HW 5

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1.) $T_{ij} = E_i E_j + B_i B_j - \delta_{ij} \frac{1}{2}(E^2 + B^2)$

$$\vec{f} = \nabla \cdot T - \frac{1}{c^2}\frac{\partial \vec{S}}{\partial t}$$

$$\vec{S} = c\vec{E} \times \vec{B}$$

J = current density

$$\vec{f} = \rho \vec{E} + \vec{J} \times \vec{B}$$

$$= (\vec{\nabla} \cdot \vec{E})\vec{E} + (\vec{\nabla} \times \vec{B}) \times \vec{B} - \frac{\partial \vec{E}}{\partial t} \times \vec{B}$$

(for aesthetics, from here on $\vec{E} = E$, etc)

$$= (\nabla \cdot E)E + (\nabla \times B) \times B - \frac{\partial}{\partial t}(E \times B)$$

$$- E \times (\nabla \times E)$$

$$= \left[(\nabla \cdot E) E - E \times (\nabla \times E) \right]$$

$$- \left[B \times (\nabla \times B) \right] - \frac{\partial}{\partial t} (E \times B)$$

$$= \left[(\nabla \cdot E) E - E \times (\nabla \times E) \right]$$

$$+ \left[(\nabla \cdot B) B - B \times (\nabla \times B) \right] - \frac{\partial}{\partial t} (E \times B)$$

$$= \left[(\nabla \cdot E) E + (E \cdot \nabla) E \right]$$

$$+ \left[(\nabla \cdot B) B + (B \cdot \nabla) B \right] - \frac{1}{2} \nabla (E^2 + B^2)$$

$$- \frac{\partial}{\partial t} (E \times B)$$

$$= \nabla \cdot T - \frac{1}{c^2} \frac{\partial \vec{S}}{\partial t}$$

2.) $F'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} F^{\alpha\beta}$

$$= (\gamma \quad \beta\gamma \quad 0 \quad 0)(\gamma \quad -\beta\gamma \quad 0 \quad 0)(0 \quad E_x \quad E_y \quad E_z)$$

$$\begin{pmatrix} -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -\beta\gamma E_x & \gamma E_x & E_y & E_z \\ \gamma E_x & -\beta\gamma E_x & B_z & -B_y \\ -\gamma E_y - \beta\gamma B_z & -\beta\gamma E_y - \gamma B_z & 0 & B_x \\ -\gamma E_z + \beta\gamma B_y & -\beta\gamma E_z + \gamma B_y & -B_x & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & E_x & \gamma(E_y + \beta B_z) & \gamma(E_z - \beta B_y) \\ -E_x & 0 & \gamma(B_z + \beta E_y) & -\gamma(B_y - \beta E_z) \\ -\gamma(E_y + \beta B_z) & -\gamma(B_z + \beta E_y) & 0 & B_x \\ -\gamma(E_z - \beta B_y) & \gamma(B_y - \beta E_z) & -B_x & 0 \end{pmatrix}$$

3.) i.

$$\frac{dU^\mu}{d\tau} = \frac{e}{mc} F^{\mu\nu} U_\nu$$

time: $\frac{dU^0}{d\tau} = \frac{e}{mc} \vec{E}_0 \cdot \vec{u}$

space: $\frac{d\vec{u}}{d\tau} = \frac{e}{mc} \vec{E}_0 U^0$

$$\frac{d^2 U^0}{d\tau^2} = \frac{e}{mc} \vec{E}_0 \cdot \frac{d\vec{u}}{d\tau} = \frac{e^2}{m^2 c^2} E_0^2 U^0$$

Solving the PDE:

$$U^0 = A e^{\frac{eE_0\tau}{mc}} + B e^{-\frac{eE_0\tau}{mc}}$$

$$\frac{d\vec{u}}{d\tau} = \frac{e}{mc} \vec{E}_0 \left(A e^{\frac{eE_0\tau}{mc}} + B e^{-eE_0\tau/mc} \right)$$

$$\vec{u} = \vec{u}_0 + \vec{E}_0 \left(A e^{eE_0\tau/mc} - B e^{-eE_0\tau/mc} \right)$$

$\frac{dU^0}{} \quad \frac{e}{} \quad \vec{E} \cdot \vec{}$

$$\frac{dU^0}{d\tau} = \frac{e}{mc} E_0 \cdot U$$

$$= \frac{e}{mc}\left[\cancel{\vec{E}_0 \cdot \vec{u}_0}^{\,0} + \vec{E}_0 \cdot \vec{E}_0 \left(\qquad \right)\right]$$

$$= \frac{e}{mc} E_0^2 \left(A e^{eE_0\tau/mc} - B e^{-eE_0\tau/mc}\right)$$

$$U^\mu U_\mu = c^2 \Rightarrow c^2 = 4AB - u_0^2$$

if $A = B$, then we can say that

$$u_\perp = 0$$

$$U^0 = A e^{eE_0\tau/mc} + A e^{-eE_0\tau/mc} = 2A\cosh\left(\frac{eE_0\tau}{mc}\right)$$

$$\vec{u} = \vec{u}_0 + 2A\vec{E}_0 \sinh\left(\frac{eE_0\tau}{mc}\right)$$

$$c^2 = 4A^2 - u_0^2 \Rightarrow 2A = c\sqrt{1 + \frac{u_0^2}{c^2}}$$

$u_0 = \gamma_0 v_0$

$U^0 = c\gamma_0 \cosh\left(\frac{eE_0\tau}{mc}\right)$

$\vec{u} = \gamma_0 \vec{V}_0 + c\gamma_0 \vec{E} \sinh\left(\frac{eE_0\tau}{mc}\right)$

$\therefore t = \int_0^\tau U^0 d\tau' = \int_0^\tau c\gamma_0 \cosh\left(\frac{eE_0\tau}{mc}\right) d\tau$

$= \frac{mc^2\gamma_0}{eE_0} \sinh\left(\frac{eE_0\tau}{mc}\right)$

$\vec{X} = \int_0^\tau \vec{u}\, d\tau' = \int_0^\tau \left[\gamma_0 \vec{V}_0 + c\gamma_0 E \sinh\left(\frac{eE_0\tau}{mc}\right)\right] d\tau'$

$= \gamma_0 \vec{V}_0 \tau + \frac{mc^2\gamma_0}{eE_0} \vec{E}_0 \left[\cosh\left(\frac{eE_0\tau}{mc}\right) - 1\right]$

$\tau = \frac{mc}{eE_0} \sinh^{-1}\left(\frac{eE_0 t}{mc\gamma_0}\right)$

$$\gamma = \gamma_0\sqrt{1+\frac{e^2E_0^2t^2}{m^2c^2\gamma_0^2}} \qquad \vec{u} = \gamma_0\left(\vec{V}_0 + \frac{eE_0t}{m\gamma_0}\right)$$

$$\vec{V}(t) = \frac{\vec{V}_0 + \frac{e\vec{E}_0 t}{m\gamma_0}}{\sqrt{1+\left(\frac{eE_0t}{mc\gamma_0}\right)^2}}$$

$$\vec{X} = \int_0^t \vec{V}(t')dt'$$

$$= \frac{mc\gamma_0\vec{V}_0}{eE_0}\sinh^{-1}\left(\frac{eE_0t}{mc\gamma_0}\right) + \frac{mc^2\gamma_0\vec{E}_0}{eE_0}\left[\sqrt{1+\left(\frac{eE_0t}{mc\gamma_0}\right)^2} - 1\right]$$

ii)

$$X_\parallel = \frac{mc^2\gamma_0}{eE_0}\left[\cosh\left(\frac{eE_0\tau}{mc}\right)-1\right] \qquad X_\perp = \gamma_0V_0\tau$$

$$= \frac{mc^2\gamma_0}{eE_0}\left[\cosh\left(\frac{eE_0X_\perp}{mc\gamma_0V_0}\right)-1\right]$$

$$\frac{mc\gamma_0}{}$$

hen $t << \frac{}{eE_0}$

$$\lim_{t\to 0} \cosh\left(\frac{eE_0 x_\perp}{mc\gamma_0 v_0}\right) = 1 + \frac{\left(\frac{eE_0 x_\perp}{mc\gamma_0 v_0}\right)^2}{2!} + \dots$$

$$x_\parallel = \frac{\cancel{m}\cancel{c^2}\cancel{\gamma_0}}{2\cancel{e}\cancel{E_0}}\left(\frac{e^{\cancel{2}} E_0^{\cancel{2}} x_\perp^2}{m^{\cancel{2}}\cancel{c^2}\gamma_0^{\cancel{2}} v_0^2}\right) = \frac{eE_0 x_\perp^2}{2m\gamma_0 v_0^2}$$

at short times, trajectory will be parabolic

hen $t >> \frac{mc\gamma_0}{eE_0}$

$$\lim_{t\to\infty} \cosh\left(\frac{eE_0 x_\perp}{mc\gamma_0 v_0}\right) = e^{\frac{eE_0 x_\perp}{mc\gamma_0 v_0}}$$

$$x_\parallel = \frac{mc^2\gamma_0}{2eE_0} e^{\frac{eE_0 x_\perp}{mc\gamma_0 v_0}}$$

t long times, trajectory becomes

hyperbolic

1.) i

$$\frac{dp^\mu}{d\tau} = \frac{1}{c} F^{\mu\nu} J_\nu \qquad U = U(\beta)$$

$$F^\mu = q\left[\partial^\mu (U\cdot A) - (U\cdot\partial) A^i\right]$$

$$\frac{dp^\mu}{d\tau} = \gamma \frac{dp^\mu}{dt} = \gamma F^\mu = (\partial^\mu A^\alpha - \partial^\alpha A^\mu) U_\alpha$$

$$\gamma[\partial^\mu A^0 - \partial^0 A^\mu - v_j(\partial^\mu A^j - \partial^j A^\mu)$$

$$- v_k(\partial^\mu A^k - \partial^k A^\mu)] \Rightarrow$$

$$\boxed{B^i = -\varepsilon_{ijk}(\partial^i A^j - \partial^j A^i)}$$

$$\Rightarrow [E^i + v_j B^k - v_k B^j] = \vec{E} + (v \times \vec{B})$$