MATH 420 - FALL 2019
ASSIGNMENT 2

Note: Most of these problems are taken from *Partial Differential Equations*, by L.C. Evans. In the following you may assume that U is a bounded open subset of $\mathbb{R}^n$ with smooth boundary ∂U, unless otherwise stated.

(1) (Existence of a partition of unity)
  (a) Let U, V be open sets with $V \subset\subset U$. Show that there exists a smooth function ζ such that $\zeta = 1$ on V and $\zeta = 0$ near ∂U.

  Hint: take $V \subset\subset W \subset\subset U$ and mollify χ_W.

  (b) Assume $U \subset\subset \cup_{i=1}^N V_i$ for bounded open sets V_i. Show that there exist C^∞ functions ζ_i with support in V_i such that $0 \leqslant \zeta_i \leqslant 1$ and
$$\sum_{i=1}^N \zeta_i(x) = 1, \quad x \in U.$$

(2) Fix $1 \leqslant p < \infty$. Show that there does *not* exist a bounded linear operator
$$T : L^p(U) \to L^p(\partial U),$$
such that $Tu = u|_{\partial U}$ whenever $u \in C(\overline{U}) \cap L^p(U)$.

(3) (Chain rule) Assume $F : \mathbb{R} \to \mathbb{R}$ is C^1 with F' bounded. Suppose $u \in W^{1,p}(U)$ for some $1 \leqslant p < \infty$. Show
$$v := F(u) \in W^{1,p}(U), \quad \text{and } v_{x_i} = F'(u)u_{x_i},\ i = 1, \dots, n.$$

(4) Assume that $1 \leqslant p < \infty$. Recall that $u^+ = \max(u, 0)$ and $u^- = -\min(u, 0)$.
  (a) Show that if $u \in W^{1,p}(U)$ then $|u| \in W^{1,p}(U)$.
  (b) Show that $u \in W^{1,p}(U)$ implies $u^+, u^- \in W^{1,p}(U)$ and
$$Du^+ = \begin{cases} Du & \text{a.e. on } \{u > 0\} \\ 0 & \text{a.e. on } \{u \leqslant 0\}. \end{cases}$$
  and
$$Du^- = \begin{cases} 0 & \text{a.e. on } \{u \geqslant 0\} \\ -Du & \text{a.e. on } \{u < 0\}. \end{cases}$$
  Hint: consider the function
$$F_\varepsilon(s) = \begin{cases} (s^2 + \varepsilon^2)^{1/2} - \varepsilon, & s \geqslant 0 \\ 0, & s < 0 \end{cases}$$

and show $u^+ = \lim_{\varepsilon\to 0} F_\varepsilon(u)$.

(c) Show that if $u \in W^{1,p}(U)$ then $Du = 0$ a.e. on $\{u = 0\}$.

(5) Consider the operator

$$Lu = -\sum_{i,j=1}^{n} (a^{ij} u_{x_i})_{x_j} + cu.$$

Prove that there exists a constant $\mu > 0$ such that the corresponding bilinear form $B[\cdot,\cdot]$ satisfies the hypotheses of the Lax-Milgram Theorem, provided

$$c(x) \geqslant -\mu, \quad x \in U.$$