① a. $a = \sqrt{\frac{m\omega}{2\hbar}}\, x + \frac{i}{\sqrt{2\hbar m\omega}}\, p \qquad a^{+} = \sqrt{\frac{m\omega}{2\hbar}}\, x - \frac{i}{\sqrt{2\hbar m\omega}}\, p$

using $[x,p] = i\hbar$, calculate $[a,a^{+}]$

$$[a,a^{+}] = \sqrt{\frac{m\omega}{2\hbar}}\left(\frac{-i}{\sqrt{2\hbar m\omega}}\right)[x,p] + \left(\frac{i}{\sqrt{2\hbar m\omega}}\right)\sqrt{\frac{m\omega}{2\hbar}}\,[p,x]$$

$$= \frac{-i}{2\hbar}[x,p] + \frac{i}{2\hbar}[p,x]$$

$$= \frac{-i}{\hbar}[x,p] = -\frac{i}{\hbar}\, i\hbar = 1 \qquad \boxed{[a,a^{+}] = 1}$$

$$[a^{+}a, a] = a^{+}aa - aa^{+}a$$

$$= (a^{+}a - aa^{+})\,a = [a^{+},a]\,a = -a \qquad \boxed{[a^{+}a, a] = -a}$$

$$[a^{+}a, a^{+}] = a^{+}aa^{+} - a^{+}a^{+}a$$

$$= a^{+}[a,a^{+}] = a^{+} \qquad \boxed{[a^{+}a, a^{+}] = a^{+}}$$

b. solve for x & p in terms of a & a^{+}:

$$a + a^{+} = 2\sqrt{\frac{m\omega}{2\hbar}}\, x \qquad a^{+} - a = -\frac{2i}{\sqrt{2\hbar m\omega}}\, p$$

$$\boxed{x = \sqrt{\frac{\hbar}{2m\omega}}\,(a+a^{+})} \qquad \boxed{p = i\sqrt{\frac{\hbar m\omega}{2}}\,(a^{+}-a)}$$

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$$

$$\frac{p^2}{2m} = -\frac{\hbar m\omega}{4m}(a^{+}-a)^2 \qquad \frac{1}{2}m\omega^2x^2 = \frac{\hbar m\omega^2}{4m\omega}(a+a^{+})^2$$

$$H = \frac{\hbar\omega}{4}\left[(a^{+}+a)^2 - (a^{+}-a)^2\right]$$

$$H = \frac{\hbar\omega}{4}\left[a^+a^+ + a^+a + aa^+ + aa - (a^+a^+ - a^+a - aa^+ + aa)\right]$$

$$H = \frac{\hbar\omega}{2}(a^+a + aa^+) \qquad [a, a^+] = 1 \text{ so } aa^+ = a^+a + 1$$

$$H = \frac{\hbar\omega}{2}(a^+a + a^+a + 1) \qquad \boxed{H = \hbar\omega(a^+a + 1/2)}$$

c. $[H, a] = \hbar\omega[a^+a, a] = -\hbar\omega a$

$[H, a^+] = \hbar\omega[a^+a, a^+] = +\hbar\omega a^+$

$$\boxed{[H, a] = -\hbar\omega a \qquad [H, a^+] = +\hbar\omega a^+}$$

d. $H|\phi\rangle = E_\phi|\phi\rangle$ so $|\phi\rangle$ is an eigenstate of the Hamiltonian. consider:

$$[H, a]|\phi\rangle = -\hbar\omega a|\phi\rangle$$

$$Ha|\phi\rangle - aH|\phi\rangle = -\hbar\omega a|\phi\rangle$$

$$H[a|\phi\rangle] = E_\phi a|\phi\rangle - \hbar\omega a|\phi\rangle$$

$$\boxed{H[a|\phi\rangle] = (E_\phi - \hbar\omega)[a|\phi\rangle]}$$

so $a|\phi\rangle$ is also an eigenstate of H, with $\hbar\omega$ less energy

$$[H, a^+]|\phi\rangle = +\hbar\omega a^+|\phi\rangle$$

$$Ha^+|\phi\rangle = a^+H|\phi\rangle + \hbar\omega a^+|\phi\rangle$$

$$\boxed{H[a^+|\phi\rangle] = (E_\phi + \hbar\omega)[a^+|\phi\rangle]}$$

so $a^+|\phi\rangle$ is also an eigenstate of H, with $\hbar\omega$ more energy

so a destroys a quanta $\hbar\omega$ of energy
a^+ creates 〃
} and they move between diff. eigenstates of H

e. $a|0\rangle = 0$ defines the vacuum.

$$H|0\rangle = \hbar\omega(a^\dagger a + 1/2)|0\rangle$$
$$= \hbar\omega a^\dagger \underbrace{a|0\rangle}_{0} + \tfrac{1}{2}\hbar\omega|0\rangle \quad \text{so} \quad \boxed{E_0 = \tfrac{1}{2}\hbar\omega}$$

$$H[a^\dagger|0\rangle] = (E_0 + \hbar\omega)[a^\dagger|0\rangle] \quad \text{so} \quad \boxed{E_{a^\dagger|0\rangle} = \tfrac{3}{2}\hbar\omega}$$

each application of $a^\dagger$ creates another unit $\hbar\omega$ of energy, so:

$$|n\rangle \sim (a^\dagger)^n|0\rangle$$

has $E_n = E_0 + n\hbar\omega$ $\boxed{E_n = \hbar\omega(n+1/2)}$

f. from $H = \hbar\omega(a^\dagger a + 1/2)$ and $E_n = \hbar\omega(n+1/2)$, can deduce:

$$a^\dagger a|n\rangle = n|n\rangle$$

$$a^\dagger|n\rangle = c_n|n+1\rangle \Rightarrow \langle n+1|c_n^* = \langle n|a$$

$$\langle n|\underbrace{aa^\dagger}_{=a^\dagger a+1}|n\rangle = |c_n|^2 \underbrace{\langle n+1|n+1\rangle}_{1}$$

$$\langle n|a^\dagger a|n\rangle + \langle n|n\rangle = |c_n|^2$$

$$|c_n|^2 = 1+n \quad \text{so} \quad \boxed{c_n = \sqrt{n+1}} \quad \boxed{a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle}$$

$$a|n\rangle = D_n|n-1\rangle \quad \langle n-1|\cdot D_n^* = \langle n|a^\dagger$$

$$\langle n|a^\dagger a|n\rangle = |D_n|^2\langle n-1|n-1\rangle$$

$$|D_n|^2 = n \quad \boxed{D_n = \sqrt{n}} \quad \boxed{a|n\rangle = \sqrt{n}\,|n-1\rangle}$$

so to find $|n\rangle$, properly normalized, from $(a^\dagger)^n|0\rangle$,

$$\boxed{|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle}$$