$\boxed{\text{Coupling of Angular Momenta}}$

eg. H atom $\quad \underbrace{|l\, m_l\rangle}_{\substack{\text{electron}\\ \text{orbital}}} \underbrace{|s\, m_s\rangle}_{\substack{\text{electron}\\ \text{spin}}} \underbrace{|i\, m_i\rangle}_{\substack{\text{nuclear}\\ \text{spin}}}$

eg. 2 Electrons in lowest orbital state of He atom

$$|s=\tfrac{1}{2}\, m_s\rangle\, |s=\tfrac{1}{2}\, m_s'\rangle$$

Matrix elements:

$$\langle l m_l| \langle s m_s| \langle i m_i| f(\vec{L}) + g(\vec{S}) + h(\vec{I}) |l' m_l'\rangle |s' m_s'\rangle |i' m_i\rangle$$

$$= \langle l m_l| f(\vec{L}) | l' m_l'\rangle\, \delta_{ss'}\, \delta_{m_s m_s'}\, \delta_{m_i m_i'}$$

$$+ \langle s m_s| g(\vec{S}) | s' m_s'\rangle\, \delta_{ll'}\, \delta_{m_l m_l'}\, \delta_{m_i m_i'}$$

$$+ \langle i m_i| h(\vec{I}) | i m_i'\rangle\, \delta_{ll'}\, \delta_{m_l m_l'}\, \delta_{ss'}\, \delta_{m_s m_s'}$$

Total angular momentum

$\vec{F} = \underbrace{\vec{L} + \vec{S}}\, + \vec{I} \quad \rightsquigarrow$ Means $\vec{L}\otimes\overleftrightarrow{1}\otimes\overleftrightarrow{1} + \overleftrightarrow{1}\otimes\vec{S}\otimes\overleftrightarrow{1} + \overleftrightarrow{1}\otimes\overleftrightarrow{1}\otimes\vec{I}$

$$\boxed{\vec{J} \equiv \vec{L} + \vec{S}}$$

$$\boxed{\vec{F} \equiv \vec{J} + \vec{I}}$$

Generic case: $\vec{J} = \vec{J}_1 + \vec{J}_2$

operate in different subspaces

rotation: $\mathcal{D} = \mathcal{D}_1 \otimes \mathcal{D}_2$

$$= \left[\overleftrightarrow{1} - \frac{i\epsilon}{\hbar}\hat{n}\cdot\vec{J}_1\right] \otimes \left[\overleftrightarrow{1} - \frac{i\epsilon}{\hbar}\hat{n}\cdot\vec{J}_2\right]$$

$$= \overleftrightarrow{1} - \frac{i\epsilon}{\hbar}\left[\vec{J}_1 \otimes \overleftrightarrow{1} + \overleftrightarrow{1} \otimes \vec{J}_2\right]$$

rotates state vector in space $|j_1\, m_1\rangle$

no rotation in space $|j_2\, m_2\rangle$

2 spaces

<u>Uncoupled base states:</u> $|j_1 m_1\, j_2 m_2\rangle = |j_1 m_1\rangle |j_2 m_2\rangle$

various notations $\left\{ \begin{array}{l} \underbrace{|j_1 m_1\, j_2 m_2\rangle} \\ |(j_1 j_2)\, m_1 m_2\rangle \\ |j_1 j_2 ; m_1 m_2\rangle \end{array} \right.$

$$J_1^2 |j_1 m_1\, j_2 m_2\rangle = \hbar^2 j_1(j_1+1) |j_1 m_1\, j_2 m_2\rangle$$

$$J_{1z} |\quad\rangle = \hbar m_1 |\quad\rangle$$

$$J_2^2 |\quad\rangle = \hbar^2 j_2(j_2+1) |j_1 m_1\, j_2 m_2\rangle$$

$$J_{2z} |\quad\rangle = \hbar m_2 |\quad\rangle$$

<u>Total angular momentum:</u> $\vec{J} \equiv \vec{J}_1 + \vec{J}_2$

$$J_z |j_1 m_1\, j_2 m_2\rangle = (J_{1z} + J_{2z}) |j_1 m_1\, j_2 m_2\rangle$$

$$= J_{1z}|j_1 m_1\rangle |j_2 m_2\rangle + |j_1 m_1\rangle J_{2z} |j_2 m_2\rangle$$

$$= \hbar(m_1 + m_2) |j_1 m_1\, j_2 m_2\rangle$$

$$\vec{J}^2 = (\vec{J}_1 + \vec{J}_2)^2 = \vec{J}_1^2 + \vec{J}_2^2 + 2\underbrace{\vec{J}_1 \cdot \vec{J}_2}$$

clearly commute

$\Rightarrow$ clearly $|j_1 m_1\, j_2 m_2\rangle$ is not an eigenstate

<u>Commutators:</u>

$$[\vec{J}^2, \vec{J}_1^2] = [\vec{J}_1^2 + \vec{J}_2^2 + 2\vec{J}_1 \cdot \vec{J}_2, \vec{J}_1^2]$$

clearly commute

$[\vec{J}_1^2, J_{1j}] = 0 \rightarrow$

$$= 2[J_{1x}J_{2x} + J_{1y}J_{2y} + J_{1z}J_{2z}, \vec{J}_1^2]$$

$$= 0$$

$\therefore [\vec{J}^2, \vec{J}_2^2] = 0$ (same argument)

But $\left. \begin{array}{l} [\vec{J}^2, J_{1z}] \neq 0 \\ [\vec{J}^2, J_{2z}] \neq 0 \end{array} \right\} \sim [J_{1x} J_{2x}, J_{1z}] \neq 0$

Also: $[\vec{J}_1^{\,2}, J_z] = [\vec{J}_1^{\,2}, J_{1z} + J_{2z}]$
$= 0$ clearly

$[\vec{J}_2^{\,2}, J_z] = 0$ similarly

$\therefore$ Simultaneous eigenkets of $\vec{J}^2, \vec{J}_1^{\,2}, \vec{J}_2^{\,2}, J_z$ are possible

$$\Rightarrow \; |(j_1 j_2)\, j\, m\rangle \quad \left\{ \begin{array}{l} \vec{J}^2 |(j_1 j_2) jm\rangle = \hbar^2 j(j+1) |(j_1 j_2) jm\rangle \\ J_z |(j_1 j_2) jm\rangle = \hbar m |(j_1 j_2) jm\rangle \\ J_1^2 |(j_1 j_2) jm\rangle = \hbar^2 j_1(j_1+1) |(j_1 j_2) jm\rangle \\ J_2^2 |(j_1 j_2) jm\rangle = \hbar^2 j_2(j_2+1) |(j_1 j_2) jm\rangle \end{array} \right.$$

coupled eigenkets

Expand the coupled eigenkets in terms of the uncoupled eigenkets

$$\boxed{|(j_1 j_2) jm\rangle = \sum_{m_1 m_2} |j_1 m_1\, j_2 m_2\rangle \underbrace{\langle j_1 m_1\, j_2 m_2 | (j_1 j_2) jm\rangle}_{\text{Clebsch-Gordon coefficients}}}$$

Relation between m, m_1 and m_2:

$$J_z = J_{1z} + J_{2z} \longrightarrow 0 = J_z - J_{1z} - J_{2z}$$

$$\therefore 0 = \langle j_1 m_1\, j_2 m_2 | (J_z - J_{1z} - J_{2z}) |(j_1 j_2) jm\rangle$$

$$= (m - m_1 - m_2) \langle j_1 m_1\, j_2 m_2 | (j_1 j_2) jm\rangle$$

$$\therefore \boxed{\begin{array}{c} \langle j_1 m_1\, j_2 m_2 | (j_1 j_2) jm\rangle = 0 \\ \text{unless } m = m_1 + m_2 \end{array}}$$

Restrictions on j

$$m = m_1 + m_2$$

$$m_{max} = m_{1\,max} + m_{2\,max}$$

$$= j_1 + j_2 \longrightarrow \therefore \underline{\underline{\max j = j_1 + j_2}}$$

(since $m = 0, \pm 1, \ldots, \pm j$)

$$\boxed{|(j_1 j_2)\; j = j_1 + j_2 \;\; m = j_1 + j_2\rangle = |j_1 \; m_1 = j_1 \;\; j_2 \; m_2 = j_2\rangle}$$

choose phase 1
(universal choice)

$$\therefore \boxed{\langle j_1 \; m_1 = j_1 \;\; j_2 \; m_2 = j_2 | (j_1 j_2) \; j = j_1 + j_2 \;\; m = j_1 + j_2\rangle = 1}$$

To get the other Clebsch-Gordon coefficients

Use $J_- = J_{1-} + J_{2-}$ ← lowering op.

$$J_- |(j_1 j_2)\; j = j_1 + j_2 \;\; m = j_1 + j_2\rangle = (J_{1-} + J_{2-}) |j_1 \; m_1 = j_1 \;\; j_2 \; m_2 = j_2\rangle$$

etc.

<u>Note</u>: all C.G. coefficients are real

$$\therefore \langle j_1 m_1 \, j_2 m_2 | (j_1 j_2) j m\rangle = \langle (j_1 j_2) j m | j_1 m_1 \, j_2 m_2\rangle$$

i.e order does not matter

Check the number of states in the two bases

Uncoupled base: $(2j_1+1)(2j_2+1)$

Coupled base:

counts backwards

$$\sum_{j=|j_1-j_2|}^{|j_1+j_2|}(2j+1) = \frac{\displaystyle\sum_{j=j_1-j_2}^{j_1+j_2}(2j+1) + \sum_{j=j_1+j_2}^{j_1-j_2}(2j+1)}{2}$$

choose $j_1 \geq j_2$

1^{st} term: $\frac{1}{2}[2(j_1-j_2)+1 + 2(j_1+j_2)+1] = 2j_1+1$

2^{nd} term: $\frac{1}{2}[2(j_1-\cancel{j_2+1})+1 + 2(j_1+\cancel{j_2-1})+1] = 2j_1+1$

$\therefore$ all terms are the same

$$= \underbrace{[(\cancel{j_1}+j_2)-(\cancel{j_1}-j_2)+1]}_{\#\text{ of terms in the sum}}(2j_1+1)$$

$$= (2j_2+1)(2j_1+1)$$

$\therefore$ Same number of states in uncoupled and coupled bases

$$(2j_1+1)(2j_2+1)$$

assumed $(j_1 j_2)$ $\rightarrow$ $\langle j_1 m_1\, j_2 m_2 | j m \rangle$

C.G. Symmetry Relations

$$\textcircled{1} \leftrightarrow \textcircled{2} \qquad \langle j_2 m_2\, j_1 m_1 | j m \rangle = (-1)^{j_1+j_2-j} \langle j_1 m_1\, j_2 m_2 | j m \rangle$$

$$\left.\begin{matrix} m_1 \to -m_1 \\ m_2 \to -m_2 \\ m_3 \to -m_3 \end{matrix}\right\} \quad \langle j_1 -m_1\, j_2 -m_2 | j -m \rangle = (-1)^{j_1+j_2-j} \langle j_1 m_1\, j_2 m_2 | j m \rangle$$

$$\left.\begin{matrix} \text{both of} \\ \text{above} \end{matrix}\right\} \quad \langle j_2 -m_2\, j_1 -m_1 | j -m \rangle = \langle j_1 m_1\, j_2 m_2 | j m \rangle$$

$$\left.\begin{matrix} j_2 m_2 \to j\, -m \\ j m \to j_2\, -m_2 \end{matrix}\right\} \quad \langle j_1 m_1\, j -m | j_2 -m_2 \rangle = \langle j_1 m_1\, j_2 m_2 | j m \rangle (-1)^{j_1-m_1} \sqrt{\frac{2j+1}{2j_2+1}}$$

$$\left.\begin{matrix} j_1 m_1 \to j\, -m \\ j m \to j_1\, -m_1 \end{matrix}\right\} \quad \langle j -m\, j_2 m_2 | j_1 -m_1 \rangle = \langle j_1 m_1\, j_2 m_2 | j m \rangle (-1)^{j_2+m_2} \sqrt{\frac{2j+1}{2j_1+1}}$$

(there are others that can be useful)

Orthogonality Relationships

$$\boxed{\delta_{jj'}\,\delta_{mm'}} = \langle (j_1 j_2) jm \,|\, (j_1 j_2) j'm' \rangle$$

$$= \sum_{\substack{j_1' j_2' \\ m_1 m_2}} \underbrace{\langle (j_1 j_2) jm \,|\, j_1'm_1'\, j_2'm_2' \rangle}_{\sim\, \delta_{j_1 j_1'}\,\delta_{j_2 j_2'}} \langle j_1'm_1'\, j_2'm_2' \,|\, (j_1 j_2) j'm' \rangle$$

change order since real

$$\boxed{= \sum_{m_1 m_2} \langle j_1 m_1\, j_2 m_2 \,|\, jm \rangle \langle j_1 m_1\, j_2 m_2 \,|\, j'm' \rangle}$$

$$\boxed{\delta_{m_1 m_1'}\,\delta_{m_2 m_2'}} = \langle j_1 m_1\, j_2 m_2 \,|\, j_1 m_1'\, j_2 m_2' \rangle$$

$$= \sum_{\substack{jm \\ j_1' j_2'}} \underbrace{\langle j_1 m_1\, j_2 m_2 \,|\, (j_1' j_2') jm \rangle}_{\sim\, \delta_{j_1 j_1'}\,\delta_{j_2 j_2'}} \langle (j_1' j_2') jm \,|\, j_1 m_1'\, j_2 m_2' \rangle$$

$$\boxed{= \sum_{jm} \langle j_1 m_1\, j_2 m_2 \,|\, jm \rangle \langle j_1 m_1'\, j_2 m_2' \,|\, jm \rangle}$$

34. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND *d* FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2	Coefficients		
m_1	m_2			
.	.			
.	.			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

1/2×1/2

m_1	m_2	J=1, M=+1
+1/2	+1/2	1

m_1	m_2	J=1, M=0	J=0, M=0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	J=1, M=−1
−1/2	−1/2	1

1×1/2

m_1	m_2	J=3/2, M=+3/2
+1	+1/2	1

m_1	m_2	J=3/2, M=+1/2	J=1/2, M=+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	J=3/2, M=−1/2	J=1/2, M=−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	J=3/2, M=−3/2
−1	−1/2	1

2×1/2

m_1	m_2	J=5/2, M=+5/2
+2	+1/2	1

m_1	m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	J=5/2, M=+1/2	J=3/2, M=+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	J=5/2, M=−1/2	J=3/2, M=−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	J=5/2, M=−5/2
−2	−1/2	1

3/2×1/2

m_1	m_2	J=2, M=+2
+3/2	+1/2	1

m_1	m_2	J=2, M=+1	J=1, M=+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	J=2, M=0	J=1, M=0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	J=2, M=−1	J=1, M=−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	J=2, M=−2
−3/2	−1/2	1

2×1

m_1	m_2	J=3, M=+3
+2	+1	1

m_1	m_2	J=3, M=+2	J=2, M=+2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

m_1	m_2	J=3, M=0	J=2, M=0	J=1, M=0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	J=3, M=−1	J=2, M=−1	J=1, M=−1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	J=3, M=−2	J=2, M=−2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	J=3, M=−3
−2	−1	1

1×1

m_1	m_2	J=2, M=+2
+1	+1	1

m_1	m_2	J=2, M=+1	J=1, M=+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	J=2, M=0	J=1, M=0	J=0, M=0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	J=2, M=−1	J=1, M=−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	J=2, M=−2
−1	−1	1

3/2×1

m_1	m_2	J=5/2, M=+5/2
+3/2	+1	1

m_1	m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	J=5/2, M=−1/2	J=3/2, M=−1/2	J=1/2, M=−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	J=5/2, M=−5/2
−3/2	−1	1

2×3/2

m_1	m_2	J=7/2, M=+7/2
+2	+3/2	1

m_1	m_2	J=7/2, M=+5/2	J=5/2, M=+5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	−3/7

m_1	m_2	J=7/2, M=+3/2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2	−1/2	1/7	16/35	2/5
+1	+1/2	4/7	1/35	−2/5
0	+3/2	2/7	−18/35	1/5

m_1	m_2	J=7/2, M=+1/2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	4/35	−27/70	2/5	−1/10

m_1	m_2	J=7/2, M=−1/2	J=5/2, M=−1/2	J=3/2, M=−1/2	J=1/2, M=−1/2
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	12/35	−5/14	0	3/10
−2	+3/2	1/35	−6/35	2/5	−2/5

m_1	m_2	J=7/2, M=−3/2	J=5/2, M=−3/2	J=3/2, M=−3/2
0	−3/2	2/7	18/35	1/5
−1	−1/2	4/7	−1/35	−2/5
−2	+1/2	1/7	−16/35	2/5

m_1	m_2	J=7/2, M=−5/2	J=5/2, M=−5/2
−1	−3/2	4/7	3/7
−2	−1/2	3/7	−4/7

m_1	m_2	J=7/2, M=−7/2
−2	−3/2	1

3/2×3/2

m_1	m_2	J=3, M=+3
+3/2	+3/2	1

m_1	m_2	J=3, M=+2	J=2, M=+2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	−1/2

m_1	m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+3/2	−1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	−2/5
−1/2	+3/2	1/5	−1/2	3/10

m_1	m_2	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4

m_1	m_2	J=3, M=−1	J=2, M=−1	J=1, M=−1
+1/2	−3/2	1/5	1/2	3/10
−1/2	−1/2	3/5	0	−2/5
−3/2	+1/2	1/5	−1/2	3/10

m_1	m_2	J=3, M=−2	J=2, M=−2
−1/2	−3/2	1/2	1/2
−3/2	−1/2	1/2	−1/2

m_1	m_2	J=3, M=−3
−3/2	−3/2	1

2×2

m_1	m_2	J=4, M=+4
+2	+2	1

m_1	m_2	J=4, M=+3	J=3, M=+3
+2	+1	1/2	1/2
+1	+2	1/2	−1/2

m_1	m_2	J=4, M=+2	J=3, M=+2	J=2, M=+2
+2	0	3/14	1/2	2/7
+1	+1	4/7	0	−3/7
0	+2	3/14	−1/2	2/7

m_1	m_2	J=4, M=+1	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2	−1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	−1/14	−3/10
0	+1	3/7	−1/5	−1/14	3/10
−1	+2	1/14	−3/10	3/7	−1/5

m_1	m_2	J=4, M=0	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	+1	8/35	−2/5	1/14	1/10	−1/5
−2	+2	1/70	−1/10	2/7	−2/5	1/5

m_1	m_2	J=4, M=−1	J=3, M=−1	J=2, M=−1	J=1, M=−1
+1	−2	1/14	3/10	3/7	1/5
0	−1	3/7	1/5	−1/14	−3/10
−1	0	3/7	−1/5	−1/14	3/10
−2	+1	1/14	−3/10	3/7	−1/5

m_1	m_2	J=4, M=−2	J=3, M=−2	J=2, M=−2
0	−2	3/14	1/2	2/7
−1	−1	4/7	0	−3/7
−2	0	3/14	−1/2	2/7

m_1	m_2	J=4, M=−3	J=3, M=−3
−1	−2	1/2	1/2
−2	−1	1/2	−1/2

m_1	m_2	J=4, M=−4
−2	−2	1

$$d^{\,j}_{m',m} = (-1)^{m-m'} d^{\,j}_{m,m'} = d^{\,j}_{-m,-m'}$$

$$d^{\,1}_{0,0} = \cos\theta \qquad d^{\,1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{\,1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^{\,1}_{1,1} = \frac{1+\cos\theta}{2} \qquad d^{\,1}_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^{\,1}_{1,-1} = \frac{1-\cos\theta}{2}$$

$$d^{\,3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,1/2} = -\sqrt{3}\,\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-1/2} = \sqrt{3}\,\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^{\,2}_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^{\,2}_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^{\,2}_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2$$

$$d^{\,2}_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{\,2}_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{\,2}_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^{\,2}_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Figure 34.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.

$\boxed{\text{3-j Symbols}}$

$$\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -M \end{pmatrix} \equiv \frac{(-1)^{j_1-j_2+M}}{\sqrt{2J+1}} \langle j_1 m_1\, j_2 m_2 | JM \rangle$$

$\therefore$ ① real

② $m_1 + m_2 = M$

③ $|j_1 - j_2| \leq J \leq j_1 + j_2$

Symmetries

- circular permutation of the 3 columns $\rightarrow$ invariant
- permutations of 2 columns $\longrightarrow (-1)^{j_1+j_2+J}$
- change signs of $m_1, m_2, M \longrightarrow (-1)^{j_1+j_2+J}$