① a. $[x,p] = i\hbar$

$[x,p^2] = xp^2 - p^2x$ add zero

$= xpp - ppx + pxp - pxp$

$= xpp - pxp + pxp - ppx$

$= (xp-px)p + p(xp-px)$

$= [x,p]p + p[x,p] = i\hbar p + p\cdot(i\hbar)$ $\boxed{[x,p^2] = 2i\hbar p}$

b. $[x,p^3] = xppp - pppx + pxpp - pxpp$

$= xppp - pxpp + pxpp - pppx$

$= [x,p]p^2 + p[x,p^2]$

$= i\hbar p^2 + p(2i\hbar p)$ $[x,p^3] = 3i\hbar p^2$

c. looks like the general form is:

$\boxed{[x,p^n] = n i\hbar p^{n-1}}$

proof: we know this is true for $n=1$, $[x,p^1] = (1)i\hbar p^0 = i\hbar$

assume it's true for $n=k$, $[x,p^k] = k i\hbar p^{k-1}$

show this implies $[x,p^{k+1}] = (k+1)i\hbar p^k$

$[x,p^{k+1}] = xp\cdot p^k - pp^kx + pxp^k - pxp^k$

$= [x,p]p^k + p[x,p^k]$

$= i\hbar p^k + p(k i\hbar p^{k-1}) = (k+1)i\hbar p^k$ //

also, very similarly:

$[p,x] = -i\hbar \Rightarrow$ try $[p,x^k] = -i\hbar\cdot k\cdot x^{k-1}$

$[p,x^{k+1}] = pxx^k - xx^kp + xpx^k - xpx^k$

$= [p,x]x^k + x[p,x^k]$

$= -i\hbar x^k - i\hbar k x^k = -i\hbar(k+1)x^k$

so $[p,x^n] = -i\hbar\cdot n x^{n-1}$

(2) a. $H = \frac{p^2}{2m} + V(x)$

use Ehrenfest's theorem $\frac{d\langle A\rangle}{dt} = \frac{1}{i\hbar}\langle [A,H]\rangle$

to calculate $\frac{d\langle x\rangle}{dt}$

$$\frac{d\langle x\rangle}{dt} = \frac{1}{i\hbar}\langle [x,H]\rangle = \frac{1}{i\hbar}\langle \underbrace{[x,p^2/2m]}_{\frac{1}{2m}[x,p^2]} + \underbrace{[x,V(x)]}_{0}\rangle$$

$$\frac{d\langle x\rangle}{dt} = \frac{1}{2i\hbar m}\langle [x,p^2]\rangle = \frac{1}{2i\hbar m}\langle 2i\hbar p\rangle \qquad \boxed{\frac{d\langle x\rangle}{dt} = \frac{\langle p\rangle}{m}}$$

b. $\frac{d\langle p\rangle}{dt} = \frac{1}{i\hbar}\langle [p,H]\rangle = \frac{1}{i\hbar}\langle [p,V(x)]\rangle$

write $V(x) = \sum_n c_n x^n \Rightarrow \frac{\partial V}{\partial x} = \sum_n n c_n x^{n-1}$

$$[p,V(x)] = \sum_n c_n [p,x^n] \qquad [p,x^n] = -i\hbar n x^{n-1}$$

$$= \sum_n c_n(-i\hbar n x^{n-1})$$

$$= -i\hbar \sum_n n c_n x^{n-1} = -i\hbar \frac{\partial V}{\partial x}$$

$$\frac{d\langle p\rangle}{dt} = \frac{1}{i\hbar}\langle [p,V(x)]\rangle = -\left\langle \frac{\partial V}{\partial x}\right\rangle \qquad \boxed{d\langle p\rangle/dt = -\langle \partial V/\partial x\rangle}$$

c. show these are consistent with Hamilton's equations:

i. $\frac{d\langle x\rangle}{dt} = \left\langle \frac{\partial H}{\partial p}\right\rangle \qquad \frac{\partial H}{\partial p} = \frac{\partial}{\partial p}\left(\frac{p^2}{2m} + V(x)\right) = p/m$

$\frac{d\langle x\rangle}{dt} = \left\langle \frac{p}{m}\right\rangle = \frac{\langle p\rangle}{m}$ ✓ consistent

ii. $\frac{d\langle p\rangle}{dt} = -\left\langle \frac{\partial H}{\partial x}\right\rangle \qquad \frac{\partial H}{\partial x} = \frac{\partial V}{\partial x}$

$\frac{d\langle p\rangle}{dt} = -\left\langle \frac{\partial V}{\partial x}\right\rangle$ ✓ consistent