# Physics 414-2 Problem Set 4

April 22, 2022

**Due: Friday, April 29 at 4 pm**

**1. Hong-Ou-Mandel Experiment with Fermions.** (a) Repeat the analysis of the Hong-Ou-Mandel effect we did in class, but now for indistinguishable fermions instead of indistinguishable bosons. What are the relative probabilities for finding both particles in the same detector vs. finding the two particles in different detectors?

(b) Comment on how your result from part (a) relates to the Pauli exclusion principle.

**2. Generalizing Hong-Ou-Mandel Interference to Many Photons.** (a) Consider a setup analogous to the Hong-Ou-Mandel interferometry experiment discussed in class, but now with 10 photons in mode a and 10 photons in mode b, as opposed to our original example of 1 photon in mode a and 1 photon in mode b. Find and plot the probability of detecting $n$ of the 20 photons in detector c for $n$ ranging from 0 to 20.

(b) Repeat part (a), but now for 100 photons in mode a and 100 photons in mode b. Find and plot the probability of detecting $n$ of the 200 photons in detector c for $n$ ranging from 0 to 200. Comment qualitatively on your results.

(c) We will now explore an interesting classical analogy to the quantum calculations you have done. Consider that there are two classical waves with a random phase $\phi$ between them incident on our beam splitter. Using the RandomReal function in Mathematica or something equivalent, make a histogram over many (about 1 million should suffice) trials of the interferometer, with a random value of $\phi$ each time, indicating the probability of finding a certain fraction of the light in detector c. Compare your result to the result of part (b). Your result should indicate a close analogy between the interference of two states with exactly $N$ photons and the interference of classical waves with a random relative phase.

(d) Repeat the calculation again, but now for 100 photons in mode a and 10 photons in mode b. Find and plot the probability of detecting $n$ of the 110 photons in detector c for $n$ ranging from 0 to 110. Comment qualitatively on your results.

**3. Number phase uncertainty relation for electromagnetic waves.** In this problem, we will explore an uncertainty relation between the number of photons in a mode of an electromagnetic field and the phase of the field. As a first step, we will consider an operator to represent the phase of the field. We will consider a given mode of the field with annihilation operator $\hat{a}$. Recall that the number operator is $\hat{N} = \hat{a}^\dagger \hat{a}$. For a state $|n\rangle$ with exactly $n$ photons in the field, $\hat{N}|n\rangle = n|n\rangle$. A phase operator $\hat{\phi}$ is sometimes defined via the following factorization, which is valid for states with photon numbers $n \gg 1$:

$$\hat{a} = \sqrt{\hat{N}+1}e^{i\hat{\phi}} \tag{1}$$

$$\hat{a}^\dagger = \sqrt{\hat{N}+1}e^{-i\hat{\phi}}. \tag{2}$$

We will consider some of the properties of this operator.

(a) Show that we can express $e^{i\hat{\phi}}$ as

$$e^{i\hat{\phi}} = \sum_{n=0}^{\infty} |n\rangle \langle n+1|. \tag{3}$$

(b) Show that the states $|\phi\rangle \equiv \sum_{n=0}^{\infty} e^{in\phi}|n\rangle$ are eigenstates of the operator $e^{i\hat{\phi}}$ with eigenvalues $e^{i\phi}$. These states can be thought of as having perfectly defined phases $\phi$. Note that these states have an infinite spread in the number of photons.

(c) By drawing an analogy between the operator pairs $(\hat{N}, \hat{\phi})$ and $(\hat{x}, \hat{p})$, argue that there is an uncertainty relation $\Delta N \Delta\phi \geq 1/2$.

(d) Comment on how the number phase uncertainty relation from part (c) relates to the result of problem 2(c).

**4. Coherent states of the electric field.** Recall from class that there is a close analogy between the electric field operator corresponding to a given mode and the position operator $\hat{x}$ of a quantum harmonic oscillator. From studying the quantum harmonic oscillator in other courses, some of you might have encountered the notion of a coherent state. A coherent state $|\alpha\rangle$ is defined to be an eigenstate of the annihilation operator $\hat{a}$ with eigenvalue $\alpha$, where $\alpha$ can be a general complex number:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \tag{4}$$

Coherent states are often described as the most classical states of a quantum harmonic oscillator, since they satisfy the lower bound of the position-momentum uncertainty relation at all times, $\Delta x \Delta p = \hbar/2$. Many electromagnetic fields, such as the electromagnetic waves produced by typical lasers, are well-described as coherent states. In this problem, we will review/study some of the properties of coherent states.

(a) Express the coherent state $|\alpha\rangle$ as a linear combination of the number eigenstates $|n\rangle$.

(b) Describe the distribution of the photon number N. Find the expectation value and standard deviation of the photon number.

(c) Determine the time evolution of $|\alpha\rangle$. Does the distribution of the photon number change in time?

(d) In the limit of large photon number $N$, find the expectation value and standard deviation of the operator $e^{i\hat{\phi}}$ in the coherent state $|\alpha\rangle$.

(e) Show that in the limit of a small phase perturbation $d\phi$, $e^{i(\phi+d\phi)} \approx e^{i\phi} + ie^{i\phi}d\phi$. Use this relation and your results from earlier in this problem to show that a coherent state with large photon number N approximately meets the lower bound of the number phase uncertainty relation–that is, $\Delta N \Delta\phi \sim 1$.

**5. Transmission microscopy of biological samples: quantum optics considerations.**

A common technique in optics is transmission microscopy. Light is sent through a given portion of a sample. Passing through the sample one time then imprints a phase shift $\phi$ on the light, depending, for instance, on the index of refraction and thickness of the sample at the point being imaged. $\phi$ is then measured by interfering the light that passed through the sample with a reference laser beam. An image of the sample can be reconstructed by measuring $\phi$ at various sub-regions across the sample. More precise measurements of $\phi$ yield images with better signal-to-noise.

Passing more photons through each point in the sample increases the signal-to-noise of the image. However, in many cases, the photon number cannot be increased without limit, because exposure to too many photons will damage the sample. This is especially true, for example, for many biological samples.

Let us say that the damage threshold for each sub-region of the sample to be imaged is $N$ photons. We will consider two different approaches to measuring the phase shift $\phi$ corresponding to each sub-region. 1. Either we send a coherent state with $\sim N$ photons once through the sample (recall that lasers typically produce coherent states). 2. Or we take a coherent state with $\sim n$ photons and pass it through the sample $m$ times, with $N = mn$ (an appropriate set of mirrors can be used to reflect the light through the sample multiple times). While approach 1 involves $m$ times more photons than approach 2, in approach 2 each photon interacts with the sample $m$ times. So the total amount of damage is the same in both cases. Which of the two approaches will yield an image of the sample with a higher signal-to-noise ratio? Justify your answer using the quantum optics concepts explored in this problem set.