Nia Burrell
(worked w/ Fangjun Zhu)

1. Brown eqns 18.87 - 89 (p. 275) :

$$[S_i, S_j] = \varepsilon_{ijk} S_k$$
$$[S_i, K_j] = \varepsilon_{ijk} K_k$$
$$[K_i, K_j] = -\varepsilon_{ijk} S_k$$

S: rotations , K: boosts

(i) $[S_i, S_j] = \varepsilon_{ijk} S_k = S_i S_j - S_j S_i$

(ii) $[S_i, K_j] = \varepsilon_{ijk} K_k = S_i K_j - K_j S_i$

(iii) $[K_i, K_j] = -\varepsilon_{ijk} S_k = K_i K_j - K_j K_i$

} matrices worked out in Mathematica (next page)

* commutators w/ repeated indices = 0

2. Correct & extend Chapter 17.2, chain rule

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i) Newton's Law: $F_j = m\ddot{x}_j$

Galilean transform: $x_1' = x_1 - vt$, take derivative

$\dot{x}_1' = \dot{x}_1 - v$, another derivative

$\ddot{x}_1' = \ddot{x}_1$, $\ddot{x}_2' = \ddot{x}_2$, $\ddot{x}_3' = \ddot{x}_3$

$\Rightarrow \boxed{F_j' = m\ddot{x}_j' = F_j \text{, invariant}}$

eqn 17.6: $\dfrac{\partial}{\partial x_1} = \dfrac{\partial}{\partial x_1'} + \dfrac{1}{v}\dfrac{\partial}{\partial t'}$, $\boxed{\text{not correct}}$

corrected version:

$$\frac{\partial\psi}{\partial t'} = \frac{\partial\psi}{\partial t} + v\frac{\partial\psi}{\partial x_1}$$

ii) Correct eqn. 17.12

eqn 17.6: $\dfrac{\partial\psi}{\partial t'} = \dfrac{\partial\psi}{\partial t} + v\dfrac{\partial\psi}{\partial x_1}$

$$\frac{\partial^2\psi}{\partial t'^2} = \frac{\partial^2\psi}{\partial t^2} + 2v\frac{\partial^2\psi}{\partial x_1\partial t} + v^2\frac{\partial^2\psi}{\partial x_1^2}$$

If $\left\{\nabla^2 - \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}\right\}\psi = 0$

then $\left\{\nabla'^2 - \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t'^2}\right\}\psi = 2v\dfrac{\partial^2\psi}{\partial x_1\partial t} + v^2\dfrac{\partial^2\psi}{\partial x_1^2} \neq 0 \Rightarrow \boxed{\text{not invariant}}$

iii) Lorentz boost

eqn. 17.23-26: $x_0' = \gamma(x_0 - \beta x_1) = \gamma x_0 - \beta\gamma x_1$

$x_1' = \gamma(x_1 - \beta x_0) = \gamma x_1 - \beta\gamma x_0$

$x_2' = x_2$

$x_3' = x_3$

$$\frac{\partial}{\partial x_0}\left(\gamma\frac{\partial}{\partial x_0'} - \beta\gamma\frac{\partial}{\partial x_1'}\right)\psi - \frac{\partial}{\partial x_1}\left(\gamma\frac{\partial}{\partial x_1'} - \beta\gamma\frac{\partial}{\partial x_0'}\right)\psi - \frac{\partial^2\psi}{\partial x_2'^2} - \frac{\partial^2\psi}{\partial x_3'^2}$$

$$= \gamma^2\frac{\partial^2\psi}{\partial x_0'^2} - \cancel{\beta\gamma^2\frac{\partial^2\psi}{\partial x_0'\partial x_1'}} - \cancel{\beta\gamma^2\frac{\partial^2\psi}{\partial x_1'\partial x_0'}} + \beta^2\gamma^2\frac{\partial^2\psi}{\partial x_1'^2} - \beta^2\gamma^2\frac{\partial^2\psi}{\partial x_0'^2} + \cancel{\beta\gamma^2\frac{\partial^2\psi}{\partial x_1'\partial x_0'}}$$

$$+ \cancel{\beta\gamma^2\frac{\partial^2\psi}{\partial x_0'\partial x_1'}} - \gamma^2\frac{\partial^2\psi}{\partial x_1'^2} - \frac{\partial^2\psi}{\partial x_2'^2} - \frac{\partial^2\psi}{\partial x_3'^2}$$

$$= \underbrace{(\gamma^2 - \beta^2\gamma^2)}_{1}\frac{\partial^2\psi}{\partial x_0'^2} - \underbrace{(\gamma^2 - \beta^2\gamma^2)}_{1}\frac{\partial^2\psi}{\partial x_1'^2} - \frac{\partial^2\psi}{\partial x_2'^2} - \frac{\partial^2\psi}{\partial x_3'^2}$$

$= 0 \Rightarrow \boxed{\text{invariant}}$

3. 4x4 matrix forms

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i) $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$, $A^\mu = (\phi, \vec{A})$, $\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right)$

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla\phi$$

$$= \left(-\frac{1}{c}\frac{\partial A_x}{\partial t} - \frac{\partial \phi}{\partial x},\ -\frac{1}{c}\frac{\partial A_y}{\partial t} - \frac{\partial \phi}{\partial y},\ -\frac{1}{c}\frac{\partial A_z}{\partial t} - \frac{\partial \phi}{\partial z}\right)$$

$$= \left(-\partial^0 A^1 + \partial^1 A^0,\ -\partial^0 A^2 + \partial^2 A^0,\ -\partial^0 A^3 + \partial^3 A^0\right) = (E_x, E_y, E_z)$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z},\ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x},\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$= \left(-\partial^2 A^3 + \partial^3 A^2,\ -\partial^3 A^1 + \partial^1 A^3,\ -\partial^1 A^2 + \partial^2 A^1\right) = (B_x, B_y, B_z)$$

$$F^{\mu\nu} = \begin{bmatrix} F^{00} & F^{01} & F^{02} & F^{03} \\ F^{10} & F^{11} & F^{12} & F^{13} \\ F^{20} & F^{21} & F^{22} & F^{23} \\ F^{30} & F^{31} & F^{32} & F^{33} \end{bmatrix}$$

$$\boxed{F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}}$$

ii) $F^*_{\mu\nu} \equiv -\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} = -\varepsilon_{\mu\nu\alpha\beta}\partial^\alpha A^\beta$

<u>each matrix element</u>:

$F^*_{00} = 0 = F^*_{11} = F^*_{22} = F^*_{33}$ b/c repeating index

$F^*_{01} = -\varepsilon_{0123}\partial^2 A^3 - \varepsilon_{0132}\partial^3 A^2 = -\partial^2 A^3 + \partial^3 A^2 = B_x$

$F^*_{02} = -\varepsilon_{0213}\partial^1 A^3 - \varepsilon_{0231}\partial^3 A^1 = \partial^1 A^3 - \partial^3 A^1 = B_y$

$F^*_{03} = -\partial^1 A^2 + \partial^2 A^1 = B_z$

$F^*_{10} = -\varepsilon_{1023}\partial^2 A^3 - \varepsilon_{1032}\partial^3 A^2 = +\partial^2 A^3 - \partial^3 A^2 = -B_x$

$F^*_{12} = -\varepsilon_{1203}\partial^0 A^3 - \varepsilon_{1230}\partial^3 A^0 = -\partial^0 A^3 + \partial^3 A^0 = E_z$

$F^*_{13} = -\varepsilon_{1302}\partial^0 A^2 - \varepsilon_{1320}\partial^0 A^2 = +\partial^0 A^2 - \partial^2 A^0 = -E_y$

$F^*_{20} = -\varepsilon_{2013}\partial^1 A^3 - \varepsilon_{2031}\partial^3 A^1 = -\partial^1 A^3 + \partial^3 A^1 = -B_y$

$F^*_{21} = -\varepsilon_{2103}\partial^0 A^3 - \varepsilon_{2130}\partial^3 A^0 = \partial^0 A^3 - \partial^3 A^0 = -E_z$

$F^*_{23} = -\varepsilon_{2301}\partial^0 A^1 - \varepsilon_{2310}\partial^1 A^0 = -\partial^0 A^1 + \partial^1 A^0 = E_x$

$F^*_{30} = -\varepsilon_{3012}\partial^1 A^2 - \varepsilon_{3021}\partial^2 A^1 = \partial^1 A^2 - \partial^2 A^1 = -B_z$

$F^*_{31} = -\varepsilon_{3102}\partial^0 A^2 - \varepsilon_{3120}\partial^2 A^0 = -\partial^0 A^2 + \partial^2 A^0 = E_y$

$F^*_{32} = \partial^0 A^1 - \partial^1 A^0 = -E_x$

$$\Longrightarrow \boxed{F^*_{\mu\nu} = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{bmatrix}}$$