ' line at $x=d'$, with d to be determined, the potential is

$$\phi(x,y,z) = -2\lambda \ln\left(\frac{\sqrt{(x-d)^2+y^2}}{r_0}\right) - 2\lambda' \ln\left(\frac{\sqrt{(x-d')^2+y^2}}{r_0}\right)$$

In order to have $\phi \to 0$ for large distances away, we need $\lambda' = -\lambda$. So:

$$\phi(x,y,z) = -\lambda\left[\ln\left(\frac{(x-d)^2+y^2}{r_0^2}\right) - \ln\left(\frac{(x-d')^2+y^2}{r_0^2}\right)\right]$$

$$= -\lambda \ln\left(\frac{(x-d)^2+y^2}{(x-d')^2+y^2}\right) = -\lambda \ln\left(\frac{x^2+y^2-2xd+d^2}{x^2+y^2-2xd'+d'^2}\right)$$

We want $x^2+y^2=R^2$ to be an equipotential, so we need

$$\frac{R^2-2xd+d^2}{R^2-2xd'+d'^2} = \text{constant}$$

One solution is $d'=d$, but then the image charge is outside the conductor, which is not physical. Note that if $d' = \frac{R^2}{d}$:

$$R^2 - 2xd' + d'^2 = R^2 - 2x\frac{R^2}{d} + \frac{R^4}{d^2}$$

$$= \frac{R^2}{d^2}\left(d^2 - 2xd + R^2\right)$$

$$\Rightarrow \frac{R^2 - 2xd + d^2}{R^2 - 2xd' + d'^2} = \frac{d^2}{R^2} = \text{constant}$$

So our image charge is a line with charge density $-\lambda$ at $x = \frac{R^2}{d}$.

b. The potential is

$$\phi(x,y) = -\lambda \ln\left(\frac{(x-d)^2 + y^2}{(x-\frac{R^2}{d})^2 + y^2}\right)$$

polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$, $r = x^2 + y^2$

$$\phi(r,\theta) = -\lambda \ln\left(\frac{r^2 - 2rd\cos\theta + d^2}{r^2 - 2\frac{R^2 r}{d}\cos\theta + \frac{R^4}{d^2}}\right)$$

The asymptotic form far from the cylinder is

$$\phi(r,\theta) \xrightarrow[r\to\infty]{} -\lambda \ln\left(\frac{1 - 2\frac{d}{r}\cos\theta + O(\frac{1}{r^2})}{1 - \frac{2R^2}{dr}\cos\theta + O(\frac{1}{r^2})}\right)$$

$$\approx -\lambda\left(\ln\left(1 - 2\frac{d}{r}\cos\theta\right) - \ln\left(1 - \frac{2R^2}{rd}\cos\theta\right)\right)$$

$$\approx -\lambda\left(-\frac{2d}{r}\cos\theta + \frac{2R^2}{rd}\cos\theta\right) \rightarrow$$ Taylor expanding $\ln(1+x) \approx x$ for small x

$$= \lambda \frac{2\cos\theta}{r}\left(d - \frac{R^2}{d}\right)$$

c. The force is just due to the field of the image charge:

$$E_x = -\frac{2\lambda}{d - \frac{R^2}{d}}$$

$$\frac{\text{Force}}{\text{length}} = E_x \lambda = -\frac{2\lambda^2}{d - \frac{R^2}{d}}$$

2. a. Conducting sphere: charge distribution will be symmetric. Potential is as if there is a point charge Q at the center so,

$$\phi = \frac{Q}{a} \Rightarrow Q = a\phi$$

so $C = a$.

b. Since the conductor of radius b is much smaller than the conductor of radius a and is held at the same potential ϕ, it will hold a much smaller amount of charge and so will not significantly perturb the charge distribution on the larger sphere, assuming that $r-a$ is not too small. The charge on the large sphere is

$q_a = a\phi$, so the potential of the small sphere is

$\phi_b = \phi = \frac{q_a}{r} + \frac{q_b}{b}$, that is, the sum of the potential contribution from the charge on the large sphere and from the charge q_b on itself. So:

$\phi = \frac{a\phi}{r} + \frac{q_b}{b} \Rightarrow q_b = b\phi\left(1 - \frac{a}{r}\right)$. The force F of interaction is the Coulomb repulsion between the charge q_b on the small sphere and the charge q_a on the large sphere:

$$F = \frac{q_a q_b}{r^2} = \frac{ab\phi^2}{r^2}\left(1 - \frac{a}{r}\right)$$

Since the charges have the same sign, the force is repulsive.

3. We will start by considering discrete charge distributions with charges located at a discrete set of positions $\vec{r}_i$. Let us denote

$r_{ij} \equiv |\vec{r}_i - \vec{r}_j|$. For distribution 1, let us say that

the charge at position $\vec{r}_i$ is q_i, and the potential at position $\vec{r}_i$ is ϕ_i. Since, for the theorem we assume that the potential arises from the charges in the distribution,

$$\phi_i = \sum_{j \neq i} \frac{q_j}{r_{ij}} \quad \left(\begin{array}{l}\text{as usual, we ignore the infinite} \\ \text{self energy of a point charge}\end{array}\right)$$

Now, if we instead put charge q_i' at $\vec{r}_i$ (corresponding to distribution 2), the new potentials are:

$\phi_i' = \sum_{j \neq i} \frac{q_j'}{r_{ij}}$. We now consider $\sum_i q_i' \phi_i$:

$$\sum_i q_i' \phi_i = \sum_i \sum_{j \neq i} q_i' \frac{q_j}{r_{ij}} = \sum_i \sum_{j \neq i} \frac{q_i' q_j}{r_{ij}} = \sum_{i,j (i \neq j)} \frac{q_i' q_j}{r_{ij}}$$

Similarly, $\sum_i q_i \phi_i' = \sum_k q_k \phi_k' = \sum_k \sum_{l \neq k} \frac{q_k q_l'}{r_{kl}} = \sum_{k,l (k \neq l)} \frac{q_l' q_k}{r_{kl}}$

where have chosen to use differen sumation indices. The sums are over all combinations of i and j (or k and l) so that $i \neq j$ (or $k \neq l$).

The two sums $\sum_i q_i' \phi_i$ and $\sum_i q_i \phi_i'$ lead to the same set of terms. To see this explicity, we can replace the index l with i and the index k with j:

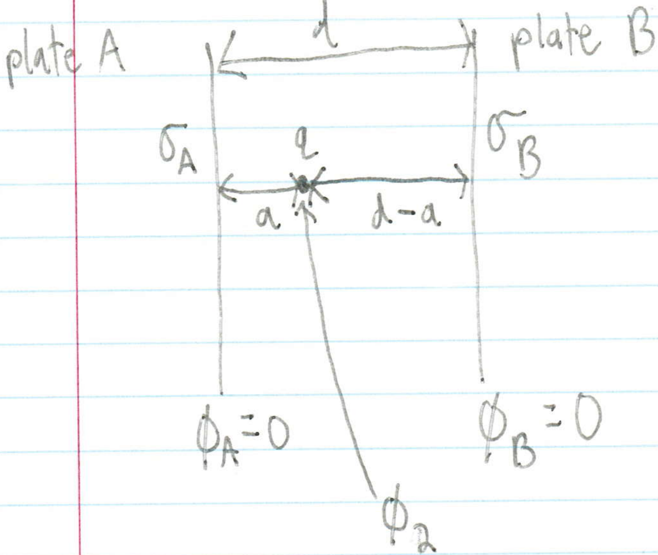
$$\sum_i q_i \phi_i' = \sum_{i,j\,(i\neq j)} \frac{q_i' q_j}{r_{ij}} = \sum q_i' \phi_i.$$

Writing the statement $\sum_i q_i \phi_i' = \sum_i q_i' \phi_i$ in terms of continuous volume and surface charge density:

$$\int_V \rho \phi' dV + \int_S \sigma \phi' dS = \int_V \rho' \phi \, dV + \int_S \sigma' \phi \, dS$$

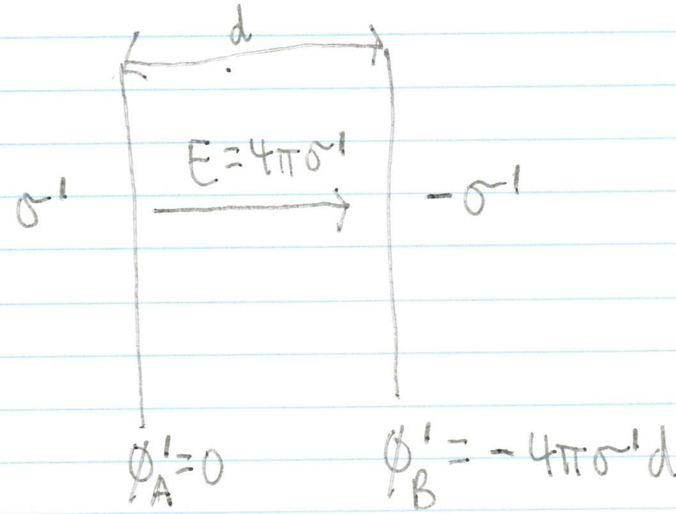
as desired.

4. a. Our physical setup is as follows:



with $\phi_A = \phi_B = 0$. ϕ_2 denotes the potential at the location of the charge q.

As an alternative setup that we can use for Green's reciprocity theorem, we take $\rho' = 0$ between the plates and surface charge densities that are uniform with values $\pm\sigma'$:



This is just the setup for a standard parallel plate capacitor with no charge in between.

ϕ_2', the potential a distance a from plate A in the alternative setup (correspinding to the position of the charge q in the original setup), is

$\phi_2' = -4\pi\sigma' a$. We now calculate the relevant integrals for Green's reciprocity theorem.

$$\int_V \rho\phi' dV = q\phi_2' = -4\pi\sigma' q a$$

$$\int_S \sigma\phi' ds = -4\pi\sigma' d \int_S \sigma_B ds = -4\pi\sigma' d q_B$$

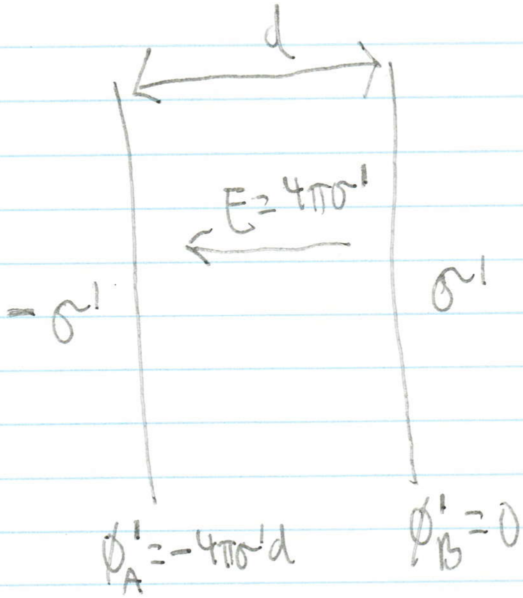
where q_B is the total charge on plate B.

$$\int_V \rho'\phi dV = 0 \qquad \text{since } \rho' = 0$$

$$\int_S \sigma'\phi ds = 0 \qquad \text{since } \phi = 0 \text{ on the plates.}$$

$$\Rightarrow -4\pi\sigma' q a - 4\pi\sigma' d q_B = 0, \text{ so } q_B = -\frac{a}{d} q.$$

We could perform an analogous calculation with the following alternative setup instead.



This would give us $q_A = -\frac{d-a}{d} q$.

b. The total induced charge on both plates is

$$q_{ind} = q_A + q_B = \frac{-a-d+a}{d} q = -q$$

5. a. Note that

$$\sin^2\theta' = 1-\cos^2\theta' = \frac{2}{3} - \frac{2}{3}\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)$$

$$= \frac{2}{3}\sqrt{4\pi}\, Y_{00}(\Omega') - \frac{2}{3}\sqrt{\frac{4\pi}{5}}\, Y_{20}(\Omega')$$

$$q_{\ell m} = \int Y^*_{\ell m}(\Omega')(r')^{\ell} \frac{1}{64\pi}(r')^2 e^{-r'} \sin^2\theta' \, (r')^2 dr' \, d\Omega'$$

$$= \frac{1}{64\pi}\int_0^\infty dr' (r')^{\ell+4} e^{-r'} \int d\Omega' Y^*_{\ell m}(\Omega') \left[\frac{2}{3}\sqrt{4\pi}\, Y_{00}(\Omega') - \frac{2}{3}\sqrt{\frac{4\pi}{5}}\, Y_{20}(\Omega')\right]$$

Because of the orthonormality of the spherical harmonics, the second integral vanishes unless $\ell=0, m=0$ or $\ell=2, m=0$. Evaluating for these two cases,

$$q_{00} = \frac{2}{3}\sqrt{4\pi}\,\frac{1}{64\pi}\int_0^\infty dr' (r')^4 e^{-r'} = \frac{1}{2\sqrt{\pi}}$$

$$q_{20} = -\frac{2}{3}\sqrt{\frac{4\pi}{5}}\,\frac{1}{64\pi}\int_0^\infty dr' (r')^6 e^{-r'} = -3\sqrt{\frac{5}{\pi}}$$

5. b. $\phi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$ (for large r compared to the distance scale over which the distribution falls off)

$$= 4\pi \frac{1}{2\sqrt{\pi}} \frac{1}{\sqrt{4\pi}} \frac{1}{r} - \frac{4\pi}{5} 3 \sqrt{\frac{5}{\pi}} \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) \frac{1}{r^3}$$

$$= \frac{1}{r} - \frac{6}{r^3}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Integrating over the charge distribution yields a total charge of

$$q = \frac{1}{64\pi} \int_0^{\infty} \int_0^{\pi} \int_0^{2\pi} r^2 e^{-r} \sin^2\theta \left(r^2 \sin\theta\right) d\phi \, d\theta \, dr = 1$$

We expect the $\frac{1}{r}$ contribution to go as $\frac{q}{r}$, which is indeed the case since $q = 1$.