$$\epsilon_{n_\alpha} = \hbar\omega_\alpha (n_\alpha + \tfrac{1}{2})$$

average energy of the α^{th} osc.

$$\bar{\epsilon}_\alpha = \frac{\sum_{n_\alpha} \epsilon_\alpha e^{-\beta e_\alpha}}{\sum e^{-\beta e_\alpha}}$$

$$= \frac{\sum_{n_\alpha} \hbar\omega_\alpha (n_\alpha + \frac{1}{2}) e^{-\beta\hbar\omega_\alpha (n_\alpha + \frac{1}{2})}}{\sum_{n_\alpha} e^{-\beta\hbar\omega_\alpha (n_\alpha + \frac{1}{2})}}$$

divide by $e^{-\beta\hbar\omega_\alpha/2}$

$$\bar{\epsilon}_\alpha = \frac{\sum_{n_\alpha} n_\alpha \hbar\omega_\alpha e^{-\beta\hbar\omega_\alpha n_\alpha}}{\sum_{n_\alpha} e^{-\beta n_\alpha \hbar\omega_\alpha}} + \frac{1}{2}\hbar\omega_\alpha$$

$e^{-\beta\hbar\omega_\alpha} = x$ on bottom have:

$$S(x) = \sum_n^N x^n \qquad S(x, N) = \sum_{n=0}^{N} x^n = \frac{1 + \cancel{x^{N+1}}}{1-x}$$

$$S_{N\to\infty}(\beta\hbar\omega_\alpha) = \frac{1}{1 - e^{-\beta\hbar\omega_\alpha}} \; ; \; \frac{\partial S(\beta\hbar\omega_\alpha)}{\partial\beta}$$

$$\frac{\partial S}{\partial\beta} = \frac{\hbar\omega_\alpha e^{-\beta\hbar\omega_\alpha}}{(1 - e^{-\beta\hbar\omega_\alpha})^2}$$

Planck distrib.

$$\boxed{\bar{\epsilon}_\alpha = \frac{\hbar\omega_\alpha}{e^{\beta\hbar\omega_\alpha} - 1} + \frac{\hbar\omega_\alpha}{2}}$$

two limits

$\beta \to 0 ; T \to \infty$

$e^{\beta\hbar\omega_\alpha} = 1 + \beta\hbar\omega_\alpha \dots$

$\bar{\epsilon}_\alpha = \frac{1}{\beta} = k_B T$

same classical result

opposite limit

$\beta \to \infty ; T \to 0$

$\bar{\epsilon}_\alpha = \hbar\omega_\alpha (e^{-\beta\hbar\omega_\alpha} + \frac{1}{2})$

extreme limit all osc. in gnd. state have only zero point