6. a. Laplace equation: $\nabla^2 \phi(\vec{r}) = -4\pi \rho(\vec{r})$

$$\nabla^2 G(\vec{r}, \vec{r}\,') = -4\pi \delta(\vec{r} - \vec{r}\,')$$

$$G(\vec{r}, \vec{r}\,') = 0 \text{ on boundaries}$$

Consider the proposed solution

$$\phi(\vec{r}) = \int_{V'} dV' \, G(\vec{r}, \vec{r}\,') \rho(\vec{r}\,')$$

For this solution:

$$\nabla^2 \phi(\vec{r}) = \int_{V'} dV' \left[\nabla^2 G(\vec{r}, \vec{r}\,')\right] \rho(\vec{r}\,')$$

$$= \int_{V'} dV' \left[-4\pi \delta(\vec{r} - \vec{r}\,')\right] \rho(\vec{r}\,')$$

$$= -4\pi \rho(\vec{r}) \checkmark$$

This solution solves the equations and matches the boundary conditions.

6. b. $\displaystyle\int_{-\infty}^{\infty} \delta(\vec{r}-\vec{r}\,')\, d^3r' = 1$

$$\int_{-\infty}^{\infty} f(\vec{r}\,')\,\delta(\vec{r}-\vec{r}\,')\,d^3r' = f(\vec{r})$$

spherical coords:

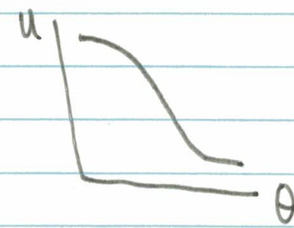
$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} f(r',\theta',\phi')\left(\overset{\delta(r-r')}{\frac{1}{r^2}}\,\delta(\cos\theta-\cos\theta')\,\delta(\phi-\phi')\right) r'^2 \sin\theta'\,dr'\,d\theta'\,d\phi'$$

$=$ (noting for $u=\cos\theta$, $u'=\cos\theta'$, $du' = -\sin\theta'\,d\theta'$, bounds $\begin{matrix}0\to 1\\ \pi\to -1\end{matrix}$, switch order to get another $-$ sign

$$= \int_0^{2\pi}\int_{-1}^{1}\int_0^{\infty} f(r',u',\phi')\,\delta(u-u')\,\delta(r-r')\,\delta(\phi-\phi')\,dr'\,du'\,d\phi'$$

$= f(r,u,\phi) = f(r,\theta,\phi)$ since over $[0,\pi]$,

there is a bijection between u and θ



6. c. For $r \neq r'$, RHS $= 0$.

For a solution $G_\ell(r,r') = r^\ell$

$$\text{LHS} = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(\ell r^{\ell+1}\right) - \ell(\ell+1)r^\ell\right)$$

$$= \frac{1}{r^2}\left(\ell(\ell+1)r^\ell - \ell(\ell+1)r^\ell\right) = 0$$

For the case of $r^{-(\ell+1)}$:

$$\text{LHS} = \frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(-(\ell+1)r^{-\ell}\right) - \ell(\ell+1)r^{-(\ell+1)}\right)$$

$$= \frac{1}{r^2}\left(\ell(\ell+1)r^{-(\ell+1)} - \ell(\ell+1)r^{-(\ell+1)}\right) = 0$$

We can take any linear combination of these solutions since our diff eqn is linear.

d. On the boundary of the sphere, since different $P_\ell(\cos\theta)$ have independent angular dependences, we need each G_ℓ to vanish. The same logic applies to the case of $r \to \infty$.

$\underline{r<r'}$: $G_\ell = a_\ell r^\ell + b_\ell r^{-(\ell+1)}$

$$0 = a_\ell R^\ell + b_\ell R^{-(\ell+1)} \Rightarrow b_\ell = -a_\ell R^{2\ell+1}$$

We can express such a solution as

$A u_1(r)$, for $u_1(r) = \left(\frac{r}{R}\right)^\ell - \left(\frac{r}{R}\right)^{-(\ell+1)}$,

A a coefficient

$\underline{r>r'}$

$0 = a_\ell r^\ell + b_\ell r^{-(\ell+1)}$ for $r \to \infty$ to meet the boundary conditions. $\Rightarrow a_\ell = 0$

Solution: $Bu_2(r)$, for $u_2(r) \equiv \left(\frac{r}{R}\right)^{-(\ell+1)}$

we have chosen to include the $\frac{1}{R}$ for convenience, will be accounted for in B

e. Second derivative of G_ℓ has a delta function at $r=r'$, so first derivative will have a step function $\Rightarrow$ it will be discontinuous. G_ℓ will be continuous.

1) G_ℓ continuous at $r=r'$:

$$A u_1(r') = B u_2(r')$$

2) $\frac{1}{r^2}\left[\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \ell(\ell+1)\right] G_\ell(r,r') = -\frac{4\pi}{r^2}\delta(r-r')$

Integrate from $r'-\epsilon$ to $r'+\epsilon$

r^2 because we are in spherical coordinates

$$\int_{r'-\epsilon}^{r'+\epsilon} r^2 dr \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} G_\ell(r,r')\right) = \int_{r'-\epsilon}^{r'+\epsilon} \frac{\partial}{\partial r}\left[r^2 \frac{\partial G_\ell(r,r')}{\partial r}\right] dr$$

$$= r^2 \frac{\partial G_\ell(r,r')}{\partial r}\Bigg|_{r'-\epsilon}^{r'+\epsilon} \underset{\epsilon\to 0}{=} r'^2\Big[\frac{\partial}{\partial r}\underbrace{G_\ell^>(r,r')}_{Bu_2(r)} - \frac{\partial}{\partial r}\underbrace{G_\ell^<(r,r')}_{Au_1(r)}\Big]_{r=r'}$$

$$= r'^2\left(Bu_2'(r') - Au_1'(r')\right)$$

2nd term: $\int_{r'-\epsilon}^{r'+\epsilon} \ell(\ell+1)\frac{1}{r^2} G_\ell(r,r') r^2 dr = \ell(\ell+1)\left[G_\ell^>(r'+\epsilon,r') - G_\ell^<(r'-\epsilon,r')\right]$

$\to 0$ as $\epsilon \to 0$

since G_ℓ is continuous.

RHS $\quad -4\pi\int_{r'-\epsilon}^{r'+\epsilon} r^2 dr \frac{1}{r^2}\delta(r-r') = -4\pi$

Conditions $\quad Au_1(r') - Bu_2(r') = 0$

$$Au_1'(r') - Bu_2'(r') = \frac{4\pi}{r'^2}$$

Two equations for A and B. In matrix form:

$$\begin{pmatrix} u_1(r') & -u_2(r') \\ u_1'(r') & -u_2'(r') \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{4\pi}{(r')^2} \end{pmatrix}$$

$$M\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{4\pi}{r'^2} \end{pmatrix}$$

Cramer's rule if $Mx = B$

$$M = \begin{pmatrix} m_1 & m_3 \\ m_2 & m_4 \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$x_1 = \frac{1}{\det M} \det M_1, \quad M_1 = \begin{pmatrix} b_1 & m_3 \\ b_2 & m_4 \end{pmatrix}$$

$$x_2 = \frac{1}{\det M} \det M_2, \quad M_2 = \begin{pmatrix} m_1 & b_1 \\ m_2 & b_2 \end{pmatrix}$$

$$\Rightarrow A = \frac{1}{\det M}\left(\frac{4\pi}{r'^2} u_2(r') \right)$$

$$B = \frac{1}{\det M}\left(\frac{4\pi}{r'^2} u_1(r') \right)$$

$$\det M = u_1' u_2 - u_1 u_2' \Big|_{r'}$$

$$= \left(\frac{r'}{R}\right)^{-(\ell+1)} \left[\ell \frac{1}{r'} \left(\frac{r'}{R}\right)^{\ell} + (\ell+1) \frac{1}{r'} \left(\frac{r'}{R}\right)^{-(\ell+1)} \right]$$

$$- \left[\left(\frac{r'}{R}\right)^{\ell} - \left(\frac{r'}{R}\right)^{-(\ell+1)} \right] \times$$

$$\times \left[-(\ell+1) \frac{1}{r'} \left(\frac{r'}{R}\right)^{-(\ell+1)} \right]$$

derivative pulls down $\frac{\text{power}}{\text{variable}}$

$$\det M = -\frac{1}{r'}\left(\frac{r'}{R}\right)^{-(l+1)}\left[-(l+1)\left(\frac{r'}{R}\right)^{l}+(l+1)\left(\frac{r'}{R}\right)^{-(l+1)} - l\left(\frac{r'}{R}\right)^{l}-(l+1)\left(\frac{r'}{R}\right)^{-(l+1)}\right]$$

$$= \frac{1}{r'}\left(\frac{r'}{R}\right)^{-l-1}\left(\frac{r'}{R}\right)^{l}(2l+1) = \frac{(2l+1)R}{r'^2}$$

$$\Rightarrow A = \frac{4\pi u_2(r')}{(2l+1)R}, \quad B = \frac{4\pi u_1(r')}{(2l+1)R}$$

$$\Rightarrow G_l^{<}(r,r') = 4\pi\frac{u_2(r')u_1(r)}{(2l+1)R} \qquad r<r'$$

$$G_l^{>}(r,r') = 4\pi\frac{u_1(r')u_2(r)}{(2l+1)R} \qquad r>r'$$

Where we define $r_> =$ larger of (r,r'), $r_< =$ smaller of (r,r')

$$G_l(r,r') = \frac{4\pi}{(2l+1)R}u_1(r_<)u_2(r_>)$$

$$= \frac{4\pi}{(2l+1)R}\underbrace{\left[\left(\frac{r_<}{R}\right)^{l}-\left(\frac{r_<}{R}\right)^{-l-1}\right]\left(\frac{r_>}{R}\right)^{-l-1}}_{\frac{R r_<^{l}}{r_>^{l+1}} - \left(\frac{R^2}{rr'}\right)^{l+1}}$$

note that $r_< r_> = rr'$
regardless of whether r or r' is larger

$$G_\ell(r,r') = \frac{4\pi}{2\ell+1}\left[\frac{r_<^\ell}{r_>^{\ell+1}} - \frac{1}{R}\left(\frac{R^2}{rr'}\right)^{\ell+1}\right] \to \text{can plug back into full sum}$$

f. If $R=0$, the second term vanishes. The Green's function for a unit charge at $\vec{r}\,'$ is $\frac{1}{|\vec{r}-\vec{r}\,'|}$, so the stated equality holds.