Local Phase and Gauge Transformations

Global phase invariance

$$\psi(\vec{x},t) \longrightarrow \tilde{\psi}(\vec{x},t) = e^{-i\chi}\psi(\vec{x},t)$$

ind. of $\vec{x},t$

Observables: $\langle\tilde{\psi}|\tilde{\theta}|\tilde{\psi}\rangle = \langle\psi|e^{i\chi}\theta e^{-i\chi}|\psi\rangle$
$= \langle\psi|\theta|\psi\rangle$

$\tilde{\theta} = \theta \rightsquigarrow$ invariant

Time evolution: use $|\psi\rangle = e^{i\chi}|\tilde{\psi}\rangle$

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = H\psi$$

$$i\hbar\frac{\partial}{\partial t}\cancel{e^{i\chi}}|\tilde{\psi}\rangle = H\cancel{e^{i\chi}}|\tilde{\psi}\rangle \longrightarrow |\tilde{\psi}\rangle \text{ evolves like } |\psi\rangle$$

Local phase transformation

$$\psi(\vec{x},t) \longrightarrow \tilde{\psi}(\vec{x},t) = e^{-i\underbrace{\chi(\vec{x},t)}}\psi(\vec{x},t)$$

depends on local location in spacetime

Observables: Use $U(\vec{x},t) = e^{-i\chi(\vec{x},t)}$

$$\langle\tilde{\psi}|\tilde{\theta}|\tilde{\psi}\rangle = \langle\psi|\theta|\psi\rangle$$

$$\langle\psi|U^{\dagger}\tilde{\theta}U|\psi\rangle =$$

$$\tilde{\theta} = U\theta U^{\dagger} = e^{i\chi(\vec{x},t)}\theta e^{-i\chi(\vec{x},t)}$$

Not all operators are invariant

$\vec{x}$ is clearly inv.

$\vec{p} = -i\hbar\vec{\nabla}$ is not inv.

Time evolution after a local phase transformation

$$H = \frac{\vec{p}^{\,2}}{2m} + q\Phi \rightsquigarrow \text{complicated because } \vec{p} \text{ operates on } \chi(\vec{x},t)$$

$|\tilde{\psi}\rangle = \mathcal{U}|\psi\rangle$

$\mathcal{U} = e^{-i\chi}$

Consider:

$$\mathcal{U}\vec{p}\,\mathcal{U}^\dagger = \mathcal{U}(-i\hbar\vec{\nabla} i\chi)\mathcal{U}^\dagger + \mathcal{U}\mathcal{U}^\dagger\vec{p}$$

$$= \vec{p} + \underbrace{\hbar(\vec{\nabla}\chi)}_{\text{offset vector function of } \vec{x},t}$$

Consider:

$$\mathcal{U}(\vec{p} - q\vec{A})\mathcal{U}^\dagger = \vec{p} + \underbrace{\hbar(\vec{\nabla}\chi) - q\vec{A}}$$

$\vec{A}(\vec{x},t)$

$$\tilde{\vec{A}} = \vec{A} - \frac{\hbar}{q}(\vec{\nabla}\chi)$$

Need generalize H and look for $\tilde{H}$ such that $i\hbar\frac{\partial}{\partial t}|\tilde{\psi}\rangle = \tilde{H}|\tilde{\psi}\rangle$

Try generalization $H = \frac{(\vec{p} - q\vec{A})^2}{2m} + q\Phi$

Thus

$$i\hbar\frac{\partial}{\partial t}\underbrace{|\psi\rangle}_{\mathcal{U}^\dagger|\tilde{\psi}\rangle} = \left[\frac{(\vec{p} - q\vec{A})^2}{2m} + q\Phi\right]\underbrace{|\psi\rangle}_{\mathcal{U}^\dagger|\tilde{\psi}\rangle}$$

$\mathcal{U}$

$$\mathcal{U}\, i\hbar \underbrace{\frac{\partial}{\partial t}\mathcal{U}^\dagger}_{i\frac{\partial\chi}{\partial t}\mathcal{U}^\dagger + \mathcal{U}^\dagger\frac{\partial}{\partial t}}|\tilde{\psi}\rangle = \frac{\mathcal{U}(\vec{p} - q\vec{A})^2\mathcal{U}^\dagger}{2m} + \mathcal{U}q\Phi\,\mathcal{U}^\dagger|\tilde{\psi}\rangle$$

$$\left[-\hbar\frac{\partial\chi}{\partial t} + i\hbar\frac{\partial}{\partial t}\right]|\tilde{\psi}\rangle = \left[\frac{(\vec{p} - q\tilde{\vec{A}})^2}{2m} + q\Phi\right]|\tilde{\psi}\rangle$$

$$i\hbar\frac{\partial}{\partial t}|\tilde{\psi}\rangle = \left[\frac{(\vec{p} - q\tilde{\vec{A}})^2}{2m} + \underbrace{q\Phi + \hbar\frac{\partial\chi}{\partial t}}_{q\tilde{\Phi}}\right]|\tilde{\psi}\rangle$$

$$\therefore \boxed{\tilde{H} = \frac{(\vec{p} - q\tilde{\vec{A}})^2}{2m} + q\tilde{\Phi}}$$

with $\boxed{\tilde{\vec{A}} = \vec{A} - \frac{\hbar}{q}\vec{\nabla}\chi}$ and $\boxed{\tilde{\Phi} = \Phi + \frac{\hbar}{q}\frac{\partial\chi}{\partial t}}$

So far: Generalize to $\boxed{H = \frac{(\vec{p} - q\vec{A})^2}{2m} + q\Phi}$

for wavefunction $|\psi\rangle$

Make a local phase transformation:

$$\boxed{|\psi\rangle \longrightarrow |\tilde{\psi}\rangle = e^{-i\chi(\vec{x},t)}|\psi\rangle}$$

The $\tilde{H}$ that governs time evolution of $|\tilde{\psi}\rangle$ has

Gauge transformation

$$\boxed{\begin{aligned} \tilde{\vec{A}} &= \vec{A} - \frac{\hbar}{q}(\vec{\nabla}\chi) \\ \tilde{\Phi} &= \Phi + \frac{\hbar}{q}\frac{\partial \chi}{\partial t} \end{aligned}}$$

Covariant Derivative — hiding in the generalized Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}\psi = \frac{(\vec{p}-q\vec{A})^2}{2m}\psi + q\Phi\psi$$

sum over repeated indices

$$\left(i\hbar\frac{\partial}{\partial t} - q\Phi\right)\psi = \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x^j} - qA_j\right)\left(-i\hbar\frac{\partial}{\partial x^j} - qA_j\right)\psi$$

$$i\hbar c\underbrace{\left(\frac{\partial}{\partial(ct)} + \frac{iq\Phi}{\hbar c}\right)}\psi = \frac{-\hbar^2}{2m}\underbrace{\left(-\frac{\partial}{\partial x^j} + \frac{iq}{\hbar}A_j\right)}\left(-\frac{\partial}{\partial x^j} + \frac{iq}{\hbar}A_j\right)\psi$$

$$\boxed{D^0 \equiv \frac{\partial}{\partial(ct)} + \frac{iq\Phi}{\hbar c}} \qquad \boxed{D^j \equiv -\frac{\partial}{\partial x^j} + \frac{iq}{\hbar}A_j}$$

$$\begin{aligned} D^\mu &= (D^0, D^j) \\ &= \left(\frac{\partial}{\partial ct} + \frac{iq\Phi}{\hbar c}, -\frac{\partial}{\partial x^j} + \frac{iq}{\hbar}A_j\right) \\ &= \underbrace{\left(\frac{\partial}{\partial ct}, -\vec{\nabla}\right)}_{\substack{\text{4-vector} \\ \partial^\mu}} + \frac{iq}{\hbar}\underbrace{\left(\frac{1}{c}\Phi, \vec{A}\right)} \end{aligned}$$

smells like a familiar 4-vector

$$A^\mu = \left(\frac{1}{c}\Phi, \vec{A}\right)$$

electric potential — vector potential

S.E. $\quad i\hbar c \, D^0 \psi = -\frac{\hbar^2}{2m} D^j D^j \psi$

$U^+ \tilde{\psi}$

U

$$i\hbar c \underbrace{U D^0 U^+}_{\tilde{D}^0} \tilde{\psi} = -\frac{\hbar^2}{2m} \underbrace{(U D^j U^+)}_{\tilde{D}^j} \underbrace{(U D^j U^+)}_{\tilde{D}^j} \tilde{\psi}$$

$\tilde{\psi} = U\psi$

$$\tilde{D}^\mu = U D^\mu U^+$$ ⇐ determines time evolution of $\tilde{\psi}$

$$\tilde{D}^\mu \tilde{\psi} = U D^\mu U^+ \tilde{\psi}$$
$$= U D^\mu U^+ (U\psi)$$
$$\boxed{\tilde{D}^\mu \tilde{\psi} = U(D^\mu \psi)}$$

compare $\quad \tilde{\psi} = U\psi$

⟹ $D^\mu \psi$ transforms like ψ

⟹ called covariant derivative

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eg. $\tilde{D}^j = -\frac{\partial}{\partial x_j} + \frac{ig}{\hbar} A_j$

$i\hbar \tilde{D}^j = -i\hbar \frac{\partial}{\partial x_j} - g A_j$

$i\hbar \vec{D} = \vec{p} - g\vec{A}$ ⇐ covariant derivative

i.e. $\vec{p} - g\vec{A}$ transforms like ψ

$$\psi \to \tilde{\psi} = e^{-i\chi} \psi$$
$$\underbrace{\vec{p} - g\vec{A}}_{} \to \vec{p} - g\tilde{\vec{A}} = e^{-i\chi} (\vec{p} - g\vec{A})$$
$$\vec{\pi} \to \tilde{\vec{\pi}} = e^{-i\chi} \vec{\pi}$$

under gauge transformation
(includes a local phase transformation)

$\boxed{\text{Invariants}}$ — $\tilde{\Phi} = \Phi + \frac{\hbar}{q}\frac{\partial \chi}{\partial t}$

$\tilde{\vec{A}} = \vec{A} - \frac{\hbar}{q}\vec{\nabla}\chi$

not invariants

① $\vec{\nabla}\tilde{\Phi} = \vec{\nabla}\Phi + \frac{\hbar}{q}\vec{\nabla}\frac{\partial}{\partial t}\chi$

same for well-behaved χ

$\frac{\partial \tilde{\vec{A}}}{\partial t} = \frac{\partial \vec{A}}{\partial t} - \frac{\hbar}{q}\frac{\partial}{\partial t}\vec{\nabla}\chi$

$\therefore \vec{\nabla}\tilde{\Phi} - \vec{\nabla}\Phi = -\left(\frac{\partial \tilde{\vec{A}}}{\partial t} - \frac{\partial \vec{A}}{\partial t}\right)$

$-\vec{\nabla}\tilde{\Phi} - \frac{\partial \tilde{\vec{A}}}{\partial t} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}$

called electric field $\therefore \boxed{\vec{E} \equiv -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}}$ invariant under gauge transformations

② $\vec{\nabla}\times\tilde{\vec{A}} = \vec{\nabla}\times\left[\vec{A} - \frac{\hbar}{q}\vec{\nabla}\chi\right] = \vec{\nabla}\times\vec{A} - \frac{\hbar}{q}\underbrace{\vec{\nabla}\times\vec{\nabla}\chi}_{=0 \text{ for any scaler } \chi(\vec{x},t)}$

called magnetic field $\therefore \boxed{\vec{B} \equiv \vec{\nabla}\times\vec{A}}$ invariant

So far: $\vec{A} \leftarrow$ requirement for Gauge transformations

$\vec{B} \leftarrow \vec{\nabla} \times \vec{A}$

$\vec{E} \equiv$

Invariants under gauge transformations

Compare to classical mechanics to establish that $\vec{E}$ and $\vec{B}$ are the familial electric and magnetic fields

i.e. show that

$$m \frac{\partial^2 \langle \vec{x} \rangle}{\partial t^2} = q \langle \vec{E} \rangle + q \left\langle \frac{\vec{\pi}}{m} \times \vec{B} \right\rangle$$

and $$m \frac{\partial \langle \vec{x} \rangle}{\partial t} = \langle \vec{\pi} \rangle$$

Uniform Magnetic Field and Magnetic Moment

Consider $\vec{A} = \frac{1}{2}\vec{B}\times\vec{r}$ with $\vec{B}$ independent of time and space

$$\boxed{H = \frac{p^2}{2m} - \underbrace{\frac{q}{2m}\overset{L_z}{\vec{L}\cdot\vec{B}}}_{-\vec{\mu}\cdot\vec{B}} + \frac{q^2B^2}{8m}(x^2+y^2)}$$

assuming $\vec{B} = B\hat{z}$

$-\vec{\mu}\cdot\vec{B}$ with $\vec{\mu} = \frac{q\vec{L}}{2m}$ magnetic moment (next page)

Orbital moment is not so surprising $q=-e$

$\mu = IA$ in SI units

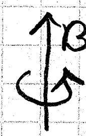
$$\vec{\mu} = \frac{-e}{\left(\frac{2\pi}{\omega_c}\right)}\cancel{\pi}\rho^2\hat{z}$$

$$= -\frac{e\omega_c}{2}\rho^2\hat{z}$$

$$= -\frac{\cancel{e}}{2}\frac{eB}{m}\cancel{\rho^2}\hat{z}\,\frac{L_z}{\cancel{eB\rho^2}}$$

$$\boxed{\vec{\mu} = \frac{q}{2m}\vec{L}}$$

$$\omega_c = \frac{eB}{m}$$



$$L_z = mv\rho$$
$$= m(\omega_c\rho)\rho$$
$$= m\omega_c\rho^2$$
$$= \cancel{m}\frac{eB}{\cancel{m}}\rho^2$$
$$= eB\rho^2$$

Orbital and spin moments

orbital $\vec{\mu} = \frac{q}{2m}\vec{L}$ ⟶ comes out naturally

postulate similar moment for spin ang. mom.

$$\vec{\mu}_s = g\frac{q}{2m}\vec{S}$$

dimensionless fudge factor (g-value or g-factor)

with $g=2$

Spin moment does come naturally from Dirac equation

$\vec{A}\cdot\vec{A}$ is often neglected — e.g. for atoms

$$\frac{\frac{q^{\cancel{2}}A^2}{\cancel{2m}}}{\frac{q}{\cancel{2m}}LB} = \frac{\frac{q}{4}B^{\cancel{2}}\rho^2}{\cancel{B}L} = \frac{qBa_B^2}{4\hbar} = 10^{-5}\frac{B}{10\,\text{Tesla}}$$

a_B^2

$\hbar$

i.e. small even for very large field