Problem Set 5

Due Thursday October 31, 9:30 AM. Submit in class or in TA's mailbox in the Physics office.

1. Consider the change of basis from $e_1 = (1,0)$, $e_2 = (0,1)$ to

$$f_1 = (\cos\theta, e^{i\alpha}\sin\theta), \quad f_2 = (-e^{-i\alpha}\sin\theta, \cos\theta) \tag{1}$$

Verify that f_1, f_2 is indeed an orthonormal basis. If an arbitrary vector can be written

$$v = \sum_i \beta_i e_i = \sum_j \gamma_j f_j, \tag{2}$$

write explicitly the matrices U and V which convert between the β_i and the γ_j:

$$\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} = U\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}, \quad \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix} = V\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}. \tag{3}$$

Show that U and V are unitary and that $U = V^{-1}$.

2. This problem concerns the properties of a quantum-mechanical bound state.

   (a) Show that, if $|\psi\rangle$ is an eigenvector of H, and A is any time-independent operator, then $\langle\psi|A|\psi\rangle$ is time-independent. Show also that implies

$$\langle\psi|[A,H]|\psi\rangle = 0. \tag{4}$$

   Does $[A,H]$ generally also vanish?

   (b) Let $H = p^2/2m + V(x)$ be the Hamiltonian for a potential in one dimension. Let $A = xp$. Compute $[A,H]$.

   (c) Using the results of parts (a) and (b), show that, if $V(x) = Cx^n$, where C is a constant, and if $|\psi\rangle$ is an eigenstate of H, then

$$\langle\psi|T|\psi\rangle = \frac{n}{2}\langle\psi|V|\psi\rangle, \tag{5}$$

   where T and V are the kinetic and potential energy. This result is called the Virial Theorem.