Hw 4

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1.) $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$

Show $ds^2 = ds'^2$ $\quad y = y' \quad z = z'$

$x = \gamma(x' - vt') \qquad t = \gamma\left(t' + \frac{vx'}{c^2}\right)$

$$ds^2 = c^2\gamma^2\left(dt' + \frac{v\,dx'}{c^2}\right)^2 - \gamma^2(dx' + v\,dt')^2$$
$$- dy'^2 - dz'^2$$

$$= c^2\gamma^2\left(dt'^2 + \frac{v^2dx'^2}{c^4} + \frac{2v\,dx'dt'}{c^2}\right)$$
$$- \gamma^2(dx'^2 + v^2dt'^2 + 2v\,dx'dt') - dy'^2 - dz'^2$$

$$= \gamma^2\left(c^2dt'^2 + \frac{v^2dx'^2}{c^2} + \cancel{2v\,dx'dt'}\right.$$
$$\left. - dx'^2 - v^2dt'^2 \cancel{- 2v\,dx'dt'}\right) - dy'^2 - dz'^2$$

$$= \gamma^2\left[dt'^2(c^2 - v^2) - dx'^2\left(1 - \frac{v^2}{c^2}\right)\right] - dy'^2 - dz'^2$$

$$= \gamma \,[\,dt\,(c \quad v\,) \quad dx\,(1 \quad c^2)\,] - dy - dz$$

$$= \gamma^2\left[c^2 dt'^2 \underbrace{\left(1-\frac{v^2}{c^2}\right)}_{1/\gamma^2} - dx'^2 \underbrace{\left(1-\frac{v^2}{c^2}\right)}_{1/\gamma^2}\right] - dy'^2 - dz'^2$$

$$= c^2 dt'^2 - dx'^2 - dy'^2 - dz'^2$$

2.) $x = \gamma(x' - vt')$ $\quad t = \gamma\left(t' - \frac{vx'}{c^2}\right)$

$ct(\sigma) = \lambda \sinh(\sigma)$ $\quad x(\sigma) = \lambda \cosh(\sigma)$

i.) $ds^2 = c^2 dt^2 - dx^2$

$$d\,ct(\sigma) = \lambda \cosh(\sigma)\, d\sigma$$

$$dx(\sigma) = \lambda \sinh(\sigma)\, d\sigma$$

$$ds^2 = \lambda^2 \sinh^2(\sigma)\, d\sigma^2 - \lambda^2 \cosh^2(\sigma)\, d\sigma$$

$$= \lambda^2 d\sigma^2$$

$$ds^2 = c^2 d\tau^2 = \lambda^2 d\sigma^2$$

$$c\,d\tau = \lambda\, d\sigma \quad \int d\sigma = \frac{c}{\lambda}\int d\tau \quad \sigma = \frac{c\tau}{\lambda}$$

ii $$\frac{dx}{d\tau} = \frac{dx}{d\sigma}\frac{d\sigma}{d\tau} = \lambda \sinh(\sigma)\frac{c}{\lambda}$$

$$= c\sinh\left(\frac{c\tau}{\lambda}\right) = u(\tau)$$

$$\int dx = \int c\sinh\left(\frac{c\tau}{\lambda}\right) d\tau$$

$$x = \lambda\cosh\left(\frac{c\tau}{\lambda}\right)$$

iii $$a(\tau) = \frac{d^2x}{d\tau^2} = \frac{d}{d\tau}\left[c\sinh\left(\frac{c\tau}{\lambda}\right)\right]$$

$$= \frac{c^2}{\lambda}\cosh\left(\frac{c\tau}{\lambda}\right)$$

iv) This is a valid trajectory since in space-time, these look like hyperbolic motion.

3.) $\vec{a} = \vec{g}$

Rindler Coordinates:

$$t' = x \sinh(at)$$
$$x' = x \cosh(at)$$

$$x = \sqrt{x'^2 - t'^2}$$
$$t = \frac{1}{a} \tanh^{-1}\left(\frac{t'}{x'}\right)$$

$$dx = \gamma(dx' - v\,dt')$$
$$dt = \gamma\left(dt' - \frac{v}{c^2}dx'\right)$$

$$u = \frac{u' - v}{1 - \frac{u'v}{\dots}}$$

u' = velocity of rest frame

$1-\frac{\;}{c^2}$ | u = vel. of moving ref. frame

$$du = \frac{du'\left(1-\frac{u'v}{c^2}\right)+(u'-v)\frac{v\,du'}{c^2}}{1-\frac{u'v}{c^2}}$$

$$= \frac{du'}{\gamma^2\left(1-\frac{u'v}{c^2}\right)^2}$$

$$\frac{du}{dt} = a = \frac{\frac{du'}{dt}}{\gamma^3\left(1-\frac{u'v}{c^2}\right)^3} = \frac{a'}{\gamma^3\left(1-\frac{u'v}{c^2}\right)^3}$$

in 1-D motion: $u' = v$

$$a = \gamma^3 a' = \frac{d(\gamma v)}{dt'} = v\frac{d\gamma}{dt'} + \gamma\frac{dv}{dt'}$$

$$= \gamma a'\left(1+\frac{\gamma^2 v^2}{c^2}\right)$$

$$\int d(\gamma v) = \int a\,dt'$$

$$\boxed{\gamma v = a t'}$$

$$\gamma = \sqrt{\frac{1}{1-\frac{v^2}{c^2}}} = \sqrt{1 + \frac{a^2 t'^2}{c^2}}$$

$$t' = \frac{\gamma v}{a} = \frac{v}{g}\sqrt{\frac{1}{1-\frac{v^2}{c^2}}}$$ $\boxed{i}$

$\boxed{ii}$ $$\frac{v}{c} = \tanh\left(\frac{g t'}{c}\right)$$

$$\frac{g t'}{c} = \tanh^{-1}\left(\frac{v}{c}\right)$$

$$t' = \frac{c}{g}\tanh^{-1}\left(\frac{v}{c}\right)$$

$\boxed{iii}$

$$t = \frac{v}{g}\sqrt{\frac{1}{1-\frac{v^2}{c^2}}} = \frac{c}{4g}\sqrt{\frac{1}{1-\frac{c^2}{4^2c^2}}}$$

$$= \frac{c}{4g}\sqrt{\frac{1}{1-\frac{1}{16}}} = \frac{c}{4g}\sqrt{\frac{1}{15/16}}$$

$$= \frac{c}{4g}\sqrt{\frac{16}{15}} = \frac{c}{g\sqrt{15}} = 7.9 \times 10^6\ s$$

$$t' = \frac{c}{g}\tanh^{-1}\left(\frac{1}{4}\right) = 7.8 \times 10^6\ s$$

4.)

i.) $F_{\mu\nu}F^{\mu\nu} =$

$$tr\left(\begin{bmatrix} 0 & E_x & E_y & E_z \\ -E & 0 & B & -B \end{bmatrix}\begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E & 0 & -B & B \end{bmatrix}\right)$$

$$\Bigg\{\begin{bmatrix} E_x & 0 & B_z & B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix}\begin{bmatrix} E_x & 0 & B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}\Bigg\}$$

$$=\left(E_x^2+E_y^2+E_z^2\right)+\left(E_x^2+B_z^2+B_y^2\right)$$

$$+\left(E_y^2+B_z^2+B_x^2\right)+\left(E_z^2+B_y^2+B_x^2\right)$$

$$=2\vec{E}^2+2\vec{B}^2=2\left(E^2+B^2\right)$$

(ii) $F^*_{\mu\nu}F^{\mu\nu}$

$$=tr\left\{\begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{bmatrix}\begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}\right\}$$

$$= (B_xE_x + B_yE_y + B_zE_z) + (B_xE_x + B_zE_z + B_yE_y)$$

$$+ (B_yE_y + B_zE_z + B_xE_x) + (B_zE_z + B_yE_y + B_xE_x)$$

$$= 4(B_xE_x + B_yE_y + B_zE_z) = 4\vec{B}\cdot\vec{E}$$

iii) A purely electric field in one inertial frame will not be seen as a purely magnetic field because the electromagnetic field is an invariant, as shown in part i.

iv) A progressive EM wave cannot be seen as purely electric + because of their or magnetic field ... it is invariant in orthogonality; it is invariant in any inertial reference frame and also in geometry.