$\boxed{\text{Arbitrary Rotations}}$ — so far rotations about only one axis

Specify an arbitrary rotation by 3 Euler angles

$$R(\alpha\beta\gamma) = R_{z'}(\gamma)\, R_{y'}(\beta)\, R_{z}(\alpha)$$

① ② ③

① rotate about $\hat{z}$

② rotate about the new body axis $\hat{y}'$

③ rotate about the new body axis $\hat{z}'$

Often more convenient to rotate about fixed axes $\hat{x}, \hat{y}, \hat{z}$

Use $R_{y'}(\beta) = R_z(\alpha)\, R_y(\beta)\, R_z^{-1}(\alpha)$

① rotate back so $\hat{y}' = \hat{y}$

② rotate about $\hat{y}$

③ rotate back

Also $R_{z'}(\gamma) = R_{y'}(\beta)\, R_z(\gamma)\, R_{y'}^{-1}(\beta)$ (similarly)

$$\therefore R(\alpha\beta\gamma) = [R_{y'}(\beta) R_z(\gamma) R_{y'}^{-1}(\beta)]\,[R_z(\alpha) R_y(\beta) \cancel{R_z^{-1}(\alpha)}]\, \cancel{R_z(\alpha)}$$

$$= [\{R_z(\alpha) R_y(\beta) \cancel{R_z^{-1}(\alpha)}\}\, R_z(\gamma)\, \underbrace{\{R_z(\alpha) R_y(\beta) R_z^{-1}(\alpha)\}^{-1}}_{\cancel{R_z(\alpha)}\,\cancel{R_y^{-1}(\beta)}\,\cancel{R_z^{-1}(\alpha)}}\, \cancel{R_z(\alpha)}\, \cancel{R_y(\beta)}$$

commute

$$= \boxed{R(\alpha\beta\gamma) = \underbrace{R_z(\alpha)\, R_y(\beta)\, R_z(\gamma)}_{\text{reverse order } \gamma \to \beta \to \alpha}}$$ ← only fixed axes

① ② ③

Use Ang. Mom. Eigenstates $\longrightarrow$ Rotate States

$$|\psi\rangle = \sum_{\alpha j m} |\alpha j m\rangle \langle \underset{\uparrow}{\alpha} j m|\psi\rangle$$

other quantum numbers (not ang. mom.)

$$\mathcal{D}|\psi\rangle = \sum_{\alpha j m} \underset{\uparrow}{\mathcal{D}} |\alpha j m\rangle \langle \alpha j m|\psi\rangle$$

$$= \sum_{\substack{\alpha' j' m' \\ \alpha j m}} |\alpha' j' m'\rangle \langle \alpha' j' m'|\mathcal{D}|\alpha j m\rangle \langle \alpha j m|\psi\rangle$$

$\delta_{\alpha'\alpha}$

$$= \sum_{\substack{\alpha j' m' \\ j m}} |\alpha j' m'\rangle \underbrace{\langle j' m'|\mathcal{D}|j m\rangle}_{\sim \delta_{j'j}} \langle \alpha j m|\psi\rangle$$

since $[\vec{J}^2, \mathcal{D}_{\hat{n}}(\alpha)]$

$\mathcal{D}_{\hat{n}}(\alpha) = e^{-\frac{i}{\hbar}\hat{n}\cdot\vec{J}\alpha}$

$[\vec{J}^2, \hat{n}\cdot\vec{J}] = 0$

$\therefore [\vec{J}^2, \underbrace{\mathcal{D}(\alpha\beta\gamma)}_{\text{constructed from } \mathcal{D}_{\hat{n}}}] = 0$

$$\therefore \mathcal{D}|jm\rangle = \sum_{j'm'} |j'm'\rangle\langle j'm'|\mathcal{D}|jm\rangle$$

$$= \sum_{m'} |jm'\rangle \langle jm'|\mathcal{D}|jm\rangle$$

$$\boxed{\mathcal{D}|\psi\rangle = \sum_{\alpha j} \sum_{m'} |\alpha j m'\rangle \underbrace{\langle j m'|\mathcal{D}|j m\rangle}_{\mathcal{D}^{(j)}_{m'm}} \langle j m|\psi\rangle}$$

Terms with each j rotate together

reason for angular momentum decomposition

Rotation matrix elements for Euler angles α, β, γ

$$D^{(j)}_{m'm}(\alpha\beta\gamma) = \underbrace{\langle jm'| e^{-\frac{i\alpha J_z}{\hbar}}}_{e^{-i\alpha m'}\langle jm'|} e^{-\frac{i\beta J_x}{\hbar}} \underbrace{e^{-\frac{i\gamma J_z}{\hbar}} |jm\rangle}_{e^{-i\gamma m}|jm\rangle}$$

$$= e^{-i(\alpha m' + \gamma m)} \underbrace{\langle jm'| e^{-\frac{i\beta J_x}{\hbar}} |jm\rangle}_{\equiv d^{(j)}_{m'm}(\beta)}$$

Already considered $j = \frac{1}{2}$

$$\boxed{d^{(\frac{1}{2})}_{m'm}(\beta)} = D_{\hat{y}}(\beta)$$

$$= \begin{pmatrix} \cos\frac{\beta}{2} - 0 & -i(0-i)\sin\frac{\beta}{2} \\ -i(0+i)\sin\frac{\beta}{2} & \cos\frac{\beta}{2} + 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}$$

Wigner's Formula

$$d^{(j)}_{m'm}(\beta) = \sum_k (-1)^{k-m+m'} \frac{\sqrt{(j+m)!\,(j-m)!\,(j+m')!\,(j-m')!}}{(j+m-k)!\,k!\,(j-k-m')!\,(k-m+m')!} \left(\cos\frac{\beta}{2}\right)^{2j-2k+m-m'} \left(\sin\frac{\beta}{2}\right)^{2k-m+m'}$$

over all values
that give well behaved factorials