Relativistic dissipative hydrodynamics and the nuclear equation of state

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The theory of dissipative, relativistic fluids due to Israel and Stewart is used to constrain the form of the nuclear equation of state. In the Israel-Stewart theory, there are conditions on the equation of state and other thermodynamic properties (the "second-order" coefficients) of a fluid which, if satisfied, guarantee that equilibria are stable and that fluid perturbations propagate causally and obey hyperbolic equations. The second-order coefficients in the Israel-Stewart theory, which are relaxation times for the dissipative degrees of freedom and coupling constants between different forms of dissipation, are derived for a free, degenerate Fermi gas. It is shown rigorously that the free, degenerate Fermi gas is stable (and hence causal) at all temperatures in this theory. These values for the second-order coefficients are then used in the stability conditions to constrain various proposed expressions for the nuclear ground-state energy. The stability conditions are found to provide significantly more stringent constraints on the proposed equations of state than the usual simple restriction that the adiabatic sound speed be less than the speed of light.

I. INTRODUCTION

One of the more productive ideas in the history of nuclear physics has been to treat nuclei within the context of fluid mechanics. In this manner, our ignorance concerning the detailed properties of nuclear interactions can be overcome by invoking our cherished beliefs concerning the generality of thermodynamics. The use of a hydrodynamic description is perhaps especially appealing in the case of relativistic heavy ion collisions.[1] In order to analyze such collisions adequately within the context of a fluid approximation, a theory of relativistic fluids incorporating the effects of dissipation is needed. The classic relativistic dissipative theories of fluids, constructed by Eckart[2-4] and Landau and Lifshitz,[5] are now known to possess pathological properties: in these theories, all equilibria are unstable[6] (independent of the fluid equation of state), and it is possible to transmit information at speeds exceeding the speed of light[7] (so that they are not, properly speaking, truly relativistic theories).

Fortunately, an alternative theory has been developed by Israel and Stewart[8-11] which does not possess these pathological properties. It is possible to prove that, within the context of the Israel-Stewart theory, if a fluid possesses stable equilibria, then linear fluid perturbations will propagate causally (at velocities less than the speed of light) and obey hyperbolic differential equations.[12,13] While there are still unanswered fundamental questions concerning this phenomenological fluid theory, it seems to be the current best candidate to fulfill the role of a relativistic theory of dissipative fluids.

The differences in the predictions of the Israel-Stewart theory and the older theories are most pronounced in situations where there is significant fluid evolution on time-scales comparable to the mean free time of the particles in the fluid. The rapid evolution of the nucleon fluid in a heavy ion collision is exactly the sort of situation where the differences between the two types of theories are most pronounced. Given that the differences are largest in this sort of case, and the pathological properties of the older theories, it seems that the Israel-Stewart theory should be the theory of choice for analyzing heavy ion collisions [as well as other highly dynamic complicated (many-nucleon) high-energy nuclear phenomena, such as supernovae].

One of the outstanding problems in nuclear physics is to adequately determine the form of the nuclear equation of state. Knowledge of the equation of state is needed for analyzing many important physical situations, including supernova explosions and neutron star formation, energetic heavy ion collisions, and the era surrounding the possible quark-hadron phase transition in the early universe. Recent developments in accelerating heavy nuclei to high energies promise a wealth of data related to the equation of state from relativistic heavy ion collisions; however, the physics involved in such a collision is exceedingly complex. It is a highly nontrivial matter to extract equation of state information from such a complex dynamical situation.

In view of our ignorance of the details of the nuclear equation of state, there has always been interest in developing constraints on possible suggested models for the equation of state. Perhaps the best known such constraint is the demand that the adiabatic sound speed of the nuclear material must be less than the speed of light. This constraint was recently analyzed for a number of suggested models by Osnes and Strottman.[14] As mentioned above, in the relativistic dissipative fluid theory of Israel and Stewart, fluid perturbations (including adiabatic sound waves) will travel at less than the speed of light if the equilibria of the fluid are stable. In order for the equilibria to be stable, the thermodynamical properties of the fluid must satisfy a number of constraints.[12] The purpose of this paper is to apply these stability constraints from the Israel-Stewart theory of relativistic dissipative fluids to several proposed models for the nuclear equation of state. Our purpose is primarily illustrative rather than attempting to be comprehensive; we wish to demonstrate how the Israel-Steward theory may be applied to the study of the properties of nuclear material. We primarily examine those candidate equations of state which have previously been analyzed in other contexts by Clare and Strottman[1] or Osnes and Strottman.[14]

A problem in the application of the Israel-Stewart fluid stability conditions to nuclear equations of state is that the stability conditions involve additional thermodynamic functions, the "second-order coefficients," which cannot be derived from the ordinary equation of state of a fluid, but are in fact separately chosen thermodynamic functions which must be specified in order to uniquely define the evolution of an Israel-Stewart fluid. Physically these coefficients are relaxation times for the various forms of dissipation and coupling constants between the different forms of dissipation. Since currently proposed nuclear equations of state do not imply any particular form for these functions, we will resort to the same technique often used in nuclear studies to handle the poorly understood thermal contribution to the internal energy: we will assume that the second-order coefficients are given by their values for a free, degenerate Fermi gas. The values of the second-order coefficients for the free, degenerate Fermi gas are derived in this paper from relativistic kinetic theory. We are thus able to only *rigorously* establish the stability of the free, degenerate Fermi gas (which is not trivial, and was not established prior to this work). Our conclusions regarding the constraints on other proposed equations of state must therefore be taken as strongly suggestive rather than certain: their domain of validity could be extended or decreased by proposing (and justifying) other forms for the second-order coefficients.

Section II of the paper is an abbreviated overview of the Israel-Stewart theory of relativistic dissipative fluids, and the principal results concerning stability and causality. The stability conditions are given here in terms of the usual thermodynamic functions used in theoretical nuclear physics. In Sec. III the second-order relaxation time and dissipation coupling coefficients are given for the free, degenerate Fermi gas for the first time. Some details of the derivations are relegated to an appendix. The second-order coefficients are then used in the stability criteria to study the free, degenerate, Fermi gas; it is found to be stable (and hence causal), for all ranges of temperature. This is our most rigorous result. In Sec. IV the Fermi gas second-order coefficients are applied in combination with various proposed nuclear equations of state in order to constrain their domain of applicability. The constraints imposed by relativistic dissipative hydrodynamic stability are found to generally be significantly more stringent than the simple adiabatic sound speed criterion.[14] We use units in which Boltzmann's constant $k=1$ and the speed of light $c=1$.

II. RELATIVISTIC DISSIPATIVE HYDRODYNAMICS: STABILITY AND CAUSALITY

The first attempts to include the effects of dissipation (thermal conductivity, bulk and shear viscosity) in a theory of relativistic fluids were the theories of Eckart[2–4] and Landau and Lifshitz.[5] These theories represent two simple ways in which the nonrelativistic Navier-Stokes-Fourier theory can be rendered into covariant form. It was long suspected that the Eckart theory was not truly relativistic, in that it was believed that thermal perturbations would propagate at speeds exceeding the speed of light. More recently, both the Eckart and Landau-Lifshitz theories have been shown to be pathological in several ways: in these theories, there are no stable equilibria[6] (regardless of the choice of equation of state of the fluid), and it has been shown rigorously that in these theories it is possible to transmit information at speeds vastly exceeding the speed of light.[7] As an example, according to the Eckart theory, water at room temperature and pressure should be unstable to exponentially growing thermal modes with growth timescales of order 10^{-35} sec; also, it should be possible to transmit information using thermal modes in the water at speeds of up to 10^6 times the speed of light. Clearly, then, these theories are not acceptable as truly relativistic theories of dissipative fluids.

An extended theory of relativistic dissipative hydrodynamics has been developed by Israel and Stewart[8–11] building on earlier nonrelativistic work of Müller[15] and Grad,[16] among others. This theory is more complicated than the simple theories of Eckart and Landau and Lifshitz: an Israel-Stewart fluid possesses 14 degrees of freedom rather than the traditional five in the simpler theories. The theory, however, has been shown to possess a number of desirable properties. Within the Israel-Stewart theory, a fluid will possess stable equilibria if and only if linear perturbations about equilibrium propagate causally (i.e., inside the light cone) and obey hyperbolic equations.[12,13] In "ordinary" situations (such as in the long wavelength limit of a simple Maxwell-Boltzmann gas), the additional nine degrees of freedom are strongly damped, and the dynamics of the remaining five are exactly what one would expect from classical theory.[7] In other physical circumstances, the nine additional degrees of freedom may not be strongly damped, and are useful in modeling quantum fluid behavior: in an appropriate limit, it is possible to use the theory as a one-fluid model of superfluids, which can be shown to be mathematically equivalent to the two-fluid Landau model.[17] While the Israel-Stewart theory still has many tests to pass, it definitely seems a much healthier candidate for a theory of relativistic dissipative fluid mechanics than the simpler Eckart and Landau-Lifshitz theories. There is some evidence[18] that the Israel-Stewart theory is nonhyperbolic and unstable in the strongly nonlinear regime; however, this breakdown of the theory does not seem to occur until the fluid is extremely far from equilibrium (e.g., when the magnitude of the heat flow is of the same order as the total mass-energy density of the fluid).

In this section we shall review the derivation of the Eckart and Israel-Stewart theories, and discuss the conditions necessary for stability of equilibria in an Israel-Stewart fluid.

The fundamental variables of a theory of relativistic dissipative hydrodynamics are the particle number current vector N^a and the fluid stress-energy tensor T^{ab} (we will consider here only a single component fluid; the generalization to multiple interacting components is simple and is discussed in Ref. 8). The most fundamental equations for the evolution of the fluid are the conservation laws for the particle number current and stress-energy:

$$\nabla_a N^a = 0 \ , \tag{1}$$

$$\nabla_a T^{ab} = 0 \ . \tag{2}$$

The four-velocity of the fluid, u^a, is not generally considered to be a fundamental variable, as its definition is ambiguous away from equilibrium. Different theories define u^a in different manners. In the Eckart theory, u^a is chosen to be parallel to N^a (follow the particles), while in the Landau-Lifshitz theory, u^a is chosen to be the (unique) timelike eigenvector of T^{ab} (follow the energy). These definitions (and other possible choices) only differ for nonequilibrium states. In this paper we will choose to always work in the Eckart frame. This implies that the particle current and stress-energy can be decomposed as follows:

$$N^a = nu^a \ , \tag{3}$$

$$T^{ab} = \rho u^a u^b + (p+\tau) q^{ab} + q^a u^b + q^b u^a + \tau^{ab} \ , \tag{4}$$

where n is the number density of particles (nucleons) as measured by a comoving observer, ρ is the total mass-energy density, p the pressure, q^a the heat flow vector, τ^{ab} the shear stress tensor, τ the bulk stress, and q^{ab} is the projection tensor constructed using u^a:

$$q^{ab} = g^{ab} + u^a u^b \ , \tag{5}$$

where g_{ab} is the space-time metric tensor. In order for the decompositions in Eqs. (3) and (4) to be unique, these quantities are constrained in the following manner:

$$q^a u_a = \tau^{ab} u_b = \tau^a{}_a = \tau_{ab} - \tau_{ba} = 0 \ . \tag{6}$$

Equations (1) and (2) are to be supplemented by the equation of state for the fluid (here taken as total internal energy density per nucleon as a function of n and T), which allows the further decomposition of ρ into the standard thermodynamic variables of nuclear physics:

$$\rho = n[m_0 + E(n,T)] \ , \tag{7}$$

where m_0 is the nucleon mass and $E(n,T)$ is the internal energy per nucleon, which is often further divided into a ground-state energy $E_0(n)$ and a thermal energy $I(n,T)$:

$$E(n,T) = E_0(n) + I(n,T) \ . \tag{8}$$

The pressure may then be written in terms of known thermodynamic functions using the first law of thermodynamics. The resulting expression is

$$p = n^2 \frac{dE_0}{dn} + n^2 \left[\frac{\partial I}{\partial n} \right]_s \ , \tag{9}$$

where s is the entropy per nucleon.

In order to complete the theory, it is necessary to implement the second law of thermodynamics and also to provide equations which determine τ, q^a, and τ^{ab}. The second law is enforced by demanding that the total entropy associated with a spacelike surface be nondecreasing into the future. The total entropy of a spacelike surface Σ is obtained by integrating the entropy current vector field over the surface:

$$S(\Sigma) = \int_\Sigma s^a d\Sigma_a \ . \tag{10}$$

The second law will hold if for any spacelike surface Σ' to the future of Σ,

$$S(\Sigma') - S(\Sigma) = \int \nabla_a s^a dV \geq 0 \ , \tag{11}$$

where the two surface integrals have been converted into a volume integral using Gauss's theorem. The integral in Eq. (11) will always be nonnegative if we assume the integrand to be nonnegative:

$$\nabla_a s^a \geq 0 \ . \tag{12}$$

Equation (12) is the local form of the second law which is used in relativistic dissipative hydrodynamics. The simple theory of Eckart now results if the entropy current vector s^a is modeled by a sum of terms up to linear order in the deviations from equilibrium (for this reason, the Eckart theory is often referred to as a "first-order" theory):

$$s^a_E = snu^a + q^a/T \ . \tag{13}$$

The better behaved theory of Israel and Stewart is obtained by modeling the entropy current vector by an expansion including all possible terms through quadratic order in the deviations from equilibrium:

$$s^a_{IS} = snu^a + \frac{q^a}{T} - \tfrac{1}{2}(\beta_0 \tau^2 + \beta_1 q_b q^b + \beta_2 \tau_{bc}\tau^{bc}) \frac{u^a}{T} + \alpha_0 \tau \frac{q^a}{T} + \alpha_1 \tau^a{}_b \frac{q^b}{T} \ , \tag{14}$$

where the α_i and β_i are new thermodynamic coefficients. The β_i describe the deviation of the physical entropy density from the thermodynamic entropy density, sn, and the α_i represent terms in the entropy current caused by heat flow–viscous couplings. These coefficients may be thought of as additional "equations of state" which are necessary to fully describe the dynamics of an Israel-Stewart fluid. Their functional form can in principal be calculated from microscopic (e.g., kinetic) theory, or they can be determined experimentally. Note that the older Eckart theory is simply the subset of the Israel-Stewart theory in which these coefficients are all chosen to be zero (however, relativistic kinetic theory gives values for simple gases which are definitely *not* zero).

The equations which determine τ, q^a, and τ^{ab} are now obtained by taking the divergence of either Eq. (13) or (14), and then enforcing Eq. (12) by using the conservation equations [Eqs. (1) and (2)] to force the divergence into the following manifestly nonnegative form:

$$T\nabla_a s^a = \frac{\tau^2}{\zeta} + \frac{q_a q^a}{\kappa T} + \frac{\tau_{ab}\tau^{ab}}{2\eta} . \tag{15}$$

The three (positive) dissipation coefficients may be identified as the bulk viscosity ζ, the thermal conductivity κ, and the shear viscosity η. In the Israel-Stewart theory, the resulting expressions for τ, q^a, and τ^{ab} are

$$\tau = -\zeta\left[\nabla_a u^a + \beta_0 u^a \nabla_a \tau - \alpha_0 \nabla_a q^a - \gamma_0 T q^a \nabla_a\left[\frac{\alpha_0}{T}\right] + \tfrac{1}{2}\tau T\nabla_a\left[\frac{\beta_0}{T}u^a\right]\right] , \tag{16}$$

$$\begin{aligned} q^a = -\kappa T q^{ab}\Bigg[& \frac{1}{T}\nabla_b T + u^c\nabla_c u_b + \beta_1 u^c \nabla_c q_b - \alpha_0 \nabla_b \tau - \alpha_1 \nabla_c \tau^c{}_b + \tfrac{1}{2} T q_b \nabla_c\left[\frac{\beta_1}{T}u^c\right] \\ & -(1-\gamma_0)\tau T \nabla_b\left[\frac{\alpha_0}{T}\right] - (1-\gamma_1) T \tau_b{}^c \nabla_c\left[\frac{\alpha_1}{T}\right] + \gamma_2 \nabla_{[b} u_{c]} q^c \Bigg] , \end{aligned} \tag{17}$$

$$\tau^{ab} = -2\eta\left\langle \nabla^a u^b + \beta_2 u^c \nabla_c \tau^{ab} - \alpha_1 \nabla^a q^b + \tfrac{1}{2} T \tau^{ab} \nabla_c\left[\frac{\beta_2}{T}u^c\right] - \gamma_1 T q^a \nabla^b\left[\frac{\alpha_1}{T}\right] + \gamma_3 \nabla^{[a} u^{c]} \tau_c{}^b \right\rangle , \tag{18}$$

where the angle brackets which appear in Eq. (18) are defined by

$$\langle A^{ab}\rangle = \tfrac{1}{2} q^a{}_c q^b{}_d (A^{cd} + A^{dc}) - \tfrac{1}{3} q^{ab} q_{cd} A^{cd} . \tag{19}$$

The expressions for τ, q^a, and τ^{ab} in the Eckart theory may be obtained from Eqs. (16)–(18) by setting all the α_i, β_i, and γ_i equal to zero.

Equations (1), (2), and (16)–(18) form the complete set of equations for the second-order Israel-Stewart theory, with the Eckart choice of four-velocity (in the "Eckart frame").

The equilibrium states of the theory are defined by the property that the entropy is unchanging in time, i.e., that $\nabla_a s^a = 0$. This implies, by Eq. (15), that $\tau = q^a = \tau^{ab} = 0$ in an equilibrium state. Equations (1), (2), and (16)–(18) may then be used to show that the Israel-Stewart equilibrium states are identical to those in the simpler Eckart theory.[12]

Once the equilibrium states have been defined, their stability may be studied by examining the evolution of linear perturbations around an equilibrium background. In the case of an Eckart fluid, it is easy to find exponentially growing solutions to the linearized equations, which indicate unstable modes. These instabilities are essentially independent of the equation of state: *all* equilibrium states in an Eckart fluid are unstable, regardless of the choice of equation of state.[6,13]

In contrast, the equilibrium states in an Israel-Stewart fluid may or may not be stable, depending on the choice of equation of state and functional form of the second-order thermodynamic coefficients α_i and β_i (the values of the γ_i coefficients do not have any influence on stability). The full analysis of stability is lengthy and will not be repeated (see Refs. 12 and 13 for full details); only a summary is given here. For an Israel-Stewart fluid, it is possible to construct an energy (Liapunov) functional from the linear perturbations around an equilibrium state.[12] Such a functional is a monotonically decreasing function of time which is quadratic in the linear perturbation variables. For the equilibrium states of a fluid to be stable, it is necessary and sufficient that the energy functional be nonnegative valued for all possible values of the perturbation variables. If a perturbation existed having an initially negative energy, then, since the energy of any perturbation decreases into the future (the energy functional is constructed so that this is true), its energy would tend to evolve towards negative infinity, suggesting the presence of an instability (this argument is made rigorous in Ref. 12). After much algebraic manipulation, the energy functional, evaluated on a spacelike surface Σ, may be written in the form

$$E(\Sigma) = \int_\Sigma e \frac{u^a}{T} d\Sigma_a , \tag{20}$$

where e is an energy density quadratic in the linear perturbations. The total energy will be nonnegative for all possible perturbations if and only if the energy density is nonnegative for all possible perturbations. The energy density may be factored into the following form:

$$e = \tfrac{1}{2}\sum_A \Omega_A (\delta Z_A)^2 , \tag{21}$$

where the Ω_A are functions of the thermodynamic variables (including the α_i and β_i), and the δZ_A are linearly independent combinations of the perturbation variables. The necessary and sufficient conditions for stability are then simply that the Ω_A must all be nonnegative for all possible choices of spacelike hypersurfaces Σ. The most restrictive form of the stability conditions, written in terms of the thermodynamic functions (E_0, I) customarily used in nuclear physics are given by

$$\begin{aligned} \Omega_1 = \Bigg[& 2n^2\frac{dE_0}{dn} + n^3\frac{d^2E_0}{dn^2} + 2n^2\left[\frac{\partial I}{\partial n}\right]_s \\ & + n^3\left[\frac{\partial^2 I}{\partial n^2}\right]_s \Bigg]^{-1} \geq 0 , \end{aligned} \tag{22}$$

$$\Omega_2 = n\left[\frac{\partial T}{\partial s}\right]_p \geq 0 , \tag{23}$$

$$\Omega_3 = n\left\{m_0 + E_0 + I - n\left[\frac{dE_0}{dn} + \left[\frac{\partial I}{\partial n}\right]_s\right] - n^2\left[\frac{d^2E_0}{dn^2} + \left[\frac{\partial^2 I}{\partial n^2}\right]_s\right]\right\} - \frac{1}{\beta_0} - \frac{2}{3\beta_2} - \frac{K^2}{\Omega_6} \geq 0 , \tag{24}$$

$$\Omega_4 = n\left[m_0 + E_0 + I + n\frac{dE_0}{dn} + n\left[\frac{\partial I}{\partial n}\right]_s\right] - \frac{2\beta_2 + \beta_1 + 2\alpha_1}{2\beta_1\beta_2 - \alpha_1^2} \geq 0 , \tag{25}$$

$$\Omega_5 = \beta_0 \geq 0 , \tag{26}$$

$$\Omega_6 = \beta_1 - \left[\frac{\alpha_0^2}{\beta_0} + \frac{2\alpha_1^2}{3\beta_2} + \frac{1}{nT}\left[\frac{\partial I}{\partial T}\right]_n^{-1}\right] \geq 0 , \tag{27}$$

$$\Omega_7 = \beta_1 - \frac{\alpha_1^2}{2\beta_2} \geq 0 , \tag{28}$$

$$\Omega_8 = \beta_2 \geq 0 , \tag{29}$$

where

$$K = 1 + \frac{\alpha_0}{\beta_0} + \frac{2\alpha_1}{3\beta_2} - \frac{n}{T}\left[\frac{\partial^2 I}{\partial n\,\partial s}\right] . \tag{30}$$

We have not written Ω_2 in terms of E_0 and $I(n,T)$ as we have been unable to derive a reasonably short expression for it in terms of those variables.

Several of the stability criteria [Eqs. (22)–(29)] are familiar: requiring Ω_1 to be nonnegative ensures that the adiabatic sound speed will be real, while the condition $\Omega_2 \geq 0$ is the relativistic Schwarzschild criterion for stability against convection.[19]

The most useful of the stability criteria in constraining the nuclear equation of state is $\Omega_3 \geq 0$, which may be viewed as providing an upper bound on the adiabatic sound speed. This bound is always less than one, and hence Ω_3 is a stronger constraint on possible equations of state than the usual perfect fluid constraint that the adiabatic sound speed be less than the speed of light.[14]

III. FREE DEGENERATE FERMI GAS IN THE ISRAEL-STEWART THEORY

In this section we will give the values of the second-order thermodynamic coefficients α_i and β_i for a free Fermi gas in the degenerate limit (the gas may be arbitrarily relativistic). We then confirm, using Eqs. (22)–(29), that a free degenerate Fermi gas is stable (and hence causal and hyperbolic) within the context of the Israel-Stewart fluid theory.

The evaluation of the second-order coefficients proceeds from relativistic kinetic theory using the Grad method of moments.[11] The history of the relativistic version of the Grad method, and the specific problem of determining the values of the α_i and β_i for simple gases, is discussed in detail in Ref. 11. In that paper Israel and Stewart develop integral expressions for the second-order coefficients for any simple gas (Maxwell-Boltzmann, Fermi, or Bose), but only explicitly evaluate the coefficients for the Maxwell-Boltzmann case. In the Appendix, we describe in some detail the evaluation of the integrals necessary to determine the coefficients for the Fermi gas case in the degenerate limit, and correct several misprints in equations in Ref. 11. Some of the second-order coefficients have previously been derived for a degenerate nonrelativistic Fermi gas within the context of a nonrelativistic version of the Israel-Stewart theory due to Müller.[20]

The second-order coefficients α_i and β_i are thermodynamic functions which we will express as functions of $\beta = mc^2/kT$, the inverse dimensionless temperature, and a dimensionless thermodynamic potential ν, defined by

$$\nu = \frac{\rho + p}{nm} - \frac{s}{\beta} - 1 . \tag{31}$$

The potential ν is equal to the nonrelativistic chemical potential per particle divided by the particle rest mass. The free Fermi gas will be strongly degenerate if $\beta\nu \gg 1$. Although we will only consider strongly degenerate Fermi gases, they may be arbitrarily relativistic. The nonrelativistic Fermi gas is obtained by taking the secondary limit $\beta \gg 1$, while the ultrarelativistic Fermi gas corresponds to $\beta \ll 1$.

The second-order coefficients for a free, degenerate Fermi gas, each expressed as an asymptotic expansion in inverse powers of $\beta\nu$, are

$$\beta_0 = \frac{405(1+\nu)^5(\beta\nu)^4}{16A_0\pi^4\nu^4(\nu^2+2\nu)^{1/2}} + O(\beta\nu)^2 , \tag{32}$$

$$\beta_1 = \frac{9(1+\nu)(\beta\nu)^2}{A_0\pi^2\nu^2(\nu^2+2\nu)^{3/2}} + O(\beta\nu)^0 , \tag{33}$$

$$\beta_2 = \frac{15(1+\nu)}{2A_0(\nu^2+2\nu)^{5/2}} + O(\beta\nu)^{-2} , \tag{34}$$

$$\alpha_0 = \frac{9(1+\nu)(\nu^2+2\nu+2)(\beta\nu)^2}{A_0\pi^2\nu^2(\nu^2+2\nu)^{3/2}} + O(\beta\nu)^0 , \tag{35}$$

$$\alpha_1 = -\frac{-6(1+\nu)}{A_0(\nu^2+2\nu)^{5/2}} + O(\beta\nu)^{-2} , \tag{36}$$

where

$$A_0 = \frac{m^4 g}{2\pi^2\hbar^3} , \tag{37}$$

and g is the spin weight. These expressions are valid as the leading terms in an asymptotic expansion at any temperature (value of β), so long as the degenerate limit, $\beta\nu \gg 1$, is satisfied.

The expressions given in Eqs. (32)–(36) may be used in the stability criteria, Eqs. (22)–(29), to determine whether a free Fermi gas is stable and causal in the degenerate limit. The following expansions of thermodynamic quantities for a free, degenerate Fermi gas are useful in evaluating the various Ω_A:

$$nm=\frac{A_0}{3}(\nu^2+2\nu)^{3/2}+O(\beta\nu)^{-2}\;, \tag{38}$$

$$\rho=\frac{A_0}{8}[-\cosh^{-1}(1+\nu)+(1+\nu)(\nu^2+2\nu)^{1/2}+2(1+\nu)(\nu^2+2\nu)^{3/2}]+O(\beta\nu)^{-2}\;, \tag{39}$$

$$p=\frac{A_0}{24}[\,3\cosh^{-1}(1+\nu)-3(1+\nu)(\nu^2+2\nu)^{1/2}+2(1+\nu)(\nu^2+2\nu)^{3/2}]+O(\beta\nu)^{-2}\;, \tag{40}$$

$$s=\frac{\pi^2(\nu^2+\nu)}{\beta\nu(\nu^2+2\nu)}+O(\beta\nu)^{-3}\;. \tag{41}$$

The various thermodynamic derivatives and second-order coefficients which appear in the stability criteria, the Ω_A, can be easily obtained from Eqs. (32)–(41). The results of substituting these expressions into the stability criteria [Eqs. (22)–(29)] are expansions of the Ω_A about the degenerate limit, $\beta\nu \gg 1$:

$$\Omega_1=\frac{9(1+\nu)}{A_0(\nu^2+2\nu)^{5/2}}+O(\beta\nu)^{-2}\;, \tag{42}$$

$$\Omega_2=\frac{A_0(\nu^2+2\nu)^{5/2}}{3\pi^2(1+\nu)}+O(\beta\nu)^{-2}\;, \tag{43}$$

$$\Omega_3=\frac{A_0(\nu^2+2\nu)^{3/2}}{15(1+\nu)}(5+4\nu+2\nu^2)+O(\beta\nu)^{-2}\;, \tag{44}$$

$$\Omega_4=\frac{A_0(\nu^2+2\nu)^{3/2}}{15(1+\nu)}(5+8\nu+4\nu^2)+O(\beta\nu)^{-2}\;, \tag{45}$$

$$\Omega_5=\frac{405(1+\nu)^5}{16\pi^4A_0\nu^4(\nu^2+2\nu)^{1/2}}(\beta\nu)^4+O(\beta\nu)^2\;, \tag{46}$$

$$\Omega_6=\frac{3(3+4\nu+2\nu^2)}{\pi^2A_0\nu^2(1+\nu)(\nu^2+2\nu)^{3/2}}(\beta\nu)^2+O(\beta\nu)^0\;, \tag{47}$$

$$\Omega_7=\frac{9(1+\nu)}{\pi^2A_0\nu^2(\nu^2+2\nu)^{3/2}}(\beta\nu)^2+O(\beta\nu)^0\;, \tag{48}$$

$$\Omega_8=\frac{15(1+\nu)}{2A_0(\nu^2+2\nu)^{5/2}}+O(\beta\nu)^{-2}\;. \tag{49}$$

Simple inspection of Eqs. (42)–(49) shows that the Ω_A are all positive definite in the degenerate limit, i.e., so long as $(\beta\nu)^{-1}$ is small. The free Fermi gas is thus stable in the degenerate limit within the context of the Israel-Stewart theory of dissipative relativistic fluids. Applying the theorem of Ref. 12, it then follows that all perturbations about equilibrium in a degenerate free Fermi gas obey hyperbolic equations and have characteristic velocities less than the speed of light. Note that the degenerate limit above does not restrict the Fermi gas to low temperatures: the proof applies to arbitrarily relativistic temperatures (β can take on arbitrarily small values) so long as the temperature is still much less than the Fermi temperature (and hence the product $\beta\nu$ is large).

IV. STABILITY CONSTRAINTS ON THE NUCLEAR EQUATION OF STATE

Theories describing the high-energy collisions of heavy nuclei require knowledge of the equation of state of dense and hot nuclear matter. The equations of state derived via relativistic field theory, from many-body theory with an effective interaction, or by phenomenological modeling are very different at high density and temperature.[1] Little information is currently available from experiment at these high densities and temperatures. Requiring the adiabatic sound speed to be less than the speed of light is one way to constrain the form of the equation of state.[14] In this section it will be shown that stability considerations, in the context of the Israel-Stewart fluid theory, usually provide more restrictive constraints.

The nuclear equation of state will be expressed in the form of Eq. (8), with the two independent thermodynamic variables taken to be the nucleon number density n and the temperature T. We now wish to apply the stability conditions of the Israel-Stewart fluid theory at zero temperature to constrain some of the proposed expressions for the ground-state energy. Unfortunately, the second-order thermodynamic coefficients, the α's and β's, are essentially additional equations of state which must be specified in any model; their exact form is unknown for the proposed nuclear equations of state. In principle, the α_i and β_i can either be calculated from microscopic theory, or determined experimentally. At this time, microscopic calculations only seem feasible for simple gas models. Ultimately, it would be hoped that the functional form of the second-order coefficients can be extracted from experimental heavy ion collision data. Here, for lack of any better values, the α_i and β_i functions for a free degenerate Fermi gas will be used. This is very much in the spirit of the usual approximations[1] made for the thermal energy of nuclear matter, $I(n,T)$; as very little is known about its true form also, the degenerate free Fermi gas value for $I(n,T)$ will also be used. The onus of proposing and justifying other values for the second-order coefficients should lie upon the defenders of a particular proposed equation of state.

When the ground-state energy $E_0(n)$, and the pressure p [Eq. (9)], are substituted into the stability conditions Eqs. (22)–(29), using the free degenerate Fermi gas values for the α_i, β_i, and $I(n,T)$ (which is proportional to T^2/m_0^2), and then the nonrelativistic limit $T/m_0\to 0$ is taken, the stability conditions which depend on the equation of state may be put in the form

$$m_0+E_0\geq 0\;, \tag{50}$$

$$\frac{dE_0}{dn}\geq 0\;, \tag{51}$$

$$3m+3E_0-n\frac{dE_0}{dn}\geq 0\;, \tag{52}$$

$$m_0+E_0-n^2\frac{d^2E_0}{dn^2}-\frac{7n}{3}\frac{dE_0}{dn}\geq 0\;, \tag{53}$$

$$n\frac{d^2E_0}{dn^2}+2\frac{dE_0}{dn}\geq 0\;. \tag{54}$$

The adiabatic sound speed, $v_s=[(\partial p/\partial\rho)_s]^{1/2}$ at $T=0$ is given by

$$v_s^2=\frac{n^2\dfrac{d^2E_0}{dn^2}+2n\dfrac{dE_0}{dn}}{m_0+E_0+n\dfrac{dE_0}{dn}} . \tag{55}$$

Equations (50) and (51) ensure that the energy density and pressure are positive. The positivity of Ω_3 requires that the function $3\Omega_8/2$ be bounded below by $(\rho+p)^{-1}$, which gives Eq. (52). Equation (53) is the Ω_3 constraint, and Eq. (54) is satisfied whenever the adiabatic sound speed is real.

The expressions for the ground-state energy considered here are reviewed in Ref. 1. Kapusta[21] considered the model

$$E_0(n)=\sum_{k=2}^{5} a_k\left[\frac{n}{n_0}\right]^{k/3} . \tag{56}$$

where $a_2=21.1$, $a_3=-38.3$, $a_4=-26.7$, and $a_5=35.9$, all in MeV. Another parametrization, due to Seyler and Blanchard,[22,23] has $a_2=19.88$, $a_3=-69.02$, $a_4=0$, and $a_5=33.46$, all in MeV. Sierk and Nix[24] proposed that

$$E_0(n)=\begin{cases} an^{2/3}-bn, & 0\le n\le n_a \\ E_0(n_0)+\dfrac{2K}{9}\left[\left[\dfrac{n}{n_0}\right]^{1/2}-1\right]^2, & n_a\le n \end{cases} \tag{57}$$

where $a=3h^2(3/2\pi)^{2/3}/40m_0$, $E_0(n_0)=-8$ MeV is the equilibrium ground-state energy, $K=9n_0^2(d^2E_0/dn^2)|_{n=n_0}=210$ MeV is the nuclear compressibility, and b and n_a are parameters chosen so that E_0 and dE_0/dn are continuous at $n=n_a$. Another model due to Siemens and Kapusta[25,26] has

$$E_0(n)=E_0(n_0)+\frac{K}{9}\left[\frac{n_0}{n}-1+\ln\left[\frac{n}{n_0}\right]\right] , \tag{58}$$

valid for $n\ge n_0$, where $E_0(n_0)=-16$ Mev and $K=210$ MeV. Scheid and Greiner[27] suggested the form

$$E_0(n)=E_0(n_0)+\frac{K}{18nn_0}(n-n_0)^2 , \tag{59}$$

where $E_0(n_0)=-16$ MeV and $K=210$ MeV.

The Skyrme force models SIII and SkM* give rise[28] to a ground-state energy of the form

$$E_0(n)=a_2\left[\frac{n}{n_0}\right]^{2/3}+a_3\left[\frac{n}{n_0}\right]+a_5\left[\frac{n}{n_0}\right]^{5/3}+a_6\left[\frac{n}{n_0}\right]^{\gamma+1} , \tag{60}$$

where for SIII $a_2=20.7$, $a_3=-61.4$, $a_5=6.42$, and $a_6=18.4$, all in MeV, and $\gamma=1$. For SkM* $a_2=22.1$, $a_3=-158.7$, $a_5=5.92$, and $a_6=114.9$, all in MeV, $\gamma=\frac{1}{6}$ and n_0 has the different value $n_0=0.160\ \mathrm{fm}^{-3}$.

Table I lists the range of values of $x=n/n_0$ for which the various expressions of the ground-state energy are stable, and therefore are valid descriptions of the nuclear ground state. Also listed, for comparison, is the value of x for which the adiabatic sound speed v_s exceeds the speed of light. The upper limit of stability is due to the Ω_3 constraint, and usually provides a tighter constraint than the adiabatic sound speed constraint. The improvement over the weaker adiabatic sound speed constraint is most dramatic for the Sierk-Nix and Scheid-Greiner equations of state, where the upper density limits have been reduced from $n=743n_0$ to $n=15.2n_0$ and from $n=\infty$ to $n=57.9n_0$, respectively.

The region of instability below $x=1$ is where the pressure goes negative. The existence of a region of stability for very small x allows for a liquid-gas phase transition to occur.[21]

These applications of the Israel-Stewart fluid stability conditions are intended to be illustrative of possible applications. Better values for the α_i and β_i, as well as better modeling of the thermal contribution to the internal energy, $I(n,T)$, will doubtless lead to better constraints on the various parts of the total nuclear equation of state. Still, we believe that the use of the degenerate limit of the free Fermi gas parameters is a good approximation to these functions for the present, until greater understanding of the relativistic fluid theory is achieved.

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TABLE I. Values of $x=n/n_0$ for which the given ground-state energy expression is stable, and the value of x for which the adiabatic sound speed v_s exceeds the speed of light.

Ground-State Energy	Regions of Stability	$v_s>1$
Kapusta	$0\le x\le 0.0193$, $1\le x\le 4.26$	5.80
Seyler-Blanchard	$0\le x\le 0.0045$, $1\le x\le 3.87$	5.35
Sierk-Nix	$0\le x\le 0.0384$, $1\le x\le 15.2$	743
Siemens-Kapusta	$1\le x$	
Scheid-Greiner	$1\le x\le 57.9$	
SIII	$0\le x\le 0.0069$, $1\le x\le 3.11$	3.93
SkM*	$0\le x\le 0.0014$, $1\le x\le 5.75$	9.29

APPENDIX

In Ref. 11, Israel and Stewart derive the values of the α_i and β_i for simple gases. Their final expressions, however, involve a number of integrations over four-momentum which Israel and Stewart only performed explicitly for the case of a Maxwell-Boltzmann gas. In this appendix we indicate how we have performed the integrals in the case of a strongly degenerate Fermi gas, and also note and correct several important misprints in key equations in Ref. 11. This appendix, and the calculation of the α_i and β_i, will only be intelligible to the reader if Ref. 11 is consulted for the full details. Our notation will generally follow Ref. 11, except where otherwise indicated.

The problem of determining the form of the α_i and β_i for a simple gas (Bose, Fermi, or Maxwell-Boltzmann) amounts to evaluating the integrals:

$$\mathcal{K}_n(\theta,\beta)=\frac{\beta^n}{(2n-1)!!}\int_0^\infty N\sinh^{2n}\chi\,d\chi\ , \tag{A1}$$

and

$$\mathcal{L}_{n+1}(\theta,\beta)=\frac{\beta^n}{(2n-1)!!}\int_0^\infty N\sinh^{2n}\chi\cosh\chi\,d\chi\ , \tag{A2}$$

where

$$N=[\exp(\beta\cosh\chi-\theta)+\epsilon]^{-1}\ , \tag{A3}$$

β is the inverse dimensionless temperature, $\beta=mc^2/kT$ (not to be confused with the second-order coefficients β_i), and θ is the thermodynamic potential whose gradient vanishes everywhere in an equilibrium state (this potential is denoted by α in Ref. 11—our notation here is in agreement with Refs. 12 and 13). The parameter ϵ takes on the values $+1$, 0, -1 for a Fermi, Maxwell-Boltzmann, and Bose gas, respectively. We will only consider the Fermi gas from this point on. The second-order coefficients α_i and β_i have been explicitly evaluated for the Maxwell-Boltzmann case in Ref. 11, and for the Bose gas in Ref. 29.

The integrals in Eqs. (A1) and (A2) can be well approximated in the degenerate limit by using the usual Sommerfeld expansion.[30] It is first necessary to change several variables in order to make the integrals have the usual form of the Sommerfeld expansion. Defining a new energy variable x by $x=\cosh\chi-1$, and a new thermodynamic potential ν by $\nu=\theta/\beta-1$, Eqs. (A1) and (A2) then take the form

$$\mathcal{K}_n(\nu,\beta)=\frac{\beta^n}{(2n-1)!!}\int_0^\infty\frac{(x^2+2x)^{(2n-1)/2}dx}{\exp[\beta(x-\nu)]+1}\ , \tag{A4}$$

$$\mathcal{L}_{n+1}(\nu,\beta)=\frac{\beta^n}{(2n-1)!!}\times\int_0^\infty\frac{(x^2+2x)^{(2n-1)/2}(x+1)dx}{\exp[\beta(x-\nu)]+1}\ . \tag{A5}$$

The integrals in Eqs. (A4) and (A5) are now in the usual form associated with the Fermi-Dirac distribution, and can be approximated by the asymptotic series obtained from the Sommerfeld expansion.[30] In order to obtain the leading terms in the degenerate expansion for the α_i and β_i, it is necessary to retain the first four terms in the Sommerfeld expansion for the integrals $\mathcal{K}_n$ and $\mathcal{L}_n$, as there are many cancellations of terms in proceeding from the $\mathcal{K}_n$ and $\mathcal{L}_n$ to the α_i and β_i.

The integrated expressions for the $\mathcal{K}_n$ and $\mathcal{L}_n$ may then be combined into the moments, I_{nq} and J_{nq}, according to the prescription of Ref. 11, Appendix A. There is a misprint in Ref. 11, Eq. (A25b); the equation should read

$$I_{n+1,q}=A_0\sum_{r=0}^{(1/2)n-q}b_{nqr}\beta^{-(q+r+1)}L_{q+r+2}\ , \tag{A6}$$

where the b_{nqr}, and the definitions of the other I_{nq} and J_{nq} are given in Appendix A of Ref. 11. There also appears to be a misprint in the definition of A_0, following Eq. (A17) of Ref. 11. The corrected definition is given in Eq. (37) of our Sec. III. Once the moments are known, the α_i and β_i may be calculated using Eqs. (7.4)–(7.8) in Ref. 11. It is important to note a misprint in Eq. (7.8a) in Ref. 11, the definition of α_0; as printed, the equation for α_0 is easily seen to not even be dimensionally correct. The corrected equation, defining α_0 in terms of the moments, should read

$$\alpha_0=(D_{41}D_{20}-D_{31}D_{30})(\Lambda\zeta\Omega J_{21}J_{31}D_{20})^{-1}\ , \tag{A7}$$

where the D_{nq}, Λ, ζ, and Ω are defined in Ref. 11.

The resulting expressions for the second-order thermodynamic coefficients, α_i and β_i, for the strongly degenerate Fermi gas, are given above in Sec. III.

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