Quantum Mechanics 412-1 Discussion

Tuesday, 5 November 2019

1. **Operator gymnastics**
   Consider generic operators X, Y, with commutator $[X, Y] = c$, where c is a c-number.
   (a) Show that $[X, Y^n] = ncY^{n-1}$
   (b) Use this to find $[X, f(Y)]$ where $f(Y)$ is defined by its power series,
   $$f(Y) \equiv \sum_n a_n Y^n \tag{1}$$
2. **The Hadamard Lemma**
   (a) By integrating the differential equation obeyed by $g(x)$,
   $$g(x) = e^{xA} B e^{-xA} \tag{2}$$
   and explicitly evaluating the first few terms of the double sum, verify the form of the Hadamard lemma,
   $$e^A B e^{-A} = B + [A, B] + \frac{1}{2}[A, [A, B]] + \ldots \tag{3}$$
   (b) If $[A, B] = c$, verify this result explicitly by making use of your answer for $[X, f(Y)]$ for $X = B$ and $f(Y) = e^{-A}$.
   (c) Use the lemma, along with $[x, p] = i\hbar$ to argue that, for constant a:
   $$e^{ipa/\hbar} f(x) e^{-ipa/\hbar} = f(x + a) \tag{4}$$
   This result indicates that the unitary operator $U = e^{ipa/\hbar}$ represents a spatial translation of distance a (so, p is the generator of translations).
3. **Hamilton's equations of motion in quantum mechanics.**
   Consider a system describing a single particle moving in a position-dependent potential $V(x)$ with Hamiltonian, $H = \frac{p^2}{2m} + V(x)$.
   (a) Calculate $\frac{d\langle x\rangle}{dt}$ using Ehrenfest's theorem,
   $$\frac{d\langle A\rangle}{dt} = \frac{1}{i\hbar}\langle [A, H]\rangle + \frac{\partial A}{\partial t} \tag{5}$$
   (b) Assuming that a power series expansion for $V(x)$ exists and using the result $[p, x^n] = -i\hbar n x^{n-1}$, calculate $\frac{d\langle p\rangle}{dt}$.
   (c) Show your answers are equivalent to Hamilton's equations of motion:
   $$\frac{d\langle x\rangle}{dt} = \left\langle \frac{\partial H}{\partial p} \right\rangle \qquad \frac{d\langle p\rangle}{dt} = -\left\langle \frac{\partial H}{\partial x} \right\rangle \tag{6}$$