$\vec{E}_{0r}$ and $\vec{E}_{0i}$ will be mutually perpendicular if

$$0 = \vec{E}_{0r} \cdot \vec{E}_{0i} = (a^2 - b^2)\sin\theta\cos\theta + \vec{a}\cdot\vec{b}\,(\cos^2\theta - \sin^2\theta)$$

$$= \tfrac{1}{2}(a^2 - b^2)\sin 2\theta + \vec{a}\cdot\vec{b}\cos 2\theta$$

This expression will indeed equal zero if

$$\tan 2\theta = -\frac{2\vec{a}\cdot\vec{b}}{a^2 - b^2}$$

We can think of $\vec{E}_{0r}$ and $\vec{E}_{0i}$ as the real and imaginary parts of the complex vector

$$\vec{e}_0 = \vec{E}_{0r} + i\vec{E}_{0i}$$

with $\vec{E}_0 = \vec{e}_0 e^{-i\theta}$. The electric field can be written as

$$\vec{E} = \vec{e}_0 e^{i(\vec{k}\cdot\vec{r} - wt - \theta)}$$

recall that $\hat{z} \perp \vec{E}_{0r} \perp \vec{E}_{0i}$

If we take $\hat{k} = \hat{z}$, we can choose $\vec{E}_{0r}$ to be in the $\hat{x}$ direction and $\vec{E}_{0i}$ to be in the $\hat{y}$ direction, with

$$\vec{E}_{0r} = E_{0x}\hat{x}, \quad \vec{E}_{0i} = E_{0y}\hat{y}$$

If we write out the x and y components of $\vec{E}$, we get, after taking the real part:

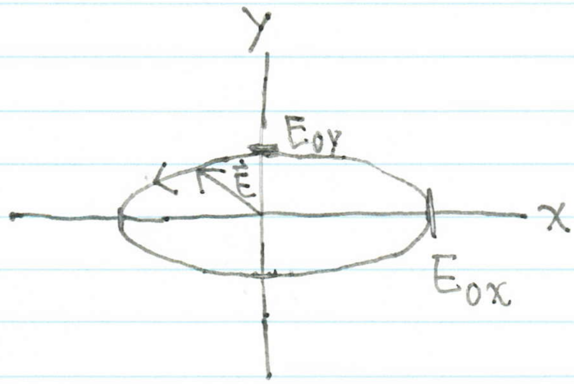
$$E_x = E_{0x}\cos(kz - wt - \theta)$$

$$E_y = -E_{0y}\sin(kz - wt - \theta)$$

Note that:

$$\frac{E_x^2}{E_{0x}^2} + \frac{E_y^2}{E_{0y}^2} = \cos^2(\cdots) + \sin^2(\cdots) = 1,$$

which is the equation of an ellipse.



At a given point $\vec{r}$ in space, the tip of the vector $\vec{E}$ rotates along an ellipse in the x-y plane with period $\frac{2\pi}{\omega}$.

For a fixed choice of x and y axes, need to specify angle of ellipse with respect to these axes.

Elliptical polarization is the most general polarization.

Some specific polarizations:

right circular polarization: $E_{0x} = E_{0y}$, $\vec{E} = E_{0x}(\hat{x} + i\hat{y})e^{i(kz - wt - \theta)}$

left circular polarization: $E_{0x} = -E_{0y}$, $\vec{E} = E_{0x}(\hat{x} - i\hat{y})e^{i(kz - wt - \theta)}$

x linear polarization: $E_{0y} = 0$, $\vec{E} = E_{0x}\,\hat{x}\,e^{i(kz - wt - \theta)}$

y linear polarization: $E_{0x} = 0$, $\vec{E} = iE_{0y}\,\hat{y}\,e^{i(kz - wt - \theta)}$

Can express any polarization as a linear combination of two linearly independent polarization states — for example, x and y linear polarization.

The Scalar Wave Equation and the Paraxial Approximation

The wave equation in terms of the electric field is

$$\vec{\nabla}^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

In general, this equation couples together the different components of the $\vec{E}$ field.

For laser beams, which tend to be well-collimated, it is often the case that there is one dominant polarization component at all points in the laser beam.

In this case, we can write:

$$\vec{E}(\vec{r},t) = \vec{E}_0\, U(\vec{r},t)$$

$\vec{E}_0$ tells us about the polarization, i.e. whether it is linear &, right circular, etc...

For a constant vector $\vec{E}_0$ and a scalar field $U(\vec{r},t)$. $U(\vec{r},t)$ then satisfies the scalar wave equation

$$\vec{\nabla}^2 U - \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = 0$$

Solving the scalar wave equation is mathematically simpler than the vector wave equation. In many situations, looking only at the scalar wave equation is sufficient to describe experiments.

plane wave example: $\vec{E}(\vec{r},t) = \vec{E}_0 \underbrace{e^{i(kz-\omega t)}}_{U(\vec{r},t)}$

As an example, let us consider a monochromatic laser beam of frequency w that is propagating in the $\hat{z}$ direction. The electric field is

$$\vec{E}(\vec{r},t) = \vec{E}_0\, U(\vec{r},t)$$

We can pull out a factor of $e^{i(kz-wt)}$ to write

$$U(\vec{r},t) = u(\vec{r})e^{i(kz-wt)}$$ $u(\vec{r})$ is an envelope function

Substituting into the scalar wave equation:

$$\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + 2ik\frac{\partial u}{\partial z} - k^2 u + \frac{w^2}{c^2}u\right]e^{i(kz-wt)} = 0$$

The last two terms cancel because $k = \frac{w}{c}$. So the wave equation for u is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + 2ik\frac{\partial u}{\partial z} = 0$$

A very common experimental situation is that u does not significantly vary along the laser beam on the length scale of one wavelength $\lambda = \frac{2\pi}{k}$.

In other words, the laser beam does not significantly diffract as it propagates over a distance λ.

The fractional change in u as the beam propagates over a distance λ is

$$\frac{\Delta u}{u} = \frac{\frac{\partial u}{\partial z}\lambda}{u}$$

Our condition that the u does not change much as the beam propagates over a distance λ can be expressed as

$$\frac{\Delta u}{u} = \frac{\frac{\partial u}{\partial z}\lambda}{u} << 1$$

$$\Rightarrow \frac{\partial u}{\partial z} << \frac{u}{\lambda} \sim ku \quad \left(\text{since } k = \frac{2\pi}{\lambda}\right)$$

Similarly, we also assume that $\frac{\partial u}{\partial z}$ does not change much over a wavelength

$$\Rightarrow \frac{\partial^2 u}{\partial z^2} << k\frac{\partial u}{\partial z}$$

Paraxial Approximation:

We drop the $\frac{\partial^2 u}{\partial z^2}$ in the scalar wave equation, because it is much smaller than the other terms.

Describes many experiments very well.

Paraxial Wave Equation:

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2ik\frac{\partial u}{\partial z} = 0}$$