Ex: What is probability of rolling at least one six in four throws of the die?

recall: $\sum_1 P(i) = 1$

$$p(\text{one six}) + p(\text{no six}) = 1$$

$$p(\text{no six}) = P(\text{not six AND not six AND not six AND not six})$$

$$= \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6}$$

$$\Rightarrow P(\text{one six}) = 1 - \left(\frac{5}{6}\right)^4 = \frac{671}{1296} \approx 0.517$$

Probability distributions can be discrete or continuous

Discrete (coin flips, dice, etc)

$P(k) \equiv$ probability of outcome k

$$\sum_k P(k) = 1$$

Continuous (ex: gaussian distribution)

$P(x) \equiv$ probability density of outcome x

$P(x)\,dx$ = probability to observe x in the interval $[x, x+dx]$

$$\int_{-\infty}^{\infty} P(x)\, dx = 1$$

$$\text{probability } x \in [x_1, x_2] = \int_{x_1}^{x_2} P(x)\, dx$$

Distributions are characterized by their moments (mean, variance, etc)

mean / expectation value

$$\mu \equiv \langle x \rangle \equiv \sum_i x_i P(i)$$

$$\mu \equiv \langle x \rangle \equiv \int_{-\infty}^{\infty} dx\, x\, P(x)$$

<u>Note</u>: ①
$$\langle x + y \rangle = \int dx\, (x+y)\, P(x)$$
$$= \int dx\, x\, P(x) + \int dx\, y\, P(x)$$
$$= \langle x \rangle + \langle y \rangle$$

②
$$\langle c x \rangle = \int dx\, c x\, P(x)$$
$$= c \int dx\, P(x)$$
$$= c \langle x \rangle$$

variance $\sigma^2 \equiv \langle (x-\mu)^2 \rangle$ σ = standard deviation

$$= \langle x^2 - 2x\mu + \mu^2 \rangle$$

$$= \int dx\, P(x) \left(x^2 - 2x\mu + \mu^2 \right)$$

$$= \int dx\, P(x)\, x^2 - \int dx\, P(x)\, 2x\mu + \underbrace{\int dx\, P(x)}_{1}\, \mu^2$$

$$= \langle x^2 \rangle - 2\mu \langle x \rangle + \mu^2$$

$$= \langle x^2 \rangle - 2\langle x \rangle\langle x \rangle + \langle x \rangle^2$$

$$\boxed{\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2}$$

Probability Distributions

There are two probability distributions that come up over and over in statistical mechanics,

$$\left.\begin{array}{c}\text{binomial distribution} \\ \rightarrow \text{discrete} \\ \\ \text{gaussian distribution} \\ \rightarrow \text{continuous}\end{array}\right\} \text{large } N \text{ limit}$$

Binomial distribution

- often introduced using coin flips, why is it relevant here?

binomial distribution = sequence of bernoulli events (like coin flips)

general chacteristics

(1) two outcomes (H / T)

(2) value of each outcome independent

Also satisfied by:

non interacting spins (up/down)

1D random walk (step left / right)

We will explore both of these as we go on

<u>Reminder</u>: permutations & combinations

<u>Ex</u>: How many ways to choose 5 letters from A - M?

A B C D E F G H I J K L M

(1) K F I B J

13 choices, 12 choices, 11 choices, 10 choices, 9 choices

$$\# \text{ ways} = 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9$$

$$= \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8!}{8!}$$

$$= \frac{13!}{8!}$$

(2) • this assumes order of choices matters

e.g. KFIBJ $\neq$ BKFJI

often this is not the case

e.g. $\uparrow\uparrow\downarrow\uparrow = \uparrow\downarrow\uparrow\uparrow = \downarrow\uparrow\uparrow\uparrow$

spin states, only matters total # up/down

If order doesn't matter, we have to divide by the number of ways to rearrange slots

So in the letter example, we have 5 letters K, F, I, B, J we want to put into the slots

$$\underline{K}\quad \underline{F}\quad \underline{I}\quad \underline{B}\quad \underline{J}$$

5 choices

4 choices (already put K)

3 choices (put K, F)

2 choices

1 choice

$\Rightarrow$ # ways to put letters into slots

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$$

(3) $\Rightarrow$ If we want # ways to choose 5 letters from A–M (this assumes order doesn't matter)

$$\# \text{ ways} = \frac{13!}{8!} \Big/ 5!$$

$\frac{13!}{8!}$ ← ways to choose letters

$5!$ ← ways to rearrange selection

$$= \frac{13!}{8!\,5!}$$

- this is the number of <u>combinations</u>, "n choose m"

$$C(n,k) = \binom{n}{m} \equiv \frac{n!}{(n-m)!\,m!}$$

<u>Back to distributions</u>:

Probability of observing m of n things in the state with probability p is

$$P_n(m) = \frac{n!}{m!\,(n-m)!} p^m (1-p)^{n-m}$$

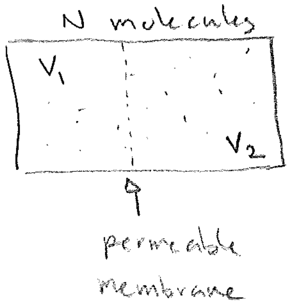
<u>ex</u>: m spins up of n spins $\uparrow\uparrow\downarrow\downarrow\uparrow\downarrow\uparrow\uparrow \ldots$

<u>mean value</u> $\langle m \rangle = np$

- this fits our intuition, spins have equal probability to be up/down, on average we expect half up

<u>Variance</u>: $\sigma^2 = np(1-p)$

Ex: Container volume V contains N gas molecules, assume gas is dilute so position of molecules independent. Although density on average is uniform, there are flucuations in density - how do they depend on relative volume?



① What is probability that a particular molecule is in V_1?

$$p = V_1 / V$$

② What is probability there are N_1 molecules in V_1, N_2 in V_2? $(N_1 + N_2 = N)$

→ molecules are either in V_1 or V_2

→ low density ⇒ noninteracting
positions independent!

⇒ binomial distribution

$$P_{N_1}(N_i) = \frac{N!}{N_1!\,(N-N_1)!}\, p^{N_1} (1-p)^{N-N_1}$$

$$= \frac{N!}{N_1!\,(N-N_1)!} \left(\frac{V_1}{V}\right)^{N_1} \left(\frac{V_2}{V}\right)^{N-N_1}$$

(3) What is average number of molecules in each part?

$$\langle N_1 \rangle = pN = \left(\frac{V_1}{V}\right) N$$

$$\langle N_2 \rangle = (1-p)N = \left(\frac{V_2}{V}\right) N$$

(4) What are the relative flucuations of the number of molecules in each part?

$$\frac{\sigma_1}{\langle N_1 \rangle} = \frac{\sqrt{Np(1-p)}}{\langle N_1 \rangle} = \frac{\sqrt{N(V_1/V)(V_2/V)}}{(V_1/V)\,N} = \frac{\sqrt{(V_2/V)}}{\sqrt{(V_1/V)}\,\sqrt{N}} = \sqrt{\frac{V_2}{V_1}}\,\frac{1}{\sqrt{N}}$$

$$\frac{\sigma_2}{\langle N_2 \rangle} = \frac{\sqrt{N(V_2/V)(V_1/V)}}{(V_2/V)\,N} = \sqrt{\frac{V_1}{V_2}}\,\frac{1}{\sqrt{N}}$$

We will see that the fact that relative flucuations scale as $1/\sqrt{N}$ is quite a generic result

Note: Often we will need to evaluate $\log N!$ in the limit $N \gg 1$

Stirling's approximation

$$\log N! \approx N\log N - N + \frac{1}{2}\log(2\pi N)$$

sometimes we will neglect