Transitions Between Two States

Many experiments deal primarily with transitions between only two states at a time

- NMR
- microwave transitions
- laser transitions

Highlights:

- Illustrates interaction representation
- Rotating transition operators "general"
- Rotating wave approximation
- Rabi "flopping"
- Rabi frequency
- Lorentzian lineshape

$\boxed{\text{Two State System}}$ — in Interaction Representation

$$|+\rangle \qquad H_0|+\rangle = \tfrac{1}{2}\hbar\omega_0|+\rangle$$
$$\hbar\omega_0$$
$$|-\rangle \qquad H_0|-\rangle = -\tfrac{1}{2}\hbar\omega_0|-\rangle$$
$$\Bigg\} \; H_0 = \tfrac{1}{2}\hbar\omega_0\sigma_z$$

Add $V(t)$ to make transitions: $H = H_0 + V(t)$

$$(V_{++} = V_{--} = 0)$$

Schrodinger equation in an interaction representation

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = (H_0 + V)|\psi\rangle \quad \leftarrow \text{exact}$$

$$H_0|n\rangle = \hbar\omega_n|n\rangle \quad \leftarrow \text{energy eigenstates}$$
$$|\psi\rangle = \sum C_m(t)e^{-i\omega_m t}|m\rangle$$

$$\langle n|i\hbar\sum_m[\dot{C}_m - i\omega_m C_m]e^{-i\omega_m t}|m\rangle$$
$$= \sum_m\langle n|\hbar\omega_m C_m e^{-i\omega_m t}|m\rangle + \sum_m\underbrace{\langle n|V|m\rangle}_{V_{nm}}C_m e^{-i\omega_m t}$$

$$i\hbar\dot{C}_n e^{-i\omega_n t} + \cancel{\hbar\omega_n C_n e^{-i\omega_n t}} = \cancel{\hbar\omega_n C_n e^{-i\omega_n t}} + \sum_m V_{nm}C_m e^{-i\omega_m t}$$

$$i\hbar\dot{C}_n = \sum_m V_{nm}C_m e^{-i\underbrace{(\omega_m-\omega_n)}_{\equiv\omega_{mn}}t}$$

$$\boxed{i\hbar\dot{C}_n = \sum_m C_m e^{-i\omega_{mn}t}V_{nm}(t)}$$

exact

S.E. Int. Rep.

Apply to 2-state:

$$i\hbar\dot{C}_+ = C_- e^{-i\omega_{-+}t}V_{+-}(t) \longrightarrow \quad i\hbar\dot{C}_+ = C_- e^{i\omega_0 t}V_{+-}(t)$$

$$i\hbar\dot{C}_- = C_+ e^{-i\omega_{+-}t}V_{-+}(t) \longrightarrow \quad i\hbar\dot{C}_- = C_+ e^{-i\omega_0 t}V_{-+}(t)$$

$\boxed{\text{Construct a "General" Transition Operator}}$

Assume $C_n(t)$ varies slowly compared to $e^{\pm i\omega_0 t}$

$$i\hbar \dot{C}_+ = C_- \underbrace{e^{i\omega_0 t}}_{\text{rapidly varying}} \underbrace{V_{+-}(t)}$$

slowly varying

① to avoid averaging $\dot{C}_+$ to zero
$\Rightarrow$ need $V_{+-}(t) \sim e^{-i\omega t}$

② need $\frac{\sigma_+}{2} e^{-i\omega t}$ to have non-zero matrix element

$\therefore V(t) \sim \frac{\sigma_+}{2} e^{-i\omega t}$ ← makes $|-\rangle \uparrow |+\rangle$

Since $V(t)$ must be Hermitian

$$\boxed{V(t) = \underbrace{\tfrac{1}{2}\hbar\Omega_R}_{\text{a constant that gives the strength of the drive}} \left[\frac{\sigma_+}{2} e^{-i\phi(t)} + \frac{\sigma_-}{2} e^{i\phi(t)} \right]}$$

$\sigma_+^{\dagger} = \sigma_-$ real

$$\boxed{\phi(t) = \phi(0) + \int_0^t \omega(t)\, dt}$$

<u>Check equation for $\dot{C}_-$</u>

$$i\hbar \dot{C}_- = C_+ \underbrace{e^{-i\omega_0 t}}_{\text{fast}} V_{-+}(t)$$

slow

need $\frac{\sigma_-}{2} e^{i\omega t}$ ∴ <u>OK</u>

$|+\rangle \downarrow |-\rangle$

When only 2-states are important
- adjust $\phi(0)$
  and/or relative phase of $|+\rangle$ and $|-\rangle$
  to make Ω_R real and positive

<u>$V(t)$ is a rotating drive</u>

$$V(t) = \tfrac{1}{2}\hbar\Omega_R \left[\frac{\sigma_x + i\sigma_y}{2}(\cos\phi - i\sin\phi) + \frac{\sigma_x - i\sigma_y}{2}(\cos\phi + i\sin\phi) \right]$$
$$= \tfrac{1}{2}\hbar\Omega_R \left[\tfrac{1}{2}\cos\phi(\sigma_x + \cancel{i\sigma_y} + \sigma_x - \cancel{i\sigma_y}) + \tfrac{1}{2}\sin\phi(\cancel{-i\sigma_x} + \sigma_y + \cancel{i\sigma_x} + \sigma_y) \right]$$

$$\boxed{V(t) = \tfrac{1}{2}\hbar\Omega_R \left[\sigma_x \cos\phi + \sigma_y \sin\phi \right]}$$

$\dot{\phi} \equiv \omega \approx \omega_0$

A bit more generally:

Hermitian drive $\quad V(t) = V e^{i\phi(t)} + V^{\dagger} e^{-i\phi(t)}$

op. ind. of time

Most general $V = a\frac{\sigma_-}{2} + b\frac{\sigma_+}{2} + \cancel{c}\,\sigma_z + \cancel{d}\,\mathbb{1}$

do not make transitions (absorb into H_0)

$\sigma_{\pm}^{\dagger} = \sigma_{\mp}$

$$V(t) = \left(a\frac{\sigma_-}{2} + b\frac{\sigma_+}{2}\right)e^{i\phi} + \left(a^*\frac{\sigma_+}{2} + b^*\frac{\sigma_-}{2}\right)e^{-i\phi}$$

$$= \underbrace{a^*\frac{\sigma_+}{2}e^{-i\phi} + a\frac{\sigma_-}{2}e^{i\phi}}_{} + \cancel{b^*\frac{\sigma_-}{2}e^{-i\phi} + b\frac{\sigma_+}{2}e^{i\phi}}$$

make real by choosing phase & time origin

rotating transition op.

$$\frac{1}{2}\hbar\Omega_R\left[\frac{\sigma_+}{2}e^{-i\phi} + \frac{\sigma_-}{2}e^{i\phi}\right]$$

these combos do not cause transitions (prev. page)

Neglect = rotating wave approx

Can have small effects e.g. Bloch-Siegert shifts

How to determine Rabi frequency Ω_R

1. Start with actual oscillating drive $V(t)$
2. Make R.W.A. $\quad V(t) = \frac{1}{2}\hbar\Omega_R\left[\frac{\sigma_+}{2}e^{-i\phi} + \frac{\sigma_-}{2}e^{i\phi}\right] + \cancel{(\quad)}$
3. Take matrix elements of what remains

$$\langle -|V(t)|+\rangle = \frac{1}{2}\hbar\Omega_R e^{-i\phi(t)}$$

$$\langle +|V(t)|-\rangle = \frac{1}{2}\hbar\Omega_R e^{i\phi(t)}$$

use either

(i.e. after RWA: $V = Ve^{i\phi} + V^{\dagger}e^{-i\phi}$ $\quad \boxed{\frac{1}{2}\hbar\Omega_R = \langle -|V|+\rangle}$)

STRONG DRIVING OF 2-STATE SYSTEM

transitions are made quickly enough to neglect decay
(i.e. closed system)

Rabi frequency – defined as amplitude of a transition operator
turns out to be frequency at which probability oscillates

π-pulses, $\pi/2$ pulses, 2π pulses, etc.

Off-resonance Rabi Flopping

Tops and a visualizing transitions – Bloch vector

Rotating reference frame

Strong Drive — transitions fast enough so we neglect decay

2 G.G.

Solve 2-State Problem

$$i\hbar\dot{C}_+ = C_- e^{i\omega_0 t} V_{+-}(t)$$

$$= C_- e^{i\omega_0 t} \tfrac{1}{2}\hbar\Omega_R e^{-i\phi}$$

$$\boxed{i\dot{C}_+ = \tfrac{1}{2}\Omega_R C_- e^{-i[\phi-\omega_0 t]}}$$

$$i\hbar\dot{C}_- = C_+ e^{-i\omega_0 t} V_{-+}(t)$$

$$= C_+ e^{-i\omega_0 t} \tfrac{1}{2}\hbar\Omega_R e^{i\phi}$$

$$\boxed{i\dot{C}_- = \tfrac{1}{2}\Omega_R C_+ e^{i[\phi-\omega_0 t]}}$$

Make decoupled, 2nd order equations

Let $\dot{\phi}(t) \equiv \omega(t) \equiv \omega_0 + \epsilon(t)$

$$i\ddot{C}_+ = \tfrac{1}{2}\Omega_R [\dot{C}_- - i\overbrace{(\dot{\phi}-\omega_0)}^{\epsilon} C_-] e^{-i[\phi-\omega_0 t]}$$

$$= \tfrac{1}{2}\Omega_R\left\{\tfrac{1}{i}\tfrac{1}{2}\Omega_R C_+ \cancel{e^{i[\phi-\omega_0 t]}}\right\}\cancel{e^{-i[\phi-\omega_0 t]}}$$

$$+\tfrac{1}{\cancel{2}}\cancel{\Omega_R}(-i\epsilon)\left\{i\dot{C}_+\frac{\cancel{2}}{\cancel{\Omega_R}}\cancel{e^{i[\phi-\omega_0 t]}}\right\}\cancel{e^{-i[\phi-\omega_0 t]}}$$

$$= -i\tfrac{1}{4}\Omega_R^2 C_+ + \epsilon\dot{C}_+$$

$$-\ddot{C}_+ = \tfrac{1}{4}\Omega_R^2 C_+ + i\epsilon\dot{C}_+$$

$$\boxed{0 = \ddot{C}_+ + i\epsilon\dot{C}_+ + \tfrac{1}{4}\Omega_R^2 C_+}$$

$\epsilon = \epsilon(t)$

$+ \leftrightarrow -$ $\quad -\phi + \omega_0 t \leftrightarrow \phi - \omega_0 t \longrightarrow \therefore \epsilon \to -\epsilon$

$$\boxed{0 = \ddot{C}_- - i\epsilon\dot{C}_- + \tfrac{1}{4}\Omega_R^2 C_-}$$

On Resonance: $\epsilon = 0$ ← ind. of time

$$\boxed{\begin{aligned} 0 &= \ddot{C}_+ + \tfrac{1}{4}\Omega_R^2 C_+ \\ 0 &= \ddot{C}_- + \tfrac{1}{4}\Omega_R^2 C_- \end{aligned}}$$

← harmonic osc. equations

↑ note: freq. $\sim \Omega_R$ is our measure of trans. op. strength

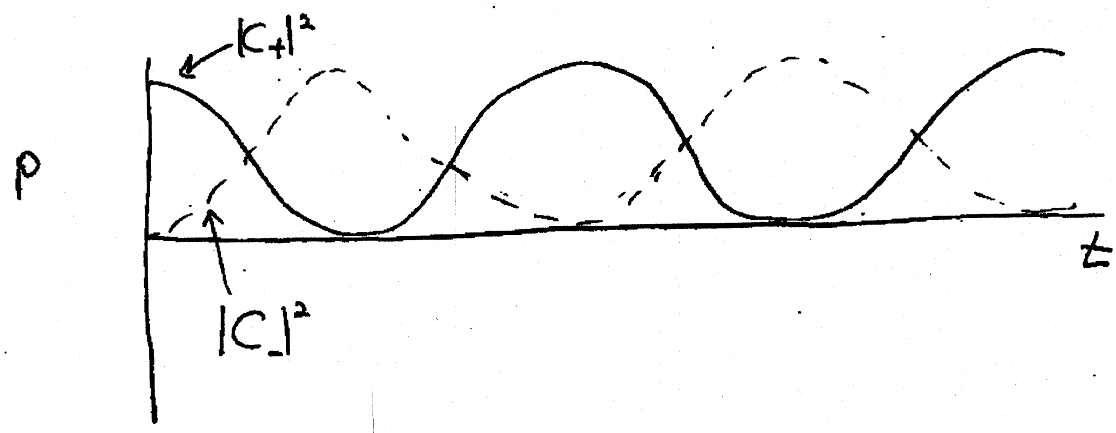
$$C_+ \sim \sin \tfrac{1}{2}|\Omega_R|t$$

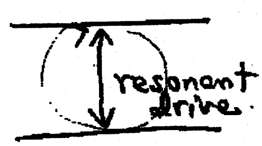
$$P_+ \sim |C_+|^2 \sim \sin^2(\tfrac{1}{2}\Omega_R t) = \tfrac{1}{2}[1 - \cos|\Omega_R|t]$$

↑ Prob. oscillates at the Rabi frequency

$$C_- \sim \sin \tfrac{1}{2}|\Omega_R|t$$

$$P_- \sim |C_-|^2 \sim \tfrac{1}{2}[1 + \cos|\Omega_R|t]$$



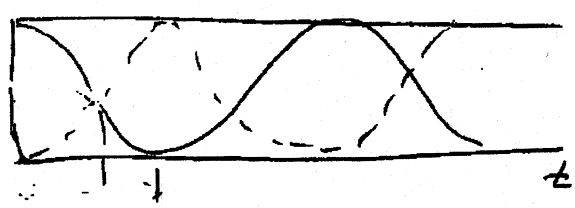


Some Often Used Terminology

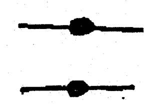
Recall: $\Omega_R = \dfrac{|V_{+-}|}{\hbar/2} = \dfrac{|V_{-+}|}{\hbar/2}$

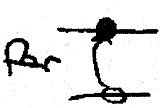
$\pi/2$ pulse: $V(t)$ is turned on for a time t

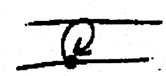
$$\Omega_R t = \pi/2$$



coherent superposition

* $\frac{\pi}{2}$ pulse $\longleftrightarrow$ $|-\rangle \rightarrow |-\rangle, |+\rangle$ 

* π pulse $\longleftrightarrow$ complete population transfer 

2π pulse no change 

etc.

Probing a Two-Level System with Weak Damping and Weak Drive

$|+\rangle$

$|-\rangle$

Weak decay: $\gamma \ll \omega_0$

$$|+\rangle \longrightarrow |+\rangle e^{-\frac{1}{2}\gamma t}$$

$$\langle +|+\rangle = 1 \longrightarrow \langle +|+\rangle e^{-\gamma t}$$ ← exponential decay

phenomenological description of decay

- easy
- not quite right
- random decay messes up phases

$\Rightarrow$ should use density operator and "master equation"

Ad hoc patch of decay into the Sch. eq.

$$e^{-i\omega_+ t} \longrightarrow e^{-i\omega_+ t - \frac{1}{2}\gamma t}$$

$$\therefore\ i\hbar \dot{C}_+ = C_- e^{i\omega_0 t - \frac{1}{2}\gamma t}\, V_{+-}(t)$$ ← $\frac{1}{2}\hbar\Omega_R e^{-i\phi}$

$$i\hbar \dot{C}_- = C_+ e^{-i\omega_0 t - \frac{1}{2}\gamma t}\, V_{-+}(t)$$ ← $\frac{1}{2}\hbar\Omega_R e^{i\phi}$

Weak drive:

$$\left.\begin{aligned} C_-(t) &\approx C_-(0) \approx 1 \\ C_+(t) &\approx C_+(0) \approx 0 \end{aligned}\right\}$$ perturbation treatment

$$\therefore\ i\hbar \dot{C}_- \sim C_+ \approx 0 \longrightarrow C_-(t) \approx 1$$

$$i\hbar \dot{C}_+ = \underbrace{C_-}_{\approx 1} e^{i\omega_0 t - \frac{1}{2}\gamma t}\, \tfrac{1}{2}\hbar\Omega_R e^{-i\phi}$$

Use $\phi = \omega t$: $\underbrace{i\hbar \dot{C}_+ = \tfrac{1}{2}\hbar\Omega_R\, e^{-i(\omega-\omega_0)t - \frac{1}{2}\gamma t}}_{\text{integrate directly}}$

"Steady-state"

long compared to $\frac{1}{\gamma}$ (ie. $T \to \infty$)

$$i\hbar\left[C_+(T) - \cancel{C_+(0)}^{\,0}\right] = \int_0^T \tfrac{1}{2}\hbar\Omega_R\, e^{-i(\omega-\omega_0)t - \frac{1}{2}\gamma t}\, dt$$

$$C_+(T) = \frac{1}{i\cancel{\hbar}}\frac{\cancel{\hbar}\Omega_R}{2}\;\frac{0-1}{-i(\omega-\omega_0) - \frac{1}{2}\gamma}$$

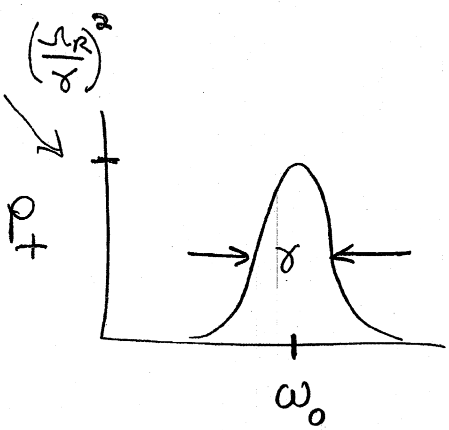
$$= -\frac{i\Omega_R}{2}\;\frac{1}{i(\omega-\omega_0)+\frac{1}{2}\gamma}$$

$$\int_0^T e^{-\lambda t}dt = \frac{1}{-\lambda}\, e^{-\lambda t}\Big|_0^T$$

Transition probability $0 \to 1$

$$\boxed{P_+ = |C_+(T)|^2 = \left(\frac{\Omega_R}{2}\right)^2 \frac{1}{(\omega-\omega_0)^2 + \left(\frac{\gamma}{2}\right)^2}}$$

Lorentzian



ω_0 = resonant freq. (max)

γ = FWHM

full width half max

Same shape as for a weakly-damped driven, classical oscillator.

Check that $P_+ \ll 1$ (i.e. to make sure drive is weak)

On res. $P_+ = \left(\frac{\Omega_R}{2}\right)^2\left(\frac{2}{\gamma}\right)^2 = \underbrace{\left(\frac{\Omega_R}{\gamma}\right)^2 \ll 1}$

$\boxed{\Omega_R < \gamma}$ weak drive