Problem Set 6

Solutions

1 (a) In the $|\varphi_1\rangle, |\varphi_2\rangle$ basis,

$$H = \begin{pmatrix} E_0 & \Delta \\ \Delta & E_0 \end{pmatrix}$$

Eigenstates of H:

$$|+\rangle = \tfrac{1}{\sqrt{2}}\left(|\varphi_1\rangle + |\varphi_2\rangle\right)$$

$$E_+ = E_0 + \Delta$$

$$|-\rangle = \tfrac{1}{\sqrt{2}}\left(|\varphi_1\rangle - |\varphi_2\rangle\right)$$

$$E_- = E_0 - \Delta$$

$$|\psi(0)\rangle = \tfrac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)$$

$$|\psi(t)\rangle = \tfrac{1}{\sqrt{2}}\left(|+\rangle e^{-i\omega_+ t} + |-\rangle e^{-i\omega_- t}\right)$$

$$P_1(t) = |\langle\varphi_1|\psi(t)\rangle|^2 = \tfrac{1}{4}\left|e^{-i\omega_+ t} + e^{-i\omega_- t}\right|^2$$

$$= \tfrac{1}{4}\left|\left(e^{-i\frac{(\omega_+ + \omega_-)}{2}t}\right)\left(e^{-i\frac{(\omega_+ - \omega_-)}{2}t} + e^{-i\frac{(\omega_- - \omega_+)}{2}t}\right)\right|^2$$

$$= \tfrac{1}{4}\left|e^{+i\frac{\Omega}{2}t} + e^{-i\frac{\Omega}{2}t}\right|^2$$

where $\Omega = \omega_- - \omega_+$

$$\boxed{P_1(t) = \cos^2 \tfrac{\Omega}{2} t}$$

$\boxed{b}$

$$H = \begin{pmatrix} E_1 & \Delta \\ \Delta & E_2 \end{pmatrix}$$

We will use the results derived in class, where the problem is expressed in terms of mixing angles Θ & ϕ.

Eigenenergies:

$$E_\pm = \frac{1}{2}(E_1 + E_2) \pm \frac{1}{2}\sqrt{(E_1 - E_2)^2 + 4\Delta^2}$$

Eigenstates:

$$|\varphi_+\rangle = \cos\frac{\Theta}{2}|\varphi_1\rangle + \sin\frac{\Theta}{2}|\varphi_2\rangle \qquad \text{where } \tan\Theta = \frac{2\Delta}{E_1 - E_2} \text{ \&}$$

$$|\varphi_-\rangle = -\sin\frac{\Theta}{2}|\varphi_1\rangle + \cos\frac{\Theta}{2}|\varphi_2\rangle \qquad \phi = 0 \text{ because } \Delta \text{ is real}$$

$$|\psi(t=0)\rangle = |\varphi_1\rangle = |\varphi_+\rangle\langle\varphi_+|\varphi_1\rangle + |\varphi_-\rangle\langle\varphi_-|\varphi_1\rangle$$

$$= \cos\frac{\Theta}{2}|\varphi_+\rangle + \sin\frac{\Theta}{2}|\varphi_-\rangle$$

$$P_1(t) = |\langle\varphi_1|\psi(t)\rangle|^2 = \left|\cos^2\frac{\Theta}{2}e^{-i\omega_+ t} + \sin^2\frac{\Theta}{2}e^{-i\omega_+ t}\right|^2$$

$$= \left|\cos^2\frac{\Theta}{2}e^{-i\frac{\Omega}{2}t} + \sin^2\frac{\Theta}{2}e^{+i\frac{\Omega}{2}t}\right|^2$$

where $\Omega = \omega_+ - \omega_-$

$$P_1(t) = \cos^4\frac{\theta}{2} + \sin^4\frac{\theta}{2} + \left(\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2}\right)\left(e^{-i\Omega t} + e^{i\Omega t}\right)$$

$$\boxed{P_1(t) = \cos^4\frac{\theta}{2} + \sin^4\frac{\theta}{2} + 2\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2}\cos\Omega t}$$

where $\tan\theta = \dfrac{2\Delta}{E_1 - E_2}$ + $\hbar\Omega = \sqrt{(E_1-E_2)^2 + 4\Delta^2}$

We can return to an earlier expression to see that $P_1(t)$ never vanishes:

$$P_1(t) = \left|\cos^2\frac{\theta}{2}e^{-i\frac{\Omega}{2}t} + \sin^2\frac{\theta}{2}e^{i\frac{\Omega}{2}t}\right|^2$$

Those are two counter-rotating phasors. The magnitude vanishes if $\sin\frac{\theta}{2} = \cos\frac{\theta}{2}$,

i.e. $\frac{\theta}{2} = 45° \Rightarrow \theta = 90° \Rightarrow \tan\theta = +\infty$.

So as long as $E_1 > E_2$, as specified in the problem, $P_1(t) > 0$.

c) $$\boxed{|\psi(t>T)\rangle = \cos\frac{\theta}{2}e^{-i\omega_+(t-T)}|\varphi_+\rangle + \sin\frac{\theta}{2}e^{-i\omega_-(t-T)}|\varphi_-\rangle}$$

where ω_- + ω_+ are given above

2a a) $[A,e^B] = A\sum_{n=0}^{\infty}\frac{1}{n!}B^n - \sum_{n=0}^{\infty}\frac{1}{n!}B^nA = \sum_{n=0}^{\infty}\frac{1}{n!}[A,B^n]$

But $[A,B^n] = [A,B^{n-1}]B + B^{n-1}[A,B] = \sum_{m=1}^{n}(B^{n-1})(c)$ for $n \geq 1$

$$\Rightarrow [A,e^B] = c\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{m=1}^{n}B^{n-1} = c\sum_{n=1}^{\infty}\frac{n}{n!}B^{n-1} = c\sum_{n=1}^{\infty}\frac{B^{n-1}}{(n-1)!}$$

$$= c\sum_{n=0}^{\infty}\frac{B^n}{n!} = \boxed{ce^B = [A,e^B]}$$

2b

$X(\lambda) \equiv e^{\lambda A}e^{B}e^{-\lambda A}$

$$\frac{\partial}{\partial\lambda}X = Ae^{\lambda A}e^{B}e^{-\lambda A} + e^{\lambda A}e^{B}(-A)e^{-\lambda A}$$

$$= (Ae^{\lambda A}e^{B}e^{-\lambda A}) + (-Ae^{\lambda A}e^{B}e^{-\lambda A} + ce^{\lambda A}e^{B}e^{-\lambda A})$$

$$= cX$$

$$\Rightarrow X(\lambda) = X(0)e^{c\lambda} \Rightarrow X(1) = X(0)e^{c}$$

$$\Rightarrow e^{A}e^{B}e^{-\lambda A} = e^{B}e^{c} = \boxed{e^{B+c\mathbb{1}}}$$

* One point here is that $e^Ae^B \neq e^{A+B}$, etc. Multiplication of exponents of operators does not generalize in this sense. The reason this doesn't work is that $[A,B] \neq 0$.

2c

$Y = e^{\lambda A}e^{\lambda B} \Rightarrow \frac{\partial}{\partial\lambda}Y = Ae^{\lambda A}e^{\lambda B} + e^{\lambda A}Be^{\lambda B}$
$= (A+B+\lambda c)Y$

(extending part a, we see $[A,e^{\lambda B}] = \lambda ce^{\lambda B}$)

$$Z = e^{\lambda(A+B)}e^{\lambda^2 c/2} \Rightarrow \frac{\partial}{\partial\lambda}Z = (A+B)Z + \frac{2\lambda c}{2}Z = (A+B+\lambda c)Z$$

So $Y'(\lambda) = Z'(\lambda)$ and also $Y(0) = Z(0)$, $\Rightarrow Y(\lambda) = Z(\lambda)$

$$\Rightarrow Y(1) = Z(1) \Rightarrow \boxed{e^Ae^B = e^{A+B+c/2}}$$