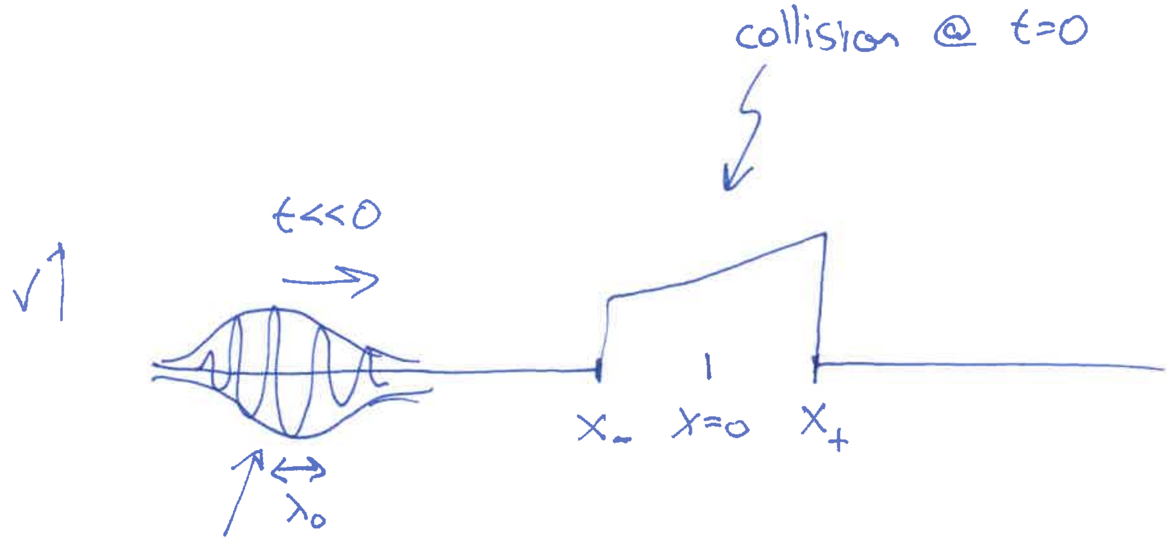
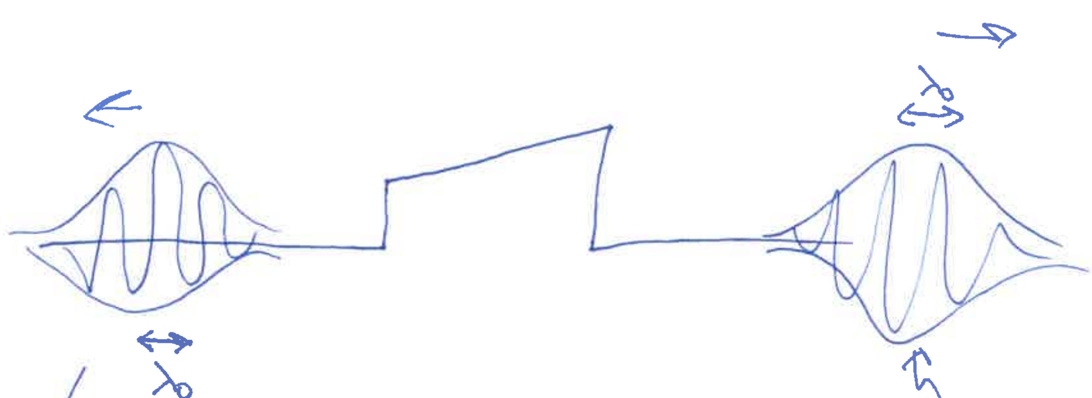
Scattering

Consider



Normalized incoming w.p.

We expect that Schr. will lead to the following at $t \gg 0$



R * normalized wave packet

T * normalized w.p.

where $\boxed{|R|^2 + |T|^2 = 1}$

Preserves probability. Can prove its true easily for non-degenerate case

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- Maybe that expectation is always true, or maybe there can be cases of resonances in (attractive) potential where w.p.s come out in bursts. (Not sure)

- In any case, turning crank on Schr always gives answer. Can do numerically.

- But behavior at boundary has to follow continuity & smoothness rules. So, we can get some results easily on ~~R~~ R, T.

- Consider these eigenfunctions of H

And here $\boxed{k>0.}$

$x < x_-$ $\quad \varphi_k = e^{ikx} + Re^{-ikx}$

$x > x_+$ $\quad \varphi_k = Te^{ikx}$

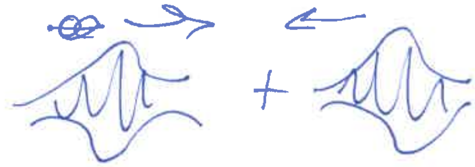
} exact solutions outside of non-zero potential

("Incoming B.C.s", because of no e^{-ikx} term for $x > x$)

- By choosing this form, we have imposed 2 B.C.s: ~~bdr~~ behavior @ $\pm\infty$. So, solutions for a given k will be unique. R + T determined by potential.

(e.g. e^{ikx} @ $x = -\infty$ would be distinctly di B.C.)

- But how can we add those e.f.s to get right initial condition @ $t \ll 0$?

Naively it looks like we have 

- Answer is that R is a fn of k.

~~Just like how we~~ added e^{ikx} functions w/ k-dependent phase to localize w.p., (and found that localization moved in time)

Now e^{ikx} part is distinct from e^{-ikx} part, k-dep phase will localize one @ $t=0$, then at much later times that one $\to 0$ & the other one is localized + moving.

k-dependence of R will lead to w.p. reflection behavior as we integrate over

$$\int \varphi_k f(k) e^{-i\omega_k t}$$

Localization in space changes position with time, like before. And now there is also a handoff from incoming $+k$ to reflected $-k$

~~$e^{ikx} + R(k)e^{-ikx}$~~

~~- Note that we cannot use $\varphi_k = e^{ikx}$ for $x < x_1$ as our basis. This is equivalent to taking $R \to 0$ in our chosen basis, and we will see that continuity + smoothness (i.e. just solving Schr) do not allow for this.~~

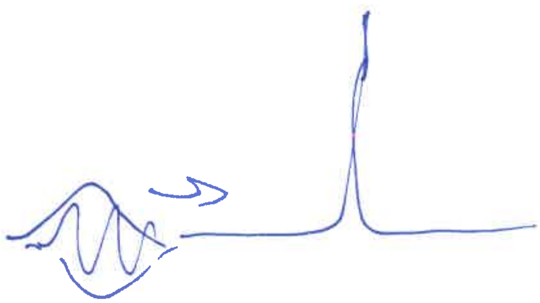
- Again $|R|^2+|T|^2=1$
- Since $\int dx\,|\psi(x)|^2$ is conserved by the Schr. eqn. & considering simplest case for a wavepacket transmission/reflection. R belongs to one w.p., T belongs to other.
- Can prove, but also statement that particles do not appear or disappear.

- $|T|^2$ is transmission prob
  $|R|^2$ is reflection prob.

$\int f(x)\delta(x)\,dx = f(0)$

So V_0 does not have same units as $V(x)$

- Example : $V(x)=V_0\,\delta(x)$



- $\varphi_k(x)=\begin{cases} e^{ikx}+Re^{-ikx} \\ Te^{ikx} \end{cases}$

  exactly for $x\neq 0$

- What happens @ $x=0$?

- Integrate Schr. eqn across potential

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\varphi_k + V_0\,\delta(x)\,\varphi_k = E\,\varphi_k$$

- Taking $\int_{-\epsilon}^{\epsilon} dx$ of above

(w.f. is not smooth b/c of inf. potential)

$$-\frac{\hbar^2}{2m}\frac{d\varphi_k}{dx}\Big|_{-\epsilon}^{\epsilon} + V_0\,\varphi_k(0) = \Theta(\epsilon)$$

$$\Rightarrow \quad \frac{\hbar^2}{2m}\,\Delta\left(\frac{d\varphi_k}{dx}\right) = V_0\,\varphi_k(0)$$

where $\Delta\left(\frac{d\varphi_k}{dx}\right) = \lim_{\epsilon\to 0}\left(\frac{d\varphi_k(x=+\epsilon)}{dx} - \frac{d\varphi_k(x=-\epsilon)}{dx}\right)$

~~If $\frac{d\varphi}{dt}$ has simple jump @ $x=0$, $\varphi(x)$ is continuous~~

	$\lim_{\epsilon\to 0}$ $x=+\epsilon$	$\lim_{\epsilon\to 0}$ $x=-\epsilon$
$\varphi_k(x)$	T	$1+R$
$\frac{d\varphi_k(x)}{dx}$	ikT	$ik(1-R)$

$$\Rightarrow \begin{cases} T = 1 + R \\ ik\left(T - (1-R)\right) = \dfrac{2mV_0}{\hbar^2}(1+R) \end{cases}$$

Solving for R (plugging 1st into second)

$$2ikR = \frac{2mV_0}{\hbar^2}(1+R)$$

$$\left(2ik - \frac{2mV_0}{\hbar^2}\right)R = \frac{2mV_0}{\hbar^2}$$

k-dep amplitude + phase shift

$$\boxed{R = -\frac{1}{1 - i\frac{\hbar^2 k}{mV_0}}} = \frac{-1}{1 - ia}$$

$$T = R + 1 \Rightarrow \boxed{T = \frac{-i\frac{\hbar^2 k}{mV_0}}{1 - i\frac{\hbar^2 k}{mV_0}}} = \frac{-ia}{1 - ia}$$

$$\left|\frac{1}{a}\right| = \left|(k e^{i\theta})^{-1}\right| = k^{-1}\left|e^{-i\theta}\right|$$

$|\alpha\beta| = |\alpha||\beta|$

- Check $|R|^2 + |T|^2 = \dfrac{1}{1+a^2} + \dfrac{a^2}{1+a^2} = 1$ ✓
- Also as k or $E \to \infty$
  $|R| \to 0$, $|T| \to 1$ as naively expected
- As $V_0 \to \infty$, $|R| \to 1$, $|T| \to 0$

– Can construct w.p.'s