Appendix A: Proving That the Spacetime Interval is an Invariant

Our proof is along the lines of that in [3] but it is somewhat simpler. Noting that when $ds = 0, ds^* = 0$ and that these elements are infinitesimals of the same (first) order, they must be proportional. On what can the proportionality factor depend? It cannot depend on where one is located in space because no point in space is privileged over any other point. Space is assumed to be homogeneous. It cannot depend upon the time because no point in time is privileged. Now we turn to the only element that relates Alicia to Beatrice; it is their relative velocity v. Now it cannot depend upon the direction of their relative velocity because no direction is favoured over any other direction; the spacetime is isotropic. All that is left is the magnitude of their relative velocity, $\mathrm{mod}(v)$. So we write

$$ds = K(\mathrm{mod}(v))\, ds^* \tag{A.1}$$

where K is the proportionality function. Suppose we were to consider the relationship from Beatrice's viewpoint. For her, Alicia has velocity $-v$ relative to Beatrice so the connection between the intervals is

$$ds^* = K(\mathrm{mod}(-v))\, ds. \tag{A.2}$$

But $\mathrm{mod}(v) = \mathrm{mod}(-v)$ so the K functions in (A.1) and (A.2) have the very same value. Thus a substitution of ds^* from (A.2) into (A.1) gives

$$ds = K(\mathrm{mod}(v))\, ds^* = K(\mathrm{mod}(v))K(\mathrm{mod}(-v))\, ds = K^2\, ds \tag{A.3}$$

and therefore $K^2 = 1$. When we square both sides of (A.1) and substitute $K^2 = 1$, we have

$$ds^2 = ds^{*2}. \tag{A.4}$$

Thus $ds = (+/-)ds^*$ and we choose the plus to preserve the direction of the flow of time. From the equality of the infinitesimals, it follows that the finite intervals are equal as well, $s = s^*$.

The important result that emerges is that the spacetime interval between any two given events is an invariant.

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Appendix B: Deriving the Einstein Field Equations

In Newtonian gravitational theory, there is one simple equation for the gravitational field that connects the gravitational potential ϕ to the distribution of matter density ρ. This equation is of the type that gives solutions for density changes that imprint upon the entire universe instantaneously. This is inconsistent with the basic premise of Special Relativity, so if Special Relativity is correct, Newtonian gravity must be replaced by a relativistic theory of gravity.

For those readers who are interested in developing a detailed understanding of the mathematics underpinning General Relativity, they would profit from reading the next level of treatment in [1] with extra tensor calculus detail in [7]. They could then follow up with more sophisticated treatments in the classic book of Landau and Lifshitz [3] as well as the various other more advanced books on General Relativity. In the present book, we are aiming at a very basic level of development with just enough mathematics to go beyond those books which rely solely upon descriptions in words.

Without getting into more advanced mathematical detail, we present a narrative that leads naturally to Einstein's General Relativity. To do so, we first consider the very basic constructs of energy and momentum. What makes them basic is that they have the valuable property of being conserved when systems undergo changes. Much of physics through the centuries has been driven by the new constructs that had been invoked to maintain this primary attribute. For example, in electromagnetism, we express the conservation of the important physical attribute that we label "charge" in the form

$$\nabla . J + \frac{\partial \rho}{\partial t} = 0 \tag{B.1}$$

where ρ here is the charge density, the amount of charge per unit volume and J is the charge-current density vector, ρv with v being the velocity of the charge at the point under investigation.

To see why (B.1) expresses the conservation of charge, we require the fundamental theorem of Gauss that the integral of the divergence of a vector over a

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given volume is equal to the flux of that vector over the surface that bounds the given volume:

$$\int (\nabla . J)\, dV = \int J.ndS \tag{B.2}$$

where n is a vector of unit length perpendicular to the bounding surface and pointing outward at every point from that surface. When we integrate (B.1) over a given volume and apply (B.2), we have

$$\frac{d}{dt}\int \rho dV = -\int J.ndS. \tag{B.3}$$

This equation states that the time-rate of change of the total charge within the volume V is given by the negative of the rate at which charge flows out through the bounding surface S. In other words, what is lost in the volume is necessarily compensated by what flows out of its bounding surface. Thus we see that (B.1) is an expression of conservation locally.

In Relativity, the natural arena for investigation is spacetime rather than space, and we generalize the three-dimensional current vector $J = (\rho v_x, \rho v_y, \rho v_z)$ to a new four-dimensional current vector J^i, $J^i = (\rho, \rho v_x, \rho v_y, \rho v_z)$, i.e. we add a fourth "time-component" ρ in going from the three-vector J to the four-vector J^i. In doing so, we are able to express the conservation Eq. (B.1) as

$$\frac{\partial J^i}{\partial x^i} = 0. \tag{B.4}$$

This is a very useful and important result: The vanishing of the divergence of a four-vector (or tensor) expresses conservation.

In a similar manner, energy and momentum conservation in Special Relativity is expressed by the vanishing of the divergence of the "energy-momentum tensor" T^{ik}:

$$\frac{\partial}{\partial x^i} T^{ik} = 0. \tag{B.5}$$

Here, instead of dealing with a vector, we have to use a tensor (of second rank-i.e. two indices i and k instead of the single index that represents a vector.) Actually a vector is also a tensor; it is a tensor of first rank. While the vector J^i has four components (i taking on the values $0, 1, 2, 3$), a second rank tensor has 16 components in general but because this tensor is symmetric, i.e. $T^{01} = T^{10}$, $T^{12} = T^{21}$..., etc., only 10 components of this energy-momentum tensor are distinct in general.

Now let us consider working in Special Relativity using non-Cartesian coordinates, for example polar coordinates, or even transforming to a reference frame that is under acceleration. It is frequently stated that the latter case is no longer one of Special Relativity, that one is dealing with General Relativity whenever acceleration enters the picture. This is quite untrue. One can apply

Special Relativity in an accelerated reference frame. In fact Bondi [2] and others have done so. General Relativity enters the discussion when *real* gravity is present, namely spacetime curvature generated by matter and/or fields, not the pseudo-gravity of accelerated reference frames.

So let us suppose that we are working within the domain of Special Relativity but not in an inertial Cartesian system of coordinates. The formalism of tensor calculus tells us how the usual partial derivative of a vector or tensor transforms (see, e.g. [1, 7]). The usual partial derivative of a vector J^i for example, changes to what is called the "covariant derivative", denoted by a semi-colon. Specifically,

$$J^i_{;k} = \frac{\partial J^i}{\partial x^k} + \Gamma^i_{lk} J^l \tag{B.6}$$

where

$$\Gamma^i_{lk} = \frac{1}{2} g^{im}(g_{ml,k} + g_{mk,l} - g_{lk,m}) \tag{B.7}$$

and we have used commas to indicate the usual partial derivatives. Thus we see that the metric tensor comes into play when we deviate from simple Cartesian coordinates.

We now have the means of expressing the conservation of charge and the conservation of energy-momentum in terms of arbitrary coordinate systems in Special Relativity. Specifically, charge conservation changes from (B.4) to

$$J^i_{;i} = 0 \tag{B.8}$$

and the expression for energy-momentum conservation changes from (B.5) to

$$T^{ik}_{;k} = 0. \tag{B.9}$$

Now locally, an accelerated reference frame is like a gravitational field (Equivalence Principle) so we are led to the expression for energy-momentum conservation in General Relativity: simply (B.9). There is really no other possibility!

We are almost there. Recall that the mass density ρ is the source of gravity in Newtonian physics and most often, when we wish to go to the limit of weak gravity and non-relativistic velocities, we should retrieve Newtonian gravity from the relativistic theory of gravity that we seek. Interestingly, the T^{00} component of the energy-momentum tensor T^{ik} is ρ in the reference frame in which the matter is at rest so we seek a relativistic gravity field equation of the form

$$G^{ik} = \text{constant}.T^{ik}. \tag{B.10}$$

Thanks to our earlier discussion regarding the conservation laws, we know what to look for in this so-far undetermined tensor G^{ik}. We know that since T^{ik} satisfies (B.9), so too must the right hand side of (B.10), i.e.

$$G^{ik}_{;k} = 0. \tag{B.11}$$

As well, G^{ik} must be constructed from g_{ik} and no higher than its second derivatives so that in the appropriate limit, the left hand side of (B.10) reduces to the "Laplacian" of the Newtonian potential ϕ, i.e. $\nabla^2\phi$ where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ when we use Cartesian coordinates.

Einstein worked diligently in search of the tensor G^{ik} and in the early stages of his search, he believed that the correct form of G^{ik} is the Ricci tensor R^{ik}. This tensor is a very complicated and lengthy combination of Christoffel symbols and their derivatives:

$$R_{ik} = \Gamma^n_{ik,n} - \Gamma^l_{il,k} + \Gamma^l_{ik}\Gamma^n_{ln} - \Gamma^n_{il}\Gamma^l_{kn}. \tag{B.12}$$

Unfortunately, this tensor does not have a vanishing covariant divergence as it must to be satisfactory. Eventually Einstein discovered that the required tensor is

$$G^{ik} = R^{ik} - \frac{1}{2}g^{ik}R^j_j, \tag{B.13}$$

now appropriately named the "Einstein tensor". Thus we have developed the Einstein field equations of General Relativity,

$$R^{ik} - \frac{1}{2}g^{ik}R^j_j = \frac{8\pi G}{c^4}T^{ik} \tag{B.14}$$

where the constant $\frac{8\pi G}{c^4}$ has been chosen in order to reduce to the Newtonian field equation for gravity

$$\nabla^2\phi = 4\pi G\rho \tag{B.15}$$

in the limit when appropriate. Rather than one simple field Eq. (B.15), we have ten very complicated non-linear partial differential equations. There are in general, ten distinct equations, counting the permutations of $i, k = 0, 1, 2, 3$ and taking into account the symmetry, $T^{10} = T^{01},\ T^{12} = T^{21}$, etc. Given the complexity, while the task of finding valuable solutions might appear hopeless, with sufficient symmetry, physicists have found some very valuable solutions. These are discussed in the text. More solutions can be found in the advanced texts.

Regarding motion in General Relativity, the Equivalence Principle again serves as a valuable guide. Recall from Newtonian physics, that when there are no forces or when the forces on a body balance out, the body moves with zero acceleration. This holds in Relativity as it does with Newton:

$$a^i = 0 \tag{B.16}$$

for zero force. Here, we have written the acceleration as having four components instead of three. This is reasonable as after all, the arena of investigation in Relativity is four-dimensional spacetime rather than three-dimensional space. You might wish to delve more deeply into the mathematics of Relativity in the suggested references where four-dimensional vectors and tensors are discussed.

Now suppose we were to examine the system from the viewpoint of an accelerated reference frame. Then the mathematical transformation of this vector a^i turns it into a different form, a^i_{intrin}, called the "intrinsic acceleration", in terms of the new coordinates, and it is equal to zero as before:

$$a^i_{\text{intrin}} = a^i + \Gamma^i_{kl} u^k u^l = 0. \quad \text{(B.17)}$$

Here, the Γ is the Christoffel symbol as discussed above and u^k is the four-vector velocity, appropriate to Relativity, just as a^i is the previous four-vector acceleration. From the Equivalence Principle, gravity is locally like being in an accelerated reference system. Therefore it is natural to take (B.17) to be the equation of motion of a free body in full-blown General Relativity, i.e. when the body is moving in curved spacetime, not only when it is moving under pseudo-gravity. But you might well object to the effect that when there is real gravity present, like the pull of the earth, it is no longer a force-free situation. This is because you are so conditioned to thinking of gravity as a force. General Relativity tells you that you must think differently about gravity. Gravity is now removed from the collection that we call "forces" and is identified with spacetime, that is, its attribute that we call "curvature". And when you ask: "How does a body move under this new expression for gravity when all the forces are absent or balanced out?", the answer is: "freely", according to (B.17), the "geodesic equation".

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