# Orbital Angular Momentum

$\vec{L}$ is a special case of a general angular momentum

$$\therefore L_z|l\,m\rangle = \hbar m|l\,m\rangle$$
$$\vec{L}^2|l\,m\rangle = \hbar^2 l(l+1)|l\,m\rangle$$

## Try a position representation

$$|\psi\rangle = \int d^3r\ |\vec{r}\rangle\underbrace{\langle\vec{r}|\psi\rangle}$$

$$\psi(\vec{r}) = \langle\vec{r}|\psi\rangle = \langle r\,\theta\,\phi|\psi\rangle$$

spherical coordinates

$$\vec{L} = \vec{r}\times\underbrace{\vec{p}}_{-i\hbar\vec{\nabla}}$$

$$\Longrightarrow\ L_z = -i\hbar\frac{\partial}{\partial\phi}$$

$$L^2 = \underbrace{-\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]}$$

$\langle\vec{r}|\psi\rangle = R(r)\,Y(\theta,\phi)$ ⟵

- no r dependence
- recognize $\vec{\nabla}^2$

$$\vec{\nabla}^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\vec{L}^2}{\hbar^2 r^2}$$

## Eigenvalue problems

$\boxed{L_z}$ $\quad L_z|lm\rangle = \hbar m|lm\rangle$

$$\langle\vec{r}|L_z|lm\rangle = \hbar m\langle\vec{r}|lm\rangle$$

$$-i\cancel{\hbar}\frac{\partial}{\partial\phi}\cancel{R(r)}Y(\theta\phi) = \cancel{\hbar}m\cancel{R(r)}Y(\theta,\phi)$$

$$\frac{\partial Y(\theta,\phi)}{\partial\phi} = im\,Y(\theta,\phi) \rightsquigarrow Y(\theta,\phi) = P_{lm}(\theta)\,e^{im\phi}$$

$$\boxed{L^2} \quad -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\underbrace{\frac{\partial^2}{\partial\phi^2}}_{(im)^2}\right]P(\theta)e^{im\phi}=\hbar^2\ell(\ell+1)P(\theta)e^{im\phi}$$

$$\boxed{\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\left\{\ell(\ell+1)-\frac{m^2}{\sin^2\theta}\right\}\right]P_{\ell m}(\theta)=0}$$

Legendre equation: avoid infinities only if

∴ half-integer values have no position rep. $\left\{ \begin{array}{l} \ell=0,1,2,\ldots \\ m=0,\pm1,\ldots,\pm\ell \end{array} \right\}$ i.e. just the values from gen. rot. conditions ☺

$P_{\ell m}(\theta) \Rightarrow$ associated Legendre functions

$Y_{\ell m}(\theta,\phi) \Longrightarrow$ spherical harmonics

normalization: $\boxed{\int d\Omega\, Y^*_{\ell' m'}(\theta,\phi)\, Y_{\ell m}(\theta,\phi) = \delta_{\ell\ell'}\,\delta_{mm'}}$

$$\langle \ell' m' | \ell m\rangle = \delta_{\ell\ell'}\,\delta_{mm'}$$

Undergrad. approach

$\vec{L}=\vec{r}\times\vec{p} \Longrightarrow$ dif. equations $\Rightarrow$ only physical solutions (no infin.)

$\ell=0,1,\ldots$

$m=0,\pm1,\ldots,\pm\ell$

This approach:

Gen. of rotations $\Longrightarrow j=0,\frac{1}{2},1,\ldots$

$m=-j,-j+1,\ldots,j$

Try a pos. rep. $\Longrightarrow$ works only for interger $j$

## Explicit functions of $\theta, \phi$

$$m \geq 0: \quad Y_{\ell m}(\theta, \phi) = \frac{(-1)^\ell}{2^\ell \ell!} \sqrt{\frac{(2\ell+1)}{4\pi} \frac{(\ell+m)!}{(\ell-m)!}} \, e^{im\phi} \frac{1}{\sin^m\theta} \frac{d^{\ell-m}}{d(\cos\theta)^{\ell-m}} (\sin\theta)^{2\ell}$$

$$m < 0: \quad Y_{\ell -m}(\theta, \phi) = (-1)^m Y^*_{\ell m}(\theta, \phi)$$

$$Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{i\phi} = -\sqrt{\frac{3}{8\pi}} \frac{x+iy}{r}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta = \sqrt{\frac{3}{4\pi}} \frac{z}{r}$$

$$Y_{1,-1} = \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{-i\phi} = \sqrt{\frac{3}{8\pi}} \frac{x-iy}{r}$$

## Relation to Legendre:

$$Y_{\ell m} = (-1)^m \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} \, P_{\ell m}(\theta) e^{im\phi}$$

$$Y_{\ell 0} = \sqrt{\frac{2\ell+1}{4\pi}} \, P_\ell(\cos\theta)$$

special case: $\theta = 0$

$$Y_{\ell m}(\theta=0, \phi) = (-1)^m \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} \, \underbrace{P_{\ell m}(1)}_{} e^{im\phi}$$

$$\boxed{(-1)^\ell \delta_{m0} \quad \begin{cases} (-1)^\ell & \text{if } m=0 \\ 0 & \text{if } m \neq 0 \end{cases}}$$

$$= \sqrt{\frac{2\ell+1}{4\pi}} \, \delta_{m0}$$

# $\boxed{\text{Parity}}$ $\qquad \vec{x} \longrightarrow -\vec{x}$

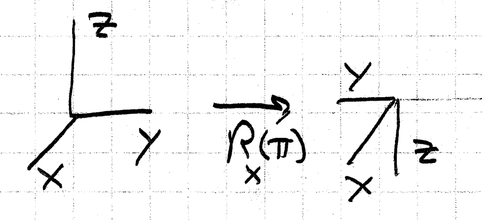
$P \qquad x, y, z \longrightarrow -x, -y, -z$

<u>Compare to reflection in a mirror</u> $\qquad x \to -x$

$$x, y, z \longrightarrow -x, y, z$$

$$\downarrow R_x(\pi)$$

$$x, y, z \xrightarrow{P} -x, -y, -z$$




$$y, z \longrightarrow -y, -z$$

$$\therefore\ P = \left\{ \begin{matrix} \text{rot.} \\ \text{by } \pi \end{matrix} \right\} \left\{ x \to -x \right\}$$

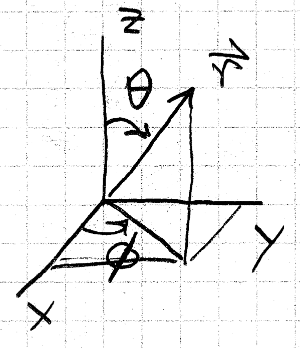
equivalent if a rotation by $\pi$ does not change the system

<u>Note:</u> Parity $\neq$ Rotation of any kind !

↳ <u>an intrinsic property of the system</u>

## Parity — If there is a position representation

$\vec{r} \rightarrow -\vec{r}$




$\theta \rightarrow \pi - \theta$

$\phi \rightarrow \phi + \pi$

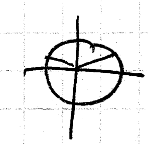
$$Y_{\ell m}(\theta,\phi) \longrightarrow Y_{\ell m}(\pi-\theta, \phi+\pi)$$

Use: $Y_{\ell m} \sim e^{im\phi} \dfrac{1}{\sin^m\theta} \dfrac{d^{\ell-m} (\sin\theta)^{2\ell}}{d(\cos\theta)^{\ell-m}}$

$$e^{im(\phi+\pi)} = e^{im\phi} e^{im\pi}$$
$$= e^{im\phi}(\underbrace{\cos m\pi}_{(-1)^m} + i \cancel{\sin m\pi})$$
$$= (-1)^m e^{im\phi}$$

$\sin(\pi-\theta) = \sin\theta$

$\cos(\pi-\theta) = -\cos\theta$



$[\cos(\pi-\theta)]^{\ell-m} = (-1)^{\ell-m}(\cos\theta)^{\ell-m}$

$\therefore (-1)^{\ell-m}$

$$Y_{\ell m}(\pi-\theta, \phi+\pi) = \cancel{(-1)^m}(-1)^{\ell-\cancel{m}} Y_{\ell m}(\theta,\phi)$$

$$\boxed{Y_{\ell m}(\pi-\theta, \phi+\pi) = (-1)^\ell Y_{\ell m}(\theta,\phi)}$$

$$\therefore P|\ell m\rangle = (-1)^\ell |\ell m\rangle$$

## Relation to Rotation Operators

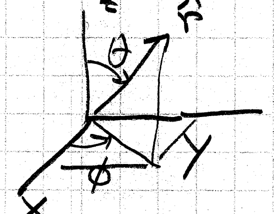
$$Y_{\ell m}(\theta,\phi) = \langle \hat{r} | \ell m \rangle = \langle \theta,\phi | \ell m \rangle$$

For $\hat{z} \Longrightarrow \theta = 0,\ \phi =$ any thing

$$Y_{\ell m}(0,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}\,\delta_{m0}$$

Rotate $|\hat{z}\rangle$ to become $|\hat{r}\rangle$

details coming in next section




$$R(\alpha\beta\gamma) = R_z(\alpha)\,R_y(\beta)\,R_z(\gamma)$$

$$\alpha \to \phi \qquad \beta \to \theta \qquad \gamma \to 0$$

$$|\hat{r}\rangle = \mathcal{D}(\phi,\theta,0)\,|\hat{z}\rangle$$

$$|\theta,\phi\rangle = \mathcal{D}(\phi,\theta,0)\,|0,0\rangle$$

could choose any $\phi$

$$\langle \ell m | \hat{r} \rangle = \langle \ell m | \mathcal{D}(\phi,\theta,0) \overbrace{|0,0\rangle}^{|\hat{z}\rangle}$$

$$Y^*_{\ell m}(\theta,\phi) = \sum_{m'} \langle \ell m | \mathcal{D}(\phi,\theta,0) | \ell m' \rangle \underbrace{\langle \ell m' | 0,0 \rangle}$$

$$Y^*_{\ell m'}(0,0) = \sqrt{\frac{2\ell+1}{4\pi}}\,\delta_{m'0}$$

$$Y_{\ell m}(\theta,\phi) = \langle \ell m | \mathcal{D}(\phi,\theta,0) | \ell 0 \rangle \sqrt{\frac{2\ell+1}{4\pi}}$$

$$\boxed{\mathcal{D}^{(\ell)}_{m0}(\phi,\theta,0) = \sqrt{\frac{4\pi}{2\ell+1}}\,Y^*_{\ell m}(\theta,\phi)}$$

Special case of $m=0$:

$$\underbrace{\mathcal{D}^{(\ell)}_{00}(\phi,\theta,0)}_{e^{-i(\phi\cdot 0+0\cdot 0)}d^{(\ell)}_{00}(\theta)} = \cancel{\sqrt{\frac{4\pi}{2\ell+1}}}\,\underbrace{Y^*_{\ell 0}(\theta,\phi)}_{\cancel{\sqrt{\frac{2\ell+1}{4\pi}}}\,P^*_\ell(\cos\theta)}$$

real

$$\boxed{d^{(\ell)}_{00}(\theta) = P_\ell(\cos\theta)}$$