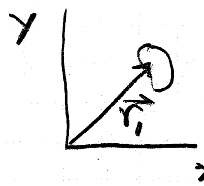
Graduate Approach to Angular Momentum

Do not start with $\vec{r} \times \vec{p}$

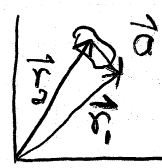
Instead start with general properties of rotations

- Generator of rotations — orbital
  — generalize
- Unitary, Hermitian
- Commutators from rotation of a vector

Infinitesmal Rotation of a Spinless Wavefunction



$\psi(\vec r) \xrightarrow{R_z(\epsilon)} \tilde\psi(\vec r)$



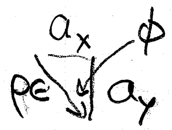
$\vec r_1 = \vec r_2 + \vec a$

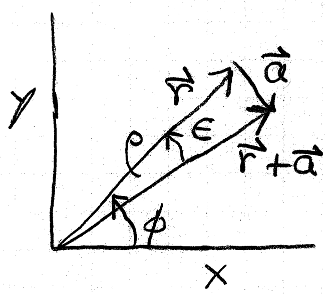
$$|\psi(\vec r)\rangle \Longrightarrow |\tilde\psi(\vec r)\rangle = D_z(\epsilon)\,|\psi(\vec r)\rangle$$

At the specific point: $|\tilde\psi(\vec r_2)\rangle = |\psi(\vec r_1)\rangle = |\psi(\vec r_2+\vec a)\rangle$

At any point: $|\tilde\psi(\vec r)\rangle = |\psi(\vec r+\vec a)\rangle$

$$\underline{D_z(\epsilon)\,|\psi(\vec r)\rangle = |\psi(\underbrace{\vec r+\vec a}_{x+a_x,\; y+a_y})\rangle}$$





$$x + a_x = x + (\underbrace{\rho\epsilon) \sin\phi}_{y} \approx x + \epsilon y$$

$$y + a_y = y - (\underbrace{\rho\epsilon) \cos\phi}_{x} \approx y - \epsilon x$$

$$\therefore D\,\psi(x,y) = \psi(x+\epsilon y,\, y-\epsilon x)$$

$$= \psi(x,y) + \epsilon y \frac{\partial\psi(x,y)}{\partial x} - \epsilon x \frac{\partial \psi(x,y)}{\partial y} + \cdots$$

$$= \Big[1 + \epsilon\big(y\underbrace{\tfrac{\partial}{\partial x}}_{\frac{p_x}{-i\hbar}} - x\underbrace{\tfrac{\partial}{\partial y}}_{\frac{p_y}{-i\hbar}}\big)\Big]\,\psi(x,y)$$

$\vec p = -i\hbar\vec\nabla$

$L_z = (\vec r \times \vec p)_z$

$= x p_y - y p_x$

$$= \Big[1 - \frac{i\epsilon}{\hbar} L_z\Big]\,\psi(x,y)$$

$$\boxed{D_z(\epsilon) = 1 - \frac{i\epsilon}{\hbar} L_z}$$

<u>Generator of an infinitesmal rotation</u>

i.e. $D_z(\epsilon)\,|\psi\rangle = \Big[1 - \frac{i\epsilon}{\hbar} L_z\Big]\,|\psi\rangle$

Other "Generators"

Translation: $T = 1 - \frac{i}{\hbar} p_x \, dx$

generator of translations

Time trans.: $\mathcal{U} = 1 - \frac{i}{\hbar} H \, dt$

gen. of time trans.

Rotation (about $\hat{z}$): $\mathcal{D} = 1 - \frac{i}{\hbar} L_z \epsilon$

gen. of rot. about $\hat{z}$

Generalize

$$\boxed{\underline{\text{Rotation:}} \quad \mathcal{D}_{\hat{n}}(\epsilon) = 1 - \frac{i}{\hbar}(\vec{J}\cdot\hat{n})\epsilon}$$

about any axis

ang. mom. is a gen. of some rotation

We showed that in the special case of a system without spin $\vec{J} \Rightarrow \vec{r} \times \vec{p} = \vec{L}$ does the job $l =$ integer

(no equiv. for half-integer j)

$\mathcal{D}$ must not change probabilities

$$\langle\alpha|\alpha\rangle = \langle\alpha|\underbrace{\mathcal{D}^\dagger \mathcal{D}}|\alpha\rangle$$

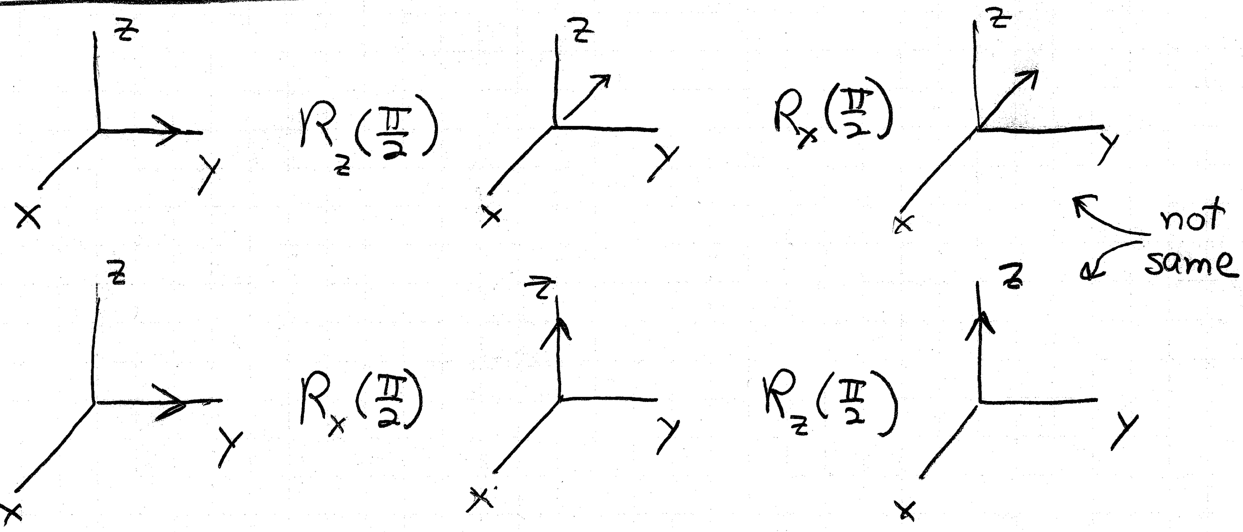
$$\begin{aligned}
1 &= \mathcal{D}^\dagger \mathcal{D} \\
&= \left[1-\frac{i}{\hbar}\epsilon\vec{J}\cdot\hat{n}\right]^\dagger\left[1-\frac{i}{\hbar}\epsilon\vec{J}\cdot\hat{n}\right] \\
&= \left[1+\frac{i}{\hbar}\epsilon\vec{J}^\dagger\cdot\hat{n}\right]\left[1-\frac{i}{\hbar}\epsilon\vec{J}\cdot\hat{n}\right] \\
&= 1+\frac{i\epsilon}{\hbar}\underbrace{\left(\vec{J}^\dagger-\vec{J}\right)}\cdot\hat{n}
\end{aligned}$$

$$\therefore \underline{\underline{\vec{J}^\dagger = \vec{J}}}$$

<u>A generator of a rotation is Hermitian</u>

Classical Rotations of a vector Do Not Commute (in general)

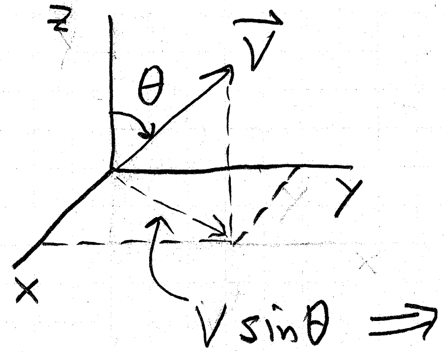
e.g.



From classical rotation of a vector

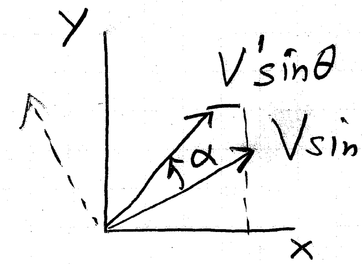
$\Longrightarrow$ deduce properties of q.m. ang. mom.

Rotate a vector about $\hat{z}$: $\vec{V}' = R_z(\alpha)\,\vec{V}$



$V_z' = V_z$ and θ stays same

In the xy plane



$\alpha = 0 \Rightarrow V_x' = V_x$
$\alpha = \frac{\pi}{2} \Rightarrow V_x' = -V_y$

$V_x' = V_x \cos\alpha - V_y \sin\alpha$

$\alpha = 0 \Rightarrow V_y' = V_y$
$\alpha = \frac{\pi}{2} \Rightarrow V_y' = V_x$

$V_y' = V_y \cos\alpha + V_x \sin\alpha$

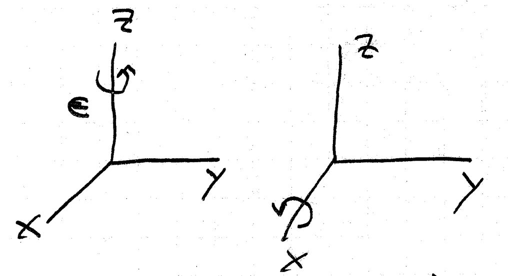
$$\overbrace{\begin{pmatrix} V_x' \\ V_y' \\ V_z' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{R_z(\alpha)}}\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

Infinitesmal rotation: $\alpha = \epsilon$

$$R_z(\epsilon) = \begin{pmatrix} 1-\frac{1}{2}\epsilon^2 & -\epsilon & 0 \\ \epsilon & 1-\frac{1}{2}\epsilon^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \cancel{\mathcal{O}(\epsilon^3)}$$

$$= (1-\tfrac{1}{2}\epsilon^2)\hat{x}\hat{x} - \epsilon\,\hat{x}\hat{y} + \epsilon\,\hat{y}\hat{x} + (1-\tfrac{1}{2}\epsilon^2)\hat{y}\hat{y} + \hat{z}\hat{z}$$

Infinitesmal rotations about $\hat{x}$



$\hat{x} \to \hat{y}$
$\hat{y} \to \hat{z}$
$\hat{z} \to \hat{x}$

$(x, y, z) \downarrow x$ $\quad$ (y, z, x) (still right handed)

$$R_z(\epsilon) = (1-\tfrac{1}{2}\epsilon^2)\hat{x}\hat{x} - \epsilon\hat{x}\hat{y} + \epsilon\hat{y}\hat{x} + (1-\tfrac{1}{2}\epsilon^2)\hat{y}\hat{y} + \hat{z}\hat{z}$$

$$\downarrow$$

$$R_x(\epsilon) = (1-\tfrac{1}{2}\epsilon^2)\hat{y}\hat{y} - \epsilon\hat{y}\hat{z} + \epsilon\hat{z}\hat{y} + (1-\tfrac{1}{2}\epsilon^2)\hat{z}\hat{z} + \hat{x}\hat{x}$$

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\tfrac{1}{2}\epsilon^2 & -\epsilon \\ 0 & \epsilon & 1-\tfrac{1}{2}\epsilon^2 \end{pmatrix}$$

Infinitesmal rotations about $\hat{y}$:

$\hat{x} \to \hat{z}$
$\hat{y} \to \hat{x}$
$\hat{z} \to \hat{y}$

$$R_y(\epsilon) = (1-\tfrac{1}{2}\epsilon^2)\hat{z}\hat{z} - \epsilon\hat{z}\hat{x} + \epsilon\hat{x}\hat{z} + (1-\tfrac{1}{2}\epsilon^2)\hat{x}\hat{x} + \hat{y}\hat{y}$$

$$R_y = \begin{pmatrix} 1-\tfrac{1}{2}\epsilon^2 & 0 & \epsilon \\ 0 & 1 & 0 \\ -\epsilon & 0 & 1-\tfrac{1}{2}\epsilon^2 \end{pmatrix}$$

See if infinitesmal rotations commute

$$R_x(\epsilon)R_y(\epsilon) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\frac{1}{2}\epsilon^2 & -\epsilon \\ 0 & \epsilon & 1-\frac{1}{2}\epsilon^2 \end{pmatrix}\begin{pmatrix} 1-\frac{1}{2}\epsilon^2 & 0 & \epsilon \\ 0 & 1 & 0 \\ -\epsilon & 0 & 1-\frac{1}{2}\epsilon^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\frac{1}{2}\epsilon^2 & 0 & \epsilon \\ +\epsilon^2 & 1-\frac{1}{2}\epsilon^2 & -\epsilon(1-\frac{1}{2}\epsilon^2) \\ -\epsilon(1-\frac{1}{2}\epsilon^2) & \epsilon & (1-\frac{1}{2}\epsilon^2)(1-\frac{1}{2}\epsilon^2) \end{pmatrix}$$

$$= \begin{pmatrix} 1-\frac{1}{2}\epsilon^2 & 0 & \epsilon \\ \epsilon^2 & 1-\frac{1}{2}\epsilon^2 & -\epsilon \\ -\epsilon & \epsilon & 1-\epsilon^2 \end{pmatrix} + \cancel{\mathcal{O}(\epsilon^3)}$$ ✓

$$R_y(\epsilon)R_x(\epsilon) = \begin{pmatrix} 1-\frac{1}{2}\epsilon^2 & 0 & \epsilon \\ 0 & 1 & 0 \\ -\epsilon & 0 & 1-\frac{1}{2}\epsilon^2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\frac{1}{2}\epsilon^2 & -\epsilon \\ 0 & \epsilon & 1-\frac{1}{2}\epsilon^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\frac{1}{2}\epsilon^2 & \epsilon^2 & \epsilon \\ 0 & 1-\frac{1}{2}\epsilon^2 & -\epsilon \\ -\epsilon & \epsilon & 1-\epsilon^2 \end{pmatrix} + \cancel{\mathcal{O}(\epsilon^3)}$$ ✓

$$R_xR_y - R_yR_x = \begin{pmatrix} 0 & -\epsilon^2 & 0 \\ \epsilon^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ ✓

$$= \underbrace{\begin{pmatrix} 1-\frac{1}{2}\cancel{(\epsilon^2)^2} & -\epsilon^2 & 0 \\ \epsilon^2 & 1-\frac{1}{2}\cancel{(\epsilon^2)^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{R_z(\epsilon^2)} - \underbrace{\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}}_{\overleftrightarrow{1}}$$

$$= R_z(\epsilon^2) - \overleftrightarrow{1}$$

how these rotations do not commute

Postulate correspondence: $D(\epsilon) \longleftrightarrow R(\epsilon)$

$$R_x(\epsilon)\, R_y(\epsilon) - R_y(\epsilon)\, R_x(\epsilon) = R_z(\epsilon^2) - 1$$

$$\updownarrow \qquad\qquad \updownarrow \qquad\qquad \updownarrow$$

$$D_x(\epsilon)\, D_y(\epsilon) - D_y(\epsilon)\, D_x(\epsilon) = D_z(\epsilon^2) - 1$$

$$\left[1 - \frac{i\epsilon}{\hbar} J_x\right]\left[1 - \frac{i\epsilon}{\hbar} J_y\right] - \left[1 - \frac{i\epsilon}{\hbar} J_y\right]\left[1 - \frac{i\epsilon}{\hbar} J_x\right] = \left[1 - \frac{i\epsilon^3}{\hbar} J_z\right] - 1$$

$$\left(\cancel{1} - \cancel{\frac{i\epsilon}{\hbar} J_y} - \cancel{\frac{i\epsilon}{\hbar} J_x} - \frac{\epsilon^2}{\hbar^2} J_x J_y\right) - \left(\cancel{1} - \cancel{\frac{i\epsilon}{\hbar} J_x} - \cancel{\frac{i\epsilon}{\hbar} J_y} - \frac{\epsilon^2}{\hbar} J_y J_x\right) = -\frac{i\epsilon^2}{\hbar} J_z$$

$$-\frac{\cancel{\epsilon^2}}{\hbar^2} J_x J_y + \frac{\cancel{\epsilon^2}}{\hbar} J_y J_x = -\frac{i\cancel{\epsilon^2}}{\hbar} J_z$$

$$-J_x J_y + J_y J_x = -i\hbar J_z$$

$$\boxed{[J_x, J_y] = i\hbar J_z}$$

Repeat for other directions

$$\boxed{[J_i, J_j] = i\hbar\, \epsilon_{ijk} J_k}$$

obtained from rotation of a vector

Finite rotations of a wave function

So far: $\mathcal{D}_z(\epsilon) = 1 - \frac{i}{\hbar}\epsilon J_z$

Finite: $\mathcal{D}_z(\phi + d\phi) = \mathcal{D}_z(d\phi)\,\mathcal{D}_z(\phi)$ second first

$$= \left[1 - \frac{i\,d\phi}{\hbar} J_z\right] \mathcal{D}_z(\phi)$$

$$\frac{\mathcal{D}_z(\phi + d\phi) - \mathcal{D}_z(\phi)}{d\phi} = \frac{-\frac{i\,\cancel{d\phi}}{\hbar} J_z\, \mathcal{D}_z(\phi)}{\cancel{d\phi}}$$

$$\frac{d\,\mathcal{D}_z(\phi)}{d\phi} = -\frac{i}{\hbar} J_z\, \mathcal{D}_z(\phi)$$ differential equation

solution $\Longrightarrow$ $\boxed{\mathcal{D}_z(\phi) = e^{-\frac{i J_z}{\hbar}\phi}}$

operator

Recall: $e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!}$ $\quad$ n=0: 1

$$\frac{de^{At}}{dt} = \frac{d}{dt}\sum_{n=0}^{\infty}\frac{(At)^n}{n!} = \sum_{n=1}^{\infty}\frac{A^n n t^{n-1}}{n!}$$

$$= A\sum_{n=1}^{\infty}\frac{(At)^{n-1}}{(n-1)!} = A\sum_{n=0}^{\infty}\frac{(At)^n}{n!}$$

$$= A e^{At}$$

All Rotation Operators D Form a Group

Group — set of elements plus an operation (a combination rule) such that

1. Combining any two elements gives a third element
2. One element I is an identity element for every element E in the group such that
$$EI = IE = E$$
3. Every element E has a unique inverse element E^{-1} such that
$$EE^{-1} = E^{-1}E = I$$
4. Combination operation is associative
$$A(BC) = (AB)C$$

U(1): Set of all phase factors $U(\theta) = e^{i\theta}$ ← real
with multiplication as the combination operation

unitary

one dimension

1. $U(\theta_1)\,U(\theta_2) = e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)} = U(\theta_1+\theta_2)$
2. $U(0) = e^{i0} = 1$ ← identity element
3. $U(\theta)\,U(-\theta) = 1 = U(0)$ → unique identity element for all elements
$U(\theta)$ is $U^{-1}(\theta) = U(-\theta)$
4. $U(\theta_1)[U(\theta_2)\,U(\theta_3)] = e^{i\theta_1}\,e^{i(\theta_2+\theta_3)} = e^{i(\theta_1+\theta_2)}\,e^{i\theta_3}$
$= [U(\theta_1)\,U(\theta_2)]\,U(\theta_3)$

O(3): 3x3 rotation matrices R — orthogonal
— 3 dimensions

$R^{-1} = R^T$ ← transpose

SU(2): special unitary group in 2 dimensions