Uncertainty Principle

- We ~~claimed~~ showed that if $G|\varphi_i\rangle = g_i|\varphi_i\rangle$ & $[G,H]=0$, then H acting on $|\varphi_i\rangle$ $(H|\varphi_i\rangle = E_i|\varphi_i\rangle)$ either ① returns $|\varphi_i\rangle$ or ② carries $|\varphi_i\rangle$ to some $|\varphi_j\rangle$ where $g_i = g_j$

- Furthermore, we claimed that basis $|\varphi_i\rangle$ can be chosen s.t. $H|\varphi_i\rangle = E_i|\varphi_i\rangle$, even if there are degeneracies in H and/or G

$\Rightarrow$ i.e. ~~states~~ if $[G,H]=0$, a full ~~states~~ basis ~~exist~~ of states exist w/ simultaneous definite values of G, H

- True in general of $[A,B]=0$

2 – On the other hand, ~~Conversely~~, if $[A,B] \neq 0$, then no states exist with simult. def. values of A & B

(Dropping $\tilde{x}$ now)

- For instance $[x,p] \neq 0$

- Obviously true :

$$[A,B]|\varphi_i\rangle = (AB - BA)|\alpha_i\rangle = AB|\varphi_i\rangle - BA|\varphi_i\rangle$$

Suppose $A|\varphi_i\rangle = a_i|\varphi_i\rangle$ + $B|\varphi_i\rangle = b_i|\varphi_i\rangle$

$$a_i b_i|\varphi_i\rangle - b_i a_i|\varphi_i\rangle = 0$$

But this is a contradiction if $[A,B] \neq 0$

$\Rightarrow$ Cannot "simultaneously diagonalize" both operators.

~~i.e. ...~~ Even stranger, cannot find a single state which has def value of both

- This leads to uncert. princ. Let's work our way there

- Expectation value $\langle O\rangle = \langle\psi|O|\psi\rangle$

$$\langle x\rangle = \langle\psi|x|\psi\rangle, \quad \langle p\rangle = \langle\psi|p|\psi\rangle$$

why is this average val. of many measurements?

Probability of measuring one value given by $|\text{coef.}|^2$

$\langle p\rangle$ is exp. value

eg. $\psi = c_1|p_1\rangle + c_2|p_2\rangle$

$$\langle p\rangle = \frac{|c_1|^2 p_1 + |c_2|^2 p_2}{|c_1|^2 + |c_2|^2} = |c_1|^2 p_1 + |c_2|^2 p_2$$

But $\langle\psi|p|\psi\rangle = c_1 p_1 |p_1\rangle + c_2 p_2 |p_2\rangle$ $(c_1^*\langle p_1| + c_2^*\langle p_2|)$

- We will first prove that for any Hermitian operators A & B

$$\langle A^2\rangle\langle B^2\rangle \geq \frac{1}{4}\left|\langle [A, B]\rangle\right|^2$$

- For any complex α

$$\langle (A+\alpha B)\psi \,|\, (A+\alpha B)\psi\rangle \geq 0 \quad \text{since } \mathcal{H} \text{ has positive norm}$$

$$\Rightarrow \langle \psi | (A+\alpha B)^\dagger (A+\alpha B)|\psi\rangle \geq 0$$

$$\langle \psi | (A+\alpha^* B)(A+\alpha B)|\psi\rangle \geq 0$$

$$\langle A^2\rangle + \alpha^*\langle BA\rangle + \alpha\langle AB\rangle + |\alpha|^2\langle B^2\rangle \geq 0$$

$$\langle BA\rangle = \langle B\psi | A\psi\rangle = \langle A\psi | B\psi\rangle^* = \langle AB\rangle^*$$

(Note that although $\langle A\rangle$ & $\langle B\rangle$ are real, $\langle AB\rangle$ is not necessarily if $[A,B]\neq 0$)

& $\langle B^2\rangle^* = \langle B^2\rangle$

$$\Rightarrow \langle A^2\rangle + \alpha\langle AB\rangle + \alpha^*\langle AB\rangle^* + |\alpha|^2\langle B^2\rangle \geq 0$$

We want to minimize the LHS wrt to α.

You get the right result if you take $\frac{\partial}{\partial\alpha}(\ldots)$ and pretend that α^* is independent, but we will use a more simple-minded approach.

~~but I don't understand why that has to work.~~

~~So, I'll do it the longer way~~ ...

$\alpha = a + ib$

$\langle AB \rangle = c + id$ where those 4 coefficients are real

$$\Rightarrow \langle A^2 \rangle + 2ac - 2bd + (a^2+b^2)\langle B^2 \rangle \geq 0$$

$$\frac{\partial}{\partial a}(\ldots) = 2c + 2a\langle B^2 \rangle = 0 \Rightarrow a = \mathrm{Re}\,\alpha = \frac{-\mathrm{Re}\langle AB \rangle}{\langle B^2 \rangle}$$

$$\frac{\partial}{\partial b}(\ldots) = -2d + 2b\langle B^2 \rangle \geq 0 \Rightarrow b = \mathrm{Im}\,\alpha = \frac{\mathrm{Im}\langle AB \rangle}{\langle B^2 \rangle}$$

$$\Rightarrow \alpha = \frac{-\langle AB \rangle^*}{\langle B^2 \rangle}$$

(From inequality in original form)

$$\langle A^2 \rangle - \frac{\langle AB \rangle\langle AB \rangle^*}{\langle B^2 \rangle} - \cancel{\frac{\langle AB \rangle^*\langle AB \rangle}{\langle B^2 \rangle}} + \cancel{\frac{|\langle AB \rangle^*|^2}{\langle B^2 \rangle}} \geq 0$$

$$\Rightarrow \boxed{\langle A^2 \rangle\langle B^2 \rangle \geq |\langle AB \rangle|^2}$$

anticommutator

Now $\langle AB \rangle = \frac{1}{2}\langle AB - BA \rangle + \frac{1}{2}\langle AB + BA \rangle$ (This is a trick to get the commutator to appear)

$$\Rightarrow |\langle AB \rangle|^2 = \frac{1}{4}|\langle AB - BA \rangle|^2 + \frac{1}{4}|\langle AB + BA \rangle|^2$$

$$+ \frac{1}{4}\Big[\underbrace{\langle AB - BA \rangle^*}_{-\langle AB - BA \rangle}\langle AB + BA \rangle + \langle AB - BA \rangle\underbrace{\langle AB + BA \rangle^*}_{\langle AB + BA \rangle}\Big]$$

So 2nd line vanishes

- ~~2nd line of above eqn is:~~

$$\cancel{\tfrac{1}{4}\{\langle AB-BA\rangle\langle AB+BA\rangle^* + \langle AB-BA\rangle^*\langle AB+BA\rangle\} \geq 0}$$

- Since $|\langle AB + BA\rangle|^2 \geq 0$, we finally have (taking only the first term + converting to inequality)

$$\underline{\langle A^2\rangle\langle B^2\rangle \geq \frac{1}{4}|\langle [A,B]\rangle|^2} \quad \text{as claimed}$$

We have specified that we are interested in cases where $[A,B] \neq 0$. $AB+BA$ might be 0 or anything else. We can also say that $\langle A^2\rangle\langle B^2\rangle \geq \frac{1}{4}|\langle AB+BA\rangle|^2$, but that is not as interesting. Doesn't carry same physical meaning as $[A,B]$ – difference obtained by reversing measurements.

~~But commutator is ≠ 0 also. Why not have inequality for anticommutator on RHS? Just not useful later in some way?~~

- Now put $A = x - c$, $B = p - d$

operators

where c & d are any constants

- $[A,B] = [x-c, p-d] = [x,p] = i\hbar$

- In particular, take $c = \langle x\rangle$, $d = \langle p\rangle$

$\langle \Delta x\rangle = 0$

$\langle (\Delta x)^2\rangle > 0$

$\Rightarrow A = x - \langle x\rangle \equiv \Delta x \qquad B = p - \langle p\rangle \equiv \Delta p$

(Deviation of given measurement from mean value) ~~≡ width of distribution~~

~~$(\Delta x)^2$ describes~~

~~RMS ... Maybe just for Gaussian?~~

$\Rightarrow \overset{A^2}{\langle(\Delta x)^2\rangle}, \overset{B^2}{\langle(\Delta p)^2\rangle}$ are positive quantities describing square of width of distribution

* These are the ~~standard deviation~~ variance of measurements of x & p

$$\Rightarrow \boxed{\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle \geq \frac{1}{4}\hbar^2} \qquad \boxed{\Delta x_{SD}\Delta p_{SD} \geq \frac{\hbar}{2}}$$

(Often a different def is used: $\Delta x_{SD} = \sqrt{\langle(\Delta x)^2\rangle}$ (just dist width)

(common ← standard deviation

$\Rightarrow$ Similar uncert. princ., just w/ different RHS

for any non-commuting observables

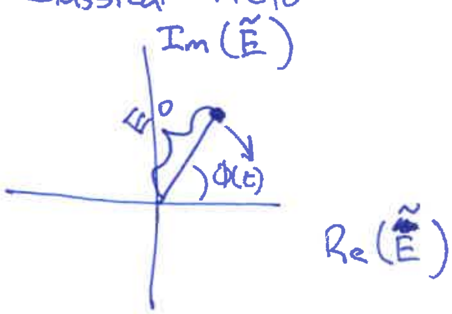
$\Rightarrow$ Spread in $x \to \infty$ as spread in $p \to 0$ & vice versa.

- Heisenberg uncertainty ~~relationship~~ principle.
- For x & p in position space, already argued qualitatively based on Fourier compositions. But more general than that.
- We made a gaussian wavepacket earlier where $\sqrt{\langle(\Delta x)^2\rangle}\sqrt{\langle(\Delta p)^2\rangle} = \frac{\hbar}{2}$. So condition can be saturated but not violated.

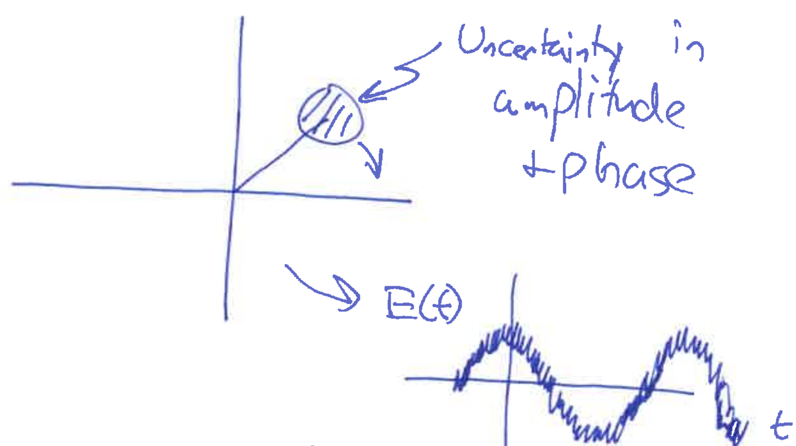
This goes for any non-commuting observables. Consider ~~a~~ the amplitude & phase of an electromagnetic wave.
From Quantum Electrodynamics, it can be shown that $[\phi, n] \neq 0$, where ϕ is the operator for phase measurements & n is the photon number (i.e. amplitude) operator. $E = \mathrm{Re}\,\tilde{E}(t)$

$$= E_0 \,\mathrm{Re}\left(e^{-i(\omega t+\phi)}\right) = E_0\,\mathrm{Re}\left(e^{-i\phi(t)}\right)$$

Classical Field



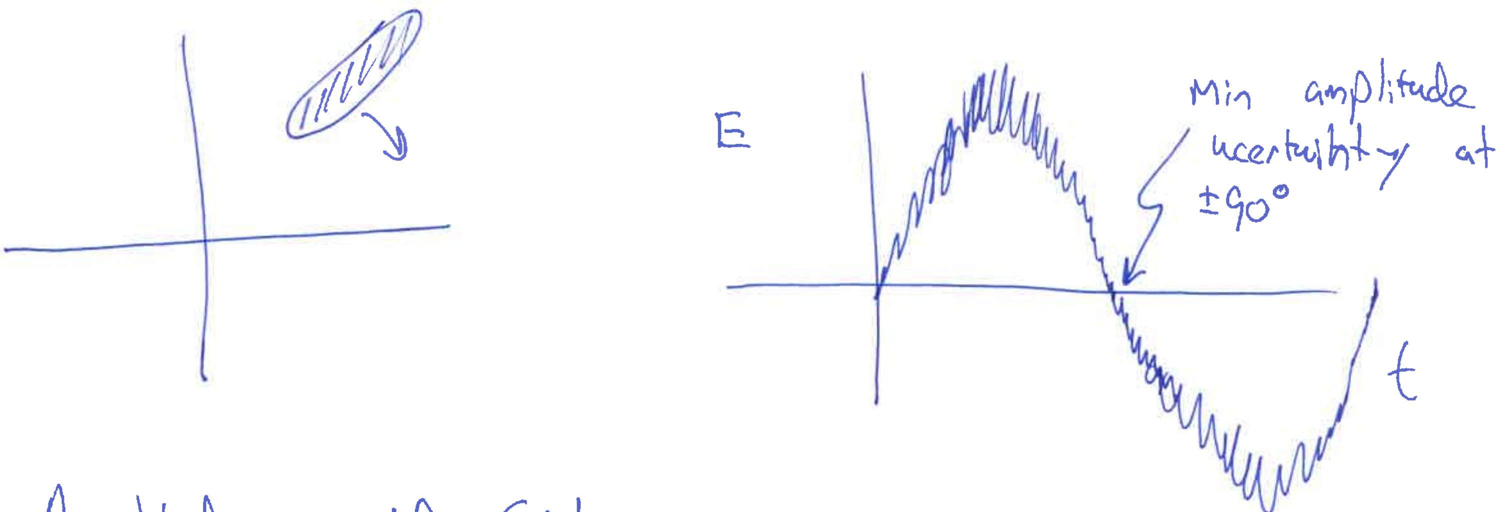
Quantum Coherent State



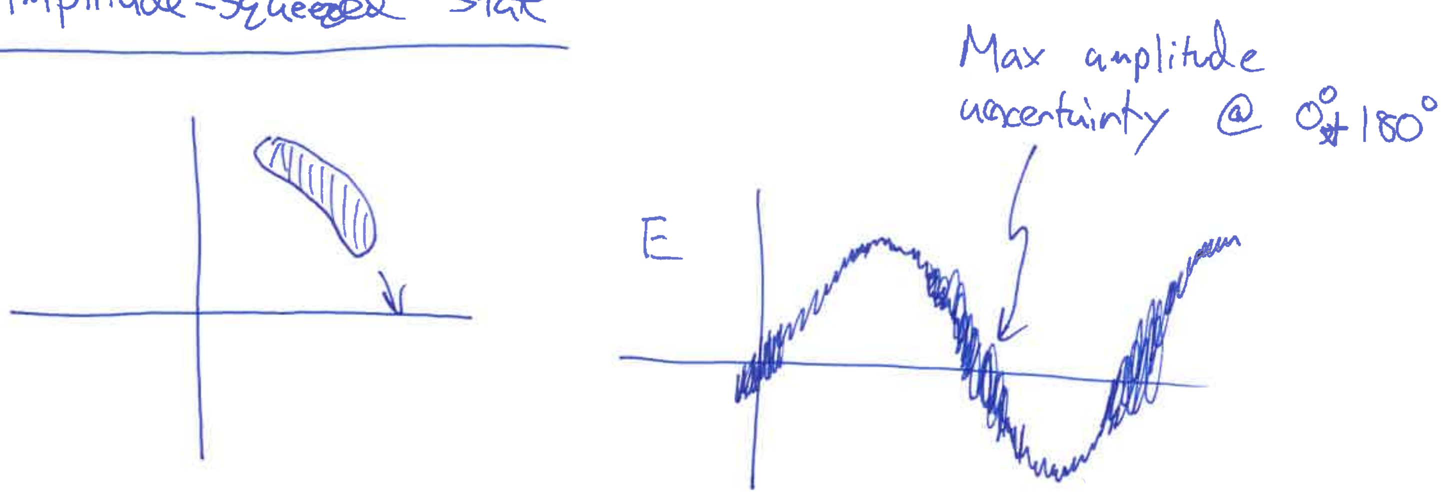
(Uniform amplitude uncertainty vs. time)

But we are allowed to squeeze the uncertainty into one or the other observable, just as we do for x + p.

Phase-squeezed State



Amplitude-squeezed State



So, for instance the amplitude-squeezed state still has amplitude uncertainty, particularly at some phases. But because its uncertainty is around E = 0, the $\langle(\Delta E)^2\rangle$ is reduced relative to the coherent state case.