Central Limit Therom

Back to die, we said P(⚄) = 1/6. What exactly does this mean?

→ If we perform N trials, ⚄ will appear approximately N/6 times

what does this mean?

Consider $S \equiv \sum_{i=1}^{N} s_i$, $s_i = \begin{cases} 1, & \text{if ⚄} \\ 0, & \text{otherwise} \end{cases}$

$$\frac{S}{N} \to \frac{1}{6} \quad \text{for large } N$$

How does it approach limit?

→ measure it! (numerically) throw N die, repeat 10,000 times to generate statistics

(show plot)

Thus, just as we showed last time, these distributions look more and more like a gaussian as $N \to \infty$

central limit theorem states for $N \to \infty$

$$p(s) = \frac{1}{\sqrt{2\pi\sigma_s^2}} e^{-(s-\bar{s})^2/2\sigma_s^2}$$

This is a <u>very</u> general statement:

probability distribution of the value of the sum of random variables $\to$ gaussian as $N \to \infty$

$\Rightarrow$ this describes the result for any sum of random variables <u>no matter what</u> the underlying probability distribution is

$\Rightarrow$ macroscopic bodies have well defined macroscopic properties even though constituent parts (microscopics) are changing rapidly

<u>ex</u>: pressure of gas

- particle positions | velocities changing much faster than measurement time

- during measurement, many collisions with wall

  pressure = sum of pressure individual particles

  has well-defined average

  $$\sigma_p \sim \frac{1}{\sqrt{N}}$$

$\Rightarrow$ pressure is well defined even though its origin is a randomly flucuating quantity

$\Rightarrow$ this is why thermodynamics is possible!

Statistical Mechanics

All stat mech problems are solved the same way:

1. Specify macrostate & accessible microstates
2. Choose the ensemble (macroscopic constraints, consistent microstates, relative prob. of microstates)
3. Calculate mean values & statistical properties (S, E, etc)

Ex: isolated system of $N=5$ spins in magnetic field B

dipoles, only point up/down

Magnetic moment μ

spin $\frac{1}{2}$

$B \uparrow$ $\quad \uparrow$ $\quad \downarrow$

$E=-\mu B$ $\quad E=+\mu B$

total energy of system: $E=-\mu B$

What is the magnetic moment of spin 1? (mean value)

① macrostate: $\begin{cases} E = -\mu B \\ N = 5 \end{cases}$

microstates: $N=5 \Rightarrow 2^5 = 32$ microstates $\uparrow\uparrow\uparrow\downarrow\uparrow$

not all allowed (accessible) as $\underline{E=-\mu B}$

ex: $\uparrow\uparrow\uparrow\downarrow\uparrow$ $\quad E=-3\mu B$

not allowed!

accessible microstates (3 up, 2 down) : ↑↑↑↓↓ ↑↑↓↑↓ ↑↓↑↑↓ ↑↑↓↓↑ ↑↓↑↓↑
↑↓↓↑↑ ↓↑↑↓↑ ↓↑↑↑↓ ↓↑↓↑↑ ↓↓↑↑↑

(2) Choose the ensemble

→ isolated system, each microstate consistent w/ $E = -\mu B$

→ each microstate equally likely

"equal a priori probabilities"

$$P_s = \frac{1}{\Omega}$$

probability system in microstate s

number of microstates

Here, $p = \frac{1}{10}$ for each microstate

(3) Calculate mean values

→ magnetic moment spin #1

mean value of spin #1 $\bar{S}_1 = \sum_{i=1}^{10} S_{1,i} P_i = \frac{1}{10}\left((+1)\,6 + (-1)\,4\right) = \frac{2}{10} = \frac{1}{5}$

magnetic moment spin #1 $\bar{M}_{S_1} = \mu \bar{S}_1 = \frac{\mu}{5}$

another way: $\overline{M} = \mu \Rightarrow$ three spins are up, two spins are down

all spins are the same, doesn't matter which one we calculate probability of being up

$$p\left(\begin{smallmatrix}\text{spin}\\ \text{up}\end{smallmatrix}\right) = \frac{3}{5}, \quad p\left(\begin{smallmatrix}\text{spin}\\ \text{down}\end{smallmatrix}\right) = \frac{2}{5}$$

$$\overline{s} = (1)\left(\tfrac{3}{5}\right) + (-1)\left(\tfrac{2}{5}\right) = \frac{1}{5} \quad \text{(all spins)}$$

Counting microstates

→ this is in some sense the hardest part, so we will practice with several examples

N noninteracting spins (general)

N nonintracting spins, spin ½, moment μ, in field B fixed on lattice = distinguishable

⇒ microstate = orientation of each spin

⇒ want $\Omega(E, B, N)$ ← microstates for specified E, B, N

Define $\begin{cases} n \equiv \#\text{ spins parallel to } B \text{ (up)} \\ N-n = \#\text{ spins antiparallel to } B \text{ (down)} \end{cases}$

$$E = n(-\mu B) + (N-n)(\mu B) = (N-2n)\mu B$$

$\Rightarrow$ given N, B, n specifies energy

$$\boxed{n = \frac{N}{2} - \frac{E}{2\mu B}}$$

$\Rightarrow$ # microstates with energy E is given by way n of N spins can be up

$$\boxed{\Omega(n, N) = \frac{N!}{n!(N-n)!}}$$

Harmonic oscillator

$$E = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

Classical: microstate specified by (x, p)

point in phase space

x, p continuous variables $\Rightarrow$ compute $\underbrace{g(E)}_{\#\text{ microstates between } E,\, E+\Delta E}\Delta E$

density of states