Physics 414-2 Problem Set 1

April 1, 2022

Due: Friday, April 8 at 4 pm

1. Interference of plane waves. Consider two plane waves propagating in opposite directions with respective electric fields $\mathbf{E}_1 = E_0 e^{i(k_1 x - \omega_1 t)}\hat{\mathbf{z}}$ and $\mathbf{E}_2 = E_0 e^{i(-k_2 x - \omega_2 t)}\hat{\mathbf{z}}$, where $k_i = \omega_i / c$ for $i = 1, 2$. The intensity $I(x,t)$ of the combined waves is given by $I(x,t) = \frac{c}{8\pi}\mathbf{E}\cdot\mathbf{E}^\star$, where $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$ is the total electric field of the two waves. (Note: In this and other relevant homework problems, I encourage you to use Mathematica for your calculations, as this can save a significant amount of time. I'm happy to provide guidance on how to use Mathematica for this purpose.)

(a) Calculate $I(x,t)$.

(b) Let us define the quantities $\Delta\omega$ and k such that $\Delta\omega = \omega_1 - \omega_2$, $\omega_1 = ck + \Delta\omega/2$, and $\omega_2 = ck - \Delta\omega/2$ (note that k is the average of k_1 and k_2). Rewrite you answer from (a) in terms of k and $\Delta\omega$.

(c) Your answer from (b) should show that the intensity $I(x,t)$ includes a constant offset plus a spatial pattern that moves with a particular velocity that depends on k and $\Delta\omega$. What is this velocity?

2. Math Review: Complex exponential theorems.

(a) Prove Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$.

(b) Let $\mathbf{a}$ and $\mathbf{b}$ be complex vectors with time dependence that goes as $e^{-i\omega t}$, for oscillation frequency ω. That is, $\mathbf{a} = \mathbf{a}_0 e^{-i\omega t}$ and $\mathbf{b} = \mathbf{b}_0 e^{-i\omega t}$, where $\mathbf{a}_0$ and $\mathbf{b}_0$ are time-independent, complex vectors. Show that the the time-averaged value $\mathrm{Re}(\mathbf{a})\cdot\mathrm{Re}(\mathbf{b})$ of the dot product of the real parts of these vectors, which we denote $\langle \mathrm{Re}(\mathbf{a})\cdot\mathrm{Re}(\mathbf{b})\rangle$, approximately satisfies $\langle \mathrm{Re}(\mathbf{a})\cdot\mathrm{Re}(\mathbf{b})\rangle = \frac{1}{2}\mathrm{Re}\left[\mathbf{a}\cdot\mathbf{b}^\star\right]$ in the limit in which the averaging time is much longer than an oscillation period.

3. Sagnac effect. Figure 1 illustrates a Sagnac interferometer. In this problem, light is split and propagates in opposite directions along a planar contour of length $4L$, enclosing area $A = L^2$.

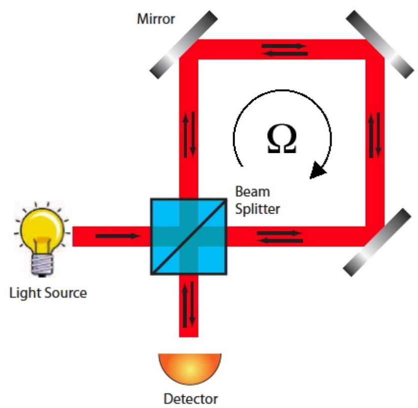

Figure 1: Illustration of a Sagnac interferometer.

If the whole apparatus (including the light source and detector) rotates with angular velocity Ω about its axis perpendicular to its plane, we will calculate the phase shift $\Delta\phi$ of one beam versus the other as they come to interfere at the detector.

(a) Calculate $\Delta\phi$ under the assumption that the angular velocity Ω is small (i.e., keep only terms to leading order in Ω). Express your answer in terms of Ω, the frequency ω of the light, the area A enclosed by the loop, and any relevant constants. (Hint: The rotation of the apparatus leads to a time difference Δt in how long it takes light to propagate around the loop clockwise versus counterclockwise. The phase difference is $\Delta\phi = \omega\Delta t$.)

(b) Does the answer in part (a) depend on the position of the rotation axis? If so, how?

(c) Using fiber optics, one can make a Sagnac interferometer in which each interferometer path winds around the loop many times. This is done by wrapping an optical fiber around the loop many times, effectively increasing the area A proportionally to the number of loops. This is the way one builds fiber optic gyroscopes, which are often used in planes for navigation purposes. Assuming a fiber with total length of 1 km that is wound multiple times around a loop with diameter 10 cm, what is the phase shift if we point the normal direction to the loop along Earth's rotation axis for light with 500 nm wavelength (Earth's rotation has a magnitude of 7×10^{-5} radians/second).

4. Fourier transforms and frequency combs. (Note: This problem will ask you to use Mathematica. Note that you can save your Mathematica notebook as a pdf.) Consider a sequence of N laser pulses that each have a Gaussian profile in time with spread σ, with the pulses evenly space by an interval τ. Specifically, the electric field amplitude as a function of time looks like:

$$f(t) = \frac{1}{\sqrt{2\pi}\sigma} \sum_{j=-N}^{j=N} e^{-\frac{(t-j\tau)^2}{2\sigma^2}} e^{-i\omega_0 t} \tag{1}$$

Here, ω_0 represents the center frequency of the laser, and $2N+1$ is the total number of pulses. We will consider the Fourier transform $F(\omega)$ of this pulse sequence, which tells us about the frequency spectrum of this pulse sequence. Specifically, the absolute value squared of $F(\omega)$, $|F(\omega)|^2$, tells us about the amount of optical power in a given frequency ω.

(a) Using Mathematica, find an expression for $|F(\omega)|^2$ in the case that $N=2$. How is $|F(\omega)|^2$ affected when ω_0 is changed?

(b) Plot your result from part (a) for several different values of σ and τ. Qualitatively describe your plots of $|F(\omega)|^2$ and how these plots are affected by changing σ or τ. (Hint: $|F(\omega)|^2$ should look like a 'comb' of regularly spaced peaks. Look at how σ and τ affect the width of the comb and the spacing of these peaks).

(c) Make plots of $|F(\omega)|^2$ for several different values of N and qualitatively describe your results. How does the width of each peak change as N is increased? (Note: For plots with higher values of N, you may want to set the option PlotPoints in Mathematica's Plot function to higher than default values so that your plots are not undersampled).

5. Optical resonator. This problem will explore the physics behind an optical resonator. An optical resonator transmits electromagnetic waves with only a narrow range of optical frequencies, making it a useful device for precisely measuring and stabilizing the frequencies of laser beams.

(a) Consider a thin, conducting metallic sheet with conductivity σ and thickness l. Later in the class, we will find that low frequency electromagnetic waves are attenuated as they propagate through a conductor. Assume that the sheet is thin enough that attenuation is a small effect. We will consider electromagnetic waves with frequency ω that propagate through the sheet in the normal direction. Under the assumption that $\sigma \gg \omega$ (also assume that the dielectric constant ϵ of the material can be approximated as an order 1 quantity), show that the effect of the conducting sheet can be replaced by a boundary condition on the tangential component of the magnetic field $\mathbf{H}$. What is this boundary condition? Use the following Maxwell's equation for electromagnetic fields in a medium:

$$\nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial D}{\partial t} + \frac{4\pi}{c}\mathbf{j}, \tag{2}$$

where $\mathbf{j} = \sigma \mathbf{E}$ is the current density and $\mathbf{D} = \epsilon \mathbf{E}$. In case you have not seen the notion of the $\mathbf{D}$ field before, I note that we will study it in further detail later in the class.

(b) Find the transmission and reflection coefficients (T and R) for an electromagnetic wave normally incident on the sheet. Show that $R+T < 1$ and explain what this means physically.

(c) A second identical sheet is placed parallel to the first a distance d from it. Find the transmission coefficient T of the combined system for normal incidence as a function of the incident wave number k. Make plots of T as a function of k for various values of σ and l and qualitatively describe how changing σ and l affects your plots. How does the distance d affect the transmission coefficient as a function of k?

(d) Qualitatively explain how an optical resonator could be used as a tool to help stabilize the frequency of a laser. Note that it is typically possible to controllably adjust a laser's frequency.