Dyson Series

more formal treatment

<u>Interaction rep:</u> $i\hbar \dfrac{\partial U_I(t,t_0)}{\partial t} = V_I(t)\, U_I(t,t_0)$

<u>Integrate</u>

$$i\hbar\Big[U_I(t,t_0) - \underbrace{U_I(t_0,t_0)}_{1}\Big] = \int_{t_0}^{t} V_I(t')\, U_I(t',t_0)\, dt'$$

integral equation

$$\boxed{U_I(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} V_I(t')\, U_I(t',t_0)\, dt'}$$

same

<u>"Solve" integral equation by iteration</u> ← assume $V_I(t)$ is "small"

$$U_I(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt_1\, V_I(t_1)\, U_I(t_1,t)$$

$$= 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt_1\, V_I(t_1)\left[1 - \frac{i}{\hbar}\int_{t_0}^{t_1} dt_2\, V_I(t_2)\, U_I(t_2,t_0)\, \cancel{dt_2}\right]$$

$$\boxed{\begin{aligned} U_I(t,t_0) = 1 &- \frac{i}{\hbar}\int_{t_0}^{t} dt_1\, V_I(t_1) + \left(\frac{-i}{\hbar}\right)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, V_I(t_1) V_I(t_2) \\ &+ \dots + \left(\frac{-i}{\hbar}\right)^k \int_{t_0}^{t} dt_1 \dots \int_{t_0}^{t_{k-1}} dt_k\, V_I(t_1) \dots V_I(t_k) + \dots \end{aligned}}$$

<u>Dyson series</u> (used in QED)

typically difficult to prove convergence

Back in the Schrodinger Picture $\quad U = U_0 U_I$

$$U_I(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt_1\, V_I(t_1)$$
$$+ \left(\frac{-i}{\hbar}\right)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, V_I(t_1) V_I(t_2) + \dots$$
$$+ \left(\frac{-i}{\hbar}\right)^k \int_{t_0}^{t} dt_1 \dots \int_{t_0}^{t_{k-1}} V_I(t_1) \dots V_I(t_k) + \dots$$

$$U = \sum_{k=0}^{\infty} U^{(k)}$$

$$U^{(0)}(t,t_0) = U_0(t,t_0)$$

$$U^{(1)}(t,t_0) = -\frac{i}{\hbar} U_0(t,t_0) \int_{t_0}^{t} dt_1\, V_I(t_1)$$
$$= \frac{1}{i\hbar} U_0(t,t_0) \int_{t_0}^{t} dt_1\, U_0^\dagger(t_1,t_0) V(t_1) U_0(t_1,t_0)$$
$$= \int_{t_0}^{t} dt_1\, \underbrace{U_0(t,t_0) U_0^\dagger(t_1,t_0)}_{U_0(t,t_1)\cancel{U_0(t_1,t_0)}} \frac{V(t_1)}{i\hbar} U_0(t_1,t_0)$$

$t_0 \quad t_1 \quad t$

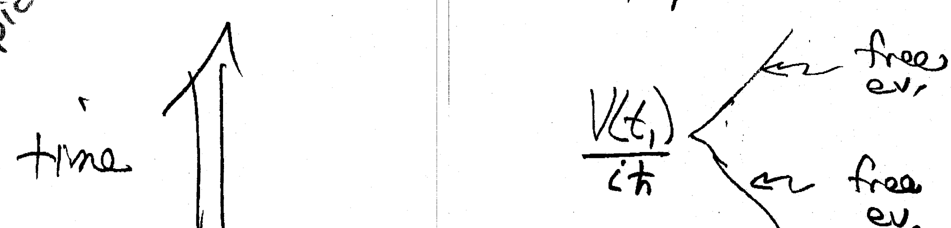
$$= \int_{t_0}^{t} dt_1\, U_0(t,t_1) \frac{V(t_1)}{i\hbar} U_0(t_1,t_0)$$

free evolution from t_1 to t

potential "acts" at t_1

free time evolution from t_0 to t_1

"funny picture"

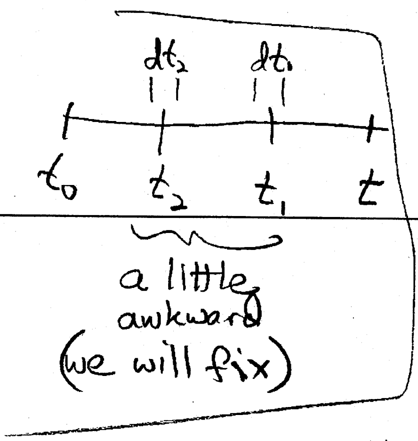


$$U^{(2)}(t,t_0) = \left(-\frac{i}{\hbar}\right)^2 U_0(t,t_0)\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, V_I(t_1)\, V_I(t_2)$$

$$= \frac{1}{(i\hbar)^2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, U_0(t t_0)\, U_0^{+}(t_1 t_0)\, V(t_1)\, U_0(t_1 t_0)\, U_0^{+}(t_2 t_0)\, V(t_2)\, U_0(t_2 t_0)$$

$$U_0(t t_0)\, U_0^{+}(t_1 t_0) = U_0(t t_1)\, \cancel{U_0(t_1 t_0)}\,\cancel{U_0^{+}(t_1 t_0)}$$

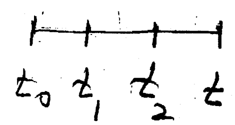
$$U_0(t_1 t_0)\, U_0^{+}(t_2 t_0) = U_0(t_1, t_2)\, \cancel{U_0(t_2 t_0)}\,\cancel{U_0^{+}(t_2 t_0)}$$

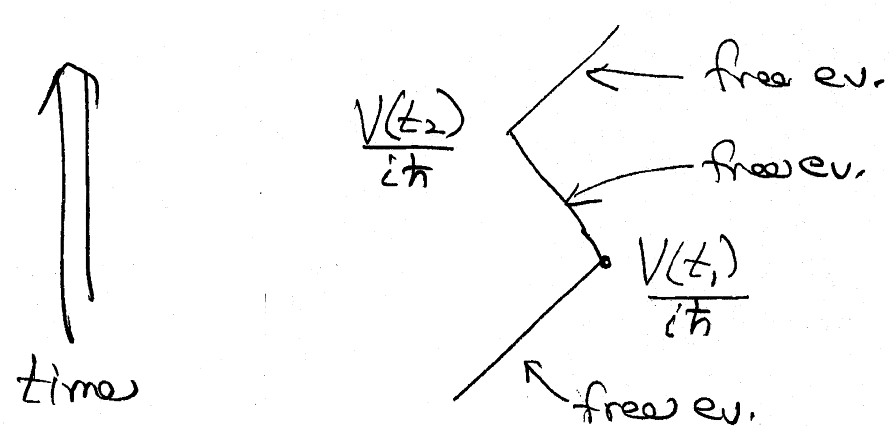


$$= \frac{1}{(i\hbar)^2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\, U_0(t t_1)\, V(t_1)\, U_0(t_1 t_2)\, V(t_2)\, U_0(t_2 t_0)$$

change $t_1 \leftrightarrow t_2$

$$= \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1\, \underbrace{U_0(t t_2)}_{\text{free ev.}}\, \underbrace{\frac{V(t_2)}{i\hbar}}_{\text{potential acts}}\, \underbrace{U_0(t_2 t_1)}_{\text{free ev.}}\, \underbrace{\frac{V(t_1)}{i\hbar}}_{\text{potential acts}}\, \underbrace{U_0(t, t_0)}_{\text{free ev.}}$$





<u>Change labels:</u> $t_k \to t_1$

$t_{k-1} \to t_2$

$\vdots$

$t_1 \to t_{k-1}$

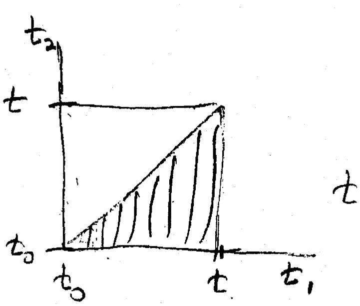
$$U^{(k)}(t\,t_0) = \int_{t_0}^{t} dt_{k-1} \cdots \int_{t_0}^{t_2} dt_1 \; U_0(t\,t_{k-1}) \frac{V(t_{k-1})}{i\hbar} U_0(t_{k-1}\,t_{k-2}) \cdots \frac{V(t_1)}{i\hbar} U_0(t_1\,t_0)$$

<u>k=0:</u> $U^{(0)}(t\,t_0) = U_0(t, t_0)$

<u>k=1:</u> $U^{(1)}(t\,t_0) = \int_{t_0}^{t} dt_1 \, U_0(t\,t_1) \frac{V(t_1)}{i\hbar} U_0(t_1\,t_0)$

<u>k=2:</u> $U^{(2)}(t\,t_0) = \int_{t_0}^{t} dt_2 \int_{t_0}^{t_1} dt_1 \, U_0(t\,t_2) \frac{V(t_2)}{i\hbar} U_0(t_2\,t_1) \frac{V(t_1)}{i\hbar} U_0(t_1\,t_0)$

$$= \int_{t_0}^{t} dt_2 \, U_0(t\,t_2) \frac{V(t_2)}{i\hbar} \int_{t_0}^{t_2} dt_1 \, U(t_2\,t_1) \frac{V(t_1)}{i\hbar} U_0(t_1\,t_0)$$



$\therefore$ k = # of times the potential "acts" (in perturbation theory)

Perturbation series $\Longrightarrow$ useful if Dyson series converges

$$|\psi(t)\rangle_I = \sum_m c_m(t)\,|m\rangle$$

$$\langle n|\psi(t)\rangle_I = c_n(t)$$

$$\boxed{c_n(t) = \langle n|\underbrace{U_I(t,t_0)}_{\text{put in Dyson series}}|\psi(t_0)\rangle_I}$$

put in Dyson series

$$c_n(t) = c_n^{(0)}(t) + c_n^{(1)}(t) + c_n^{(2)}(t) + \dots$$

$$c_n^{(0)} = \langle n|1|\underbrace{\psi(t_0)\rangle_I}_{|\psi(t_0)\rangle}$$

$\Leftarrow$ call this $|i\rangle$

$$\boxed{c_n^{(0)} = \delta_{ni}}$$

$$c_n^{(1)} = \langle n| -\frac{i}{\hbar}\int_{t_0}^{t} dt_1\, V_I(t_1)\,|i\rangle$$

$$= -\frac{i}{\hbar}\int_{t_0}^{t} dt_1\,\langle n|\underbrace{V_I(t_1)}_{U_0^\dagger V U_0}|i\rangle$$

$$\boxed{c_n^{(1)} = -\frac{i}{\hbar}\int_{t_0}^{t} dt_1\, V_{ni}(t_1)\, e^{i\omega_{ni}(t_1-t_0)}}$$

$\Leftarrow$ will av. to zero unless

$V_{ni}(t) \sim e^{-i\omega t}$

$\underline{\omega \approx \omega_{ni}}$

$$C_n^{(2)} = \langle n | \left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 \, \underbrace{V_I(t_2)}_{} \overset{\sum_m |m\rangle\langle m|}{\downarrow} V_I(t_1) \, | i \rangle$$

$$= \left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 \; e^{i\omega_n(t_2-t_0)} V_{nm}(t_2) e^{-i\omega_m(t_2-t_0)} \; e^{i\omega_m(t_1-t_0)} V_{mi}(t_1) e^{-i\omega_i(t_1-t_0)}$$

$$= \left(-\frac{i}{\hbar}\right)^2 \int dt_2 \int dt_1 \; \underbrace{e^{i\omega_{nm}(t_2-t_0)} V_{nm}(t_2)}_{\text{averages to zero unless near res.}} \; \underbrace{e^{i\omega_{mi}(t_1-t_0)} V_{mi}(t_1)}_{\text{av. to zero unless near res.}}$$

etc.