① Find the eigenvalues by solving the characteristic equation:

$$C\vec{v} = \lambda\vec{v} \Rightarrow (C - \lambda I)\vec{v} = \vec{0} \qquad (\star)$$
$$\det(C - \lambda I) = 0$$

$$C - \lambda I = \begin{pmatrix} -\Lambda & 1 & 0 \\ 1 & -\Lambda & 1 \\ 0 & 1 & -\Lambda \end{pmatrix} \frac{1}{\sqrt{2}} \qquad \text{where} \quad \Lambda/\sqrt{2} = \lambda$$

$$\det(C - \lambda I) = -\Lambda(\Lambda^2 - 1) + 1(0 + \Lambda) + 0 = 0$$
$$= -\Lambda^3 + 2\Lambda = 0 \qquad \Rightarrow \Lambda = 0 \text{ or } \Lambda^2 = 2$$

so $\Lambda = 0, \pm\sqrt{2}$ so $\boxed{\lambda = 0, \pm 1}$

substitute back into $(\star)$ to find the eigenvectors $\vec{v}$:

i. $\lambda = \pm 1$ solution, write $\vec{v}_{\pm 1} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \mp\sqrt{2} & 1 & 0 \\ 1 & \mp\sqrt{2} & 1 \\ 0 & 1 & \mp\sqrt{2} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \pm\sqrt{2}\,a = b \qquad \pm\sqrt{2}\,c = b \qquad \boxed{\vec{v}_{\pm 1} = \frac{1}{2}\begin{pmatrix} 1 \\ \pm\sqrt{2} \\ 1 \end{pmatrix}}$$
$$a = c = \pm b/\sqrt{2}$$

ii. $\lambda = 0$ solution:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{aligned} b &= 0 \\ a + c &= 0 \end{aligned} \qquad \boxed{\vec{v}_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}$$

eigenvalues are $-1, 0, +1$ (quantized in integer steps and run from $-j$ to $+j$, where $j = 1$) so conclude this is an angular momentum operator for a spin-one system (specifically it's J_x in the basis of J_z eigenstates) (but that is not something you're expected to guess!)

② $A = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}$ $B = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}$

a. to see if B is degenerate, check to see if it has any repeated eigenvalues:

$$\det(B - \lambda I) = 0$$

$$\begin{vmatrix} b-\lambda & 0 & 0 \\ 0 & -\lambda & -ib \\ 0 & ib & -\lambda \end{vmatrix} = (b-\lambda)(\lambda^2 - b^2) = 0$$

$b - \lambda = 0$ or $\lambda^2 - b^2 = 0$

$b = \lambda$ or $(\lambda - b)(\lambda + b) = 0$

$\lambda = b$ or $\lambda = -b$

repeated!

$\boxed{\text{yes, degenerate}}$

b. check $[A,B] = AB - BA = 0$

$$[A,B] = ab\left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right\}$$

$$[A,B] = ab\left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}\right\} = 0$$

$\boxed{[A,B] = 0}$

c. so, naively one might look at A and just write down its eigenstates by inspection as: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
which is a perfectly correct set of eigenstates! but $\lambda = +a$ $\lambda = -a$ $\lambda = -a$

they are clearly not eigenstates of B, even though A & B commute. this subtlety is due to the degeneracy of the system — since degeneracy means that neither A nor B have unique eigenstates, there are some e'states of A that are not e'states of B, and vice versa. (you may recall that degeneracy is the exceptional case we must pay attention to when arguing that commuting observables share eigenstates,

$$A|\vec{v}_a\rangle = \alpha|\vec{v}_a\rangle$$

$$[A,B] = AB - BA = 0$$

$$BA|\vec{v}_a\rangle = AB|\vec{v}_a\rangle$$

$$A[B|\vec{v}_a\rangle] = \alpha[B|\vec{v}_a\rangle]$$

$\Rightarrow$ either $B|\vec{v}_a\rangle \sim |\vec{v}_a\rangle$, shared eigenstates

or $B|\vec{v}_a\rangle$ is not proportional to $|\vec{v}_a\rangle$, but is a distinct eigenstate of A with the same (degenerate!) eigenvalue α).

let's try to find a simultaneous eigenbasis for these two observables by writing down a general set of eigenvectors for A:

$$\lambda = \pm a$$

for $\lambda = +a$,

$$(A-\lambda I)|\vec{v}_a\rangle = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2a & 0 \\ 0 & 0 & -2a \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad y = z = 0$$

$$\vec{v}_{+a} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

check compatibility with B:

$$B|\vec{v}_a\rangle = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = b\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = b|\vec{v}_a\rangle$$

already an e'state of B!

so $\boxed{\vec{v}_{+a,+b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$

for $\lambda = -a$,

$$(A-\lambda I)|\vec{v}_{-a}\rangle = \begin{pmatrix} 2a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad x = 0$$

so generally the $\lambda = -a$ eigenvalue is described by a two-dimensional degenerate subspace,

$$|\vec{v}_{-a}\rangle = \begin{pmatrix} 0 \\ y \\ z \end{pmatrix} \quad \text{valid } \forall\, y,z$$

let's fix y & z by demanding compatibility with B.
since we already found $\begin{pmatrix}1\\0\\0\end{pmatrix}$ corresponds to:

$$B\begin{pmatrix}1\\0\\0\end{pmatrix} = b\begin{pmatrix}1\\0\\0\end{pmatrix}$$

we have covered one of the two $\lambda = b$ eigenvalues, and must find an eigenvector for the other $\lambda = b$ case & for $\lambda = -b$

i. $\lambda = b$

$$(B - bI)\begin{pmatrix}0\\x\\y\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}0 & 0 & 0\\0 & -b & -ib\\0 & ib & -b\end{pmatrix}\begin{pmatrix}0\\x\\y\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \quad x = -iy, \quad ix = y$$

$x = 1/\sqrt{2}$, $y = i/\sqrt{2}$ for normalization

so $\boxed{|\vec{V}_{-a,+b}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\i\end{pmatrix}}$

ii. $\lambda = -b$

$$(B + bI)\begin{pmatrix}0\\x\\y\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}2b & 0 & 0\\0 & b & -ib\\0 & ib & b\end{pmatrix}\begin{pmatrix}0\\x\\y\end{pmatrix} = \begin{pmatrix}0\\b(x-iy)\\b(ix+y)\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \quad x = iy$$

$$\boxed{|\vec{V}_{-a,-b}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-i\end{pmatrix}}$$

(3) consider the ket: $|v\rangle = |\alpha\rangle + \lambda|\beta\rangle$

by the Hilbert space axioms, $\langle v|v\rangle \geq 0$

$$\langle v|v\rangle = \left[\langle\alpha| + \lambda^*\langle\beta|\right]\left[|\alpha\rangle + \lambda|\beta\rangle\right]$$
$$\langle v|v\rangle = \langle\alpha|\alpha\rangle + \lambda\langle\alpha|\beta\rangle + \lambda^*\langle\beta|\alpha\rangle + \lambda^*\lambda\langle\beta|\beta\rangle$$
$$\langle v|v\rangle = |\alpha|^2 + \lambda\langle\alpha|\beta\rangle + \lambda^*\langle\alpha|\beta\rangle^* + \lambda^*\lambda|\beta|^2$$

from $\langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle^*$

so:
$$|\alpha|^2 + \lambda\langle\alpha|\beta\rangle + \lambda^*\langle\alpha|\beta\rangle^* + \lambda\lambda^*|\beta|^2 \geq 0 \qquad (\star)$$

to choose λ, exploit the fact that the ~~RHS~~ LHS is always greater than or equal to zero & extremize it with respect to λ, meaning:

$$f(\lambda,\lambda^*) \equiv |\alpha|^2 + \lambda\langle\alpha|\beta\rangle + \lambda^*\langle\alpha|\beta\rangle^* + \lambda\lambda^*|\beta|^2$$
$$f(\lambda,\lambda^*) \geq 0$$

$$\frac{df}{d\lambda} = \langle\alpha|\beta\rangle + \lambda^*|\beta|^2 = 0$$

$\frac{d^2f}{d\lambda^* d\lambda} = |\beta|^2 > 0$ so minimum at:

$$\lambda^* = -\frac{1}{|\beta|^2}\langle\alpha|\beta\rangle \quad \text{so} \quad \lambda = -\frac{1}{|\beta|^2}\langle\alpha|\beta\rangle^*$$

is the value of λ that minimizes $f(\lambda,\lambda^*)$, getting it as close as possible to the lower bound of zero (can think of this as a method for getting the most information out of an inequality as possible, like in variational methods)

sub λ into $(\star)$

$$|\alpha|^2 - \frac{1}{|\beta|^2}|\langle\alpha|\beta\rangle|^2 - \frac{1}{|\beta|^2}|\langle\alpha|\beta\rangle|^2 + \left(\frac{1}{|\beta|^2}\right)^2|\langle\alpha|\beta\rangle|^2|\beta|^2 \geq 0$$
$$|\alpha|^2 - \frac{2}{|\beta|^2}|\langle\alpha|\beta\rangle|^2 + \frac{1}{|\beta|^2}|\langle\alpha|\beta\rangle|^2 \geq 0$$
$$|\alpha|^2 \geq \frac{1}{|\beta|^2}|\langle\alpha|\beta\rangle|^2$$

we recover the Schwarz inequality: $\boxed{|\alpha||\beta| \geq |\langle\alpha|\beta\rangle|}$

4 consider the volume integral over all of 3D space:

$$I = \int d^3r \; \frac{\partial |\psi|^2}{\partial t} = \int d^3r \left[\frac{\partial}{\partial t}(\psi \cdot \psi^*)\right]$$

$$I = \int d^3r \left[\psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi^*}{\partial t}\right]$$

from the Schrödinger equation,

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\cdot\psi \qquad \text{assume potential is real}$$

$$\frac{\partial \psi}{\partial t} = \frac{i\hbar}{2m}\nabla^2\psi - \frac{i}{\hbar}V\psi \qquad \frac{\partial \psi^*}{\partial t} = -\frac{i\hbar}{2m}\nabla^2\psi^* + \frac{i}{\hbar}V\psi^*$$

$$I = \int d^3r \left[\frac{i\hbar}{2m}\psi^*\nabla^2\psi - \cancel{\frac{i}{\hbar}V\psi^*\psi} - \frac{i\hbar}{2m}\psi\nabla^2\psi^* + \cancel{\frac{i}{\hbar}V\psi\psi^*}\right]$$

$$I = \frac{i\hbar}{2m}\int d^3r \underbrace{(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*)}_{\equiv K}$$

$$K = \psi^*\partial_i\partial_i\psi - \psi\partial_i\partial_i\psi^* \qquad \text{product rule/IBP}$$

$$K = \partial_i(\psi^*\partial_i\psi) - (\partial_i\psi^*)(\partial_i\psi) - [\partial_i(\psi\partial_i\psi^*) - (\partial_i\psi)(\partial_i\psi^*)]$$

$$K = \vec\nabla\cdot(\psi^*\vec\nabla\psi) - (\vec\nabla\psi^*)\cdot(\vec\nabla\psi) - \vec\nabla\cdot(\psi\vec\nabla\psi^*) + (\vec\nabla\psi)\cdot(\vec\nabla\psi^*)$$

$$K = \vec\nabla\cdot(\psi^*\vec\nabla\psi - \psi\vec\nabla\psi^*)$$

$$I = \frac{i\hbar}{2m}\int d^3r \; \vec\nabla\cdot(\psi^*\vec\nabla\psi - \psi\vec\nabla\psi^*)$$

$$I = -\int d^3r \; \vec\nabla\cdot\left[\frac{i\hbar}{2m}(\psi\vec\nabla\psi^* - \psi^*\vec\nabla\psi)\right] = \int d^3r \; \frac{\partial|\psi|^2}{\partial t}$$

so may identify: $\partial|\psi|^2/\partial t = -\vec\nabla\cdot\vec j$

$$\boxed{\vec j = \frac{i\hbar}{2m}(\psi\vec\nabla\psi^* - \psi^*\vec\nabla\psi)}$$

as $|\psi|^2$ is the probability density, may interpret:
(volume V, rigid, bounded by surface S)

$$\int_V d^3r \; \frac{\partial|\psi|^2}{\partial t} = -\int_V d^3r \; \vec\nabla\cdot\vec j \qquad \text{div. theorem}$$

$$\underbrace{\int_V d^3r \; \frac{\partial|\psi|^2}{\partial t}}_{\text{rate of change of probability that particle is within some volume } V} = -\underbrace{\oint_S d\vec S\cdot\vec j}_{\text{flow of probability out of the surface}}$$

rate of change of probability that particle is within some volume V = − (flow of probability out of the surface)

so $\vec j$ represents a probability current density calculate $\vec{j}$ for $\psi(\vec{r}) = A e^{i\vec{p}\cdot\vec{r}/\hbar} + B e^{-i\vec{p}\cdot\vec{r}/\hbar}$

$+\vec{p}$ component $\quad -\vec{p}$ component

$$\psi^*(\vec{r}) = A^* e^{-i\vec{p}\cdot\vec{r}/\hbar} + B^* e^{i\vec{p}\cdot\vec{r}/\hbar}$$

$$\vec{\nabla}\psi = \frac{i\vec{p}}{\hbar}\left(A e^{i\vec{p}\cdot\vec{r}/\hbar} - B e^{-i\vec{p}\cdot\vec{r}/\hbar}\right)$$

$$\vec{\nabla}\psi^* = -\frac{i\vec{p}}{\hbar}\left(A^* e^{-i\vec{p}\cdot\vec{r}/\hbar} - B^* e^{i\vec{p}\cdot\vec{r}/\hbar}\right)$$

writing $\phi \equiv i\vec{p}\cdot\vec{r}/\hbar$ for brevity, $\vec{\phi} \equiv i\vec{p}/\hbar$ (sorry this notation sucks)

$$\psi\vec{\nabla}\psi^* = -\vec{\phi}\cdot\left(A e^{\phi} + B e^{-\phi}\right)\left(A^* e^{-\phi} - B^* e^{\phi}\right)$$

$$= -\vec{\phi}\left(|A|^2 - AB^* e^{2\phi} + A^* B e^{-2\phi} - |B|^2\right)$$

$$\psi^*\vec{\nabla}\psi = +\vec{\phi}\cdot\left(A^* e^{-\phi} + B^* e^{\phi}\right)\left(A e^{\phi} - B e^{-\phi}\right)$$

$$= +\vec{\phi}\left(|A|^2 - A^* B e^{-2\phi} + B^* A e^{2\phi} - |B|^2\right)$$

$$\vec{j} = \frac{i\hbar}{2m}\left(\psi\vec{\nabla}\psi^* - \psi^*\vec{\nabla}\psi\right)$$

$$\vec{j} = -\frac{i\hbar}{2m}\vec{\phi}\Big[|A|^2 - \cancel{AB^* e^{2\phi}} + \cancel{A^* B e^{-2\phi}} - |B|^2$$

$$+ |A|^2 - \cancel{A^* B e^{-2\phi}} + \cancel{B^* A e^{2\phi}} - |B|^2\Big]$$

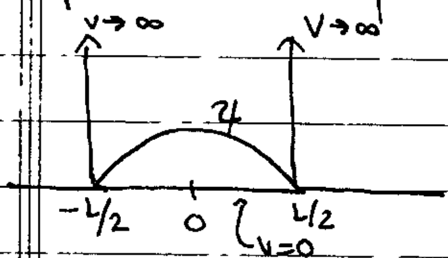
$$\vec{j} = -\frac{i\hbar}{m}\vec{\phi}\left(|A|^2 - |B|^2\right) = -\frac{i\hbar}{m}\cdot\left(\frac{i\vec{p}}{\hbar}\right)\left(|A|^2 - |B|^2\right)$$

$$\boxed{\vec{j} = \frac{\vec{p}}{m}\left(|A|^2 - |B|^2\right)}$$

so $|A|^2$ amount moves with $+\vec{p}/m$ and
$|B|^2$ // $-\vec{p}/m$

may interpret this as probability flowing in diff. directions from the two momentum components, with speed $|\vec{p}/m|$, effectively

(5) particle initially in ground state of a well of width L centered on the origin, with wavefunction:



$$\psi_{0,L}(x) = \begin{cases} \sqrt{\frac{2}{L}}\cos(\pi x/L) & -L/2 \le x \le L/2 \\ 0 & \text{otherwise} \end{cases}$$

the well suddenly expands to span from $-L$ to L.
can find the ground state of the new well by sending $L \to 2L$ in $\psi_{0,L}(x)$ above:

$$\psi_{0,2L}(x) = \begin{cases} \sqrt{\frac{1}{L}}\cos\left(\frac{\pi x}{2L}\right) & -L \le x \le L \\ 0 & \text{otherwise} \end{cases}$$

the particle's wavefunction is unchanged. the probability it will be observed in the ground state of the new well is:

$$P = |\langle \psi_{0,2L} | \psi_{0,L}\rangle|^2$$

$$\langle \psi_{0,2L} | \psi_{0,L}\rangle = \int_{-\infty}^{+\infty} dx\, \psi_{0,2L}^*(x)\psi_{0,L}(x)$$

the integral is only non-zero between $-L/2$ & $L/2$ due to $\psi_{0,L}$:

$$\langle \psi_{0,2L} | \psi_{0,L}\rangle = \int_{-L/2}^{+L/2} dx \sqrt{\tfrac{1}{L}}\cos\left(\tfrac{\pi x}{2L}\right)\sqrt{\tfrac{2}{L}}\cos\left(\tfrac{\pi x}{L}\right)$$

let $\theta = \pi x/2L$ $\quad dx = \frac{2L}{\pi}d\theta$ $\quad \theta: -\pi/4 \to \pi/4$

$$\langle \psi_{0,2L} | \psi_{0,L}\rangle = \frac{\sqrt{2}}{L}\cdot\frac{2L}{\pi}\int_{-\pi/4}^{+\pi/4} d\theta\, \cos(\theta)\cos(2\theta) \qquad \text{even}$$

$$= \frac{4\sqrt{2}}{\pi}\int_0^{\pi/4} d\theta\, \cos\theta\cdot[1-2\sin^2\theta]$$

let $u = \sin\theta$ $\quad du = \cos\theta\, d\theta$ $\quad u: 0 \to 1/\sqrt{2}$

$$\langle \psi_{0,2L} | \psi_{0,L}\rangle = \frac{4\sqrt{2}}{\pi}\int_0^{1/\sqrt{2}} du\cdot(1-2u^2)$$

$$= \frac{4\sqrt{2}}{\pi}\left[u - \tfrac{2}{3}u^3\right]_0^{1/\sqrt{2}}$$

$$= \frac{4\sqrt{2}}{\pi}\left(\frac{1}{\sqrt{2}} - \frac{2}{3}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{2}}\right) = \frac{4}{\pi}(1-1/3) = \frac{4}{\pi}\cdot\frac{2}{3}$$

$\langle \psi_{0,2L} | \psi_{0,L}\rangle = 8/3\pi$ so $\boxed{P = \left(\frac{8}{3\pi}\right)^2}$

(6) $[J,Q] \neq 0$ write this non-zero quantity as some α, (generally a q-number)

$$JQ = QJ + \alpha$$

$[J,H] = 0$ $[Q,H] = 0$

consider an energy eigenstate,

$$H|\chi\rangle = E|\chi\rangle$$

$$[J,H] = 0 \Rightarrow JH = HJ$$

$$HJ|\chi\rangle = JH|\chi\rangle$$

$$H[J|\chi\rangle] = E[J|\chi\rangle]$$ either ~~J~~

$J|\chi\rangle \propto |\chi\rangle$ or H is degenerate ($J|\chi\rangle$ not proportional to $|\chi\rangle$ but a distinct eigenstate w/ same E)

assume NOT degenerate, so $J|\chi\rangle = j|\chi\rangle$

similarly,

$$[Q,H] = 0 \Rightarrow H[Q|\chi\rangle] = E[Q|\chi\rangle]$$

so if H not degenerate, $Q|\chi\rangle \propto |\chi\rangle$, write:

$$Q|\chi\rangle = q|\chi\rangle$$

then:

$$JQ|\chi\rangle = J[q|\chi\rangle] = qj|\chi\rangle$$

$$QJ|\chi\rangle = Q[j|\chi\rangle] = qj|\chi\rangle$$

so $(JQ - QJ)|\chi\rangle = 0$

$[J,Q]|\chi\rangle = 0$ but $[J,Q] \neq 0$!

could claim $\alpha|\chi\rangle = 0$ for $\alpha \neq 0$, so α annihilates 0, but would have to hold for all energy eigenstates of H; since ~~[illegible]~~ these form a complete basis, α annihilates every state in the Hilbert space, and the only operator that does this is 0, and we assumed $\alpha \neq 0$, contradiction!

so H must be degenerate

example: $J = L_x$, $Q = L_z$, for simple hydrogen atom H (all different m_ℓ states for a given n & ℓ are degenerate).