Applied Nonlinear Dynamics 322

Spring 2022

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Problem Set 3 **Saturday, April 16, 2022** **due Friday, April 22, 2022**

1. **SNIC in the Morris-Lecar Model for Spiking Neurons**
SNIC bifurcations occur also in two- and higher-dimensional systems. The Morris-Lecar model is a classic example. It has been introduced to describe voltage oscillations in muscle fibers [2]. Subsequently, it has been used in a much wider context to describe neurons that spike periodically over a wide range of frequencies depending on the magnitude of the excitation that they receive [1]. The model describes the dynamics of the voltage V of the neuron and of the gating variable w for the potassium channel, which varies between $w = 0$ when all ion channels are closed and $w = 1$ when all channels are open,

$$\frac{dV}{dt} = -g_l\,(V - V_L) - g_{Ca} m_\infty(V)\,(V - V_{Ca}) - g_K w\,(V - V_K) + I_{inj} \tag{1}$$

$$\frac{dw}{dt} = \frac{1}{\tau_w(V)}\,(w_\infty(V) - w)\,. \tag{2}$$

Here I_{inj} is an external input, which serves for us as a control parameter. The steady-state value w_∞ for w depends on the voltage,

$$w_\infty(V) = \frac{1}{2}\left\{1 + \tanh\left(\frac{V - V_3}{V_4}\right)\right\}.$$

The gating variable w relaxes towards w_∞ on the time scale τ_W,

$$\tau_W(V) = \frac{3}{\cosh\left(\frac{V-V_3}{2V_4}\right)}.$$

The gating variable m for the calcium channel is assumed to follow the voltage so fast that it can be replaced by its steady-state value $m_\infty(V)$,

$$m_\infty(V) = \frac{1}{2}\left\{1 + \tanh\left(\frac{V - V_1}{V_2}\right)\right\}.$$

Use the following parameters

$$g_l = 0.5, \quad g_K = 2, \quad g_{Ca} = 1.33$$

$$V_L = -0.5, \quad V_{Ca} = 1, \quad V_K = -0.7$$

$$V_1 = -0.01, \quad V_2 = 0.15, \quad V_3 = 0.1, \quad V_4 = 0.145.$$

To analyze this model use the new incarnation of the matlab program pplane, which was originally written by J. Polking[1]. It is called phaseplane and available at `https://github.com/MathWorks-Teaching-Resources/Phase-Plane-and-Slope-Field`.
Note: To avoid retyping the equations later you can save your system.

[1] http://math.rice.edu/~polking/

(a) Start with $I_{inj} = 0.05$. Investigate the trajectories one obtains for initial conditions (w_0, V_0) with $w_0 = 0$ and $-0.2 \leq V_0 \leq 0.4$. To do so select under 'Options' 'Solution Direction' 'forward' and plot all the trajectories in the same plot. Describe the trajectories; what do they have in common? Find two fixed points (equilibrium points). Pplane can do that for you under 'Solutions'. By choosing suitable initial conditions determine their stability.

(b) Now increase I_{inj} in small steps to $I_{inj} = 0.07$. What happens to the fixed points? What kind of bifurcation do you observe? Describe the long-term solution. Plot the solution also in terms of $V(t)$ and $w(t)$ (using the option 'Graph').

(c) Now investigate the transition quantitatively. Measure the period T of the oscillation as a function of I_{inj} in the range $[0.0695, 0.09]$. In order to obtain sufficiently many cycles choose 2 plot steps per computation step (under settings). To read off the period, plot also $V(t)$ (under graph) and in the figure that appears use the data tips under View/Figure Toolbar. Based on part 1b, what dependence do you expect for $T(I_{inj})$? Plot $T(I_{inj})$ suitably so that the expected behavior would appear as a straight line on the graph. Do your simulations confirm your expectations?

(d) Exploration: for $I_{inj} = 0.075$ study the behavior of the system for general initial conditions (within $-0.5 \leq V \leq 0.5$ and $0 \leq w \leq 0.8$). Do all initial conditions converge to the periodic orbit arising from the SNIC? Can you detect a fixed point? Can you detect another periodic orbit? Assess the stability of the objects you can find.
Note: For this task you may find it useful to use also the backward solution under 'Options'.

2. Sketch a two-dimensional phase portrait containing the following features:

   (a) exactly five fixed points: four saddle-points and one unstable spiral point.

   (b) exactly three fixed points. Two are saddle-points without a trajectory connecting them. One is a node.

3. Give a two-dimensional dynamical system for which the stable and unstable manifolds of a saddle fixed point coincide with the x- and y-nullclines.

4. Consider the dynamical system

$$\begin{aligned} \dot{x} &= \mu x + 2y - \alpha x^3 \,, \\ \dot{y} &= -x + y - \alpha y^3 \,. \end{aligned}$$

   (a) For which values of μ is it guaranteed that the linearization around the fixed point $(0, 0)$ provides the correct topology of the dynamics in the vicinity of the fixed point?

   (b) For the values of μ for which the topology obtained by the linearization is not guaranteed to be correct, illustrate that fact with a couple of simulations (using again the successor of pplane to plot a few solutions). Describe the difference in the phase portraits .

References

[1] H. Lecar. The Morris-Lecar model. *Scholarpedia*, 2(10), 2007.

[2] C. Morris and H. Lecar. Voltage oscillations in the barnacle giant muscle fiber. *Biophys. J.*, 35:193, 1981.