Theory of PDE Homework 2

Adrienne Sands

March 22, 2017

1. Let Φ be the fundamental solution of the Laplace equation in $\mathbb{R}^n, n \geq 2$. Prove that $-\Delta\Phi = \delta_0$ in the sense of distributions, that is for each $\psi \in C_c^\infty(\mathbb{R}^n)$ (smooth functions with compact support), one has
$$-\int_{\mathbb{R}^n} \Phi(x)\Delta\psi(x)dx = \psi(0)$$

Proof. Since Φ blows up at 0, we isolate the singularity inside a ball of fixed radius $\epsilon > 0$.
$$\begin{aligned}\int_{\mathbb{R}^n} \Phi(x)\Delta\psi(x)\, dx &= \int_{\mathbb{R}^n - B(0,\epsilon)} \Phi(x)\Delta\psi(x)\, dx + \int_{B(0,\epsilon)} \Phi(x)\Delta\psi(x)\, dx \\ &=: I_\epsilon + J_\epsilon\end{aligned}$$

We claim $I_\epsilon \to -\psi(0)$ and $J_\epsilon \to 0$ as $\epsilon \to 0^+$. First, apply Green's formula to I_ϵ.
$$I_\epsilon = \int_{\mathbb{R}^n - B(0,\epsilon)} \Delta\Phi(x)\psi(x)\, dx + \int_{\partial(\mathbb{R}^n - B(0,\epsilon))} \Phi(x)\frac{\partial\psi(x)}{\partial\nu} dS - \int_{\partial(\mathbb{R}^n - B(0,\epsilon))} \psi(x)\frac{\partial\Phi(x)}{\partial\nu} dS$$

The first term vanishes since $\Delta\Phi(x) = 0$ away from 0, and we need only consider the boundary $\partial B(0,\epsilon)$ for ψ compactly supported. Thus,
$$\begin{aligned}I_\epsilon &= \int_{\partial B(0,\epsilon)} \Phi(x)\frac{\partial\psi(x)}{\partial\nu} dS - \int_{\partial B(0,\epsilon)} \psi(x)\frac{\partial\Phi(x)}{\partial\nu} dS \\ &=: I_1 + I_2\end{aligned}$$

We claim $I_1 \to 0$ and $I_2 \to -\psi(0)$ as $\epsilon \to 0^+$. By definition of the fundamental solution,
$$\int_{\partial B(0,\epsilon)} |\Phi(x)|\, dS = \begin{cases} \frac{1}{2\pi}\int_{\partial B(0,\epsilon)} |\log|x||\, dS = \frac{|\log\epsilon|}{2\pi}\int_{\partial B(0,\epsilon)} dS = \epsilon|\log\epsilon| & (n=2) \\ \frac{1}{n(n-2)\alpha(n)}\int_{\partial B(0,\epsilon)} |x|^{2-n} dS = \frac{\epsilon^{2-n}}{n(n-2)\alpha(n)}\int_{\partial B(0,\epsilon)} dS = \frac{\epsilon}{n-2} & (n \geq 3)\end{cases}$$

For ψ continuous and compactly supported,
$$|I_1| = \left|\int_{\partial B(0,\epsilon)} \Phi(x)\frac{\partial\psi(x)}{\partial\nu} dS\right| \leq \|D\psi\|_{L^\infty}\int_{\partial B(0,\epsilon)} |\Phi(x)|\, dS \leq \begin{cases} C\epsilon|\log\epsilon| & (n=2) \\ C\epsilon & (n\geq 3)\end{cases} \to 0$$

Next we compute the (inward) normal derivative $\dfrac{\partial\Phi}{\partial\nu}$ at $x \in \partial B(0,\epsilon)$
$$\frac{\partial\Phi}{\partial\nu} = D\Phi(x)\cdot\nu = \frac{-x}{n\alpha(n)\epsilon^n}\cdot\frac{-x}{\epsilon} = \frac{|x|^2}{n\alpha(n)\epsilon^{n+1}} = \frac{1}{n\alpha(n)\epsilon^{n-1}}$$

By continuity of $\psi(x)$,

$$I_2 = -\int_{\partial B(0,\epsilon)} \psi(x)\frac{\partial \Phi}{\partial \nu} dS = \frac{-1}{n\alpha(n)\epsilon^{n-1}} \int_{\partial B(0,\epsilon)} \psi(x) dS = -\fint_{\partial B(0,\epsilon)} \psi(x) dS \to -\psi(0)$$

Consider the final term J_ϵ. For ϵ sufficiently close to 0 and $n = 2$, we have the following:

$$\begin{aligned}\left|\int_{B(0,\epsilon)} \Phi(x)dx\right| &= \frac{1}{2\pi}\left|\int_{B(0,\epsilon)} \log|x|dx\right| = \frac{1}{2\pi}\left|\int_0^{2\pi}\int_0^{\epsilon} r\log r dr d\theta\right| \\ &= \left|\int_0^\epsilon r\log r dr\right| = \left|\frac{2\epsilon^2\log\epsilon - \epsilon^2}{4}\right| \leq \frac{\epsilon^2|\log\epsilon|}{2}\end{aligned}$$

By a similar computation for $n \geq 3$,

$$\begin{aligned}\left|\int_{B(0,\epsilon)} \Phi(x)dx\right| &= \frac{1}{n(n-2)\alpha(n)}\left|\int_{B(0,\epsilon)} |x|^{2-n}dx\right| \\ &= \frac{1}{n(n-2)\alpha(n)}\left|\int_0^\epsilon r^{2-n}dr\int_{\partial B(0,r)} dS\right| \\ &= \frac{1}{n-2}\left|\int_0^\epsilon r dr\right| = \frac{\epsilon^2}{2(n-2)}\end{aligned}$$

Thus,

$$|J_\epsilon| = \left|\int_{B(0,\epsilon)} \Phi(x)\Delta\psi(x)\, dx\right| \leq \|\Delta\Phi(x)\|_{L^\infty}\left|\int_{B(0,\epsilon)} \Phi(x)dx\right| \leq \begin{cases} C\epsilon^2|\log\epsilon| & (n=2) \\ C\epsilon^2 & (n\geq 3)\end{cases} \to 0$$

Therefore as $\epsilon \to 0^+$

$$\int_{\mathbb{R}^n} \Phi(x)\Delta\psi(x)\, dx = I_\epsilon + J_\epsilon \to 0 - \psi(0) + 0 = -\psi(0)$$

□

2. Let $U = \{(x_1, x_2) \in \mathbb{R}^2 : -1 < x_2 < 1\}$.

   (a) Prove that there is no nontrivial bounded solution of the boundary value problem

$$\begin{aligned} \Delta u &= 0 \quad \text{in } U \\ u &= 0 \quad \text{on } \partial U \end{aligned}$$

   Remark: Here we are talking about classical solutions, that is, solutions in $C^2(U)\cap C(\bar{U})$
   Hint: One way to prove this is by using a reflection principle as in [Evans, 2.5/10]

   Proof. Suppose u is a bounded solution to this boundary value problem, and let

$$v(x_1,x_2) = \begin{cases} u(x_1,x_2) & |x_2| \leq 1 \\ -u(x_1, 2-x_2) & 1 < x_2 < 2 \end{cases}$$

   By construction, $u(x_1, 1) = -u(x_1, 1) = 0$, so $v \in C(V)$. We claim v is a harmonic extension of u to the larger strip $V = \{(x_1, x_2) : -1 \leq x_2 < 2\}$, and it suffices to show

v satisfies the mean value property in V. Since u is harmonic in U, v satisfies the mean value property for $|x_2| < 1$ and $1 < x_2 < 2$. Let B^+ and B^- denote the upper and lower halves of the ball. For $x = (x_1, 1)$ arbitrary, we have

$$\begin{aligned}
⨍_{B(x,r)} v(y)dy &= ⨍_{B^-(x,r)} u(y_1, y_2)dy - ⨍_{B^+(x,r)} u(y_1, 2 - y_2)dy \\
&= ⨍_{B^-(x,r)} u(y_1, y_2)dy - ⨍_{B^-(x,r)} u(y_1, y_2)dy = 0 = v(x)
\end{aligned}$$

Thus v is harmonic in V. Reflecting infinitely to larger strips, we can extend u to a (bounded) harmonic function v on R^2 which is 0 on the boundary of each strip. By Liouville v is trivial, and since $u = v\big|_U$ by construction, u is trivial. □

(b) Is there a nontrivial solution of this problem if the boundedness requirement is removed?

Proof. Consider $u(x, y) = e^{\pi x} \sin \pi y$ on U. By an easy computation,

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \pi^2 e^{\pi x} \sin \pi y - \pi^2 e^{\pi x} \sin \pi y = 0$$

Since u is harmonic and $u(x, \pm 1) = 0$, u solves the boundary value problem. □

3. The Kelvin transform of a function u on $\mathbb{R}^n$ is defined by

$$Ku(x) = |x|^{2-n} u\left(\frac{x}{|x|^2}\right)$$

(a) Prove that if u is harmonic in $B(0, 1)$, then its Kelvin transform is harmonic in $\mathbb{R}^n - B(0, 1)$ (cp. [Evans, 2.5/11]).

Proof. To simplify the proof, we use polar coordinates such that

$$u = u(r, \theta) \quad \text{ and } \quad \Delta = \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^{n-1}}\right)$$

Since $x \mapsto \frac{x}{|x|^2}$ is conformal and scales r to $\frac{1}{r}$, we write the Kelvin transform as follows:

$$Ku(r, \theta) = r^{2-n} u\left(\frac{1}{r}, \theta\right) = r^{n-2} u(y), \qquad y = \frac{1}{r}$$

(reasonably) suppressing θ for convenience. By the chain rule,

$$\begin{aligned}
\frac{n-1}{r}\frac{\partial(Ku)}{\partial r} &= (2-n)(n-1)r^{-n}u(y) - (n-1)y^{-n-1}u'(y) \\
\frac{\partial^2(Ku)}{\partial r^2} &= (2-n)(1-n)r^{-n}u(y) + 2(n-1)r^{-n-1}u'(y) + r^{-n-2}u''(y)
\end{aligned}$$

Thus,

$$\begin{aligned}
\Delta(Ku) &= \frac{\partial^2(Ku)}{\partial r^2} + \frac{n-1}{r}\frac{\partial(Ku)}{\partial r} + \Delta_{S^{n-1}}(Ku) \\
&= r^{-n-2}\left(u''(y) + r(n-1)u'(y) + r^2 \Delta_{S^{n-1}} u(y)\right) \\
&= y^{n+2}\left(u''(y) + \frac{(n-1)}{y}u'(y) + \frac{1}{y^2}\Delta_{S^{n-1}}u(y)\right) = 0
\end{aligned}$$

□

(b) Using this, find a solution of the exterior problem

$$\begin{aligned} \Delta u &= 0, \quad \text{in } \mathbb{R}^n - B(0,1) \\ u &= g, \quad \text{on } \partial B(0,1) \end{aligned}$$

where g is a continuous function on $\partial B(0,1)$.

Proof. By Poisson's formula for the ball,

$$v(x) = \frac{1-|x|^2}{n\alpha(n)} \int_{\partial B(0,1)} \frac{g(y)}{|x-y|^n} dS(y)$$

is harmonic in $B(0,1)$ and satisfies $v = g$ on $\partial B(0,1)$. We claim $u(x) = Kv(x)$ solves the exterior problem. By part (a) u is harmonic on $\mathbb{R}^n - B(0,1)$, and for $x \in \partial B(0,1)$,

$$Kv(x) = |x|^{2-n} u(\frac{x}{|x|^2}) = u(x) = g(x)$$

□

4. [Evans 2.5/5] We say $v \in C^2(\bar{U})$ is *subharmonic* if

$$-\Delta v \le 0 \quad \text{in } U$$

(a) Prove for subharmonic v that

$$v(x) \le \fint_{B(x,r)} v(y)\, dy \quad \text{for all } B(x,r) \subset U$$

Proof. We adapt the proof of the mean value formula for harmonic functions. Define

$$\phi(r) = \fint_{\partial B(x,r)} v(y) dS(y) = \fint_{\partial B(0,1)} v(x+rz) dS(z)$$

We claim $\phi(r)$ is monotonically increasing. Differentiating,

$$\phi'(r) = \fint_{\partial B(0,1)} z \cdot Dv(x+rz) dS(z) = \fint_{\partial B(x,r)} \frac{y-x}{r} \cdot Dv(y) dS(y)$$

For $y \in \partial B(x,r)$, the (outward) unit normal vector is $\nu = \frac{y-x}{r}$, so we are integrating the directional derivative $\nu \cdot Dv(y) = \frac{\partial v(y)}{\partial \nu}$. By Green's formula,

$$\begin{aligned} \phi'(r) &= \fint_{\partial B(x,r)} \frac{\partial v(y)}{\partial \nu} dS(y) = \frac{n}{r} \frac{1}{\alpha(n) r^n} \int_{\partial B(x,r)} \frac{\partial v(y)}{\partial \nu} dS(y) \\ &= \frac{n}{r} \frac{1}{\alpha(n) r^n} \int_{B(x,r)} \Delta v\, dy = \frac{n}{r} \fint_{B(x,r)} \Delta v\, dy \ge 0 \end{aligned}$$

for v subharmonic. Since $\phi(r)$ is monotonically increasing,

$$v(x) = \lim_{r \to 0^+} \phi(r) \le \phi(r) = \fint_{\partial B(x,r)} v(y)\, dS(y) = \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(x,r)} v(y)\, dS(y)$$

Polar coordinates yield our result:

$$\begin{aligned}\fint_{B(x,r)} v(y)dy &= \frac{1}{\alpha(n)r^n}\int_{B(x,r)} v(y)dy = \frac{1}{\alpha(n)r^n}\int_0^r \int_{\partial B(x,\rho)} v(y)dS(y)d\rho \\ &\geq \frac{1}{\alpha(n)r^n}\int_0^r n\alpha(n)\rho^{n-1}v(x)d\rho = \frac{v(x)}{r^n}\int_0^r n\rho^{n-1}d\rho = v(x)\end{aligned}$$

□

(b) Prove that therefore $\max_{\bar{U}} v = \max_{\partial U} v$.

Proof. Since $v \in C^2(\bar{U})$, v achieves a maximum on $\bar{U}$. Suppose there is some $x_0 \in U$ such that $v(x_0) = \max_{\bar{U}} v = M$. For all $0 < r < \operatorname{dist}(x_0, \partial U)$, part (a) asserts

$$M = v(x_0) \leq \fint_{B(x_0,r)} v(y)dy \leq \fint_{B(x_0,r)} M dy = M$$

By equality of the terms above, $v \equiv M$ in every $B(x_0, r) \subset U$ and $\{x \in U : v(x) = M\}$ is relatively open in U. For continuous v, this set is also relatively closed in U and therefore equal to U by connectedness (i.e. $v \equiv M$ in U). Finally, take any $x \in \partial U$ and sequence $\{x_k\} \subset U$ converging to x. By continuity of v,

$$v(x) = \lim_{x_k \to x} v(x_k) = \lim_{x_k \to x} M = M$$

Thus, $v \equiv M$ in $\bar{U}$ and $\max_{\bar{U}} v = \max_{\partial U}$. □

(c) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u is harmonic and $v := \phi(u)$. Prove v is subharmonic.

Proof. Since $\phi''(u) \geq 0$ for convex ϕ, our result follows by direct computation:

$$\begin{aligned}D^2 v &= \phi''(u)|Du|^2 + \phi'(u)D^2 u \\ \Delta v &= \operatorname{tr}(D^2 v) = \phi''(u)|Du|^2 + \phi'(u)\Delta u \\ &= \phi''(u)|Du|^2 \geq 0\end{aligned}$$

□

(d) Prove $v := |Du|^2$ is subharmonic, whenever u is harmonic.

Proof. Since u is harmonic, u_{x_i} is harmonic. Thus $\phi(u_{x_i}) = (u_{x_i})^2$ is subharmonic by (c) and $|Du|^2 = \sum_{i\leq n}(u_{x_i})^2$ is subharmonic as a sum of subharmonic functions. □

5. [Evans 2.5/7] Use Poisson's formula for the ball to prove

$$r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}u(0) \leq u(x) \leq r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}}u(0)$$

whenever u is positive and harmonic in $B^\circ(0, r)$. This is an explicit form of Harnack's inequality.

Proof. Let $x \in B^\circ(0,r)$. We freely use Poisson's formula for the ball:

$$u(x) = \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{u(y)}{|x-y|^n} dS(y) = r^{n-2}(r^2 - |x|^2) ⨍_{\partial B(0,r)} \frac{u(y)}{|x-y|^n} dS(y)$$

By the triangle inequality (with $|x| < |y| = r$ and $n \geq 1$),

$$(r - |x|)^n \leq |x-y|^n \leq (r + |x|)^n \iff \frac{1}{(r+|x|)^n} \leq \frac{1}{|x-y|^n} \leq \frac{1}{(r-|x|)^n}$$

Notice $\dfrac{1}{(r \pm |x|)^n}$ are independent of y. By Poisson's formula and the mean value formula for u harmonic, we have our result:

$$r^{n-2} \frac{r^2 - |x|^2}{(r+|x|)^n} ⨍_{\partial B(0,r)} u(y) dS(y) \leq u(x) \leq r^{n-2} \frac{r^2 - |x|^2}{(r-|x|)^n} ⨍_{\partial B(0,r)} u(y) dS(y)$$

$$\Updownarrow$$

$$r^{n-2} \frac{r - |x|}{(r+|x|)^{n-1}} u(0) \leq u(x) \leq r^{n-2} \frac{r + |x|}{(r-|x|)^{n-1}} u(0)$$

$\square$