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Transient Relativistic Thermodynamics and Kinetic Theory

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The paper develops, from both the phenomenological and kinetic points of view, a generalized formulation of irreversible thermodynamics applicable to the description of thermal phenomena in the presence of strong gravitational fields, fast rotation and rapid fluctuations. The coefficients in the generalized transport equations are evaluated explicitly for a relativistic quantum gas.

1. INTRODUCTION

One of the most annoying paradoxes which have plagued thermodynamical theory has been the parabolic character of the differential equations of heat flow. Even in classical theory, instantaneous propagation of heat is an offense to intuition, which expects propagation at about the mean molecular speed; in a consistent relativistic theory it ought to be completely prohibited.

Although it was recognized that the origin of this problem must reside in some deficiency of conventional thermodynamics when applied to the description of transient effects, the nature of this deficiency was not pinpointed for a long time. In 1949, Grad [1] showed how transient effects could be effectively treated within the framework of classical kinetic theory by employing a method of moments instead of the Chapman–Enskog normal solution. Suitable truncation of the moment equations gave a closed system of differential equations which turned out to be hyperbolic, with propagation speeds of the order of the speed of sound. About 1969–1970, relativistic versions of the Grad method taking account of transient effects were developed by Stewart [2, 3], Anderson and Stewart [4] and independently by Marle [5] and Kranyš [6]. Subsequent detailed calculations [6, 7] which will be summarized in Sec. 8,

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showed that $(3/5)^{1/2}c$ is an upper bound to the wave-front speed of thermal disturbances in a relativistic gas, attained in the limit of infinitely high temperature. So the theory complies with causality. The upper bound $(3/5)^{1/2}c$ is first mentioned in Stewart [3].

In the context of phenomenological theory, instantaneous propagation remained for many years a puzzle that makeshift devices, like the addition of ad hoc relaxation terms to Fourier's law [8], could not resolve in a logically satisfying way. However, in a 1967 paper on nonrelativistic thermodynamics, Müller [9] showed that the difficulty lies in the conventional theory's neglect of terms of second order in heat flow and viscosity in the expression for the entropy. Restoring these terms, Müller derived a modified system of phenomenological equations which was consistent with the linearized form of Grad's kinetic equations. Müller's theory was rediscovered and extended to relativistic fluids by Israel [10] in 1976. Since then, a number of papers on this subject have appeared [7, 11-14, 19].

In this account, which we have tried to make reasonably self-contained, our purpose is first, to draw together the phenomenological and kinetic approaches, and secondly, to provide the explicit form of the coefficients in the generalized transport equations for a relativistic quantum gas. A preliminary report of this work was given in ref. [7]. Quite apart from questions of principle, these results will ultimately be of practical interest in astrophysical and cosmological situations involving fast rotation, strong gravitational fields or rapid fluctuations (neutron stars, black hole accretion, early universe), although it will probably be some time before the state of the art in these fields makes such refinements necessary.

In Section 2, it is shown how virtually the entire phenomenological theory can be developed from a single postulate, a covariant form of the Gibbs relation. The development, which is presented more inductively in Section 2, can be compactly summarized as follows. Associated with an arbitrary state of a simple fluid are a conserved particle flow vector N^μ and energy tensor $T^{\lambda\mu}$, and an entropy flux S^μ with nonnegative divergence. Equilibrium states $(N^\mu_{(0)}, T^{\lambda\mu}_{(0)}, S^\mu_{(0)})$ comprise a 5-dimensional subspace parametrized by a thermal potential α and an inverse-temperature 4-vector β_λ; one has the relationship

$$S^\mu_{(0)} = P\beta^\mu - \alpha N^\mu_{(0)} - \beta_\lambda T^{\lambda\mu}_{(0)} \tag{1.1}$$

(where P is the thermodynamical pressure) and the covariant Gibbs relation

$$dS^\mu = -\alpha dN^\mu - \beta_\lambda dT^{\lambda\mu}, \tag{1.2}$$

in which the velocity is treated as a full-fledged thermodynamical variable. Equations (1.1) and (1.2) imply

$$d(P\beta^\mu) = N^\mu_{(0)}\, d\alpha + T^{\lambda\mu}_{(0)}\, d\beta_\lambda\,. \tag{1.3}$$

The transition from equilibrium to non-equilibrium thermodynamics is effected by the single assumption that (1.2) holds for *arbitrary* virtual displacements from an equilibrium state (α, β_λ), not just displacements to neighbouring equilibrium states. By addition of (1.1) and (1.2), we obtain for arbitrary states near equilibrium,

$$S^\mu = P\beta^\mu - \alpha N^\mu - \beta_\lambda T^{\lambda\mu} - Q^\mu, \tag{1.4}$$

where Q^μ is of second order in the deviations from equilibrium. Taking the divergence, and noting (1.3) and the conservation laws, yields

$$0 \leqslant S^\mu_{|\mu} = -(N^\mu - N^\mu_{(0)})(\partial_\mu \alpha) - (T^{\lambda\mu} - T^{\lambda\mu}_{(0)})\, \beta_{(\lambda|\mu)} - Q^\mu{}_{|\mu}\,. \tag{1.5}$$

Conventional theory assumes that $Q^\mu = 0$; (1.5) then leads directly to Fourier's law and the Navier-Stokes equation. However, these is no justification for this assumption, and kinetic theory shows that it is false (Sec. 5). In general, $Q^\mu{}_{|\mu}$ is as large as the other terms in (1.5), unless the space-time gradients of the deviations from equilibrium (i.e. heat flux and viscous stress) are negligible on the scale of mean-free-path/mean time between collisions, i.e. quasistationary conditions. Retention of Q^μ leads to generalized phenomenological equations.

Sections 3-6 develop kinetic theory along parallel lines. Technical details are relegated to a series of Appendices. Section 7 summarizes the main results, including the final form of the generalized transport equations. Finally, in Sec. 8 we study the characteristics of the equations and show that the propagation is causal.

2. Phenomenological Theory

In this review of phenomenological transient thermodynamics we shall limit ourselves to the case of a simple fluid in a gravitational field. At the end of the section we shall indicate briefly how the theory can be extended to more general situations.

An arbitrary local state of the fluid is specified phenomenologically by "primary variables" S^μ (entropy flux), N^μ (particle flow vector), $T^{\lambda\mu}$ (symmetric energy tensor) and by additional variables. For complete specification of a non-equilibrium state the set of additional variables needed is, in general, infinite. (In the case of a gas, they might correspond to the complete set of moments of the microscopic distribution function, the first two of which are N^μ and $T^{\lambda\mu}$.) The phenomenological theory postulates that the primary variables satisfy

$$N^\mu{}_{|\mu} = 0, \qquad T^{\lambda\mu}{}_{|\mu} = 0, \qquad S^\mu{}_{|\mu} \geqslant 0, \tag{2.1}$$

expressing the usual conservation laws and the positivity of entropy production.

(a) *Equilibrium*

Equilibrium states $S^\mu_{(0)}$, $N^\mu_{(0)}$, $T^{\lambda\mu}_{(0)}$ $\cdots$ are distinguished by subscript 0. They are assumed to be characterized by the following four conditions:

(i) The entropy production vanishes:

$$S_{(0)}{}^{\mu}{}_{|\mu} = 0. \tag{2.2}$$

(ii) In the absence of external fields other than gravity (but permitting arbitrarily strong inertial forces due to gravity and rotation), the primary variables are spatially isotropic; i.e. there is a unique 4-velocity u^{μ} ($u_{\mu}u^{\mu} = -1$) such that

$$S^{\mu}_{(0)} = Su^{\mu}, \qquad N^{\mu}_{(0)} = nu^{\mu}, \qquad T^{\lambda\mu}_{(0)} = \rho u^{\lambda}u^{\mu} + P\Delta^{\lambda\mu} \tag{2.3}$$

where $\Delta^{\lambda\mu}(u) = g^{\lambda\mu} + u^{\lambda}u^{\mu}$ denotes the spatial projection tensor orthogonal to u^{μ}.

(iii) Each equilibrium state is assumed to be specifiable completely by just n, ρ and u^{μ}. Thus, the set of equilibrium states forms a 5-dimensional subspace Σ_0 of the (infinite-dimensional) space of all states. In particular, a given fluid has a characteristic equation of state

$$S = S(\rho, n) \tag{2.4}$$

which determines the entropy density S, and from which the pressure $P(\rho, n)$ can be derived by the relation

$$S = (\rho + P)/T - \alpha n. \tag{2.5}$$

Here, inverse temperature T^{-1} and thermal potential α [=(relativistic chemical potential)$/T$] are defined as partial derivatives of $S(\rho, n)$:

$$dS = T^{-1}\, d\rho - \alpha dn. \tag{2.6}$$

From (2.5) and (2.6) we derive the identity

$$d(P/T) = n\, d\alpha - \rho d(T^{-1}) \tag{2.7}$$

as well as the standard form of the Gibbs relation

$$Td(S/n) = d(\rho/n) + Pd(1/n). \tag{2.8}$$

Thus, the postulated relation (2.5) is equivalent to an identification of the isotropic stress components P in $T^{\lambda\mu}_{(0)}$ with work done in a virtual isentropic expansion:

$$P = -[\partial(\rho/n)/\partial(1/n)]_{S/n}\,. \tag{2.9}$$

(iv) Finally, it is assumed that the equilibrium flow u^{μ} is expansion-free and shear-free, and that the thermal potential is constant:

$$\Delta_{\lambda}{}^{\alpha}\Delta_{\mu}{}^{\beta}u_{(\alpha|\beta)} = 0, \qquad \partial_{\mu}\alpha = 0. \tag{2.10}$$

As we shall see below, these are the conditions for absence of viscous stresses and heat flow.

It will be useful to re-express the thermodynamical relations (2.5)-(2.7) in a covariant form. Let us define

$$\beta_\mu = T^{-1} u_\mu \, . \tag{2.11}$$

Equations (2.10), in conjunction with (2.1)-(2.3) and (2.7), imply that β_μ is a Killing vector,

$$\beta_{(\lambda|\mu)} = 0, \tag{2.12}$$

so the gravitational field of a fluid in thermal equilibrium is necessarily stationary.

The space Σ_0 of equilibrium states may be parametrized by α, β_μ. For an arbitrary virtual displacement in Σ_0 we easily derive from (2.7) and (2.3),

$$d(P\beta^\mu) = N^\mu_{(0)}\, d\alpha + T^{\lambda\mu}_{(0)}\, d\beta_\lambda \tag{2.13}$$

while (2.5) yields

$$S^\mu_{(0)} = P\beta^\mu - \alpha N^\mu_{(0)} - \beta_\lambda T^{\lambda\mu}_{(0)} \, . \tag{2.14}$$

These equations imply

$$dS^\mu_{(0)} = -\alpha\, dN^\mu_{(0)} - \beta_\lambda\, dT^{\lambda\mu}_{(0)} \, , \tag{2.15}$$

a covariant extension of (2.6) in which arbitrary virtual changes of 4-velocity are now permitted.

(b) *Off-equilibrium*

In (2.15) the differentials are constrained by the requirement that the displacement be tangent to Σ_0. We now make the fundamental assumption ("release of variations") that (2.15) stays valid for a virtual displacement from a point (α, β_μ) of Σ_0 to an *arbitrary* neighbouring state:

$$dS^\mu = -\alpha dN^\mu - \beta_\lambda dT^{\lambda\mu}; \tag{2.16}$$

i.e., it is assumed that the differentials are now unconstrained and that no extra differentials (of variables that vanish for equilibrium) enter. This is the simplest consistent generalization of (2.15), and leads to the generally accepted connection between entropy flux and heat flux for infinitesimal departures from equilibrium [see (2.26) below]. For a dilute gas it can be rigorously justified by kinetic theory (Sec. 5).

The postulate of released variations enables us to write an expression for the entropy S^μ of an arbitrary state (N^μ, $T^{\lambda\mu}$,...) that is close to an equilibrium state. (The dots represent the additional variables needed to define such a state.) By addition of (2.14) and (2.16) we obtain the key equation

$$S^\mu = P(\alpha, T)\,\beta^\mu - \alpha N^\mu - \beta_\lambda T^{\lambda\mu} - Q^\mu. \tag{2.17}$$

Here, α, β_λ are parameters of a nearby equilibrium state, $P(\alpha, T)$ is the corresponding equilibrium pressure and Q^μ represents an undetermined quantity of second order ("O_2") in the deviations $N^\mu - N^\mu_{(0)}$, $T^{\lambda\mu} - T^{\lambda\mu}_{(0)}$,.... For a given state (N^μ, $T^{\lambda\mu}$,...) the choice of nearby equilibrium state is quite arbitrary up to first order ("O_1"). Under a change

$$\alpha \to \alpha' = \alpha + \delta\alpha, \qquad \beta_\lambda \to \beta_\lambda' = \beta_\lambda + \delta\beta_\lambda, \qquad (|\,\delta\alpha\,|, |\,\delta\beta_\lambda\,|) \leqslant O_1 \tag{2.18}$$

(2.13) shows that (2.17) remains unchanged in form, which only the second-order term Q^μ requiring adjustment:

$$S^\mu = P(\alpha', T')\, u^{\mu'} - \alpha' N^\mu - \beta_\lambda' T^{\lambda\mu} - Q^{\mu'}, \tag{2.19}$$

where (up to terms linear in $\delta\alpha$, $\delta\beta_\lambda$)

$$Q^\mu - Q^{\mu'} = (N^\mu - N^\mu_{(0)})\,\delta\alpha + (T^{\lambda\mu} - T^{\lambda\mu}_{(0)})\,\delta\beta_\lambda\,. \tag{2.20}$$

(c) *Fitting conditions*

In (2.17) the parameters α, β_λ are not connected to the actual state (N^μ, $T^{\lambda\mu}$,...). It is important to emphasize in particular that $P(\alpha, T)$ generally differs (by a quantity of first order in the deviations) from the "actual" thermodynamical pressure. The latter can be defined (unambiguously to first order, see below) as work done in an isentropic expansion, thus distinguishing it from the bulk viscous term in $T^{\lambda\mu}$. This lack of connection is hardly surprising, since α, β_λ have a meaning only on Σ_0. To extend their definition off Σ_0 is unnecessary, arbitrary and without physical significance. However, it is mathematically convenient. Specifically, it would economize symbols if we were able to identify $P(\alpha, T)$ with the actual thermodynamical pressure (at least to first order). We now show how this can be achieved by fitting a fictitious "local" equilibrium state (α, β_λ) to each state in a suitable way.

The 4-velocity u^μ_N parallel to N^μ, and u^μ_E, the normalized timelike eigenvector of $T^{\lambda\mu}$, coincide for an equilibrium state and differ by a small angle (order O_1) for a state (N^μ, $T^{\lambda\mu}$,...) close to equilibrium. We choose an arbitrary 4-velocity u^μ lying within a cone of angle $\chi = O_1$ and containing these two vectors. The particle and energy densities measured by an observer with 4-velocity u^μ are

$$n(u) = -u_\mu N^\mu, \; \rho(u) = u_\mu u_\nu T^{\mu\nu}. \tag{2.21}$$

These quantities are unchanged to first order if u^μ is tilted through an angle $\epsilon \leqslant O_1$. (This is because both depend on u^μ through $\sim$cosh χ, so the changes are $\sim\epsilon$ sinh $\chi \leqslant$

ϵO_1.) As an "equilibrium reference state" (α, $T^{-1}u_\lambda$) for the given state we choose a state with hydrodynamical velocity u^μ and whose particle and energy densities agree with $n(u)$, $\rho(u)$, i.e.

$$u_\mu(N^\mu - N^\mu_{(0)}) = u_\lambda u_\mu(T^{\lambda\mu} - T^{\lambda\mu}_{(0)}) = 0. \tag{2.22}$$

The thermodynamical variables $S(u)$, $\alpha(u)$, $T(u)$ and $P(u)$ of the equilibrium state are then defined by the equation of state $S(u) = S(\rho(u), n(u))$, (2.5) and (2.6); all are invariant to first order under a change of u^μ. We do not pin down u^μ, since it is useful to have a formulation which includes both of the obvious choices $u^\mu = u^\mu_E$ and $u^\mu = u^\mu_N$ as special cases.

From (2.17) and (2.22),

$$-u_\mu(S^\mu - S^\mu_{(0)}) = u_\mu Q^\mu, \tag{2.23}$$

so the actually observed entropy density $-u_\mu S^\mu$ differs from the equilibrium entropy $S(u)$ only in second order. It follows that the equilibrium pressure $P(u) = P(\alpha, T)$ which enters (2.17), and is precisely defined by (2.5) or (2.9), agrees, to first order, with the change of (actual) energy per particle $[\rho(u)/n(u)]$ when the (actual) entropy per particle is held fixed, i.e. with the actual thermodynamical pressure. We also infer that the form of the equation of state relating $\rho(u)$, $n(u)$ and $(-u_\mu S^\mu)$ remains invariant to first order for small departures from equilibrium. For the sake of brevity we shall sometimes refer to $P(u)$, $T(u)$,... as the actual pressure, temperature,... even though this description is inaccurate or ill-defined when second-order terms are taken into account, and, indeed, the values of all these quantities depend, in second order, on the choice of u^μ. Insofar as the appearance of P, T, α in the linear phenomenological laws is concerned these niceties are of no importance, but they do play a role in (2.17) where we must keep careful track of second order terms. Finally, it may not be superfluous to remark that, since equilibrium reference states are fitted to the actual states independently point-by-point, there are no restrictions, like (2.2) or (2.10), on the way in which α, β_λ may vary from point to point.

With any given choice of u^μ and the above definitions of p, n, S and P, we can decompose N^μ and $T^{\lambda\mu}$ uniquely as follows:

$$N^\mu = nu^\mu + n^\mu, \qquad u_\mu n^\mu = 0 \tag{2.24a}$$

$$T^{\lambda\mu} = \rho u^\lambda u^\mu + P\Delta^{\lambda\mu} + 2h^{(\lambda}u^{\mu)} + \tau^{\lambda\mu} \tag{2.24b}$$

$$u_\lambda h^\lambda = u_\lambda \tau^{\lambda\mu} = 0, \qquad \tau^{\lambda\mu} = \Pi\Delta^{\lambda\mu} + \pi^{\lambda\mu}, \qquad \pi^\lambda_\lambda = 0. \tag{2.24c}$$

The bulk and shear viscous stresses Π, $\pi^{\lambda\mu}$ are invariant to first order under a change of u^μ, but n^μ, h^μ undergo first-order changes. However, the spatial vector

$$q^\mu = h^\mu - n^\mu(\rho + P)/n \tag{2.25}$$

changes only in second order. It represents the heat flux (energy flow relative to particle stream) correctly to first order, since it is exactly equal to the heat flux in the frame $u^\mu = u^\mu_N$, where $n^\mu = 0$. By substituting (2.24) and (2.25) and recalling (2.5) we find that (2.17) can be rewritten

$$S^\mu = (S/n)\, N^\mu + T^{-1} q^\mu - Q^\mu, \tag{2.26}$$

in agreement (to first order) with the conventional decomposition into an entropy flow convected with the particles and an irreversible flow associated with heat conduction.

(d) *Entropy production and phenomenological equations*

Taking the divergence of (2.17) and employing the conservation laws (2.1) and the thermodynamical identity (2.13) yields

$$S^\mu{}_{|\mu} = -(N^\mu - N^\mu_{(0)})(\partial_\mu \alpha) - (T^{\lambda\mu} - T^{\lambda\mu}_{(0)})\, \beta_{(\lambda|\mu)} - Q^\mu{}_{|\mu}\,. \tag{2.27}$$

Insertion of (2.24) yields

$$S^\mu{}_{|\mu} = h^\mu[\partial_\mu(T^{-1}) + T^{-1}\dot{u}_\mu] - n^\mu \partial_\mu \alpha - T^{-1}\tau^{\lambda\mu} u_{\lambda|\mu} - Q^\mu{}_{|\mu} \tag{2.28}$$

where $\dot{u}_\mu = u_{\mu|\nu} u^\nu$ is the acceleration vector.

In order to proceed further, we must now confront the question: what is the form of the second-order term Q^μ in the expression (2.17) for S^μ? For the derivation of the linear phenomenological laws, third-order contributions to Q^μ may be neglected, and it is sufficiently general to assume that Q^μ is a quadratic function of all those independent, irreducible quantities

$$\{X^{\mu_1\cdots\mu_n}_{(A,n)}\} = \{\Pi,\, q^\mu,\, \pi^{\lambda\mu},\ldots\} \tag{2.29}$$

needed for a complete description of a non-equilibrium state, which vanish in equilibrium. In general, it is to be expected (and kinetic theory confirms in the case of a dilute gas) that the set $\{X_{(A,n)}\}$ is infinite. Typical terms of Q^μ have the form (A, B, C,... label different tensors)

$$\alpha_{AB} X^{\mu_1\cdots\mu_n\mu}_{(A,n+1)} X_{(B,n)\mu_1\cdots\mu_n} \qquad \text{or} \qquad \beta_{AC} u^\mu X^{\mu_1\cdots\mu_n}_{(A,n)} X_{(C,n)\mu_1\cdots\mu_n} \tag{2.30}$$

where α_{AB}, β_{AC} are undetermined functions of ρ and n. If the resulting expression for Q^μ is inserted into (2.28), a bilinear form results, whose positivity together with the usual assumption of linear relations between stresses and gradients leads to a complete system of linear, first-order differential equations for the set $\{X_{(A,n)}\}$. This system exhibits Onsager-like symmetries in the couplings α_{AB} and β_{AC} ($A \neq C$): for example, $X_{(A,n+1)}$ is coupled to a gradient of $X_{(B,n)}$ with coefficient α_{AB}, while $X_{(B,n)}$ is coupled to the divergence of $X_{(A,n+1)}$ with the same coefficient. The coefficients β_{AC} are associated with relaxation terms (comoving time-derivatives). We postulate that

$$u_\mu Q^\mu \leqslant 0, \tag{2.31}$$

which means, according to (2.23), that of all states with given (ρ, n) the equilibrium state has the largest entropy. This guarantees that the system is dissipative (relaxation times positive).

To obtain a manageable system of equations the infinite set $\{X_{(A,n)}\}$ has to be truncated. In the quasistationary theory ($Q^\mu = 0$) this truncation completely empties the set. We shall adopt a less drastic approximation, the simplest which still leads to a hyperbolic system. This retains only $\{\Pi, \pi^{\mu\nu}, h^\mu, n^\mu\}$. It may be called the *hydrodynamical description* inasmuch as it assumes that an arbitrary state is adequately specified by the hydrodynamical variables N^μ, $T^{\lambda\mu}$ alone, in particular that S^μ is expressible as a function of N^μ and $T^{\lambda\mu}$. For a gas, it corresponds to the Grad 14-moment approximation of kinetic theory (Sec. 6).

In the hydrodynamical description the most general form for $Q^\mu(u)$ is

$$TQ^\mu = \tfrac{1}{2}u^\mu(\beta_0\Pi^2 + \beta_1 q_\lambda q^\lambda + \beta_2\pi_{\kappa\lambda}\pi^{\kappa\lambda}) - \alpha_0\Pi q^\mu - \alpha_1\pi_\lambda{}^\mu q^\lambda + TR^\mu. \tag{2.32}$$

In the expression

$$R^\mu = \gamma_1 u^\mu h^\alpha h_\alpha + \gamma_2\tau^{\lambda\mu}h_\lambda + \gamma_3\Pi h^\mu \tag{2.33}$$

we have lumped together those terms of Q^μ which are not invariant to second order under the transformations $u_\mu \to u_\mu' = u_\mu + \delta u_\mu$. Choosing δu_μ so that $|\delta u_\mu| \ll O_1$ and linearizing in δu_μ, we see from (2.20) that

$$Q^\mu - Q^{\mu\prime} = R^\mu - R^{\mu\prime} = T^{-1}(\tau^{\lambda\mu} + h^\lambda u^\mu)\,\delta u_\lambda \tag{2.34}$$

On the other hand, the equation

$$u_E^\mu = u^\mu + (\rho + P)^{-1}\,h^\mu + O_2 \tag{2.35}$$

(immediately obtainable from (2.24b)) shows that $h^\mu - h^{\mu\prime} = (\rho + P)\,\delta u^\mu$. Substitution in (2.33) and comparison with (2.34) yields definite values for the coefficients γ_i:

$$R^\mu = [T(\rho + P)]^{-1}\,(\tfrac{1}{2}u^\mu h^\alpha h_\alpha + \tau^{\lambda\mu}h_\lambda). \tag{2.36}$$

The hydrodynamical description therefore requires just five new phenomenological coefficients $\alpha_i(\rho, n)$, $\beta_j(\rho, n)$ ($i = 1, 2; j = 1, 2, 3$) beyond the quasistationary theory.

Substitution of (2.32) and (2.36) into (2.28) leads to an expression for the entropy production $S^\mu{}_{|\mu}$ which is bilinear in Π, q^λ, $\pi^{\lambda\mu}$ and the thermodynamical and velocity gradients. If a linear relation between these variables is assumed, positivity of $S^\mu{}_{|\mu}$

leads to phenomenological laws in the usual way. However, there are two complicating factors. The expression for $S^{\mu}{}_{|\mu}$ contains terms involving gradients of α_i, β_j multiplying second-order quantities; in the presence of strong gravitational fields and rapid rotation the thermodynamical gradients do not vanish even in equilibrium and so cannot be considered "small". One is thus faced with the problem of deciding how to "share" the bilinear term $(\partial_\mu \alpha_1)\, q_\lambda \pi^{\lambda\mu}$ between q_λ and $\pi^{\lambda\mu}$ and similarly for the term $(\partial_\mu \alpha_0)\, q^\mu \Pi$. Secondly, the existence of preferred directions in equilibrium means that the Curie principle is not valid and that Π, q^λ and $\pi^{\lambda\mu}$ can couple to each other through the equilibrium thermodynamical gradients (all parallel to β^μ) and the vorticity tensor $\omega_{\alpha\beta}$. These two factors greatly increase the complexity of the equations and the number of undetermined phenomenological coefficients that occur in them. The ensuing kinetic-theoretical analysis will take full account of these factors. We shall see (Sec. 7) that a simple gas takes advantage of relatively few of the coupling opportunties available to it. Whether this is a symptom of some underlying chastity principle applicable to all fluids is an open question. In any event, for the remainder of this section it is best to assume that equilibrium gradients are sufficiently small that we may neglect their products with first order quantities in the phenomenological equations. This leads uniquely to the following results. We consider in turn the two most obvious choices for u^μ.

(i) *Energy frame*, $u^\mu = u^\mu_E$. We substitute

$$h^\mu = 0,\ n^\mu = -nq^\mu/(\rho + P) \tag{2.37}$$

in the expressions (2.24) for N^μ and $T^{\lambda\mu}$, and in (2.29) and obtain

$$\Pi = -\tfrac{1}{3}\zeta_V(u^\mu_{E|\mu} + \beta_0 \dot{\Pi} - \alpha_0 q^\mu{}_{|\mu}) \tag{2.38a}$$

$$q^\lambda = \kappa T \Delta^{\lambda\mu}[(\partial_\mu \alpha)\, nT/(\rho + P) - \beta_1 \dot{q}_\mu + \alpha_0\, \partial_\mu \Pi + \alpha_1 \pi^\nu_{\mu|\nu}] \tag{2.38b}$$

$$\pi_{\lambda\mu} = -2\zeta_S(u^E_{\langle\lambda|\mu\rangle} + \beta_2 \dot{\pi}_{\lambda\mu} - \alpha_1 q_{\langle\lambda|\mu\rangle}). \tag{2.38c}$$

The phenomenological coefficients κ, ζ_V and ζ_S (thermal conductivity, bulk and shear viscosities) are necessarily positive. Angular brackets enclosing indices denote (for a given unit time-like vector u^μ) the symmetric, trace-free part of the spatial projection, i.e.

$$A_{\langle\lambda\mu\rangle} = (\Delta^\alpha_{(\lambda} \Delta^\beta_{\mu)} - \tfrac{1}{3}\Delta_{\lambda\mu}\Delta^{\alpha\beta})\, A_{\alpha\beta}\,. \tag{2.39}$$

If the coefficients α_i, β_j are set equal to zero, equations (2.24) with $h^\mu = 0$ and (2.38) reduce to the quasistationary equations in the form given by Landau and Lifshitz [15].

(ii) *Particle frame*, $u^\mu = u^\mu_N$. We substitute

$$h^\mu = q^\mu,\ n^\mu = 0 \tag{2.40}$$

in (2.24) and in (2.29), (2.32) and (2.36), and obtain

$$\Pi = -\tfrac{1}{3}\zeta_V(u^{\mu}_{N|\mu} + \beta_0\dot{\Pi} - \bar{\alpha}_0 q^{\mu}{}_{|\mu}) \tag{2.41a}$$

$$q^{\lambda} = -\kappa T\Delta^{\lambda\mu}(T^{-1}\,\partial_{\mu}T + \dot{u}^{N}_{\mu} + \bar{\beta}_1\dot{q}_{\mu} - \bar{\alpha}_0\,\partial_{\mu}\Pi - \bar{\alpha}_1\pi^{\nu}_{\mu|\nu}) \tag{2.41b}$$

$$\pi_{\lambda\mu} = -2\zeta_S(u^{N}_{\langle\lambda|\mu\rangle} + \beta_2\dot{\pi}_{\lambda\mu} - \bar{\alpha}_1 q_{\langle\lambda|\mu\rangle}) \tag{2.41c}$$

where

$$\bar{\alpha}_0 - \alpha_0 = \bar{\alpha}_1 - \alpha_1 = -(\bar{\beta}_1 - \beta_1) = -[(\rho + P)\,T]^{-1}. \tag{2.42}$$

Equations (2.41) and (2.24) with $n^{\mu} = 0$ reduce to the equations originally given by Eckart [16] if one sets $Q^{\mu} = 0$, which is equivalent to setting $\bar{\alpha}_0 = \bar{\alpha}_1 = \beta_0 = \bar{\beta}_1 = \beta_2 = 0$.

The generalization to the case of fluid mixtures is straightforward. For a mixture the primary variables are S^{μ}, $T^{\lambda\mu}$ satisfying (2.1) and, for each component A, a particle flux N^{μ}_{A} (not conserved if there are chemical or nuclear reactions). Equations (2.16) and (2.17) generalize to

$$dS^{\mu} = -\sum_{A} \alpha_A\,dN^{\mu}_{A} - \beta_{\lambda}\,dT^{\lambda\mu} \tag{2.43}$$

$$S^{\mu} = P(\alpha_A\,,\,T)\,\beta^{\mu} - \sum_{A} \alpha_A N^{\mu}_{A} - \beta_{\lambda}T^{\lambda\mu} - Q^{\mu}, \tag{2.44}$$

where α_A = (chemical potential)/T for component A (constant in equilibrium) and the quadratic expression Q^{μ} is generalized appropriately. The resulting equations are given in reference [10].

Elastic solids have been treated by Kranyš [12], and polarized media in electromagnetic fields by Israel [14] and Israel and Stewart [19].

In the following pages we shall present a kinetic-theoretical derivation of the equations (2.38) and (2.41) which yields the explicit form of the new coefficients α, β as thermodynamical functions in the case of a gas.

3. Relativistic Transport Equation for a Simple Quantum Gas

We consider a distribution of identical particles in a Riemannian space-time. The particles interact by short-range forces, idealized as point collisions, and via the gravitational field, treated as a self-consistent background. Synge's invariant distribution function $N(x, p)$ is defined by the statement that

$$Nm^{-1}p^{\alpha}d\Sigma_{\alpha}\,d\omega \tag{3.1}$$

is the number of world-lines cutting an element of 3-surface $d\Sigma_{\alpha}$ and having 4-momenta p^{α} which terminate on a cell of 3-area $m\,d\omega$ on the mass shell $p_{\alpha}p^{\alpha} = -m^2$.

For the nett number of particles in the momentum range (p^α, $d\omega$) which are created by collisions in the 4-volume $(-g)^{1/2}\, d^4x$ we write

$$m^{-1}\mathscr{C}(x^\alpha, p_\beta)\, d\omega(-g)^{1/2}\, d^4x. \tag{3.2}$$

Equating this to the number leaving the boundary, and assuming the world-lines between collisions to be geodesic yields the transport equation in the form

$$p^\mu \partial_\mu N(x, p) = \mathscr{C}, \tag{3.3}$$

where

$$\partial_\mu \equiv \partial/\partial x^\mu + p_\kappa \Gamma^\kappa_{\lambda\mu}\, \partial/\partial p_\lambda \tag{3.4}$$

is the space-time gradient operator for "fixed" (i.e. parallelly-propagated) p_λ.

We shall never need to write down an explicit form for the collision term $\mathscr{C}$, as this is irrelevant for the determination of the relaxation and coupling terms in the macroscopic transport equations. We require only the following general properties:

(i) $\mathscr{C}$ is a purely local function or functional of N, independent of $\partial_\mu N$.

(ii) The form of $\mathscr{C}$ is consistent with conservation of 4-momentum and number of particles at collisions [see (3.6)].

(iii) $\mathscr{C}$ yields a non-negative expression for the entropy production [see (3.12)] and does not vanish unless N has the form of a local equilibrium distribution (see beginning of Sec. 4).

These requirements are of course met by Boltzmann's ansatz for 2-particle collisions, and, indeed, one may hope that they hold somewhat more generally, although the locality assumption (i) is a powerful restriction.

Multiplying (3.3) by an arbitrary (tensor) function of momentum $\Psi(p)$ and integrating over the mass-shall yields the general moment equation

$$\left(\int N(x, p)\, \Psi(p)\, p^\mu\, d\omega\right)_{|\mu} = \int \Psi\mathscr{C}\, d\omega. \tag{3.5}$$

The right-hand side gives the rate of production per unit 4-volume of the property Ψ due to collisions. This should vanish if Ψ is summationally conserved. Thus, with the choices $\Psi = 1$ and p^λ, the requirement

$$\int \mathscr{C}\, d\omega = \int \mathscr{C} p^\lambda\, d\omega = 0 \tag{3.6}$$

leads to the conservation laws

$$N^\mu{}_{|\mu} = 0,\ T^{\lambda\mu}{}_{|\mu} = 0 \tag{3.7}$$

where

$$N^{\mu}(x) = m^{-1} \int N p^{\mu}\, d\omega, \qquad T^{\lambda\mu} = m^{-1} \int N p^{\lambda} p^{\mu}\, d\omega. \tag{3.8}$$

From (3.3) we also obtain the identity

$$\left(\int \phi(N)\, p^{\mu}\, d\omega\right)_{|\mu} = \int \mathscr{C} \phi'(N)\, d\omega \tag{3.9}$$

valid for an arbitrary scalar function $\phi(N)$ which goes to zero for large momenta. We define the entropy vector by

$$S^{\mu} = -m^{-1} \int \phi(N)\, p^{\mu}\, d\omega \tag{3.10}$$

where

$$\phi(N) = (N \ln N - \epsilon^{-1} \Delta \ln \Delta),\ \Delta = 1 + \epsilon N \tag{3.11}$$

so that

$$y(N) = \phi'(N) = \ln(N/\Delta). \tag{3.12}$$

We adopt units in which Boltzmann's constant $k = 1$, $c = 1$, and $h^3 = g$, where g is the spin-weight (number of available states per quantum phase-cell). Key results will be quoted in conventional units. We define ϵ to be $+1$ for bosons, -1 for fermions and the formulas are written so that the non-quantum limit corresponds to $\epsilon \to 0$. Equation (3.10) is equivalent to

$$S^{\mu} = -m^{-1} \int N y p^{\mu}\, d\omega + \Omega^{\mu} \tag{3.13}$$

$$\Omega^{\mu} = m^{-1} \epsilon^{-1} \int (\ln \Delta)\, p^{\mu}\, d\omega. \tag{3.14}$$

We require the form of $\mathscr{C}[N]$ to be such that the entropy production is given by a positive-definite integral for arbitrary N:

$$S^{\mu}{}_{|\mu} = -m^{-1} \int \mathscr{C}[N]\, y(N)\, d\omega \geqslant 0. \tag{3.15}$$

The infinitesimal change of S^{μ} under an arbitrary variation of $N(x, p)$ is

$$\delta S^{\mu} = -m^{-1} \int y(\delta N)\, p^{\mu}\, d\omega. \tag{3.16}$$

From (3.12), we have

$$dN = N \Delta\, dy, \tag{3.17}$$

so the corresponding change of the nth moment

$$I^{\alpha_1\cdots\alpha_n}(x) = m^{1-n} \int N p^{\alpha_1} \cdots p^{\alpha_n} \, d\omega \tag{3.18}$$

is

$$\delta I^{\alpha_1\cdots\alpha_n}(x) = m^{1-n} \int N\Delta(\delta y)\, p^{\alpha_1} \cdots p^{\alpha_n} \, d\omega \tag{3.19}$$

4. Equilibrium

"Local equilibrium" is said to prevail at an event x if $S^{\alpha}{}_{|\alpha} = 0$ at x. According to (3.15) and (3.6) this happens if either

$$\mathscr{C}[N] = 0 \tag{4.1}$$

or

$$y(N) = y_0 = \alpha(x) + \beta_\lambda(x)\, p^\lambda, \tag{4.2}$$

a linear combination of the collision-invariants $1, p^\lambda$. We shall assume (as in fact happens in the case of the Boltzmann ansatz) that either of these conditions implies the other and that each is a necessary condition for local equilibrium, so that there is a unique local equilibrium distribution

$$N_0(x, p) = gh^{-3}[\exp(-\alpha - \beta_\lambda p^\lambda) - \epsilon]^{-1} \tag{4.3}$$

which has the form of the standard local Boltzmann, Bose and Fermi distributions. The constants h and g have been momentarily restored for explicitness. The vector

$$\beta_\lambda = m^{-1}\beta u_\lambda\,,\ u_\lambda u^\lambda = -1,\ \beta = m/kT \tag{4.4}$$

must be timelike to make $N_0 \to 0$ for large momenta.

According to (3.3) and (4.1), $p^\mu \partial_\mu y_0 = 0$ for all p^μ. Thus, if there is local equilibrium at all points of some region,

$$\partial_\mu \alpha = \beta_{(\lambda|\mu)} = 0 \tag{4.5}$$

in agreement with the phenomenological equilibrium conditions (2.10), (2.12).

For a function N which, at a given point, has the form of a local equilibrium distribution $N_0(\alpha + \beta_\lambda p^\lambda)$, the moments $I_{(0)}^{\alpha_1\cdots\alpha_n}$ defined by (3.18) can depend only on $g^{\lambda\mu}$, u^λ and scalar functions of α and β. Thus, for example,

$$N^\lambda_{(0)} m = I^\lambda_{(0)} = I_{10} u^\lambda \tag{4.6a}$$

$$T^{\lambda\mu}_{(0)} = I^{\lambda\mu}_{(0)} = I_{20} u^\lambda u^\mu + I_{21} \Delta^{\lambda\mu} \tag{4.6b}$$

$$I^{\lambda\mu\nu}_{(0)} = I_{30} u^\lambda u^\mu u^\nu + 3 I_{31} \Delta^{(\lambda\mu} u^{\nu)}. \tag{4.6c}$$

The coefficients $I_{nq}(\alpha, \beta)$ are labelled by n, the order of the moment, and by $q \leqslant [\frac{1}{2}n]$, the number of projection factors $\Delta^{\lambda\mu}$ in the term concerned. Comparing (4.6) with the phenomenological expressions (2.3), we can identify

$$I_{10} = nm,\ I_{20} = \rho,\ I_{21} = P. \tag{4.7a}$$

We shall also write

$$I_{31} = \zeta,\ I_{30} = nm + 3\zeta. \tag{4.7b}$$

The last equation is obtained by contracting (4.6c).

Variation of the moments involves, according to (3.19), the auxiliary moments

$$J^{\alpha_1 \cdots \alpha_n} = m^{1-n} \int N\Delta p^{\alpha_1} \cdots p^{\alpha_n}\, d\omega. \tag{4.8}$$

The equilibrium moments $J_{(0)}^{\alpha_1 \cdots \alpha_n}$ have expansions analogous to (4.6), with new coefficients J_{nq} replacing the I_{nq}. The equilibrium moments are studied in detail in Appendix A, where it is shown, for example, that

$$J_{21} = nm/\beta, \qquad J_{31} = (\rho + P)/\beta \tag{4.9}$$

$$J_{42} = \zeta/\beta, \qquad J_{41} = (nm + 5\zeta)/\beta. \tag{4.10}$$

In the special case of a Boltzmann gas ($\Delta = 1$) one has $J_{nm} = I_{nm}$. Comparison of (4.7) and (4.10) then yields

$$P = nm/\beta,\ \zeta = (\rho + P)/\beta \tag{4.11}$$

the first of which is Boyle's law.

5. Entropy

The entropy vector for an equilibrium distribution $N_0(\alpha + \beta_\lambda p^\lambda)$ is obtainable from (3.13) and (4.2).

$$S^\mu_{(0)} = \Omega^\mu_{(0)} - \alpha N^\mu_{(0)} - \beta_\lambda T^{\lambda\mu}_{(0)}\,. \tag{5.1}$$

It can be shown (Appendix B) that

$$\Omega^\mu_{(0)} = P(\alpha, \beta)\, \beta^\mu. \tag{5.2}$$

Now consider an arbitrary variation δN of the distribution function. This may be a displacement to another equilibrium state, involving merely changes $\delta\alpha$, $\delta\beta_\lambda$ in the equilibrium parameters; or it may be a transition from an equilibrium distribution

$N_0(\alpha + \beta_\lambda p^\lambda)$ to a neighbouring non-equilibrium distribution $N(x, p)$. According to (3.16), (4.2) and (4.6), the resulting change of entropy is given quite generally to first order in δN, by

$$\delta S^\mu \equiv S^\mu - S^\mu_{(0)} = -\alpha\, \delta N^\mu - \beta_\lambda\, \delta T^{\lambda\mu}. \tag{5.3}$$

This is the covariant Gibbs relation (2.16) with released variations which is the heart of the phenomenological approach. In the present case it is easy to go further and explicitly evaluate S^μ to second order in the deviation $N - N_0$ (Appendix B). One recovers (2.17), with

$$Q^\mu = \tfrac{1}{2} \int (N_0 \Delta_0)^{-1} (N - N_0)^2\, p^\mu\, d\omega. \tag{5.4}$$

6. Grad 14-Moment Approximation

We seek a microscopic equivalent of the hydrodynamical description. As discussed in Sec. 2, this is a linearized theory of small departures from equilibrium in which it is assumed that the hydrodynamical variables N^μ, $T^{\lambda\mu}$ continue to give a complete description even of non-equilibrium states. Thus, deviations from equilibrium are completely specified by the 14 variables

$$N^\mu - N^\mu_{(0)}, \qquad T^{\lambda\mu} - T^{\lambda\mu}_{(0)}. \tag{6.1}$$

Here, $N^\mu_{(0)}$, $T^\mu_{(0)}$ refer to some nearby equilibrium state; they are determined by 5 variables $\alpha(x)$, $\beta_\lambda(x)$ which are entirely arbitrary up to terms of first order in the deviations. Hence only 9 of the 14 variables (6.1) are physically significant: they are the 3 independent components of heat flux q^μ and the 6 components of viscous stress $\tau^{\lambda\mu}$, which are (invariant) measures of "deviation from equilibrium", though not from any one equilibrium state in particular.

The microscopic counterpart of this description is one in which the function $y(x, p) = \ln(N/\Delta)$ differs from an equilibrium value $y_0 = \alpha(x) + \beta_\lambda(x)\, p^\lambda$ by a function of momenta specifiable by 14 parameters, 5 of which will be arbitrarily adjustable. This is accomplished by postulating that $y - y_0$ can be approximated by a quadratic function:

$$y - y_0 = \epsilon(x) + m^{-1}\epsilon_\lambda(x)\, p^\lambda + m^{-2}\epsilon_{\lambda\mu}(x)\, p^\lambda p^\mu, \tag{6.2}$$

or

$$y(x, p) = (\alpha + \epsilon) + (\beta_\lambda + m^{-1}\epsilon_\lambda)\, p^\lambda + m^{-2}\epsilon_{\lambda\mu} p^\lambda p^\mu, \tag{6.3}$$

where the functions ϵ, ϵ_λ, $\epsilon_{\lambda\mu}$ are small of first order. Without loss of generality, we may assume $\epsilon_{\lambda\mu}$ to be trace-free

$$g^{\lambda\mu}\epsilon_{\lambda\mu} = 0. \tag{6.4}$$

The actual distribution function $N(x, p)$ is completely specified by the 14 variables $\alpha + \epsilon$, $\beta_\lambda + m^{-1}\epsilon_\lambda$ and $\epsilon_{\lambda\mu}$, and only those combinations have physical significance; the 5 parameters ϵ and ϵ_λ are arbitrary, and their arbitrariness reflects our freedom in choosing the local equilibrium parameters α and β_λ up to terms of first order.

As explained in Sec. 2, it is convenient to limit this freedom by imposing the two fitting conditions (2.22). This leaves u^μ arbitrary to first order, but fixes α and β by the requirement that the equilibrium densities $n(\alpha, \beta)$, $\rho(\alpha, \beta)$ be equal to the actual densities—$u_\mu N^\mu$, $u_\mu u_\nu T^{\mu\nu}$ in the frame u_μ. It then follows that the equilibrium entropy density and thermo-dynamical pressure $P(\alpha, \beta)$ differ only in second order from the actual quantities. Fixing α and β means that ϵ and $\epsilon_\lambda u^\lambda$ are also fixed, while the spatial components $\Delta^{\lambda\mu}\epsilon_\mu$ remain arbitrary. We then expect that the remaining 9 variables $\epsilon_{\lambda\mu}$ should be expressible in terms of $\tau_{\lambda\mu}$ and q_μ. This is worked out in Appendix C, where it is found that

$$\epsilon_{\lambda\mu} = B_0(\Delta_{\lambda\mu} + 3u_\lambda u_\mu)\Pi + B_1 u_{(\lambda}q_{\mu)} + B_2\pi_{\lambda\mu} \tag{6.5a}$$

$$m\beta_\lambda + \epsilon_\lambda = \beta u_\lambda^E + D_0 u_\lambda \Pi + D_1 q_\lambda \tag{6.5b}$$

$$\epsilon = E_0\Pi. \tag{6.5c}$$

The coefficients B_i, D_i and E_0 are complicated thermodynamical functions.

To determine the evolution of the 14 independent hydrodynamical variables N^μ, $T^{\lambda\mu}$ we require 14 equations, of which 5 are provided by the conservation laws $N^\mu{}_{|\mu} = T^{\lambda\mu}{}_{|\mu} = 0$. The moment equation (3.5), with $\Psi = p^\alpha p^\beta$:

$$I^{\alpha\beta\mu}{}_{|\mu} = m^{-2}\int \mathscr{C} p^\alpha p^\beta \, d\omega \tag{6.6}$$

provides exactly the 9 additional equations required. (Contraction of (6.6) merely reproduces $N^\mu{}_{|\mu} = 0$.) Since $\mathscr{C}$ vanishes if N has a local equilibrium form, i.e. if $\epsilon_{\lambda\mu} = 0$ in (6.3), we may write (to linear order in the deviations)

$$m^{-2}\int \mathscr{C} p^\alpha p^\beta \, d\omega = -X^{\alpha\beta\lambda\mu}\epsilon_{\lambda\mu}. \tag{6.7}$$

The "collision tensor" $X^{\lambda\mu\alpha\beta}$ depends on the nature of the microscopic interactions (collision cross-sections, etc.) and on the *local* form of $N(x, p)$ (cf assumption (i) of Sec. 3). To evaluate the right-hand side of (6.7) to first order, it is permissible to replace N by N_0, which is spatially isotropic. It follows that $X^{\lambda\mu\alpha\beta}$ is a spatially isotropic tensor, constructed out of u^λ, $\Delta^{\alpha\beta}$ and scalar functions. From the symmetry and trace-free character of (6.7), and recalling (6.4),

$$X^{\alpha\beta\lambda\mu} = X^{(\alpha\beta)(\lambda\mu)} \tag{6.8}$$

$$g_{\alpha\beta}X^{\alpha\beta\lambda\mu} = g_{\lambda\mu}X^{\alpha\beta\lambda\mu} = 0. \tag{6.9}$$

From (3.15), (6.3) and (3.6) we obtain, correct to *second* order,

$$0 \leqslant mS^{\mu}{}_{|\mu} = -\int \mathscr{C} y \, d\omega = \epsilon_{\alpha\beta} X^{\alpha\beta\lambda\mu} \epsilon_{\lambda\mu} \tag{6.10}$$

so that $X^{(\alpha\beta)(\lambda\mu)}$, considered as a 9×9 matrix, is positive definite. It is also symmetric:

$$X^{\alpha\beta\lambda\mu} = X^{\lambda\mu\alpha\beta} \tag{6.11}$$

since the only skew expression satisfying (6.8),

$$\Delta^{\alpha\beta} u^{\lambda} u^{\mu} - u^{\alpha} u^{\beta} \Delta^{\lambda\mu},$$

violates (6.9). These properties imply that $X^{\alpha\beta\lambda\mu}$ is fully determined by three positive scalars A, B, C:

$$\begin{aligned} X^{\kappa\lambda\mu\nu} = {} & \tfrac{1}{3} A (\tfrac{1}{3} \Delta^{\kappa\lambda} \Delta^{\mu\nu} + \Delta^{\kappa\lambda} u^{\mu} u^{\nu} + \Delta^{\mu\nu} u^{\kappa} u^{\lambda} + 3 u^{\kappa} u^{\lambda} u^{\mu} u^{\nu}) \\ & + \tfrac{1}{5} B \Delta^{\kappa\langle\mu} \Delta^{\nu\rangle\lambda} + 4 C u^{(\kappa} \Delta^{\lambda)(\mu} u^{\nu)}. \end{aligned} \tag{6.12}$$

Turning now to the left-hand side of (6.6), we have from (3.19), (6.3) and (4.8)

$$\begin{aligned} I^{\alpha\beta\mu}{}_{|\mu} &= m^{-2} \int N \Delta(\partial_{\mu} y) \, p^{\alpha} p^{\beta} p^{\mu} \, d\omega \\ &= J^{\alpha\beta\mu} (\alpha + \epsilon)_{|\mu} + J^{\alpha\beta\lambda\mu} (m\beta_{\lambda} + \epsilon_{\lambda})_{|\mu} + J^{\alpha\beta\kappa\lambda\mu} \epsilon_{\kappa\lambda|\mu} . \end{aligned} \tag{6.13}$$

Now, $\alpha_{|\mu}$ and $\beta_{(\lambda|\mu)}$ are small of first order, since they vanish for equilibrium. Hence, to linear order, we may replace $J^{\cdots}$ by the equilibrium moments $J^{\cdots}_{(0)}$ on the right-hand side of (6.13). We thus obtain the basic system of 9 equations

$$-X^{\alpha\beta\lambda\mu} \epsilon_{\lambda\mu} = J_{(0)}^{\alpha\beta\mu} (\alpha + \epsilon)_{|\mu} + J_{(0)}^{\alpha\beta\lambda\mu} (m\beta_{\lambda} + \epsilon_{\lambda})_{|\mu} + J_{(0)}^{\alpha\beta\kappa\lambda\mu} \epsilon_{\kappa\lambda|\mu} , \tag{6.14}$$

which relate $\epsilon_{\lambda\mu}$ and its first derivatives linearly to the thermodynamical and velocity gradients $\alpha_{|\mu}$, $\beta_{[|\mu}$. It is a straightforward matter to recast these into the form of transport equations for Π, q_{λ} and $\pi_{\lambda\mu}$, using (6.5), (6.12) and the conservation laws. In performing this reduction, it has to be borne in mind that $\beta_{[\lambda|\mu]}$ does not generally vanish in equilibrium, hence $\dot{u}_{\mu}$, the angular velocity $\omega_{\lambda\mu}$ and spatial thermodynamical gradients (apart from $\alpha_{|\mu}$) cannot be considered to be small.

7. The Macroscopic Transport Equations

The result of this reduction of (6.14) is (see Appendix D)

$$\Pi = -\tfrac{1}{3} \zeta_{V} (u^{\mu}_{E|\mu} + \beta_0 \dot{\Pi} - \alpha_0 q^{\mu}{}_{|\mu} - a_0' q^{\mu} \dot{u}_{\mu}) \tag{7.1a}$$

$$(\kappa T)^{-1}\, q_\lambda = \Delta^\mu_\lambda(\alpha_{|\mu}/\eta\beta - \beta_1 \dot{q}_\mu + \alpha_0 \Pi_{|\mu} + \alpha_1 \pi^\nu_{\mu|\nu}) + a_0 \Pi \dot{u}_\lambda + a_1 \pi_\lambda{}^\mu \dot{u}_\mu + \beta_1 \omega_{\lambda\mu} q^\mu \tag{7.1b}$$

$$\pi_{\lambda\mu} = -2\zeta_S(\Delta^\alpha_{\langle\lambda}(u_E)\, \Delta^\beta_{\mu\rangle}(u_E)\, u^E_{\alpha|\beta} - a_1 q_{\langle\lambda|\mu\rangle} + \beta_2(\dot{\pi})_{\langle\lambda\mu\rangle} - a_1' q_{\langle\lambda}\dot{u}_{\mu\rangle} - 2\beta_2 \pi^\alpha_{\langle\lambda}\omega_{\mu\rangle\alpha}). \tag{7.1c}$$

In the case where one can neglect products of $\dot{u}_\mu$ and $\omega_{\lambda\mu}$ with the viscous stresses and heat flux, these equations reduce to the simpler equations (2.38), The 4 additional coefficients a_0 , a_0' , a_1 , a_1' which appear in (7.1) are subject to 2 constraints

$$a_0 + a_0' = \frac{\partial}{\partial\beta}(\beta\alpha_0), \qquad a_1 + a_1' = \frac{\partial}{\partial\beta}(\beta\alpha_1), \tag{7.2}$$

where the partial derivatives are evaluated for fixed thermal potential α. The relations (7.2) have a general validity, not limited to a gas, which can be traced to the fact that the terms involving the a's originate from terms

$$(\beta\alpha_0 \Pi q^\mu + \beta\alpha_1 q^\lambda \pi_\lambda{}^\mu)_{|\mu} \tag{7.3}$$

in the expression for the entropy production [see (2.28) and (2.32)].

In order to list the explicit form of the coefficients in (7.1), we require some preliminary definitions:

$$\eta = (\rho + P)/nm,\ \Lambda = 1 + 5(\zeta/nm) - \eta^2 \tag{7.4}$$

$$\Omega = 3(\partial \ln \zeta/\partial \ln n)_{(S/n)} - 5 \tag{7.5}$$

$$D_{nq} = J_{n+1,q} J_{n-1,q} - (J_{nq})^2. \tag{7.6}$$

The transport coefficients are given by

$$\zeta_V = 3(\zeta\Omega)^2/\beta A,\ \zeta_S = 10\zeta^2/\beta B,\ \kappa = (\Lambda nm)^2/mC. \tag{7.7}$$

For the explicit evaluation of A, B, C in terms of the collision cross-section, see refs. [17, 18].

The coupling and relaxation coefficients are given by

$$\alpha_0 = (D_{41}D_{20} - D_{31}D_{30})/\Lambda\zeta\Omega J_{21}J_{31} \tag{7.8a}$$

$$\alpha_1 = (J_{41}J_{42} - J_{31}J_{52})/\Lambda\zeta J_{21}J_{31} \tag{7.8b}$$

$$\beta_0 = (3\beta/\zeta^2\Omega^2)\{5J_{52} - (3/D_{20})[J_{31}(J_{31}J_{30} - J_{41}J_{20}) + J_{41}(J_{41}J_{10} - J_{31}J_{20})]\} \tag{7.8c}$$

$$\beta_1 = D_{41}/\Lambda^2 nm J_{21}J_{31},\ \beta_2 = \tfrac{1}{2}\beta J_{52}/\zeta^2 \tag{7.8d}$$

$$a_1 = (\beta/\zeta^2\Lambda)(\eta J_{42} - J_{52}). \tag{7.8e}$$

The expression for a_0 is long, and is given in Appendix D, equation (D15).

In the special case of a *Boltzmann gas*, the coefficients (7.8) can all be expressed as (functions of β) divided by P. It is convenient to write the formulae in terms of the ratio of specific heats γ, as this is a slowly varying function of β alone, increasing monotonically from $\gamma = \frac{4}{3}$ ($\beta = 0$) to $\gamma = \frac{5}{3}$ ($\beta = \infty$). With the standard notation $K_n(\beta)$ for the modified Bessel functions of the second kind, we have (Appendix E)

$$\eta = K_3(\beta)/K_2(\beta), \qquad \gamma/(\gamma - 1) = \beta^2(1 + 5\eta/\beta - \eta^2) \tag{7.9}$$

$$\alpha_0 = (\gamma - 1)\,\Omega^{**}/\gamma\Omega P, \qquad \alpha_1 = -(\gamma - 1)/\gamma P \tag{7.10a}$$

$$\beta_0 = \frac{3\Omega^*}{\eta^2\Omega^2 P}, \qquad \beta_1 = \left(\frac{\gamma - 1}{\gamma}\right)^2 \frac{\beta}{\eta P}\left(5\eta^2 - \frac{\gamma}{\gamma - 1}\right) \tag{7.10b}$$

$$\beta_2 = \tfrac{1}{2}(1 + 6\eta/\beta)/\eta^2 P, \qquad a_1 = -[1 + \eta\beta(\gamma - 1)/\gamma]/\eta^2 P \tag{7.10c}$$

where we have defined

$$\Omega = 3\gamma - 5 + 3\gamma/\eta\beta, \; \Omega^* = 5 - 3\gamma + 3(10 - 7\gamma)\,\eta/\beta \tag{7.11a}$$

$$\Omega^{**} = 5 - 3\gamma + 3\gamma^2/(\gamma - 1)\,\eta^2\beta^2. \tag{7.11b}$$

(The formula for Ω^{**} given in refs. [7, 10] contains an error of sign.) Asymptotic forms are:

(i) *Nonrelativistic limit* ($\beta \to \infty$).

$$\begin{aligned} &\alpha_0 \approx \tfrac{4}{5}\beta P^{-1}, && \alpha_1 \approx -\tfrac{2}{5}P^{-1}, && \\ &\beta_0 \approx \tfrac{6}{5}\beta^2 P^{-1}, && \beta_1 \approx \tfrac{2}{5}\beta P^{-1}, && \beta_2 \approx \tfrac{1}{2}P^{-1} \\ & && a_1 \approx -\tfrac{2}{5}\beta P^{-1}. && \end{aligned} \tag{7.12}$$

These results are consistent with the linearized form of Grad's "13-moment equations" (equation (5.18) of ref. [1]).

(ii) *Ultrarelativistic limit* ($\beta \to 0$).

$$\begin{aligned} &\alpha_0 \approx 6\beta^{-2}P^{-1}, && \alpha_1 \approx -\tfrac{1}{4}P^{-1}, && \\ &\beta_0 \approx 216\beta^{-4}P^{-1}, && \beta_1 \approx \tfrac{5}{4}P^{-1}, && \beta_2 \approx \tfrac{3}{4}P^{-1} \\ &a_0 \approx 24\beta^{-2}P^{-1}, && a_1 \approx -\tfrac{1}{8}\beta^2 P^{-1}. && \end{aligned} \tag{7.13}$$

8. Characteristics and Propagation Speeds

The propagation of transient effects in a relativistic gas has been studied in detail in refs. [11, 13]. Here, we shall merely summarize the most interesting results, assuming that equilibrium anisotropies are unimportant, so that we may use the simpler form of the transient equations (2.38). This system of 9 equations for 14 unknowns $y_A =$

$(\alpha, \beta, u_E^\mu, \Pi, q_\lambda, \pi_{\lambda\mu})$ is to be supplemented by the 5 conservation laws $N^\mu{}_{|\mu} = T^{\lambda\mu}{}_{|\mu} = 0$. The complete system, which is of first order and quasilinear, may be written

$$C^{AB\mu}(\alpha, \beta, u_E)\, \partial_\mu y_B = D^A(y) \; (A, B = 1,..., 14) \tag{8.1}$$

where the right-hand side contains all collision terms and the coefficients $C^{AB\mu}$ are purely thermodynamical functions. A characteristic surface, $\phi(x^\mu) = \text{const.}$, of (8.1) is a 3-space Σ across which the y_A are continuous but their first derivatives are permitted to have discontinuities $[\partial_\mu y_A]$, necessarily normal to Σ, i.e.

$$[y_A] = 0, \; [\partial_\mu y_A] = Y_A(\partial_\mu \phi). \tag{8.2}$$

From (8.1) and (8.2) we obtain the compatibility condition

$$\det[C^{AB\mu}(\partial_\mu \phi)] = 0 \tag{8.3}$$

which determines ϕ and hence the wave-front speed. These characteristic velocities are (like the speed of sound) functions of α, β only and independent of the details of the collision process; for a Boltzmann gas they depend only on β.

To analyze equations (8.1), it is sufficient to focus attention on a single point and to choose a Lorentz frame with time-axis along u_E^μ and such that the wave-front coincides locally with the yz-plane. The 14 variables y_A are then found to split into 5 subsets, or modes, each of which can be excited independently. The different modes are:

(i) Two transverse shear modes $\sigma_{yy} - \sigma_{zz}$ and σ_{yz}. Their wave-front speeds are zero, i.e. they decay (on the scale of a mean collision-time) without propagating.

(ii) Two longitudinal-transverse modes (q_y, σ_{xy}, u_y) and (q_z, σ_{xz}, u_z). In a Boltzmann gas, their wave-front speeds increase monotonically from $(\frac{7}{5}kT/m)^{1/2}$ at low temperatures to $(\frac{1}{5})^{1/2}c$ in the ultrarelativistic limit.

(iii) One longitudinal mode $(\alpha, \beta, u_x, q_x, \Pi, \pi_{xx})$ associated with three distinct wave fronts with speeds $(0, v_1, v_2)$. For a Boltzmann gas v_1^2 varies monotonically between $[(49 - (826)^{1/2}/15](kT/m) = 1.35\, kT/m$ at low temperatures to $\frac{1}{3}c^2$ in the high-temperature limit; the corresponding limits for v_2^2 are $[49 + (826)^{1/2}/15](kT/m) = 5.18\, kT/m$ and $\frac{3}{5}c^2$. Although calculations are incomplete, it is reasonable to expect that this mode will decay, after a few collision times, to an adiabatic sound wave.

It thus appears that the predictions of the equations of transient thermodynamics conform with relativistic causality and, generally, with intuitive ideas about the propagation of thermal disturbances.

APPENDIX A. Moments of The Equilibrium Distribution Function

We consider equilibrium distributions only, omitting subscript 0 for simplicity. The moments (3.18), (4.8), can be expanded in terms of symmetrized tensor products of u^λ and $\Delta^{\mu\nu}$. We define, for $2q \leq n$,

$$U_{(q)}^{\alpha_1\cdots\alpha_n} = \Delta^{(\alpha_1\alpha_2} \cdots \Delta^{\alpha_{2q-1}\alpha_{2q}}u^{\alpha_{2q+1}} \cdots u^{\alpha_n)}. \tag{A1}$$

The full expansion of this symmetrized product contains $n!$ terms, each of which makes $2^q q!\,(n-2q)!$ appearances, differing only by trivial permutations like $\Delta^{\lambda\mu} \to \Delta^{\mu\lambda}$, $\Delta^{\alpha\beta}\Delta^{\lambda\mu} \to \Delta^{\lambda\mu}\Delta^{\alpha\beta}$, $u^\alpha u^\beta \to u^\beta u^\alpha$. If these trivial repetitions are lumped together, the expansion contains

$$a_{nq} = \binom{n}{2q}(2q-1)!! \tag{A2}$$

terms. For example,

$$a_{31}U_{(1)}^{\lambda\mu\nu} = u^\lambda\Delta^{\mu\nu} + u^\mu\Delta^{\nu\lambda} + u^\nu\Delta^{\lambda\mu}, \tag{A3}$$

$$a_{42}U_{(2)}^{\kappa\lambda\mu\nu} = \Delta^{\kappa\lambda}\Delta^{\mu\nu} + \Delta^{\kappa\mu}\Delta^{\nu\lambda} + \Delta^{\kappa\nu}\Delta^{\lambda\mu}. \tag{A4}$$

We note the useful recursion formulas

$$nU_{(q)}^{\alpha_1\cdots\alpha_{n-1}\lambda} = (n-2q)\,U_{(q)}^{\alpha_1\cdots\alpha_{n-1}}u^\lambda + 2qU_{(q-1)}^{(\alpha_1\cdots\alpha_{n-2}}\Delta^{\alpha_{n-1})\lambda} \tag{A5}$$

$$U_{(q+1)}^{\alpha_1\cdots\alpha_n} = U_{(q)}^{(\alpha_1\cdots\alpha_{n-2}}\Delta^{\alpha_{n-1}\alpha_n)} \tag{A6}$$

and the orthogonality relations

$$\binom{n}{2q} U_{(q')}^{\alpha_1\cdots\alpha_n}U_{(q)\alpha_1\cdots\alpha_n} = (-1)^n(2q+1)\,\delta_{qq'}\,. \tag{A7}$$

The moments defined by (3.18) and (4.8) are expanded in the form

$$I^{\alpha_1\cdots\alpha_n} = \sum_{q=0}^{[\frac{1}{2}n]} a_{nq}I_{nq}U_{(q)}^{\alpha_1\cdots\alpha_n} \tag{A8}$$

$$J^{\alpha_1\cdots\alpha_n} = \sum_{q=0}^{[\frac{1}{2}n]} a_{nq}J_{nq}U_{(q)}^{\alpha_1\cdots\alpha_n} \tag{A9}$$

so that

$$(2q+1)!!\,J_{nq} = (-1)^n\,J^{\alpha_1\cdots\alpha_n}U_{(q)\alpha_1\cdots\alpha_n}\,. \tag{A10}$$

Contraction of (A9) yields

$$J_{n+2,q} = J_{nq} + (2q+3)\,J_{n+2,q+1}\,(2q \leqslant n) \tag{A11}$$

$$J^{\alpha_1\cdots\alpha_{n-2}**} = \sum_{q=0}^{[\frac{1}{2}n-1]} a_{n-2,q}J_{nq}U_{(q)}^{\alpha_1\cdots\alpha_{n-2}} \tag{A12}$$

$$J^{\alpha_1\cdots\alpha_{n-2}\lambda*}\Delta_\lambda^\mu = -(n-2)\sum_{q=0}^{[\frac{1}{2}(n-3)]} a_{n-3,q}J_{n,q+1}U_{(q)}^{(\alpha_1\cdots\alpha_{n-3}}\Delta^{\alpha_{n-2})\mu} \tag{A13}$$

$$n(n-1)\, J^{\alpha_1\cdots\alpha_{n-2}\langle\lambda\mu\rangle} = 4 \sum_{q=0}^{[\frac{1}{2}n]} q(q-1)\, a_{nq} J_{nq} \Delta^{\lambda\rangle(\alpha_1} U^{\alpha_2\cdots\alpha_{n-3}} \Delta^{\alpha_{n-2})\langle\mu}. \tag{A14}$$

Formulas analogous to (A10-14) hold with J replaced by I. The asterisk denotes projection onto u^α, $X^{*\cdots} = u_\alpha X^{\alpha\cdots}$, and the angular bracket notation was defined in (2.39).

To obtain explicit integrals for I_{nq} and J_{nq}, we choose hyperspherical co-ordinates (χ, θ, ϕ) on the pseudo-sphere $p_\alpha p^\alpha = -m^2$ with polar axis along u^μ, so that $p_\mu u^\mu = -m \cosh \chi$ and $d\omega = 4\pi m^2 \sinh^2 \chi\, d\chi$. Then

$$(2q+1)!!\, I_{nq} = A_0 \int_0^\infty N \sinh^{2(q+1)}\chi \cosh^{n-2q} \chi\, d\chi \tag{A15}$$

$$(2q+1)!!\, J_{nq} = A_0 \int_0^\infty N\Delta \sinh^{2(q+1)} \chi \cosh^{n-2q} \chi\, d\chi \tag{A16}$$

where

$$N = 1/[\exp(\beta \cosh \chi - \alpha) - \epsilon],\ \Delta = 1 + \epsilon N \tag{A17}$$

and $A_0 = 4\pi m^3$ ($=4\pi m^3 g/h^3$ in conventional units with $c = 1$).

Derivatives of $I_{nq}(\alpha, \beta)$ and $J_{nq}(\alpha, \beta)$ of arbitrary order can be expressed in terms of the J_{nq}:

$$dI_{nq} = J_{nq}\, d\alpha - J_{n+1,q}\, d\beta \tag{A18}$$

$$\beta dJ_{nq} = [(n+1)\, J_{n-1,q} + J_{n-3,q-1}]\, d\alpha - [(n+2)\, J_{nq} + J_{n-2,q-1}]\, d\beta. \tag{A19}$$

The first formula follows directly from (3.19), the second is obtainable from (A16) by integration by parts.

The integrals (A15), (A16) can be reduced to standard functions $\mathscr{K}_n(\alpha, \beta)$, $\mathscr{L}_{n+1}(\alpha, \beta)$ defined (for $n \geqslant 0$) by

$$(2n-1)!!\, \mathscr{K}_n(\alpha, \beta) = \beta^n \int_0^\infty N \sinh^{2n} \chi\, d\chi \tag{A20}$$

$$(2n-1)!!\, \mathscr{L}_{n+1}(\alpha, \beta) = \beta^n \int_0^\infty N \sinh^{2n} \chi \cosh \chi\, d\chi. \tag{A21}$$

These functions have the following properties (verifiable by integration by parts). For $n = 1, 2, \ldots$

$$\partial\mathscr{K}_n/\partial\alpha = \mathscr{L}_n \tag{A22}$$

$$\partial\mathscr{L}_{n+1}/\partial\alpha = -\beta^n \partial(\beta^{-n}\mathscr{K}_n)/\partial\beta = \mathscr{K}_{n-1} + (2n/\beta)\, \mathscr{K}_n \tag{A23}$$

and for $n = 2, 3, \ldots$

$$-\beta^n \partial(\beta^{-n}\mathscr{L}_n)/\partial\beta = \mathscr{L}_{n-1} + (2n/\beta)\, \mathscr{L}_n\,. \tag{A24}$$

In the special case of a Boltzmann gas,

$$\mathscr{L}_n = \mathscr{K}_n = e^{\alpha} K_n(\beta) \ (\epsilon = 0)$$

where $K_n(\beta)$ are modified Bessel functions of the second kind.

Reduction of (A15) yields, for $n = 0, 1,\ldots,$ and $q \leqslant \frac{1}{2}n$,

$$I_{nq} = A_0 \sum_{r=0}^{\frac{1}{2}n-q} b_{nqr} \beta^{-(q+r+1)} \mathscr{K}_{q+r+1} \tag{A25a}$$

$$I_{n+1,q} = A_0 \sum_{r=0}^{\frac{1}{2}n-q} b_{nqr} \beta^{-(q+r+1)} \mathscr{L}_{q+r+1} \tag{A25b}$$

and differentiation of these results with respect to α then gives

$$J_{nq} = A_0 \sum_{r=0}^{\frac{1}{2}n-q} b_{nqr} \beta^{-(q+r+1)} \mathscr{L}_{q+r+1} \qquad (n = 0, 2,\ldots; q \leqslant \tfrac{1}{2}n) \tag{A26a}$$

$$J_{nq} = A_0 \sum_{r=0}^{\frac{1}{2}(n+1)-q} c_{nqr} \beta^{-(q+r+1)} \mathscr{K}_{q+r} \qquad (n = 1, 3,\ldots; q \leqslant \tfrac{1}{2}(n-1)). \tag{A26b}$$

The numerical coefficients are

$$b_{nqr} = \frac{(2q+2r+1)!!}{(2q+1)!!} \binom{\frac{1}{2}n - q}{r} \qquad (n = 0, 2,\ldots) \tag{A27}$$

$$c_{nqr} = \frac{(2q+2r+1)!!}{(2q+1)!!} \left\{ (n+1) \binom{\frac{1}{2}(n-1) - q}{r-1} + (2q+1) \binom{\frac{1}{2}(n-1) - q}{r} \right\}$$

$$(n = 1, 3,\ldots). \tag{A28}$$

Explicit values:

$$\begin{array}{lll}
b_{00r} = (1) & & \\
b_{20r} = (1, 3), & b_{21r} = (1) & \\
b_{40r} = (1, 6, 15), & b_{41r} = (1, 5), & b_{42r} = (1) \\
c_{10r} = (1, 2) & & \\
c_{30r} = (1, 5, 12), & c_{31r} = (1, 4) & \\
c_{50r} = (1, 8, 39, 90), & c_{51r} = (1, 9, 30), & c_{52r} = (1, 6)
\end{array}$$

Entries in each row-vector list values for $r = 0, 1,\ldots, [\frac{1}{2}n] - q$. Particular results which are used constantly:

$$nm = I_{10} = A_0 \beta^{-1} \mathscr{L}_2 \,,\ \rho = I_{20} = A_0(\beta^{-1}\mathscr{K}_1 + 3\beta^{-2}\mathscr{K}_2) \tag{A29a}$$

$$P = I_{21} = A_0 \beta^{-2} \mathscr{K}_2 \,,\ \zeta = I_{31} = A_0 \beta^{-2} \mathscr{L}_3 \,,\ I_{30} = I_{10} + 3I_{31} \tag{A29b}$$

$$J_{21} = nm/\beta,\ J_{31} = (\rho + P)/\beta = \eta J_{21}\,,\ J_{42} = \zeta/\beta \tag{A29c}$$

$$J_{52} = A_0(\beta^{-3}\mathscr{K}_2 + 6\beta^{-4}\mathscr{K}_3) \tag{A29d}$$

$$J_{41} = \beta^{-1}(nm + 5\zeta),\ J_{51} = J_{31} + 5J_{52}\,. \tag{A29e}$$

We shall also require formulas for certain thermodynamical derivatives. From (A18) we have

$$d(nm) = J_{10}d\alpha - J_{20}d\beta,\ d\rho = J_{20}d\alpha - J_{30}d\beta \tag{A30a}$$

$$\beta dP = nm(d\alpha - \eta d\beta),\ \beta d\zeta = nm[\eta d\alpha - (\Lambda + \eta^2)\, d\beta] \tag{A30b}$$

where

$$\Lambda = D_{31}/J_{21}^2 = J_{41}/J_{21} - \eta^2 = 1 + 5\zeta/nm - \eta^2 \tag{A31}$$

and the symbols D_{nq} are defined by (7.6). It follows that

$$(\Lambda + \eta^2)\, dP + \eta\, d\zeta = \Lambda(nm/\beta)\, d\alpha \tag{A32}$$

and also

$$nTd(S/n) = d\rho - \eta d(nm) \tag{A33}$$

$$= (J_{20} - \eta J_{10})\, d\alpha - (J_{30} - \eta J_{20})\, d\beta. \tag{A34}$$

From (A30) and (A34),

$$D_{20}\, d\alpha = (\eta J_{20} - J_{30})\, d(nm) - nTJ_{20}d(S/n) \tag{A35a}$$

$$D_{20}\, d\beta = (J_{20} - \eta J_{10})\, d(nm) - nTJ_{10}d(S/n) \tag{A35b}$$

$$(\partial\zeta/\partial nm)_{S/n} = (D_{20})^{-1}\,\{J_{31}(J_{30} - \eta J_{20}) - J_{41}(J_{20} - \eta J_{10})\}. \tag{A36}$$

The specific heats per particle are defined by

$$C_P = -\beta[\partial(S/n)/\partial\beta]_P\,,\ C_V = -\beta[\partial(S/n)/\partial\beta]_n\,,\ \gamma = C_P/C_V\,. \tag{A37}$$

Evaluating with the aid of (A34) and (A30), we obtain

$$C_P - C_V = \beta^2(\eta J_{10} - J_{20})^2/nmJ_{10} \tag{A38}$$

$$\gamma - 1 = (\eta J_{10} - J_{20})^2/D_{20} \tag{A39}$$

$$(\partial \ln P/\partial \ln n)_{S/n} = \gamma(nm)^2/\beta P J_{10}\,. \tag{A40}$$

APPENDIX B. ENTROPY

We shall verify equations (2.17) and (5.4), which give the entropy of an arbitrary distribution N to second order in $(N - N_0)$, where N_0 is any nearby equilibrium distribution.

S^μ is given by (3.10), with

$$\phi(N) = -\epsilon^{-1} \ln \Delta + Ny, \; \phi'(N) = y, \; \phi''(N) = (N\Delta)^{-1} \tag{B1}$$

so that

$$\phi(N) = -\epsilon^{-1} \ln \Delta_0 + (\alpha + \beta_\lambda p^\lambda) N + \tfrac{1}{2}(N_0\Delta_0)^{-1} (N - N_0)^2 + \cdots \tag{B2}$$

$$S^\mu = \Omega^\mu_{(0)} - \alpha N^\mu - \beta_\lambda T^{\lambda\mu} - \tfrac{1}{2} \int (N_0\Delta_0)^{-1} (N - N_0)^2 p^\mu \, d\omega \tag{B3}$$

where

$$\Omega^\mu_{(0)} = m^{-1}\epsilon^{-1} \int (\ln \Delta_0) \, p^\mu \, d\omega. \tag{B4}$$

An elegant way to evaluate $\Omega^\mu_{(0)}$ is to employ the covariant integration-by-parts formula

$$\int p_{[\lambda} \, \partial\Phi/\partial p_{\mu]} \, d\omega = 0 \tag{B5}$$

in which Φ is an arbitrary (tensor) function that goes to zero for large momenta. To prove (B5), one rewrites the left-hand side as

$$\int \cdots d\omega = 2 \int \cdots \delta(p_\alpha p^\alpha + m^2) \, \theta(p^4)(-g)^{1/2} \, d^4p \tag{B6}$$

and integrates by parts, obtaining

$$-\int \Phi \, \partial(\delta \cdot \theta \cdot p_{[\mu})/\partial p_{\lambda]} \, (-g)^{1/2} \, d^4p = 0. \tag{B7}$$

If we choose $\Phi = \epsilon^{-1} \ln \Delta_0$, and note

$$\epsilon^{-1} d(\ln \Delta) = N \, dy, \tag{B8}$$

then (B5) yields

$$\Omega_{(0)[\lambda}\delta^\kappa_{\mu]} + T_{(0)[\lambda}^{\kappa}\beta_{\mu]} = 0 \tag{B9}$$

which contracts to

$$\Omega^\lambda_{(0)} = P(\alpha, \beta) \, \beta^\lambda \tag{B10}$$

where P is the pressure associated with the equilibrium distribution.

APPENDIX C. RELATION BETWEEN MICROSCOPIC AND HYDRODYNAMICAL DEVIATIONS FROM EQUILIBRIUM

If the deviation of the distribution function from equilibrium is given by the quadratic expression (6.2) for $y - y_0$, the corresponding linearized deviations of the moments $I^{\alpha_1 \cdots \alpha_n}$ are determined by (3.19) to be

$$\delta I^{\alpha_1 \cdots \alpha_n} = I^{\alpha_1 \cdots \alpha_n} - I_{(0)}^{\alpha_1 \cdots \alpha_n} = \epsilon J_{(0)}^{\alpha_1 \cdots \alpha_n} + \epsilon_\lambda J_{(0)}^{\alpha_1 \cdots \alpha_n \lambda} + \epsilon_{\lambda\mu} J_{(0)}^{\alpha_1 \cdots \alpha_n \lambda\mu}. \tag{C1}$$

For $n = 1, 2$ this yields, with the aid of (A13, 14),

$$\begin{aligned} mn^\alpha &= m\Delta^\alpha_\beta \, \delta N^\beta = \Delta^\alpha_\beta(\epsilon_\lambda J_{(0)}^{\lambda\beta} + \epsilon_{\lambda\mu} J_{(0)}^{\lambda\mu\beta}) \\ &= \Delta^\alpha_\beta(J_{21}\epsilon^\beta + 2J_{31}\epsilon^{*\beta}) \end{aligned} \tag{C2}$$

$$\pi^{\alpha\beta} = \delta T^{\langle\alpha\beta\rangle} = \epsilon_{\lambda\mu} J_{(0)}^{\lambda\mu\langle\alpha\beta\rangle} = 2J_{42}\epsilon^{\langle\alpha\beta\rangle} \tag{C3}$$

$$h^\alpha = -\Delta^\alpha_\beta \, \delta T^{*\beta} = \Delta^\alpha_\beta(J_{31}\epsilon^\beta + 2J_{41}\epsilon^{*\beta}). \tag{C4}$$

Hence, by (2.25) and (A31)

$$q^\alpha = h^\alpha - \eta mn^\alpha = 2(J_{41} - \eta J_{31}) \, \Delta^\alpha_\beta \epsilon^{*\beta} = 2\Lambda J_{21} \Delta^\alpha_\beta \epsilon^{*\beta}. \tag{C5}$$

So far, all results are independent (to first order) of the choice of local equilibrium distribution N_0. But now, if we wish to obtain the bulk stress Π conveniently as the trace $\frac{1}{3}\Delta_{\alpha\beta}\delta T^{\alpha\beta}$, we must impose the fitting conditions (2.22) for the reasons explained in Sec. 2. These conditions give, with the aid of (C1) and (A12),

$$0 = -u_\alpha m \delta N^\alpha = \epsilon J_{10} + \epsilon_* J_{20} + \epsilon_{**}(J_{30} + J_{31}) \tag{C6}$$

$$0 = u_\alpha u_\beta \delta T^{\alpha\beta} = \epsilon J_{20} + \epsilon_* J_{30} + \epsilon_{**}(J_{40} + J_{41}) \tag{C7}$$

whose solution is

$$\epsilon = C_1 \epsilon_{**}, \; C_1 = -[1 + 4(D_{20})^{-1} (J_{31}J_{30} - J_{41}J_{20})] \tag{C8a}$$

$$\epsilon_* = C_2 \epsilon_{**}, \; C_2 = 4(D_{20})^{-1} (J_{31}J_{20} - J_{41}J_{10}) \tag{C8b}$$

If these conditions hold, we can set

$$\Pi = \tfrac{1}{3}\Delta_{\alpha\beta}\delta T^{\alpha\beta} = \epsilon J_{21} + \epsilon_* J_{31} + \epsilon_{**}(J_{41} + \tfrac{5}{3}J_{42}). \tag{C9}$$

Substituting (C8) into (C9), and recalling (A36), we find

$$\Pi = -4(\zeta\Omega/\beta) \, \epsilon_{**} \tag{C10}$$

where Ω is defined by (7.5).

Equations (C2-5) and (C8, 10) can be solved for $\epsilon_{\alpha\beta}$, ϵ_α, ϵ in terms of $\pi_{\alpha\beta}$, q_α, π and h_α. The results are expressed by equations (6.5) with the coefficients given by

$$B_0 = -\beta/4\zeta\Omega,\ B_1 = -\beta/\Lambda nm,\ B_2 = \beta/2\zeta \tag{C11a}$$

$$D_1 = B_1 J_{41}/J_{31},\ D_0 = -3B_0C_2,\ E_0 = 3B_0C_1. \tag{C11b}$$

APPENDIX D. DERIVATION OF THE MACROSCOPE TRANSPORT EQUATIONS

We shall outline the deduction of equations (7.1) and (6.14).

From (6.5) we have

$$\begin{aligned}\epsilon_{(\lambda\mu|\nu)} = (\Delta_{(\lambda\mu} + 3u_{(\lambda}u_{\mu})\,\partial_{\nu)}(B_0\Pi) + u_{(\lambda}(B_1q_\mu)_{|\nu)} + (B_2\pi_{(\lambda\mu})_{|\nu)}\\ - (8B_0\Pi u_{(\lambda} + B_1q_{(\lambda})\,\dot{u}_\mu u_{\nu)}\end{aligned} \tag{D1}$$

$$(m\beta_{(\lambda} + \epsilon_{(\lambda)})_{|\mu)} = (\beta u^E_{(\lambda})_{|\mu)} + (D_1q_{(\lambda})_{|\mu)} + u_{(\lambda}\partial_{\mu)}(D_0\Pi) - D_0\Pi\dot{u}_{(\lambda}u_{\mu)} \tag{D2}$$

where we have used

$$u_{\alpha|\beta} = -(\dot{u}_\alpha u_\beta + \omega_{\alpha\beta}) + O_1 \tag{D3}$$

and the vorticity $\omega_{\alpha\beta}$ is defined by

$$\omega_{\alpha\beta} = \Delta^\lambda_\alpha\Delta^\mu_\beta\partial_{[\lambda}u_{\mu]}. \tag{D4}$$

In general, $\omega_{\alpha\beta}$ does not vanish in equilibrium and changes in first order when u^μ is changed. It is necessary to bear in mind that the term $(\beta u^E_{(\lambda})_{|\mu)}$ in (D2) is small of order O_1, but the terms into which it decomposes are not in general small.

To derive (7.1c), we extract the spatial trace-free components $\langle\alpha\beta\rangle$ of equation (6.14). According to (A14) we have

$$J_{(0)}^{\langle\alpha\beta\rangle\lambda\mu} = 2J_{42}\Delta^{\lambda\langle\alpha}\Delta^{\beta\rangle\mu} \tag{D5}$$

$$J_{(0)}^{\langle\alpha\beta\rangle\lambda\mu\nu} = 2J_{52}(2u^{(\lambda}\Delta^{\mu)\langle\alpha}\Delta^{\beta\rangle\nu} + \Delta^{\lambda\langle\alpha}\Delta^{\beta\rangle\mu}u^\nu). \tag{D6}$$

In (D5), we can replace $\Delta^{\alpha\beta}(u)$ by $\Delta^{\alpha\beta}(u_n)$ with an error or order O_1. Two of the terms in (6.14) thus reduce to

$$J_{(0)}^{\langle\alpha\beta\rangle\lambda\mu}(m\beta_\lambda + \epsilon_\lambda)_{|\mu} = 2J_{42}(\Delta_E^{\lambda\langle\alpha}\Delta_E^{\beta\rangle\mu}\beta u^E_{\lambda|\mu} + D_1q^{\langle\alpha|\beta\rangle}) \tag{D7}$$

$$J_{(0)}^{\alpha\beta\langle\lambda\mu\nu\rangle}\epsilon_{\lambda\mu|\nu} = 2J_{52}[-(B_1q^{\langle\alpha})^{|\beta\rangle} - 2B_2\pi_\mu^{\langle\alpha}\omega^{\beta\rangle\mu} + (B_2\pi)^{\cdot\langle\alpha\beta\rangle} + B_1q^{\langle\alpha}\dot{u}^{\beta\rangle}]. \tag{D8}$$

The notation $\pi^{\cdot\langle\alpha\beta\rangle}$ indicates the spatial trace-free part of the derivative $\dot{\pi}^{\alpha\beta}$. In the first term on the right-hand side of (D7) it is necessary to specify the projector as

$\Delta_{\alpha\beta}(u_E)$ as this term is sensitive to changes of frame. From (6.12) and (6.5a) we immediately find

$$X^{\langle\alpha\beta\rangle\lambda\mu}\epsilon_{\lambda\mu} = -\tfrac{1}{5}BB_2\pi^{\alpha\beta}. \tag{D9}$$

By combining (D7-9), and making use of (C11, 8), we arrive at (7.1c) with coefficients given by (7.7), (7.8).

Similarly, (7.1b) is derived by multiplying (6.14) by $u_\alpha \Delta^\gamma_\beta$. This involves evaluation (to first order) of

$$u^\lambda \Delta^{\mu\gamma}(\beta u^E_{(\lambda)})_{|\mu)} = \tfrac{1}{2}\beta u^\gamma_{E|\lambda} u^\lambda_E - \tfrac{1}{2}\Delta^{\mu\gamma}_E\,\partial_\mu\beta. \tag{D10}$$

The right-hand side can be transformed with the aid of the conservation identity $T^{\lambda\mu}{}_{|\mu} = 0$, evaluated in the E-frame and the thermodynamical identity (2.7) or (A30b). This leads to

$$u^\lambda \Delta^{\mu\gamma}(\beta u^E_{(\lambda)})_{|\mu)} = -\tfrac{1}{2}\Delta^{\gamma\mu}[\eta^{-1}\,\partial_\mu\alpha + (\beta\tau^\nu_{\mu|\nu}/(\rho + P)] \tag{D11}$$

and also to the constantly used result

$$\partial_\mu\beta = \beta\dot{u}_\mu + O_1\,. \tag{D12}$$

Two of the terms in (6.14) now become

$$\begin{aligned}\Delta^\gamma_\beta J^{*\beta\lambda}(m\beta_\lambda + u_\lambda)_{|\mu} = \Delta^{\gamma\mu}J_{41}[\eta^{-1}\,\partial_\mu\alpha + \beta\tau^\nu_{\mu|\nu}/(\rho + P) - D_1\dot{q}_\mu \\ - D_1 q^\lambda\omega_{\lambda\mu} + \partial_\mu(D_0\Pi) - D_0\Pi\dot{u}_\mu]\end{aligned} \tag{D13}$$

$$\begin{aligned}\Delta^\gamma_\beta J^{*\beta\lambda\mu\nu}\epsilon_{\lambda\mu|\nu} = \Delta^{\gamma\mu}J_{51}[B_1\dot{q}_\mu + 2B_2\pi_{\mu\nu}\dot{u}^\nu + 8B_0\Pi\dot{u}_\mu - 3\partial_\mu(B_0\Pi) + B_1 q^\lambda\omega_{\lambda\mu}] \\ - \Delta^{\gamma\mu}J_{52}[5\partial_\mu(B_0\Pi) + 2\Delta^{\lambda\nu}(B_2\pi_{\lambda\mu})_{|\nu}].\end{aligned} \tag{D14}$$

The rest of the derivation is straightforward. We obtain (7.1b) with a_0 given by

$$\begin{aligned}\Lambda J_{21}a_0 = (\rho + P)^{-1}J_{41} + \beta^{-1}(8B_0J_{51} - D_0J_{41}) + J_{41}(\partial D_0/\partial\beta) \\ - J_{31}(\partial E_0/\partial\beta) - (3J_{51} + 5J_{52})(\partial B_0/\partial\beta)\end{aligned} \tag{D15}$$

where the partial derivatives are evaluated for constant α. It is not immediately evident that the coefficient α_0 in (7.1b) has the form stated in (7.8a); however it is not difficult to demonstrate equality by showing that the difference between the two expressions is linear in $(J_{41}J_{10} - J_{31}J_{20})$ and $(J_{41}J_{20} - J_{31}J_{30})$ with vanishing coefficients.

Finally, (7.1a) is obtained as the $**$ component of (6.14). This requires evaluating

$$J^{**\mu}_{(0)}\,\partial_\mu(\alpha + \epsilon) = J_{30}[\dot{\alpha} + (E_0\Pi)^{\cdot}].$$

The time-derivative of any function of α, β is immediately obtained from (A35) and the equations

$$u_N^\mu\,\partial_\mu n = -nu^\mu_{N|\mu}\,, \qquad u_N^\mu\,\partial_\mu(S/n) = -n^{-1}(T^{-1}q^\mu)_{|\mu} \tag{D16}$$

which follow (to first order) from $N^\mu_{|\mu} = 0$, $S^\mu_{|\mu} = O_2$ and (2.26). We thus obtain

$$J_{(0)}^{**\mu}\,\partial_\mu(\alpha + \epsilon) = (D_{20})^{-1}\,J_{30}\{(\eta J_{20} - J_{30})\,nmu^\mu_{N|\mu} + \beta^{-1}J_{20}(\beta q^\mu)_{|\mu}\} + J_{30}E_0\dot{\Pi}. \tag{D17}$$

The expression

$$u^\lambda u^\mu(\beta u_\lambda^E)_{|\mu} = -[u_N^\mu + q^\mu/(\rho + P)]\,\partial_\mu\beta + O_2 \tag{D18}$$

is evaluated similarly, giving

$$\begin{aligned} J_{(0)}^{**\lambda\mu}(m\beta_\lambda + \epsilon_\lambda)_{|\mu} = {} & -(D_{20})^{-1}\,J_{40}\{(\eta J_{10} - J_{20})\,nmu^\mu_{N|\mu} + \beta^{-1}J_{10}(\beta q^\mu)_{|\mu}\} \\ & - \frac{J_{40}}{\rho + P}\,q^\mu\,\partial_\mu\beta + J_{41}\beta u^\alpha_{E|\alpha} - D_1(J_{40} + J_{41})\,q_\alpha\dot{u}^\alpha \\ & + J_{41}(D_1 q^\alpha)_{|\alpha} - J_{40}D_0\dot{\Pi}. \end{aligned} \tag{D19}$$

The last term of (6.14) takes the form

$$J_{(0)}^{**\lambda\mu\nu}\epsilon_{\lambda\mu|\nu} = 3(J_{50} + J_{51})\,B_0\dot{\Pi} + (J_{50} + 2J_{51})\,B_1 q_\alpha\dot{u}^\alpha - J_{51}(B_1 q^\alpha)_{|\alpha}\,. \tag{D20}$$

The coefficient of $u^\mu_{N|\mu}$ in the sum of (D17), (D19) and (D20) is

$$nm(D_{20})^{-1}\,\{J_{30}(\eta J_{20} - J_{30}) - J_{40}(\eta J_{10} - J_{20})\} + \beta J_{41} = -\zeta\Omega \tag{D21}$$

by virtue of (A36) and (7.5). We thus obtain (7.1a), with

$$\begin{aligned} \zeta\Omega a_0' = {} & (D_{20})^{-1}\,(J_{20}J_{30} - J_{40}J_{10}) - (J_{40}/J_{31}) + (J_{50} + 2J_{51})\,B_1 - (J_{40} + J_{41})\,D_1 \\ & + \beta(\zeta\Omega + \beta J_{41})(\partial/\partial\beta)[1/(\rho + P)] - \beta(J_{51}\,\partial B_1/\partial\beta - J_{41}\,\partial D_1/\partial\beta). \end{aligned} \tag{D22}$$

The partial derivatives in (D15) and (D22) can be evaluated with the aid of (A19), but that does not seem to shorten the expressions.

APPENDIX E. Formulae for a Boltzmann Gas

We list a number of useful formulae which hold for a classical relativistic gas ($\epsilon = 0$ in (4.3)).

$$\mathscr{L}_n(\alpha, \beta) = \mathscr{K}_n(\alpha, \beta) = e^\alpha K_n(\beta);\; J_{nq} = I_{nq} \tag{E1}$$

$$\eta = K_3(\beta)/K_2(\beta),\; C_P - C_V = k/m, \tag{E2}$$

$$-d\eta/d\beta = \Lambda = \gamma/(\gamma - 1)\,\beta^2 = 1 + 5\eta/\beta - \eta^2 \tag{E3}$$

$$nm = \beta P,\ \zeta = \eta P,\ J_{52} = (P/\beta)(1 + 6\eta/\beta) \tag{E4}$$

$$D_{20} = P^2/(\gamma - 1),\ D_{31} = \Lambda P^2,\ D_{41} = P^2(5\eta^2/\beta^2 - \Lambda) \tag{E5}$$

$$J_{31}J_{30} - J_{41}J_{20} = P^2(\Lambda - \eta^2),\ J_{41}J_{10} - J_{31}J_{20} = P^2\beta(\Lambda + \eta/\beta) \tag{E6}$$

$$J_{52}J_{31} - J_{41}J_{42} = (\eta P/\beta)^2 \tag{E7}$$

$$(\partial \ln P/\partial \ln n)_{S/n} = \gamma,\ (\partial \ln \zeta/\partial \ln n)_{S/n} = \gamma(1 + 1/\eta\beta) \tag{E8}$$

Asymptotic formulae. $\beta \to \infty$:

$$\eta = 1 + \tfrac{5}{2}\beta^{-1} + \tfrac{15}{8}(\beta^{-2} - \beta^{-3}) + \tfrac{135}{128}\beta^{-4} + \tfrac{45}{32}\beta^{-5} + \cdots \tag{E9}$$

$$\gamma = \tfrac{5}{3}(1 - \beta^{-1} + 4\beta^{-2} - \tfrac{119}{8}\beta^{-3} + \tfrac{425}{8}\beta^{-4} - \cdots) \tag{E10}$$

$\beta \to 0$:

$$\eta\beta = 4(1 + \tfrac{1}{8}\beta^2 + \tfrac{1}{16}\beta^4 \ln \beta + \tfrac{1}{32}(\gamma_E - 2 \ln 2)\,\beta^4 + \cdots) \tag{E11}$$

$$\gamma = \tfrac{4}{3}(1 + \tfrac{1}{24}\beta^2 + \tfrac{1}{16}\beta^4 \ln \beta + \tfrac{1}{32}(\gamma_E + \tfrac{8}{9} - 2 \ln 2)\,\beta^4 + \cdots) \tag{E12}$$

where γ_E is Euler's constant.

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