Physics 412-2 (Spring 2020) **Final Exam (4 pages)** Professor Gerald Gabrielse
Distributed: 9:30 AM on Tuesday, 10 March 2020 Due: 10 AM on Thursday, 12 March 2020

Signed Certification

Exam solutions will only be graded if they are handed to Laura before the specified time, accompanied with the following signed certification. Please note carefully the three required certifications before you start the exam.

I certify that during this exam period I have not communicated with any person about the exam or any material even remotely related to quantum mechanics or the exam material. I further certify that I have referenced in my solutions any books, web sources, or other materials that I have used to make these solutions. (Such sources are allowed with attribution.) Finally, I certify that I have reported any evidence of cheating that I am aware of, and will do so if such evidence emerges later.

Signature: ____________________ Printed Name: ____________________

To get maximum credit, please make sure that I can read your writing, can understand your answer and can clearly see how you arrived at the answer. An unsupported "right answer" does not generally get much credit. If you do not attempt any part of this exam, I will not have any way to give you partial credit. Bluffing is not positively rewarded, neither are many words that communicate inefficiently or say little. Different answers that do not agree with each other give evidence of a lack of understanding. Your answer to each of the listed items is worth 5 points unless specifically designated differently. You are responsible for updates and clarifications posted on the web page.

Exam Problems

The master equation for a two-state system is given in the Schrodinger picture by

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar}[H_0 + V(t), \rho] - \frac{\gamma}{2}\left(a^\dagger a \rho - 2a\rho a^\dagger + \rho a^\dagger a\right) \tag{1}$$

with $H_0 = \frac{1}{2}\hbar\omega_0\sigma_z$ and $V(t) = \frac{1}{2}\hbar\Omega_R\left[a^\dagger e^{-i\omega t} + a e^{i\omega t}\right]$.

- The angular frequencies ω_0, Ω_R and ω are real and positive, as is the rate γ.
- The decay terms on the right of the master equation I introduced without proof in class.
- To make it easier for me to deduce what you do, please use H_0 eigenstates $|+\rangle = a^\dagger|-\rangle$ and $|-\rangle = a|+\rangle$, and let ρ_{++} be the upper left element in a matrix representation of the density operator.
- Use the interaction representation whenever possible unless specifically asked to do differently. This will save you lots of time and effort. The reference time is $t_0 = 0$ to simplify the notation.
- The many itemized questions are there to guide you and minimize the possibility that you get stuck – not because this exam is long and difficult.

This exercise will allow you to be more general than I was in class because you use a density operator. It will also allow you to acquire an idea of how a master equation allows you to treat unstable states in quantum mechanics without cheating in the ad hoc way I did in class.

1. **Master Equation and its Matrix Representation in the Interaction Representation**

   (a) Explain very succinctly how a density operator is a generalization of a wavefunction and when this generalization is needed.

   (b) The time evolution operator $U_0(t)$ is the time evolution operator for time evolution governed by H_0. Relate U_0 and H_0.

   (c) What does $V(t)$ do, in what sense it is "general," and in what ways it is not?

   (d) Show how to write $V(t)$ in terms of sines and cosines rather than complex exponentials. What does this version tell you about $V(t)$?

   (e) Write ρ, the density operator in the Schrodinger picture, as a matrix. What are the relationships between these matrix elements, if any?

   (f) Write ρ^I, the density operator in the interaction picture, as a matrix. What are the relationships between these matrix elements, if any?

   (g) Starting from the master equation in the Schrodinger picture, obtain the master equation in the interaction representation in terms of the transition operator V_I and the raising and lowering operators, $a_I^\dagger$ and a_I, that are appropriate in the interaction picture. *(15 points)*

   (h) Write the time evolution operators in the Schrodinger picture, $U_0(t)$ and $U_0(t)^\dagger$, as matrices.

   (i) Find the relationships between the matrix components of ρ and ρ^I.

   (j) What does the requirement of probability conservation mean for the density operator matrix components in both pictures?

   (k) Relate the raising and lowering operators in the interaction picture to their counterparts in the Schrodinger picture.

   (l) Show how to write a and $a^\dagger$ as matrices.

   (m) Show how to write a_I and $a_I^\dagger$ as matrices.

   (n) Show how to write $a^\dagger a$ and $a_I^\dagger a_I$ as matrices.

   (o) Show how to write the transition operator in the interaction representation, $V_I(t)$ as a matrix.

   (p) Use your matrices to write the master equation in the interaction representation as a matrix equation.

   (q) Write down the 4 component equations that come from the master equation. How many of these equations are linearly independent? *(15 points)*

   (r) Use the component equations to show that probability is conserved.

2. **Unstable State Solution**

   (a) Turn off the drive (i.e. set the transition operator to zero) and solve for the time dependence of each of the 4 density operator components in the interaction representation.

   (b) What are the corresponding time dependent solutions for the 4 density operator components in the Schrodinger picture?

   (c) Show how the probabilities for being in each of the two states depend upon time.

(d) Compare your solutions with what you obtain by patching decay into a two state wave function in the ad hoc way we used in class. Highlight the differences, if any. *(10 points)*

3. **Strong Resonant Drive Solutions**

   (a) Explain how $\Omega_R \gg \gamma$ is a strong drive limit. What can then be neglected in the master equation?

   (b) In the interaction picture, show how to use the component equations and probability conservation to derive a second order differential equation for ρ^I_{++} that is independent from the other density operator components. *(10 points)*

   (c) In the interaction picture, show how to get the general solution to this differential equation for ρ^I_{++}.

   (d) In the interaction picture, show how to get the values of the constants in the general solution for ρ^I_{++} in terms of the density operator components (not their derivatives) at $t = 0$.

   (e) Obtain the corresponding general solution for this density operator component in the Schrodinger picture.

   (f) What is the initial, $t = 0$, density operator matrix for an incoherent state with a probability p to be in the upper state? Explain each entry.

   (g) What is the $\rho^I_{++}(t)$ that evolves from the incoherent $t = 0$ state?

   (h) Compute the density operator at time t for the general pure state with real α and δ:

$$|\psi(0)\rangle = e^{i\alpha}\left[\sqrt{p}\,|+\rangle + \sqrt{1-p}\,e^{i\delta}|-\rangle\right]. \tag{2}$$

   (i) Compare the density operator components for this coherent initial state with the components for the incoherent $t = 0$ density operator considered just above. Explain differences, if any.

   (j) Show how to obtain the time dependence of the probability to be in the upper state that evolves from this initial pure state.

   (k) Identify and explain differences, if any, for the incoherent and coherent density operators at time t.

   (l) Compare the probabilities to be in the upper and lower states at time t that evolve when the coherent and incoherent initial states at $t = 0$ have equal populations in the two energy eigenstates.

4. **Weak Drive, Approximate Solutions:** A weak drive $V(t)$ is applied, which means that $\Omega_R \ll \gamma$. It is applied at angular drive frequency $\omega = \omega_0 + \epsilon$, where ϵ is the detuning of the drive from resonance.

   (a) The $t = 0$ state is the lower energy eigenstate of H_0. What is the $t = 0$ density operator matrix in the interaction picture?

   (b) Take the system to be only in the lower energy eigenstate at time $t = 0$. For a weak drive, what are the relative sizes of the diagonal density operator components at time t?

   (c) For this case, make an equation for an off-diagonal component of the density operator at time t in the interaction representation.

(d) Solve this equation assuming making perturbation theory assumptions based upon the drive being weak.

(e) Use this solution to determine $\partial \rho^I_{++}(t)/\partial t$ for the given initial state in the weak drive limit.

(f) For a weak drive we expect that the probabilities for being in the upper and lower states reach an unchanging, steady state. Explain why.

(g) What is then the equation for the steady state ρ^I_{++}?

(h) Solve this equation for to determine ρ^I_{++}. What then is the solution in Schrodinger picture?

(i) Sketch the resulting probability for being in the upper state as a function of the drive frequency detuning.

(j) Determine the full-width-half-maximum (FWHM) for this lineshape.

(k) What is the largest possible steady-state probability for this weak drive, and at which drive frequency does it occur?

5. How many hours did you work on this exam?

6. Provide any comments that you wish on the exam.