Problem Set 6

Due Thursday November 7, 9:30 AM.
Submit in class or in TA's mailbox in the Physics office.

1. Consider a 2-state system, for concreteness, two states on a metal surface separated by an insulating barrier. Let $|\varphi_1\rangle$, $|\varphi_2\rangle$ represent the two states. Assume that these two states are orthonormal. Let the transition matrix element for transitions between these two states (by tunnelling) be
$$\langle\varphi_1|H|\varphi_2\rangle = \Delta. \tag{1}$$
   (a) Let $\langle\varphi_1|H|\varphi_1\rangle = \langle\varphi_2|H|\varphi_2\rangle = E_0$. Consider the Schrödinger equation
$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle \tag{2}$$
   restricted to this 2-state subspace (i.e., with $|\psi(t)\rangle$ a linear combination of $|\varphi_1\rangle$ and $|\varphi_2\rangle$. Solve this equation with the initial condition:
$$|\psi(0)\rangle = |\varphi_1\rangle. \tag{3}$$
   Find the probability for the particle to be found in the state $|\varphi_1\rangle$ as a function of time.

   (b) Let $\langle\varphi_1|H|\varphi_1\rangle = E_1$, $\langle\varphi_2|H|\varphi_2\rangle = E_2$, with $E_1 > E_2$. Again solve the Schrödinger equation with the same initial conditions. Find the probability for the particle to be found in the state $|\varphi_1\rangle$ as a function of time. Show that this probability never vanishes despite the fact that $|\varphi_2\rangle$ gives lower energy.

   (c) Assuming the conditions of part (b), suppose that using a very sensitive detector which does not remove the electron, we measure that the particle is in the state $|\varphi_1\rangle$ at $t = T$. Write an equation for the subsequent evolution of the system.

2. Let A and B be operators such that $[A,B] = c$ is a number. (Often, when we talk about operators, we call a number a 'c-number' to emphasize that it commutes with all operators.)

   (a) Show that
$$[A, e^B] = ce^B. \tag{4}$$
   (b) Using this equation, show that
$$e^A e^B e^{-A} = e^{B+c}. \tag{5}$$
   One possible strategy is the following. Consider the object
$$X(\lambda) = e^{\lambda A} e^B e^{-\lambda A}. \tag{6}$$
   Show that this object satisfies the differential equation
$$\frac{d}{d\lambda}X(\lambda) = cX(\lambda), \tag{7}$$
   with initial condition $X(0) = e^B$.

(c) Under the same conditions, prove that

$$e^A e^B = e^{A+B+c/2}. \tag{8}$$

One possible strategy is to write the differential equations for

$$Y(\lambda) = e^{\lambda A} e^{\lambda B}, \tag{9}$$

$$Z(\lambda) = e^{\lambda(A+B)+\lambda^2 c/2}. \tag{10}$$