Physics 414-2 Problem Set 3

April 15, 2022

Due: Friday, April 22 at 4 pm

1. Acceleration of an object by a Gaussian beam. Consider a Gaussian beam of the form discussed in lecture, with coordinates chosen so that the transverse size of the Gaussian beam is minimized for $z = 0$ (as in lecture, we assume that the beam propagates along the z axis). The Gaussian beam is generated by light with wavelength 500 nm, and we take the parameter w_0 (the waist size) to be $w_0 = 1$ mm. The peak intensity of the beam is 10 Watts/cm^2. We consider an object in the beam that acts as a perfect absorber of light from the beam, with an effective absorption cross sectional area of $(100\ \mu\text{m})^2$. The object has a mass of 10^{-9} kg. Ignore the influence of blackbody radiation emitted by the object.

(a) Assuming that the object starts from rest at $z = 0$ in the center of the beam (i.e., $x = 0$, $y = 0$), find the position of the object as a function of time. To do this, you can use Mathematica's NDSolve function to solve the equations of motion and then plot the solution. The Mathematica tutorial on Canvas gives an example of how to use NDSolve. Please let me know if you need any help with this.

(b) Using your solution from part (a), plot the velocity as a function of time. Does the velocity keep increasing in an unbounded manner, or does it approach a steady-state value?

2. Acceleration of an object by a Gaussian beam: the off-axis case. Repeat problem 1, but for the object now starting 500 μm away from the z-axis in the x-direction. In other words, the object starts from rest at $x = 500\ \mu$m, $y = 0$, $z = 0$.

3. Influence of laser beam perturbations on a laser beam trap for polarizable objects. The aim of problems 3 and 4 is to illustrate how laser beam perturbations which are nominally small can still have important effects in modern experiments. As we will see later in this class, a laser beam can be used to create an attractive potential for polarizable objects, with the potential equal to $U(\vec{r}) = -aI(\vec{r})$. Here, $I(\vec{r})$ denotes the local intensity of the laser field, and a is a constant of proportionality.

(a) Consider a Gaussian beam that propagates along the z axis. For $z = 0$, the beam intensity is given as a function of x and y by $I(\vec{r}) = I_0 \exp\left(-\frac{2(x^2+y^2)}{w_0^2}\right)$, where I_0 is a constant. If x and y are small compared to w_0, $U(\vec{r})$ can be approximated as a harmonic potential with spring constant k plus a constant offset c: $U(\vec{r}) = c + \frac{1}{2}k(x^2 + y^2)$. In this situation, the motion of an object in this potential will therefore correspond to simple harmonic motion. Find expressions for k and c in terms of a, I_0 and w_0.

(b) Now, we consider the case in which a small intensity perturbation of size δ and spatial frequency κ is added to the laser beam, so that $I(\vec{r}) = I_0 \exp\left(-\frac{2(x^2+y^2)}{w_0^2}\right)(1 + \delta \cos \kappa x)$. If $\delta \ll 1$, will the the force on the object still approximately correspond to linear restoring force toward the origin? Why or why not?

4. Effect of beam perturbations on an atomic gravimeter. (a) Atom interferometers, which split and recombine the wavefunctions of atoms using laser pulses, are able to make very accurate gravimeters. For simplicity, let us assume that the laser pulses correspond to vertically propagating plane waves with wave number k and frequency ω. A schematic of such an atom interferometer is illustrated in Fig. 1. First, a laser pulse that acts as a beam splitter (often called a $\pi/2$-pulse) splits the wavefunction into a superposition of two different trajectories with momentum differing by $\hbar k$ ($\hbar k$ corresponds to the momentum of a photon). Specifically, one part of the wavefunction (the upper path) receives a momentum kick of $\hbar k$, while the other part of the wavefunction (the lower path) receives no momentum kick. The trajectories then freely separate from each other over a time T, reaching a maximum separation of $\Delta z = \frac{\hbar k}{m}T$, where m is the mass of each atom. Subsequently, a second laser pulse that acts a mirror (often called a π-pulse) redirects the two trajectories back toward each other via additional momentum kicks (the upper trajectory gets a $-\hbar k$ momentum kick, and the lower trajectory gets $+\hbar k$ momentum kick). When the trajectories once again overlap, a final beam splitter interaction is applied to make them interfere. The phase $\Delta\phi$ of the interference pattern, which is equal to the phase difference between the two trajectories, then provides information about the gravitational acceleration g experienced by the atoms.

We will calculate $\Delta\phi$ using the following set of rules that emerge from a full quantum mechanical calculation of the atom interferometer (note: a more detailed study of atom interferometers could make a good final project topic). The rules are as follows:

- **Rule 1:** Let us say that the vertical trajectory of a given path within the interferometer (either the upper or lower path) is defined by the function $z(t)$, and let us say that the laser is fixed to the floor of the lab at height

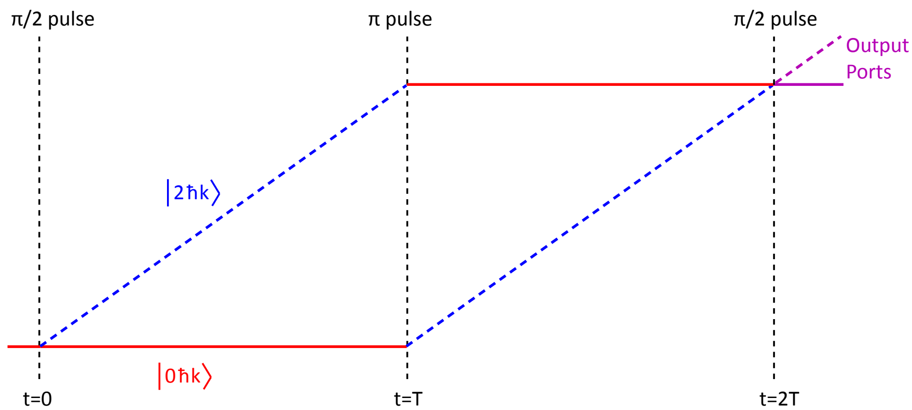

Figure 1: Illustration of an atomic gravimeter. The trajectories separate along the vertical axis. For simplicity, the gravitational free fall of the trajectories is not illustrated here. In general, the momentum difference between the two trajectories can be made equal to an integer number of photon momentum kicks $\hbar k$.

z_0 (in the frame of reference of the lab). Whenever our trajectory receives a momentum kick of $\pm\hbar k$ from the laser beam, the trajectory receives an extra phase factor of

$$\phi_{\text{kick}} = \pm \left[k \left(z(t) - z_0 \right) - \omega t \right], \tag{1}$$

where the kick occurs at time t.

- **Rule 2:** In addition to the phases ϕ_{kick} associated with momentum kicks from the laser beam, each trajectory has a phase factor equal to the classical action of the trajectory (in other words, the Lagrangian of the atom integrated over the trajectory).

Using the above rules, calculate the phase difference $\Delta\phi$ between the two trajectories assuming a vertical gravitational acceleration of g. (Hint: You can simplify this calculation to a few lines by doing the calculation in a conveniently chosen frame of reference as opposed to the lab frame).

(b) In part (a), we have assumed that the laser beam is well-approximated by a plane wave with wave number k. We will now consider what happens when there are high spatial frequency perturbations on the laser beam. Let us consider a perturbed beam of the form that we studied in Problem 4 of Problem Set 2. In this problem, we looked at a laser field of the form $u(x, y, z)e^{i(kz-\omega t)}$, with

$u(x, y, 0) = 1 + \delta \cos(\kappa x)$. We calculated $u(x, y, z)$ and expressed it in the form $u(x, y, z) = |u(x, y, z)| \, e^{i\phi(x,y,z)}$. We found that $\phi(x, y, z) = -\delta \cos(\kappa x) \sin\left(\frac{\kappa^2}{2k} z\right)$.

In the presence of the perturbed beam described above, it can be rigorously shown in a full quantum mechanical treatment that for an atom located in the region near the point (x, y, z), k should be replaced with $k + \frac{\partial \phi(x,y,z)}{\partial z}$ in the expression for the interferometer phase $\Delta\phi$, assuming that the motion of the atom during the interferometer is small compared to the scale of spatial variations in $\phi(x, y, z)$. Intuitively this makes sense, because the effective local z-component of the k-vector is equal to the z component of the local gradient of the spatial phase $kz + \phi(x, y, z)$, as discussed in Lecture 7. The z component of this gradient is exactly $k + \frac{\partial \phi(x,y,z)}{\partial z}$.

Find the correction to the phase shift as a function of the location (x,y,z) of the atom, for $u(x, y, z)$ as described above (see also problem 4 of problem set 2).

(c) For $\delta = 0.1$, 780 nm laser light, and $\kappa = 3 \times 10^3$ m^{-1}, what is the fractional size of the correction to the phase shift as compared to the unperturbed phase shift? State-of-the-art atomic gravimeters can achieve an accuracy of better than 1 part per billion. At this level of accuracy, is the influence of beam perturbations an important effect to consider?