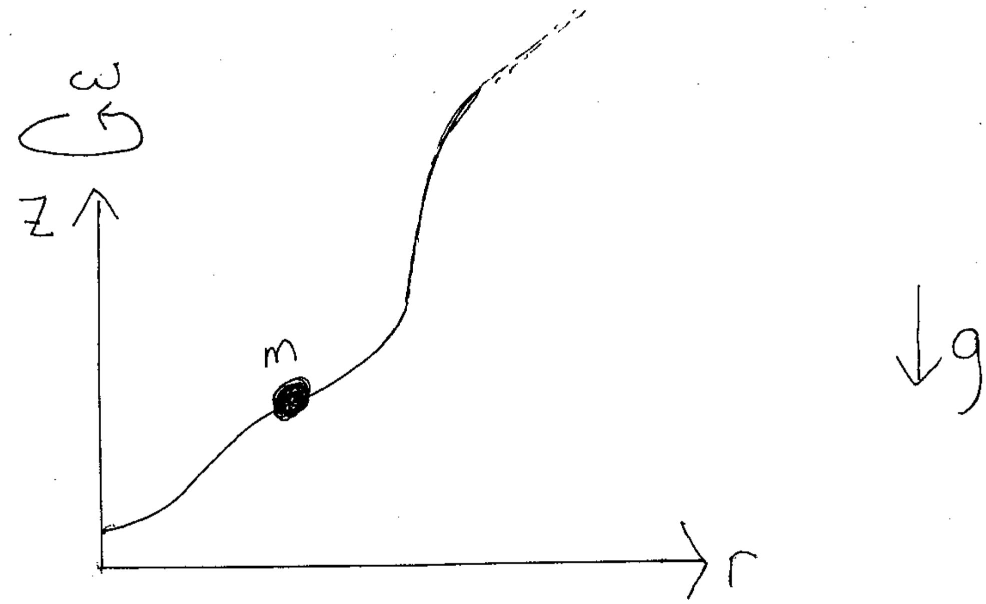
Final Exam for PHYS 411, Fall 2016. Th. Dec 1, 3:30–5:30

1. [**5pts/35**] Consider a wire that lies in a 2-D plane, of functional form $z = f(r)$. A bead of mass m slides frictionlessly along the wire, with gravity acting downwards. The wire rotates about the vertical at constant angular frequency ω.



(a) What is the Lagrangian?
(b) Derive an equation for the value of r at which the bead is in equilibrium.
(c) What is the Hamiltonian? And, is it conserved?

(a) Since $z = f(r)$, we have $\dot{z} = f'\dot{r}$ and

$$L = \frac{m}{2}\left(\dot{r}^2(1 + f'^2) + r^2\omega^2\right) - mgf(r) \tag{1}$$

(b) At equilibrium, the left side of the eom is 0. So, $dL/dr = 0$. i.e., $\omega^2 r = gf'$
(c)

$$H = \frac{p^2}{2m(1 + f'^2)} - \frac{mr^2\omega^2}{2} + mgf \tag{2}$$

Yes, it's conserved.

2. **[10pts/35]** Consider the following Lagrangian that describes 2D motion for a particle of mass 1 and charge q in a uniform B-field:

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + qBx\dot{y} \tag{3}$$

(i) What is the solution S to the Hamilton-Jacobi equation? (You should reduce S to quadratures–i.e., to integrals in a single variable. But you do not need to do the integral).
(ii) You should have two constants of motion in your solution S. What are they, and what are their physical interpretations?
(iii) Use your solution S to determine the equation for the orbit-in-space, i.e., for the curve in the $x-y$ plane that the particle traces. (There will be an integral, but it is trivial to perform.) Describe qualitatively, and briefly, your result for the orbit-in-space.

(i)

$$p_x = \dot{x} \tag{4}$$

$$p_y = \dot{y} + qBx \tag{5}$$

$$H = \frac{p_x^2}{2} + \frac{\dot{y}^2}{2} \tag{6}$$

$$= \frac{p_x^2}{2} + \frac{(p_y - qBx)^2}{2} \tag{7}$$

h-jac is:

$$\frac{1}{2}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{1}{2}(\frac{\partial S}{\partial y} - qBx)^2 + \frac{\partial S}{\partial t} = 0 \tag{8}$$

solution is

$$S = -Et + S_x(x) + p_y y \tag{9}$$

where

$$\frac{1}{2}\left(\frac{dS_x}{dx}\right)^2 + \frac{1}{2}(p_y - qBx)^2 = E \tag{10}$$

And so,

$$S_x = \int dx \sqrt{2E - (p_y - qBx)^2} \tag{11}$$

(ii) Constants are E (energy) and p_y (y-momentum)
(iii) Equation for $y(x)$ comes from:

$$\beta_y = \partial S/\partial p_y = y + \int dx \frac{-(p_y - qBx)}{\sqrt{2E - (p_y - qBx)^2}} \tag{12}$$

The integral is trivial, since

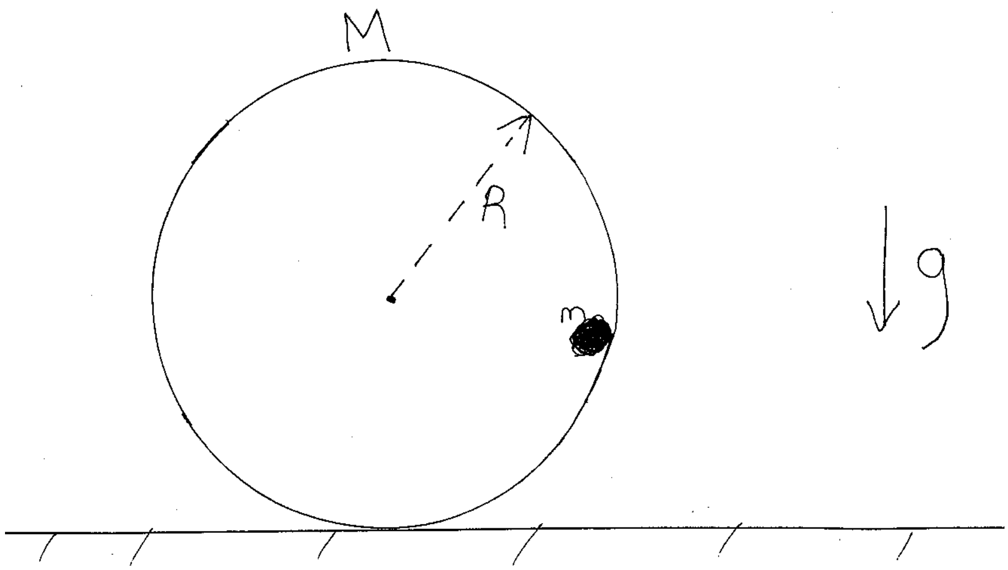
$$\frac{d}{dx}\sqrt{2E - (p_y - qBx)^2} = \frac{-qB(p_y - qBx)}{\sqrt{2E - (p_y - qBx)^2}} \tag{13}$$

So,

$$\beta_y = y - \frac{1}{qB}\sqrt{2E - (p_y - qBx)^2} \tag{14}$$

A circle!

3. [10pts/35] Consider a bead of mass m that slides without friction inside a ring of mass M. The ring rolls without slipping on a horizontal surface. All motion is confined to a vertical plane and gravity acts downwards.



(a) What is the Lagrangian?
(b) Initially, the ring and bead are held stationary, with the bead at its rightmost position relative to the ring (at height R above the surface). After the ring and bead are released, what is the translational speed of the ring when the bead is at its lowest position?

(a)

$$X = R\theta \tag{15}$$
$$x = R\theta + R\sin\phi \tag{16}$$
$$y = -R\cos\phi \tag{17}$$

So, using fact that ring has $T = T_{cm} + T_{rcm}$,

$$L = MR^2\dot{\theta}^2 + \frac{m}{2}\left(R^2\dot{\theta}^2 + R^2\dot{\phi}^2 + 2R^2\dot{\theta}\dot{\phi}\cos\phi\right) + mgR\cos\phi \tag{18}$$

(b) constants of motion are:

$$p_\theta = R^2(2M+m)\dot{\theta} + mR^2\dot{\phi}\cos\phi \tag{19}$$
$$E = MR^2\dot{\theta}^2 + \frac{m}{2}\left(R^2\dot{\theta}^2 + R^2\dot{\phi}^2 + 2R^2\dot{\theta}\dot{\phi}\cos\phi\right) - mgR\cos\phi \tag{20}$$

Initially, $\cos\phi = 0$, so

$$p_\theta = 0 \tag{21}$$
$$E = 0 \tag{22}$$

Finally $\cos\phi = 1$, (setting $R = m = 1$ for ease of units):

$$p_\theta = (2M+1)\dot{\theta} + \dot{\phi} = 0 \tag{23}$$
$$E = (M + \frac{1}{2})\dot{\theta}^2 + \frac{1}{2}\dot{\phi}^2 + \dot{\theta}\dot{\phi} - g = 0 \tag{24}$$

So, subbing the former into the latter gives

$$\dot{\theta}^2 \left(M + \frac{1}{2} + \frac{(2M+1)^2}{2} - (2M+1) \right) = g \tag{25}$$
$$\dot{\theta}^2 \left(2M^2 + M \right) = g \tag{26}$$

Putting back dimensions,

$$\dot{X} = R\dot{\theta} = \frac{(gR)^{1/2} m}{\sqrt{(2M+m)M}} \tag{27}$$

4. [10pts/35] Consider the following transformation:

$$Q = \alpha q^a p \tag{28}$$
$$P = \beta q^b \tag{29}$$

(a) Determine a and b required to make the transformation canonical.
(b) Use this transformation to solve the following Hamiltonian:

$$H = \frac{q^4 p^2}{2} + \frac{\lambda}{q^2} \tag{30}$$

where λ is a constant, i.e., what are the solutions $q(t)$ and $p(t)$ to this Hamiltonian?
Hint: transform the Hamiltonian into a very simple one that you know how to solve.
Optional: you can check your solution by inserting it into the equations of motion for q and p.

(a) Need $[Q, P] = 1$. So:

$$-\alpha q^a \beta b q^{b-1} = 1 \tag{31}$$

Need:

$$a - 1 + b = 0 \tag{32}$$
$$\alpha\beta b = -1 \tag{33}$$

So:

$$b = -\frac{1}{\alpha\beta} \tag{34}$$
$$a = 1 + \frac{1}{\alpha\beta} \tag{35}$$

(b) Inverting the transformation:

$$q = \left(\frac{P}{\beta}\right)^{-\alpha\beta} \tag{36}$$
$$p = \frac{Q}{\alpha}\left(\frac{P}{\beta}\right)^{\alpha\beta(a)} \tag{37}$$
$$= \frac{Q}{\alpha}\left(\frac{P}{\beta}\right)^{\alpha\beta+1} \tag{38}$$

If I set $\alpha\beta = 1$, I will turn it into an SHO. I can, e.g., set $\alpha = \beta = 1$. Then,

$$H = \frac{1}{2}Q^2 + \lambda P^2 \tag{39}$$

The eom is

$$\dot{Q} = 2\lambda P \tag{40}$$
$$\dot{P} = -Q \tag{41}$$

So, $\ddot{Q} = -2\lambda Q$

$$Q = A\cos(\sqrt{2\lambda}t + \phi) \tag{42}$$
$$P = -\frac{A}{\sqrt{2\lambda}}\sin() \tag{43}$$

In terms of which,

$$q = 1/P = \frac{\sqrt{2\lambda}}{A}\frac{1}{\sin()} \tag{44}$$
$$p = QP^2 = \frac{-A^3}{2\lambda}\cos()\sin()^2 \tag{45}$$

Check: should have $\dot{q} = pq^4$.

$$LHS = -\frac{\cos}{\sin^2}\frac{2\lambda}{A} \tag{46}$$
$$RHS = -(2\lambda)\frac{1}{A}\frac{\cos}{\sin^2} \tag{47}$$