Postulates of QM, Part VI

Let's check whether the generator for spatial translations, which is now the definition of the momentum operator, matches de Broglie's version, $\underset{\sim}{p} = -i\hbar \frac{d}{dx}$. (Maybe new? Instead of defining $\underset{\sim}{p}$ that way, we ask what operator is needed to generate translation.)

- Recall $U(a)\underset{\sim}{x} \neq \underset{\sim}{x}\, U(a)$ ← Measure then translate vs reverse

$$U(a)\underset{\sim}{x} = (\underset{\sim}{x} + a)\,U(a)$$

+ $\underset{\sim}{p}$ is defined by

$$U(a) = \mathbb{1} + \frac{ia\underset{\sim}{p}}{\hbar} + \mathcal{O}(a^2)$$

Plugging in & keeping only order a:

$$\frac{ia\underset{\sim}{p}}{\hbar}\underset{\sim}{x} = a + \underset{\sim}{x}\left(\frac{ia\underset{\sim}{p}}{\hbar}\right)$$

$\Rightarrow$ $\left[\underset{\sim}{x}, \frac{i\underset{\sim}{p}}{\hbar}\right] = -1$ or $[\underset{\sim}{x}, \underset{\sim}{p}] = i\hbar$

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- This ~~expr~~ relation expresses antagonistic relationship between x & p.

(~~p generates translations. The ... saying~~ measuring position then ~~that~~ a moving particle gives different result from moving & then measuring position. Of course.) But realizing p is generator for translations in QM connection now makes p measurements antagonistic to x measurements.

- Lets' carefully normalize our states of def. x.

Recall $\mathbb{1} = \sum_i |\psi_i\rangle\langle\psi_i|$ ← If a complete orthonormal basis

$\Rightarrow$ We choose a a position space version to include a dx prefactor for

$$\mathbb{1} = \int dx\, |x\rangle\langle x|$$

$$\mathbb{1}|y\rangle = \int dx\, |x\rangle\langle x|y\rangle = |y\rangle$$

if $\langle x|y\rangle = \delta(x-y)$

So this ~~is~~ gives proper normalization for that def. of $\mathbb{1}$

- We can now define Schr. w.f.

$$\psi(x) \equiv \langle x|\psi\rangle$$

- This is state vector in Hilbert space (now considering to be the fundamental description) projected onto the position-state basis (one of many bases we could use)

- $|\langle x|\psi\rangle|^2$ gives probability of finding particle at position x, ~~as does $|\psi(x)|^2$~~

~~Lets check normalization for each way~~

- In same way we can define a state of def p:

$$\hat{p}\,|p_0\rangle = p_0\,|p_0\rangle$$

- Convenient to normalize s.t.

$$\langle p|q\rangle = 2\pi\,\delta\!\left(\frac{p-q}{\hbar}\right) \leftarrow \text{Integration variable} \propto \frac{p}{\hbar}$$

Completeness relation is

$$\Rightarrow \quad \mathbb{1} = \int \frac{dp}{2\pi\hbar} |p\rangle\langle p|$$

- Momentum space w.f. $\tilde{\psi}(p) = \langle p|\psi\rangle$

  Normalization?

  ~~Satisfies normalization~~

$$\cancel{\int \frac{dp}{2\pi\hbar} |\tilde{\psi}(p)|^2 = 1}$$

(~~Can show that by~~ ...)

$$1 = \langle\psi|\psi\rangle = \langle\psi|\mathbb{1}|\psi\rangle$$

$$= \int \frac{dp}{2\pi\hbar} \langle\psi|p\rangle\langle p|\psi\rangle = \boxed{\int \frac{dp}{2\pi\hbar} |\tilde{\psi}(p)|^2 = 1}$$

(Matches earlier definition w/ $p = \hbar k$)

- Now what is ~~action of~~ $\hat{p}$ in position space? ~~...~~

- Since $\langle x|\hat{x}|\psi\rangle = x\,\psi(x)$

$\Downarrow$

We can then prove that $\langle x|\hat{p}|\psi\rangle = -i\hbar \frac{d}{dx}\psi(x)$, i.e.

$= -i\hbar \frac{d}{dx}\langle x|\psi\rangle$

$\langle x|\hat{p} = -i\hbar \frac{d}{dx}\langle x|$

Proof: If the commutator is correct, then the definition is correct. Let's check commutator

$$\langle x | [\underset{\sim}{x}, \underset{\sim}{p}] | \psi \rangle = i\hbar \langle x | \psi \rangle = i\hbar \psi(x)$$

(using our guess)
$$= \langle x | \underset{\sim}{x}\underset{\sim}{p} | \psi \rangle - \langle x | \underset{\sim}{p}\,\underset{\sim}{x} | \psi \rangle$$

$$= x \langle x | \underset{\sim}{p} | \psi \rangle - \left(-i\hbar \frac{d}{dx}\right) \langle x | \underset{\sim}{x} | \psi \rangle$$

$$= -i\hbar x \frac{d}{dx} \psi(x) + \left(i\hbar \frac{d}{dx}\right)\left(x\, \psi(x)\right)$$

$$= \cancel{-i\hbar x \frac{d\psi}{dx}} + i\hbar \psi(x) + \cancel{i\hbar x \frac{d\psi}{dx}} = i\hbar \psi(x) \checkmark$$

Note: It is better to say $\langle x | \underset{\sim}{p} | \psi \rangle = -i\hbar \frac{d}{dx} \psi(x)$ rather than $\underset{\sim}{p} = -i\hbar \frac{d}{dx}$. Everyone knows what we mean by latter, but it is a little confused, since $\underset{\sim}{p}$ is an operator acting in $\mathcal{H}$, and the $\frac{d}{dx}$ action is what we get : position ...

Summary: (1) Needed $[\underset{\sim}{x}, \underset{\sim}{p}] = i\hbar$ for $\underset{\sim}{p}$ to generate translations.
(2) This action of $\underset{\sim}{p}$ on position states gives that commutator.

- So now we can compute $\langle x | p \rangle$, the x-space ~~wavefunction~~ representation of a particle w/ def. momentum.

$$\langle x | \underset{\sim}{p} | p \rangle = p \langle x | p \rangle = -i\hbar \frac{d}{dx} \langle x | p \rangle$$

operating to right — operating to left

This is a dif. eq. for $\langle x|p\rangle$.

Its solution is

$$\langle x|p\rangle = e^{i\frac{px}{\hbar}}$$

That eqn does not determine normalization, but we can check that it's right:

$$\langle p|q\rangle = \langle p|\mathbb{1}|q\rangle = \int dx \langle p|x\rangle\langle x|q\rangle$$

$$= \int dx\, e^{\frac{-ipx}{\hbar}} e^{\frac{iqx}{\hbar}} = 2\pi\delta\left(\frac{p}{\hbar} - \frac{q}{\hbar}\right) \checkmark$$

— Now we can prove interesting relationship between pos-space + mom-space representations postulated earlier (namely that they are F.T.'s of each other)

$$\tilde{\psi}(p) = \langle p|\psi\rangle = \int dx \langle p|x\rangle\langle x|\psi\rangle$$

$$= \int dx\, e^{-ipx/\hbar}\psi(x)$$

$$\psi(x) = \langle x|\psi\rangle = \int \frac{dp}{2\pi\hbar}\langle x|p\rangle\langle p|\psi\rangle$$

$$= \int \frac{dp}{2\pi\hbar}\, e^{\frac{ipx}{\hbar}}\,\tilde{\psi}(p)$$

- Next, let's write the eigenvalue eqn for energy $H|\varphi\rangle = E|\varphi\rangle$ with

$$H = \frac{\tilde{p}^2}{2m} + V(x)$$

- Project ~~both~~ both sides onto $\langle x|$:

$$\langle x|\, \frac{\hat{p}^2}{2m} + V(x)\, |\varphi\rangle = E\langle x|\varphi\rangle$$

($\tilde{p}$ acting backward on $\langle x|$ twice)

$$\left[(-i\hbar)^2 \frac{1}{2m}\frac{d^2}{dx^2} + V(x)\right]\varphi(x) = E\,\varphi(x)$$

~~Which is Schroedinger's version of Schrodinger's equation!~~

This tells us how H acts on an eigenfunction when viewed in position space. ~~So more generally, where $\psi(x) = \sum a_n \psi_n(x)$, $H\psi(x) = \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\psi(x)$.~~

This is Schrodinger's version of the Schr eqn!

Powerful!

All we have said is

① A system is described by a state vector in a complex $\mathcal{H}$

② (Geometric) Symmetry transformations are described by unitary operators in $\mathcal{H}$ (which is very gentle postulate)

Results:

- If time-translation is a symmetry
  $\Rightarrow$ Conserved quantity we call energy
  $\Rightarrow i\hbar \frac{d}{dt}|\psi\rangle = H|\psi\rangle$ energy operator
- Action of $\underset{\sim}{p}$ in position space
- ~~Et~~ Projected onto position space, Schr. looks like $i\hbar \frac{d}{dt}\psi(x,t) = \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\psi(x,t)$
- States of definite momentum are plane waves
- $\psi(x)$ & $\tilde{\psi}(p)$ are F.T.s of each other
- $[\underset{\sim}{x}, \underset{\sim}{p}] = i\hbar$ Important one!

- And, BTW, if space trans is a symmetry, then p is conserved, but that's not required for any of above
- Now jump from how to go from a $|\psi\rangle$ ~~$\psi(x)$ (or a $|\psi\rangle$)~~ to the outcome of a measurement remains a postulate of QM.
- It just turns out that we don't actually need to take Schr, or deBr waves, or $[\underset{\sim}{x}, \underset{\sim}{p}]$ as postulates.