Momentum Distributions

$$\tilde{\psi}(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx}\, \psi(x)$$

$$\Updownarrow \qquad (p = \hbar k)$$

$$\psi(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx}\, \tilde{\psi}(k)$$

Let's find an exact statement for finding some momentum when we make that measurement.

$$1 = \int dx\, |\psi(x)|^2 = \int dx\, \psi^*(x)\psi(x)$$

$$= \int dx \left[\int \frac{dk}{2\pi} e^{-ikx}\, \tilde{\psi}^*(k)\right]\left[\int \frac{d\ell}{2\pi} e^{i\ell x}\, \tilde{\psi}(\ell)\right]$$

$$= \int \frac{dk}{2\pi} \int \frac{d\ell}{2\pi}\, 2\pi\, \delta(k-\ell)\, \tilde{\psi}^*(k)\, \tilde{\psi}(\ell)$$

$$= \int \frac{dk}{2\pi}\, |\tilde{\psi}(k)|^2 \implies$$

$\tilde{\psi}(k)$ is normalized by this definition – integration parameter must be $\frac{dk}{2\pi}$

$$P_{x,x+dx} = |\psi(x)|^2 dx$$

$$P_{k,k+dk} = |\tilde{\psi}(k)|^2 \frac{dk}{2\pi}$$

$$\Rightarrow P_{p,p+dp} = |\tilde{\psi}(p)|^2 \frac{dp}{2\pi\hbar}$$

where, again, $\tilde{\psi}(k)$ is just f.t. of $\psi(x)$

- Interesting. We have assigned 2 meanings to spatial wavefunction $\psi(x)$: ① Related to position ~~probability~~ distribution of QM particle

② Plane-wave composition is related to momentum distribution ~~composition~~ for particle

$\Rightarrow$ Position + momentum entertwined somehow for QM particles

$\Rightarrow$ $\Delta x \Delta p$ uncertainty relation

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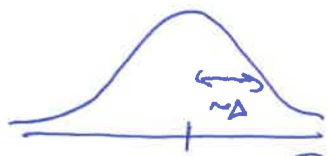
Gaussian math

– $\displaystyle\int_{-\infty}^{\infty} dx\, e^{-x^2} = \sqrt{\pi}$

– $\displaystyle\int_{-\infty}^{\infty} dx\, e^{-\lambda x^2}$ $\qquad z = \sqrt{\lambda}\, x$

$$= \frac{1}{\sqrt{\lambda}} \int_{-\infty}^{\infty} dz\, e^{-z^2} = \sqrt{\frac{\pi}{\lambda}}$$



$$\frac{\psi(\sqrt{2}\Delta)}{\psi(0)} = e^{-1} \approx \frac{1}{3}$$

– $\psi(x) = c\, e^{-x^2/2\Delta^2}$

$$1 = \int dx\, |\psi|^2 = c^2 \int dx\, e^{-x^2/\Delta^2} = c^2 \sqrt{\pi \Delta^2}$$

$$\Rightarrow \quad c = \frac{1}{(\pi \Delta^2)^{1/4}}$$

$\Rightarrow \quad \psi(x) = \dfrac{1}{(\pi \Delta^2)^{1/4}}\, e^{-x^2/2\Delta^2}$ is properly normalized

$$\tilde{\psi}(k) = \int dx\, e^{-ikx}\, \psi(x)$$

$$= \int dx \frac{1}{(\pi\Delta^2)^{1/4}}\, e^{-ikx}\, e^{-x^2/2\Delta^2}$$

~~Do~~ Don't know how to integrate. Compl. sq. to make it one gaussian.

Completing square in exp

$$= \int dx \frac{1}{(\pi\Delta^2)^{1/4}}\, e^{-\frac{1}{2\Delta^2}(x + ik\Delta^2)^2}\, e^{-\frac{\Delta^2}{2}k^2}$$

(Using $z = \frac{x + ik\Delta^2}{\sqrt{2\Delta^2}} \Rightarrow dz = \frac{dx}{\sqrt{2\Delta^2}}$ ~~we know~~ value (like before w/out i))

$$= \frac{1}{(\pi\Delta^2)^{1/4}}\sqrt{2\pi\Delta^2}\, e^{-\frac{\Delta^2}{2}k^2}$$

(F.T. of Gaussian is Gaussian)

$$\cancel{\tilde{\psi}(k) = (4\pi\Delta^2)^{1/4}\, e^{-\frac{\Delta^2}{2}k^2} \Longleftrightarrow \psi(x)}$$

$$\Rightarrow \quad \psi(x) = \frac{1}{(\pi\Delta^2)^{1/4}}\, e^{-x^2/2\Delta^2} \quad \Longleftrightarrow \quad \tilde{\psi}(k) = (4\pi\Delta^2)^{1/4}\, e^{-\frac{\Delta^2}{2}k^2}$$

$$\langle f \rangle = \int P(x)\, f(x)\, dx$$

$|\psi(x)|^2$ describes probability distribution

$$\Rightarrow \int_{-\infty}^{\infty} x^2 |\psi(x)|^2 \, dx \quad \text{describes} \quad \langle x^2 \rangle$$

For probability dist

$$\psi^2 \propto e^{-\lambda x^2}$$

$$\langle x^2 \rangle = \frac{\int dx\, x^2 e^{-\lambda x^2}}{\int dx\, e^{-\lambda x^2}} = \frac{\int dx \frac{-d}{d\lambda} e^{-\lambda x^2}}{\int dx\, e^{-\lambda x^2}}$$

take care of
To normalization

$$= \frac{-\frac{d}{d\lambda} \int dx\, e^{-\lambda x^2}}{\int dx\, e^{-\lambda x^2}} = \frac{-\frac{d}{d\lambda}\sqrt{\frac{\pi}{\lambda}}}{\sqrt{\frac{\pi}{\lambda}}} = \frac{\frac{1}{2}\lambda^{-3/2}}{\lambda^{-1/2}}$$

$$= \frac{1}{2\lambda}$$

No factor of 2 because we squared w.f.

So for $\lambda = \frac{1}{\Delta^2}$

$$\langle x^2 \rangle = \Delta^2/2$$

Similarly, for $\tilde{\psi}(k)$, $\lambda = \Delta^2$

$$\langle k^2 \rangle = \frac{1}{2\Delta^2}$$

$$\langle x^2\rangle\langle p^2\rangle = \hbar^2\langle x^2\rangle\langle k^2\rangle = \frac{\hbar^2}{4}$$

(Really just more formal way to look at widths)

$\Rightarrow$ Spread in x distribution times spread in p dist is constant for Gaussian distribution.

$\Rightarrow$ Makes sense. Tighter bunching in x means higher k's (shorter λ's) need to be included to get sharper features

$\Rightarrow$ In a few lectures, we will prove a theorem that

$$\left[\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle\right]^{1/2} \geq \frac{1}{2}\hbar$$

where $\Delta x = x - \langle x\rangle$, $\Delta p = p - \langle p\rangle$