MATH 420 - FALL 2019
ASSIGNMENT 4

Note: Some of these problems are taken from *Partial Differential Equations*, by L.C. Evans. Assume that U is a bounded domain in $\mathbb{R}^n$ with smooth boundary ∂U, unless otherwise stated.

(1) Recall that Young's inequality, which we proved in class, states that for real numbers $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ then for $a, b > 0$,

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}.$$

(a) Prove Hölder's inequality that for $u \in L^p(U)$ and $v \in L^q(U)$, with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\int_U uv dx \leqslant \|u\|_p \|v\|_q,$$

where we are writing $\|u\|_p = \left(\int_U |u|^p dx\right)^{1/p}$ for the $L^p(U)$ norm.

(b) Prove that if $\mathrm{Vol}(U) = 1$ and $u \in L^p(U)$ then

$$\|u\|_r \leqslant \|u\|_p,$$

if $1 \leqslant r \leqslant p$.

(c) Prove the generalized Hölder's inequality

$$\int_U u_1 \cdots u_m dx \leqslant \|u_1\|_{p_1} \cdots \|u_m\|_{p_m},$$

for $u_i \in L^{p_i}(U)$ and $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = 1$.

(2) The goal of this problem is to give a direct proof of the *Poincaré inequalities*: for every $u \in W_0^{1,p}(U)$ with $1 \leqslant p < \infty$ we have

$$\int_U |u|^p dx \leqslant (\mathrm{diam}(U))^p \int_U |Du|^p dx.$$

(a) Explain briefly why we may assume without loss of generality that

$$U \subset [-a, a] \times \mathbb{R}^{n-1},$$

for $a = \mathrm{diam}$ (U)/2.

(b) Suppose that $u \in C_c^1(U)$ (extend by 0 outside U) and show that

$$|u(x)| \leqslant \int_{-a}^{a} |Du(t, x_2, \ldots, x_n)| dt.$$

Then integrate in x to prove the Poincaré inequality for $p = 1$.

(c) *Jensen's inequality* says that for a convex function $\varphi : \mathbb{R} \to \mathbb{R}$ and (X, Σ, μ) a measure space with $\int_X d\mu = 1$ we have

$$\varphi\left(\int_X g d\mu\right) \leqslant \int_X \varphi(g) d\mu,$$

for all integrable functions $g : X \to \mathbb{R}$. Using this inequality, prove the Poincaré inequality for general $1 \leqslant p < \infty$ by obtaining a pointwise bound for $|u(x)|^p$ and then integrating in x.

(d) Is there a version of Poincaré's inequality for $p = \infty$? If so what is it?

(3) For $k = 0, 1, 2, \ldots$ and $\gamma \in (0, 1]$, prove that $C^{k,\gamma}(\overline{U})$ is a Banach space.

(4) Assume $0 < \beta < \gamma \leqslant 1$. For $u \in C^1(\overline{U})$, prove

$$\|u\|_{C^{0,\gamma}(\overline{U})} \leqslant \|u\|_{C^{0,\beta}(\overline{U})}^{\frac{1-\gamma}{1-\beta}} \|u\|_{C^{0,1}(\overline{U})}^{\frac{\gamma-\beta}{1-\beta}}.$$