Physics 414-2 Problem Set 2

April 8, 2022

Due: Friday, April 15 at 4 pm

1. Fourier optics for Gaussian beams. Using Fourier optics, show that you can reproduce the expression from lecture regarding how a Gaussian beam propagates in z. Specifically, from lecture, we recall that the amplitude of a Gaussian beam has the form (with a factor of $e^{i(kz-\omega t)}$ pulled out as in lecture)

$$u(x,y,z) = u_0 \frac{w_0}{w(z)} \exp\left[-\frac{x^2+y^2}{w(z)^2}\right] \exp\left[i\left(k\frac{x^2+y^2}{2R(z)} - \arctan\left(\frac{z}{z_R}\right)\right)\right], \quad (1)$$

where $z_R = kw_0^2/2$, $w(z) = w_0\sqrt{1+(z/z_R)^2}$, and $R(z) = z(1+(z_R/z)^2)$. Starting with the amplitude at $z=0$, $u(x,y,0)$, show that you can reproduce the above expression for $u(x,y,z)$ using Fourier optics. I suggest that you use Mathematica for this, because the math will be a bit tedious to do by hand.

2. Fourier transforms with rotational symmetry and Bessel beams. (a) Consider a function $f(x,y)$. We can choose to use polar coordinates, so that $x = r\cos\theta$, $y = r\sin\theta$, and $r = \sqrt{x^2+y^2}$. In an ordinary two-dimensional Fourier transform, we express the result in terms of the components of the $\vec{k}$-vector in the x and y directions, k_x and k_y. We can also express the $\vec{k}$-vector components in polar coordinates, using $k_x = q\cos\phi$, $k_y = q\sin\phi$, and $q = \sqrt{k_x^2+k_y^2}$.

We will consider the special case in which f is rotationally symmetric, so that it depends only on r. In this case, we can express it as $f(r)$. Using polar coordinates as described above, show that the Fourier transform of $f(r)$ can be expressed as $F(q)$, with

$$F(q) = \int_0^\infty f(r) J_0(qr) r dr, \quad (2)$$

where J_0 is a function known as a zeroth order Bessel function of the first kind. You can use the relation $\int_0^{2\pi} e^{-irq\cos\theta} d\theta = 2\pi J_0(qr)$.

(b) Find an expression for $f(r)$ in terms of $F(q)$ (this corresponds to the inverse Fourier transform).

(c) Consider a laser beam that is an equal superposition of plane waves that have the following $\vec{k}$-vector components: z-component $k_z = k\cos\beta$, x-componet $k_x = k\sin\beta\cos\phi$, and y-component $k_y = k\sin\beta\sin\phi$ for all angles ϕ (from 0 to 2π). In polar coordinates, the distribution of the transverse components k_x and k_y forms an infinitely thin ring, which we can represent in terms of a Dirac delta function as

$$F(q) = \frac{\delta(q - k_r)}{k_r}, \tag{3}$$

with $k_r \equiv k\sin\beta$. What is the field amplitude (leaving out the factor of $e^{-i\omega t}$) as a function of the radial distance r from the z-axis and z? A laser beam of this type is called a Bessel beam.

(d) What happens to the transverse shape of the Bessel beam as z changes?

(e) Is it possible to physically create a Bessel beam? Why or why not?

3. Speed of shaped beams. (a) In the first lecture, we had considered a laser pulse directed from one satellite to another separated by a baseline L. For an ideal plane wave, each photon would take a time $\Delta t = L/c$ to travel over the baseline. For a Gaussian beam of waist w_0 and wave number k, what is the expectation value of the photon travel time Δt? What is the standard deviation of the photon travel time Δt? Assume the Gaussian beam form expressed in Eq. (1) with $z = 0$ located at the satellite emitting the photon.

(b) Repeat part (a) for a Bessel beam corresponding to an angle β from the baseline direction (see the previous problem).

4. Paraxial propagation of beam perturbations. We consider a scalar laser field of the form $u(x, y, z)e^{i(kz-\omega t)}$. For a perfect plane wave, $u(x, y, z) = 1$. Imagine that at $z = 0$, there is a small amplitude perturbation to the field that varies sinusoidally as a function of x, so that $u(x, y, 0) = 1 + \delta\cos(\kappa x)$. We assume that δ is real. Here, κ denotes the transverse spatial frequency of the perturbation.

(a) Using paraxial beam propagation, find the field $u(x, y, z)$ as a function of z.

(b) It can be useful to write $u(x, y, z)$ as a product of an amplitude and a phase factor: $u(x, y, z) = |u(x, y, z)|\, e^{i\phi(x,y,z)}$. To first order in δ, find $|u(x, y, z)|$ and $\phi(x, y, z)$.

(c) At $z = 0$, the perturbation only affects the amplitude while leaving the phase unchanged. Describe qualitatively how the respective effects of the perturbation on the amplitude and phase evolve as a function of z.

5. Gaussian beam with an amplitude perturbation. We consider a Gaussian beam with a sinusoidal amplitude perturbation so that at $z = 0$,

$$u(x, y, 0) = u_0 \frac{w_0}{w(z)} \exp\left[-\frac{x^2 + y^2}{w(z)^2}\right] \exp\left[i\left(k\frac{x^2 + y^2}{2R(z)} - \arctan\left(\frac{z}{z_R}\right)\right)\right](1+\delta \cos(\kappa x)), \tag{4}$$

where as in the previous problem, we assume that δ is real. Again, κ denotes the transverse spatial frequency of the perturbation.

Using paraxial beam propagation, find the field $u(x, y, z)$ as a function of z. You will likely find it useful to use Mathematica for this. The Manipulate function in Mathematica can be used to make animations which are helpful visualization tools. Use the Manipulate and Plot3D functions in Mathematica to make an animation of how $|u(x, y, z)|$ as a function of x and y changes as z evolves. Specifically, in your animation, make 3D plots of $|u(x, y, z)|$ vs. x and y for values of z ranging from 0 to 300. For your animation, plug in the values $u_0 = 1$, $w_0 = 1$, $\kappa = 30$, $k = 1000$, and $\delta = 0.5$.

Describe qualitatively what you see in your animation, and provide a simple physical description of why you see the observed behavior.

The Wolfram documentation that comes with Mathematica provides useful examples of the Manipulate and Plot3D functions. Aslan and I are happy to help you with your Mathematica code and with your interpretation of the results.