Midterm for PHYS 411, Fall 2016. Fri. Oct 28, 9-9:50

1: [10pts/30] A test particle is orbiting a star of mass M on an eccentric orbit (with eccentricity e and semimajor axis a). You may use the following relations for its orbital radius, energy, and angular momentum:

$$r = \frac{a(1-e^2)}{1+e\cos(\theta-\varpi)} \tag{1}$$

$$E = -\frac{GM}{2a} \tag{2}$$

$$l = \sqrt{GMa(1-e^2)} \tag{3}$$

For simplicity, we have set the particle's mass to be $m=1$.
DO NOT assume that e is small.

(a) [2pts] Use the above relations to calculate its speed at periapse (in terms of its orbital elements and GM)
(b) [8pts] When the particle is at periapse, it suddenly gets an infinitesimal kick δv_θ to its velocity in the tangential direction (in the direction of its motion). What are the resulting changes in a and e?
(a) We know $l = rv_\theta = a(1-e)v_\theta$. And eq 3 then implies:

$$a(1-e)v_\theta = \sqrt{GMa}\sqrt{1-e^2}$$

So,

$$v_\theta = \sqrt{\frac{GM}{a}}\sqrt{\frac{1+e}{1-e}}$$

(b) First use the energy relation to get δa.

$$E = -\frac{GM}{2a} = v_r^2/2 + v_\theta^2/2 - GM/r \tag{4}$$

$$\delta E = \frac{GM}{2a^2}\delta a = v_\theta \delta v_\theta \tag{5}$$

$$\delta a = \frac{2a^2}{GM} v_\theta \delta v_\theta \tag{6}$$

$$= \frac{2a^{3/2}}{\sqrt{GM}}\sqrt{\frac{1+e}{1-e}}\delta v_\theta \tag{7}$$

Next, use angular momentum to get δe:

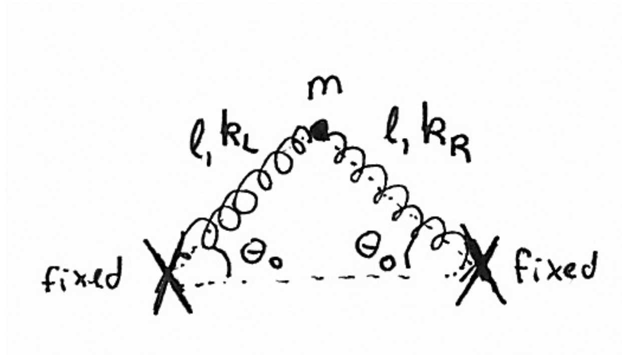
$$l = \sqrt{GMa(1-e^2)} = rv_\theta \tag{8}$$

$$\frac{\sqrt{GM}}{2\sqrt{a}}\delta a\sqrt{1-e^2} + \sqrt{GM}\frac{\sqrt{a}}{\sqrt{1-e^2}}(-e\delta e) = a(1-e)\delta v_\theta \tag{9}$$

$$a(1+e)\delta v_\theta + \sqrt{GM}\frac{\sqrt{a}}{\sqrt{1-e^2}}(-e\delta e) = a(1-e)\delta v_\theta \tag{10}$$

$$\delta e = \sqrt{\frac{a}{GM}}2\delta v_\theta\sqrt{1-e^2} \tag{11}$$

2: [10pts/30] Consider a system with this equilibrium configuration:



The mass m is connected to two springs (with spring constants k_L and k_R, and length l for both springs.). The other side of each spring is fixed. When perturbed, the mass's motion is confined to the plane. Note that θ_0 is not small.
Determine the $\boldsymbol{T}$ and $\boldsymbol{V}$ matrices.

Take x and y to denote m's displacement from equilibrium. Then,

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) \tag{12}$$

The mass is at (x, y) and the other end of the left spring is at $(-l\cos\theta_0, -l\sin\theta_0)$ Therefore, the total length of the left spring is

$$\begin{aligned} l_{spr} &= \left((l\cos\theta_0 + x)^2 + (l\sin\theta_0 + y)^2\right)^{1/2} && (13)\\ &\approx \left(l^2 + 2xl\cos\theta_0 + 2yl\sin\theta_0\right)^{1/2} && (14)\\ &\approx l\left(1 + \frac{x}{l}\cos\theta_0 + \frac{y}{l}\sin\theta_0\right) && (15)\\ &= l + (x\cos\theta_0 + y\sin\theta_0) && (16)\end{aligned}$$

where we keep terms to linear order in x and y. Therefore, V of the left spring is

$$\begin{aligned} V_L &= \frac{k_L}{2}\left(l_{spr} - l\right)^2 && (17)\\ &= \frac{k_L}{2}\left(x^2\cos^2\theta_0 + y^2\sin^2\theta_0 + 2xy\cos\theta_0\sin\theta_0\right) && (18)\end{aligned}$$

And by symmetry, it is the same for the right spring, except with $x \to -x$ (and $k_L \to k_R$). Hence

$$V = \frac{x^2}{2}(k_L + k_R)\cos^2\theta_0 + \frac{y^2}{2}(k_L + k_R)\sin^2\theta_0 + \cos\theta_0\sin\theta_0 xy(k_L - k_R) \tag{19}$$

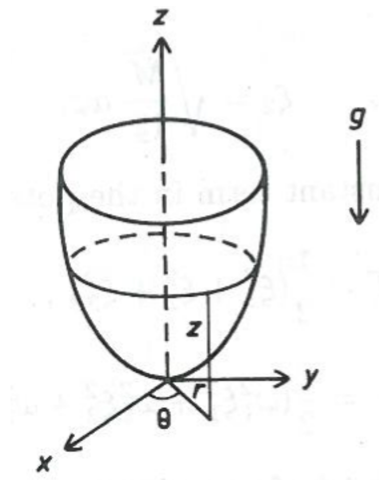
And in terms of matrices,

$$\begin{aligned} \boldsymbol{T} &= m\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} && (20)\\ \boldsymbol{V} &= \begin{pmatrix} (k_L + k_R)\cos^2\theta_0 & (k_L - k_R)\cos\theta_0\sin\theta_0 \\ (k_L - k_R)\cos\theta_0\sin\theta_0 & (k_L + k_R)\sin^2\theta_0 \end{pmatrix} && (21)\end{aligned}$$

3: [10pts/30] A particle moves on the inside wall of bowl whose height is given by

$$z = \frac{k}{2}(x^2 + y^2) = \frac{kr^2}{2} \tag{22}$$

where r is the cylindrical radius. The gravitational acceleration is g.



(a) [4pts] What are the equations of motion?
(b) [6pts] Solve the system by reducing it to a quadrature, i.e., to a single integral. The integral should only contain constants (in addition to the integration variable). Hint: first find the conserved quantities.

Will set mass=1.

(a)

$$T = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) \tag{23}$$

$$= \frac{1}{2}\left(\dot{r}^2(1 + k^2r^2) + r^2\dot{\theta}^2\right) \tag{24}$$

and $V = gz = g\frac{kr^2}{2}$, so

$$L = \frac{1}{2}\left(\dot{r}^2(1 + k^2r^2) + r^2\dot{\theta}^2\right) - g\frac{kr^2}{2} \tag{25}$$

And, the equations of motion are

$$\frac{dp_r}{dt} = \frac{d}{dt}\left(\dot{r}(1 + k^2r^2)\right) = k^2\dot{r}^2r + r\dot{\theta}^2 - gkr \tag{26}$$

$$\frac{dp_\theta}{dt} = \frac{d}{dt}\left(r^2\dot{\theta}\right) = 0 \tag{27}$$

(b) There are two constants. The first constant is angular momentum $l = r^2\dot{\theta}$, whose symmetry is rotation.

The second constant is energy, whose symmetry is time-translation:

$$\begin{aligned} E &= p_r\dot{r} + p_\theta\dot{\theta} - L && (28)\\ &= \dot{r}^2(1+k^2r^2) + r^2\dot{\theta}^2 - \frac{1}{2}\left(\dot{r}^2(1+k^2r^2) + r^2\dot{\theta}^2\right) + g\frac{kr^2}{2} && (29)\\ &= \frac{1}{2}\left(\dot{r}^2(1+k^2r^2) + r^2\dot{\theta}^2\right) + g\frac{kr^2}{2} && (30) \end{aligned}$$

The energy is written in terms of l as

$$E = \frac{1}{2}\left(\dot{r}^2(1+k^2r^2) + \frac{l^2}{r^2}\right) + g\frac{kr^2}{2} \tag{31}$$

So,

$$\dot{r}^2 = \left(2E - gkr^2 - \frac{l^2}{r^2}\right) / \left(1+k^2r^2\right) \tag{32}$$

And

$$t - T_0 = \int dr \frac{\sqrt{1+k^2r^2}}{\sqrt{2E - gk^2 - l^2/r^2}} \tag{33}$$