HW 1

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worked c
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1.) $\vec{x} = \vec{a} \times (\vec{x} + \vec{b})$

$= (\vec{a} \times \vec{x}) + (\vec{a} \times \vec{b})$

$= \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_i & a_j & a_k \\ x_i & x_j & x_k \end{pmatrix} + \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_i & a_j & a_k \\ b_i & b_j & b_k \end{pmatrix}$

$= \hat{i}(a_j x_k - a_k x_j) - \hat{j}(a_i x_k - a_k x_i)$

$+ \hat{k}(a_i x_j - a_j x_i) + \hat{i}(a_j b_k - a_k b_j)$

$- \hat{j}(a_i b_k - a_k b_i) + \hat{k}(a_i b_j - a_j b_i)$

$= \hat{i}[a_j(x_k + b_k) - a_k(x_j + b_j)]$

$- \hat{j}[a_i(x_k + b_k) - a_k(x_i + b_i)]$

$+ \hat{k}[a_i(x_j + b_j) - a_j(x_i + b_i)]$

2.) $v = \begin{pmatrix} 2xy - z \\ x^2+y^2+z^2 \\ V_z \end{pmatrix}$ $\vec{\nabla} \times \vec{v} = 0$

i.) $\vec{\nabla} \times \vec{v} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \nabla_x & \nabla_y & \nabla_z \\ 2xy-z & x^2+y^2+z^2 & V_z \end{pmatrix}$

$$= \hat{i}\left[\frac{\partial v_z}{\partial y} - \frac{\partial}{\partial z}\left(x^2+y^2+z^2\right)\right]$$

$$-\hat{j}\left[\frac{\partial v_z}{\partial x} - \frac{\partial}{\partial z}(2xy-z)\right]$$

$$-\hat{k}\left[\frac{\partial}{\partial x}(x^2+y^2+z^2) - \frac{\partial}{\partial y}(2xy-z)\right]$$

$$= \hat{i}\left(\frac{\partial v_z}{\partial y} - 2z\right) - \hat{j}\left(\frac{\partial v_z}{\partial x} + 1\right) + \hat{k}(2x-2x) = 0$$

$\frac{\partial V_z}{\partial y} = 2z$ $\int \partial v_z = \int 2z\,\partial y$ $V_{z,i} = 2zy$

∂v_z $\int \partial v_z = \int -\partial x$ $V_z = -X$

$$\frac{\partial v_z}{\partial x} = -1 \quad \int dv_z \quad z,j$$

$$V_z = 2zy - x$$

ii) $\vec{V} = \vec{\nabla}\phi$

$$\phi_x = \int_0^x (2x'y - z)\,dx' = x^2 y - xz$$

$$\phi_y = \int_0^y (x^2 + y'^2 + z^2)\,dy' = x^2 y + \frac{y^3}{3} + z^2 y$$

$$\phi_z = \int_0^z (2z'y - x)\,dz' = z^2 y - xz$$

$$\phi = x^2 y - xz + \frac{y^3}{3} + z^2 y$$

3.) i.)

$$\vec{\nabla} \times (\nabla\phi) = 0$$

$$[\nabla \times (\nabla \phi)]_i = \varepsilon_{ijk} \nabla_k \nabla_j \phi = -\varepsilon_{ikj} \nabla_k \nabla_j \phi$$

$$= -[\nabla \times (\nabla \phi)]_i$$

$$= [\nabla \times (\nabla \phi)]_i$$

$$[\nabla \times (\nabla \phi)]_i = -[\nabla \times (\nabla \phi)]_i$$

only when $[\nabla \times (\nabla \phi)]_i = 0$

ii.) $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{a}) = 0$

$$\nabla_i \hat{e}_i (\varepsilon_{ijk} \nabla_j a_k) \hat{e}_k = \varepsilon_{ijk} \nabla_i \nabla_j a_k \hat{e}_i \cdot \hat{e}_k$$

$$= \varepsilon_{ijk} \nabla_i \nabla_j a_k \delta_{ik} \Rightarrow \boxed{\delta_{ik} = \begin{cases} 1 & \text{if } i = k \\ 0 & i \neq k \end{cases}}$$

$\Rightarrow 0$ since $i \neq k$

iii) $\vec{\nabla} \times (\vec{\nabla} \times \vec{a}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{a}) - \nabla^2 \vec{a}$

$$\varepsilon_{ijk} \nabla_j [\vec{\nabla} \times \vec{a}]_k = \varepsilon_{ijk} \nabla_j \varepsilon_{klm} \nabla_l a_m$$

$$= \varepsilon_{ijk} \varepsilon_{klm} \nabla_j \nabla_l a_m = \varepsilon_{ijk} \varepsilon_{lmk} \nabla_j \nabla_l a_m$$

$$= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \nabla_i \nabla_l a_m$$

$$= \delta_{il}\delta_{jm}\nabla_j\nabla_l a_m - \delta_{im}\delta_{jl}\nabla_j\nabla_l a_m$$

$$= (\delta_{il}\nabla_l)(\delta_{jm}\nabla_j a_m) - (\delta_{jl}\nabla_j\nabla_l)(\delta_{im} a_m)$$

$$= \nabla_i(\nabla_m a_m) - (\nabla_j\nabla_j) a_i = \vec{\nabla}(\vec{\nabla}\cdot\vec{a}) - \nabla^2\vec{a}$$

iv.) $\vec{\nabla}\cdot(\vec{a}\times\vec{b}) = \vec{b}\cdot(\vec{\nabla}\times\vec{a}) - \vec{a}\cdot(\vec{\nabla}\times\vec{b})$

$$\nabla_i(\varepsilon_{ijk} a_j b_k) = \varepsilon_{ijk}\nabla_i a_j b_k = \varepsilon_{ijk} a_j \nabla_i b_k$$

$$\varepsilon_{ijk}\nabla_i a_j b_k - \varepsilon_{ijk} a_j \nabla_i b_k = (\vec{\nabla}\times\vec{a})\cdot\vec{b} - (\vec{\nabla}\times\vec{b})\cdot\vec{a}$$

$$= \vec{b}\cdot(\vec{\nabla}\times\vec{a}) - \vec{a}\cdot(\vec{\nabla}\times\vec{b})$$