Path so far

Define generator of rotations

$$\mathcal{D}_{\hat{n}}(\epsilon) \equiv \overleftrightarrow{1} - \frac{i}{\hbar}\left(\vec{J}\cdot\hat{n}\right)\epsilon$$

defines $\vec{J}$

(use $\vec{L}$ as an <u>analogy</u> only)

Consider general rotation in space

$$\Rightarrow \underbrace{[J_i, J_j] = i\hbar\epsilon_{ijk} J_k}$$

<u>N=1:</u> cannot realize

<u>N=2:</u> $S_i = \frac{1}{2}\hbar\sigma_i$ satisfy

Pauli matrices

$$S_z\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}1\\0\end{pmatrix}, \quad S_z\begin{pmatrix}0\\1\end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix}0\\1\end{pmatrix}$$

spin $\frac{1}{2}$

spin up spin down

$$\vec{S}^2 = \underbrace{\tfrac{3}{4}\hbar^2}\overleftrightarrow{1}$$

eigenvalue for all vectors in the 2x2 space

<u>N≥3:</u> consider this now

(actually consider N≥2 all at once, i.e. including spin $\frac{1}{2}$)

Eigenkets and Eigenvalues of Angular Momentum

- Use no differential equations
- Use only $[J_i, J_j] = i\epsilon_{ijk} J_k$

$\underline{[J_i, J_j] \neq 0 \text{ for } i \neq j} \longrightarrow$ ∴ cannot make simultaneous eigenkets of J_x, J_y

$$[\vec{J}^2, J_z] = [J_x^2 + J_y^2 + J_z^2, J_z]$$

$$= J_x J_x J_z - J_z J_x J_x + J_y J_y J_z - J_z J_y J_y + \cancel{[J_z^2, J_z]}$$

add 0 $\rightsquigarrow$

$$- J_x J_z J_x + J_x J_z J_x - J_y J_z J_y + J_y J_z J_y$$

$$= J_x \underbrace{[J_x, J_z]}_{-i\hbar J_y} + \underbrace{[J_x, J_z]}_{-i\hbar J_y} J_x + J_y \underbrace{[J_y, J_z]}_{i\hbar J_x} + J_y \underbrace{[J_z, J_y]}_{i\hbar J_x}$$

$$= -i\hbar J_x J_y - i\hbar J_y J_x + i\hbar J_y J_x + i\hbar J_y J_x$$

$$= 0$$

$\longrightarrow$ ∴ can make simultaneous eigenkets of $\vec{J}^2$ and J_z $\leftarrow$ or J_x, J_y

Make simultaneous eigenkets of $\vec{J}^2$ and J_z

$$\vec{J}^2 |a\, b\rangle = a |a\, b\rangle$$

$$J_z |a\, b\rangle = b |a\, b\rangle$$

a, b real since $\vec{J}^2, J_z$ are Hermitian

Ladder Operators: $J_\pm = J_x \pm i J_y$

$$[J_+, J_-] = [J_x + iJ_y, J_x - iJ_y]$$

$$= i \underbrace{[J_y, J_x]}_{-i\hbar J_z} - i \underbrace{[J_x, J_y]}_{i\hbar J_z}$$

$$\boxed{[J_+, J_-] = 2\hbar J_z}$$

$$\boxed{[J_z, J_\pm] = [J_z, J_x \pm iJ_y] = [J_z, J_x] \pm i[J_z, J_y] = i\hbar J_y \pm \hbar J_x = \pm\hbar[J_x \pm iJ_y] = \pm\hbar J_\pm}$$

$$[J_z, J_x \pm iJ_y] = \underbrace{[J_z, J_x]}_{i\hbar J_y} \pm i\underbrace{[J_z, J_y]}_{-i\hbar J_x}$$

$$\boxed{[\vec{J}^2, J_\pm] = [\vec{J}^2, J_x \pm iJ_y] = 0}$$

Consider the state vectors $J_\pm |a\, b\rangle$

operator makes one state vector from another

$$J_z\{J_\pm |a\, b\rangle\} = \Big(J_\pm J_z + \underbrace{[J_z, J_\pm]}_{\pm\hbar J_\pm}\Big)|a\, b\rangle$$

$$= J_\pm b\, |a\, b\rangle \pm \hbar J_\pm |a b\rangle$$

$$= \underbrace{(b \pm \hbar)}_{J_z \text{ eigenvalue increased by } \hbar}\{J_\pm |a b\rangle\}$$

$$\vec{J}^2\{J_\pm |a b\rangle\} = J_\pm \vec{J}^2 |a b\rangle$$

$$= a\{J_\pm |a\, b\rangle\}$$

$\vec{J}^2$ eigenvalue unchanged

For any state vector: $\langle\phi|\phi\rangle \geq 0$ ← postulate of Q.M.

Use $|\phi_\pm\rangle = J_\pm^\dagger|ab\rangle \rightsquigarrow \langle\phi_\pm| = \langle ab|J_\pm$

$\therefore \langle ab|J_\pm J_\pm^\dagger|ab\rangle \geq 0$

$\therefore \langle ab|\underbrace{J_+J_+^\dagger + J_-J_-^\dagger}|ab\rangle \geq 0$

$$\boxed{\begin{aligned} J_+J_+^\dagger + J_-J_-^\dagger &= (J_x+iJ_y)(J_x-iJ_y) + (J_x-iJ_y)(J_x+iJ_y) \\ &= 2(J_x^2+J_y^2) + i(\cancel{J_yJ_x} - \cancel{J_xJ_y} - \cancel{J_yJ_x} + \cancel{J_xJ_y}) \\ &= 2(\vec{J}^2 - J_z^2) \end{aligned}}$$

$\langle ab|\cancel{2}(\vec{J}^2 - J_z^2|ab\rangle \geq 0$

$a - b^2 \geq 0$

$\underline{b^2 \leq a} \rightsquigarrow a$ is non-neg.

Since $b^2 \leq a \longrightarrow$ cannot raise or lower b forever

$J_+|a\,b_{max}\rangle = 0$ and $J_-|a\,b_{min}\rangle = 0$

$\underbrace{J_-J_+}|a\,b_{max}\rangle = 0$ | $\underbrace{J_+J_-}|a\,b_{min}\rangle = 0$

$$\boxed{\begin{aligned} J_-J_+ &= (J_x-iJ_y)(J_x+iJ_y) = J_x^2+J_y^2+i\underbrace{[J_x,J_y]}_{i\hbar J_z} = J_x^2+J_y^2-\hbar J_z \\ &= \vec{J}^2 - J_z^2 - \hbar J_z \\ J_+J_- &= (J_x+iJ_y)(J_x-iJ_y) = J_x^2+J_y^2+i\underbrace{[J_y,J_x]}_{-i\hbar J_z} \\ &= \vec{J}^2 - J_z^2 + \hbar J_z \end{aligned}}$$

$(\vec{J}^2 - J_z^2 - \hbar J_z)|a\,b_{max}\rangle = 0$ | $(\vec{J}^2 - J_z^2 + \hbar J_z)|a\,b_{min}\rangle = 0$

$a = b_{max}(b_{max}+\hbar)$ | $a = b_{min}(b_{min}-\hbar) = (-b_{min})(-b_{min}+\hbar)$

$b_{max} = -b_{min}$

Repeatedly raise $|a\, b_{min}\rangle$ with J_+

$$b_{min} \to b_{min} \to b_{min} + \hbar \to \cdots$$

pos. integer

To stop from raising forever need $b_{max} = b_{min} + n\hbar$

$$b_{max} = -b_{max} + n\hbar$$

$$\boxed{\begin{aligned} b_{max} &= \tfrac{1}{2} n\hbar \\ b_{min} &= -\tfrac{1}{2} n\hbar \end{aligned}}$$

$n = 0, 1, 2 \cdots$

Let $j \equiv \frac{1}{2} n$

$\longrightarrow$ possible j values

$$j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots$$

$$\begin{aligned} a &= b_{max}(b_{max} + \hbar) \\ &= \tfrac{1}{2} n\hbar \left(\tfrac{1}{2} n\hbar + \hbar\right) \\ &= j\hbar (j\hbar + \hbar) \\ &= \hbar^2 j(j+1) \end{aligned}$$

$\leftarrow$ eigenvale of $\vec{J}^2$

Let $b \equiv m\hbar$:

$$b_{min} \le b \le b_{max}$$

$$-j\cancel{\hbar} \le m\cancel{\hbar} \le j\cancel{\hbar}$$

$$-j \le m \le j$$

$$\therefore\ m = -j, -j+1, \ldots, j$$

Normal notation:

$$\boxed{\begin{aligned} &\vec{J}^2 |jm\rangle = \hbar^2 j(j+1) |jm\rangle \\ &J_z |jm\rangle = \hbar m |jm\rangle \\ &j = 0, \tfrac{1}{2}, 1, \ldots \qquad m = -j, -j+1, \ldots, j \end{aligned}}$$

Normalization for $J_\pm$ Matrix Elements

$$J_\pm |jm\rangle = C_{jm}^{(\pm)} |j\ m\pm1\rangle \rightsquigarrow \underline{C_{jm}^{\pm} = \langle j\ m\pm1 | J_\pm | jm\rangle}$$

Get the constant $C_{jm}^{(+)}$

$$J_+|jm\rangle = C_{jm}^{(+)}|j\ m+1\rangle \rightsquigarrow \langle jm|J_+^\dagger = C_{jm}^{(+)*}\langle j\ m+1|$$

$$\langle jm|\underbrace{J_+^\dagger J_+}_{\vec{J}^2 - J_z^2 - \hbar J_z \text{ (showed earlier)}}|jm\rangle = |C_{jm}^{(+)}|^2$$

$$\hbar^2 j(j+1) - \hbar^2 m^2 - \hbar\,\hbar m$$

$$|C_{jm}^{(+)}|^2 = \hbar^2[j(j+1) - m(m+1)]$$
$$= \hbar^2[(j-m)(j+m+1)]$$

$j(j+1) - m(m+1) = j^2 + j - m^2 - m = j^2 + mj + j - mj - m^2 - m = j(j+m+1) - m(j+m+1)$

Free to choose phase, ∴ choose simple $C_{jm}^{+} = \hbar\sqrt{(j-m)(j+m+1)}$

$$J_+|jm\rangle = \hbar\sqrt{(j-m)(j+m+1)}\,|j\ m+1\rangle$$

Relate to $C_{jm}^{(-)}$
$$= \langle j\ m-1|J_-|jm\rangle$$
$$= \langle jm|J_-^\dagger|j\ m-1\rangle^*$$
$$= \langle jm|J_+|j\ m-1\rangle^*$$
$$= (C_{j\,m-1}^{+})^*$$

$$= \hbar\sqrt{(j-\{m-1\})(j+\{m-1\}+1)}$$
$$= \hbar\sqrt{(j-m+1)(j+m)}$$

Together

$$J_\pm|jm\rangle = \hbar\sqrt{(j\mp m)(j\pm m+1)}\,|j\ m\pm1\rangle$$