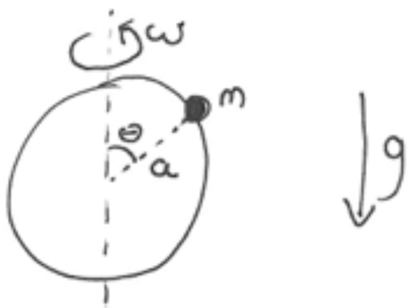
PHYS 411, Fall 2014
Midterm
Mon. Nov 3, 9-9:50

1: [15pts/100] Write the Lagrangian for a bead (mass m) that is constrained to move on a circular hoop of radius a in the presence of gravity (g). The hoop rotates with constant angular speed ω around a vertical axis that coincides with the diameter of the hoop:



$$L = \frac{m}{2}(a^2\dot{\theta}^2 + a^2 \sin^2\theta\omega^2) - mga\cos\theta \tag{1}$$

2: [20pts/100] Consider the following Lagrangian,

$$L = \frac{1}{2}T_{ij}\dot{q}_i\dot{q}_j - \frac{1}{2}V_{ij}q_iq_j + C_{ij}\dot{q}_iq_j \ , \tag{2}$$

where $\boldsymbol{T}$, $\boldsymbol{V}$, and $\boldsymbol{C}$ are constant matrices, and the summation convention is being used. The frequencies of normal mode oscillations, ω, are determined by setting det($\boldsymbol{M}$)=0 for some matrix $\boldsymbol{M}$. Determine $\boldsymbol{M}$ (in terms of quantities used in this question).

The momentum and force are

$$p_i = T_{ij}\dot{q}_j + C_{ij}q_j \tag{3}$$

$$\frac{\partial L}{\partial q_i} = -V_{ij}q_j + C_{ji}\dot{q}_j \tag{4}$$

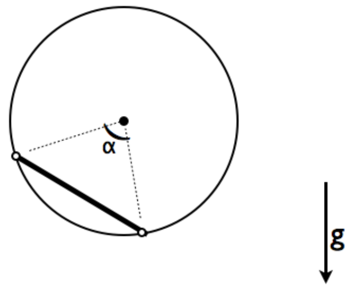
Then

$$T_{ij}\ddot{q}_j + (C_{ij} - C_{ji})\dot{q}_j + V_{ij}q_j = 0 \tag{5}$$

So,

$$\boldsymbol{M} = -\omega^2\boldsymbol{T} - i\omega(\boldsymbol{C} - \tilde{\boldsymbol{C}}) + \boldsymbol{V} \tag{6}$$

3: [30pts/100] A uniform rod slides with its ends on a smooth vertical circle. The rod subtends an angle α from the center of the circle. The circle is fixed in a uniform gravitational field (gravitational acceleration = g); the rod's mass is m; and the radius of the circle is R.



What is the Lagrangian? It is suggested that you use as the independent degree of freedom the angle between the vertical direction and the line that connects the center of the circle to the center of the rod (θ).

Set $T = T_{cm} + T_{rcm}$, with $T_{rcm} = \frac{ml^2}{24}\dot{\theta}^2$.

$$x_{cm} = R\cos(\alpha/2)\sin\theta \tag{7}$$
$$y_{cm} = -R\cos(\alpha/2)\cos\theta \tag{8}$$

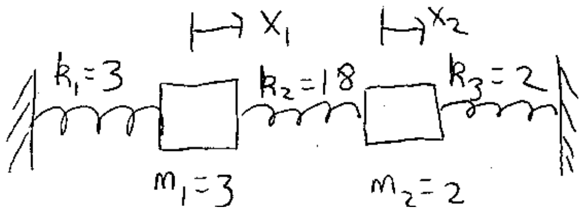
So, using $R\sin(\alpha/2) = l/2$,

$$\begin{aligned} T &= \frac{ml^2}{24}\dot{\theta}^2 + \frac{m}{2}R^2\cos^2(\alpha/2)\dot{\theta}^2 &(9)\\ &= \frac{mR^2}{6}\sin^2(\alpha/2)\dot{\theta}^2 + \frac{m}{2}R^2\cos^2(\alpha/2)\dot{\theta}^2 &(10)\\ \frac{L}{mR^2} &= \frac{1}{6}\sin^2(\alpha/2)\dot{\theta}^2 + \frac{1}{2}\cos^2(\alpha/2)\dot{\theta}^2 + \frac{g}{R}\cos\theta\cos(\alpha/2) &(11)\\ &= (\frac{1}{2} - \frac{1}{3}\sin^2(\alpha/2))\dot{\theta}^2 + \frac{g}{R}\cos\theta\cos(\alpha/2) &(12) \end{aligned}$$

So, frequency of oscillation and period is

$$\omega^2 = \frac{g}{R}\frac{\cos\frac{\alpha}{2}}{1 - \frac{2}{3}\sin^2\frac{\alpha}{2}} \tag{13}$$
$$P = 2\pi\sqrt{\frac{R}{g}}\left(\frac{1 - \frac{2}{3}\sin^2\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}\right)^{1/2} \tag{14}$$

4: [35pts/100] (a) Find the $\boldsymbol{T}$ and $\boldsymbol{V}$ matrices for the following system.



(b) Find the normal modes and frequencies
(c) Initially, $x_1 = x_2 = 0$ and $v_1 = 1$ and $v_2 = 0$. What are $x_1(t)$ and $x_2(t)$?

(a) $L = \frac{1}{2}\dot{x}_i\dot{x}_j T_{ij} - \frac{1}{2}x_i x_j V_{ij}$, with

$$\boldsymbol{T} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \tag{15}$$

$$\boldsymbol{V} = \begin{pmatrix} 3+18 & -18 \\ -18 & 2+18 \end{pmatrix} \tag{16}$$

So,

$$-\omega^2 \boldsymbol{T} + \boldsymbol{V} = \begin{pmatrix} -3\omega^2 + 21 & -18 \\ -18 & -2\omega^2 + 20 \end{pmatrix} \tag{17}$$

With characteristic equation:

$$\omega^4 - 17\omega^2 + 16 = 0 \tag{18}$$

and roots and corresponding e-vectors:

$$\omega^2 = 1 \tag{19}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{20}$$

and

$$\omega^2 = 16 \tag{21}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \tag{22}$$

(c)

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (A\cos t + B\sin t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (C\cos 4t + D\sin 4t)\begin{pmatrix} -2 \\ 3 \end{pmatrix} \tag{23}$$

where $A = 0$, $B = 3/5$, $C = 0$, $D = -1/20$.