Exponential Function

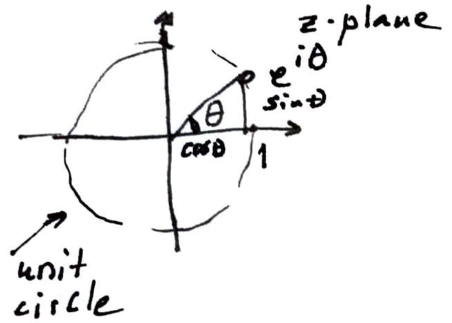
$$z = x+iy \Rightarrow e^z = e^{x+iy} \overset{def}{=} e^x(\cos y + i\sin y)$$

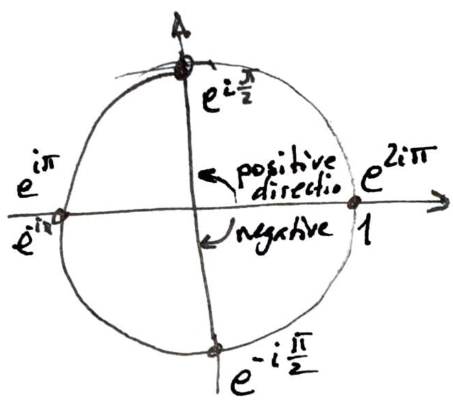
(we will also have later another, equivalent, definition)

If z is pure imaginary, say, $z = i\theta$, then

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{– Euler's formula.}$$

Note that $|e^{i\theta}| = |\cos\theta + i\sin\theta| = \sqrt{\cos^2\theta + \sin^2\theta} = 1 \quad \forall$ real θ





Also,

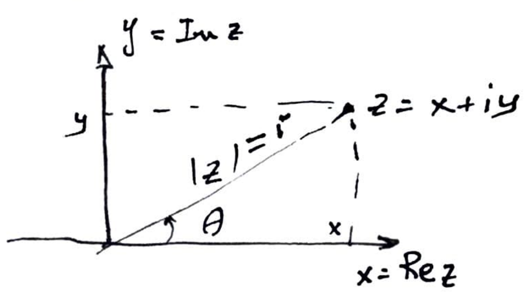
$$e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$$

because

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)$$

$$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) = e^{i(\theta_1+\theta_2)}$$

Trigonometric (Polar) form of complex numbers



$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = x+iy = r(\cos\theta + i\sin\theta) = re^{i\theta}, \quad r = |z|$$

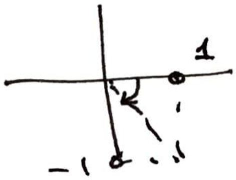
θ – argument z, $\theta = \arg z$; infinitely many arguments for the same z, they differ by a multiple of 2π.

By $\arg z$ we mean the entire set of θ

The value of θ between $\pm\pi$ is called the principal argument,

$\text{Arg}\, z$, $\quad -\pi < \text{Arg}\, z \le \pi$

Ex. $\operatorname{Arg}(1-i) = -\frac{\pi}{4}$



$\arg(1-i) = -\frac{\pi}{4} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$

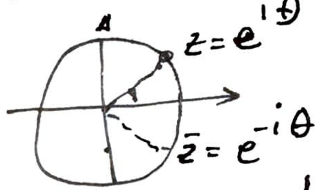
$1 - i = \sqrt{2}\, e^{-i\frac{\pi}{4}} = \sqrt{2}\, e^{-i(\frac{\pi}{4} + 2\pi n)}, \quad n = 0, \pm 1, \pm 2 \ldots$

Note: If $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$ and $z_1 = z_2 \Rightarrow r_1 = r_2$ & $\theta_1 - \theta_2 = 2\pi n$ for some n

Products and Quotients (Polar form)

$z_1 = r_1 e^{i\theta_1}, \quad z_2 = r_2 e^{i\theta_2} \Rightarrow$

$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}, \quad \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$



$z = re^{i\theta} \Rightarrow z^n = r^n e^{in\theta}$, n-integer

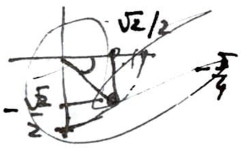
$z\bar{z} = |z|^2 = 1$

$\bar{z} = \frac{1}{z}$

Ex. $(1+i)^7 = \left[\sqrt{2}\, e^{i\frac{\pi}{4}}\right]^7 = 2^{\frac{7}{2}} e^{7i\frac{\pi}{4}}$

$= 8\sqrt{2}\, e^{-i\frac{\pi}{4}} = 8\sqrt{2}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right).$

$= 8(1-i)$



Note: $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2$ – may or may not be true

$z_1 = e^{i\frac{3\pi}{4}} = z_2 \Rightarrow z_1 z_2 = e^{i\frac{3\pi}{2}}$

$\operatorname{Arg} z_1 = \operatorname{Arg} z_2 = \frac{3\pi}{4}, \quad \operatorname{Arg} z_1 z_2 = -\frac{\pi}{2}.$

but $\arg(z_1 z_2) = \arg z_1 + \arg z_2$ is true if understood in the sense of sets

Roots of Complex Numbers

Given a complex number z_0 we want to find all complex numbers z such that $z^n = z_0$ (n-integer). Those z will be denoted $z_0^{1/n}$ and as we will see, there are exactly n such numbers, so that $z_0^{1/n}$ denotes all of them.

It is convenient to use the polar form of these numbers:

$z_0 = r_0 e^{i\theta_0}$, $z = r e^{i\theta}$ and $z^n = z_0 \Rightarrow r^n e^{in\theta} = r_0 e^{i\theta_0}$

$\Rightarrow r^n = r_0$ & $n\theta = \theta_0 + 2k\pi$, $k = 0, \pm 1, \pm 2, \dots$

Thus, $r = r_0^{1/n}$, $\theta = \frac{1}{n}\theta_0 + 2\frac{k}{n}\pi$, $k = 0, \pm 1, \pm 2 \dots \Rightarrow$

$z = \sqrt[n]{r_0}\, e^{i\left(\frac{\theta_0}{n} + 2\frac{k}{n}\pi\right)}$, $k = 0, \pm 1, \pm 2, \dots$

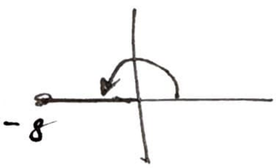
Only n of these roots are different: $k = 0, 1, \dots, n-1$ because $e^{2\pi i} = 1$.

A convenient formal way to get the final answer:

$z_0 = r e^{i(\theta_0 + 2\pi k)} \Rightarrow z_0^{1/n} = r^{1/n} e^{i\left(\frac{\theta_0}{n} + 2\frac{k}{n}\pi\right)}$, $k = 0, 1, \dots, n-1$

Ex. $(-8)^{1/3} = (8e^{i\pi})^{1/3} = \left[8e^{i(\pi + 2\pi k)}\right]^{1/3}$

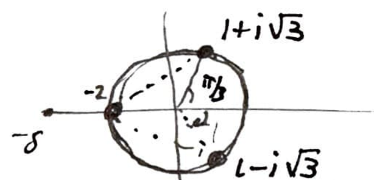
$= 2e^{i\frac{\pi}{3} + \frac{2}{3}\pi k i}$, $k = 0, 1, 2$



$k=0$: $2e^{i\frac{\pi}{3}} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 1 + i\sqrt{3}$

$k=1$: $2e^{i\pi} = -2$

$k=2$: $2e^{i\frac{\pi}{3} + i\frac{4\pi}{3}} = 2e^{\frac{5\pi}{3}i} = e^{-\frac{\pi}{3}i} = 1 - i\sqrt{3}$



equilateral triangle

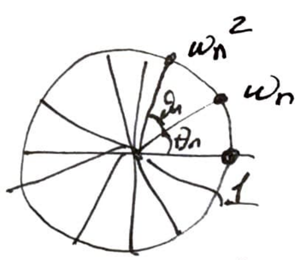
Ex. $1^{1/n} = ?$ (n - positive integer)

$1 = 1 \cdot e^{i \cdot 2k\pi}$, $k = 0, \pm 1, \pm 2, \dots$

$1^{1/n} = \sqrt[n]{1}\, e^{i\frac{2k\pi}{n}}$, $k = 0, 1, \dots, n-1$

$1, e^{\frac{2\pi}{n}i}, e^{2\cdot\frac{2\pi}{n}\cdot i}, e^{3\cdot\frac{2\pi}{n}i}, \dots, e^{(n-1)\frac{2\pi}{n}i}$

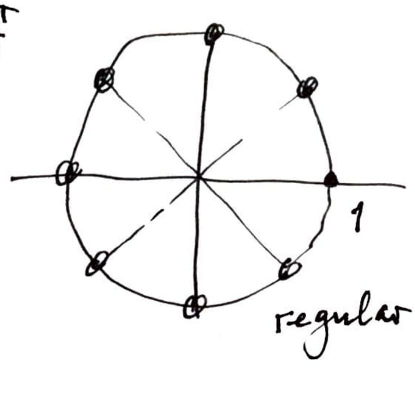
$1, \overset{\shortparallel}{\omega_n}, \omega_n^2, \omega_n^3, \dots, \omega_n^{n-1}$ - n roots



$\theta_n = \frac{2\pi}{n}$

Vertices of n-sided regular polygon

e.g. $n = 8 \Rightarrow \theta_n = \frac{\pi}{4}$



regular octagon

Functions of a Complex Variable

Def. A complex function $f(z)$ defined on a set of complex numbers S is a rule that assigns to each $z \in S$ a complex number $f(z)$
(S - domain of definition)

Ex. $f(z) = z^2$, $\quad z = x+iy \Rightarrow f(z) = (x+iy)^2 = x^2 - y^2 + 2ixy$
$$= u(x,y) + iv(x,y)$$
$u(x,y) = \mathrm{Re}(f(z))$, $\quad v(x,y) = \mathrm{Im}\, f(z)$

In general, $\quad f(z) = u(x,y) + iv(x,y)$, where $z = x+iy$
Sometimes we deal real-valued function of complex variable, e.g.
$f(z) = |z|^2 = x^2 + y^2$ or $f(z) = z^2 + \bar{z}^2$

Sometimes it is convenient to use the polar form of z, $\quad z = re^{i\theta}$,
then $\quad u = u(r,\theta)$ & $v = v(r,\theta)$

Ex. $f(z) = z^2 = (re^{i\theta})^2 = r^2 e^{2i\theta} = r^2(\cos 2\theta + i \sin 2\theta) \Rightarrow$
$\mathrm{Re}\, f(z) = r^2 \cos 2\theta$, $\quad \mathrm{Im}\, f(z) = r^2 \sin 2\theta$

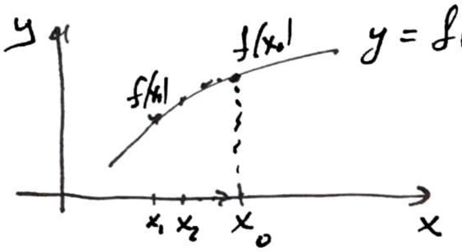
Generalization of the concept of a function: rule that assigns more than one value to each $z \in S$ (domain of definition) – multi-valued function

Ex. $z = re^{i\theta} \Rightarrow z^{1/2} = \sqrt{r}\, e^{i\frac{\theta}{2} + \pi i k}$, $\quad k = 0, 1$
$z^{1/2} = \pm \sqrt{r}\, e^{i\frac{\theta}{2}}$ - multi-valued function
Often times we choose one of the possible values, e.g.
$z^{1/2} = \sqrt{r}\, e^{i\theta}$

Limits and Continuity

Def. A function $f(z)$ is continuous at $z = z_0$ if
$f(z_0)$ exists and $\lim_{z \to z_0} f(z) = f(z_0)$

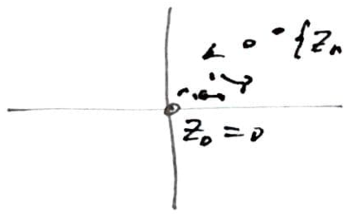
We need to discuss the meaning of the limit. In calculus



$x_n \to x_0$, $n \to \infty$
$f(x_n) \to f(x_0)$ for any sequence $\{x_n\}$ (There is also the ε-δ definition which is equivalent to this one)

Same for functions of complex variables:
If for any $\{z_n\} \to z_0$, we have $f(z_n) \to f(z_0)$
then $f(z)$ is continuous at $z = z_0$.
We say that $z_n \to z_0$ as $n \to \infty$ if $|z_n - z_0| \to 0$
$f(z_n) \to f(z_0)$ if $|f(z_n) - f(z_0)| \to 0$

Geometrically,



Ex. $f(z) = z^2$ is continuous at $z = z_0$. We have to prove that
i.e. $|z_n^2 - z_0^2| \to 0$ if $|z_n - z_0| \to 0$.
We have $|z_n^2 - z_0^2| = |z_n - z_0|\,|z_n + z_0| \le k\,|z_n - z_0| \to 0$

$|z_n + z_0| = |z_n - z_0 + 2z_0| \le$
$\le |z_n - z_0| + 2|z_0| \le \quad \Rightarrow |z_n^2 - z_0^2| \le \text{const} \cdot |z_n - z_0|$