1.a. The magnetic field only couples currents and $\vec{E}$ fields in orthogonal directions, so it will not affect ρ_{xx}, ρ_{yy}, and ρ_{zz}.

Thus, $\rho_{xx} = \rho_{yy} = \rho_{zz} = \frac{1}{\sigma(\omega)} = \frac{1 - i\omega\tau}{\sigma(0)} \rightarrow$ from lecture

The magnetic field will couple j_x and E_y (and j_y and E_x) via the Hall effect. Using the formula for the Hall effect:

$$E_y = \frac{-j_x B_0}{n_0 e c} = \rho_{yx} j_x$$

$$\Rightarrow \rho_{yx} = -\frac{B_0}{n_0 e c}$$

$$E_x = \frac{j_y B_0}{n_0 e c} = \rho_{xy} j_y$$

$$\Rightarrow \rho_{xy} = \frac{B_0}{n_0 e c}$$

$$\rho_{ik} = \begin{pmatrix} \rho_{xx} & \rho_{xy} & 0 \\ -\rho_{xy} & \rho_{xx} & 0 \\ 0 & 0 & \rho_{xx} \end{pmatrix}$$

Since the xz and yz elements are zero by properties of the cross product.

1. b. The conductivity tensor is σ_{ij} is just the inverse of the resistivity tensor:

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & -\sigma_{yx} & 0 \\ \sigma_{yx} & \sigma_{xx} & 0 \\ 0 & 0 & \sigma_{zz} \end{pmatrix}$$

where:

$$\sigma_{xx} = \frac{\rho_{xx}}{\rho_{xx}^2 + \rho_{xy}^2}, \quad \sigma_{yx} = \frac{\rho_{xy}}{\rho_{xx}^2 + \rho_{xy}^2}, \quad \sigma_{zz} = \frac{1}{\rho_{xx}}$$

For $\omega\tau \ll 1 \ll \omega_c\tau$ $\left(\omega_c = \frac{eB_0}{mc}\right)$

$$\rho_{xx} = \frac{1}{\sigma_0} = \frac{m}{n_0 e^2 \tau}, \quad \rho_{xy} = \frac{B_0}{n_0 e c} = \frac{m}{n_0 e^2} \omega_c$$

$$\rho_{xx}^2 + \rho_{xy}^2 = \left(\frac{m}{n_0 e^2}\right)^2 \left(\frac{1}{\tau^2} + \omega_c^2\right) = \left(\frac{m}{n_0 e^2}\right)^2 \left(\frac{\overbrace{1 + \omega_c^2\tau^2}^{\approx \omega_c^2\tau^2 \text{ in this limit}}}{\tau^2}\right)$$

$$\approx \left(\frac{m}{n_0 e^2}\right)^2 \omega_c^2$$

$$\Rightarrow \sigma_{xx} \approx \frac{\frac{m}{n_0 e^2 \tau}}{\left(\frac{m}{n_0 e^2}\right)^2 \omega_c^2} = \frac{n_0 e^2}{m\tau} \frac{m^2 c^2}{e^2 B_0^2} = \frac{m n_0 c^2}{B_0^2 \tau}$$

$$\sigma_{yx} \approx \frac{\frac{m}{n_0 e^2} \omega_c}{\left(\frac{m}{n_0 e^2}\right)^2 \omega_c^2} = \frac{n_0 e^2}{m} \frac{mc}{eB_0} = \frac{n_0 e c}{B_0}$$

$$\sigma_{zz} \approx \frac{n_0 e^2}{m}\tau = \sigma_0$$

As $B_0 \to \infty$, all terms in the conductivity tensor go to zero, except σ_{zz} which goes to σ_0.

c. The situation is similar to HW 7 Prob. 6, but now $\omega_0 = 0$ and the $\omega\omega_c$ term in the denominator of $n_\pm^2$ dominates

$$n_\pm^2 \approx 1 + \frac{4\pi n_0 e^2/m}{\pm\omega\omega_c} \approx \pm \frac{4\pi n_0 e^2/m}{\omega\omega_c}$$

for sufficiently small ω. The dispersion relation is

$$\omega^2 = \frac{c^2 k^2}{n_\pm^2} \Rightarrow \omega = \pm \frac{c^2 k^2 \omega_c}{4\pi n_0 e^2/m}$$ k is real, so there is no attenuation

In the absence of a magnetic field, we have the situation in the low frequency limit described in lecture:

$\epsilon(\omega) \approx 4\pi \dfrac{i n e^2 \tau}{m\omega}$. Then $n(\omega) = \sqrt{\epsilon(\omega)}$ will be complex, since $\sqrt{i} = \left(e^{i\frac{\pi}{2}}\right)^{\frac{1}{2}} = e^{i\frac{\pi}{4}} \propto 1+i$.

Thus, k will be complex, and so there will be some attenuation from the imaginary part.

assume this is real part of k

2. a. $\frac{w}{k} = \frac{c}{Re[n(w)]}$

$\Rightarrow k = \frac{w}{c} Re[n(w)]$

$\frac{dk}{dw} = \frac{1}{c} \frac{d}{dw}\left[w\, Re[n(w)]\right] = \frac{1}{c} Re\left[\frac{d}{dw}\left(w\, Re[n(w)]\right)\right]$

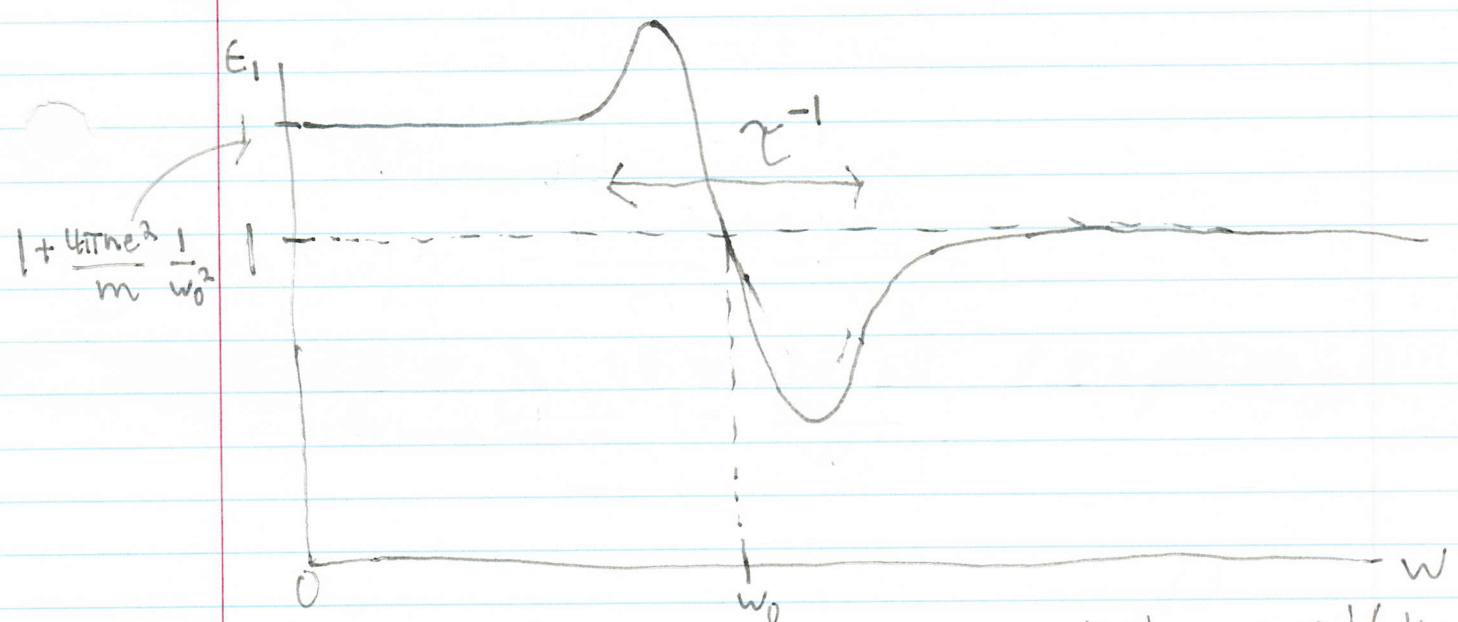
We can pull the Re outside of the expression since taking a derivative and multiplying by w does not affect which part of the expression is real. The group velocity is

$v_g = \frac{dw}{dk} = \left(\frac{dk}{dw}\right)^{-1} = \frac{c}{Re\left[\frac{d}{dw}(w n(w))\right]}$

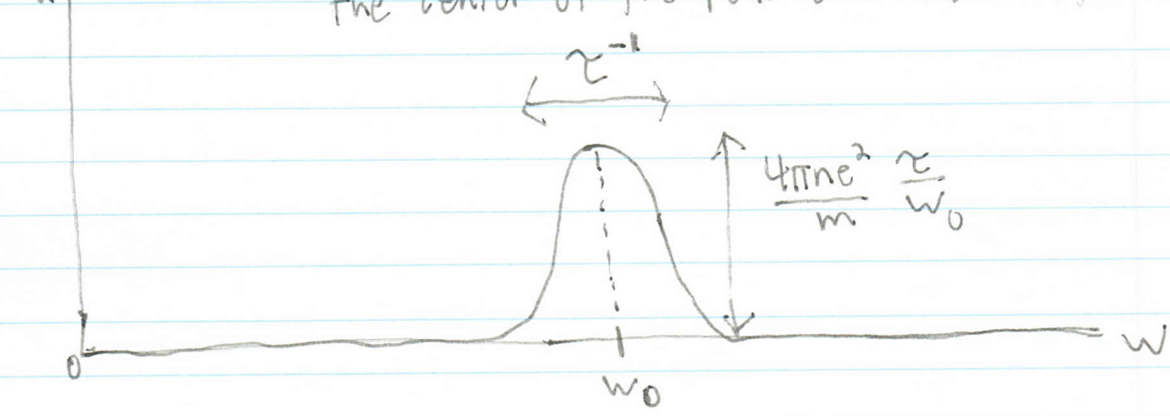
b. see Mathematica

4. a.

Let us consider how ϵ_1 and ϵ_2 behave in the vicinity of w_0. At $w=w_0$, $\epsilon_1=1$, while $\epsilon_2 = \frac{4\pi n e^2}{m}\frac{\tau}{w_0}$. To learn more about the behavior of ϵ_1 and ϵ_2, we can plot these quantities.



The plot for ϵ_2 is in the limit $w_0\tau \gg 1$ (otherwise the center of the peak can be shifted.)



There are features near w_0 with characteristic width τ^{-1}. The behaviors for negative w follow from even or odd symmetry as outlined in point 1 above.

As we had show before (pg. 3-4), for $w \ll w_0$, $\frac{d\epsilon_1}{dw} > 0$. In a region of width $\sim \tau^{-1}$ around w_0,

$\frac{d\epsilon_1}{dw} < 0 \Rightarrow$ the dispersion changes sign.

As continue to increase w well past w_0, we once again have $\frac{d\epsilon_1}{dw} > 0$.

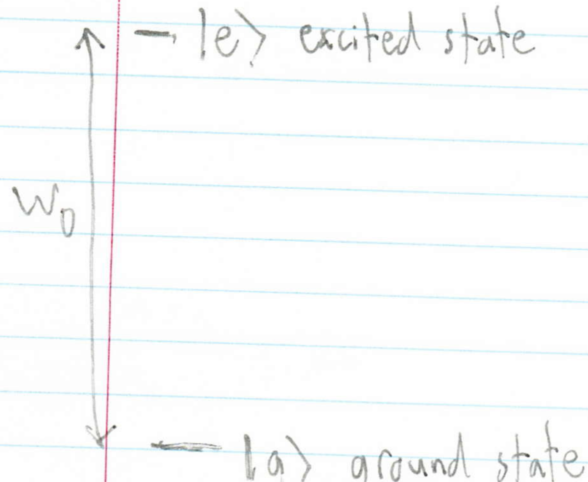
ϵ_2 is approximately 0 except for a window of width τ^{-1} around w_0. Since nonzero w_0 corresponds to absorption, we see that the absorption is concentrated around the resonance frequency of the bound electrons in the medium which is what we would have expected.

The width of the absorption feature, often called the "linewidth" is approximately equal to τ^{-1}.

b. Transition Linewidth

We see that the absorption feature has width τ^{-1}. This "linewidth" corresponds to the range Δw of frequencies that are significantly absorbed by the medium. τ corresponds to the damping time of the electron motion.

It turns out that this feature that has come out of our simple classical model is a general feature of atomic transitions that can be derived formally using a full quantum mechanical treatment. We can also see this behavior qualitatively arise from the energy-time uncertainty principle in quantum mechanics. Let us consider, for example, an atom with a ground state and excited state with transition frequency w_0. The lifetime of the excited state is denoted τ: after this time, the excited state spontaneously decays to the ground state.



excited state lifetime: τ
↓
analogous to the damping time in our model

We let $\Delta\omega$ denote the range of frequencies that can be absorbed by the transition.

Assume that the transition is excited by an electromagnetic field with a spread in frequencies larger than $\Delta\omega$:

1. The atom is initially in the ground state $|g\rangle$.

2. The atom absorbs a photon to transition to the excited state $|e\rangle$. Because of the finite width $\Delta\omega$ of the absorption probability vs. frequency, the uncertainty in the energy absorbed from the field is $\hbar\Delta\omega$ (since the energy of a photon is $\hbar\omega$).

3. After a characteristic time τ, the excited state spontaneously emits a photon to decay back down to the ground state.

If we assume that the energy-time uncertainty principle is near its lower bound:

$$\Delta E \Delta t \sim \hbar$$

$\Delta E = \hbar\Delta\omega$: uncertainty in energy absorbed from field

$\Delta t = \tau$: characteristic time it takes for atom's state to change

$$\Rightarrow \Delta\omega \sim \tau^{-1}$$

just as we had found from our classical model.

5. At very high frequencies, the electrons can be approximated as unbound, since the driving dynamics happen much more quickly than any restoring force can respond. They can also be approximated as undamped, since the driving dynamics happen much more quickly than any damping. Therefore, for high frequencies, the system behaves like the high frequency limit of the Drude model for unbound electrons, so:

$$\epsilon(w) \to 1 - \frac{4\pi n e^2}{m w^2} \quad \text{as } w \to \infty$$

Taking the limit $w \to \infty$ of our model:

$$\epsilon(w) \to 1 + \frac{4\pi n e^2}{m} \sum_i \frac{f_i}{-w^2} = 1 - \frac{4\pi n e^2}{m w^2} \sum_i f_i$$

Equating the two expressions for $\epsilon(w)$,

$$\sum_i f_i = 1.$$

6. a. $\chi(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, e^{iwt} \chi(t)$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \cos(wt) \chi(t) + i \int_{-\infty}^{\infty} dt \sin(wt) \chi(t)$$

Since $\chi(t)$ is real, given that it is a physical electric susceptibility, we can identify

$$\chi_1(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \cos(wt) \chi(t)$$

as the real part of $\chi(w)$ and

$\chi_2(w)$ as the imaginary part of $\chi(w)$, where

$$\chi_2(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \sin(wt) \chi(t)$$

Note that $\chi_1(-w) = \chi_1(w)$, and $\chi_2(-w) = -\chi_2(w)$.

Recall that $\epsilon_1(w) - 1 = 4\pi \chi_1(w)$, and $\epsilon_2(w) = 4\pi \chi_2(w)$,

so $\epsilon_1(w) - 1$ is an even function of w, and $\epsilon_2(w)$ is an odd function of w. The relevant Kramers-Kronig relation states that

$$\epsilon_1(w) - 1 = \frac{1}{\pi} P \int_{-\infty}^{\infty} dw' \frac{\epsilon_2(w')}{w' - w}$$

We multiply the numerator and denominator by $w'+w$ to get

$$\epsilon_1(w) - 1 = \frac{1}{\pi} P \int_{-\infty}^{\infty} dw' \frac{w' \epsilon_2(w')}{(w')^2 - w^2} + \frac{w}{\pi} P \int_{-\infty}^{\infty} dw' \frac{\epsilon_2(w')}{(w')^2 - w^2}$$

Since $\frac{\epsilon_2(w')}{(w')^2 - w^2}$ is an odd function of w', the second term is zero. On the other hand,

$\frac{w' \epsilon_2(w')}{(w')^2 - w^2}$ is an even function of w',

so:

$$\epsilon_1(w) - 1 = \frac{2}{\pi} P \int_0^{\infty} dw' \frac{w' \epsilon_2(w')}{(w')^2 - w^2}$$

b. For $w \to \infty$, $\epsilon(w) \approx \epsilon_1(w) \approx 1 - \frac{4\pi n e^2}{m w^2}$.

For large w, the integrand is

$$\frac{w' \epsilon_2(w')}{(w')^2 - w^2} \approx -\frac{w' \epsilon_2(w')}{w^2}$$

except for near $w' = w$. For $w' \approx w$, it can be written as

$$\frac{w' \epsilon_2(w')}{(w'-w)(w'+w)}$$

A principle value integral of $\frac{1}{w'-w}$ does not lead to divergences (it is like a principle value integral of $\frac{1}{x}$), and $\epsilon_2(w') \to 0$ faster than $\frac{1}{w'}$ for $w' \to w \to \infty$. So for large w', the contribution to the integral can be neglected in the limit $w \to \infty$. Thus, for $w \to \infty$,

$$\epsilon_1(w) - 1 = -\frac{4\pi n e^2}{m w^2} = -\frac{2}{\pi w^2} \int_0^\infty dw' \, w' \epsilon_2(w')$$

$$\Rightarrow \int_0^\infty dw' \, w' \epsilon_2(w') = \frac{2\pi^2 n e^2}{m}$$

c. For the multiresonance Drude-Lorentz model,

$$\frac{2\pi^2 n e^2}{m} = \int_0^\infty dw' \, w' \epsilon_2(w')$$

$$= \frac{4\pi n e^2}{m} \sum_i f_i \int_0^\infty dw' \frac{(w')^2/\tau}{((w')^2 - w_i^2)^2 + \frac{(w')^2}{\tau^2}}$$

$$= \frac{2\pi^2 n e^2}{m} \sum_i f_i$$

$$\Rightarrow \sum_i f_i = 1.$$

1. a. Recall from lecture that

$$\frac{d^2 E}{d\omega\, d\Omega} = 2\,|a(\vec{r},\omega)|^2 \qquad \text{for}$$

$$a(\vec{r},\omega) = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{q^2}{4\pi c}}\, e^{i\omega r/c} \,\cdot$$

$$\cdot \int_{-\infty}^{\infty} d\bar{t}\, e^{i\omega[\bar{t} - \hat{r}\cdot\vec{r}_0(\bar{t})/c]} \left(\frac{\hat{r}\times\left((\hat{r}-\vec{\beta})\times\dot{\vec{\beta}}\right)}{(1-\hat{r}\cdot\vec{\beta})^2} \right)$$

We have parallel acceleration, so $\vec{\beta}\times\dot{\vec{\beta}} = 0$. So

$$\hat{r}\times\left((\hat{r}-\vec{\beta})\times\dot{\vec{\beta}}\right) = \hat{r}\times(\hat{r}\times\dot{\vec{\beta}})$$

$$= \left(\hat{r}\times(\hat{r}\times\hat{z})\right)\frac{a\,\omega_0^2}{c}\cos(\omega_0\bar{t})$$

for $|\bar{t}| < \frac{N\pi}{\omega_0}$. Also, since it is assumed that

$\frac{a\omega_0}{c} \ll \bar{\beta}_0$, we can approximate $\vec{r}_0(\bar{t})$ as

$\vec{r}_0(\bar{t}) \approx \bar{\beta}_0 c\,\bar{t}\,\hat{z}$ (a constant offset in $\vec{r}_0(\bar{t})$ would just give a time-independent phase factor that could be pulled out of the integral and would cancel when we take $|a(\vec{r},\omega)|^2$). We can also approximate

$$(1-\hat{r}\cdot\vec{\beta})^2 \approx (1-(\hat{r}\cdot\hat{z})\bar{\beta}_0)^2.$$

Using the notation from class,

$\hat{r}\cdot\hat{z} = \cos\theta$, and $|\hat{r}\times(\hat{r}\times\hat{z})| = \sin\theta$. The acceleration occurs from $-\frac{N\pi}{w_0} < \bar{t} < \frac{N\pi}{w_0}$, which will define our bounds of integration for $a(\vec{r},w)$. We can pull the $\hat{r}\times(\hat{r}\times\hat{z})$ out of the integral to get

$$a(\vec{r},w) = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{q^2}{4\pi c}}\,\frac{a w_0^2}{c}\,e^{iwr/c}\left(\hat{r}\times(\hat{r}\times\hat{z})\right)\frac{1}{(1-\bar{\beta}_0\cos\theta)^2}\int_{-\frac{N\pi}{w_0}}^{\frac{N\pi}{w_0}} d\bar{t}\, e^{iw(1-\bar{\beta}_0\cos\theta)\bar{t}}\cos(w_0\bar{t})$$

Evaluating the integral in Mathematica yields

$$\int_{-\frac{N\pi}{w_0}}^{\frac{N\pi}{w_0}} d\bar{t}\, e^{iw(1-\bar{\beta}_0\cos\theta)\bar{t}}\cos(w_0\bar{t}) = \frac{2(-1)^N \frac{w}{w_0}(1-\bar{\beta}_0\cos\theta)\sin\left(N\pi\frac{w}{w_0}(1-\bar{\beta}_0\cos\theta)\right)}{w_0\left(\frac{w^2}{w_0^2}(1-\bar{\beta}_0\cos\theta)^2-1\right)}$$

$$\Rightarrow \frac{d^2E}{d\omega\, d\Omega} = 2\,|a(\vec{r},\omega)|^2$$

$$= 2\,\frac{q^2}{8\pi^2}\,\frac{a^2\omega_0^4}{c^3}\,\frac{\sin^2\theta}{(1-\bar{\beta}_0\cos\theta)^4}\cdot$$

$$\cdot\,\frac{4}{\omega_0^2}\left[\frac{\frac{\omega}{\omega_0}(1-\bar{\beta}_0\cos\theta)\sin\left(N\pi\frac{\omega}{\omega_0}(1-\bar{\beta}_0\cos\theta)\right)}{\left[\frac{\omega}{\omega_0}(1-\bar{\beta}_0\cos\theta)\right]^2-1}\right]^2$$

$$= \frac{(qa\omega_0)^2 N\sin^2\theta}{2\pi c^3(1-\bar{\beta}_0\cos\theta)^4}\underbrace{\left[\frac{2}{N\pi}\left(\frac{\frac{\omega}{\omega_0}(1-\bar{\beta}_0\cos\theta)\sin\left(N\pi\frac{\omega}{\omega_0}(1-\bar{\beta}_0\cos\theta)\right)}{\left[\frac{\omega}{\omega_0}(1-\bar{\beta}_0\cos\theta)\right]^2-1}\right)^2\right]}_{f_N\left[\frac{\omega}{\omega_0}(1-\bar{\beta}_0\cos\theta)\right]}$$

$$= \frac{(qa\omega_0)^2 N\sin^2\theta}{2\pi c^3(1-\bar{\beta}_0\cos\theta)^4}\, f_N\left[\frac{\omega}{\omega_0}(1-\bar{\beta}_0\cos\theta)\right]$$

b. See Mathematica file.

c. For $\bar{\beta}_0 \ll 1$, we can approximate $1-\bar{\beta}_0\cos\theta \approx 1$:

$$\frac{d^2E}{d\Omega\, dw} \approx \frac{(qaw_0)^2 N \sin^2\theta}{2\pi c^3} f_N\left(\frac{w}{w_0}\right)$$

$$= \frac{(qaw_0)^2 \sin^2\theta}{\pi^2 c^3} \frac{\frac{w^2}{w_0^2}\sin^2\left(N\pi \frac{w}{w_0}\right)}{\frac{w^2}{w_0^2} - 1}$$

There is a complete factorization between the θ dependence and the w dependence. For fixed angle, the frequency distribution looks like

$f_N\left(\frac{w}{w_0}\right)$, with $f_N(x)$ plotted in part b.

For all angles, the peak in the frequency distribution is at $w = w_0$.

d. For $\bar{\beta}_0 \to 1$. ($\bar{\gamma} >> 1$), the domiant radiation occurs for $\theta << 1$, since the denominator becomes very large. As shown in lecture, in this limit,

$$1 - \bar{\beta}_0 \cos\theta \approx \frac{1+\bar{\gamma}^2\theta^2}{2\bar{\gamma}^2}, \quad \sin\theta \approx \theta$$

$$\Rightarrow \frac{d^2E}{d\omega\, d\Omega} \approx \frac{(qa\omega_0)^2}{2\pi c^3} \frac{16\bar{\gamma}^8\theta^2}{(1+\bar{\gamma}^2\theta^2)^4} N f_N\left(\frac{\omega}{\omega_0}\left(\frac{1+\bar{\gamma}^2\theta^2}{2\bar{\gamma}^2}\right)\right)$$

Angular dependence mainly determined by

$$\frac{\bar{\gamma}^8\theta^2}{(1+\bar{\gamma}^2\theta^2)^4} \text{ factor}$$

For a given θ, peak frequency is at

$$\frac{\omega}{\omega_0}\left(\frac{1+\bar{\gamma}^2\theta^2}{2\bar{\gamma}^2}\right) = 1$$

$$\Rightarrow \omega_{peak} = \omega_0 \frac{2\bar{\gamma}^2}{1+\bar{\gamma}^2\theta^2}$$

For small angles $\bar{\gamma}\theta \lesssim 1$, $\omega_{peak} >> \omega_0$.