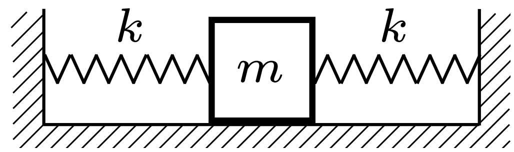
Physics 411 Quiz (Goldstein Chapters 1-2)

Monday, October 14th, 2019

This quiz is worth 3% of the normal problem set

1. A weight of mass m can slide horizontally without friction, squeezed between two identical springs of spring constant k. The weight is confined to the vertical plane shown. How many degrees of freedom does the system have? Determine the Lagrangian and obtain Euler-Lagrange equations of motion. What is the period of oscillations about the equilibrium point?



Solution. (1 point) Let x be the horizontal displacement of the weight. Then:

$$T = \frac{m\dot{x}^2}{2}, V = \frac{2kx^2}{2} \tag{1}$$

$$L = T - V = \frac{1}{2}m\dot{x}^2 - kx^2. \tag{2}$$

Equations of motion:

$$m\ddot{x} + 2kx = 0 \tag{3}$$

have the following solution:

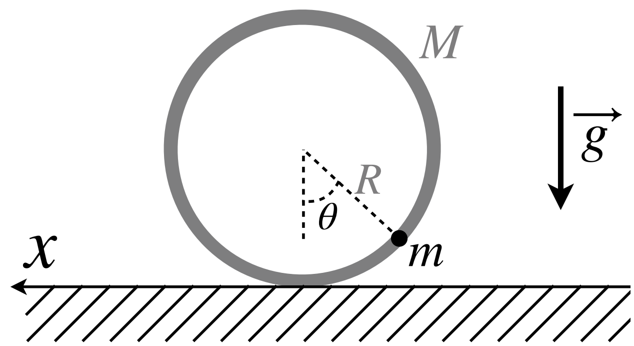
$$x = \frac{v_0}{\omega}\sin\omega t + x_0, \tag{4}$$

where $\omega = \sqrt{\frac{2k}{m}}$, and x_0 and v_0 are the initial displacement and velocity (at $t = 0$). The period of motion is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{2k}}. \tag{5}$$

As you can see, the two springs combine to effectively act as a single spring with twice as large spring constant, $2k$. This is not surprising since the forces of the two springs add up.

2. A uniform ring of radius R and mass M can roll without friction and without slipping on a horizontal surface. The ring is confined to the vertical plane shown. Point mass m is attached to the ring. How many degrees of freedom does the system have? Determine the Lagrangian and obtain Euler-Lagrange equations of motion. Linearize the equations about the equilibrium point and obtain the period of oscillations.



Solution. (2 points) Since the ring cannot slip, the value of θ and the coordinate of its center, x, are related: $x = R\theta$. Hence, the system has 1 degree of freedom. We can write the kinetic energy as:

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}R^2\dot{\theta}^2 + \frac{1}{2}mv_\mathrm{m}^2, \tag{6}$$

where the first term on the right gives the kinetic energy of linear motion of the ring, the second term gives the ring's kinetic energy relative to the ring's center of mass, and the third term gives the kinetic energy of the point mass. Because the velocity of the point of contact, at which the ring touches the surface, is zero (no slippage), the whole system undergoes instantaneous rotation about the point of contact. Hence, the velocity of the point mass can be written as $v_\mathrm{m} = l\dot{\theta}$, where $l = 2R\sin\frac{\theta}{2}$ is the distance between the point of contact and the point mass. This gives $v_\mathrm{m}^2 = 4R^2\dot{\theta}^2\sin^2\frac{\theta}{2}$.

While elegant, this approach for computing v_m is admittedly not the most transparent. Here is how we can obtain v_m in a standard, cleaner way. Let us direct the x-axis to the left, as shown in the figure: this is a somewhat unconventional direction, but this allows us to write $x = R\theta$ (with the plus sign) and $\dot{x} = R\dot{\theta}$. For the coordinates of the mass and their time derivatives, we get:

$$x_\mathrm{m} = x - R\sin\theta, \qquad y_\mathrm{m} = y - R\cos\theta. \tag{7}$$

$$\dot{x}_\mathrm{m} = \dot{x} - R\dot{\theta}\cos\theta, \quad \dot{y}_\mathrm{m} = \dot{y} + R\dot{\theta}\sin\theta = R\dot{\theta}\sin\theta, \tag{8}$$

because $y = R = \text{const}$ and $\dot{y} = 0$. The square of the velocity of the mass therefore is

$$v_\mathrm{m}^2 = \dot{x}_\mathrm{m}^2 + \dot{y}_\mathrm{m}^2 = R^2\dot{\theta}^2(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta) \tag{9}$$

$$= 2R^2\dot{\theta}^2(1 - \cos\theta) \tag{10}$$

$$= 4R^2\dot{\theta}^2\sin^2\frac{\theta}{2}, \tag{11}$$

which is the same as what we obtained above and where we made use of the trigonometric identify, $1 - \cos\theta = 2\sin^2\frac{\theta}{2}$.

Therefore,

$$T = R^2\dot{\theta}^2 \left[M + 2m \sin^2 \frac{\theta}{2} \right]. \tag{12}$$

The potential energy of the ring does not change as it rolls since its center of mass always remains at the same height. Thus, the changes in potential energy are associated only with the motion of the point mass:

$$V = -mgR\cos\theta. \tag{13}$$

The Euler-Lagrange equations of motion are that the following two expressions are equal:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left[2R^2\dot{\theta}\left(M + 2m\sin^2\frac{\theta}{2}\right)\right] = 2R^2\ddot{\theta}\left(M + 2m\sin^2\frac{\theta}{2}\right) + 2R^2\dot{\theta}\left(4m\sin\frac{\theta}{2}\cos\frac{\theta}{2}\frac{\dot{\theta}}{2}\right), \tag{14}$$

$$\frac{\partial L}{\partial \theta} = R^2\dot{\theta}^2 4m\sin\frac{\theta}{2}\cos\frac{\theta}{2}\frac{1}{2} - mgR\sin\theta. \tag{15}$$

The equilibrium point is $\theta = 0$. To linearize the system, we drop all non-linear terms, e.g., θ^2, $\dot{\theta}^2$, etc., and set $\sin\theta \approx \theta$. We are left with the following for the linearized equation of motion:

$$2R^2\ddot{\theta}M = -mgR\theta, \tag{16}$$

which implies that the period is $T = 2\pi\sqrt{\frac{2MR}{mg}}$.