$\boxed{\text{Angular Momentum}}$: Undergrad

"Typical" undergrad approach: $\vec{L} = \vec{r} \times \vec{p}$

$\vec{p} \leftarrow -i\hbar\vec{\nabla}$ in pos. rep.

- $\vec{r}, \vec{p}$ Hermitian $\Longrightarrow$ $\vec{L}$ Hermitian
- $[x, p_x] = i\hbar \Longrightarrow [L_x, L_y] = i\hbar L_z$, cyc. perm.

  $[L_i, L_j] = i\hbar\epsilon_{ijk} L_k$
- $[\vec{L}^2, L_z] = 0 \Longrightarrow$ simultaneous eigenkets $|l\, m\rangle$

  $\vec{L}^2 |l\, m\rangle = \hbar^2 l(l+1) |l\, m\rangle$

  $L_z |l\, m\rangle = \hbar m |l\, m\rangle$

etc.

- In position rep.

$$\vec{L} = \vec{r} \times \vec{p}$$

$$= -i\hbar\left[\hat{\phi}\frac{\partial}{\partial\theta} - \hat{\theta}\,\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right] \leftarrow \text{spherical coord.}$$

$$\Downarrow$$

$$L_z = -i\hbar\frac{\partial}{\partial\phi}$$

$$\vec{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

- Make and solve differential equations for ang. mom. eigenfunctions $Y_{lm}(\theta, \phi) \leftarrow$ pos. rep.

$$L_z Y_{lm} = \hbar m Y_{lm} \longrightarrow -i\hbar\frac{\partial}{\partial\phi} Y_{lm} = \hbar m Y_{lm}$$

$$\Rightarrow Y_{lm} = e^{im\phi} P_{lm}(\theta)$$

$$\vec{L}^2 Y_{\ell m} = \hbar^2 \ell(\ell+1) Y_{\ell m}$$

$$\longrightarrow \underbrace{0 = \sin\theta \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta} P_{\ell m}\right) + [\ell(\ell+1)\sin^2\theta - m^2] P_{\ell m}}_{\text{well-studied equation}}$$

<u>Solution outline</u> $P_{\ell m} = \sin^{|m|}\theta \sum_k c_k \cos^k\theta$

series diverges at $\theta = 0$ and $\theta = \pi$
unless truncated

$\Longrightarrow$ gives conditions on ℓ, m

$$\boxed{\begin{array}{l} \ell = 0, 1, 2, \ldots \leftarrow \text{integer only} \\ m = -\ell, -\ell+1, \ldots, \ell-1, \ell \end{array}}$$