Physics 412-1 Problem Set 4 Solutions

Problem 1

We must solve the equation

$$(-\frac{\hbar}{2m}\frac{d^2}{dx^2} - V_0\delta(x))\phi(x)) = E\phi(x) \tag{1}$$

for $E < 0$.
For $x > 0$, the euqation is

$$-\frac{\hbar}{2m}\frac{d^2}{dx^2}\phi(x)) = E\phi(x). \tag{2}$$

Therefore the solution is

$$\phi_>(x) = e^{-\kappa x}, \tag{3}$$

where $\kappa = \sqrt{\frac{-2mE}{\hbar^2}}$. Similarly for $x < 0$, the solution is

$$\phi_<(x) = e^{\kappa x}. \tag{4}$$

.
To see how to match these solution, integrate the full equation over a small interval around $x = 0$, we have

$$-\frac{\hbar}{2m}\frac{d\phi}{dx}\bigg|_{-\epsilon}^{+\epsilon} - V_0\phi(0) = O(\epsilon). \tag{5}$$

So as $\epsilon \to 0$,

$$-\frac{\hbar}{2m}(\frac{d\phi_>}{dx}(0) - \frac{d\phi_<}{dx}(0)) = V_0\phi(0). \tag{6}$$

And therefore $\kappa = \frac{mV_0}{\hbar^2}$ and $E = -\frac{mV_0^2}{2\hbar^2}$.
To sum up, $\phi(x) = \frac{\sqrt{mV_0}}{\hbar}e^{-mV_0|x|/\hbar^2}$ and energy $E = -\frac{mV_0^2}{2\hbar^2}$, which is a unique answer by the procedure.

Problem 2

(a)

Let the wavefunction to be an eigenstate over all sapce with $E = \frac{\hbar^2 k^2}{2m}$. Inside the well, we have $\phi_a(x) = T_0 e^{ik_0 x} + R_0 e^{-ik_0 x}$, with energy $\frac{\hbar^2 k_0^2}{2m} - V_0$ Therefore

$$k_0 = \sqrt{k^2 + \frac{2mV_0}{\hbar^2}} \tag{7}$$

Consider the boundary conditions, at $x = 0$ we have

$$\phi_L = \phi_R \to 1 + R = R_0 + T_0, \tag{8}$$

$$\phi_R' = \phi_L' \to k(1 - R) = k_0(T_0 - R_0). \tag{9}$$

At $x = a$, we have

$$R_0 e^{-ik_0 a} + T_0 e^{ik_0 a} = T e^{ika}, \tag{10}$$

$$k_0(T_0 e^{ik_0 a} - R_0 e^{-ik_0 a}) = kTe^{ika}. \tag{11}$$

Solve the above equations, we have

$$T_0 = \frac{2(1+c)}{(1+c)^2 - (1-c)^2 e^{i\theta}} \tag{12}$$

$$R_0 = \frac{-2e^{-i\theta}(1-c)}{(1+c)^2 - (1-c)^2 e^{i\theta}} \tag{13}$$

$$R = \frac{(e^{i\theta} - 1)(c^2 - 1)}{(1+c)^2 - (1-c)^2 e^{i\theta}} \tag{14}$$

$$T = \frac{4e^{i(k_0 - k)a}}{(1+c)^2 - (1-c)^2 e^{i\theta}} \tag{15}$$

where $c = \frac{k_0}{k}$ and $\theta = 2k_0 a$

(b)

As $E \to \infty$, $c \to 1$, so $R \to \frac{(e^{i\theta}-1)(0)}{1} = 0$.

(c)

As $V_0 \to \infty$, $c \to \infty$. So

$$R \to \frac{(e^{i\theta} - 1)c^2}{c^2(1 - e^{i\theta})} \tag{16}$$

If $e^{i\theta} - 1 \neq 0$, $R \to -1$.
But if $e^{i\theta} = 1$, we need to go back and calculate

$$R = T_0 + R_0 - 1 = \frac{2(1+c) - 2(1-c)}{1 + 2c + c^2 - 1 + 2c - c^2} - 1 = \frac{4c}{4c} - 1 = 0 \tag{17}$$

To sum up, as $V_0 \to \infty$, $R \to -1$(totally reflected with π phase shift), unless $k_0 a = n\pi$, in which case $R \to 0$. This phenomenon is called a resonance, and has no analog in motion of classical particles.