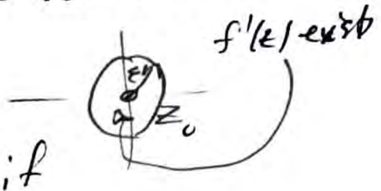
Analytic Functions

Def. A function $f(z)$ is analytic at $z = z_0$ if $f'(z)$ exists for all z,
$0 \le |z - z_0| < \varepsilon$ for some ε



Def. A function $f(z)$ is analytic in a domain if it is analytic at every point of the domain

Def. A function $f(z)$ is an entire function if it is analytic at every point of the complex z-plane

Examples (1) $P_n(z)$ - n^{th} degree polynomial - entire function

(2) $f(z) = |z|^2$ has the derivative at $z=0$, but not analytic at $z=0$

(3) $f(z) = e^z$ - entire function

(4) $f(z) = (x-y)^2 + 2i(x+y) \Rightarrow u(x,y) = (x-y)^2,\ v(x,y) = 2(x+y)$

C-R conditions?

$$u_x = 2(x-y) \quad \Big| \quad v_x = 2 \quad \Big| \quad u_x \overset{?}{=} v_y \quad \Big| \quad x-y=1$$
$$u_y = -2(x-y) \quad \Big| \quad v_y = 2 \quad \Big| \quad u_y \overset{?}{=} -v_x \quad \Big| \quad x-y=1$$

$\Rightarrow f'(z)$ exists only on the line $x-y=1$ (and is equal to
$f'(z) = u_x + i v_x = 2(x-y) + 2i = 2+2i$) but not analytic anywhere

Th If $f_1(z)$ & $f_2(z)$ are analytic in ~~D~~ some domain D, then

(1) $c_1 f_1(z) + c_2 f_2(z)$ - analytic in D (c_1, c_2 - constants)

(2) $f_1(z) f_2(z)$ - analytic in D

(3) $f_1(z) / f_2(z)$ - analytic in D (except where $f_2(z) = 0$)

(4) Superposition of analytic functions is an analytic function

Def. A real function $u(x,y)$ is harmonic if u_{xx}, u_{xy}, u_{yy} exist and are continuous and

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{(Laplace equation)}$$

Th. If $f(z) = u(x,y) + iv(x,y)$, $z = x+iy$, is analytic in some domain D, then u & v are harmonic in D

Proof:
$$u_x = v_y \qquad\qquad u_y = -v_x$$
$$\Downarrow \tfrac{\partial}{\partial x} \qquad\qquad \Downarrow \tfrac{\partial}{\partial y}$$
$$u_{xx} = v_{xy} \quad \oplus \quad u_{yy} = -v_{xy} \implies u_{xx} + u_{yy} = 0$$

Similarly for v, $v_{xx} + v_{yy} = 0$

(Here we implicitly assumed that u_x, u_y, v_x, v_y are differentiable – will be proved later, not using this theorem.

Thus, we observe that an innocent definition of derivative $f'(z)$ implies extremely strong conditions on u and v.

Ex. Does there exist an analytic function $f(z)$ such that its real part is equal to $\frac{x^2+ay^2}{2}$ for some a?

$f(z) = u(x,y) + iv(x,y) \implies u(x,y) = \frac{1}{2}(x^2+ay^2) \implies$

$u_{xx} + u_{yy} = 1 + a$ – must be zero $\implies \underline{\underline{a = -1}}$

Thus, $u(x,y) = \frac{1}{2}(x^2-y^2)$. What's $f(z)$? – Could guess that for $z = x+iy \implies z^2 = x^2-y^2+2ixy \implies$

$f(z) = \frac{1}{2}z^2 = \frac{1}{2}(x^2-y^2) + ixy$. $\implies \underline{v(x,y) = xy}$

~~[A systematic way to derive $f(z)$: use $z \leftrightarrow x = \frac{1}{2}(z+\bar{z})$, $y = \frac{1}{2i}(z-\bar{z})$~~

~~$\implies u(x,y) = \frac{1}{2}(x^2-y^2) = \frac{1}{8}\left[z^2+\bar{z}^2+2z\bar{z} + z^2+\bar{z}^2-2z\bar{z}\right] = \frac{1}{4}(z^2+\bar{z}^2)$~~

Def. If $u(x,y)$ and $v(x,y)$ are harmonic in D and satisfy the C-R condition $u_x = v_y$, $u_y = -v_x$ then v is called a harmonic conjugate of u.

<u>Th.</u> A function $f(z) = u(x,y) + iv(x,y)$ is analytic in D if and only if v is a harmonic conjugate of u.

<u>Ex.</u> Check that $u(x,y) = 2x(1-y)$ is harmonic. Find $v(x,y)$ such that $f(z) = u(x,y) + iv(x,y)$ is analytic and write f in terms of z (not just x and y)

$u_x = +2(1-y)$; $u_{xx} = 0$; $u_{yy} = 0 \Rightarrow u_{xx} + u_{yy} = 0 \Rightarrow u$ - harmonic

$\cancel{v_y = u_x = -2(1-y)} \Rightarrow \cancel{v(x,y) = \ldots}$

$$v_y = +u_x = +2(1-y) \Longrightarrow v(x,y) = 2y - y^2 + h(x)$$

constant of integration may be a function of x

$$v_x = -u_y = 2x \quad \& \quad v_x = h'(x) \Rightarrow h'(x) = 2x, \ h(x) = x^2 + c$$

real

$$\Rightarrow v(x,y) = 2y - y^2 + x^2 + c \Rightarrow$$

$$f(x,y) = 2x(1-y) + i\left(2y - y^2 + x^2 + c\right) =$$

$$= \underbrace{2x + i2y}_{2z} \underbrace{- 2xy + i(x^2 - y^2)}_{iz^2} + ic = 2z + iz^2 + ic$$

A systematic way: $x = \frac{1}{2}(z + \bar{z})$, $y = \frac{1}{2i}(z - \bar{z})$ (take $c=0$)

$$f(x,y) = (z + \bar{z})\left[1 - \frac{1}{2i}(z - \bar{z})\right] + i\left[\frac{1}{i}(z - \bar{z}) + \frac{1}{4}(z - \bar{z})^2 + \frac{1}{4}(z + \bar{z})^2\right]$$

$$= \underbrace{z - \bar{z}}_{} - \frac{1}{2i}(z^2 - \bar{z}^2) + \underbrace{(z - \bar{z})}_{} + \cancel{\tfrac{1}{4}(z-\bar{z})^2} + \frac{i}{2}(z^2 + \bar{z}^2)$$

$$= 2z + iz^2$$

Elementary functions

Exponential function

The exponential function has already been defined as

$$e^z = e^x e^{iy} = e^x(\cos y + i \sin y),$$

This is an entire function with $\frac{d}{dz}(e^z) = e^z$.

We have also seen that $e^{z_1} e^{z_2} = e^{z_1 + z_2}$.

In addition,

$$|e^z| = e^x, \quad \arg e^z = y + 2\pi n \quad (n = 0, \pm 1, \pm 2, \dots)$$

Note that for x-real, $e^x > 0$, while for z-complex, e^z -complex, in particular, can take on negative values, e.g., $e^{i\pi} = -1$

Also, since $e^{2\pi i} = 1 \Rightarrow e^{z + 2\pi i} = e^z$, so that e^z - periodic, with a pure imaginary period $2\pi i$ (and multiplies of $2\pi i$)

Trigonometric functions

As we have discussed, for x-real

$$\left.\begin{aligned} e^{ix} &= \cos x + i \sin x \\ e^{-ix} &= \cos x - i \sin x \end{aligned}\right| \Longrightarrow \begin{aligned} \cos x &= \tfrac{1}{2}(e^{ix} + e^{-ix}) \\ \sin x &= \tfrac{1}{2i}(e^{ix} - e^{-ix}) \end{aligned}$$

The same equations are taken as definitions of sine and cosine of complex argument z:

$$\cos z = \tfrac{1}{2}(e^{iz} + e^{-iz}), \quad \sin z = \tfrac{1}{2i}(e^{iz} - e^{-iz}) -$$

Both are entire functions

Usual derivatives and trig identities:

$$\frac{d}{dz}\sin z = \cos z, \quad \frac{d}{dz}\cos z = -\sin z ;$$

$$\sin(-z) = -\sin z, \quad \cos(-z) = \cos z$$

$$\sin(z_1+z_2) = \sin z_1 \cos z_2 + \sin z_2 \cos z_1$$

$$\cos(z_1+z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$

$$\ldots$$

everything can be proved using the definitions in terms of exponentials and properties of exponentials

What are $Re(\sin z)$ & $Im(\sin z)$?

For y - real we have

$$\sin(iy) = \frac{1}{2i}\left[e^{i(iy)} - e^{-i(+iy)}\right] = \frac{1}{2i}(e^{-y}-e^{y}) =$$

$$= i\,\frac{e^{y}-e^{-y}}{2} = i\sinh(y)$$

$$\cos(iy) = \frac{1}{2}\left[e^{i(iy)} + e^{-i(iy)}\right] = \frac{1}{2}(e^{y}+e^{-y}) = \cosh y$$

Thus, for $z = x+iy$ we have

$$\sin(x+iy) = \sin x\cos(iy) + \cos x \sin(iy) =$$

$$= \sin x \cosh y + i\cos x \sinh y \implies$$

$$Re(\sin z) = \sin x \cosh y, \quad Im(\sin z) = \cos x \sinh y$$

In a similar way,

$$\cos(x+iy) = \cos x\cosh y - i\sin x \sinh y \implies$$

$$Re(\cos z) = \cos x \cosh y, \quad Im(\cos z) = -\sin x \sinh y$$

Ex. $\cos z = 0 \implies Re(\cos z) = \cos x \cosh y = 0$ &

$$x = \frac{\pi}{2} + \pi n, \; n = 0, \pm 1, \ldots$$

$$Im(\sin z) = -\sin x \sinh y = 0$$

$$\implies y = 0$$

Thus, $z = \frac{\pi}{2} + \pi n, \; n = 0, \pm 1, \pm 2, \ldots$ — real (no non-real solutions)

Ex. 2π is a period of $\sin z$:

$$\sin(z+2\pi) = \frac{1}{2i}\left[e^{i(z+2\pi)} - e^{-i(z+2\pi)}\right] = \frac{1}{2i}\left[e^{iz} - e^{-iz}\right] = \sin z$$

An important difference between the real and complex cases:
$\sin z$ & $\cos z$ (z - complex) are not bounded, e.g., for $z = iy$

$\cos(iy) = \cosh y = \frac{1}{2}(e^{y}+e^{-y})$

Other trig functions are defined in terms of $\sin z$ and $\cos z$ in the usual way, e.g. $\tan z = \frac{\sin z}{\cos z}$, with the usual derivatives and trig identities. These functions are not entire but analytic, where the denominator $\neq 0$.

Hyperbolic Functions ← did not have time to discuss, next time

Def $\sinh z = \frac{1}{2}(e^{z}-e^{-z})$, $\cosh z = \frac{1}{2}(e^{z}+e^{-z})$ - entire

$\tanh z = \frac{\sinh z}{\cosh z}$, $\operatorname{sech} z = \frac{1}{\cosh z}$, $\operatorname{csch} z = \frac{1}{\sinh z}$, ...

- analytic where denom $\neq 0$

Relation between trig and hyperbolic functions:

$\cosh(iz) = \frac{e^{iz}+e^{-iz}}{2} = \cos z \Rightarrow \cos(iz) = \cosh z$

$\sinh(iz) = \frac{e^{iz}-e^{-iz}}{2} = i\sin z \Rightarrow \sin(iz) = i\sinh z$

can be used to derive identities for ~~the~~ hyperbolic functions using trig identities, e.g.,

$\sinh(z_1+z_2) = -i\sin i(z_1+z_2) = -i\left[\sin(iz_1)\cos(iz_2) + \sin(iz_2)\cos(iz_1)\right] =$

$= -i\left[i\sinh z_1 \cosh z_2 + i\sinh z_2 \cosh z_1\right] = \sinh z_1 \cosh z_2 + \sinh z_2 \cosh z_1$

or $\sin^2 z + \cos^2 z = 1 \Rightarrow \sin^2(iz) + \cos^2(iz) = 1 \Rightarrow \cosh^2 z - \sinh^2 z = 1$

Ex. Solve $\cosh z = 0 \Rightarrow \frac{e^{z}+e^{-z}}{2} = 0 \Rightarrow e^{2z} = -1 = e^{\pi i + 2\pi i k} \Rightarrow$

$z = \frac{1}{2}\pi i + \pi i k$, $k = 0, \pm 1, \pm 2, \ldots$