Problem Set 8

Solutions

$\boxed{1}$

$$\Pi |\psi(t)\rangle = \Pi e^{-\frac{iH}{\hbar}t} |\psi(t=0)\rangle$$

If $[\Pi, H]=0$ then

$$= e^{-i\frac{H}{\hbar}t}\Pi|\psi(t=0)\rangle = \pi_i e^{-i\frac{H}{\hbar}t}|\psi(t=0)\rangle$$

$$= \pi_i |\psi(t)\rangle$$

, where π_i is the initial definite parity

$\boxed{2}$

No, because a non-flat potential has $[p, H] \neq 0$.

Proof for this case:

$$[p, H] = [p, \overset{\text{kinetic}}{T} + \overset{\text{potential}}{V(x)}] = [p, V(x)]$$

$$\langle x|[p, V(x)]|\psi\rangle = -i\hbar\frac{d}{dx}V(x)\langle x|\psi\rangle + i\hbar V(x)\frac{d}{dx}\langle x|\psi\rangle$$

$$= -i\hbar\left(\frac{d}{dx}V(x)\right)\psi(x) \neq 0$$

$\boxed{3}$ $\Pi^\dagger \vec{S}\Pi = \vec{S}$, $\Pi^\dagger\vec{p}\Pi = -\vec{p}$ $\Rightarrow$ $\Pi^\dagger\vec{S}\cdot\vec{p}\Pi = -\vec{S}\cdot\vec{p}$

since it is an odd-parity operator.

The initial state of the isolated particle is non-degenerate, so if $[\Pi, H]=0$, it must have a definite parity, and $|\psi(t)\rangle$ should have the same definite parity. In which case $\langle\vec{S}\cdot\vec{p}\,(t)\rangle = 0$ for all time

4. (a)

$$H = \begin{pmatrix} \epsilon & -\Delta & 0 & -\Delta & -\Delta \\ -\Delta & \epsilon & -\Delta & 0 & -\Delta \\ 0 & -\Delta & \epsilon & -\Delta & -\Delta \\ -\Delta & 0 & -\Delta & \epsilon & -\Delta \\ -\Delta & -\Delta & -\Delta & -\Delta & \epsilon \end{pmatrix} \tag{1}$$

The Hamiltonian above is the same of problem 7.2 so we refer to the eigenvectors and eigenvalues of that problem's solution.

$$\Pi_x = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

The reflection does not change site 5, so Π_x act as the identity in that state (eigenvalue is unity). More generally $\Pi_x^2 = \mathbb{1}$ so the reflection has eigenvalues ± 1. Eigenstates are found by explicitly writing the 5-dimensional matrix equation:

$$\Pi_x |\pi\rangle = \pm |\pi\rangle\,, \tag{3}$$

and are:

$$|\pi_1\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \qquad |\pi_2\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad |\pi_5\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{4}$$

with eigenvalue $+1$ and:

$$|\pi_3\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad |\pi_4\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \tag{5}$$

with eigenvalue -1.

$[\Pi_x, H] = 0$ so we can expect the eigenvectors to be shared in general. However, testing the $|\pi\rangle$ eigenvectors in the Hamiltonian will reveal off-diagonal elements. More specifically:

i. testing those eigenvectors in the Hamiltonian (1), or
ii. making reference to problem 7.2, or
iii. inspecting H in the $|\pi\rangle$ space, i.e. the matrix elements $\langle\pi|H|\pi\rangle$, or
iv. stating H in $|\pi\rangle$ space is block-diagonal,

we find $|\pi_2\rangle, |\pi_3\rangle$ and $|\pi_4\rangle$ to be eigenvectors of H, with eigenvalues:

$$e_2 = e_4 = \epsilon \qquad \text{and} \qquad e_3 = e_+ = \epsilon + 2\Delta \tag{6}$$

The others are found diagonalizing the subspace spanned by $|\pi_1\rangle$ and $|\pi_5\rangle$:

$$\begin{pmatrix} \langle\pi_1|H|\pi_1\rangle & \langle\pi_1|H|\pi_5\rangle \\ \langle\pi_5|H|\pi_1\rangle & \langle\pi_5|H|\pi_5\rangle \end{pmatrix} = \begin{pmatrix} \epsilon - 2\Delta & -2\Delta \\ -2\Delta & \epsilon \end{pmatrix} \tag{7}$$

which leads to eigenvalues

$$E_\pm = \epsilon - (1 \mp \sqrt{5})\Delta\,, \tag{8}$$

with eigenvectors

$$|E_\pm\rangle = \frac{1}{\sqrt{10 \pm 2\sqrt{5}}}\left[-2|\pi_1\rangle + (1 \pm \sqrt{5})|\pi_5\rangle\right]. \tag{9}$$

(b) The system is symetric under $\pi/2$ rotations about site 5. This shifts every site to its counterclockwise neighbour. So, by inspection we could have:

$$R_{\pi/2} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{10}$$

By checking that $(R_{\pi/2})^4 = \mathbb{1}$ or by solving the eigenvalue problem you should find the four eigenvalues (which need not to be real) to be $e^{\pm i\pi/2}$ and ± 1.

Eigenvectors with eigenvalue $+1$ are:

$$|+1\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad \text{and} \qquad |+1'\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}; \tag{11}$$

with imaginary eigenvalues we have:

$$|+i\rangle = \frac{1}{2}\begin{pmatrix} i \\ -1 \\ -i \\ 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad |-i\rangle = \frac{1}{2}\begin{pmatrix} -i \\ -1 \\ i \\ 1 \\ 0 \end{pmatrix} \tag{12}$$

and finally with eigenvalue -1:

$$|-1\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \tag{13}$$

The Hamiltonian is again block-diagonal on these states. By operating with H, we find:

$$H|\pm i\rangle = \epsilon|\pm i\rangle \qquad \text{and} \qquad H|-1\rangle = (\epsilon + 2\Delta)|-1\rangle\,. \tag{14}$$

and the degenerate subspace is:

$$\begin{pmatrix} \langle 1|H|1\rangle & \langle 1|H|1'\rangle \\ \langle 1'|H|1\rangle & \langle 1'|H|1'\rangle \end{pmatrix} = \begin{pmatrix} \epsilon - 2\Delta & -2\Delta \\ -2\Delta & \epsilon \end{pmatrix} \tag{15}$$

which is the same as (7), leading to the same eigensystem.

Also the eigenvectors are consistent. Within the degenerate subspace with energy ϵ, the $\pi + R$ basis vectors are superpositions of one another. For instance, $|+i\rangle = \left(\frac{i-1}{2}\right)|\pi_2\rangle + \left(\frac{i+1}{2}\right)|\pi_4\rangle$.

5. (a) You should find:

$$h = \frac{1}{2}(p^2 + x^2) + \Lambda x^4 = \frac{1}{2}\frac{d^2}{dx^2} + \frac{x^2}{2} + \Lambda x^4\,. \tag{16}$$

If $h\psi = e\psi$, and if Λ is small enough, then the dependence of eigenvalues are on the form $e = f(\Lambda)$.

(b)

$$E_{try} = \frac{\hbar\omega}{2} + \Lambda\langle 0|x^4|0\rangle \tag{17}$$

To compute $\langle 0|x^4|0\rangle$ you can do $(a + a^\dagger)^4$ by brute force or:

$$\frac{1}{4}\Big[\langle 0|(a + a^\dagger)^2\Big]\Big[(a + a^\dagger)^2|0\rangle\Big]. \tag{18}$$

Since $(a + a^\dagger)^2 = a^2 + a^{\dagger 2} + 2N + \mathbb{1}$, where $N = a^\dagger a$ we have

$$(a + a^\dagger)^2|0\rangle = \sqrt{2}|1\rangle + |0\rangle \tag{19}$$

and

$$\langle 0|x^4|0\rangle = 3 \tag{20}$$

in the appropriate units. The energy is:

$$E_{try} = \left(\frac{1}{2} + \frac{3}{4}\Lambda\right)\hbar\omega\,. \tag{21}$$

(c) Suppose $H|\alpha\rangle = E_\alpha|\alpha\rangle$ is the exact eigenproblem for the Hamiltonian given, with $\alpha = g$ being the exact ground state energy. Then, if $\{|\alpha\rangle\}$ is a complete basis:

$$|0\rangle = \sum_\alpha |\alpha\rangle\langle\alpha|0\rangle \tag{22}$$

and

$$E_{try} = \sum_\alpha |\langle\alpha|0\rangle|^2 E_\alpha \tag{23}$$

which clearly leads to

$$E_{try} \geqslant E_g\,, \tag{24}$$

since all $E_\alpha \geqslant E_g$.

(d) Why not $|1\rangle$? Because the integrand in $\langle 0|x^4|1\rangle$ will be odd, leading to no correction.