The Wave Equation

In vacuum, with no sources ($\rho = 0$, $\vec{j} = 0$), Maxwell's equations for the electromagnetic field are:

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{H} = 0$$

$$\vec{\nabla} \times \vec{H} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{E} = 0$$

We also recall that:

$$\vec{H} = \vec{\nabla} \times \vec{A}$$

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi$$

ϕ is scalar potential, $\vec{A}$ is vector potential

To simplify the equations, we can make the choice of gauge:

$$\phi = 0 \quad \text{and} \quad \vec{\nabla} \cdot \vec{A} = 0 \quad \text{(Coulomb gauge)}$$

Then:

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

$$\frac{\partial \vec{E}}{\partial t} = -\frac{1}{c}\frac{\partial^2 \vec{A}}{\partial t^2}$$

Plug into equation for $\vec{\nabla}\times\vec{H}$:

$$c\,\vec{\nabla}\times\vec{H} = -\frac{1}{c}\frac{\partial^2\vec{A}}{\partial t^2}$$

Also:

$$\vec{\nabla}\times\vec{H} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\overbrace{\vec{\nabla}\cdot\vec{A}}^{\to 0 \text{ by choice of gauge}}) - \vec{\nabla}^2\vec{A} = -\vec{\nabla}^2\vec{A}$$

so: $$\boxed{\vec{\nabla}^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = 0}$$

To get an equation for $\vec{E}$, take another time derivative:

$$\vec{\nabla}^2\frac{\partial\vec{A}}{\partial t} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\frac{\partial\vec{A}}{\partial t}\right) = 0 \qquad \left(\text{recall } \vec{E} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right)$$

$$\Rightarrow \boxed{\vec{\nabla}^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0}$$

To get an equation for $\vec{H}$, take the curl:

$$\vec{\nabla}\times(\vec{\nabla}^2\vec{A}) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\vec{\nabla}\times\vec{A}) = 0$$

From above, $\vec{\nabla}^2\vec{A} = -\vec{\nabla}\times\vec{H}$, so:

$$\vec{\nabla}\times(\vec{\nabla}^2 A) = -\vec{\nabla}\times(\vec{\nabla}\times\vec{H}) = -\left[\vec{\nabla}(\overbrace{\vec{\nabla}\cdot\vec{H}}^{0 \text{ from Maxwell's equations}}) - \vec{\nabla}^2\vec{H}\right] = \vec{\nabla}^2\vec{H}$$

$$\Rightarrow \boxed{\nabla^2\vec{H} - \frac{1}{c^2}\frac{\partial^2\vec{H}}{\partial t^2} = 0}$$

The wave equations for $\vec{E}$ and $\vec{H}$ have some resemblance to the 1D scalar wave equation:

$$\frac{\partial^2 f}{\partial t^2} - c^2 \frac{\partial^2 f}{\partial z^2} = 0$$

where z is chosen as the one spatial dimension to consider. It is instructive to examine solutions to this equation. Note that:

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial z}\right) f =$$

$$= \frac{\partial^2 f}{\partial t^2} - c\frac{\partial f}{\partial t \partial z} + c\frac{\partial f}{\partial t \partial z} - c^2\frac{\partial^2 f}{\partial z^2} = \frac{\partial^2 f}{\partial t^2} - c^2\frac{\partial^2 f}{\partial z^2} = 0$$

The 1D scalar wave equation can therefore be written as:

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial z}\right) f = 0$$

Let us consider functions of the form

$$f = f_1\left(t - \frac{z}{c}\right) + f_2\left(t + \frac{z}{c}\right)$$

for arbitrary functions f_1 and f_2. Let $\alpha = t - \frac{z}{c}$.

$$\frac{\partial f_1(\alpha)}{\partial t} = \frac{\partial f_1(\alpha)}{\partial \alpha}\frac{\partial \alpha}{\partial t} = \frac{\partial f_1(\alpha)}{\partial \alpha}$$

$$c\frac{\partial f_1(\alpha)}{\partial z} = c\frac{\partial f_1(\alpha)}{\partial \alpha}\frac{\partial \alpha}{\partial z} = c\frac{\partial f_1(\alpha)}{\partial \alpha}\left(-\frac{1}{c}\right) = -\frac{\partial f_1(\alpha)}{\partial \alpha} = -\frac{\partial f_1(\alpha)}{\partial t}$$

so $\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial z}\right) f_1\left(t - \frac{z}{c}\right) = 0$, so $f_1\left(t - \frac{z}{c}\right)$ satisfies the wave equation. A similar argument shows that $f_2\left(t + \frac{z}{c}\right)$ also satisfies the wave equation, because $\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial z}\right) f_2\left(t - \frac{z}{c}\right) = 0$.

The sum $f = f_1(t - \frac{z}{c}) + f_2(t + \frac{z}{c})$ is also a solution

<u>Example:</u> Consider the case where $f_2 = 0$. Then our solution is $f_1(t - \frac{z}{c})$. This solution means that the field f_1 has the same value each time the argument $t - \frac{z}{c} = A$, for some constant A.

This corresponds to:

$$z = \text{constant} + ct$$

$\Rightarrow$ the values of the field propagate in space along the positive direction of z at the speed of light c.

Similarly, in the case where $f_1 = 0$, the solution $f_2(t + \frac{z}{c})$ corresponds to a wave traveling along the negative direction of z with speed c.

Plane Waves

Plane waves are a special case in which the fields depend on only one spatial coordinate and on time.

Let us choose this spatial coordinate to be z for the moment. The wave equation for $\vec{A}$ then is:

$$\vec{\nabla}^2 \vec{A}(z,t) - \frac{1}{c^2} \frac{\partial^2 \vec{A}(z,t)}{\partial t^2} = 0$$

Since there is no dependence of $\vec{A}$ on x or y, this reduces to:

$$\frac{\partial^2 A_i(z,t)}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 A_i(z,t)}{\partial t^2} = 0$$

$$\Rightarrow \quad \frac{\partial^2 A_i(z,t)}{\partial t^2} - c^2 \frac{\partial^2 A_i(z,t)}{\partial z^2} = 0$$

For $i = x, y, z$. That is, we get a 1D wave equation for each of the three components A_x, A_y, and A_z of $\vec{A}$.

Recall that we chose our gauge so that

$$\phi = 0, \quad \vec{\nabla} \cdot \vec{A} = 0$$

In this case, where $\vec{A}$ only depends on z and t,

$$\vec{\nabla} \cdot \vec{A} = 0 \Rightarrow \frac{\partial A_z}{\partial z} = 0$$

Therefore, the wave equation for A_z reduces to

$$\frac{\partial^2 A_z}{\partial t^2} = 0 \implies \frac{\partial A_z}{\partial t} = \text{constant}$$

In our gauge, $\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}$, so $E_z = \text{constant}$.

A constant background field has no relation to the electromagnetic wave. In this case, a nonzero A_z can just give a constant background E_z component, so for the purpose of considering plane waves <u>we set $A_z = 0$.</u> Since A_z has no dependence on x or y, it will not contribute to the curl $\vec{\nabla} \times \vec{A} = \vec{H}$, so it will not influence the magnetic field.

We consider solutions to the wave equation that move in the positive z direction, so

$$\vec{A} = \vec{A}\left(t - \frac{z}{c}\right) = \vec{A}(\alpha) \quad \text{where } \alpha = t - \frac{z}{c}$$

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial \alpha}\underbrace{\frac{\partial \alpha}{\partial t}}_{=1} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial \alpha}$$

$$\vec{H} = \vec{\nabla} \times \vec{A} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times \vec{A} = \left(\overbrace{\frac{\partial \alpha}{\partial x}}^{0}\frac{\partial}{\partial \alpha}, \overbrace{\frac{\partial \alpha}{\partial y}}^{0}\frac{\partial}{\partial \alpha}, \frac{\partial \alpha}{\partial z}\frac{\partial}{\partial \alpha}\right) \times \vec{A}$$

$$= \left(0, 0, -\frac{1}{c}\frac{\partial}{\partial \alpha}\right) \times \vec{A} = -\frac{1}{c}\,\hat{z} \times \frac{\partial \vec{A}}{\partial \alpha}$$

($\hat{z}$ is unit vector in +z direction)

Comparing the equations for $\vec{E}$ and $\vec{H}$, we see that:

$\boxed{\vec{H} = \hat{z} \times \vec{E}}$ and we assume $\boxed{\vec{E} \perp \hat{z}}$ from above

$\vec{E}$, $\hat{z}$, and $\vec{H}$ are mutually perpendicular

$\vec{E}$ and $\vec{H}$ point in x-y plane

This implies that the electric field, the magnetic field, and the direction of propagation point in mutually perpendicular directions.

The energy flux is given by the Poynting vector:

$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{H} = \frac{c}{4\pi}\vec{E}\times(\hat{z}\times\vec{E}) = \left[(\vec{E}\cdot\vec{E})\hat{z} - \underbrace{(\vec{E}\cdot\hat{z})}_{0}\vec{E}\right]\frac{c}{4\pi}$$

$$= \frac{c}{4\pi}E^2\hat{z}$$

As expected, energy flows along the propagation direction $\hat{z}$.

Monochromatic Plane Waves and Polarization

In many experimentally relevant situations, we are interested in waves that oscillate in time at a single frequency w.

For a plane wave propagating in the $\hat{z}$ direction, we are therefore interested in solutions of the form

$$\vec{A} \propto \cos(kz - wt - \theta),$$ where $k = \frac{w}{c}$ and θ is a constant representing the phase offset of the oscillation

It is mathematically nicer to work with complex exponentials and take the real part at the end. Note that:

$$e^{i(kz - wt - \theta)} = \cos(kz - wt - \theta) + i\sin(kz - wt - \theta)$$

When we take the real part, we get

$$\text{Re}\left[e^{i(kz - wt - \theta)}\right] = \cos(kz - wt - \theta)$$

Using the complex exponential approach, our solution for $\vec{A}$ takes the form

$$\vec{A} = \vec{A}_0 e^{i(\vec{k}\cdot\vec{r} - wt)} \quad ; \quad \vec{r} = (x, y, z) \; ; \; \vec{k} = \frac{w}{c}\hat{n}$$

$\vec{A}_0$ is a complex valued constant. Here, we have absorbed the factor $e^{-i\theta}$ into $\vec{A}_0$ and generalized to the case of propagation along a direction $\hat{n}$:

$\hat{n}$ is a unit vector pointing in the direction of propagation

Note also that $\hat{n} = \hat{k}$.

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} = i\frac{w}{c}\vec{A} = ik\vec{A}$$

$$\vec{H} = \vec{\nabla}\times\vec{A} = i\vec{k}\times\vec{A}$$

For the electric field, we can write

$$\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-wt)}$$

$\vec{E}_0$ is a complex vector with amplitude and phase. In general, we note that $\vec{E}^2$ is also complex.

We will write $\vec{E}_0$ as

$$\vec{E}_0 = \left(\vec{E}_{0r} + i\vec{E}_{0i}\right)e^{-i\theta}$$

The ultimate goal is to get a convenient picture of direction which $\vec{E}$ points as a function of time

where $\vec{E}_{0r}$ and $\vec{E}_{0i}$ are real vectors, and θ is real.

We will choose θ so that $\vec{E}_{0r}$ and $\vec{E}_{0i}$ are mutually orthogonal. Let $\vec{a}$ and $\vec{b}$ denote the real and imaginary parts of $\vec{E}_0$:

$$\vec{a} \equiv Re[\vec{E}_0], \quad \vec{b} \equiv Im[\vec{E}_0]$$

Since $\vec{E}_{0r} + i\vec{E}_{0i} = \vec{E}_0 e^{i\theta} = (\vec{a}+i\vec{b})e^{i\theta}$:

$$\vec{E}_{0r} = \vec{a}\cos\theta - \vec{b}\sin\theta$$

$e^{i\theta} = \cos\theta + i\sin\theta$

$$\vec{E}_{0i} = \vec{a}\sin\theta + \vec{b}\cos\theta$$