Midterm for PHYS 411, Fall 2017. Fri. Oct 27, 12-1

1: [10pts/20] A particle moves in a central force field given by

$$\boldsymbol{F} = -\frac{k\hat{\boldsymbol{r}}}{r^n}$$

where $k > 0$; $n \neq 1$; and r is the particle's distance from the center of the force field.

(a) What is the Lagrangian in cylindrical co-ordinates (r, θ)?

(b) Now, assume that the particle's orbit is nearly circular, with $r(t) = r_0 + r_1(t)$ (for constant r_0 and $r_1 \ll r_0$). What are the linearized equations of motion for r_1 and θ_1 (where θ_1 is the perturbed θ)?

(c) What is the frequency of oscillation, and what is the criterion for stable oscillations?

(a)

$$V = kr^{-n+1}\frac{1}{-n+1} \tag{1}$$

$$L = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - kr^{-n+1}\frac{1}{-n+1} \tag{2}$$

(b) eom are:

$$\ddot{r} = r\dot{\theta}^2 - kr^{-n} \tag{3}$$

$$\frac{d}{dt}r^2\dot{\theta} = 0 \tag{4}$$

The circular orbit has:

$$\dot{\theta}_0^2 = kr_0^{-n-1} \tag{5}$$

So, the linearized equations are:

$$\ddot{r_1} = r_1\dot{\theta}_0^2 + r_0 2\dot{\theta}_0\dot{\theta}_1 + nkr_0^{-n-1}r_1 \tag{6}$$

$$r_0^2\dot{\theta}_1 + \dot{\theta}_0 2r_0r_1 = 0 \tag{7}$$

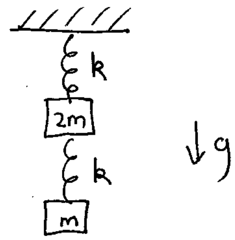
(c)

Combining the above, we have

$$
\begin{aligned}
-\omega^2 &= \dot\theta_0^2 + 2\dot\theta_0^2(-2) + nkr_0^{-n-1} && (8) \\
&= kr_0^{-n-1}(1-4+n) = (-3+n)kr_0^{-n-1} && (9)
\end{aligned}
$$

Stable if $n < 3$.

2: [10pts/20] Consider the following system. Motion is constrained to one dimension, along the vertical axis.



(a) Determine the Lagrangian L and the $\boldsymbol{T}$ and $\boldsymbol{V}$ matrices.

(b) Find the eigenfrequencies.

(c) Find the eigenvectors. (You do not need to normalize them).

Solution:

(a)

$$T = \frac{1}{2}(2m)\dot{z}_1^2 + \frac{m}{2}\dot{z}_2^2 \tag{10}$$

$$V = \frac{1}{2}kz_1^2 + \frac{k}{2}(z_1 - z_2)^2 \tag{11}$$

So,

$$T = \begin{pmatrix} 2m & 0 \\ 0 & m \end{pmatrix} \tag{12}$$

$$V = \begin{pmatrix} 2k & -k \\ -k & k \end{pmatrix} \tag{13}$$

(b)

$$-\omega^2 T + V = \begin{pmatrix} -2m\omega^2 + 2k & -k \\ -k & -m\omega^2 + k \end{pmatrix} . \tag{14}$$

Determinant gives (setting $m = k = 1$):

$$\omega^4 + 2\omega^2 - \frac{1}{2} = 0 \tag{15}$$

So, reinserting k and m,

$$\omega^2 = \frac{k}{m}(1 \pm \frac{1}{\sqrt{2}}) \tag{16}$$

(c) Eigenvectors are

$$\begin{pmatrix} -\omega^2 \frac{m}{k} + 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \mp 1 \\ \sqrt{2} \end{pmatrix} \tag{17}$$