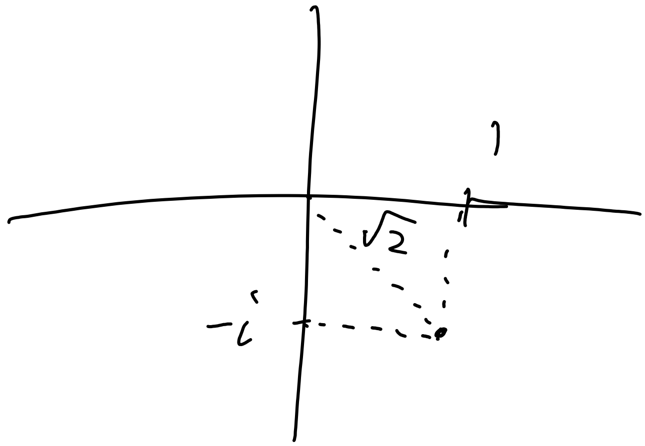
HW 1

Tuesday, April 13, 2021 6:15 PM

1.) $(1-i)^3$



$$= \left(\sqrt{2}\, e^{-i\frac{\pi}{4}}\right)^3$$

$$= 2^{3/2} e^{-i3\frac{\pi}{4}} = 2^{3/2}\left[\cos\left(-\frac{\pi}{4}\right) - i\sin\left(-\frac{\pi}{4}\right)\right]$$

$$= 2^{3/2}\left[\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right] = 2\sqrt{2}\left[\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}\right]$$

$$= 2 + i2$$

$$\frac{2+3i}{3-2i} \cdot \frac{3+2i}{3+2i} = \frac{6+4i+9i-6}{9+6i-6i+6}$$

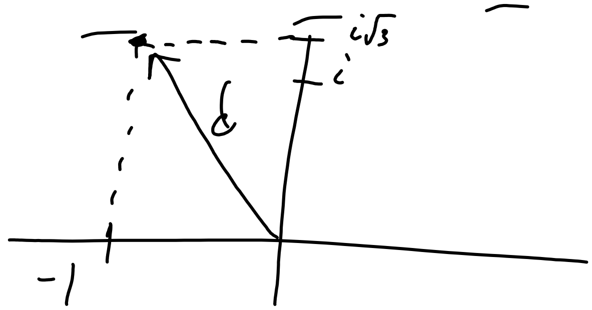
$$= \frac{13i}{15}$$

$$(1+i)^{25} = \left(\sqrt{2}\, e^{i\frac{\pi}{4}}\right)^{25} = 2^{\frac{25}{2}} e^{i25\pi/4}$$

$$= 2^{25/2} e^{i\frac{\pi}{4}} = 2^{25/2}\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]$$

$$= 2^{25/2}\left[\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right] = 2^{25/2}\frac{\sqrt{2}}{2} + i\,2^{25/2}\frac{\sqrt{2}}{2}$$

$$\frac{(-1+i\sqrt{3})^{15}}{(1-i)^{20}} \Rightarrow$$



$$d^2 = (-1)^2 + (i\sqrt{3})^2 = 1 + (-3) = 4$$

$$d = 2$$

$$\boxed{\begin{aligned} \cos\theta &= \frac{-1}{2} \\ \theta &= \cos^{-1}\left(\frac{-1}{2}\right) = \frac{2\pi}{3} \end{aligned}}$$

$$\Rightarrow \frac{\left(2e^{i2\frac{\pi}{3}}\right)^{15}}{\left(\sqrt{2}e^{i3\pi/4}\right)^{20}}$$

$$\frac{(-1-i\sqrt{3})^{15}}{(1+i)^{20}} = \frac{\left(2e^{i4\frac{\pi}{3}}\right)^{15}}{\left(\sqrt{2}e^{i\pi/4}\right)^{20}}$$

$$\frac{\left(2e^{i2\frac{\pi}{3}}\right)^{15}}{\left(\sqrt{2}\,e^{i3\frac{\pi}{4}}\right)^{20}} + \frac{\left(2e^{i4\frac{\pi}{3}}\right)^{15}}{\left(\sqrt{2}\,e^{i\frac{\pi}{4}}\right)^{20}}$$

$$\frac{2^{15}e^{i10\pi}}{2^{10}e^{i15\pi}} + \frac{2^{15}e^{i20\pi}}{2^{10}e^{i5\pi}}$$

$$2^5e^{-i5\pi} + 2^5e^{i15\pi}$$

$$2^5\{[\cos(5\pi) - i\sin(5\pi)] + [\cos(15\pi) + i\sin(15\pi)]\}$$

$$2^5[-1 - 0 + (-1) + 0]$$

$$2^5(-2) = -2^6$$

2.

$$z^3 = (x+iy)^3 = (x+iy)(x+iy)(x+iy)$$

$$= (x+iy)(x^2 - y^2 + 2ixy)$$

$$x^3 - xy^2 + 2ix^2y + ix^2y - iy^2 - 2xy^2$$

$= x^3 - xy^2 + 2ix^2 y + \ldots$

$= (x^3 - 3xy^2) + i(3x^2 y - y^2)$

$$\frac{\bar{z}}{z} = \frac{x - iy}{x + iy}\left(\frac{x - iy}{x - iy}\right) = \frac{x^2 - y^2 - 2ixy}{x^2 + y^2}$$

$$= \frac{x^2 - y^2}{x^2 + y^2} - i\left(\frac{2xy}{x^2 + y^2}\right)$$

$$\frac{z - i}{1 - i\bar{z}} = \frac{x + iy - i}{1 - i(x - iy)} = \frac{x + iy - i}{1 - ix - y}\left(\frac{1 - y + ix}{1 - y + ix}\right)$$

$$= \frac{x - xy + ix^2 + iy - iy^2 - xy - i + iy + x}{1 - y + \cancel{ix} - \cancel{ix} + \cancel{ixy} + x - y + y^2 - \cancel{ixy}}$$

$$= \frac{2x - 2xy + ix^2 + 2iy - iy^2 - i}{1 - 2y + x + y^2}$$

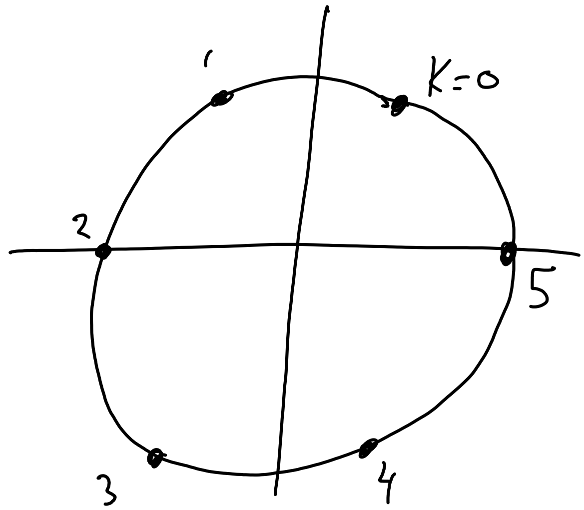
$$= \frac{2x - 2xy}{1 - 2y + x + y^2} + i\left(\frac{x^2 + 2y - y^2 - 1}{1 - 2y + x + y^2}\right)$$

3. $(1)^{1/6} = \left(1e^{i2\pi}\right)^{1/6} = \left(1e^{i(2\pi+2\pi k)}\right)^{1/6}$

$= 1e^{i\left(\frac{\pi}{3}+\frac{\pi}{3}k\right)}$ $\quad K = 0, 1, 2, 3, 5$

$K=0 \quad e^{i\pi/3} = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + \frac{i\sqrt{3}}{2}$

$K=1 \quad e^{i2\pi/3} = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$



$K=2 \quad e^{i\pi} = -1$

$K=3 \quad e^{i4\pi/3} = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$

$K=4 \quad e^{i5\pi/3} = \frac{1}{2} - i\sqrt{3}/2$

$K=5 \quad e^{i2\pi} = 1$

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4. $(-1)^{1/3} = \left(e^{i\pi}\right)^{1/3} = \left(e^{i(\pi+2\pi k)}\right)^{1/3}$

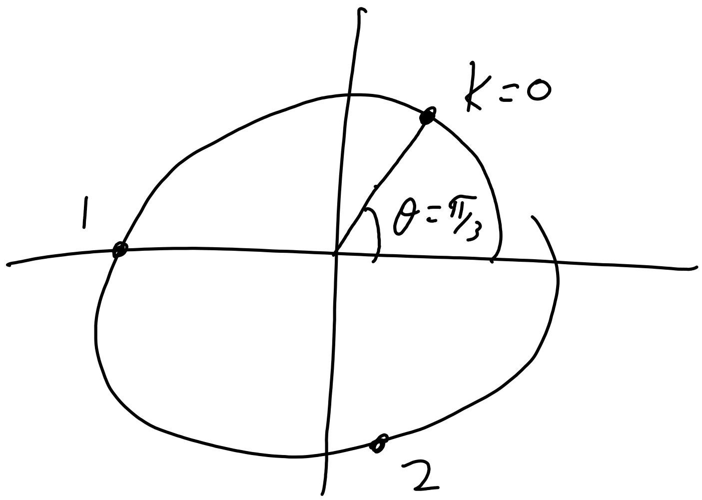
$= e^{i\left(\frac{\pi}{3}+\frac{2\pi}{3}k\right)}$ $\quad K = 0, 1, 2$

$K=0 \quad e^{i\pi/3} = \frac{1}{2} + \frac{i\sqrt{3}}{2}$

$K=1 \quad e^{i\pi} = -1$

$i5\pi/3$

$$k=2 \qquad e \qquad = \frac{1}{2} - \frac{i\sqrt{3}}{2}$$



$$(-16)^{1/4} = \left(16e^{i\pi}\right)^{1/4} = \left[16e^{i(\pi+2\pi k)}\right]^{1/4}$$

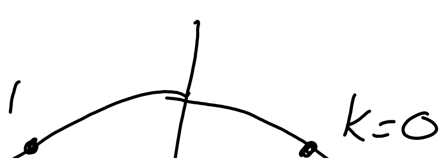
$$= 2e^{i\left(\frac{\pi}{4}+\frac{\pi}{2}k\right)} \qquad k = 0,1,2,3$$

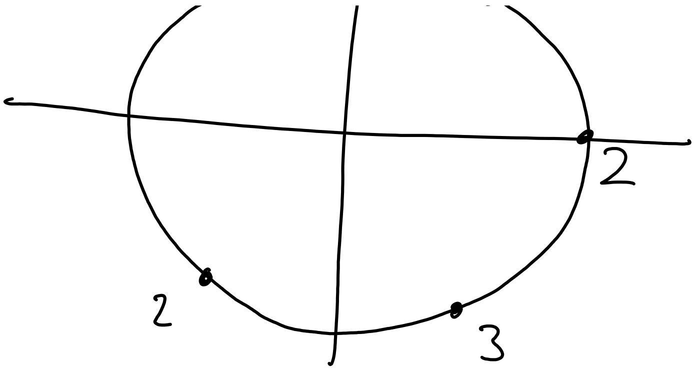
$$k=0 \qquad 2e^{i\frac{\pi}{4}} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

$$k=1 \qquad 2e^{i3\pi/4} = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

$$k=2 \qquad 2e^{i5\pi/4} = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

$$k=3 \qquad 2e^{i7\pi/4} = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$





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5) $P(z) = z^n + z^{n-1} + \dots + z^2 + z + 1$

$$P(z) - zP(z) = (z^n + z^{n-1} + \dots 1) - (z^{n+1} + z^n + \dots + z)$$

$$= 1 - z^{n+1} = P(z)(1-z)$$

$$P(z) = \frac{1 - z^{n+1}}{1-z} = 0$$

$$P = 1 - e^{i\theta(n+1)} = 0$$

$$e^{i\theta(n+1)} = 1 = \left| e^{i\theta(n+1)} \right|$$

6] $\operatorname{Im}\left(z + \frac{1}{z}\right) = 0$

$iy + \frac{1}{iy} = 0 \qquad iy = \frac{-1}{iy} = \frac{i}{y}$

$y^2 = 1 \qquad y = \pm 1$

7] $|z-1| = |z+i|$

$|x+iy-1| = |x+iy+i|$

$|x+i(y+i)| = |x+i(y+1)|$

$\cancel{x^2} + (y+i)^2 = \cancel{x^2} + (y+1)^2$

$y^2 + 1 + 2iy = y^2 + 1 + 2y$

$2iy = 2y \Rightarrow y = 0$

$z = x$ any real number

8] $z = x + iy = \mathrm{Re}\, z + \mathrm{Im}\, z$

from triangle inequality,

$$|z| \leq |\mathrm{Re}\, z| + |\mathrm{Im}\, z|$$