Review of Complex Numbers

Def. (formal) A complex number $z=(x,y)$ is an ordered pair of 2 real numbers, $x = \mathrm{Re}\, z$, $y = \mathrm{Im}\, z$ subject to the following rules

$$\left.\begin{array}{l} z_1=(x_1,y_1) \\ z_2=(x_2,y_2)\end{array}\right| \Rightarrow \begin{array}{l} z_1+z_2 = (x_1+x_2,\, y_1+y_2) \\ z_1 z_2 = (x_1x_2-y_1y_2,\, x_1y_2+x_2y_1)\end{array}$$

How do we arrive from this to the conventional $z=x+iy$?
Complex numbers - extension of real numbers, so that real numbers must be part of complex numbers, and, indeed, ~~Note~~ - complex # $(x,0)$ - (zero imaginary part) can be/is treated as just a real number.

Ex. If $z=(x,y)$ and c - real $\therefore$ $c=(c,0)$. Then

$$cz = (c,0)(x,y) = (cx, cy) \quad \text{- as it should be}$$

Thus, $z=(x,y) = \underbrace{(x,0)}_{x} + y\underbrace{(0,1)}_{i} = x+iy$

what's i?

$$i^2 = (0,1)\cdot(0,1) = (-1,0) \Rightarrow \underline{i^2=-1}$$

Then addition and multiplication rules become

$$(x_1+iy_1)+(x_2+iy_2) = (x_1+x_2)+i(y_1+y_2)$$

$$(x_1+iy_1)(x_2+iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2+x_2y_1)$$

We observe that the r.h.s. can be obtained by formally manipulating the l.h.s. as if they involved only real numbers and replacing i^2 by -1 when it occurs.
From now on this is what we are going to do in our calculations involving complex numbers

Ex. $z = x+iy$, $\bar{z} = x-iy$ - complex conjugate

$$\underline{\underline{z\bar{z}}} = (x+iy)(x-iy) = x^2+y^2 = |z|^2, \quad |z| = \sqrt{x^2+y^2}$$

$|z|$ - absolute value of z, a real number

Ex. (division)

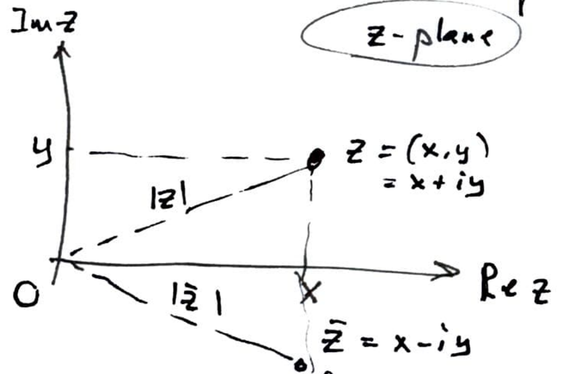
$z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2 \neq 0$ then

$$\frac{z_1}{z_2} = \frac{x_1+iy_1}{x_2+iy_2} = \frac{(x_1+iy_1)(x_2-iy_2)}{(x_2+iy_2)(x_2-iy_2)} = \frac{x_1x_2 - y_1y_2}{x_2^2+y_2^2} + i\,\frac{x_2y_1 - x_1y_2}{x_2^2+y_2^2}$$

also, $\operatorname{Re} z = \dfrac{z+\bar{z}}{2}$, $\operatorname{Im} z = \dfrac{z-\bar{z}}{2i}$

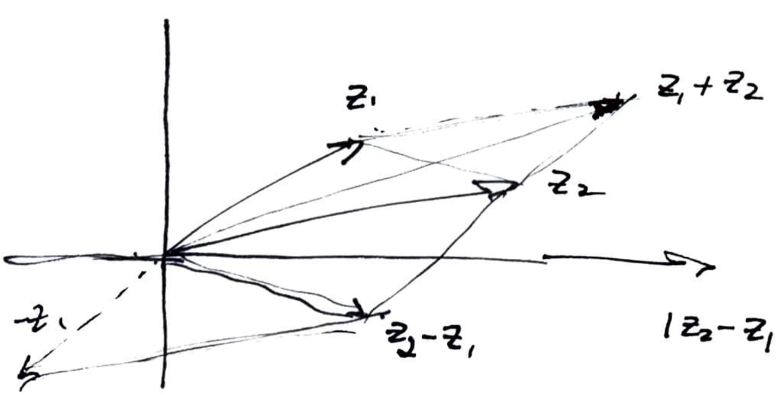
Geometrical Interpretation

Since a complex number is represented by 2 real numbers ($\operatorname{Re} z$ & $\operatorname{Im} z$), it is natural to think of it as a point in the plane:



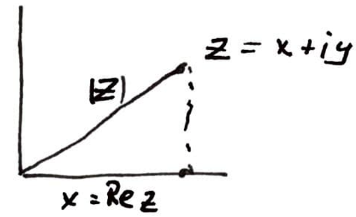
There are many analogies between Complex numbers and 2D vectors. However, they are not identical, e.g. the scalar product $x_1x_2 + y_1y_2$ is very different from $z_1 \cdot z_2$

$|z|$ - distance from z to the origin



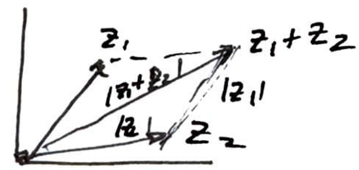
$|z_2 - z_1|$ - distance between z_1 & z_2

Ex. ① $\operatorname{Re} z \le |\operatorname{Re} z| \le |z|$



$x^2 \le x^2 + y^2$

② $\operatorname{Im} z \le |\operatorname{Im} z| \le |z|$



③ $|z_1+z_2| \le |z_1| + |z_2|$

triangle inequality

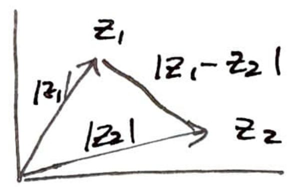
$|z_1+z_2|^2 \le |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$

$(x_1+x_2)^2 + (y_1+y_2)^2 \le x_1^2 + y_1^2 + x_2^2 + y_2^2 + 2|z_1||z_2|$

$x_1x_2 + y_1y_2 \le |z_1||z_2|$, $(x_1x_2 + y_1y_2)^2 \le (x_1^2+y_1^2)(x_2^2+y_2^2)$

$2x_1x_2y_1y_2 \le x_1^2y_2^2 + y_1^2x_2^2$ $\qquad (x_1y_2 - y_1x_2)^2 \ge 0$

④ $|z_1| - |z_2| \le |z_1 - z_2|$



⑤ $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$

⑥ $\bar{\bar{z}} = z$

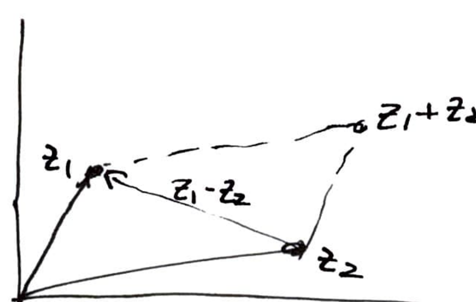
⑦ $\overline{z_1z_2} = \bar{z}_1 \cdot \bar{z}_2 \Rightarrow |z_1z_2|^2 = |z_1|^2 \cdot |z_2|^2$

$x_1x_2 - y_1y_2 \mp i(x_1y_2 + x_2y_1) = (x_1 \mp iy_1)(x_2 \mp iy_2)$

⑧ $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$ $\qquad \frac{\overline{z_1\bar{z}_2}}{|z_2|^2} = \frac{\bar{z}_1z_2}{|z_2|^2}$, $\frac{\bar{z}_1}{\bar{z}_2} = \frac{\bar{z}_1z_2}{|z_2|^2}$

⑨ $|z_1+z_2|^2 + |z_1-z_2|^2 = 2|z_1|^2 + 2|z_2|^2$

the sum of the squares of all parallelogram sides equals the sum of the squares of the diagonals

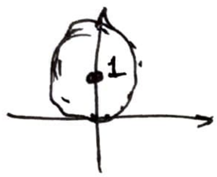


$(z_1+z_2)(\bar{z}_1+\bar{z}_2) + (z_1-z_2)(\bar{z}_1-\bar{z}_2) =$

$= 2|z_1|^2 + 2|z_2|^2$

10 $|z-i|=1$ – circle of radius 1 centered at $z=i$

– as discussed $|z_1-z_2|$ – distance between z_1 and z_2

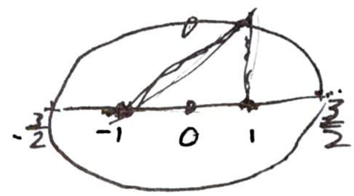


11 $|2z-i|=1$? $\left|z-\frac{1}{2}i\right|=\frac{1}{2} \Rightarrow$

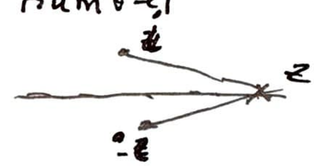
12 $|z-1|+|z+1|=3$

$|z-1|+|z+1|=2$

$|z-1|+|z+1|=1$



13 $|z-i|=|z+i| \Rightarrow z$ - any real number



formal proof

$z=x+iy \Rightarrow z-i=x+i(y-1)$

$z+i=x+i(y+1)$

$\cancel{x^2}+(y-1)^2=\cancel{x^2}+(y+1)^2 \Rightarrow y=0 \Rightarrow \underline{z=x}$