Extra Credit Problem Set #3 PHYS-411
[due 10am, Friday, October 25th online in Canvas]

This problem set is meant to give you experience in computational physics solving numerically for the dynamics of the systems in our class. This problem set is worth 50% of the points for a regular problem set.

Prologue

For this assignment, you will need four formulas describing the Keplerian orbit of a test particle around the Sun:

i. Its radius r versus angle θ is given by the ellipse:
$$r(\theta) = \frac{a(1-e^2)}{1+e\cos(\theta-\theta_E)}$$
where a is the semimajor axis, e is the eccentricity, and θ_E is the orientation of the ellipse; specifically, it gives the angle of closest approach to the Sun ("periapse").

ii. Its energy per unit mass is
$$u = -\frac{GM_S}{2a}$$
where M_S is the Sun's mass.

iii. Its angular momentum per unit mass is
$$h = \sqrt{GM_S a(1-e^2)}$$

iv. Its orbital period is
$$T = 2\pi\left(\frac{a^3}{GM_S}\right)^{1/2}$$

Note that energy and angular momentum per unit mass are defined as
$$u = \frac{v^2}{2} - \frac{GM_S}{r}$$
$$h = xv_y - yv_x$$
where v_x and v_y are the Cartesian components of the velocity, $v = (v_x^2 + v_y^2)^{1/2}$, and $x = r\cos(\theta)$, and $y = r\sin(\theta)$

Problem 1:

Write a code integrating the equations of motion for a test particle orbiting the Sun.
$$\ddot{x} = -GM_S x/r^3$$
$$\ddot{y} = -GM_S y/r^3$$

Use 4th-order Runge-Kutta. You should measure time in units of years, and length in units of AU; so, what should you set GM_S equal to? Initialize a particle (Earth) at x=1, y=0, v_x=0, v_y=2π, and integrate.

- Plot its x vs. y. To make the x vs. y plot look like a circle in gnuplot, first type "set size square", and then something like "plot [-1.5:1.5] [-1.5:1.5] 'out.dat' ".
- What is the orbital period, and what should it be?
- Repeat simulation, but initialized with v_x=1 and v_y=2π+1. Plot its x vs y on the same graph as before.

2. Write two functions to transform between the Cartesian variables used in your integrator and the particle's position relative to an ellipse. One of the functions, "ellipse_to_xy", will take as input the four quantities a, e, θ, and θ_E, and output the four quantities x, v_x, y, v_y. The second function, "xy_to_ellipse" will do the opposite. To test (and show me) that your code works, choose 4 random numbers for a, e, θ, and θ_E (.1,.2,.4,.3), apply ellipse_to_xy and print out x,vx,y,vy. Then apply xy_to_ellipse to recover the original values.

   Note: it is straightforward algebra to use formulas i-iii above (together with the definitions of u and h) to derive the transformation laws. A few hints:

   - In "ellipse_to_xy", to calculate v_x and v_y, you can define α such that v_x=v cosα and v_y=v sinα. Then one can write $h=xv_y-yv_x= rv\sin(\alpha-\theta)$, and use that to solve for α.
   - In "xy_to_ellipse", to get θ, check out the function atan2 in "math.h"
   - In "xy_to_ellipse", getting θ_E can be a little tricky. (And you will need it below.) There are two things to note. If you use formula i above, you will need to do an "acos" (inverse cosine). But this always returns an angle between 0 and π . To account for when θ -θ_E lies between -π and 0, you will need an if statement: check if the radial velocity $v_r= (x\, v_x + y\, v_y)/v$ is positive, i.e. if the particle is going from its periapse (point of closest approach) to apoapse (point of furthest distance). If so, then θ - θ_E should lie between 0 and π and you're done. If not, flip the sign of θ - θ_E. The second tricky thing is that your expression for θ_E will likely blow up when e=0. I will let you figure out how to handle this.

3. Using your functions from question 2, initialize a test particle ("Mercury") with a=0.39AU and e=0.206, θ_E = -3π/4 and its initial angle θ = θ_E. Integrate, and plot y vs. x and r vs. θ , as output from your simulation. Also superimpose on these plots the analytic formula for an ellipse (formula i in Prologue above). Make sure you (and I) can see that the theory matches the simulation.
4. Calculate the effect of Jupiter on the precession of Mercury's perihelion. Do not bother integrating Jupiter's equation of motion. Just force Jupiter to have a

circular orbit, with radius equal to its semimajor axis. Start Mercury as in question 3. Use the function "xy_to_ellipse" to output Mercury's θ_E as a function of time, and hand in the plot. From that, what is the rate at which Jupiter causes Mercury's perihelion to precess? Quote your answer both in radians per year, and in arcseconds per century, and estimate the error in your precession rate.

5. Repeat question 4, but now include the planets Venus, Earth, Mars, Jupiter, and Saturn (Uranus and Neptune are unimportant.). Treat all of those planets as you did Jupiter. What is your result for the precession rate, and with what error? How does this compare with the pre-general-relativity result of 531 arcseconds per century? (When I did this question, I got a slightly different answer. The observed result is 574 arcseconds per century.)

Problem 2

1. (a) Numerically solve for the evolution of a two-dimensional double compound pendulum, which consists of two uniform rods of equal mass m and length l attached to each other and to a pivot point via frictionless joints, as shown in Figure 1.

   (b) Initial conditions: both rods are horizontal and at rest, i.e., $\theta_1 = \theta_2 = \pi/2$, $\dot{\theta}_1 = \dot{\theta}_2 = 0$. See the left panel of Figure 2.

   (c) Please see the attached Wikipedia article for the equations of motion.

   (d) You can use one of the Runge-Kutta methods to obtain the evolution of the pendulum in time, writing out θ_1, θ_2, $\dot{\theta}_1$ and $\dot{\theta}_2$ to a text file. RK4 is the best for this problem due to high accuracy.

   (e) Make a plot of the trajectory traced out by the tip of the double pendulum similar to the right panel of Figure 2. No need to show the pendulum itself.

   (f) Make a movie of the pendulum similar to that in Figure 2 and in the wikipedia article on the double pendulum.

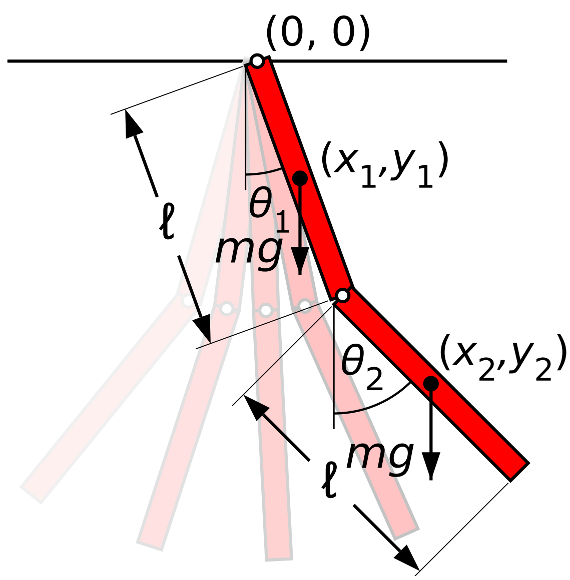


Figure 1: Double compound pendulum consists of two uniform rods each of mass m and length l. It is parameterized via two coordinates, θ_1 of the top rod and θ_2 of the bottom rod. [source: Wikipedia, public domain].

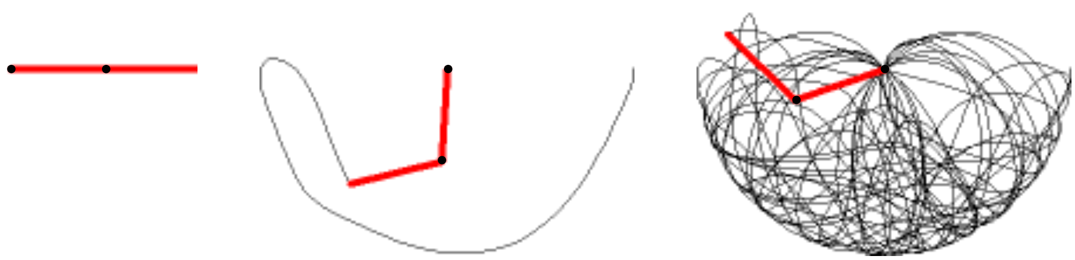
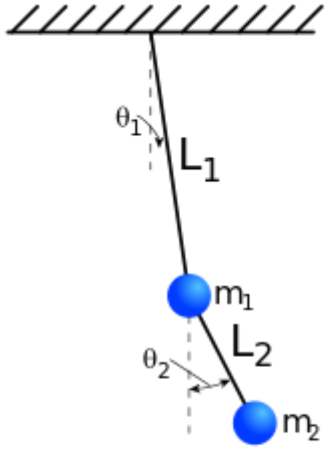


Figure 2: A sequence of snapshots of the double pendulum showing the trajectory of the tip of the pendulum. [left panel] The pendulum starts out at the horizontal position and zero velocity. [middle panel] After it is released, the trajectory of its tip is traced out in grey color. [right panel] The trajectory after a long time. [source: Wikipedia, public domain]

Wikipedia

Double pendulum

In physics and mathematics, in the area of dynamical systems, a **double pendulum** is a pendulum with another pendulum attached to its end, and is a simple physical system that exhibits rich dynamic behavior with a strong sensitivity to initial conditions.[1] The motion of a double pendulum is governed by a set of coupled ordinary differential equations and is chaotic.



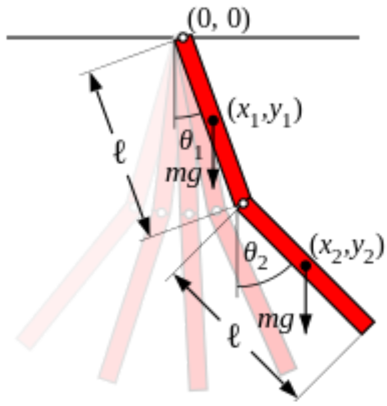
A double pendulum consists of two pendulums attached end to end.

Contents

Analysis and interpretation

Several variants of the double pendulum may be considered; the two limbs may be of equal or unequal lengths and masses, they may be simple pendulums or compound pendulums (also called complex pendulums) and the motion may be in three dimensions or restricted to the vertical plane. In the following analysis, the limbs are taken to be identical compound pendulums of length l and mass m, and the motion is restricted to two dimensions.

In a compound pendulum, the mass is distributed along its length. If the mass is evenly distributed, then the center of mass of each limb is at its midpoint, and the limb has a moment of inertia of $I = \frac{1}{12}ml^2$ about that point.



Double compound pendulum

It is convenient to use the angles between each limb and the vertical as the generalized coordinates defining the configuration of the system. These angles are denoted θ_1 and θ_2. The position of the center of mass of each rod may be written in terms of these two coordinates. If the origin of the Cartesian coordinate system is taken to be at the point of suspension of the first pendulum, then the center of mass of this pendulum is at:

$$\begin{aligned} x_1 &= \frac{l}{2}\sin\theta_1 \\ y_1 &= -\frac{l}{2}\cos\theta_1 \end{aligned}$$

and the center of mass of the second pendulum is at

$$\begin{aligned} x_2 &= l\left(\sin\theta_1 + \tfrac{1}{2}\sin\theta_2\right) \\ y_2 &= -l\left(\cos\theta_1 + \tfrac{1}{2}\cos\theta_2\right) \end{aligned}$$

This is enough information to write out the Lagrangian.

Lagrangian

The Lagrangian is

Motion of the double compound pendulum (from numerical integration of the equations of motion)

Trajectories of a double pendulum

$$\begin{aligned}L &= \text{kinetic energy} - \text{potential energy} \\ &= \tfrac{1}{2}m\left(v_1^2 + v_2^2\right) + \tfrac{1}{2}I\left({\dot\theta_1}^2 + {\dot\theta_2}^2\right) - mg\left(y_1 + y_2\right) \\ &= \tfrac{1}{2}m\left({\dot x_1}^2 + {\dot y_1}^2 + {\dot x_2}^2 + {\dot y_2}^2\right) + \tfrac{1}{2}I\left({\dot\theta_1}^2 + {\dot\theta_2}^2\right) - mg\left(y_1 + y_2\right)\end{aligned}$$

The first term is the *linear* kinetic energy of the center of mass of the bodies and the second term is the *rotational* kinetic energy around the center of mass of each rod. The last term is the potential energy of the bodies in a uniform gravitational field. The dot-notation indicates the time derivative of the variable in question.

Substituting the coordinates above and rearranging the equation gives

$$L = \tfrac{1}{6}ml^2\left({\dot\theta_2}^2 + 4{\dot\theta_1}^2 + 3{\dot\theta_1}{\dot\theta_2}\cos(\theta_1 - \theta_2)\right) + \tfrac{1}{2}mgl\left(3\cos\theta_1 + \cos\theta_2\right).$$

There is only one conserved quantity (the energy), and no conserved momenta. The two momenta may be written as

$$\begin{aligned}p_{\theta_1} &= \frac{\partial L}{\partial\dot\theta_1} = \tfrac{1}{6}ml^2\left(8\dot\theta_1 + 3\dot\theta_2\cos(\theta_1 - \theta_2)\right) \\ p_{\theta_2} &= \frac{\partial L}{\partial\dot\theta_2} = \tfrac{1}{6}ml^2\left(2\dot\theta_2 + 3\dot\theta_1\cos(\theta_1 - \theta_2)\right).\end{aligned}$$

These expressions may be inverted to get

$$\begin{aligned}\dot\theta_1 &= \frac{6}{ml^2}\frac{2p_{\theta_1} - 3\cos(\theta_1 - \theta_2)p_{\theta_2}}{16 - 9\cos^2(\theta_1 - \theta_2)} \\ \dot\theta_2 &= \frac{6}{ml^2}\frac{8p_{\theta_2} - 3\cos(\theta_1 - \theta_2)p_{\theta_1}}{16 - 9\cos^2(\theta_1 - \theta_2)}.\end{aligned}$$

The remaining equations of motion are written as

$$\begin{aligned}
\dot{p}_{\theta_1} &= \frac{\partial L}{\partial \theta_1} = -\tfrac{1}{2} m l^2 \left(\dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + 3 \frac{g}{l} \sin \theta_1 \right) \\
\dot{p}_{\theta_2} &= \frac{\partial L}{\partial \theta_2} = -\tfrac{1}{2} m l^2 \left(-\dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + \frac{g}{l} \sin \theta_2 \right).
\end{aligned}$$

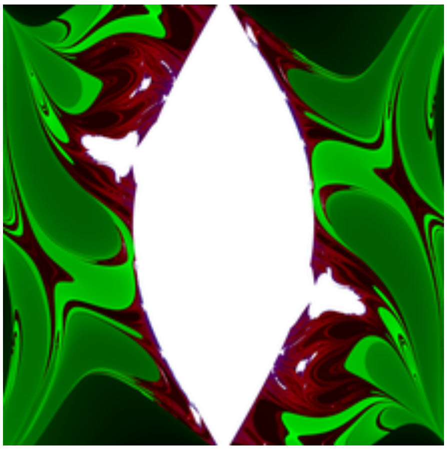
These last four equations are explicit formulae for the time evolution of the system given its current state. It is not possible to go further and integrate these equations analytically, to get formulae for θ_1 and θ_2 as functions of time. It is, however, possible to perform this integration numerically using the Runge Kutta method or similar techniques.

Chaotic motion

The double pendulum undergoes chaotic motion, and shows a sensitive dependence on initial conditions. The image to the right shows the amount of elapsed time before the pendulum flips over, as a function of initial position when released at rest. Here, the initial value of θ_1 ranges along the x-direction from −3 to 3. The initial value θ_2 ranges along the y-direction, from −3 to 3. The colour of each pixel indicates whether either pendulum flips within:

- $10\sqrt{l/g}$ (green)
- $100\sqrt{l/g}$ (red)
- $1000\sqrt{l/g}$ (purple) or
- $10000\sqrt{l/g}$ (blue).

Initial conditions that do not lead to a flip within $10000\sqrt{l/g}$ are plotted white.



Graph of the time for the pendulum to flip over as a function of initial conditions

The boundary of the central white region is defined in part by energy conservation with the following curve:

$$3\cos\theta_1 + \cos\theta_2 = 2.$$

Within the region defined by this curve, that is if

$$3\cos\theta_1 + \cos\theta_2 > 2,$$

then it is energetically impossible for either pendulum to flip. Outside this region, the pendulum can flip, but it is a complex question to determine when it will flip. Similar behavior is observed for a double pendulum composed of two point masses rather than two rods with distributed mass.[2]

Long exposure of double pendulum exhibiting chaotic motion (tracked with an LED)

The lack of a natural excitation frequency has led to the use of double pendulum systems in seismic resistance designs in buildings, where the building itself is the primary inverted pendulum, and a secondary mass is connected to complete the double pendulum.

See also

- Double inverted pendulum
- Pendulum (mathematics)
- Mid-20th century physics textbooks use the term "double pendulum" to mean a single bob suspended from a string which is in turn suspended from a V-shaped string. This type of pendulum, which produces Lissajous curves, is now referred to as a Blackburn pendulum.

Notes

1. Levien, R. B.; Tan, S. M. (1993). "Double Pendulum: An experiment in chaos". *American Journal of Physics* **61** (11): 1038. Bibcode:1993AmJPh..61.1038L (http://adsabs.harvard.edu/abs/1993AmJPh..61.1038L). doi:10.1119/1.17335 (https://doi.org/10.1119%2F1.17335).
2. Alex Small, *Sample Final Project: One Signature of Chaos in the Double Pendulum (https://12d82b32-a-62cb3a1a-s-sites.googlegroups.com/site/physicistatlarge/Computational%20Physics%20Sample%20Project-Alex%20Small-v1.pdf)*, (2013). A report produced as an example for students. Includes a derivation of the equations of motion, and a comparison between the double pendulum with 2 point masses and the double pendulum with 2 rods.

References

- Meirovitch, Leonard (1986). *Elements of Vibration Analysis* (2nd ed.). McGraw-Hill Science/Engineering/Math. ISBN 0-07-041342-8.
- Eric W. Weisstein, *Double pendulum* (2005), ScienceWorld *(contains details of the complicated equations involved)* and "Double Pendulum" by Rob Morris, Wolfram Demonstrations Project, 2007 (animations of those equations).
- Peter Lynch, *Double Pendulum*, (2001). *(Java applet simulation.)*
- Northwestern University, *Double Pendulum*, *(Java applet simulation.)*
- Theoretical High-Energy Astrophysics Group at UBC, *Double pendulum*, (2005).

External links

- Animations and explanations of a double pendulum and a physical double pendulum (two square plates) by Mike Wheatland (Univ. Sydney)
- Interactive Open Source Physics JavaScript simulation with detailed equations double pendulum
- Interactive Javascript simulation of a double pendulum
- Double pendulum physics simulation from www.myphysicslab.com using open source JavaScript code
- Simulation, equations and explanation of Rott's pendulum
- Comparison videos of a double pendulum with the same initial starting conditions on YouTube
- Double Pendulum Simulator - An open source simulator written in C++ using the Qt toolkit.
- Online Java simulator of the Imaginary exhibition.