Gaussian Beams

General solutions to the paraxial wave equations can be written as linear combinations of a discrete set of functions called Hermite-Gaussian modes.

Hermite-Gaussian modes are labeled by two indices, l and m. A family of Hermite-Gauss modes exists for any choice $w_0 > 0$ of the parameter w_0.

To construct a basis for solutions of the paraxial wave equation, we choose some $w_0 > 0$. Our Hermite-Gaussian modes are then of the forms

$$u_{l,m}(x,y,z) = u_0 \frac{w_0}{w(z)} H_l\left(\frac{\sqrt{2}x}{w(z)}\right) H_m\left(\frac{\sqrt{2}y}{w(z)}\right) \times$$

$$\times \exp\left(-\frac{x^2+y^2}{w(z)^2}\right) \exp\left(-i\Psi(z)\right) \exp\left(ik\frac{x^2+y^2}{2R(z)}\right)$$

For l, m integers ≥ 0 and:

$$w(z) \equiv w_0 \sqrt{1 + \left(\frac{z}{z_R}\right)^2}$$

$$z_R \equiv \frac{\pi w_0^2}{\lambda} \quad \left(\lambda = \frac{2\pi}{k}\right)$$

$$R(z) \equiv z\left(1 + \left(\frac{z_R}{z}\right)^2\right)$$

$$\Psi(z) = \arctan\left(\frac{z}{z_R}\right) \cdot (l+m+1)$$

H_i is the ith Hermite polynomial, common special functions that also show up in the solutions of the quantum harmonic oscillator. The first few Hermite polynomials are

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

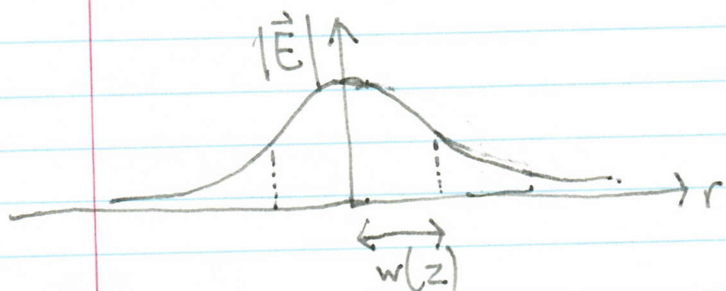
$$H_2(x) = 4x^2 - 2$$

Many laser beams are, in practice, well approximated by solutions of the form $u_{0,0}(x, y, z)$, for $l = 0$ and $m = 0$. The $l=0$, $m=0$ solution is called a Gaussian beam and has the form

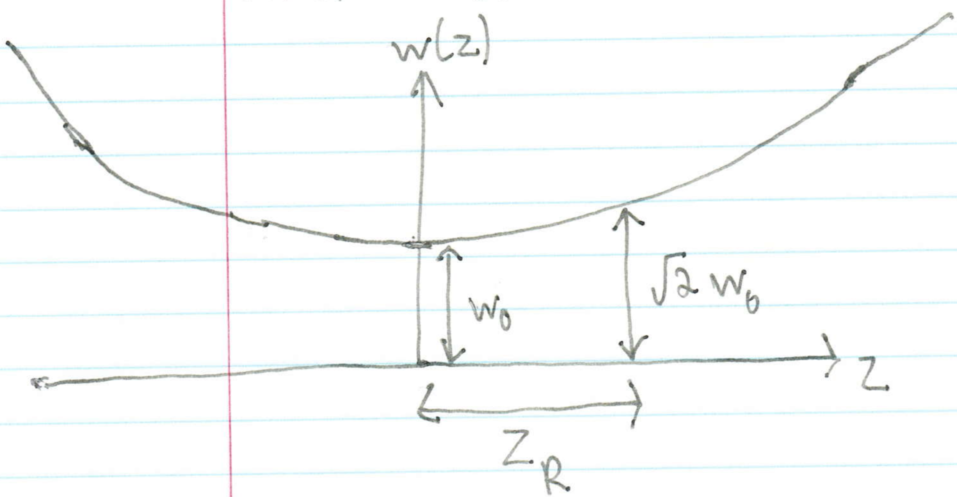
$$u_{0,0}(x, y, z) = u_0 \frac{w_0}{w(z)} \exp\left(-\frac{r^2}{w(z)^2}\right) \exp\left(-i\Psi(z)\right) \exp\left(ik\frac{r^2}{2R(z)}\right)$$

for $r^2 = x^2 + y^2$. Note that the Gaussian beam has cylindrical symmetry about the z-axis.

$\vec{E}$ field amplitude has Gaussian profile:



Size $w(z)$ of the beam looks like this as a function of z:



$z_R = \frac{\pi w_0^2}{\lambda}$ is called the Rayleigh range of the beam. At $z = \pm z_R$, $w(z)$ has expanded by a factor of $\sqrt{2}$.

Behavior of $w(z)$ in two different regimes:

$z \ll z_R$

$$w(z) \approx \left(1 + \frac{z^2}{2 z_R^2}\right) w_0$$

For z well within the Rayleigh range z_R, the beam size $w(z)$ stays approximately constant, slowly growing as the term $\frac{z^2}{2 z_R^2} w_0$.

$z \gg z_R$

$$w(z) \approx w_0 \frac{z}{z_R}$$

$w(z)$ grows linearly in z.

We will come back to the phase factors and their physical significance.

Fourier Optics

Idea: If we know $u(\vec{r})$ for some value of z, we want to be able to determine how $u(\vec{r})$ evolves as z changes. Let us choose coordinates so that we know $u(\vec{r})$ for $z=0$, which we express as $u(x,y,0)$. We can Fourier transform $u(x,y,0)$ to

write it as a superposition of plane waves. We can then solve for how each plane evolves with z. Finally, we can perform an inverse Fourier transform to determine $u(x,y,z)$.

Paraxial propagation of a plane wave

General plane wave solution:

$$U_{\vec{k}}(\vec{r},t) = e^{i(\vec{k}\cdot\vec{r}-wt)} = e^{i(k_x x + k_y y + k_z z - wt)}$$

$$k = \frac{w}{c} = \sqrt{k_x^2 + k_y^2 + k_z^2}$$

$$\Rightarrow k_z = \sqrt{k^2 - (k_x^2 + k_y^2)}$$

Recall that we carried out the factorization

$$U(\vec{r},t) = u(\vec{r})\, e^{i(kz - wt)}$$

$$u_{\vec{k}}(\vec{r}) = e^{i(k_x x + k_y y)}\, e^{i(k_z - k)z}$$

To obtain the paraxial wave equation, we made the assumption that $\frac{\partial u}{\partial z} \ll ku$.

Is this approximation valid for $u_{\vec{k}}(\vec{r})$? Under what assumptions is it valid?

Let us check:

$$\frac{\partial u_{\vec{k}}}{\partial z} = i(k_z - k)\vec{u}_k$$

To have $\frac{\partial u_{\vec{k}}}{\partial z} \ll k\vec{u}_{\vec{k}}$, we need:

$$|k_z - k| \ll k$$

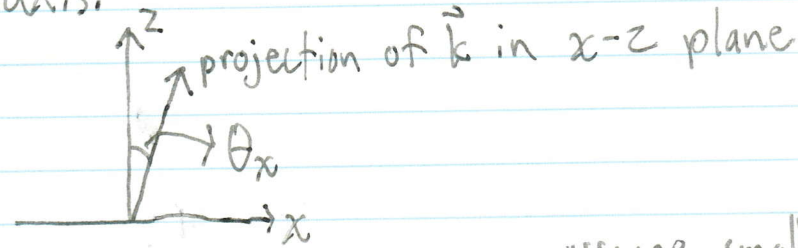
$$\Rightarrow k - \sqrt{k^2 - (k_x^2 + k_y^2)} \ll k$$

$$\Rightarrow k\left(1 - \sqrt{1 - \frac{(k_x^2 + k_y^2)}{k^2}}\right) \ll k$$

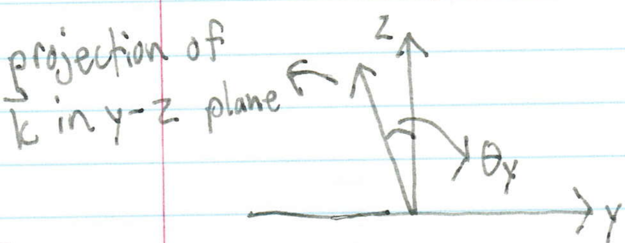
$$\Rightarrow 1 - \sqrt{1 - \frac{(k_x^2 + k_y^2)}{k^2}} \ll 1$$

This condition will hold if $\frac{k_x^2}{k^2} + \frac{k_y^2}{k^2} \ll 1$.

Let us define the angles θ_x and θ_y that tell us the angular deviation of $\vec{k}$ away from the z axis.



assume small angles



$k_x \approx k\theta_x$, $k_y \approx k\theta_y$ for small angles

For the paraxial wave equation to hold, $k_x, k_y \ll k$, so $\theta_x, \theta_y \ll 1$.

Par → parallel

axial → to the axis

approximation

The paraxial approximation is valid for laser beams that contain Fourier components $u_{\vec{k}}$ that are close to parallel with the main axis of propagation (we have chosen coordinates so that this is the z axis).

If we consider plane wave components for which $\frac{k_x^2}{k^2} + \frac{k_y^2}{k^2} \ll 1$ is a valid approximation,

then we can make the Taylor expansion

$$k_z = \sqrt{k^2 - (k_x^2 + k_y^2)} \approx k - \frac{k}{2}\left(\frac{k_x^2}{k^2} + \frac{k_y^2}{k^2}\right)$$

We then have $k_z - k = -\frac{k}{2}\left(\frac{k_x^2 + k_y^2}{k^2}\right)$, so

$$u_{\vec{k}}(x, y, z) = e^{i(k_x x + k_y y)} e^{-i\left(\frac{k_x^2 + k_y^2}{2k}\right) z}$$

Check that $u_{\vec{k}}$ satisfies paraxial wave equation:

$$\frac{\partial^2 u_{\vec{k}}}{\partial x^2} = -k_x^2 u_{\vec{k}}$$

$$\frac{\partial^2 u_{\vec{k}}}{\partial y^2} = -k_y^2 u_{\vec{k}}$$

$$2ik \frac{\partial u_{\vec{k}}}{\partial z} = 2ik\left(-i \frac{k_x^2 + k_y^2}{2k}\right) u_{\vec{k}} = (k_x^2 + k_y^2) u_{\vec{k}}$$

so $\frac{\partial^2 u_{\vec{k}}}{\partial x^2} + \frac{\partial^2 u_{\vec{k}}}{\partial y^2} + 2ik \frac{\partial u_{\vec{k}}}{\partial z} = 0$ indeed holds. ✓

Fourier beam propagation:

Say that we know $u(\vec{r})$ for $z=0$: $u(x,y,0)$

We can take a Fourier transform in the x and y directions:

$$\tilde{u}(k_x,k_y,z=0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u(x,y,0)\,e^{-i(k_x x+k_y y)}\,dx\,dy$$

The Fourier components $\tilde{u}(k_x,k_y,z=0)$ represent coefficients in a plane wave decomposition of $u(x,y,0)$:

inverse Fourier transform

$$u(x,y,0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{u}(k_x,k_y,z=0)\,e^{i(k_x x+k_y y)}\,dk_x\,dk_y$$

$\tilde{u}(k_x,k_y,z=0)$ is the coefficient of the plane wave $e^{i(k_x x+k_y y)}$.

How do we add z dependence? We've just seen how the z dependence comes in for a plane wave with x-component k_x and y-component k_y:

$$e^{i(k_x x+k_y y)}\longrightarrow e^{i(k_x x+k_y y)}\,e^{-i\left(\frac{k_x^2+k_y^2}{2k}\right)z}$$

So $\tilde{u}(k_x,k_y,z)=e^{-i\left(\frac{k_x^2+k_y^2}{2k}\right)z}\,\tilde{u}(k_x,k_y,z=0)$.