1. This is analogous to the problem of the point charge near a plane interface between dielectrics that we solved in lecture. We can think of the line charge as an infinite line of point charges, each of charge dq. To solve the problem for $z>0$, we can place an image charge dq' corresponding to each dq, as we did in lecture.

We recall from lecture that the image charge dq' is

$$dq' = \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}$$ and is located at $z = -d$.

If we add up all the contributions dq' into an image line charge λ' at $z=-d$, we have

$$\frac{\lambda'}{\lambda} = \frac{dq'}{dq} = \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} \Rightarrow \lambda' = \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}\lambda$$

At the line charge λ, the electric field from the image charge λ' is

$$\vec{E} = \frac{2\lambda'}{2d\epsilon_1}\hat{z} = \frac{\lambda'}{d\epsilon_1}\hat{z}$$

So the force per unit length on the line charge is

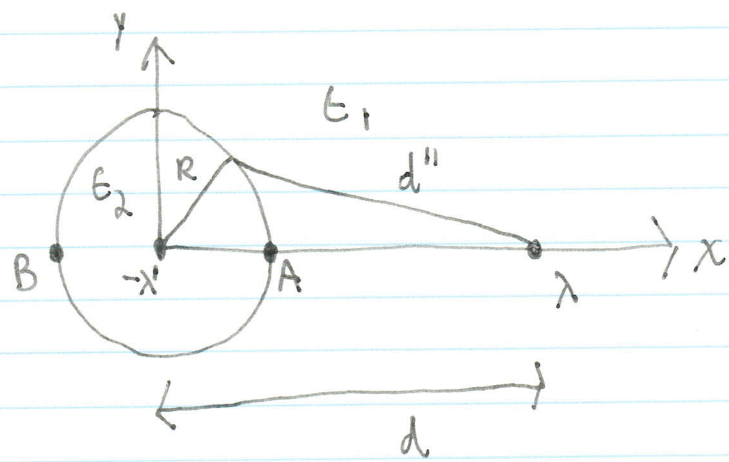
$$\frac{\vec{F}}{L} = \frac{\lambda\lambda'}{d\epsilon_1}\hat{z} = \frac{\lambda^2}{d}\frac{\epsilon_1 - \epsilon_2}{\epsilon_1(\epsilon_1 + \epsilon_2)}\hat{z}$$

For $\epsilon_1 > \epsilon_2$ the force is repulsive, for $\epsilon_1 < \epsilon_2$ it is attractive.

2. a. To determine the potential, let us first try an image line charge $-\lambda'$ that lies at $x=0$, $y=0$ for the potential outside the cylinder. On the outside boundary, the potential is then

$$\phi_1(R) = -\frac{2\lambda}{\epsilon_1} d'' + \frac{2\lambda'}{\epsilon_1} \ln R$$

with distances defined as follows:



Note that d'' is different for different points on the boundary. For the potential inside the cylinder, let us try an image line charge λ'' located at the position of the real line charge λ. On the inner boundary, the potential is:

$$\phi_2(R) = -\frac{2\lambda''}{\epsilon_2} \ln d'' + C$$

where C is a constant (from the choice of where we start our integral over the electric field)

Since d'' takes on different values over the surface, the only way to satisfy the boundary condition

$\phi_1(r=R) = \phi_2(r=R)$ at all points on the boundary is to set

$$\frac{2\lambda}{\epsilon_1} = \frac{2\lambda''}{\epsilon_2} \Rightarrow \lambda'' = \frac{\epsilon_2}{\epsilon_1}\lambda$$

But we also need to satisfy

$$\epsilon_1 \frac{\partial \phi_1}{\partial n} = \epsilon_2 \frac{\partial \phi_2}{\partial n}$$ at all points on the boundary

At point A shown on the diagram, this becomes

$$\frac{2\lambda}{d-R} + \frac{2\lambda'}{R} = \frac{\epsilon_2}{\epsilon_1}\frac{2\lambda}{d-R}$$ (after plugging in for λ'')

$$\Rightarrow \lambda' = \frac{R}{d-R}\left(\frac{\epsilon_2}{\epsilon_1} - 1\right)$$

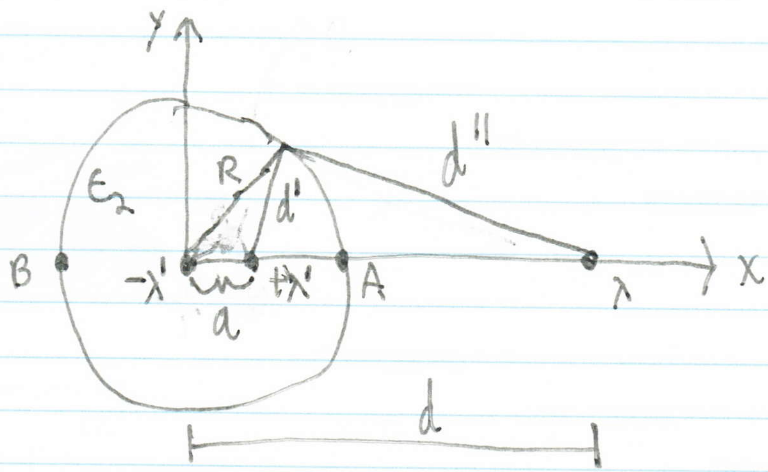
However, at point B, the condition becomes

$$\frac{2\lambda}{d+R} - \frac{2\lambda'}{R} = \frac{\epsilon_2}{\epsilon_1}\frac{2\lambda}{d+R}$$

$$\Rightarrow \lambda' = \frac{R}{d+R}\left(1 - \frac{\epsilon_2}{\epsilon_1}\right)$$

which will in general require a different value of λ'. So our one point charge at the origin is not sufficient to meet all boundary conditions.

So we will add another image charge along the x axis for the case of the potential outside the sphere to give us another degree of freedom for matching the boundary conditions. To be able to oppose the effect of the charge $-\lambda'$ at the origin, we can choose the new charge to be $+\lambda'$ and hope that the additional degree of freedom of the position of this charge is sufficient to meet our boundary conditions. The setup is as follows:



$$\Rightarrow \phi_1(r=R) = -\frac{2\lambda}{\epsilon_1}\ln d'' - \frac{2\lambda'}{\epsilon_1}\ln d' + \frac{2\lambda'}{\epsilon_1}\ln R$$

$$\phi_2(r=R) = -\frac{2\lambda''}{\epsilon_2}\ln d'' + C$$

We need this to hold at all points on the boundary

$$\Rightarrow -\frac{2\lambda'}{\epsilon_1}\ln d' = \left(\frac{2\lambda}{\epsilon_1} - \frac{2\lambda''}{\epsilon_2}\right)\ln d'' + \left(C - \frac{2\lambda'}{\epsilon_1}\ln R\right)$$

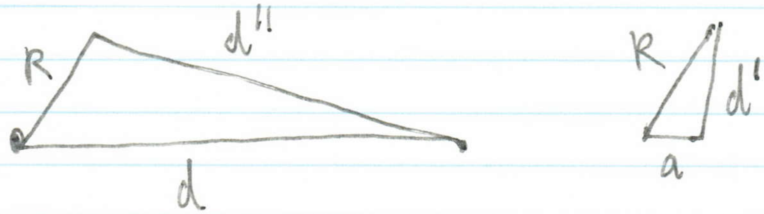
Note that if $\frac{2\lambda}{\epsilon_1} - \frac{2\lambda''}{\epsilon_2} = -\frac{2\lambda'}{\epsilon_1}$, then

we have

$$-\frac{2\lambda'}{\epsilon_1} \ln\left(\frac{d'}{d''}\right) = C - \frac{2\lambda'}{\epsilon_1} \ln R$$

This will be the case if $\epsilon_1 \lambda'' = \epsilon_2 (\lambda + \lambda')$

$\Rightarrow \lambda'' = \frac{\epsilon_2}{\epsilon_1}(\lambda + \lambda')$. To make the boundary condition hold for all points on the boundary, we would like to have $\frac{d'}{d''}$ have a constant ratio at all boundary points. This will be the case if the following triangles are similar:



Given that they share an angle, this will be the case if $\frac{R}{d} = \frac{a}{R} \Rightarrow a = \frac{R^2}{d}$

Then $\frac{d'}{d''} = \frac{R}{d}$ by taking side length ratios

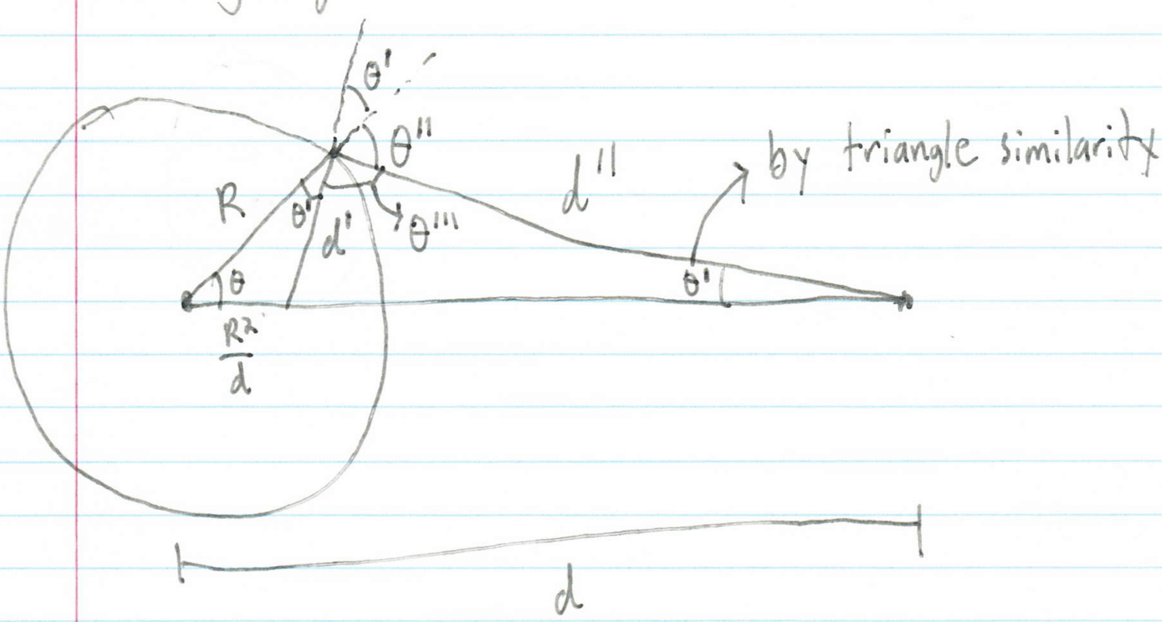
The boundary condition becomes:

$$-\frac{2\lambda'}{\epsilon_1}\ln\left(\frac{R}{d}\right) = -\frac{2\lambda'}{\epsilon_1}\ln R + \frac{2\lambda'}{\epsilon_1}\ln d = C - \frac{2\lambda'}{\epsilon_1}\ln R$$

$$\Rightarrow C = \frac{2\lambda'}{\epsilon_1}\ln d$$

So $\phi_1 = \phi_2$ on the boundary is satisfied. Now we need to check our boundary condition in terms of the normal fields. We need:

$\epsilon_1 E_{1_\perp} = \epsilon_2 E_{2_\perp}$ where $E_{i_\perp}$ is the normal field in region i at the boundary. It helps to draw the following angles:

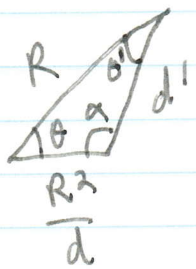


$\theta''' = \pi - \theta - 2\theta'$ since angles of the big triangle must add to π

The contribution to $E_{1\perp}$ from $-\lambda'$ is $-\frac{2\lambda'}{\epsilon_1 R}\hat{r}$.

The contribution from $+\lambda'$ is $\frac{2\lambda'}{\epsilon_1 d'}\cos\theta'\,\hat{r}$.

To find θ' and d' in terms of θ, we look at the following triangle:



$$\alpha = \pi - \theta - \theta'$$

The law of cosines says:

$$d'^2 = R^2 + \frac{R^4}{d^2} - 2\frac{R^3}{d}\cos\theta$$

$$\Rightarrow d' = R\sqrt{1 + \frac{R^2}{d^2} - 2\frac{R}{d}\cos\theta}$$

It also says:

$$\frac{R^4}{d^2} = R^2 + d'^2 - 2d'R\cos\theta'$$

$$\Rightarrow -2R^2 + 2\frac{R^3}{d}\cos\theta = -2R^2\sqrt{1 + \frac{R^2}{d^2} - 2\frac{R}{d}\cos\theta}\,\cos\theta'$$

$$\Rightarrow \cos\theta' = \frac{1 - \frac{R}{d}\cos\theta}{\sqrt{1 + \frac{R^2}{d^2} - 2\frac{R}{d}\cos\theta}}$$

The contribution to $E_{1\perp}$ from $+\lambda'$ is then

$$\frac{2\lambda'}{\epsilon_1 d'}\cos\theta' = \frac{2\lambda'}{\epsilon_1 R}\,\frac{1-\frac{R}{d}\cos\theta}{1+\frac{R^2}{d^2}-2\frac{R}{d}\cos\theta}$$

Now, the contribution from λ is:

$$-\frac{2\lambda}{\epsilon_1 d''}\cos\theta''\,\hat{r}$$

Since $\theta''+\pi-\theta-2\theta'+\theta'=\pi$ from looking at our figure with the angles, $\theta''=\theta+\theta'$, so

$$\cos\theta''=\cos(\theta+\theta')=\cos\theta\cos\theta'-\sin\theta\sin\theta'$$

By the law of sines:

$$\frac{\sin\theta'}{\frac{R^2}{d}}=\frac{\sin\theta}{d'} \Rightarrow \sin\theta'=\frac{\frac{R^2}{d}}{d'}\sin\theta$$

2nd term in angle addition formula gives a $\sin^2\theta$

$$\text{so } \cos\theta''=\frac{\cos\theta-\frac{R}{d}\cos^2\theta}{\sqrt{1+\frac{R^2}{d^2}-2\frac{R}{d}\cos\theta}}-\frac{\frac{R}{d}(1-\cos^2\theta)}{\sqrt{1+\frac{R^2}{d^2}-2\frac{R}{d}\cos\theta}}$$

$$=\frac{\cos\theta-\frac{R}{d}}{\sqrt{1+\frac{R^2}{d^2}-2\frac{R}{d}\cos\theta}}=\frac{d\left(\cos\theta-\frac{R}{d}\right)}{\sqrt{R^2+d^2-2Rd\cos\theta}}$$

Another application of the law of cosines gives

$$d'' = \sqrt{R^2 + d^2 - 2Rd\cos\theta}$$

So the contribution to $E_{1\perp}$ from λ is

$$-\frac{2\lambda}{\epsilon_1 R}\frac{dR\cos\theta - R^2}{R^2 + d^2 - 2dR\cos\theta} = \frac{2\lambda}{\epsilon_1 R}\frac{1 - \frac{d}{R}\cos\theta}{1 + \frac{d^2}{R^2} - 2\frac{d}{R}\cos\theta}$$

Now, the contribution from λ'' to $E_{2\perp}$, which is the only contribution to $E_{2\perp}$, is

$$E_{2\perp} = \frac{2\lambda''}{\epsilon_2 R}\frac{1 - \frac{d}{R}\cos\theta}{1 + \frac{d^2}{R^2} - 2\frac{d}{R}\cos\theta}$$ ($\hat{r}$ direction defined to be positive)

The boundary condition $\epsilon_1 E_{1\perp} = \epsilon_2 E_{2\perp}$ the becomes

$$-\frac{2\lambda'}{R} + \frac{2\lambda'}{R}\frac{1 - \frac{R}{d}\cos\theta}{1 + \frac{R^2}{d^2} - 2\frac{R}{d}\cos\theta} + \frac{2\lambda}{R}\frac{1 - \frac{d}{R}\cos\theta}{1 + \frac{d^2}{R^2} - 2\frac{d}{R}\cos\theta}$$

$$= \frac{2\lambda''}{R}\frac{1 - \frac{d}{R}\cos\theta}{1 + \frac{d^2}{R^2} - 2\frac{d}{R}\cos\theta}$$

We recall that $\lambda'' = \frac{\epsilon_2}{\epsilon_1}(\lambda + \lambda') \Rightarrow \lambda' = \frac{\epsilon_1}{\epsilon_2}\lambda'' - \lambda$

$$\Rightarrow \frac{2}{R}\left(\lambda''\frac{\epsilon_1}{\epsilon_2}-\lambda\right)\left(-1+\frac{1-\frac{R}{d}\cos\theta}{1+\frac{R^2}{d^2}-2\frac{R}{d}\cos\theta}\right)$$

$$+\frac{2\lambda}{R}\frac{1-\frac{d}{R}\cos\theta}{1+\frac{d^2}{R^2}-2\frac{d}{R}\cos\theta}=\frac{2\lambda''}{R}\frac{1-\frac{d}{R}\cos\theta}{1+\frac{d^2}{R^2}-2\frac{d}{R}\cos\theta}$$

Note that:

$$-1+\frac{1-\frac{R}{d}\cos\theta}{1+\frac{R^2}{d^2}-2\frac{R}{d}\cos\theta}=-1+\frac{\frac{d^2}{R^2}-\frac{d}{R}\cos\theta}{1+\frac{d^2}{R^2}-2\frac{d}{R}\cos\theta}$$

$$=-\frac{1-\frac{d}{R}\cos\theta}{1+\frac{d^2}{R^2}-\frac{2d}{R}\cos\theta}$$

$$\Rightarrow -\left(\lambda''\cdot\frac{\epsilon_1}{\epsilon_2}-\lambda\right)+\lambda=\lambda'' \Rightarrow \lambda''(\epsilon_1+\epsilon_2)=2\lambda\epsilon_2$$

$$\Rightarrow \lambda''=\frac{2\epsilon_2}{\epsilon_1+\epsilon_2}\lambda$$

If this additional condition is met, the the second boundary condition is satisfied at all points on the boundary.

Plugging back in for λ':

$$\lambda' = \left(\frac{2\epsilon_1}{\epsilon_1+\epsilon_2} - 1\right)\lambda = \frac{\epsilon_1 - \epsilon_2}{\epsilon_1+\epsilon_2}\lambda$$

So the potential outside the cylinder is:

$$\phi_1(\vec{r}) = -\frac{2\lambda}{\epsilon_1}\ln(|\vec{r}-d\hat{x}|) - \frac{2\lambda'}{\epsilon_1}\ln\left(\left|\vec{r}-\frac{R^2}{d}\hat{x}\right|\right)$$

$$+\frac{2\lambda'}{\epsilon_1}\ln(|\vec{r}|)$$

The potential inside the cylinder is

$$\phi_2(\vec{r}) = -\frac{2\lambda''}{\epsilon_2}\ln(|\vec{r}-d\hat{x}|) + \frac{2\lambda'}{\epsilon_1}\ln(d)$$

for $\lambda' = \frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\lambda$, $\lambda'' = \frac{2\epsilon_2}{\epsilon_1+\epsilon_2}\lambda$.

2. b. The force per unit length is

$$\frac{\vec{F}}{L} = \lambda\left(-\frac{2\lambda'}{\epsilon_1 d} + \frac{2\lambda'}{\epsilon_1\left(d-\frac{R^2}{d}\right)}\right)\hat{r}$$

$$= \frac{2\lambda^2}{\epsilon_1}\left[\frac{(\epsilon_1-\epsilon_2)d}{(\epsilon_1+\epsilon_2)(d^2-R^2)} - \frac{\epsilon_1-\epsilon_2}{(\epsilon_1+\epsilon_2)d}\right]\hat{r}$$

$$= \frac{2\lambda^2(\epsilon_1-\epsilon_2)R^2}{\epsilon_1 d(d^2-R^2)(\epsilon_1+\epsilon_2)}\hat{r}$$

The force is repulsive for $\epsilon_1 > \epsilon_2$ and attractive for $\epsilon_1 < \epsilon_2$.

3. a. The potential must be uniform on each of the conducting spherical shells, so the electric field at each conducting surface must be normal to the surface. To satisfy these conditions and Gauss' law, we consider an electric field of the form

$\vec{E} = \frac{A}{r^2}\hat{r}$ between the two spheres, where A is to be determined by the charge on the inner spherical surface. Since $\vec{E}$ is in the $\hat{r}$ direction and continuous, the tangential and perpendicular field boundary conditions at the interface between the dielectric and the empty space are satisfied.

$\vec{D}_1 = \frac{\epsilon A}{r^2}\hat{r}$ for the dielectric portion and $\vec{D}_2 = \vec{E} = \frac{A}{r^2}\hat{r}$

for the empty portion. Since the surface charge density on the conducting surfaces is excess charge, and because $\vec{D}$ satisfies Gauss' law if we only look at excess charge, the surface charge density σ_1 on the inner conductor in the dielectric half of the problem is

$$\sigma_1 = \frac{D_1}{4\pi}\bigg|_{r=a} = \frac{\epsilon A}{4\pi a^2}$$

The surface charge density σ_2 on the inner conductor in the empty half of the problem is

$$\sigma_2 = \frac{D_2}{4\pi}\bigg|_{r=a} = \frac{A}{4\pi a^2}$$

Since the total charge on the inner conductor is Q:

$$2\pi a^2\sigma_1 + 2\pi a^2\sigma_2 = Q \Rightarrow \frac{\epsilon A}{2} + \frac{A}{2} = Q$$

$$\Rightarrow A = \frac{2Q}{\epsilon+1}$$

So the field between the spheres is

$$\vec{E} = \frac{2Q}{\epsilon+1}\frac{1}{r^2}\hat{r}$$

b. As we found above, in the region with the dielectric, it is

$$\sigma_1 = \frac{\epsilon A}{4\pi a^2} = \frac{\epsilon Q}{2\pi(\epsilon+1)a^2}$$

In the region without the dielectric, it is

$$\sigma_2 = \frac{Q}{2\pi(\epsilon+1)a^2}$$

c. $\vec{D}$ satisfies Gauss' law if we only include excess charge, while $\vec{E}$ satisfies Gauss' law if we include the total charge, which includes the polarization charge density on the surface of the dielectric (which is not excess charge).

$$\Rightarrow 4\pi(\sigma_1 + \sigma_p) = E_1\Big|_{r=a} = \frac{2Q}{\epsilon+1}\frac{1}{a^2}$$

$$\Rightarrow \sigma_p = \frac{Q}{2\pi(\epsilon+1)a^2} - \sigma_1$$

$$= \frac{Q(1-\epsilon)}{2\pi(\epsilon+1)a^2}$$

4. External field in absence of spherical shell:

$\vec{B}_0 = \vec{H}_0 = B_0 \hat{z}$

Asymptotic form: $\phi_0 = -B_0 z = -B_0 r\cos\theta$

$\Rightarrow$ only $\frac{1}{r^{l+1}}$ terms survive, in addition to r term

$r > r_2$: $\phi_M = -B_0 r\cos\theta + \sum_{l=0}^{\infty} \frac{a_l}{r^{l+1}} P_l(\cos\theta)$

note that we have azimuthal symmetry

$r_1 < r < r_2$: $\phi_M = \sum_{l=0}^{\infty} \left(b_l r^l + c_l \frac{1}{r^{l+1}}\right) P_l(\cos\theta)$

$r < r_1$: Potential cannot blow up at the origin, so no $\frac{1}{r^{l+1}}$ terms:

$$\phi_M = \sum_{l=0}^{\infty} d_l r^l P_l(\cos\theta)$$

Boundary conditions: ϕ_m continuous across boundaries so fields don't blow up

$\mu \frac{\partial \phi_m}{\partial r}$ continuous across boundaries ; implies condition 3

$\frac{\partial \phi_m}{\partial \theta}$ continuous across boundaries

Boundary at r_2:

$$B_0 r_2 \sin\theta + \sum_{l=0}^{\infty} \underbrace{\left(\frac{a_l}{r_2^{l+1}} - b_l r_2^l - \frac{c_l}{r_2^{l+1}} \right)}_{\text{must be 0 for } l \neq 1 \text{ to get correct angular dependence}} \frac{\partial P_l(\cos\theta)}{\partial \theta} = 0$$

$$-B_0 \cos\theta + \sum_{l=0}^{\infty} \underbrace{\left(-(l+1) \frac{a_l}{r_2^{l+2}} - \mu l r_2^{l-1} b_l + \mu \frac{c_l(l+1)}{r_2^{l+2}} \right)}_{\text{must be 0 for } l \neq 1} P_l(\cos\theta) = 0$$

2nd term actually 0 for $l=0$

at $r=r_1$ and conditions

to meet both conditions, want $a_l = b_l = c_l = 0$ for $l \neq 1$

For $l=1$: $\frac{\partial P_1(\cos\theta)}{\partial \theta} = -\sin\theta$ ($P_1 \to \cos\theta$)

$$\Rightarrow a_1 - r_2^3 b_1 - c_1 = B_0 r_2^3$$

$$2a_1 + \mu r_2^3 b_1 - 2\mu c_1 = -B_0 r_2^3$$

4 linear eqns all equal to zero in 4 variables, want all 4 variables to be 0

Boundary at r_1

Will just have $l=1$ terms to match functional form.

$$b_1 r_1 + \frac{c_1}{r_1^2} - d_1 r_1 = 0 \Rightarrow r_1^3 b_1 + c_1 - r_1^3 d_1 = 0$$

$$\mu b_1 - 2\mu c_1 \frac{1}{r_1^3} - d_1 = 0 \Rightarrow \mu r_1^3 b_1 - 2\mu c_1 - d_1 r_1^3 = 0$$

We now have 4 equations in 4 variables. Solving for d_1 in Mathematica:

$$d_1 = -\left[\frac{9 B_0 \mu}{(2\mu+1)(\mu+2) - 2\frac{r_1^3}{r_2^3}(\mu-1)^2}\right]$$

For $\mu \gg 1$,

$$d_1 \approx -\frac{9\mu}{2\mu^2 - 2\frac{r_1^3}{r_2^3}\mu^2} B_0 = -\frac{9}{2\mu\left(1-\frac{r_1^3}{r_2^3}\right)} B_0$$

For large μ, significant shielding. Shield works better for larger $r_2 - r_1$.

Inside the shell, $\vec{B} = -\vec{\nabla}\phi_m$, where $\phi_m = d_1 z$

$$\vec{B} = -d_1 \hat{z} = \frac{9}{2\mu\left(1-\frac{r_1^3}{r_2^3}\right)} B_0$$