$\boxed{\text{Time Reversal Symmetry}}$ — start with antilinear operators

Symmetry operations that take $|\psi\rangle \to |\tilde{\psi}\rangle$ must satisfy $|\langle\tilde{\varphi}|\tilde{\psi}\rangle|^2 = |\langle\varphi|\psi\rangle|^2$.

<u>There are only two possibilities:</u> Bargmann, J. Math. Phys. 5, 862 (1964)

① $\boxed{\langle\tilde{\phi}|\tilde{\psi}\rangle = \langle\phi|\psi\rangle}$

$$\underbrace{\begin{array}{l}\text{Linear operator: } L\left[a|\alpha\rangle + b|\beta\rangle\right] = aL|\alpha\rangle + bL|\beta\rangle \\ \text{Unitary: } L^{+}L = LL^{+} = \hat{I}\end{array}}_{\text{all sym. operations so far}}$$

— rotations
— translations
— time translations

we will show equivalence

② $\boxed{\langle\tilde{\phi}|\tilde{\psi}\rangle = \langle\phi|\psi\rangle^*}$

$$\underbrace{\text{Antilinear operator: } A\left[a|\alpha\rangle + b|\beta\rangle\right] = a^*A|\alpha\rangle + b^*A|\beta\rangle}_{\text{eg. } T = \text{time reversal operator}}$$

<u>Antilinear operators</u> → ambiguous when operating in two directions

$$\langle\psi|\overrightarrow{A}|\left[a|\alpha\rangle + b|\beta\rangle\right] = a^*\langle\psi|\overrightarrow{A}|\alpha\rangle + b^*\langle\psi|\overrightarrow{A}|\beta\rangle,$$
$$\langle\psi|\overleftarrow{A}|\left[a|\alpha\rangle + b|\beta\rangle\right] = a\langle\psi|\overleftarrow{A}|\alpha\rangle + b\langle\psi|\overleftarrow{A}|\beta\rangle$$

not clearly the same

$\boxed{\text{To avoid ambiguity we will operate all antilinear operators only to the right}}$

depends on the choice of basis

$$\boxed{\text{Claim: Antiunitary } A = \mathcal{U}K}$$

$\mathcal{U}$: unitary operator operates on base kets

K: operates only to right; complex conjugation of coefficients (no effect on base kets)

Check if antilinear

$$A\left[a|\alpha\rangle + b|\beta\rangle\right] = \mathcal{U}K\, a|\alpha\rangle + \mathcal{U}K\, b|\beta\rangle$$
$$= a^* \underbrace{\mathcal{U}K}_{A}|\alpha\rangle + b^* \underbrace{\mathcal{U}K}_{A}|\beta\rangle$$

$\therefore$ antilinear ✓

Check if $\langle\tilde{\phi}|\tilde{\psi}\rangle = \langle\phi|\psi\rangle^*$

$$|\psi\rangle = \sum_a \underbrace{|a\rangle}_{\text{choice of base kets}}\langle a|\psi\rangle \longrightarrow |\tilde{\psi}\rangle = \mathcal{U}K|\psi\rangle$$
$$= \mathcal{U}K \sum_a |a\rangle\langle a|\psi\rangle$$
$$= \sum_a \mathcal{U}|a\rangle\langle a|\psi\rangle^*$$

$$|\phi\rangle = \sum_a |a\rangle\langle a|\phi\rangle \longrightarrow |\tilde{\phi}\rangle = \sum_a \mathcal{U}|a\rangle\langle a|\phi\rangle^*$$

$$\langle\tilde{\phi}| = \sum_a \langle a|\mathcal{U}^\dagger \langle a|\phi\rangle$$

$$\langle\tilde{\phi}|\tilde{\psi}\rangle = \sum_a \langle a|\mathcal{U}^\dagger\langle a|\phi\rangle \sum_b \mathcal{U}|b\rangle\langle b|\psi\rangle^*$$
$$= \sum_{ab} \underbrace{\langle a|\underbrace{\mathcal{U}^\dagger\mathcal{U}}_{1}|b\rangle}_{\delta_{ab}} \langle a|\phi\rangle\langle b|\psi\rangle^*$$
$$= \sum_a \langle a|\phi\rangle\langle a|\psi\rangle^* = \sum_a \langle\psi|a\rangle\langle a|\phi\rangle$$
$$= \langle\psi|\phi\rangle$$
$$= \langle\phi|\psi\rangle^* \checkmark$$

$\boxed{\text{Time Reversal Symmetry}}$ — better called motion reversal

$$|\alpha\rangle = \text{state} \Longrightarrow |\tilde{\alpha}\rangle = T|\alpha\rangle$$

$\uparrow$ time-reversed state

If motion is reversed

$$\left.\begin{aligned} \langle\alpha|\vec{x}|\alpha\rangle &= \langle\tilde{\alpha}|\vec{x}|\tilde{\alpha}\rangle \\ \langle\alpha|\vec{p}|\alpha\rangle &= -\langle\tilde{\alpha}|\vec{p}|\tilde{\alpha}\rangle \\ \langle\alpha|\vec{J}|\alpha\rangle &= -\langle\tilde{\alpha}|\vec{J}|\tilde{\alpha}\rangle \end{aligned}\right\} \text{expect these etc.}$$

If H is invariant under "motion reversal" $\leftarrow$ assumes this

$\mathcal{U}$ = time ev. operator

$$THT^{-1} = H \longrightarrow TH = HT$$
$$[H, T] = 0 \quad \text{equiv.}$$

Also expect: $\underbrace{\mathcal{U}(dt)}_{\text{forward in time}}\underbrace{T|\alpha\rangle}_{\text{time reversed state}} = \underbrace{T}_{\text{time reversed}}\underbrace{\mathcal{U}(-dt)|\alpha\rangle}_{\text{reversed in time}}$

$|\alpha\rangle$
$T|\alpha\rangle$
$\mathcal{U}(dt)T|\alpha\rangle$

$|\alpha\rangle$
$-dt \quad \mathcal{U}(-dt)|\alpha\rangle$
$T \quad T\mathcal{U}(-dt)|\alpha\rangle$

$$\left[\cancel{1} - \frac{i}{\cancel{\hbar}}H(\cancel{dt})\right]T|\alpha\rangle = T\left[\cancel{1} - \frac{i}{\cancel{\hbar}}H(-\cancel{dt})\right]|\alpha\rangle$$

$$-iHT|\alpha\rangle = TiH|\alpha\rangle$$

$$-iHT = TiH \qquad \leftarrow |\alpha\rangle \text{ is arb.}$$

$$-i\cancel{H} = TiHT^{-1}$$
$$= TiT^{-1}\underbrace{THT^{-1}}_{\cancel{H}} \qquad \leftarrow \text{since inv.}$$

we always operated T and T^{-1} to the right

$$\underline{\underline{TiT^{-1} = -i}} \longrightarrow \underline{\text{T is antilinear}}$$

$\therefore$ choose antilinear alternative ~~for time reversal~~

Time Reversal and Schrodinger Equation

H time reversal invariant

S.E. $\quad H|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle$

$T^{-1}T$

T

$$\underbrace{THT^{-1}}_{H}\, T|\psi(t)\rangle = -i\hbar\frac{\partial}{\partial t}T|\psi(t)\rangle$$

$$H\left[T|\psi(t)\rangle\right] = -i\hbar\frac{\partial}{\partial t}\left[T|\psi(t)\rangle\right]$$

$$= i\hbar\frac{\partial}{\partial(-t)}\left[T|\psi(t)\rangle\right]$$

<u>Let $\tau \equiv -t$</u>

$$H\left[T|\psi(-\tau)\rangle\right] = i\hbar\frac{\partial}{\partial \tau}\left[T|\psi(-\tau)\rangle\right]$$

Schrodinger equation

$T|\psi(-t)\rangle$ is a solution

> If $|\psi(t)\rangle$ is a solution to S.E.
> and $THT^{-1} = H$
> then $T|\psi(-t)\rangle$ is also a solution

<u>In the position representation</u>

$\psi(\vec{x},t) = \langle\vec{x}|\psi(t)\rangle$ is a solution to S.E.

$$|\psi(-t)\rangle = \int d^3x\, |\vec{x}\rangle\langle\vec{x}|\psi(-t)\rangle$$

$$T|\psi(-t)\rangle = \int d^3x\, T|\vec{x}\rangle\langle\vec{x}|\psi(-t)\rangle$$

$$= \int d^3x\, |\vec{x}\rangle\langle\vec{x}|\psi(-t)\rangle^* \longrightarrow \therefore\ \psi^*(\vec{x},-t) \text{ is also a solution}$$

$\boxed{\text{Orbital Angular Momentum Base}}$

Use a base $|\vec{x}\rangle$ such that $T|\vec{x}\rangle = |\vec{x}\rangle$

$$T|\psi\rangle = \int d^3x \, T|\vec{x}\rangle\langle\vec{x}|\psi\rangle^* = \int d^3x \, \underbrace{\langle x|\psi\rangle^*}_{\psi(x)^*} =$$

$$\therefore \text{ Under } T, \quad \psi(\vec{x}) = \langle\vec{x}|\psi\rangle \implies \psi^*(\vec{x}) = \langle\vec{x}|\psi\rangle^*$$

We are free to choose $\langle\vec{x}|lm\rangle = e^{if(l,m)} Y_{lm}(\hat{x})$

usual choice $\rightarrow$ ① If we choose $\langle\vec{x}|lm\rangle = Y_{lm}(\vec{x})$

$$\underset{T}{\Longrightarrow} \langle\vec{x}|lm\rangle^* = Y^*_{lm}(\hat{x}) = (-1)^m Y_{l-m}(\hat{x})$$
$$= (-1)^m \langle\vec{x}|l-m\rangle$$

$$\therefore \underline{\underline{T|lm\rangle = (-1)^m|l-m\rangle}}$$

not usual choice $\Longrightarrow$ ② If we instead choose $\langle\vec{x}|lm\rangle = i^l Y_{lm}(\hat{x})$

$$\underset{T}{\Longrightarrow} \langle\vec{x}|lm\rangle^* = i^l(-1)^l Y^*_{lm}(\vec{x})$$
$$= i^l(-1)^l(-1)^m Y_{l-m}(\hat{x})$$
$$= (-1)^{l \pm m}\langle\vec{x}|l-m\rangle$$

$$\underline{\underline{T|lm\rangle = (-1)^{l-m}|l-m\rangle}}$$

Both choices have the same rotation properties

$\boxed{\text{For a general angular momentum operator}}$

$$T J_z T^{-1} = -J_z \quad \leftarrow \text{how we started}$$

$$T J_z = -J_z T$$

$$T J_z |jm\rangle = -J_z T |jm\rangle$$

$$T \hbar m |jm\rangle =$$

$$J_z [T|jm\rangle] = \hbar(-m)[T|jm]$$

$$\therefore T|jm\rangle \sim |j-m\rangle$$

$$\boxed{T = \mathcal{U}K} \longrightarrow \text{choose } \mathcal{U} = \underbrace{e^{i\delta} \mathcal{D}_y(\pi)}_{\text{unitary}}$$

$$\text{since } \mathcal{D}_y(\pi)|jm\rangle = (-1)^{j-m}|j-m\rangle$$

$$\therefore \boxed{T|jm\rangle = e^{i\delta}(-1)^{j-m}|j\,-m\rangle}$$

<u>Apply to orbital case:</u> $T|\ell m\rangle = e^{i\delta}(-1)^{\ell-m}|\ell-m\rangle$

① Usual choice: $\langle\vec{x}|\ell m\rangle = Y_{\ell m}(\hat{x})$

$$|\ell m\rangle = \int d^3x\, |\vec{x}\rangle\langle\vec{x}|\ell m\rangle$$

$$T|\ell m\rangle = \int d^3x\, |\vec{x}\rangle \underbrace{\langle\vec{x}|\ell m\rangle^*}$$

$$e^{i\delta}(-1)^{\ell-\cancel{m}}|\ell-m\rangle = (-1)^{\cancel{m}}|\ell-m\rangle \qquad Y^*_{\ell m} = (-1)^m Y_{\ell-m}$$

$\underline{T = (-1)^\ell \mathcal{D}_y(\pi)K} \longleftarrow \quad \therefore e^{i\delta} = (-1)^\ell$ for this choice

② Another choice: $\langle\vec{x}|\ell m\rangle = i^\ell Y_{\ell m}(\hat{x})$

$$\langle\vec{x}|\ell m\rangle^* = (-1)^{\ell-m} i^\ell Y_{\ell -m}(\hat{x})$$

$\underline{T = \mathcal{D}_y(\pi)K} \longleftarrow \quad \therefore$ Need $e^{i\delta} = 1 \longrightarrow \underline{\delta = 0}$ for this choice

Show $D_y(\pi)\,|jm\rangle = (-1)^{j-m}\,|j\,{-m}\rangle$ and $D_y(2\pi)\,|\psi_j\rangle = (-1)^{2j}\,|\psi_j\rangle$ for our conventions for $|jm\rangle$ and $D_y(\alpha)$

$\boxed{\text{Symmetries and Energy Levels}}$ (quick review)

<u>If H is invariant under some unitary symmetry operation</u>

$\boxed{\text{Parity}}$ $\quad \mathcal{P} H \mathcal{P} = H \quad \leftarrow \mathcal{P} = \mathcal{P}^\dagger$

$$H\mathcal{P} = \mathcal{P}H \longrightarrow [H, \mathcal{P}] = 0 \leftarrow \text{equiv.}$$

Can make simultaneous eigenkest of H and $\mathcal{P}$

$$H|E_p\rangle = E|E_p\rangle \text{ and } \mathcal{P}|E_p\rangle = p|E_p\rangle$$

$\boxed{\text{Rotations}}$

$$J_z H J_z^\dagger = H \iff [H, J_z] = 0$$

$$\therefore e^{-\frac{i\alpha J_z}{\hbar}} H e^{\frac{i\alpha J_z}{\hbar}} = H \iff [H, e^{-\frac{i\alpha}{\hbar}J_z}] = 0$$

$$\vec{J}^2 H \vec{J}^2 = H \iff [H, \vec{J}^2] = 0$$

Also need $[J_z, \vec{J}^2] = 0$

$\therefore$ $|\alpha j m\rangle$ is an energy eigenstate

$\boxed{\text{Time-reversal symmetry}}$ — a bit more intricate

In general:

$$T|jm\rangle = e^{i\delta} D_y(\pi) K |jm\rangle$$
$$= e^{i\delta} (-1)^{j-m} |j\,-m\rangle$$
$$T^2|jm\rangle = e^{i\delta} D_y(\pi) K \left[e^{i\delta} (-1)^{j-m} |j\,-m\rangle \right]$$
$$= \cancel{e^{i\delta}} \cancel{e^{-i\delta}} (-1)^{j-\cancel{m}} (-1)^{j+\cancel{m}} |jm\rangle$$
$$= (-1)^{2j} |jm\rangle$$

Apply to more general wavefunctions

$$|\psi\rangle_j = \sum_m a_m |jm\rangle \quad \leftarrow \text{only one } j \text{ value}$$
$$T|\psi\rangle_j = e^{i\delta} D_y(\pi) K = e^{i\delta} \sum_m a_m^* (-1)^{j-m} |j\,-m\rangle$$
$$\boxed{T^2|\psi\rangle_j = e^{i\delta} D_y(\pi) K \left[e^{i\delta} \sum_m a_m^* (-1)^{j-m} |j\,-m\rangle \right] = \cancel{e^{i\delta}} (-1)^{j+\cancel{m}} \cancel{e^{-i\delta}} \sum_m a_m (-1)^{j-\cancel{m}} |jm\rangle = (-1)^{2j} |\psi\rangle_j}$$
$\leftarrow$ any w.f. with one j satisfies

$$|\psi\rangle = \sum_{jm} a_{jm} |jm\rangle$$
$$T|\psi\rangle = \sum_{jm} e^{i\delta} (-1)^{j-m} a_{jm}^* |j\,-m\rangle$$
$$T^2|\psi\rangle = \sum_{jm} \cancel{e^{i\delta}} (-1)^{j+m} \cancel{e^{-i\delta}} (-1)^{j-m} a_{jm} |jm\rangle$$
$$= \sum_j (-1)^{2j} \sum_m a_{jm} |jm\rangle \neq \underline{\underline{(-1)^{2j}|\psi\rangle}}$$
in general

$$\therefore T^2|\psi\rangle = \begin{cases} (-1)^{2j}|\psi\rangle, & \text{if one } j \text{ only} \\ (-1)^{2j}|\psi\rangle, & \text{if } j \text{ is all integer} \\ (-1)^{2j}|\psi\rangle, & \text{if } j \text{ is all half-integer} \end{cases}$$
$\neq (-1)^{2j}|\psi\rangle$ mixed int. & half int. j

<u>Consider a N-spin system with H invariant under T</u>

① $THT^{-1} = H$

$TH = HT$

② $H|E\rangle = E|E\rangle$

③ $T^2|E\rangle = (-1)^N|E\rangle$

$$\underbrace{TH|E\rangle}_{E|E\rangle} = HT|E\rangle$$

$$H[T|E\rangle] = E[T|E\rangle]$$

and $H\,|E\rangle = E|E\rangle$

<u>Are $|E\rangle$ and $T|E\rangle$</u> → $\boxed{1}$ same state, i.e. $T|E\rangle = e^{i\alpha}|E\rangle$

→ $\boxed{2}$ deg. states, i.e. $T|E\rangle \neq e^{i\alpha}|E\rangle$

Assume that they are the same state

i.e. Assume $T|E\rangle = e^{i\alpha}|E\rangle$

$$T^2|E\rangle = T e^{i\alpha}|E\rangle$$

$$(-1)^N|E\rangle = e^{-i\alpha}T|E\rangle$$

$$\underline{T|E\rangle = e^{+i\alpha}(-1)^N|E\rangle}$$

Contradiction for $(-1)^N = -1$ → <u>N odd</u>

∴ not same state

∴ deg. states

eg• central field

• spin-orbit $\vec{l}\cdot\vec{s}$

• electric field $-\vec{E}\cdot q\vec{r}$

NOT magnetic field $\sim \vec{B}\cdot\vec{J}$

<u>Kramer's degeneracy</u>

For H invariant under T, a system of N=odd spin $\frac{1}{2}$ has at least doubly-degenerate energy eigenstates.

[redacted]