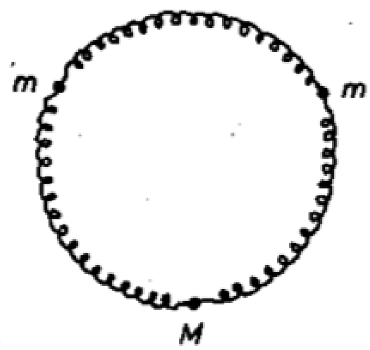
PHYS 411, Fall 2014
Final Exam
Thurs. Dec. 11, 3pm

1: [10 pts] Consider the following system:



The masses are confined to a horizontal circle of radius r and the 3 springs have the same spring constant k and equal length in equilibrium.

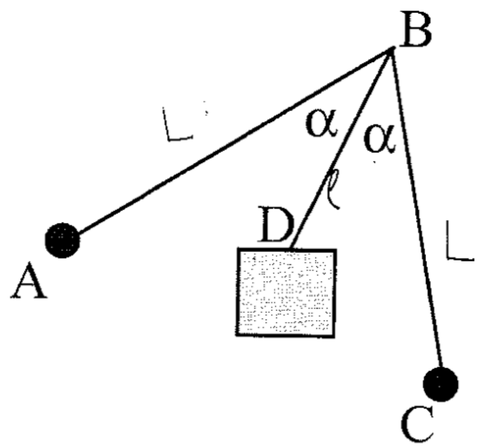
Find the T and V matrices for small perturbations about equilibrium. (You do not need to solve the system)

$$\boldsymbol{T} = r^2 \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix} \tag{1}$$

$$V = \frac{kr^2}{2}\left((\theta_1-\theta_2)^2 + (\theta_2-\theta_3)^2 + (\theta_3-\theta_1)^2\right) \tag{2}$$

$$\boldsymbol{V} = kr^2 \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \tag{3}$$

2: [25 pts] A rigid structure consists of three massless rods joined at B and two point masses (mass m) attached at A and C:



The lengths and angles are as shown (L, l, α). The rigid system is supported at the fixed point D and rocks back and forth with a small amplitude of oscillation. Motion is confined to a vertical plane, and gravity acts downwards.

(a) What is the Lagrangian? (Hint: it might be helpful to first use the distance DC as a parameter, before re-expressing that in terms of given quantities)

(b) What is the frequency of oscillation?

(c) What is the criterion on l for stable oscillations to be possible?

(a) We have

$$\begin{aligned} T &= mb^2\dot{\theta}^2 && (4) \\ V &= mgb(\cos(\beta+\theta)+\cos(\beta-\theta)) && (5) \\ &= 2mgb\cos\beta\cos\theta && (6) \end{aligned}$$

after defining the hinted distance:

$$b = \sqrt{L^2 + l^2 - 2Ll\cos\alpha} \tag{7}$$

and the angle β as BDC, i.e.

$$b\cos\beta + L\cos\alpha = l \tag{8}$$

So, the Lagrangian is (dividing through by $2mb^2$)

$$L = \frac{1}{2}\dot{\theta}^2 - \frac{g}{b}\cos\beta\cos\theta \tag{9}$$

(b)

$$\ddot{\theta} = \frac{g}{b}\cos\beta\sin\theta \tag{10}$$

$$\omega^2 = -\frac{g}{b^2}(l - L\cos\alpha) \tag{11}$$

(c)

$$l < L\cos\alpha \tag{12}$$

3: [25 pts]

(a) What is the Lagrangian for a symmetrical top that is placed on a horizontal frictionless table in a gravitational field. Take your co-ordinates to be the Euler angles (θ, ϕ, ψ) and the horizontal co-ordinates of the top's center of mass (x, y), and define clearly any quantities you use that do not appear in this question.

Hint: Think about how this system differs from a top with one point fixed, which we showed in class has the following Lagrangian

$$L = \frac{I_1}{2}(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{I_3}{2}(\dot{\phi}\cos\theta + \dot{\psi})^2 - mgl_{cm}\cos\theta$$

where I_1 and I_3 are the principle moments of inertia about the fixed point and l_{cm} is the distance of the center of mass from the fixed point.

(b) You should find 4 cyclic co-ordinates. What are the physical interpretations of the corresponding conjugate momenta?

(c) Is the problem integrable? Explain your reasoning

(a) Note that the height of the center of mass is $l_{cm}\cos\theta$. So,

$$T_{cm} = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + l_{cm}^2\dot{\theta}^2\sin^2\theta\right) \tag{13}$$

$$L = \frac{1}{2}\left(I_1(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + I_3'(\dot{\phi}\cos\theta + \dot{\psi})^2\right) - mgl_{cm}\cos\theta + \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + l_{cm}^2\sin^2\theta\dot{\theta}^2) \tag{14}$$

where

$$I_3' = I_3 - ml_{cm}^2 \tag{15}$$

(b) p_x is x-momentum; p_y is the y-momenum; $p_\phi = L_z$, $p_\psi = L_3$.

(c) Yes. 5 degrees of freedom, and 4 constant momena plus constant energy. So 5 conserved quantities.

4: [15 pts] Consider a test particle that feels a rotating potential of the form $V(r, \theta - \alpha t)$, where α is constant, and r and θ are polar co-ordinates. Motion is confined to the $r - \theta$ plane, and you may set the test particle's mass $m = 1$. Define all quantities you introduce clearly.

(a) What is the Hamiltonian?

(b) Find a canonical transformation that makes the new (transformed) Hamiltonian independent of time. What is the new Hamiltonian?

(c) Use your result from part (b) to determine a conserved quantity in terms of $\{r, \dot{r}, \theta, \dot{\theta}, \alpha\}$.
(a)

$$L = \frac{\dot{r}^2}{2} + \frac{r^2\dot{\theta}^2}{2} - V(r, \theta - \alpha t) \tag{16}$$
$$p_r = \dot{r} \tag{17}$$
$$p_\theta = r^2\dot{\theta} \tag{18}$$
$$H = \frac{p_r^2}{2} + \frac{p_\theta^2}{2r^2} + V(r, \theta - \alpha t) \tag{19}$$

(b)

$$F_2 = P_1 r + P_2(\theta - \alpha t) \tag{20}$$
$$R = r \tag{21}$$
$$\Theta = \theta - \alpha t \tag{22}$$
$$p_r = P_1 \tag{23}$$
$$p_\theta = P_2 \tag{24}$$
$$K = \frac{P_1^2}{2} + \frac{P_2^2}{2R^2} + V(R, \Theta) - \alpha P_2 \tag{25}$$

(c)

$$K = \frac{\dot{r}^2}{2} + \frac{r^2\dot{\theta}^2}{2} - \alpha r^2\dot{\theta} + V(r, \theta - \alpha t) \tag{26}$$

5: [25 pts]

(a) Show using Poisson brackets that the following transformation for a system of one degree of freedom is canonical:

$$\begin{aligned} Q &= q\cos\alpha - p\sin\alpha && (27)\\ P &= q\sin\alpha + p\cos\alpha\ , && (28) \end{aligned}$$

where α is constant.

(b) Find a generating function of the form $F_2(q, P)$ that generates the above transformation to linear order in α.

(c) If the Hamiltonian is invariant under the transformation of equations (27)–(28), what function of q and p is conserved? Use your result from part (b).

Solution

(a) $[Q, Q] = 0$ and $[P, P] = 0$ and

$$[Q, P] = \cos^2\alpha + \sin^2\alpha = 1 \qquad (29)$$

$$\qquad (30)$$

(b) We want transformation

$$Q = q - p\alpha \qquad (31)$$

$$P = q\alpha + p \qquad (32)$$

In other words, expressing things in terms of q and P:

$$Q = q - P\alpha = \frac{\partial F_2}{\partial P} \qquad (33)$$

$$p = P - q\alpha = \frac{\partial F_2}{\partial q} \qquad (34)$$

So:

$$F_2 = qP - \frac{\alpha}{2}(P^2 + q^2) \qquad (35)$$

(c) To linear order in α,

$$F_2 = qP - \frac{\alpha}{2}(p^2 + q^2) \qquad (36)$$

Therefore the infinitesimal generating function is $G = p^2 + q^2$. Therefore $p^2 + q^2$ is conserved.