Physics 411 Problem Set 1 Solutions (Goldstein Chapters 1-2)

Due at 10 am, Friday, October 4th, 2019

Instructor: Sasha Tchekhovskoy

E-mail: atchekho@northwestern.edu

1. What is the Lagrangian for a projectile launched radially outwards from the surface of Earth, and what is the equation of motion? Show that energy is conserved. Deduce Earth's escape speed, i.e. the minimum speed required to escape Earth's gravity from its surface. Give both the symbolic expression for the escape speed and how much faster it is than a Boeing 747.

Solution. Lagrangian is

$$L = \frac{m\dot{r}^2}{2} + \frac{GMm}{r}.$$

The equation of motion is

$$\ddot{r} + GM/r^2 = 0.$$

To find the conserved energy, multiply both sides of this equation by $\dot{r}$ and integrate in time, giving us

$$E = \frac{m\dot{r}^2}{2} - \frac{GMm}{r}.$$

To find the escape speed, set $E = 0$, giving us

$$v_{\rm esc} = \left(\frac{2GM}{r}\right)^{1/2} = 11.2 \text{ km/s}.$$

The speed of Boeing 747 is around Mach 0.85, i.e., it travels at 85% of the sound speed, or 920 km/h = 0.26 km/s. That is, the escape speed exceeds that of the Boeing by a factor of around 44.

2. Determine the Lagrangian and equation of motion for a pendulum whose pivot point (i.e. the point at the top) is shaken horizontally so that its position is $x_{\rm piv}(t) = A\sin(\omega t)$?

Solution. The Lagrangian for a pendulum is

$$L = \frac{m(\dot{x}^2 + \dot{y}^2)}{2} + mgl\cos\theta,$$

where $x = x_{\rm piv}(t) + l\sin\theta$, $y = -l\cos\theta$. Therefore,

$$L = glm\cos\theta + \frac{m}{2}\left(\dot{\theta}^2 l^2 \sin^2\theta + \left(A\omega\cos\omega t + \dot{\theta} l\cos\theta\right)^2\right).$$

This gives us the equation of motion:

$$\ddot{\theta} l + g\sin\theta = A\omega^2 \sin\omega t\cos\theta.$$

3. (from Goldstein chapter 1): Obtain the Lagrange equations of motion for a spherical pendulum, i.e. a mass suspended by a rigid weightless rod (or a string) that can swing left-right and back-forth.

Solution. In spherical polar coordinates, we can write:

$$L = \frac{ml^2(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta)}{2} + mgl\cos\theta.$$

Equations of motion in θ and φ directions:

$$\ddot{\theta} = -\frac{g}{l}\sin\theta + \dot{\varphi}^2 \sin\theta\cos\theta, \tag{1}$$

$$\ddot{\varphi} = 0. \tag{2}$$

4. (from Goldstein chapter 1): Two mass points of mass m_1 and m_2 are connected by a string passing through a hole in a smooth table so that m_1 rests on the table surface and m_2 hangs suspended. Assuming m_2 moves only in a vertical line, what are the generalized coordinates for the system? Write down the Lagrange equations for the system and, if possible, discuss the physical significance any of them might have. Reduce the problem to a single second-order differential equation and obtain a first integral of the equation. What is its physical significance? (Consider the motion only so long as neither m_1 nor m_2 passes through the hole.) Notes: (i) m_1 can move in two dimensions. (ii) A "first integral" means: the equation d/dt(first integral) = 0 is equivalent to the equation of motion.

Solution. Because the two masses are connected by a string of a fixed length, d, the two masses cannot move independently, and there are only two degrees of freedom in the system: those describing the position of the first mass. For instance, we can pick the polar coordinates, r and θ, of the first mass on the table as the generalized coordinates of the system. Then, the displacement of the second mass relative to the table is given via $y = r - d$. The Lagrangian takes the form:

$$L = \frac{m_1(\dot{r}^2 + r^2\dot{\theta}^2)}{2} + \frac{m_2\dot{r}^2}{2} - m_2 gr. \tag{3}$$

The equation of motion in the θ-direction reduces to the angular momentum conservation for the first mass, $dl/dt = 0$, where the angular momentum is given by the usual expression:

$$l = m_1 r^2\dot{\theta}. \tag{4}$$

The radial equation of motion:

$$(m_1 + m_2)\ddot{r} = -m_2 g + m_1 r\dot{\theta}^2 = -m_2 g + \frac{l^2}{m_1 r^3}. \tag{5}$$

As you can see, the force of gravity acting on the second mass is counteracted by the centrifugal force acting on the first mass; the combination of these two forces is what accelerates both masses. What kind of integral of motion can we expect? The first one, we found already: it's the angular momentum. The second one is the energy. To obtain it, we use the usual trick: multiply eq. (5) with $\dot{r}$ and integrate in time:

$$\frac{(m_1 + m_2)\dot{r}^2}{2} = -m_2 gr - \frac{l^2}{2m_1 r^2} + E. \tag{6}$$

Here, the integration constant E is the conserved energy, the integral of motion we have been looking for.