Logarithmic Function is defined as the inverse of the exponential function.

i.e., $\quad w = \log z \iff e^{w} = z \quad (z \neq 0)$.

Let us find the real and imaginary parts of $w = \log z$.

We have

$$w = u + iv, \qquad z = \underbrace{r}_{=|z|} e^{i\theta}$$

take it to be the principal value, $\quad -\pi < \theta \leq \pi$.

Then $\quad e^{w} = z \Rightarrow e^{u+iv} = re^{i\theta} \Rightarrow e^{u}e^{iv} = re^{i\theta} \Rightarrow$

$e^{u} = r = |z| \Rightarrow u = \ln|z|$ and

$v = \theta + 2\pi n \quad (n = 0, \pm 1, \pm 2, \dots)$

Thus, $\quad \log z = \ln|z| + i(\theta + 2\pi n) \quad , n = 0, \pm 1, \pm 2, \dots$

$$\Updownarrow$$

$$\boxed{\log z = \ln|z| + i \arg z}$$

We see that $\log z$ is a multi-valued function.

Ex. $\quad \log i = \ln|i| + i\arg(i) = 0 + i\left(\frac{\pi}{2} + 2\pi n\right)$

$\log i = i\left(\frac{\pi}{2} + 2\pi n\right), \quad n = 0, \pm 1, \pm 2, \dots$

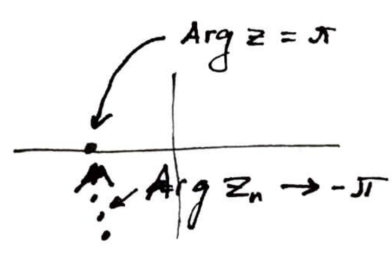
The principal value of $\log z$ (denoted $\operatorname{Log} z$) is

$$\operatorname{Log} z = \ln|z| + i \, \cancel{\operatorname{Arg} z} \operatorname{Arg} z$$

i.e., when the principal value of the argument is used, e.g.

$$\boxed{\operatorname{Log} i = i\frac{\pi}{2}}$$

The function $\operatorname{Arg} z$ is discontinuous at the points of negative real axis, same is true for $\operatorname{Log} z$.



If we consider $\operatorname{Arg} z$ for $-\pi < \theta < \pi$ (i.e., do not include the negative real axis in the domain of definition), then $\operatorname{Arg} z$ is continuous in this domain. Therefore

$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z, \quad -\pi < \operatorname{Arg} z < \pi$$

is also continuous. Moreover, it is analytic in this domain:

$u = \ln r \Rightarrow u_r = \frac{1}{r}, \quad u_\theta = 0$

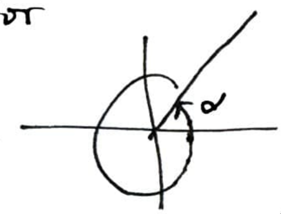
$v = \theta \Rightarrow v_r = 0, \quad v_\theta = 1$

C-R conditions in polar form
$u_r = \frac{1}{r} v_\theta$
$v_r = -\frac{1}{r} u_\theta$
are satisfied

and

$$\boxed{\frac{d}{dz} \log z = e^{-i\theta}(u_r + i v_r) = e^{-i\theta} \cdot \frac{1}{r} = \frac{1}{re^{i\theta}} = \frac{1}{z}}$$

We can also consider $z = re^{i\theta}$, $\alpha < \theta < \alpha + 2\pi$ for any α, then $\log z = \ln r + i\theta$ is continuous and analytic



<u>Def.</u> A branch of a multi-valued function $f(z)$ is a function that takes on <u>one</u> of the values of $f(z)$ and is analytic in some domain

Equation

$$\log z = \ln r + i\theta, \quad \alpha < \theta < \alpha + 2\pi$$

defines a branch of $\log z$; if $\alpha = -\pi \Rightarrow$ the principal branch.

Properties of Logs:

$\log(z_1 z_2) = \log z_1 + \log z_2$ (to be understood as equation for multi-valued functions)

Indeed, if $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2} \Longrightarrow$

$$\log(z_1 z_2) = \log\left(r_1 r_2 e^{i(\theta_1+\theta_2)}\right) = \ln(r_1 r_2) + i(\theta_1 + \theta_2) =$$

$$= (\ln r_1 + i\theta_1) + (\ln r_2 + i\theta_2) = \ln z_1 + \ln z_2$$

but $\operatorname{Log}(z_1 z_2) = \operatorname{Log}(z_1) + \operatorname{Log}(z_2)$ is not necessarily true:

e.g. take $z_1 = z_2 = e^{i\frac{3\pi}{4}}$