Problem Set 2

Due Tuesday October 15, 9:30 AM. Submit in TA's mailbox in the Physics office.

1. We will discuss this in detail soon, but every measurable quantity can be associated with some operator O. The average value of repeated measurements of that quantity is given by the time-dependent expectation value $\langle O\rangle(t) = \int \psi^*(x,t)\ O\ \psi(x,t)dx$. Here, O has no explicit time-dependence, and since it corresponds to an observable must be self-adjoint (a.k.a. Hermitian in physicist-speak).

   In class we have or will demonstrate a few points about the time dependence of probability distributions associated with the position of a quantum particle. In particular:

   - (a) For any $V(x)$, if the system is prepared in an eigenstate of the Hamiltonian H, then the probability distribution does not change in time.
   - (b) For the harmonic oscillator potential, preparing the system in any superposition of two eigenstates of H will result in a probability distribution which changes in time, with a periodicity $T = 2\pi/\omega$.
   - (c) For a different confining potential, preparing the system in a superposition of two eigenstates of H will result in a probability distribution which changes in time, with a periodicity which depends on the states involved.

   Now, instead of considering the time-dependence of the probability distribution, demonstrate the equivalent points for the time-dependence of a measurement expectation value. Note that the eigenstates of H are generally different from the eigenstates of O. *To ensure familiarity with the basic concepts, for this problem, do *not* use Dirac notation.*

2. Consider a particle in a 1D infinite square well. For simplicity, set the length of the box to be $a = \pi$ and set $\hbar^2/2m = 1$.

   (a) According to Fourier's theorem, any function which vanishes at $x = 0, \pi$ can be expanded in a series:
   $$f(x) = \left(\frac{2}{\pi}\right)^{1/2} \sum_{n=1}^{\infty} c_n \sin(nx). \tag{1}$$
   Determine the coefficients c_n for the specific choice
   $$f(x) = \begin{cases} (6/\pi)^{1/2} & \pi/6 < x < \pi/3 \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$
   Note that this function is normalized like a Schrödinger wavefunction.

   (b) Plot the Fourier series approximations to $f(x)$ with 2, 5, 10, and 20 terms. Make sure your plots demonstrate the numerical convergence of the series (except just at the points of discontinuity in $f(x)$).

(c) Construct the time-dependent solution $\psi(x,t)$ of the Schrödinger equation for the particle in a box with initial condition $f(x)$. You could numerically integrate the Schrodinger equation to obtain an answer, but you would have trouble with errors creeping in at large times. Instead, find the time dependence by using the wavefunction composition in terms of the first N energy eigenstates, where $N \gg 1$. Plot this solution for a sequence of times, to get an idea of its behavior.

(d) In a real atom, electrons in excited states decay down to the ground state by the emission of radiation. A simple model of this behavior is given by replacing

$$c_n \to c_n e^{-nt/8} \tag{3}$$

for $n \geq 1$ and adding a term

$$a(t)|\phi_1(x)|^2 = a(t)\frac{2}{\pi}\sin^2(x) \tag{4}$$

to $|\psi(x,t)|^2$ to preserve the condition $\int \mathrm{d}x|\psi(x,t)|^2 = 1$. Find a formula for $a(t)$. Plot $|\psi(x,t)|^2$ using this model and the same initial condition used above, and describe the evolution of this function.