Physics 414-2 Problem Set 5

April 27, 2022

Due: Friday, May 6 at 4 pm

1. Image charges with two-dimensional symmetry. A line charge with linear charge density λ is placed parallel to, and a distance d away from, the axis of a conducting cylinder of radius R held at a fixed potential. The cylinder and line charge are both taken to be infinitely long. The potential away from the cylinder vanishes at infinity. Find:

(a) The magnitude and position of the image charge(s).

(b) The potential outside the cylinder, including the asymptotic form far from the cylinder. Express the potential in cylindrical coordinates with the origin at the axis of the cylinder and the direction from the origin to the line charge as the x axis.

(c) The force per unit length on the line charge.

2. Conductors joined by a wire.
(a) What is the capacitance C of a sphere of radius a, defined so that if the sphere is at potential ϕ, the charge on the sphere is $Q = C\phi$?

(b) A small conducting sphere with radius b is at a distance r from the center of a spherical conductor with large radius a ($a \gg b$), with $r > a$. The distance $r - a$ from the small sphere to the surface of the large sphere is assumed to be large compared with b and comparable in size to a. The two conductors are joined by a thin wire, so that they are at the same potential ϕ. Determine the mutual force of the small and large conducting spheres to leading order in b, assuming that the sphere of radius b is sufficiently small so that it does not significantly perturb the charge distribution on the sphere of radius a. Is the force attractive or repulsive?

3. Green's reciprocity theorem. Green's reciprocity theorem provides a useful way to solve some electrostatics problems, as we will see in problem 4.

The statement of the theorem is as follows. Let ϕ be a potential arising from a volume charge density ρ within a volume V and a surface charge density σ on the surface S that forms the boundary of this volume. Let ϕ' be the potential arising from a different charge distribution (ρ' volume charge density, σ' surface charge density) within the same volume V and surface S. Then the following general relation holds:

$$\int_V \rho\phi' dV + \int_S \sigma\phi' dS = \int_V \rho'\phi dV + \int_S \sigma'\phi dS. \tag{1}$$

Here, $\int_V f dV$ is the volume integral over V of a quantity f, and $\int_S f dS$ is a surface integral over S of a quantity f. Prove Green's reciprocity theorem. (Hint: you may find it useful to first consider a charge distribution that is nonzero at a discrete set of points and then take the limit of a continuous charge distribution at the end.)

4. Parallel plate capacitors with a charge between them: an application of Green's reciprocity theorem. Consider two infinite, grounded (that is, held at zero potential) parallel conducting planes separated by a distance d. A point charge q is placed in between the two planes at a distance a from one of the plates.

(a) Use Green's reciprocity theorem to find the total charge on each of the plates.

(b) What is the total induced charge on both plates?

5. Multipole expansion. Consider a localized charge distribution with charge density

$$\rho(\mathbf{r}) = \frac{1}{64\pi} r^2 e^{-r} \sin^2\theta. \tag{2}$$

In lecture, we showed that we can express the potential at large distances from this charge distribution as a multipole expansion in the following way:

$$\phi(\mathbf{r}) = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Y_{lm}(\Omega)}{2l+1} \frac{1}{r^{l+1}} q_{lm}, \tag{3}$$

where

$$q_{lm} \equiv \int d^3r' Y^{\star}_{lm}(\Omega')(r')^l \rho(\mathbf{r}'). \tag{4}$$

We have used the notation that Ω represents the angular dependence, with $d\Omega = \sin\theta d\theta d\phi$.

(a) Find all the coefficients q_{lm}. You can make use of the orthornormality of spherical harmonics, which means that $\int Y^\star_{lm}(\Omega')Y_{l'm'}(\Omega')d\Omega' = \delta_{ll'}\delta_{mm'}$ (δ_{ij} is the Kronkecker delta function, which is equal to 1 if $i = j$ and 0 otherwise). Recall from lecture that $Y_{00}(\Omega) = \frac{1}{\sqrt{4\pi}}$ and $Y_{20}(\Omega) = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$.

(b) What is the potential at large distances from the localized charge distribution? You should find that your expression includes a term that goes as $\frac{1}{r}$. Give an interpretation of this $\frac{1}{r}$ contribution in terms of the total charge of the distribution.

6. **Solving boundary value problems using Green's functions.** This problem will illustrate a powerful tool for solving boundary value problems: the use of Green's functions. For an electrostatics problem with a given geometry and boundary conditions, Green's functions provide a way to solve for the potential for arbitrary charge distributions. By the end of the problem, we will also derive the expansion for $\frac{1}{|\mathbf{r}-\mathbf{r}'|}$ in terms of spherical harmonics that we used in class when discussing the multipole expansion.

(a) We will now consider the problem of a grounded conducting sphere of radius R, where we also assume that the potential falls to zero at infinity, which is a physically reasonable assumption. For an arbitrary charge distribution $\rho(\mathbf{r}')$ outside the sphere, we want to be able to determine the potential at all points outside the sphere (inside the sphere, the potential will just be zero). Show that if we consider a function $G(\mathbf{r}, \mathbf{r}')$ that satisfies the boundary conditions ($G(\mathbf{r}, \mathbf{r}') = 0$ for $\mathbf{r}$ on the sphere and at infinity) and also satisfies the equation

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = -4\pi\delta(\mathbf{r} - \mathbf{r}'), \tag{5}$$

then the solution for the potential outside the sphere is

$$\phi(\mathbf{r}) = \int_{V'} dV'\rho(\mathbf{r}')G(\mathbf{r}, \mathbf{r}'), \tag{6}$$

where $\int_{V'} dV'$ represents an integral over the volume outside the sphere (using the coordinates $\mathbf{r}'$). The function $G(\mathbf{r}, \mathbf{r}')$ is referred to as a Green's function.

(b) Show that the dirac delta function in spherical coordinates is given by $\delta(\mathbf{r} - \mathbf{r}') = \frac{1}{r^2}\delta(r - r')\delta(\Omega - \Omega')$, where $\delta(\Omega - \Omega') = \delta(\cos\theta - \cos\theta')\delta(\phi - \phi')$.

(c) Using the result from part (b) and one of the properties of the spherical harmonics, we can write

$$\delta(\mathbf{r} - \mathbf{r}') = \frac{1}{r^2}\delta(r - r')\sum_{lm} Y_{lm}(\Omega)Y^\star_{lm}(\Omega'). \tag{7}$$

This suggests that we try a form for $G(\mathbf{r}, \mathbf{r}')$ that has a similar form, so that we can try to reduce Eq. 5 to an equation involving only r and r'. Let us try the following:

$$G(\mathbf{r}, \mathbf{r}') = \sum_{lm} G_{lm}(r, r') Y_{lm}(\Omega) Y^{\star}_{lm}(\Omega'). \tag{8}$$

Applying the Laplacian in spherical coordinates to the lefthand side of Eq. 5, recalling from quantum mechanics that the angular part of the Laplacian (which corresponds to the square of the quantum angular momentum operator) acting on $Y_{lm}(\Omega)$ yields a factor of $l(l+1)Y_{lm}(\Omega)$, and substituting Eq. 7 into Eq. 5, we get

$$\sum_{lm} \frac{1}{r^2}\left(\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - l(l+1)\right) G_{lm}(r, r') Y_{lm}(\Omega) Y^{\star}_{lm}(\Omega') = -\frac{4\pi}{r^2}\delta(r-r') \sum_{lm} Y_{lm}(\Omega) Y^{\star}_{lm}(\Omega') \tag{9}$$

Because the spherical harmonics are an orthogonal set of functions that form a complete basis, and because this relation must hold for all Ω, the equality individually holds for each value of l and m. Therefore, for each l, m,

$$\frac{1}{r^2}\left(\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - l(l+1)\right) G_{lm}(r, r') = -\frac{4\pi}{r^2}\delta(r - r') \tag{10}$$

In the case that $r \neq r'$, verify that for a given l, expressions of the form $a_l(r')r^l + b_l(r')r^{-(l+1)}$ are solutions to equation (10), where a_l and b_l are coefficients that can in general depend on r'.

(d) For r and r' outside the sphere, find the general forms of $G_{lm}(r, r')$ in the separate cases that $r < r'$ and $r > r'$. For each case, determine the form of the solution up to a coefficient that can depend on r'. We will determine these coefficients in part (e).

(e) What conditions on $G_{lm}(r, r')$ and its derivative at $r = r'$ can you use to determine the constant coefficients in part (d) and construct a complete solution (Hint: integrate equation (10) to find the discontinuity of the derivative of $G_{lm}(r, r')$ at $r = r'$.)? Show that the solution is

$$G_{lm}(r, r') = \frac{4\pi}{(2l+1)}\left[\left(\frac{r_<^l}{r_>^{l+1}}\right) - \frac{1}{R}\left(\frac{R^2}{rr'}\right)^{l+1}\right], \tag{11}$$

where $r_>$ is the greater of (r, r') and $r_<$ is the lesser of (r, r').

(f) Using equation (8), we can express the full Green's function as

$$G(\mathbf{r}, \mathbf{r}') = 4\pi \sum_{lm} \frac{Y_{lm}(\Omega) Y^{\star}_{lm}(\Omega')}{2l+1}\left[\left(\frac{r_<^l}{r_>^{l+1}}\right) - \frac{1}{R}\left(\frac{R^2}{rr'}\right)^{l+1}\right]. \tag{12}$$

If we take the limit $R \to 0$, it is as if we are calculating the potential due to a unit charge at $\mathbf{r}'$ without a conducting sphere present. Use this fact to explain why the following relation that we've used in lecture holds:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = 4\pi \sum_{lm} \frac{Y_{lm}(\Omega) Y^{\star}_{lm}(\Omega')}{2l+1} \left(\frac{r^l_<}{r^{l+1}_>} \right). \tag{13}$$