Time Evolution

H is the time translation operator

$$T(dt) \equiv 1 - \frac{i\,dt}{\hbar} H$$

$$T(dt)\,|\psi(t)\rangle = |\psi(t+dt)\rangle$$

$$|\psi(t)\rangle - \frac{i}{\hbar}\,dt\,H\,|\psi(t)\rangle = |\psi(t+dt)\rangle$$

$$\frac{|\psi(t+dt)\rangle - |\psi(t)\rangle}{dt} = \frac{1}{i\hbar}\,\frac{\cancel{dt}\,H\,|\psi(t)\rangle}{\cancel{dt}}$$

$$\boxed{i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = H\,|\psi(t)\rangle} \quad \text{S.E.}$$

Time evolution operator $U(t,t_0)$

$$|\psi(t)\rangle = U(t,t_0)\,|\psi(t_0)\rangle$$

$$i\hbar \frac{\partial}{\partial t} U(t,t_0)\,|\psi(t_0)\rangle = H\,U(t,t_0)\,|\psi(t_0)\rangle$$

$$\boxed{i\hbar \frac{\partial U(t,t_0)}{\partial t} = H\,U(t,t_0)} \rightarrow -i\hbar\frac{\partial U^\dagger}{\partial t} = U^\dagger H$$

Solution for H ind. of time

$$U(t,t_0) = e^{-\frac{i}{\hbar}H(t-t_0)}$$

Two pictures (for $\mathcal{O}$ ind. of time) → Review

Schrodinger: $|\psi(t_0)\rangle \longrightarrow |\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle$

$\mathcal{O} \longrightarrow \mathcal{O}$

Heisenberg: $|\psi(t_0)\rangle \longrightarrow |\psi(t_0)\rangle$

$\mathcal{O}_H(t_0) \longrightarrow \mathcal{O}_H(t)$

Insist that observables are the same

$t=t_0$: $\langle\psi(t_0)|\mathcal{O}_H(t_0)|\psi(t_0)\rangle = \langle\psi(t_0)|\mathcal{O}|\psi(t_0)\rangle$

$\therefore \boxed{\mathcal{O}_H(t_0) = \mathcal{O}}$

$t=t$: $\langle\psi(t_0)|\mathcal{O}_H(t)|\psi(t_0)\rangle = \langle\psi(t)|\mathcal{O}|\psi(t)\rangle$

$= \langle\psi(t_0)|U^\dagger(t,t_0)\mathcal{O}U(t,t_0)|\psi(t_0)\rangle$

$$\boxed{\mathcal{O}_H(t) = U^\dagger(t,t_0)\,\mathcal{O}\,U(t,t_0)}$$

Equation of motion for $\mathcal{O}_H(t)$

$$i\hbar\frac{\partial \mathcal{O}_H(t)}{\partial t} = i\hbar\frac{\partial}{\partial t}\left[U^\dagger \mathcal{O} U\right]$$

$$= i\hbar\frac{\partial U^\dagger}{\partial t}\mathcal{O}U + U\mathcal{O}\, i\hbar\frac{\partial U}{\partial t}$$

$$= -U^\dagger H\mathcal{O}U + U\mathcal{O}HU$$

$$= -\underbrace{U^\dagger H U}_{H_H}\underbrace{U^\dagger \mathcal{O}U}_{\mathcal{O}_H} + \underbrace{U\mathcal{O}U^\dagger}_{\mathcal{O}_H}\underbrace{UHU}_{H_H}$$

$$\boxed{i\hbar\frac{\partial \mathcal{O}_H(t)}{\partial t} = [\mathcal{O}_H, H_H]}$$

For H ind. of t $\Longrightarrow$ $[U,H]=H$ $\longrightarrow$ $\boxed{i\hbar\frac{\partial \mathcal{O}_H}{\partial t} = [\mathcal{O}_H, H]}$

Heisenberg eq. of motion

$\boxed{\text{Interaction Picture}}$ $\quad H = H_0 + V(t)$

H_0 ← independent of t

$\underline{H_0 \text{ has eigenstates:}}$ $\quad H_0 |n\rangle = \hbar\omega_n |n\rangle$

$\underline{\text{Time evolution from } H_0 \text{ alone:}}$

$$i\hbar \frac{\partial U_0(t,t_0)}{\partial t} = H_0 U_0(t,t_0) \longrightarrow U_0(t,t_0) = e^{-\frac{iH_0(t-t_0)}{\hbar}}$$

$\underline{V(t) \text{ makes transitions between } |n\rangle}$

$$|\psi(t)\rangle = \sum_n \underbrace{c_n(t)}_{\text{transitions between states}} \underbrace{e^{-i\omega_n(t-t_0)}}_{\text{time evolution of state } |n\rangle \text{ due to } H_0} |n\rangle$$

$\underline{\text{Focus on the time evolution due to interaction}}$

$$\boxed{|\psi(t)\rangle_I \equiv \underbrace{U_0^\dagger(t,t_0)}_{\text{undoes the evolution due to } H_0} |\psi(t)\rangle}$$

$$= \sum_n c_n(t) |n\rangle$$

$$= U_0^\dagger \sum_n c_n e^{-i\omega_n(t-t_0)} |n\rangle$$

$$= \sum_n c_n \cancel{e^{-i\omega_n(t-t_0)} e^{i\omega_n(t-t_0)}} |n\rangle$$

$$= \sum_n c_n(t) |n\rangle$$

$\underline{\text{wavefunction in the interaction representation}}$

$\underline{\text{At } t=t_0:}$ $\quad |\psi(t_0)\rangle_I = \underbrace{U_0^\dagger(t_0,t_0)}_{1} |\psi(t_0)\rangle$

$= |\psi(t_0)\rangle$ ← same in Schr. and Int. rep.

Observables in the Interaction Representation

Require ~~Want~~ $\underbrace{{}_I\langle\psi(t)|}_{\langle\psi(t)|\,\mathcal{U}_0}\, \mathcal{O}_I \,\underbrace{|\psi(t)\rangle_I}_{\mathcal{U}_0^+|\psi(t)\rangle} = \langle\psi(t)|\mathcal{O}|\psi(t)\rangle$

Sch. rep.

$$\therefore\ \mathcal{U}_0\,\mathcal{O}_I\,\mathcal{U}_0^+ = \mathcal{O}$$

$$\boxed{\mathcal{O}_I(t) = \mathcal{U}_0^+(t,t_0)\,\mathcal{O}\,\mathcal{U}_0(t,t_0)}$$

Evolution of $\mathcal{O}_I$ (for $\mathcal{O}$ ind. of time) and H_0 ind. of time

$i\hbar\frac{\partial \mathcal{U}_0}{\partial t} = H_0\mathcal{U}_0$

$-i\hbar\frac{\partial \mathcal{U}_0^+}{\partial t} = \mathcal{U}_0^+H_0$

$$i\hbar\frac{\partial \mathcal{O}_I}{\partial t} = i\hbar\frac{\partial}{\partial t}\,\mathcal{U}_0^+\mathcal{O}\,\mathcal{U}_0$$

$$= \underbrace{i\hbar\frac{\partial \mathcal{U}_0^+}{\partial t}}_{-\mathcal{U}_0^+H_0}\mathcal{O}\,\mathcal{U}_0 + \mathcal{U}_0^+\mathcal{O}\,\underbrace{i\hbar\frac{\partial \mathcal{U}_0}{\partial t}}_{H_0\mathcal{U}_0}$$

$$= \underbrace{\mathcal{U}_0^+\mathcal{O}\,\mathcal{U}_0}_{\mathcal{O}_I}H_0 - H_0\underbrace{\mathcal{U}_0^+\mathcal{O}\,\mathcal{U}_0}_{\mathcal{O}_I}$$

$$\boxed{i\hbar\frac{\partial \mathcal{O}_I}{\partial t} = [\mathcal{O}_I, H_0]}$$

not for av. values

not so useful

$$i\hbar\frac{\partial}{\partial t}\langle\mathcal{O}\rangle = i\hbar\frac{\partial}{\partial t}\,{}_I\langle\psi(t)|\mathcal{O}_I(t)|\psi(t)\rangle_I$$

$$= i\hbar\left(\frac{\partial}{\partial t}\,{}_I\langle\psi(t)|\right)\mathcal{O}_I(t)|\psi(t)\rangle_I + i\hbar\,{}_I\langle\psi(t)|\mathcal{O}_I(t)\frac{\partial}{\partial t}|\psi(t)\rangle_I$$

$$+\,{}_I\langle\psi(t)|\,i\hbar\frac{\partial \mathcal{O}_I}{\partial t}|\psi(t)\rangle_I$$

Time evolution of w.f. in interaction rep.

$$\underbrace{|\psi(t)\rangle_I}_{\mathcal{U}_0^{+}\underbrace{|\psi(t)\rangle}_{\mathcal{U}|\psi(t_0)\rangle}} = \mathcal{U}_I(t,t_0)\underbrace{|\psi(t_0)\rangle_I}_{|\psi(t_0)\rangle}$$

$$\therefore \boxed{\mathcal{U}_0^{+}\mathcal{U} = \mathcal{U}_I} \longrightarrow \boxed{\mathcal{U} = \mathcal{U}_0\mathcal{U}_I}$$

$i\hbar\dfrac{\partial \mathcal{U}}{\partial t} = H\mathcal{U}$

$-i\hbar\dfrac{\partial \mathcal{U}^{+}}{\partial t} = \mathcal{U}^{+}H$

$$i\hbar\frac{\partial \mathcal{U}_I}{\partial t} = i\hbar\frac{\partial}{\partial t}\mathcal{U}_0^{+}\mathcal{U}$$

$$= \underbrace{i\hbar\frac{\partial \mathcal{U}_0^{+}}{\partial t}}_{-\mathcal{U}_0^{+}H_0}\mathcal{U} + \mathcal{U}_0^{+}\underbrace{i\hbar\frac{\partial \mathcal{U}}{\partial t}}_{H\mathcal{U}}$$

$$= \underbrace{\mathcal{U}_0^{+}\underbrace{(H-H_0)}_{V}\mathcal{U}_0}_{V_I}\underbrace{\mathcal{U}_0^{+}\mathcal{U}}_{\mathcal{U}_I}$$

exact

$$\boxed{i\hbar\frac{\partial \mathcal{U}_I}{\partial t} = V_I\,\mathcal{U}_I} \xrightarrow{|\psi(t_0)\rangle} \boxed{i\hbar\frac{\partial |\psi(t)\rangle_I}{\partial t} = V_I(t)\,|\psi(t)\rangle_I}$$

<u>In terms of $c_n(t)$:</u> $|\psi(t)\rangle_I = \sum_n c_n(t)|n\rangle$

$V_I = \mathcal{U}_0^{+}\,\theta\,\mathcal{U}_0$

$$i\hbar\frac{\partial}{\partial t}\sum_m c_m|m\rangle = V_I\sum_m c_m|m\rangle$$

$\langle n| \rightarrow$ $$i\hbar\dot{c}_n = \sum_m \langle n|\mathcal{U}_0^{+}V\mathcal{U}_0|m\rangle c_m$$

$$\boxed{i\hbar\dot{c}_n = \sum_m c_m V_{nm}e^{i\omega_{nm}t}}$$ exact

quick first order

$\boxed{\text{Time Dependent Perturbation Theory}}$

$$H = H_0 + V(t) \longrightarrow \underbrace{\text{causes transitions}}_{\text{"small"}}$$

$$H_0 |n\rangle = \hbar\omega_n |n\rangle$$

Exact Schrodinger equation: (interaction picture)

$$i\hbar \dot{C}_n = \sum_m C_m V_{nm}(t)\, e^{-i\omega_{mn} t}$$

First order perturbation theory

$\underline{t=0:}$ $\quad C_n(0) = \delta_{ni}$

$\underline{t=t:}$ $\quad C_i(t) \approx 1$ and $C_{f\neq i}(t) \approx 0$

Plug into Schrodinger equation

$\underline{n=i:}$ $\quad i\hbar \dot{C}_i \approx 0 \rightsquigarrow C_i(t) \approx 1$

$\underline{n=f\neq i:}$ $\quad i\hbar \dot{C}_f \approx \underbrace{C_i}_{\approx 1} V_{fi}(t)\, e^{-i\omega_{if} t}$

$$= V_{fi}(t)\, e^{-i\omega_{if} t}$$

$$C_f(t) - C_f(0) \approx -\frac{i}{\hbar} \int_0^t V_{fi}(t')\, e^{-i\omega_{if} t'} dt'$$

$$\boxed{C_f^{(1)}(t) = -\frac{i}{\hbar} \int_0^t V_{fi}(t')\, e^{-i\omega_{if} t'} dt'}$$

1st order in the interaction

$$C_f(t) = C_f^{(1)}(t) + C_f^{(2)}(t) + \dots$$