Problem Set 5

Solutions

1 We showed in class that a unitary transformation carries one orthonormal basis into another.

This means that

$$|f_1\rangle = U|e_1\rangle \qquad |e_1\rangle = V|f_1\rangle$$
$$|f_2\rangle = U|e_2\rangle \qquad |e_2\rangle = V|f_2\rangle$$

for some unitary operators U, V.

Let's make sure that these definitions of U & V match those given in the problem.

$$|v\rangle = \gamma_1|f_1\rangle + \gamma_2|f_2\rangle = \beta_1|e_1\rangle + \beta_2|e_2\rangle \quad \text{from problem}$$

$$= \gamma_1 U|e_1\rangle + \gamma_2 U|e_2\rangle \quad \text{from my definitions above}$$

Projecting onto $\langle e_1|$, we get

$$\gamma_1\langle e_1|U|e_1\rangle + \gamma_2\langle e_1|U|e_2\rangle = \beta_1$$

And onto $\langle e_2|$, we get

$$\gamma_1\langle e_2|U|e_1\rangle + \gamma_2\langle e_2|U|e_2\rangle = \beta_2$$

Up to this point we have not chosen a basis to work in. We will do that now, using the one specified in the problem.

$$|e_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |e_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad U = \begin{pmatrix} \langle e_1|U|e_1\rangle & \langle e_1|U|e_2\rangle \\ \langle e_2|U|e_1\rangle & \langle e_2|U|e_2\rangle \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$$

From above projection equations, we then have

$$\Rightarrow \quad \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = U \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$$

, which matches Eq (3) of the problem.

Note that U transforms the basis vectors $|e_i\rangle$ into $|f_i\rangle$, but that it transforms the coefficients of the $|f\rangle$ coordinate system into those of the $|e\rangle$ coordinate system. The at first confusing reversed directionality in those two transformations is a manifestation of the well-known equivalence of transformations: you can rotate axes one way or rotate the vectors the opposite way.

Back to

$$|v\rangle = \gamma_1 |f_1\rangle + \gamma_2 |f_2\rangle = \beta_1 |e_1\rangle + \beta_2 |e_2\rangle$$

$$\downarrow$$

$$= \beta_1 V|f_1\rangle + \beta_2 V|f_2\rangle$$

Projecting onto $\langle f_1|$ & $\langle f_2|$, we have

$$\gamma_1 = \beta_1 V_{11} + \beta_2 V_{12}$$

$$\gamma_2 = \beta V_{21} + \beta_2 V_{22}$$

$$\Rightarrow \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = V \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \quad \checkmark$$

$$\langle f_1 | f_2 \rangle = (\cos\theta)(-e^{-i\alpha}\sin\theta) + (e^{-i\alpha}\sin\theta\cos\theta)$$

$$= 0 \quad \checkmark \quad \Rightarrow f \text{ basis is orthogonal}$$

$$\langle f_1 | f_1 \rangle = \langle f_2 | f_2 \rangle = \cos^2\theta + e^{i\alpha}\sin\theta \, e^{-i\alpha}\sin\theta = 1 \quad \checkmark$$

$\Rightarrow f$ basis is normalized

~~Does $U = V^\dagger$?~~

~~In the given basis, just from the $|e_i\rangle, |f_i\rangle$ definitions:~~

$$\cancel{U = \begin{pmatrix} \cos\theta & e^{-i\alpha}\sin\theta \\ -e^{i\alpha}\sin\theta & \cos\theta \end{pmatrix} \quad V = \begin{pmatrix} \cos\theta \\ \\ \end{pmatrix}}$$

From the definitions we wrote at the very beginning, we can easily construct the matrix elements of U & V in the given basis:

$$\boxed{\begin{array}{l} U = \begin{pmatrix} \cos\theta & -e^{-i\alpha}\sin\theta \\ e^{i\alpha}\sin\theta & \cos\theta \end{pmatrix} \\ \\ V = \begin{pmatrix} \cos\theta & e^{-i\alpha}\sin\theta \\ -e^{i\alpha}\sin\theta & \cos\theta \end{pmatrix} \end{array}}$$

$\Leftrightarrow |f_i\rangle = U|e_i\rangle$

$\Leftrightarrow |e_i\rangle = V|f_i\rangle$

Does $U = V^{-1}$?

$$UV = \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

$$\Rightarrow \boxed{U = V^{-1}}$$

Does $U^\dagger = U^{-1}$?

$$U^\dagger \equiv (U^T)^* = \begin{pmatrix} \cos\theta & e^{-i\alpha}\sin\theta \\ -e^{i\alpha}\sin\theta & \cos\theta \end{pmatrix} = V \text{ which is } U^{-1}$$

$\Rightarrow$ $\boxed{U \text{ is unitary}}$

And same goes for V

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a) $|\psi(t)\rangle = e^{-i\omega t}|\psi\rangle$ where $H|\psi\rangle = \hbar\omega|\psi\rangle$

$$\Rightarrow \langle\psi(t)|[A,H]|\psi(t)\rangle = \langle\psi|e^{i\omega t}[A,H]e^{-i\omega t}|\psi\rangle$$

$$= \langle\psi|[A,H]|\psi\rangle \Rightarrow \text{time-independent}$$

regardless of whether $[A,H]$ vanishes

b) We will use

$$[AB,C] = ABC - CAB = ABC - CAB - ACB + ACB$$

$$= A[B,C] + [A,C]B$$

$$[xp,p^2] = p\cancel{[p,p^2]}^{0} + [x,p^2]p$$

$$= -[p^2,x]p = -(p[p,x]+[p,x]p)p$$

$$= 2i\hbar p^2$$

$$[xp,V] = x[p,V] + \cancel{[x,V]}^{0}p$$

Now $\langle x|x[p,V]|\psi\rangle = x\langle x|pV|\psi\rangle - x\langle x|Vp|\psi\rangle$

$$= x(-i\hbar)\frac{d}{dx}(V\langle x|\psi\rangle) - xV(-i\hbar)\frac{d}{dx}\langle x|\psi\rangle$$

$$= -i\hbar x V'(x)\psi(x)$$

$$\Rightarrow [xp, V(x)] = -i\hbar x V'(x)$$

$$\boxed{\Rightarrow [xp,H] = \frac{i\hbar p^2}{m} - i\hbar x V'(x)}$$

c) $\langle\psi|\,[xp, H]\,|\psi\rangle = 0$

$$\Rightarrow \langle\psi|\,(2i\hbar T - i\hbar x V'(x))\,|\psi\rangle = 0$$

$$\Rightarrow \langle 2T\rangle = \langle x V'(x)\rangle = \langle n V(x)\rangle$$

$$\Rightarrow \boxed{\langle T\rangle = \frac{n}{2}\langle V(x)\rangle}$$