Wigner-Eckart Theorem

$$\langle \underbrace{\alpha'}_{\text{other quantum numbers}} \underbrace{j'm'}_{\text{ang. mom.}} | \underbrace{T_{kq}}_{\text{spherical tensor}} | \alpha j m \rangle = \underbrace{\langle jm\, kq | j'm' \rangle}_{\text{Clebsch-Gordan coefficient}} \overbrace{\underbrace{f(\alpha' j' k \alpha j)}_{\text{no } m', m \text{ dependences}}}^{\text{some function}}$$

$$\boxed{j' = |j-k|, |j-k+1|, \ldots, j+k}$$

$$\boxed{m' = m+q}$$

$$\boxed{\langle \alpha' j' m' | T_{kq} | \alpha j m \rangle = \langle jm\, kq | j'm' \rangle \frac{\overbrace{\langle \alpha' j' \| T^{(k)} \| \alpha j \rangle}^{\text{reduced matrix element}}}{\sqrt{2j'+1}}}$$

$$\boxed{\langle \alpha' j' m' | T_{kq} | \alpha j m \rangle = (-1)^{j-k+m'} \underbrace{\begin{pmatrix} j & k & j' \\ m & q & -m' \end{pmatrix}}_{\text{3-j symbol}} \langle \alpha' j' \| T^{(k)} \| \alpha j \rangle}$$

$$\langle jm\, kq | j'm' \rangle = (-1)^{j-k+m'} \sqrt{2j'+1} \begin{pmatrix} j & k & j' \\ m & q & -m' \end{pmatrix}$$

(Proof in most Q.M. books)

$\boxed{\text{Expand an obvious spherical tensor}}$

problem?

Define $\hat{t}_{kq}(\alpha' j' \alpha j) = \sum_{m'm} |\alpha' j' m'\rangle \underbrace{\langle \alpha' j' m' | \hat{t}_{kq}(\alpha' j' \alpha j) | \alpha j m\rangle}_{} \langle \alpha j m|$

choose: $(-1)^{j-k+m'} \underbrace{\begin{pmatrix} j & k & j' \\ m & q & -m' \end{pmatrix}}_{}$

cyclic $\rightsquigarrow \begin{pmatrix} j' & j & k \\ -m' & m & q \end{pmatrix}$

Invert using orthogonality

$$\underbrace{\sum_{kq} (-1)^k (2k+1) \begin{pmatrix} j' & j & k \\ -M' & M & q \end{pmatrix} \hat{T}_{kq}}_{} = \sum_{m'm} |\alpha' j' m'\rangle (-1)^{j'+m'} \underbrace{\sum_{kq} (2k+1) \begin{pmatrix} j' & j & k \\ -M' & M & q \end{pmatrix} \begin{pmatrix} j' & j & k \\ -m' & m & q \end{pmatrix}}_{\delta_{M'm'}\,\delta_{Mm}}$$

$$= (-1)^{j'+M'} |\alpha' j' M'\rangle \langle \alpha j M|$$

$$|\alpha' j' m'\rangle \langle \alpha j m| = (-1)^{j'+m'} \sum_{kq} (-1)^k (2k+1) \begin{pmatrix} j' & j & k \\ -m' & m & q \end{pmatrix} \hat{T}_{kq}$$

General operator

$$\mathcal{O} = \sum_{\substack{\alpha' j' m' \\ \alpha j m}} |\alpha' j' m'\rangle \langle \alpha' j' m' | \mathcal{O} | \alpha j m\rangle \langle \alpha j m|$$

$$= \sum_{\substack{\alpha' j' m' \\ \alpha j m}} (-1)^{j'+m'} \sum_{kq} (-1)^k (2k+1) \begin{pmatrix} j' & j & k \\ -m' & m & q \end{pmatrix} \hat{t}_{kq}(\alpha' j' \alpha j) \langle \alpha' j' m' | \mathcal{O} | \alpha j m\rangle$$

$$= \sum_{\substack{\alpha' j' \\ \alpha j}} \sum_{kq} \hat{t}_{kq}(\alpha' j' \alpha j) \underbrace{\sum_{m'm} (-1)^{j'-k+m'} (2k+1) \begin{pmatrix} j' & j & k \\ -m' & m & q \end{pmatrix} \langle \alpha' j' m' | \mathcal{O} | \alpha j m\rangle}_{f(\alpha' j' \alpha j\, kq)}$$

$$\boxed{\mathcal{O} = \sum_{\substack{\alpha' j' \\ \alpha j}} \sum_{kq} f(\alpha' j' \alpha j\, kq)\, \hat{t}_{kq}(\alpha' j' \alpha j)}$$

Like: $|\psi\rangle = \sum_{\alpha j m} |\alpha j m\rangle \langle \alpha j m | \psi\rangle$

Consequences:

① Get all m', m, q matrix elements from one

(e.g. $\langle \alpha' j' \| T^{(k)} \| \alpha j \rangle = \dfrac{\sqrt{2j'+1}}{\langle j0\, k0 | j'0 \rangle} \underbrace{\langle \alpha' j' 0 | T_{10} | \alpha j 0 \rangle}_{\text{calc. just this one}}$

$\Rightarrow$ get rest from W.E.

$\langle \alpha' j' m' | T_{kq} | \alpha j m \rangle = \underbrace{\langle jm\, kq | j'm' \rangle}_{\text{only dif. between dif. } m', q, m} \dfrac{\langle \alpha' j' \| T^{k} \| \alpha j \rangle}{\sqrt{2j'+1}}$)

② All spherical tensors with a given kq are proportional (actually at matrix elements of sph. tensors)

$\langle \alpha' j' m' | U_{kq} | \alpha j m \rangle = \langle \alpha' j' m' | T_{kq} | \alpha j m \rangle \underbrace{\dfrac{\langle \alpha' j' \| T^{(k)} \| \alpha j \rangle}{\langle \alpha' j' \| U^{(k)} \| \alpha j \rangle}}_{\text{a number}}$

$\therefore U_{kq} \sim T_{kq} \leftarrow$ in a particular spaces

(e.g. The spin $\frac{1}{2}$ electron has a magnetic moment (edm)

$\therefore \vec{S} \propto \vec{\mu} \rightarrow$ for subspace $|\alpha j m\rangle$

$\vec{\mu} = \mu \dfrac{\vec{S}}{\hbar/2}$ — const.

If the spin $\frac{1}{2}$ electron has an electric dipole moment $\vec{d}$

$\vec{d} = d \dfrac{\vec{S}}{\hbar/2}$)

Caution: the proportionality constant may be zero

(e.g. $d=0$ means the electron has no edm)

③ <u>Projection Theorem</u>: W.E. for a vector operator $\vec{V}$ when $j'=j$

$$\langle \alpha' j m | V_{1q} | \alpha j m \rangle = \frac{\langle \alpha' j m | \vec{J}\cdot\vec{V} | \alpha j m \rangle}{\hbar^2 j(j+1)} \langle j m' | J_{1q} | j m \rangle$$

note: $\vec{J}\cdot\vec{V} = \sum_q J_{1q} V_{1-q}$

Coupling of Spherical Tensors

For ang. mom. states:

$$|(k_1 k_2)\, kq\rangle = \sum_{q_1 q_2} |k_1 q_1\rangle |k_2 q_2\rangle \langle k_1 q_1\, k_2 q_2 | kq\rangle$$

For spherical tensors:

$$\boxed{T_{kq} = \sum_{q_1 q_2} A_{k_1 q_1} B_{k_2 q_2} \langle k_1 q_1\, k_2 q_2 | kq\rangle \text{ is a spherical tensor}}$$

Coupling Theorem

eg. Illustrate by coupling 2 vector $\vec{V}, \vec{W}$

$$\begin{matrix} k_1 = 1 \longrightarrow \\ k_2 = 1 \longrightarrow \end{matrix} \quad k = 0, 1, 2$$

$$\boxed{T_{00} = -\frac{1}{\sqrt{3}} \vec{V} \cdot \vec{W}} \leftarrow \text{scalar}$$

$$\boxed{T_{1q} = -\frac{i}{\sqrt{2}} (\vec{V} \times \vec{W})_{1q}} \leftarrow \text{vector}$$

$$\boxed{\begin{aligned} T_{2\pm2} &= V_{1\pm1} W_{1\pm1} \\ T_{2\pm1} &= \frac{1}{\sqrt{2}}\left[V_{1\pm1} W_{10} + V_{10} W_{1\pm1}\right] \\ T_{20} &= \frac{1}{\sqrt{6}}\left[V_{11} W_{1-1} + 2V_{10} W_{10} + V_{1-1} W_{11}\right] \end{aligned}}$$

rank 2 tensor

$\boxed{\text{Selection Rules}}$ – a very important example

Electric dipole radiation: (E1)

$$\underbrace{\text{Transition operator}}_{Q_{1q}} \sim \vec{p} \sim \vec{r} \leftarrow \text{W.E. theorem}$$

$$\langle j'm' | Q_{1q} | jm \rangle \longrightarrow |j'-j| \le 1 \le j+j'$$
(no $j=0$ to $j'=0$)

Magnetic dipole radiation: (M1)

dipole $\Longrightarrow k=1$

$$\langle j'm' | M_{1q} | jm \rangle \longrightarrow |j'-j| \le 1 \le j+j'$$
(no $j=0$ to $j'=0$)

Rotation properties give the same rotational selection rules $\longrightarrow$ What is different?

$Q_{1q} \leftarrow$ odd parity

$M_{1q} \leftarrow$ even parity

For states $|\alpha \ell j m\rangle$

odd parity: $(-1)^{\ell'+\ell} = -1 \longrightarrow \Delta\ell = \text{odd}$

even parity: $(-1)^{\ell'+\ell} = 1 \longrightarrow \Delta\ell = \text{even}$

E1: $\lvert j'-j\rvert \le 1 \le j+j'$ (no $j=0$ to $j'=0$) parity change $\Delta\ell = \text{odd}$	M1: $\lvert j'-j\rvert \le 1 \le j+j'$ (no $j=0$ to $j=0$) no parity change $\Delta\ell = \text{even}$

Electric quadupole transition (E2): $\longrightarrow$ E2

$$\langle \alpha' j' m' | Q_{2q} | \alpha j m \rangle$$

$$|j'-j| \leq 2 \leq j'+j$$

no $j=0$ to $j'=0$

no $j=0$ to $j'=1$

no $j'=1$ to $j=0$

Even parity operator $\longrightarrow$ no parity change

$\Delta l = \text{even}$