WKB Method

useful approximation method for time-indep Schrödinger equation in 1-D (also applicable for radial equation in 3D).

- Bound state energies
- Tunneling rates through a barrier

If V is constant, consider $E > V$.

$$\psi(x) = Ae^{\pm ikx}, \quad k = \sqrt{2m(E-V)/\hbar^2} \text{ is a}$$

good solution to Schrödinger equation

Looks sinusoidal. If V is not constant, but varies slowly w.r.t. $\frac{1}{k}$, then we can treat it as approximately sinusoidal, with a changing wavelength + amplitude. (fast oscillations w/ an envelope)

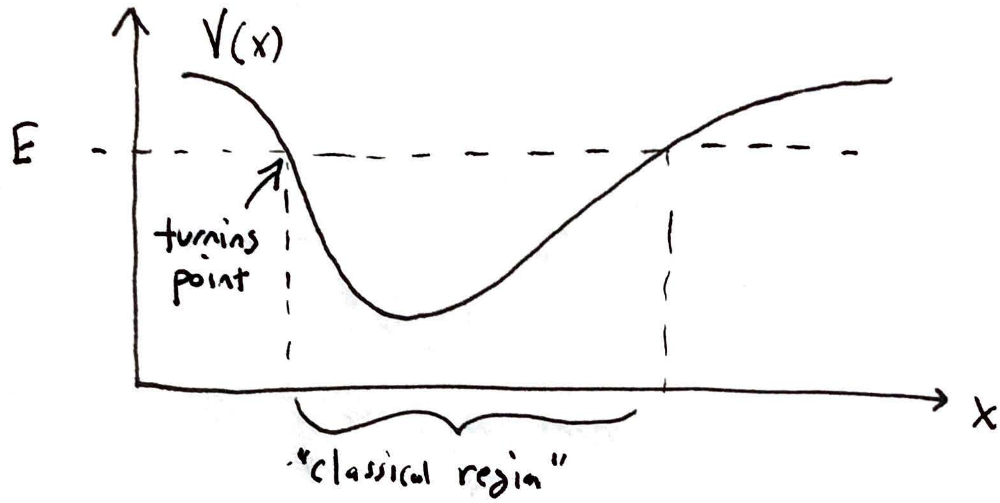
If $E < V$

$$\psi(x) = Ae^{\pm \rho x} \qquad \rho = \sqrt{2m(V-E)/\hbar^2}$$

exponential decay.

If V is not constant, varies slowly w.r.t. $\frac{1}{\rho}$, then we have an exponential, modulated w/ slowly varying function of x.

This picture encounters a difficulty if $E \approx V$ at a classical "turning point", then $\frac{1}{p}$ or $\frac{1}{k}$ are large, V can't vary slowly.



Consider Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi = E\psi$$

$$\Rightarrow \quad \frac{d^2\psi}{dx^2} = -\frac{p^2(x)}{\hbar^2}\psi \qquad p = \sqrt{2m\,(E-V(x))}$$

Let's assume in classical zone, so p is real we can express $\psi(x) = A(x)\,e^{i\phi(x)}$ as amplitude & phase.

put into Schrödinger equation:

$$\frac{d^2\psi}{dx^2} = A'' + 2iA'\phi' + iA\phi'' - A\phi'^2$$

$$A'' + 2iA'\phi' + iA\phi'' - A\phi'^2 = \frac{-p^2}{\hbar^2}A$$

Real part: $A'' = A\left[\phi'^2 - \frac{p^2(x)}{\hbar^2}\right]$

Imag. part: $2A'\phi' + A\phi'' = 0$

equivalent to $\left(A^2(x)\,\phi'(x)\right)' = 0$

The imag. part has a solution

$$A(x) = \frac{C}{\sqrt{\phi'(x)}}$$ for some constant C.

To solve real part, we make slowly varying envelope approximation. $\Rightarrow$ A'' is negligible.

$$(\phi'(x))^2 = \frac{p^2(x)}{\hbar^2} \Rightarrow \frac{d\phi}{dx} = \pm\frac{p(x)}{\hbar}$$

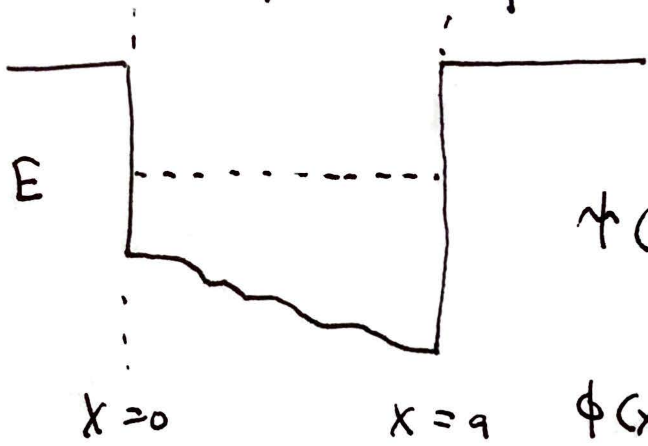
$$\Rightarrow \phi(x) = \pm\frac{1}{\hbar}\int p(x)\,dx$$

$$\psi(x) \cong \frac{C}{\sqrt{p(x)}}\, e^{\pm\frac{i}{\hbar}\int p(x)\,dx}$$

$$\Rightarrow \psi(x) = \frac{C_+}{\sqrt{p(x)}}\, e^{+\frac{i}{\hbar}\int p(x)\,dx} + \frac{C_-}{\sqrt{p(x)}}\, e^{-\frac{i}{\hbar}\int p(x)\,dx}$$

Note $|\psi(x)|^2$ is $\propto \frac{|C|^2}{|p(x)|}$ gets small if moving fast, less likely to be found where it's moving fast.

Example : Square well potential



If $E > V(x)$

$$\psi(x) \cong \frac{1}{\sqrt{p(x)}}\left[C_1 \sin\phi(x) + C_2 \cos\phi\right.$$

$$\phi(x) = \frac{1}{\hbar}\int_0^x p(x')\,dx'$$

Here $\psi = 0$ at $x=0$ $\Rightarrow$ $C_2 = 0$

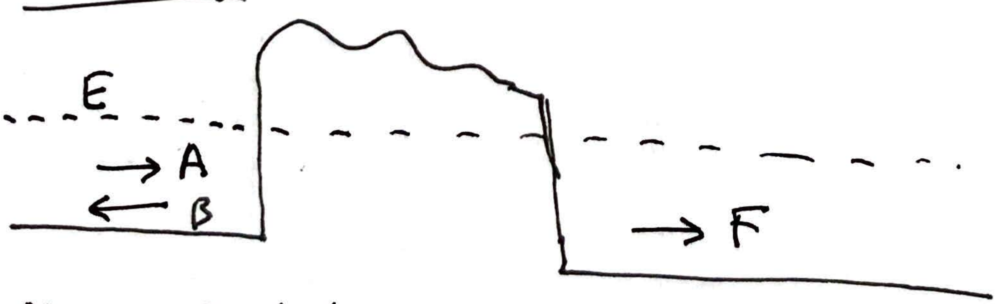
$\psi(x) = 0$ at $x = a$ $\Rightarrow$ $\phi(a) = n\pi$, $n = 1, 2, \ldots$

$$\boxed{\int_0^a p(x)\,dx = n\pi\hbar} \leftarrow \text{energy quantization}$$

If we had a flat bottom, $V=0$, $p(x) = \sqrt{2mE}$

$\Rightarrow$ $E_n = \dfrac{n^2\pi^2\hbar^2}{2ma^2}$ get precisely energies for a square well.

Tunneling



Now $E < V$ inside, so $\psi(x) \cong \dfrac{C}{\sqrt{|p(x)|}}\, e^{\pm\frac{1}{\hbar}\int |p(x)|\,dx}$

Left side $\psi(x) = Ae^{ikx} + Be^{-ikx}$

Right side $\psi(x) = Fe^{ikx}$

Transmission coefficient $T = \dfrac{|F|^2}{|A|^2}$

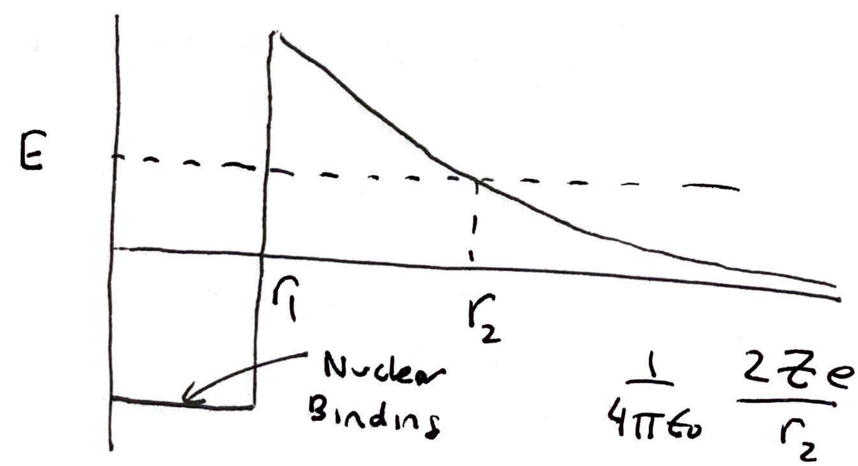
Inside Barrier

$$\psi(x) \cong \frac{C}{\sqrt{|p(x)|}} e^{\frac{1}{\hbar}\int_0^x |p(x')|dx'} + \frac{D}{\sqrt{|p(x)|}} e^{-\frac{1}{\hbar}\int_0^x |p(x')|dx'}$$

$$\Rightarrow \frac{|F|}{|A|} \cong e^{\frac{1}{\hbar}\int_0^a |p(x')|dx'}$$

$$T \simeq e^{-2\gamma} \qquad \gamma = \frac{1}{\hbar}\int_0^a |p(x)|dx$$

Ex: can be used to estimate radioactive decay: α-decay from a radioactive nucleus.



$$\frac{1}{4\pi\epsilon_0}\frac{2Ze^2}{r_2} = E$$

The exponent γ is

$$\gamma = \frac{1}{\hbar}\int_{r_1}^{r_2} \sqrt{2m\left(\frac{1}{4\pi\epsilon_0}\frac{2Ze^2}{r} - E\right)}\, dr = \frac{\sqrt{2mE}}{\hbar}\int_{r_1}^{r_2}\sqrt{\frac{r_2}{r} - 1}\, dr$$

$$= \frac{\sqrt{2mE}}{\hbar}\left[r_2 \cos^{-1}\sqrt{\frac{r_1}{r_2}} - \sqrt{r_1(r_2 - r_1)} \right]$$

If $r_1 << r_2$, then $\cos^{-1}\sqrt{\frac{r_1}{r_2}} \approx \frac{\pi}{2} - \sqrt{\frac{r_1}{r_2}}$

$$\gamma \simeq K_1 \frac{Z}{\sqrt{E}} - K_2\sqrt{Z r_1}$$

$$K_1 = \frac{e^2}{4\pi\epsilon_0}\frac{\pi\sqrt{2m}}{\hbar} = (1.98\ \text{MeV})^{1/2}$$

$$K_2 = \left(\frac{e^2}{4\pi\epsilon_0}\right)^{1/2}\frac{4\sqrt{m}}{\hbar} = (1.485\ \text{fm})^{-1/2}$$

α-particle is confined "rattles" around, time between collisions is $\frac{2r_1}{v}$, if v is average velocity. each collision has probability $e^{-2\gamma}$ to escape.

<u>lifetime</u> $\boxed{\tau = \left(\frac{2r}{v}\right) e^{2\gamma}}$

Remark :

there are techniques for treating region near classical turning points. "patching" wave function where you can linearize the potential.