# Phys 411, Fall 2019

Midterm solutions

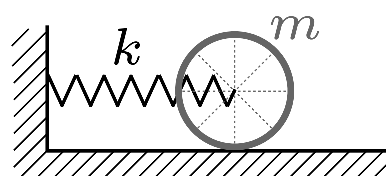
Wednesday, October 30th, 10-10:50 am.

1. [20 points] A bicycle wheel of mass $m$ can roll on a horizontal surface without slippage and resistance. A spring attaches the axis of the wheel to a wall (you can neglect the mass of the spokes). The spring constant is $k$. The wheel is vertical and the motion of its axis is one-dimensional along the horizontal axis.

   (a) Determine the Lagrangian.

   (b) Obtain Euler-Lagrange equations of motion.

   (c) What is the period of oscillations about the equilibrium point?




**Solution.** The no-slip condition implies that $x = r\theta$ and $\dot{x} = r\dot{\theta}$, where $r$ is the radius of the wheel. Then, we have

$$T = T_{\rm cm} + T_{\rm rcm} = \frac{m\dot{x}^2}{2} + \frac{mr^2\dot{\theta}^2}{2}, \quad V = \frac{kx^2}{2}. \tag{1}$$

Note that the rotation associated with the center of mass motion ($T_{\rm cm}$) equals the rotational energy relative to the center of mass ($T_{\rm rcm}$): both contributions to the kinetic energy need to be included. This results in the Lagrangian:

$$L = T - V = \frac{2m\dot{x}^2}{2} - \frac{kx^2}{2}. \tag{2}$$

The Euler-Lagrange equation is:

$$2m\ddot{x} = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}\right] = \frac{\partial L}{\partial x} = -kx. \tag{3}$$

Thus, the frequency of oscillations and the corresponding period are

$$\omega = \left(\frac{k}{2m}\right)^{1/2}, \quad P = \frac{2\pi}{\omega} = 2\pi\left(\frac{2m}{k}\right)^{1/2}. \tag{4}$$

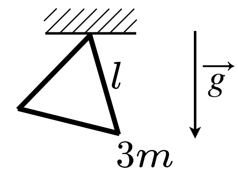
This is the same frequency as for a point mass on a spring, but with twice as large mass. As we see from eq. (1), this difference comes about from the kinetic energy of the wheel mimicking the higher mass.

2. [40 points] An equilateral triangle made of three identical uniform rods, each of length $l$ and mass $m$ (that is, the total mass of the triangle is $3m$), can swing about one of its vertices in a uniform gravitational field (gravitational acceleration = $g$). The motion is constrained to the vertical plane.

    (a) Determine the Lagrangian.

    (b) Obtain Euler-Lagrange equations of motion.

    (c) Linearize the equations about the equilibrium point and obtain the period of oscillations.




**Solution.** We use the polar angle $\theta$ of the center of mass of the triangle as the generalized coordinate. Enumerating the rods 1 through 3 counter-clockwise starting with the top right rod, we can write the kinetic energy of rod $i$ as $T_i = T_{i,\mathrm{cm}} + T_{i,\mathrm{rcm}}$. We have $T_{\mathrm{cm},i} = \frac{1}{2} m r_{\mathrm{cm},i}^2 \dot{\theta}^2$, where $r_{\mathrm{cm},i}$ is the distance to the center mass of the $i$-th rod from the suspension point:

$$T_{\mathrm{cm},1} = T_{\mathrm{cm},2} = \frac{1}{2} m \left(\frac{l}{2}\right)^2 \dot{\theta}^2, \quad T_{\mathrm{cm},3} = \frac{1}{2} m \left(\frac{l\sqrt{3}}{2}\right)^2 \dot{\theta}^2. \tag{5}$$

Here, $r_{\mathrm{cm},1} = r_{\mathrm{cm},2} = l/2$ and $r_{\mathrm{cm},3} = l\sqrt{3}/2$. The rotational energy relative to the center of mass is the same for all rods:

$$T_{\mathrm{rcm},i} = \frac{1}{2} \frac{ml^2}{12} \dot{\theta}^2, \quad i = 1, 2, 3. \tag{6}$$

The potential energy of the triangle can be written as

$$V = -3mgy_{\mathrm{cm}} = -3mgr_{\mathrm{cm}} \cos\theta = -\sqrt{3} mgl \cos\theta, \tag{7}$$

where $r_{\mathrm{cm}} = l/\sqrt{3}$ is the distance to the triangle's center of mass from the suspension point. Therefore, the Lagrangian is

$$L = T - V = \frac{3}{4} ml^2 \dot{\theta}^2 + \sqrt{3} mgl \cos\theta. \tag{8}$$

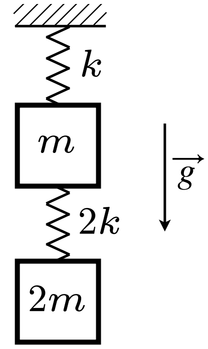
Euler-Lagrange equations are

$$\frac{3}{2} ml^2 \ddot{\theta} = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\theta}}\right] = \frac{\partial L}{\partial \theta} = -\sqrt{3} mgl \sin\theta. \tag{9}$$

Linearizing this equation by approximating $\sin\theta \approx \theta$, we get the oscillation frequency and period:

$$\omega = \left(\frac{2g}{\sqrt{3} l}\right)^{1/2}, \quad P = \frac{2\pi}{\omega} = 2\pi \left(\frac{\sqrt{3} l}{2g}\right)^{1/2}. \tag{10}$$

3. [40 points] Consider the following system. Motion is constrained to one dimension, along the vertical axis.




   (a) Determine the Lagrangian $L$ and the $\boldsymbol{T}$ and $\boldsymbol{V}$ matrices.
   (b) Find the eigenfrequencies.
   (c) Find the eigenvectors. (You do not need to normalize them).

**Solution.** The kinetic and potential energies are

$$T = \frac{m\dot{y}_1^2}{2} + \frac{2m\dot{y}_2^2}{2}, \tag{11}$$

$$V = \frac{ky_1^2}{2} + \frac{2k(y_2 - y_1)^2}{2}. \tag{12}$$

Thus, the $\boldsymbol{T}$ and $\boldsymbol{V}$ matrices are:

$$\boldsymbol{T} = \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix}, \qquad \boldsymbol{V} = \begin{bmatrix} 3k & -2k \\ -2k & 2k \end{bmatrix}. \tag{13}$$

The condition for eigenvectors to be non-trivial is $\left|-\omega^2\boldsymbol{T} + \boldsymbol{V}\right| = 0$, or:

$$\left|\begin{bmatrix} 3k - m\omega^2 & -2k \\ -2k & 2k - 2m\omega^2 \end{bmatrix}\right| = 0 \tag{14}$$

Thus, we obtain the following equation for $\omega^2$:

$$2m^2\omega^4 - 8km\omega^2 + 2k^2 = 0, \tag{15}$$

which gives

$$\omega_\pm^2 = \frac{4km \pm \sqrt{16k^2m^2 - 4k^2m^2}}{2m^2} = (2 \pm \sqrt{3})\frac{k}{m}. \tag{16}$$

Thus, we get two solutions: $\omega_\pm^2 = (2 \pm \sqrt{3})k/m$, and the corresponding (unnormalized) eigenvectors we can obtain by demanding that

$$(-\omega^2\boldsymbol{T} + \boldsymbol{V})\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0, \tag{17}$$

which gives $(3k - m\omega_\pm^2)a_1 - 2ka_2 = 0$, and the (unnormalized) eigenvectors:

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -1 \mp \sqrt{3} \\ 1 \end{bmatrix} \tag{18}$$