Physics 414-2 Section 1

March 30, 2021

1. Use Mathematica to find the Fourier transform $F(k)$ of the Gaussian distribution $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/(2\sigma^2)}$. As the spread σ of $f(x)$ is made narrower, what happens to the spread of $F(k)$?

2. Use Mathematica to find the Fourier transform $F(k)$ of the square pulse function defined to be $f(x) = 1$ for $-a < x < a$ and $f(x) = 0$ elsewhere. Plot $F(k)$ for some example values of a. How does the width of $F(k)$ depend on a?

3. The Dirac Delta function $\delta(x - x_0)$ is defined by integration, so that for any function $f(x)$,

$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x) = f(x_0) \tag{1}$$

Find the Fourier transform of $\delta(x - x_0)$ by hand. Does Mathematica agree? Note that Mathematica uses a different sign convention for the Fourier transform (you can look this up by searching FourierTransform in the Mathematica help menu).

4. Prove Plancherel's theorem, which states that

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |F(k)|^2 \, dk. \tag{2}$$

5. Use Mathematica and Fourier Transforms to solve the problem of the freely expanding Gaussian wave packet in quantum mechanics in one dimension (solve for the position space wavefunction as a function of time).

6. If the Fourier transform of $f(x)$ is $F(k)$, what is the Fourier transform of the derivative $f'(\mathrm{x})$?

7. For a normalized function $f(x)$ such that $\int_{-\infty}^{\infty} |f(x)|^2 \, dx = 1$, we can consider $|f(x)|^2$ as a probability distribution in x. By Plancherel's theorem, $F(k)$ is also normalized, and we can consider $|F(k)|^2$ as a probability distribution in k. We consider the spreads

$$\Delta x = \left[\int_{-\infty}^{\infty} (x - \langle x \rangle)^2 \left|f(x)\right|^2 dx\right]^{1/2} \tag{3}$$

$$\Delta k = \left[\int_{-\infty}^{\infty} (k - \langle k \rangle)^2 \left|F(k)\right|^2 dk\right]^{1/2}, \tag{4}$$

where the means $\langle x \rangle$ and $\langle k \rangle$ are given by

$$\langle x \rangle = \int_{-\infty}^{\infty} x \left|f(x)\right|^2 dx \tag{5}$$

$$\langle k \rangle = \int_{-\infty}^{\infty} k \left|F(k)\right|^2 dk. \tag{6}$$

We can prove an uncertainty relation $\Delta x \Delta k \geq \frac{1}{2}$. We will do this in steps.

(a) Define a function $g(x) \equiv e^{-i\langle k \rangle x} f(x)$. Using integration by parts and the Cauchy-Schwarz inequality, $(\int_{-\infty}^{\infty} a(x)b^{\star}(x))dx \leq (\int_{-\infty}^{\infty} \left|a(x)\right|^2 dx)^{1/2}(\int_{-\infty}^{\infty} \left|b(x)\right|^2 dx)^{1/2}$ for well-behaved functions $a(x)$ and $b(x)$, where $b^{\star}(x)$ is the complex conjugate of $b(x)$, show that

$$\Delta x \times \left[\int_{-\infty}^{\infty} \left|g'(x)\right|^2 dx)\right]^{1/2} \geq \frac{1}{2}. \tag{7}$$

(b) Using our previous result for the Fourier transform of a derivate, as well as Plancherel's theorem, show that

$$\int_{-\infty}^{\infty} \left|g'(x)\right|^2 dx) = \int_{-\infty}^{\infty} k^2 \left|G(k)\right|^2 dk, \tag{8}$$

where $G(k)$ is the Fourier transform of $g(x)$.

(c) Show that $G(k) = F(k + \langle k \rangle)$.

(d) Use the result of (c) to show that $\Delta x \Delta k \geq \frac{1}{2}$. How does this relate to the Heisenberg uncertainty principle? We could just as well have carried out the same proof with x replaced by t and k replaced by ω. For a very short laser pulse, what does this tell us about the distribution of frequencies in the pulse?