Quantum Mechanics 412-1 Discussion

Tuesday, 26 November 2019

1. **Quantum Harmonic Oscillator**
   Consider the Hamiltonian for the quantum harmonic oscillator,

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \tag{1}$$

(a) Consider the creation and annihilation operators,

$$a = \sqrt{\frac{m\omega}{2\hbar}}x + \frac{i}{\sqrt{2\hbar m\omega}}p \tag{2}$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}x - \frac{i}{\sqrt{2\hbar m\omega}}p \tag{3}$$

Calculate the commutators $[a, a^\dagger]$, $[a^\dagger a, a]$, and $[a^\dagger a, a^\dagger]$.

(b) Invert these definitions to find x and p in terms of a and $a^\dagger$, and use this to rewrite the Hamiltonian in terms of the creation and annihilation operators as follows:

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) \tag{4}$$

(c) Use this to calculate $[H, a]$ and $[H, a^\dagger]$.

(d) Consider some eigenstate of the Hamiltonian, $H\left|\phi\right\rangle = E_\phi\left|\phi\right\rangle$. What are the energies of the states $a\left|\phi\right\rangle$ and $a^\dagger\left|\phi\right\rangle$?

(e) Interpreting a as an annihilation operator that takes a state to one of lower energy, we must have some condition on how the annihilation operator acts on the ground state $\left|0\right\rangle$, which has the lowest possible energy of any eigenstate:

$$a\left|0\right\rangle = 0 \tag{5}$$

What is the energy of $\left|0\right\rangle$? How about the energy of $a^\dagger\left|0\right\rangle$? Write down a formula for E_n where (for normalization c):

$$H\left|n\right\rangle = E_n\left|n\right\rangle \tag{6}$$

$$\left|n\right\rangle = c(a^\dagger)^n\left|0\right\rangle \tag{7}$$

(f) Find the action of a and $a^\dagger$ on a general normalized eigenstate $\left|n\right\rangle$ by solving for the constants C_n and D_n:

$$a^\dagger\left|n\right\rangle = C_n\left|n+1\right\rangle \tag{8}$$

$$a\left|n\right\rangle = D_n\left|n-1\right\rangle \tag{9}$$

Use these results to evaluate the normalization constant c in $\left|n\right\rangle = c(a^\dagger)^n\left|0\right\rangle$.