Problem Set 7

Solutions

$\boxed{1}$

$$H = \begin{array}{cc} |1\rangle & |2\rangle \\ \end{array}$$
$$H = \begin{pmatrix} E_0 - \frac{3\Delta}{4} & -\Delta \\ -\Delta & E_0 + \frac{3\Delta}{4} \end{pmatrix}$$

$\Delta > 0$

$|1\rangle$ is the ground state in limit $|u| \gg \Delta$

$$= E_0 \mathbb{1} + h$$

Find eigenvalues:

$$h - \lambda \mathbb{1} = 0$$

$$\begin{vmatrix} -\frac{3\Delta}{4} - \lambda & -\Delta \\ -\Delta & \frac{3\Delta}{4} - \lambda \end{vmatrix} = 0 \Rightarrow (-\lambda)^2 - \left(\frac{3\Delta}{4}\right)^2 - \Delta^2 = 0$$

$$\Rightarrow \lambda_{a,g} = \pm \frac{5\Delta}{4}$$

Find eigenvectors:

$$\begin{pmatrix} -\frac{3\Delta}{4} & -\Delta \\ -\Delta & \frac{3\Delta}{4} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \pm \frac{5\Delta}{4}$$

Top one:

$$-\frac{3}{4}\alpha - \beta = \pm\frac{5}{4}\alpha$$

$$\Rightarrow \left(\pm\frac{5}{4}+\frac{3}{4}\right)\alpha = -\beta$$

Using + sign:

$$\frac{\alpha}{\beta} = -\frac{1}{2} \Rightarrow$$

Using − sign:

$$\frac{\alpha}{\beta} = 2 \Rightarrow$$

$$\boxed{|\varphi_u\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix}1\\-2\end{pmatrix},\ E_u = E_0 + \frac{5\Delta}{4} \qquad |\varphi_g\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix}2\\1\end{pmatrix},\ E_g = E_0 - \frac{5\Delta}{4}}$$

a) $$|\psi(t)\rangle = e^{-i\omega_g t}|\varphi_g\rangle$$

$$P_L(t) \approx P_1(t) = |\langle 1|\psi(t)\rangle|^2 = \left(\frac{1}{\sqrt{5}}\cdot 2\right)^2 = \boxed{\frac{4}{5}}$$

b) $$\langle\psi(t)|O|\psi(t)\rangle = \langle\psi_g|e^{+i\omega_g t}\,O\,e^{-i\omega_g t}|\psi_g\rangle = \langle\psi_g|O|\psi_g\rangle$$

c) $$|\psi(t=0)\rangle = |1\rangle = \frac{1}{\sqrt{5}}\left(|\varphi_u\rangle + 2|\varphi_g\rangle\right)$$

$$\boxed{|\psi(t)\rangle = \frac{1}{\sqrt{5}}\left(|\varphi_u\rangle e^{-i\omega_u t} + 2|\varphi_g\rangle e^{-i\omega_g t}\right)}$$

d) $P_g(t) = |\langle \varphi_g | \psi(t) \rangle|^2 = \frac{4}{5}$

e) $P_L(t) \approx P_1(t) = |\langle 1 | \psi(t) \rangle|^2$

$$= \left(\frac{1}{25}\right) \left| e^{-i\omega_u t} + 4e^{-i\omega_g t} \right|^2$$

$$= \left(\frac{1}{25}\right) \left| 1 + 4e^{i\frac{5}{2}\frac{\Delta}{\hbar}t} \right|^2$$

It is now time-dependent because neither the initial state nor the measurement state are eigenstates of energy.

f)

$$\Pi = \begin{array}{cc} |1\rangle & |2\rangle \\ \end{array}$$
$$\Pi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Pi = \begin{array}{cc} |\varphi_g\rangle & |\varphi_u\rangle \end{array}$$
$$\Pi = \begin{pmatrix} \langle \varphi_g | \Pi | \varphi_g \rangle & \langle \varphi_g | \Pi | \varphi_u \rangle \\ \langle \varphi_u | \Pi | \varphi_g \rangle & \langle \varphi_u | \Pi | \varphi_u \rangle \end{pmatrix}$$

$$\Pi | \varphi_g \rangle = \left(\frac{1}{\sqrt{5}}\right)(|1\rangle + 2|2\rangle)$$

$$\Pi | \varphi_u \rangle = \frac{1}{\sqrt{5}}(-|2\rangle + |1\rangle)$$

$$\begin{array}{cc} |\varphi_g\rangle & |\varphi_u\rangle \end{array}$$
$$\Pi = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} \\ -\frac{3}{5} & -\frac{4}{5} \end{pmatrix}$$

$$\langle \Pi(t)\rangle = \langle \psi(t)|\Pi|\psi(t)\rangle \quad , \quad |\psi(t)\rangle = \frac{1}{\sqrt{5}}\left(2|\varphi_g\rangle e^{-i\omega_g t} + |\varphi_u\rangle e^{-i\omega_u t}\right)$$

Working in $|\varphi_g\rangle, |\varphi_u\rangle$ basis:

$$\langle \Pi(t)\rangle = \frac{1}{5}\left(2e^{+i\omega_g t}, e^{+i\omega_u t}\right)\begin{pmatrix} \frac{4}{5} & -\frac{3}{5} \\ -\frac{3}{5} & -\frac{4}{5} \end{pmatrix}\begin{pmatrix} 2e^{-i\omega_g t} \\ e^{-i\omega_u t} \end{pmatrix}$$

$$= \frac{1}{25}\left(2e^{i\omega_g t}, e^{i\omega_u t}\right)\begin{pmatrix} 8e^{-i\omega_g t} & -3e^{-i\omega_u t} \\ -6e^{-i\omega_g t} & -4e^{-i\omega_u t} \end{pmatrix}$$

$$= \left(\frac{1}{25}\right)\left(16 - 6e^{-i(\omega_u-\omega_g)t} - 6e^{+i(\omega_u-\omega_g)t} - 4\right)$$

$$\boxed{\langle \Pi(t)\rangle = \left(\frac{1}{25}\right)\left(12 - 12\cos[(\omega_u-\omega_g)t]\right)}$$

Using that expression $\langle \Pi(t=0)\rangle = 0$

But going back to $\Pi|1\rangle = |2\rangle$, since $|\psi(t=0)\rangle = |1\rangle$, we can immediately say that $\langle\psi(t=0)|\Pi|\psi(t=0)\rangle$

$$= \langle 1|\Pi|1\rangle = \langle 1|2\rangle = 0.$$

So our $t=0$ $\langle\Pi\rangle$ result agrees using either basis.

g) Parity is not a conserved quantity because the external field breaks parity symmetry, i.e. $[\Pi, H] \neq 0$.

If $U \ll \Delta$, we expect to recover parity symmetry. (The tunneling interaction dominates the field interaction).

Now $|\varphi_g\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$, $\Pi|\varphi_g\rangle = +|\varphi_g\rangle$

$|\varphi_u\rangle = |-\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$, $\Pi|\varphi_u\rangle = -|\varphi_u\rangle$

$\Rightarrow |1\rangle = \left(\frac{1}{\sqrt{2}}\right)(|\varphi_g\rangle + |\varphi_u\rangle)$

$|\psi(t)\rangle = \left(\frac{1}{\sqrt{2}}\right)\left(|\varphi_g\rangle e^{-i\omega_g t} + |\varphi_u\rangle e^{-i\omega_u t}\right)$

$\langle \Pi(t)\rangle = \frac{1}{2}(1 - 1) = 0$, which is time-independent

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I will work this problem at first by pretending I don't know about the rotation symmetry.

a)

$$H = \begin{array}{cc} \begin{pmatrix} \epsilon & -\Delta & 0 & -\Delta \\ -\Delta & \epsilon & -\Delta & 0 \\ 0 & -\Delta & \epsilon & -\Delta \\ -\Delta & 0 & -\Delta & \epsilon \end{pmatrix} & \begin{array}{c} |1\rangle \\ |2\rangle \\ |3\rangle \\ |4\rangle \end{array} \end{array}$$

Guessing eigenstates:

$$|\varphi_1\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad E_1 = \epsilon - 2\Delta$$

$$|\varphi_2\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad E_2 = \epsilon$$

~~Note that $\langle \varphi_1 | \varphi_2 \rangle = 0$ as needed for a non-degenerate H desired~~ for a degenerate H

$$|\varphi_3\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad E_3 = \epsilon + 2\Delta$$

$$|\varphi_4\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad E_4 = \epsilon$$

We know that if $E_n \neq E_m$ then $\langle \varphi_n | \varphi_m \rangle = 0$.

Let's check the one degeneracy. Did we pick states such that $\langle \varphi_2 | \varphi_4 \rangle = 0$? Yes b)

$$H = \begin{pmatrix} \epsilon & -\Delta & 0 & -\Delta & -\Delta \\ -\Delta & \epsilon & -\Delta & 0 & -\Delta \\ 0 & -\Delta & \epsilon & -\Delta & -\Delta \\ -\Delta & 0 & -\Delta & \epsilon & -\Delta \\ -\Delta & -\Delta & -\Delta & -\Delta & \epsilon \end{pmatrix}$$

Now including the center site

$$|\varphi_1\rangle \to |\chi_1\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad |\varphi_2\rangle \to |\chi_2\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \text{ etc.}$$

$$\boxed{H|\chi_1\rangle \neq E|\chi_1\rangle} \qquad \boxed{\begin{matrix} H|\chi_2\rangle = \epsilon|\chi_2\rangle \quad H|\chi_3\rangle = (\epsilon+2\Delta)|\chi_3\rangle \\ H|\chi_4\rangle = \epsilon|\chi_4\rangle \end{matrix}}$$

Eigenvectors with same eigenvalues as before

c) Now diagonalize H on the $|\varphi_1\rangle, |5\rangle$ subspace

$$\langle\varphi_1|H|\varphi_1\rangle = \langle\varphi_1| \cdot \frac{1}{2}\begin{pmatrix} \epsilon-2\Delta \\ \epsilon-2\Delta \\ \epsilon-2\Delta \\ \epsilon-2\Delta \\ -4\Delta \end{pmatrix} = \left(\frac{1}{4}\right) \cdot 4(\epsilon-2\Delta) = \epsilon - 2\Delta$$

$$\langle 5|H|\varphi_1\rangle = \langle 5| \cdot \checkmark = -2\Delta = \langle\varphi_1|H|5\rangle$$ since we know in this case Δ is real

$$\langle 5|H|5\rangle = \epsilon$$

$$\Rightarrow \quad H = \begin{array}{c} \begin{matrix} |\varphi_1\rangle & |5\rangle \end{matrix} \\ \begin{pmatrix} \epsilon-2\Delta & -2\Delta \\ -2\Delta & \epsilon \end{pmatrix} \end{array} =$$

$$= (\epsilon-\Delta)\mathbb{1} + \begin{pmatrix} -\Delta & -2\Delta \\ -2\Delta & +\Delta \end{pmatrix}$$

$$= (\epsilon - \Delta)\mathbb{1} - \Delta\begin{pmatrix} 1 & +2 \\ +2 & -1 \end{pmatrix}$$

Find eisenvalues λ

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0 \Rightarrow -1 + \lambda^2 - 4 = 0 \Rightarrow \lambda = \pm\sqrt{5}$$

Find eigenvectors

$$\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \pm\sqrt{5}\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow a + 2b = \pm\sqrt{5}\, a \Rightarrow \frac{a}{b} = \frac{2}{\pm\sqrt{5}-1}$$

Eigenvectors | ~~Eigenvalues~~ | Eigenvalues

$|\psi_1\rangle \quad (N_1)\left(2|\chi_1\rangle + (\sqrt{5}-1)|5\rangle\right)$

$$\boxed{|\psi_1\rangle = N_1\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \sqrt{5}-1 \end{pmatrix}}$$

(Using the basis $|1\rangle \dots |5\rangle$)

$$E_{\psi_1} = \epsilon - \Delta - \Delta(+\sqrt{5})$$

$$= \epsilon - (\sqrt{5}+1)\Delta$$

$$\boxed{|\psi_2\rangle = N_2\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -\sqrt{5}-1 \end{pmatrix}}$$

$$E_{\psi_2} = \epsilon - \Delta + \sqrt{5}\Delta$$

$$= \epsilon + (\sqrt{5}-1)\Delta$$

The eigenvectors of the full H are

$|\chi_2\rangle, |\chi_3\rangle, |\chi_4\rangle, |\psi_1\rangle, |\psi_2\rangle$

d) Rotation by $90°$ is a symmetry of the problem

$$R = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- If we checked, we would find $[R, H] = 0$

~~Ignoring~~

- With only four sites on a square, ~~we showed~~ rotation in class that R is non-degenerate. So eigenvectors ~~of the rotated H (ignoring site 5)~~ must be $|\varphi_1\rangle .. |\varphi_4\rangle$ from part (a), which we could easily verify explicitly
- Adding the 5th site, there is now a degeneracy in R in the subspace spanning $|\varphi_1\rangle + |5\rangle$
- So eigenstates of H must be

$$|\varphi_2\rangle, |\varphi_3\rangle, |\varphi_4\rangle, \alpha|\chi_1\rangle + \beta|5\rangle, \gamma|\chi_1\rangle + \delta|5\rangle$$

$$\begin{matrix} \| & \| & \| \\ |\chi_2\rangle & |\chi_3\rangle & |\chi_4\rangle \end{matrix}$$

which is what we found