$\boxed{\text{Vector Operator}}$

<u>Ordinary vector:</u> $\begin{pmatrix}\tilde{V}_x\\ \tilde{V}_y\\ \tilde{V}_z\end{pmatrix} = R\begin{pmatrix}V_x\\ V_y\\ V_z\end{pmatrix} \leftarrow \tilde{V}_i = \sum_j R_{ij}V_j$

with $R_x = \begin{pmatrix}1 & & \\ & 1 & -\epsilon\\ & \epsilon & 1\end{pmatrix}$, $R_y = \begin{pmatrix}1 & & \epsilon\\ & 1 & \\ -\epsilon & & 1\end{pmatrix}$, $R_z = \begin{pmatrix}1 & -\epsilon & \\ \epsilon & 1 & \\ & & 1\end{pmatrix}$.

3 components that rotate by a prescribed rule

<u>Vector Operators</u> — when we rotate only the system

<u>System:</u> $|\alpha\rangle \longrightarrow |\tilde{\alpha}\rangle = \mathcal{D}|\alpha\rangle$

<u>Operator</u> - does not change when only the system is rotated

eg. $\vec{x}$ does not change

but matrix elements change

$$\underbrace{\begin{pmatrix}\langle\alpha|x|\alpha\rangle\\ \langle\alpha|y|\alpha\rangle\\ \langle\alpha|z|\alpha\rangle\end{pmatrix}}_{\text{transforms like an ord. vector}} \longrightarrow \begin{pmatrix}\langle\tilde{\alpha}|x|\tilde{\alpha}\rangle\\ \langle\tilde{\alpha}|y|\tilde{\alpha}\rangle\\ \langle\tilde{\alpha}|z|\tilde{\alpha}\rangle\end{pmatrix}$$

∴ equal

$$R\begin{pmatrix}\langle\alpha|x|\alpha\rangle\\ \langle\alpha|y|\alpha\rangle\\ \langle\alpha|z|\alpha\rangle\end{pmatrix}$$

<u>Generally:</u> $\underbrace{\langle\tilde{\alpha}|V_i|\tilde{\alpha}\rangle}_{} = \sum_j R_{ij}\langle\alpha|V_j|\alpha\rangle$

$\langle\alpha|\mathcal{D}^\dagger V_i \mathcal{D}|\alpha\rangle =$

vector operator is 3 component op. that "rotate" in this way

$$\therefore \boxed{\mathcal{D}^\dagger V_i \mathcal{D} = \sum_j R_{ij}V_j}$$

Equivalent definition of vector operator

Infinitesmal rotation: $D_{\hat{n}}(\epsilon) = \overleftrightarrow{1} - \frac{i}{\hbar}\epsilon \vec{J}\cdot\hat{n}$

$$D^\dagger V_i D = \left[1 + \frac{i}{\hbar}\epsilon \vec{J}\cdot\hat{n}\right] V_i \left[1 - \frac{i}{\hbar}\epsilon \vec{J}\cdot\hat{n}\right]$$

$$= V_i - \frac{i\epsilon}{\hbar}\left[V_i, \vec{J}\cdot\hat{n}\right] + \cancel{\mathcal{O}(\epsilon^2)}$$

Vector operator: $D^\dagger V_i D = R_{ij} V_j$

$$V_i - \frac{i\epsilon}{\hbar}\left[V_i, \vec{J}\cdot\hat{n}\right] = R_{ij} V_j$$

$\boxed{z}$ Rotations about $\hat{z}$

$$\begin{pmatrix} V_x - \frac{i\epsilon}{\hbar}[V_x, J_z] \\ V_y - \frac{i\epsilon}{\hbar}[V_y, J_z] \\ V_z - \frac{i\epsilon}{\hbar}[V_z, J_z] \end{pmatrix} = \begin{pmatrix} 1 & -\epsilon & 0 \\ \epsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} V_x - \epsilon V_y \\ V_y + \epsilon V_x \\ V_z \end{pmatrix}$$

$$\therefore\ [V_x, J_z] = -i\hbar V_y$$
$$[V_y, J_z] = i\hbar V_x$$
$$[V_z, J_z] = 0$$

$\boxed{x}$

$$\begin{pmatrix} V_x - \frac{i\epsilon}{\hbar}[V_x, J_x] \\ V_y - \frac{i\epsilon}{\hbar}[V_y, J_x] \\ V_z - \frac{i\epsilon}{\hbar}[V_z, J_x] \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & -\epsilon \\ & \epsilon & 1 \end{pmatrix}\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} V_x \\ V_y - \epsilon V_z \\ V_z + \epsilon V_y \end{pmatrix}$$

$$\therefore\ [V_x, J_x] = 0$$
$$[V_y, J_x] = -i\hbar V_z$$
$$[V_z, J_x] = i\hbar V_y$$

$\boxed{\hat{y}}$

$$\begin{pmatrix} V_x - \frac{i\epsilon}{\hbar}[V_x, J_y] \\ V_y - \frac{i\epsilon}{\hbar}[V_y, J_y] \\ V_z - \frac{i\epsilon}{\hbar}[V_z, J_y] \end{pmatrix} = \begin{pmatrix} 1 & & \epsilon \\ & 1 & \\ -\epsilon & & 1 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} V_x + \epsilon V_z \\ V_y \\ V_z - \epsilon V_x \end{pmatrix}$$

$$\therefore [V_x, J_y] = i\hbar V_z$$
$$[V_y, J_y] = 0$$
$$[V_z, J_y] = -i\hbar V_x$$

$\boxed{\hat{x}}$, $[\hat{y}]$, $[\hat{z}]$ together

$$\boxed{[V_i, J_j] = i\hbar \epsilon_{ijk} V_k}$$

equiv. def. of a vector operator

Check for [illegible] examples

$\vec{J}$ — get comm. relation for $\vec{J}$ ✓

$\vec{X}$ Use $L_j = \epsilon_{kij} x_k p_i$

$L_z = \epsilon_{123} x_1 p_2 + \epsilon_{213} x_2 p_1$

$L_z = x p_y - y p_x$

$$[x_m, L_j] = [x_m, \epsilon_{kij} x_k p_i]$$
$$= \epsilon_{kij} x_k \underbrace{[x_m, p_i]}_{i\hbar \delta_{mi}}$$
$$= \epsilon_{kmj} x_k$$

$$[x_i, L_j] = \epsilon_{kij} x_k$$

$$\boxed{[x_i, L_j] = \epsilon_{ijk} x_k}$$ ✓

$= [illegible]$

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[illegible] $= [illegible]$

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