Entanglement & Quantum information science

- Wigner formalism - Wigner functions in QM
  - tool used to determine degree of "non-classical" nature in a quantum state / system.
  "quantum state tomography"

In Q.M. there is probability distribution

$$P(x) = |\psi(x)|^2$$

$\psi(x) = \langle x|\psi\rangle$. In momentum space

$$\langle p|\psi\rangle = \varphi(p) = \frac{1}{\sqrt{\hbar}}\int e^{-ixp/\hbar}\,\psi(x)\,dx$$

$$\langle p|\psi\rangle = \int dx\,\langle p|x\rangle\langle x|\psi\rangle$$

$|\varphi(p)|^2$ is a probability density in momentum variable.

Wigner sought a way to display probability distribution in x and p simultaneously.

Also you can establish a connection w/ ensemble of trajectories in classical phase space. ⟷ density distributions in $x + p$ or Wigner function in Q.M case.
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The expectation value of operator $\hat{A}$:

$$\langle A \rangle = \int \psi^*(x)\, A\, \psi(x) = \langle \psi | \hat{A} | \psi \rangle$$

$\hat{A}$ is a function of position and momentum

$$\hat{A} = A(\hat{x}, \hat{p})$$

Define a function $\tilde{A}$ of operator A by

$$\tilde{A}(x,p) \equiv \int e^{-ipy/\hbar} \underbrace{\langle x + \tfrac{y}{2} | \hat{A} | x - \tfrac{y}{2} \rangle}_{\substack{\text{matrix element of} \\ \text{operator in } x\text{-basis.}}} dy$$

Called Weyl transform of operator; function of x & p.

(Note: also could express it in terms of matrix elements in momentum basis.

$$\tilde{A}(x,p) \equiv \int e^{ixu/\hbar} \langle p + \tfrac{u}{2} | \hat{A} | p - \tfrac{u}{2} \rangle du \Big)$$

Trace of product of two operators

$$\mathrm{Tr}\left[\hat{A}\hat{B}\right] = \frac{1}{\hbar} \iint \tilde{A}(x,p)\, \tilde{B}(x,p)\, dx\, dp$$

(Homework)

Consider a pure state $|\psi\rangle$

the density matrix $\hat{\rho} = |\psi\rangle\langle\psi|$

this can be expressed in position basis

$$\rho(x, x') = \langle x|\hat{\rho}|x'\rangle = \psi(x)\psi^*(x')$$

$$\langle A\rangle = \mathrm{Tr}\left[\hat{\rho}\hat{A}\right]$$

$$= \frac{1}{\hbar}\int \tilde{\rho}\,\tilde{A}\,dx\,dp$$

The Wigner function $W(x,p)$ is prop. to the Weyl transform of the density operator:

$$W(x,p) \equiv \frac{\tilde{\rho}}{\hbar} = \frac{1}{\hbar}\int e^{-ipy/\hbar}\,\psi\left(x+\frac{y}{2}\right)\psi^*\left(x-\frac{y}{2}\right)dy$$

So expectation value can be written:

$$\langle A\rangle = \iint W(x,p)\,\underbrace{\tilde{A}(x,p)}\,dx\,dp$$

like an average over phase space of physical quantity represented by $\tilde{A}$

If we integrate Wigner function over p

and use $\int e^{ipx/\hbar}\,dp = \hbar\,\delta(x)\,2\pi$

$$\Rightarrow \int W(x,p)\,dp = \psi^*(x)\psi(x) \leftarrow \text{prob. distribution for } x$$

$$\int W(x,p)\,dx = \varphi^*(p)\varphi(p) \leftarrow \text{prob. distribution for } p$$

Remark: we can't really interpret $W(x,p)$ as a simple probability distribution — can be negative .. connected w/ non classical behavior and entanglement.

(Note we can also express Wigner function in momentum representation:

$$W(x,p) = \frac{\tilde{\rho}}{\hbar} = \frac{1}{\hbar}\int e^{ixu/\hbar} \langle p+\tfrac{u}{2}|\psi\rangle\langle\psi|p-\tfrac{u}{2}\rangle\,du$$

$$= \frac{1}{\hbar}\int e^{ixu/\hbar}\,\varphi^*(p+\tfrac{u}{2})\,\varphi(p-\tfrac{u}{2})\,du \Big)$$

- The Weyl transform of identity $\mathbb{1}$ operator

$$\tilde{\mathbb{1}}(x,p) = \int e^{-ipy/\hbar} \langle x+\tfrac{y}{2}|\mathbb{1}|x-\tfrac{y}{2}\rangle\,dy$$

$$= \int e^{-ipy/\hbar}\,\delta(x+\tfrac{y}{2}-(x-\tfrac{y}{2}))\,dy$$

$$= 1$$

$$\iint W(x,p)\,dx\,dp = \mathrm{Tr}(\hat{\rho}) = 1$$

$W(x,p)$ is thus normalized in (x,p) space.

Other useful properties:

- If a wave function $\psi(x)$ yeilds a Wigner function $W(x,p)$, then the wave function $\psi(x-b)$ will give $W(x-b,p)$

- If wave function shifts as $\psi(x)e^{-ixb/\hbar}$ the new wigner function is $W(x, p-b)$.

- We can recover wave function in following way:

$$\int W(x,p)\, e^{ipx'/\hbar}\, dp = \psi^*\left(x-\frac{x'}{2}\right)\psi\left(x+\frac{x'}{2}\right)$$

set $x = \frac{x}{2}$, $x' = x$

$$\psi(x) = \frac{1}{\psi^*(0)}\int W\left(\frac{x}{2}, p\right) e^{ipx/\hbar}\, dp$$

up to normalization we can set $\psi(x)$ from W.

Consider Weyl tranforms of operators correspondng to physical observables: $\hat{A}$

Assume operator $\hat{A}$ is only a function of $\hat{x}$

$$\tilde{A} = \int e^{-ipy/\hbar} \langle x + \tfrac{y}{2} | \hat{A}(\hat{x}) | x - \tfrac{y}{2} \rangle \, dy$$

$$= \int e^{-ipy/\hbar} A(x - \tfrac{y}{2}) \, \delta(y) \, dy = A(x)$$

(Note similar result for $\hat{A}(\hat{p})$ in momentum space)

We can make an extended argument for sums of operators where each term is purely a function of $\hat{x}$ or $\hat{p}$.

e.g. given a Hamiltonian $\hat{H}(\hat{x}, \hat{p})$

$$= T(\hat{p}) + u(\hat{x})$$, this

becomes $$H(x,p) = T(p) + u(x)$$

We can look at expectation values of observables:

$$\langle x \rangle = \iint W(x,p) \, x \, dx \, dp$$

$$\langle p \rangle = \iint W(x,p) \, p \, dx \, dp$$

$$\langle T \rangle = \iint W(x,p) \, T(p) \, dx \, dp$$

$$\langle u \rangle = \iint W(x,p)\, u(x)\, dx\, dp$$

$$\langle H \rangle = \iint W(x,p)\, H(x,p)\, dx\, dp$$

Remark

The Wigner function acts like a probability distribution in phase space, except that W can be negative. Expectation values of other operators $\hat{A}(\hat{x},\hat{p})$ w/ terms e.g. $\hat{x}\hat{p}$ are not as simple.

Example problem : Harmonic Oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 \hat{x}^2$$

Lowest energy eigenstates:

$$\psi_0(x) = \frac{1}{\pi^{1/4}} \frac{1}{a^{1/2}} e^{-x^2/2a^2} \quad ; a^2 = \frac{\hbar}{m\omega}$$

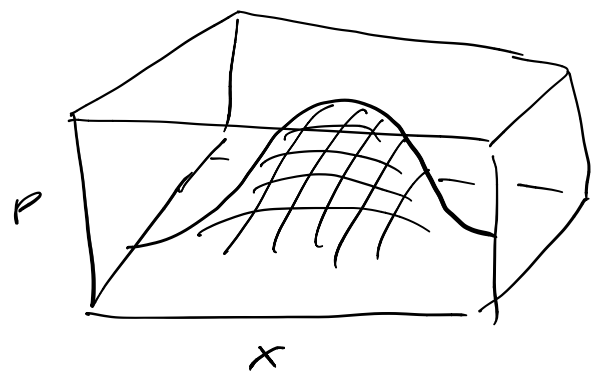
$$\psi_1(x) = \frac{1}{\pi^{1/4}} \sqrt{\frac{2}{a}}\, \frac{x}{a}\, e^{-x^2/2a^2}$$

We can write wigner functions for these states, using

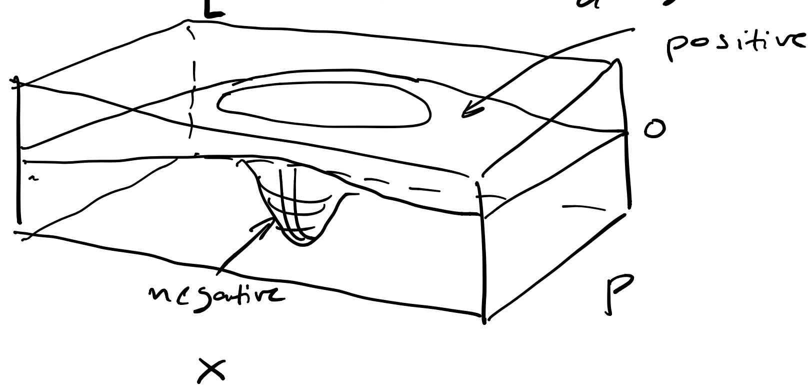
$$W(x,p) = \frac{1}{h} \int e^{-ipy/\hbar} \psi\left(x+\frac{y}{2}\right) \psi^*\left(x-\frac{y}{2}\right) dy$$

We find

$$W_0(x,p) = \frac{2}{\hbar} \exp\left(-\frac{a^2 p^2}{\hbar^2} - \frac{x^2}{a^2}\right)$$



$$W_1(x,p) = \frac{2}{\hbar}\left(-1 + 2\left(\frac{ap}{\hbar}\right)^2 + 2\left(\frac{x}{a}\right)^2\right) \cdot \exp\left[-\frac{a^2 p^2}{\hbar^2} - \frac{x^2}{a^2}\right]$$



This is a Fock state, pure state $|n=1\rangle$ in simple harmonic oscillator. Has no classical equivalent to motion of mass oscillating attached by a spring.

Consider ground state energy:

$$\langle H \rangle = \iint W_0(x,p)\left(\frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}\right)dx\,dp$$

$$= \frac{\hbar\omega}{2}.$$

Consider energy spread in ground state

$$\Delta = \langle H^2 \rangle - \underbrace{\langle H \rangle^2}_{= \left(\frac{\hbar\omega}{2}\right)^2}$$

we expect to find $\Delta = 0$

$$\langle H^2 \rangle = \iint W_0(x,p)\,\tilde{H}^2\,dx\,dp$$

this contains cross terms depending on $\hat{x}$ and $\hat{p}$ so Weyl transform is non-trivial.

HW exercise will be to show

$\langle H^2 \rangle = \left(\frac{\hbar\omega}{2}\right)^2$ using Weyl transform.

<u>Remark</u>: negative wigner function $\Rightarrow$ behavior that is quantum no classical analog, but

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a wigner function positive everywhere doesn't mean that we can exhibit only classical phenomena.

Time-dependence of Wigner function

$$\frac{\partial W}{\partial t} = \frac{1}{h}\int e^{-ipy/\hbar}\left[\frac{\partial \psi^*(x-\frac{y}{2})}{\partial t}\psi(x+\frac{y}{2}) + \frac{\partial \psi(x+\frac{y}{2})}{\partial t}\psi^*(x-\frac{y}{2})\right]dy$$

Use schrödinger equation:

$$\frac{\partial \psi(x,t)}{\partial t} = \frac{-\hbar}{2im}\frac{\partial^2\psi(x,t)}{\partial x^2} + \frac{1}{i\hbar}U(x)\psi(x,t)$$

We can write

$$\frac{\partial W}{\partial t} = \frac{\partial W_T}{\partial t} + \frac{\partial W_U}{\partial t}$$

Find that $\displaystyle \frac{\partial W_T}{\partial t} = \frac{1}{4\pi i m}\int e^{-ipy/\hbar}\left[\frac{\partial^2\psi^*(x-\frac{y}{2})}{\partial x^2}\psi(x+\frac{y}{2}) - \psi^*(x-\frac{y}{2})\frac{\partial^2\psi(x+\frac{y}{2})}{\partial x^2}\right]dy$

$$= -\frac{p}{m}\frac{\partial W}{\partial x}(x,p)$$

$$\frac{\partial W_u}{\partial t} = \sum_{s=0} (-\hbar^2)^s \frac{1}{(2s+1)!}\left(\frac{1}{2}\right)^{2s} \frac{\partial^{2s+1}}{\partial x^{2s+1}} u(x)$$

$$\times \left(\frac{\partial}{\partial p}\right)^{2s+1} W(x,p)$$

Note if $u(x)$ has vanishing derivatives above 2nd-order, e.g. free particle, harmonic oscillator, constant force, then simplifies to

$$\Rightarrow \frac{\partial W_u}{\partial t} = \frac{\partial u(x)}{\partial x}\frac{\partial W(x,p)}{\partial p}$$

Looks like classical Liouville equation:

$$\frac{\partial W}{\partial t}(x,p) = -\frac{p}{m}\frac{\partial W(x,p)}{\partial x} + \frac{\partial u(x)}{\partial x}\frac{\partial W(x,p)}{\partial p}$$

Looks like classical physics under influence of potential. (If higher derivative terms are nonzero gives a diffusion-like behavior).

For the Harmonic oscillator, in (x,p) classically the time evolution is

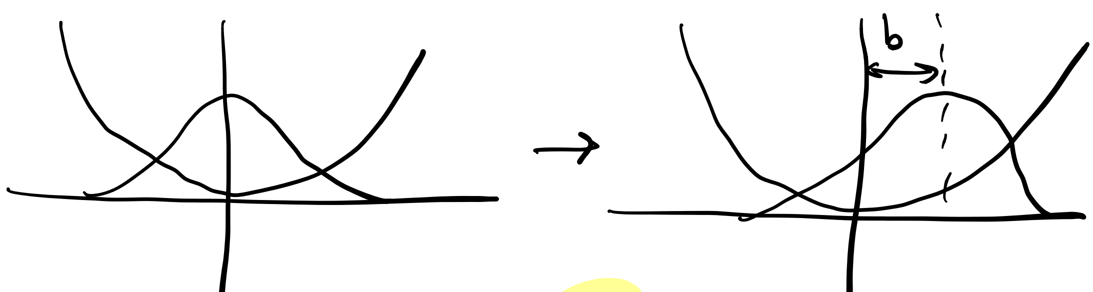
$$X_{cl} = x\cos(\omega t) - \frac{p}{m\omega}\sin(\omega t)$$

$$P_{cl} = p\cos(\omega t) + m\omega x\sin(\omega t)$$

So for Wigner function, we can write time evolution as : for initial Wigner function $W(x, p, 0)$,

$$W(x,p,t) = W\left(x\cos(\omega t) - \frac{p}{m\omega}\sin(\omega t),\; p\cos(\omega t) + m\omega x\sin(\omega t),\; 0\right)$$

<u>Example</u> : consider a Harmonic oscillator state with ground state shifted by a displacement "b".



$$W(x,p,0) = \frac{2}{\hbar}\exp\left[-\frac{a^2 p^2}{\hbar^2} - \frac{(x-b)^2}{a^2}\right]$$

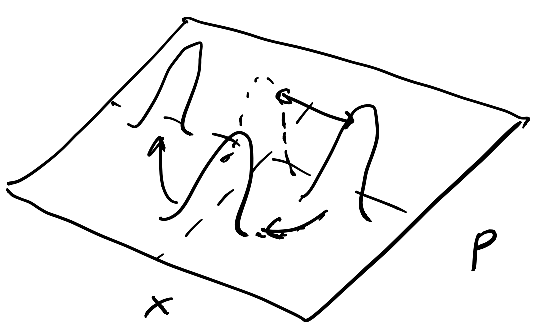
$$W(x,p,t) = \frac{2}{\hbar}\exp\left[-\frac{a^2}{\hbar^2}\left(p\cos\omega t + m\omega x\sin\omega t\right)^2 - \frac{1}{a^2}\left(x\cos\omega t - \frac{p}{m\omega}\sin\omega t - b\right)^2\right]$$

This corresponds to a <u>coherent state</u>

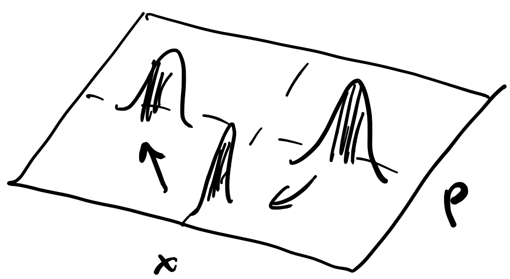
Close classical correspondence to a mass oscillating on a spring. More difficult (or less transparent) to study time evolution in Schrödinger-picture Q.M.

Note: if we relax the constraint $a^2 = \frac{\hbar}{m\omega}$ we can also study time evolution of "squeezed states"

<u>Coherent state</u>



<u>squeezed state</u>

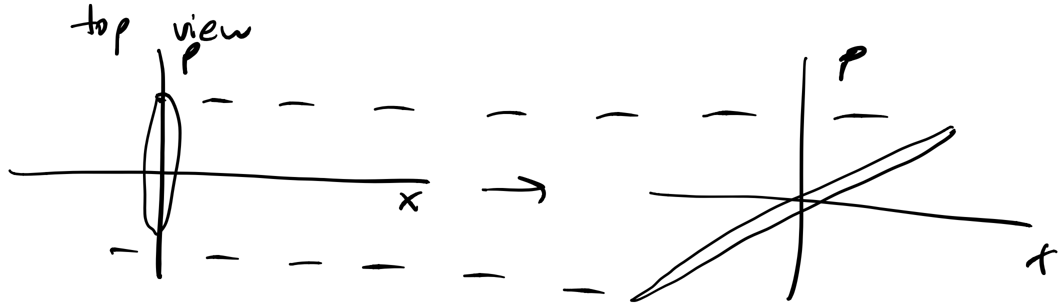


useful for understanding time evolution of coherent, squeezed states.
(useful in quantum optics)

<u>Example</u>: free particle

$$W(x,p,t) = W\left(x - \frac{p}{m}t,\, p,\, 0\right)$$

Looks like a "shear" of the distribution.



Many of these results would be more difficult to obtain starting from Schrödinger equation.

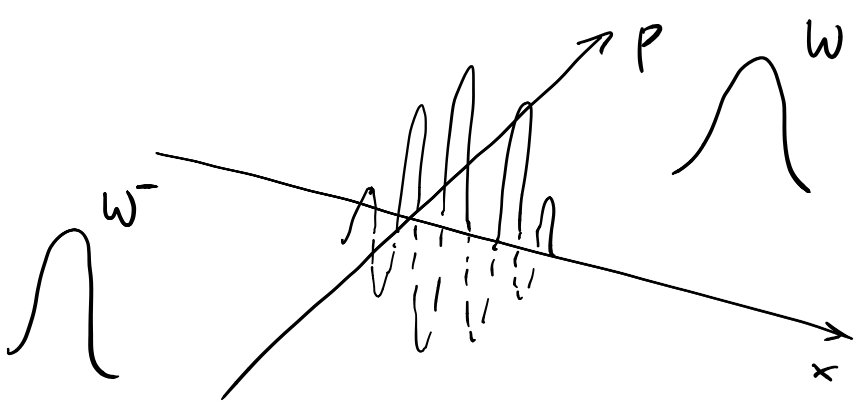
Wigner function for entangled states

Consider a wave-function with two Gaussian wave packets which have interacted and formed a composite single wave function

$$\psi = \frac{1}{\sqrt{2}}\left[\psi_1(x - x_0, p_0) + \psi_2(x + x_0, -p_0)\right]$$

If we generate the Wigner function it has terms:

$$W(x,p,t) = \frac{1}{4}\Big[W^+(x,p,t) + W^-(x,p,t) + \underbrace{2e^{-\frac{x^2}{2\sigma^2} - 2\sigma^2 p^2}}_{\text{arises from entanglement between wave packets.}} \cos\left(2\frac{p_0}{\hbar}x\right)\Big]$$



Entanglement (negative values of Wigner function) exhibited around origin $x = p = 0$.

This is a Schrödinger cat state.

Wigner function is useful in Quantum information, Quantum optics, quantum sensing/metrology...