

Postulates of QM, Part II

Postulate 2

Every measurable quantity corresponds to a linear, self-adjoint operator \hat{O} acting on states in \mathcal{H} . If $| \psi \rangle$ is an eigenvector of \hat{O} :

(physicists use "self-adjoint" interchangeably, even though mathematicians would distinguish)
"Hermitian"

$$\hat{O} | \psi \rangle = \lambda | \psi \rangle$$

then the measurement of \hat{O} yields a definite value λ if the system is prepared in state $| \psi \rangle$.

- Case where system is not in an eigenstate is covered in Postulates 3-5
- Self-adjoint: \hat{O} is s.-a. if

$$\langle v_1 | \hat{O} v_2 \rangle = \langle \hat{O} v_1 | v_2 \rangle$$

"Self-adjoint" \approx "Hermitian". Mathematically these are different, related to questions of boundedness, but for our QM purposes, they are the same.

- More generally the "Hermitian adjoint" of operator Φ is defined by equ.

$$\langle \psi_1 | \Phi \psi_2 \rangle = \langle \Phi^\dagger \psi_1 | \psi_2 \rangle \quad \text{for all } \psi_1, \psi_2 \in \mathcal{H}$$

\uparrow
 Hermitian adjoint.

Φ is self-adjoint if $\Phi^\dagger = \Phi$

- For vector space of functions, self-adjoint means

$$\int dx \psi_1^*(x) \Phi \psi_2(x) = \int dx (\Phi \psi_1(x))^* \psi_2(x),$$

which is def. we used for Sturm-Liouville

- What does s-a mean for n-tuple model of \mathcal{H} ?
- For n-tuple model of Hilbert space:

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

most general
linear

Mapping from a to b given by

$$m = \begin{pmatrix} m_{11} & m_{12} & \dots \\ m_{21} & & \\ \vdots & & \end{pmatrix}$$

$$b_i = \sum_{j=1}^n m_{ij} a_j, \quad m_{ij} \in \mathbb{C}$$

- If we think of m_{ij} as a $n \times n$ matrix,

$$b = \underset{\sim}{m} \cdot a$$

- Condition of self-adjointness is

$$\langle a | \sigma b \rangle = \langle \sigma a | b \rangle$$

(From def. of inner product)

$$\Rightarrow \sum_{ij} a_i^* (m_{ij} b_j) = \sum_{ij} (m_{ij} a_j)^* b_i$$

~~Interchanging~~ \downarrow

- Interchanging i, j

$$= \sum_{ij} a_i^* m_{ji}^* b_j$$

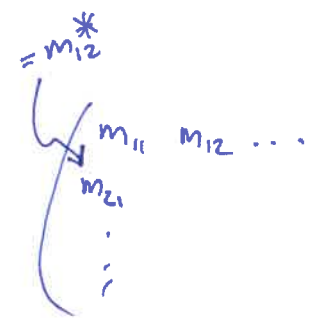
- Since this must hold (for s-a operator) for any pair of vectors $|a\rangle, |b\rangle$, we must have

$$m_{ij} = m_{ji}^*$$

- In terms of matrices

$$\underset{\sim}{m} = (\underset{\sim}{m}^T)^*$$

Only true if m is Hermitian



- We can define Hermitian adjoint of a matrix \tilde{n} s.t.

$$\tilde{n}^+ \equiv (\tilde{n})^T*$$

H.A.

Observables never look like this

These can correspond to observables

(0 i) is Hermitian

(0 1), (0 i) are ~~not~~ not

We call \tilde{n} "self-adjoint" or "Hermitian" if $\tilde{n} = \tilde{n}^+$

- We saw from Sturm-Liouville that if an operator \hat{O} was self-adjoint, eigenfunctions of \hat{O} satisfied certain properties. Now generalize to any Hilbert space.

- Theorem:

Let \hat{O} be a self-adjoint operator on \mathcal{H} . Then the eigenvectors of \hat{O} form a complete basis for \mathcal{H} .

- For space of functions, proof is S-L theorem.

- For n-tuples you prove by diagonalizing arbitrary matrix & using well-known results from lin. algebra

- From this result, some consequences follow:

① Eigenvalues of σ are real

ψ_i is e.v. of σ :

$$\cancel{\sigma \psi_i = \lambda_i \psi_i} \quad \sigma \psi_i = \lambda_i \psi_i$$

$$\Rightarrow \langle \psi_i | \sigma \psi_i \rangle = \langle \psi_i | \lambda_i \psi_i \rangle = \lambda_i \langle \psi_i | \psi_i \rangle$$

But also

$$\langle \psi_i | \sigma \psi_i \rangle = \langle \sigma \psi_i | \psi_i \rangle = \langle \lambda_i^* \psi_i | \psi_i \rangle = \lambda_i^* \langle \psi_i | \psi_i \rangle$$

$$\Rightarrow \lambda_i = \lambda_i^*$$

* comes from the def. of inner product, ~~conjugate in first term~~

② If ψ_i, ψ_j are e.v.s of σ with diff e.v.s

$$\Rightarrow \langle \psi_i | \psi_j \rangle = 0$$

$$\langle \psi_i | \sigma \psi_j \rangle = \lambda_j \langle \psi_i | \psi_j \rangle$$

$$= \langle \sigma \psi_i | \psi_j \rangle = \lambda_i \langle \psi_i | \psi_j \rangle \quad (\text{since } \lambda_i = \lambda_i^*)$$

- This can be true one of two ways:
~~is a contradiction~~ either $\lambda_i = \lambda_j$ or

$$\langle \psi_i | \psi_j \rangle = 0$$

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③ Let $|v_i\rangle$ be an orthonormal basis of H .

$$\Rightarrow \sum_i |v_i\rangle\langle v_i| = \mathbb{1}$$

Where $\mathbb{1}$ is unit operator $\mathbb{1}|v\rangle = |v\rangle$

* - This will be true if $|v_i\rangle$ are normalized e.v.s of \mathcal{O}

Proof:

If $\{|v_i\rangle\}$ is orthonormal basis, any $|v\rangle$ is

$$|v\rangle = \sum_j \alpha_j |v_j\rangle$$

$$\begin{aligned} \Rightarrow \left(\sum_i |v_i\rangle\langle v_i| \right) |v\rangle &= \sum_{i,j} |v_i\rangle \underbrace{\langle v_i | v_j \rangle}_{\delta_{ij} \text{ since orthonormal}} \alpha_j \\ &= \sum_j |v_j\rangle \alpha_j = |v\rangle \end{aligned}$$

$$\Rightarrow \sum_i |v_i\rangle\langle v_i| = \mathbb{1} \quad \checkmark$$

Just projects onto basis & writes ~~it as~~ as sum of basis vectors

- On ~~the~~ Hilbert space of n -tuples,

a basis vector $|v_i\rangle$ is an n -tuple with

k th entry $(v_i)_k$

- Using linear algebra formalism, $|v_i\rangle$ is a column vector and from def. of inner product $\langle v_i|$ is a row vector (v_i, v_i, \dots)

- Then above formula reads
(In component form) $\sum_i (v_i)_k (v_i)_l^* = \delta_{kl}$ (In matrix form)

$$\sum_i (v_i)_k (v_i)_l^* = \delta_{kl}$$

$$\sum_i v_i (v_i)^T = \begin{pmatrix} 1 & 0 \\ 0 & \ddots & 0 \end{pmatrix}$$

(easy to see for diagonalized basis but here is an example from a different basis)

where I have represented v_i as a column vector that order is necessary to get an $n \times n$ matrix

Example

$$v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

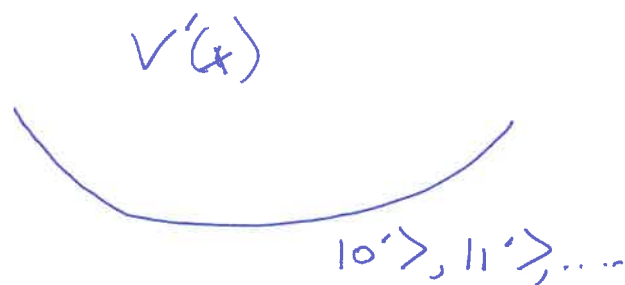
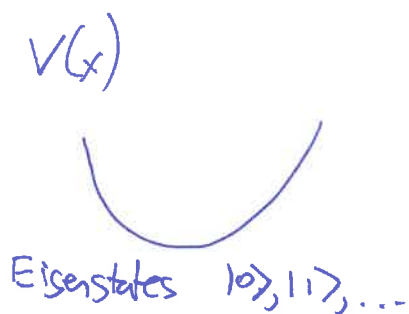
$$= (v_1, v_2) \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} + (v_1, v_2) \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$$

- We usually like to think of the identity operator $\sum |u_i\rangle\langle u_i|$ as just an arbitrary state / some choice of projecting \wedge onto ~~not different~~ basis.

- For instance we can use the eigenstates of H with either of these potentials to represent any complex-valued function



Then we can represent for instance $|0\rangle$ as

$$\begin{aligned}
 |0\rangle &= c_0 |0'\rangle + c_1 |1'\rangle + \dots \\
 &= \underbrace{\langle 0'|0\rangle}_{\uparrow} |0'\rangle + \langle 1'|0\rangle |1'\rangle + \dots \\
 &= |0'\rangle \langle 0'|0\rangle + |1'\rangle \langle 1'|0\rangle + \dots \\
 &= \underbrace{\sum_n |n'\rangle \langle n'|0\rangle}_{\uparrow}
 \end{aligned}$$

* Unitary Operators - Different class from Hermitian

- Preserve inner product: $\langle v|w \rangle = \langle Uv|Uw \rangle$
- These are generalization of 3D rotation.

which leaves lengths & dot products of vectors invariant.

- Using def of Hermitian adjoint

$$\langle u_0 | u_w \rangle = \langle v | u^\dagger u_w \rangle$$

- Since $\langle u|w \rangle$ this equals $\langle u|w \rangle$ by def of unitary,

$$\Rightarrow u^+ u = 1$$

\Rightarrow

$$\begin{aligned} \textcircled{1} \quad u^+ &= u^{-1}, & u u^{-1} &= 1 \\ \textcircled{2} \quad u u^+ &= 1 & \leftarrow \end{aligned}$$

~~(adjoint)~~
 adjoint :

- Note that these are not self-adjoint

$u^+ \neq u$ except for special cases ($u=1, u=\pi$)
where $uu=1$

is a matrix mult. which

$\sum_i (u_{ik})^2 = \sum_k (u_{ki})^2 = 1$

Hilbert space of n -tuples, this can be written as

$\sum_i (u_{ik})^2 = \sum_k (u_{ki})^2 = 1$

or

$\sum_i (u_{ik})^2 = \sum_k (u_{ki})^2 = 1$

Schrodinger equation

$H \psi = E \psi$

or

$H \psi = E \psi$

How could you tell?

- It turns out that the columns of U are an orthonormal basis for n -tuples

Proof

- U preserves orthogonality + normalization, by def.
 - Consider U acting on the orthonormal basis

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \end{pmatrix}, \text{ etc}$$

$$\begin{pmatrix} \vdots & u_{12} & \vdots \\ \vdots & u_{22} & \vdots \\ \vdots & u_{32} & \vdots \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} u_{12} \\ u_{22} \\ u_{32} \end{pmatrix}$$

U

And having U act on a dif. basis vector gives a different column.

\Rightarrow The columns of U are an orthonormal basis, since that ^(simplest) basis vector ~~set~~ ^{set} is

Special Operator Summary

Any Operator A

Hermitian adjoint A^\dagger :

$$\langle \psi | A \chi \rangle = \langle A^\dagger \psi | \chi \rangle$$

Inverse A^{-1} :

$$A A^{-1} = A^{-1} A = \mathbb{1}$$

Hermitian Operator Θ

$$\Theta = \Theta^\dagger:$$

$$\Rightarrow \langle \psi | \Theta \chi \rangle = \langle \Theta \psi | \chi \rangle$$

\Rightarrow Eigenvalues of Θ are real

\Rightarrow Eigenvectors of Θ can be chosen such that they form an orthonormal basis for \mathcal{H}

- Requires normalizing

- Requires making right care in degenerate subspace

e.g. $\Theta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $|\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\psi_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

both are eigenvectors, but not orthogonal.

Infinite # of choices, with $\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

or $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ being popular choices

Unitary Operator U

~~$$U^\dagger U = 1$$~~

$$U^\dagger U = 1 :$$

$$\Rightarrow \langle U\psi | U\chi \rangle = \langle \psi | \chi \rangle$$

$$\Rightarrow U^\dagger = U^{-1}$$

$\Rightarrow U$ rotates $|\psi\rangle$ in \mathcal{H} .

Equivalently, U transforms one orthonormal basis for \mathcal{H} into another