Astro 411 Problem Set 7 (Goldstein Chapters 8)

Due at 10 am, Wednesday, November 27th, 2019

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1. [50 pts] Heavy symmetrical top (Exercise 29, p. 367 third edition or Exercise 22, p. 376 second edition): Obtain the Hamiltonian of a heavy symmetrical top with one point fixed, and from it the Hamilton's equations of motion. Relate these to the equations of motion discussed in Section 5.7 and, in particular, show how the solution may be reduced to quadratures. ~~Also use the Routhian procedure to eliminate the cyclic coordinates.~~ ← no need to do the crossed-out part.

2. (a) [5 pts] Show that the Lagrangian for a particle in a rotating frame is

$$L = \frac{m}{2}\boldsymbol{v}^2 + m\boldsymbol{v}\cdot(\boldsymbol{\Omega}\times\boldsymbol{r}) + \frac{m}{2}(\boldsymbol{\Omega}\times\boldsymbol{r})^2 - V(r), \tag{1}$$

   where $\boldsymbol{v}$ is the velocity in the rotating frame, and $\boldsymbol{\Omega}$ is the frame's angular frequency.

   (b) [20 pts] Determine the conjugate momentum, Hamiltonian, and Hamilton's equations of motion. Describe the physical interpretation of the conjugate momentum in this case. Use your Hamilton equations of motion to determine the form of the Coriolis and centrifugal forces.

   (c) [25 pts] Consider a moon on a circular orbit around Saturn, with an orbital angular frequency of Ω. A "ring particle" (i.e. one of the balls of ice that compose Saturn's rings) flies by the moon. Use your Hamiltonian from part (b) to argue that the change in the ring particle's energy from before it encounters the moon to after (δE) is related to the change in its vertical angular momentum (δl) via the relation

$$\delta E = c\delta l, \tag{2}$$

   for constant c. What is c?

   Note 1: Before and after the ring particle flies by the moon, you can neglect the moon's gravity, i.e., consider the ring particle to be only affected by Saturn's gravity.

   Note 2: Energy (E) and angular momenta (l) are measured in the "space frame" (i.e. not in the rotating frame), and vertical angular momentum means the component of angular momentum perpendicular to the moon's (and ring's) orbital plane. You should make use of the conservation of energy in the rotating frame. This is sometimes called Jacobi's constant.

   Note 3: For a bonus 10 points, use equation (2) to relate the change in the ring particle's a to its change in e, and use that to show that if the ring particle is initially on a circular orbit, then after the encounter it will always be pushed away from the moon. This effect is called orbital repulsion. In the 1970's it was known that there are gaps in Saturn's rings that should have been filled in by diffusion. The puzzle was solved by Goldreich & Tremaine, who predicted that there were unobserved moons in the middle of the gaps that repelled the rings. These moons were later discovered by the Voyager spacecraft when it visited Saturn.