ES_APPM 312-0 "Complex Variables"

Homework 2 (DUE TUESDAY, 4/20/2021)

Exercise 2-1 (9 pts). Which of the following satisfy the Cauchy-Riemann conditions?

(a) $f(z) = x^2 - y^2 - 2ixy$,

(b) $f(z) = x^3 - 3y^2 + 2x + i(3x^2y - y^3 + 2y)$,

(c) $f(z) = \frac{1}{2}\ln(x^2 + y^2) + i \text{ arccot}\left(\frac{x}{y}\right)$.

Exercise 2-2 (5 pts). Determine where $f'(z)$ exists and provide an expression for it:

$$f(z) = r^2(\cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta), \quad z = re^{i\theta}.$$

Exercise 2-3 (17 pts). Which of the following functions are harmonic?

(a) $v(x, y) = x^2 - y^2 + y$,

(b) $v(x, y) = x^3 - y^3$,

(c) $v(x, y) = 3x^2y - y^3 + xy$,

(d) $v(x, y) = x^4 - 6x^2y^2 + y^4 + x^3y - xy^3$.

For those that are harmonic, assume that they give the imaginary part of an analytic function and find both the real part and the analytic function itself as a function of z (not x and y).

Exercise 2-4 (5 pts). Show in an easy way that

$$(x^2 + y^2)^{1/4}\cos\left(\frac{1}{2}\arctan\frac{y}{x}\right)$$

is harmonic. (Hint: use polar coordinates and by inspection find an analytic function which has this as its real part.) What is this function's harmonic conjugate?

Exercise 2-5 (4 pts). Find all analytic functions $f(z)$ such that $\text{Re}f(z) = \text{Im}f(z)$.