Method of Partial Waves

Assume a central potential $V(r)$.

$\vec{L}$ is a constant of the motion.

There are eigenstates common to H, L^2, L_z.

the wave functions of these states are partial waves.

$H_0 = \frac{p^2}{2\mu}$ e.g. for a free particle.

a complete set of commuting observables would be

H_0, P_x, P_y, P_z

this is like being in a central potential w/ $V = 0$.

H, L^2, L_z form a complete set of commuting

observables also.

These are 2 separate bases: $[\vec{P}, \vec{L}] \neq 0$

For example spin-0 particle: $\hat{p}|p\rangle = p|p\rangle$

$H_0|p\rangle = \frac{p^2}{2\mu}|p\rangle$

$|p| = \sqrt{2\mu E}$ an infinite # of kets satisfy this

each energy E is infinitely degenerate.

$\langle r|p\rangle = \left(\frac{1}{2\pi\hbar}\right)^{3/2} e^{i\vec{p}\cdot\vec{r}/\hbar}$ w.f. of momentum

ket. (plane waves)

$$H_0 |k\rangle = \frac{\hbar^2 k^2}{2\mu} |k\rangle \qquad p|k\rangle = \hbar k |k\rangle$$

$$\langle k|k'\rangle = \delta(k-k')$$

$$\int d^3k \; |k\rangle\langle k| = \mathbb{1}$$

$$\langle r|k\rangle = \left(\frac{1}{2\pi}\right)^{3/2} e^{i\vec{k}\cdot\vec{r}}$$

The stationary states with well-defined angular momentum

$$\varphi^{(0)}_{k,\ell,m} = \sqrt{\frac{2k^2}{\pi}}\, j_\ell(kr)\, Y_\ell^m(\theta,\varphi)$$

$j_\ell(\rho)$ is a spherical Bessel function

$$j_\ell(\rho) = (-1)^\ell \rho^\ell \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^\ell \left(\frac{\sin\rho}{\rho}\right)$$

The eigenvalues of

H_0, L^2, and L_z are $\frac{\hbar^2 k^2}{2\mu}$, $\ell(\ell+1)\hbar^2$, $m\hbar$

these <u>free spherical waves</u> are orthonormal in the sense

$$\langle \varphi^{(0)}_{k,\ell,m} | \varphi^{(0)}_{k',\ell',m'} \rangle = \frac{2}{\pi} k k' \int_0^\infty j_\ell(kr)\, j_{\ell'}(k'r)\, r^2 dr \int d\Omega\, Y_\ell^{m*} Y_{\ell'}^{m'}$$

$$= \delta(k-k')\,\delta_{\ell\ell'}\,\delta_{mm'}$$

$$\int_0^\infty dk \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} |\varphi^{(0)}_{k,\ell,m}\rangle\langle \varphi^{(0)}_{k,\ell,m}| = 1$$