① $[X,Y] = c$

a. prove result by induction. $[X,Y^n] = ncY^{n-1}$

$[X,Y^1] = (1)c\cdot Y^{1-1} = c$ ✓ $\quad [X,Y^0] = [X,I] = 0$

$= (0)\cdot c\cdot Y^{0-1} = 0$ ✓

assume $[X,Y^n]$ & find $[X,Y^{n+1}]$

$$
\begin{aligned}
[X,Y^{n+1}] &= XY^{n+1} - Y^{n+1}X \\
&= XY\cdot Y^n - YY^nX + YXY^n - YXY^n \\
&= XYY^n - YXY^n + YXY^n - YY^nX \\
&= [X,Y]Y^n + Y[X,Y^n] \\
&= cY^n + Y\cdot ncY^{n-1} \\
&= (n+1)cY^{n+1} \checkmark
\end{aligned}
$$

so $\boxed{[X,Y] = cnY^{n-1}}$

b. $f(Y) \equiv \sum_n a_n Y^n$

$$[X,f(Y)] = \sum_n a_n [X,Y^n] = c\sum_n a_n \cdot \underbrace{nY^{n-1}}_{= \frac{d}{dY}(Y^n)}$$

$$[X,f(Y)] = c\cdot \frac{d}{dY}\left(\sum_n a_n Y^n\right)$$

$$\boxed{[X,f(Y)] = c\cdot \frac{df}{dY}}$$

(2) a. $g(x) = e^{xA} B e^{-xA}$

$$\frac{dg}{dx} = e^{xA} A B e^{-xA} - e^{xA} B A e^{-xA}$$

$$\frac{dg}{dx} = e^{xA}[A,B]e^{-xA} \quad (\cancel{*})$$

rewrite $e^{\pm xA}$ as series:

$$\frac{dg}{dx} = \sum_{n=0}^{\infty} \frac{A^n x^n}{n!} [A,B] \sum_{m=0}^{\infty} \frac{(-A)^m x^m}{m!}$$

$$\frac{dg}{dx} = \sum_n \sum_m \frac{1}{n!}\frac{1}{m!} A^n [A,B](-A)^m x^{n+m}$$

$$g(x) = \int_0^x \frac{dg(y)}{dy} dy + g(0) \qquad g(0) = e^{0 \cdot A} B e^{-0 \cdot A} = IBI = B$$

$$g(x) = B + \int_0^x \cancel{dy} \sum_n \sum_m \frac{1}{n!}\frac{1}{m!} A^n [A,B](-A)^m y^{n+m} dy \qquad n+m \geq 0$$

$$g(x) = B + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{1}{n!}\frac{1}{m!} A^n [A,B](-A)^m \frac{1}{n+m+1} x^{n+m+1}$$

use this to find the first few terms in the expansion of:

$$e^A B e^{-A} = g(1)$$

$$e^A B e^{-A} = B + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{1}{n!}\frac{1}{m!} A^n [A,B](-A)^m \frac{1}{n+m+1}$$

$$= B + \underbrace{[A,B]}_{n=m=0} + \underbrace{A[A,B]\tfrac{1}{2}}_{n=1\ m=0} - \underbrace{[A,B]A\tfrac{1}{2}}_{n=0,\ m=1} + \cdots$$

$$\boxed{e^A B e^{-A} = B + [A,B] + \frac{1}{2}[A,[A,B]] + \underbrace{\cdots}_{\text{all remaining terms } \propto [A,B]}}$$

b. if $[A,B]=c$, the LHS is:

$$e^A B e^{-A} = e^A\left(e^{-A}B + Be^{-A} - e^{-A}B\right)$$
$$= e^A e^{-A} B + e^A [B, e^{-A}]$$
$$= B + e^A [B, e^{-A}]$$

find $[B, e^{-A}]$ using:

if $[X,Y]=\alpha$, $[X, f(Y)] = \alpha \frac{df}{dY}$

$X=B \quad Y=A \quad \alpha=-c$

$$[B, e^{-A}] = -c\frac{d}{dA}(e^{-A}) = +ce^{-A}$$

$$e^A B e^{-A} = B + e^A(ce^{-A}) = B + c \checkmark$$

from the Hadamard lemma,

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2}[A,[A,B]] + \cdots$$
$$= B + c + \underbrace{\frac{1}{2}[A,c]}_{0} + \cdots \quad \text{c is a c number}$$

$$e^A B e^{-A} = B + c \checkmark$$

in agreement!

or, from (✱),

$$\frac{dg}{dx} = e^{xA}[A,B]e^{-xA} = c \cdot e^{xA}e^{-xA} = c$$

$$g(x) = cx + g(0) = B + cx$$

$$e^A B e^{-A} = g(1) = B + c \checkmark$$

c. $e^{A}Be^{-A} = B + [A,B] + \frac{1}{2}[A,[A,B]] + \cdots$

consider $Q \equiv e^{ipa/\hbar} f(x) e^{-ipa/\hbar}$

$B = f(x) \quad A = ipa/\hbar$

$$Q = f(x) + \left[\frac{ipa}{\hbar}, f(x)\right] + \frac{1}{2}\left[\frac{ipa}{\hbar}, \left[\frac{ipa}{\hbar}, f(x)\right]\right] + \cdots$$

$$Q = f(x) + \left(\frac{ia}{\hbar}\right)[p, f(x)] + \frac{1}{2}\left(\frac{ia}{\hbar}\right)^2 [p,[p,f(x)]] + \cdots$$

evaluate : $[p, f(x)]$ from $[p,x] = -i\hbar$

and $[X, f(Y)] = c\frac{df}{dY}$ for $[X,Y] = C$

$X = p \quad x = Y \quad c = -i\hbar$

$$[p, f(x)] = -i\hbar \frac{df}{dx}$$

$$Q = f(x) + \frac{ia}{\hbar}(-i\hbar)\frac{df}{dx} + \frac{1}{2}\left(\frac{ia}{\hbar}\right)^2(-i\hbar)\left[p, \frac{df}{dx}\right] + \cdots$$

$$Q = f(x) + a \cdot f'(x) + \frac{1}{2} \cdot \left(\frac{ia}{\hbar}\right) a \cdot \underbrace{[p, f'(x)]}_{-i\hbar f''(x)} + \cdots$$

$$Q = f(x) + a \cdot f'(x) + \frac{1}{2}a^2 f''(x) + \cdots$$

recognize Taylor series for $f(x+a)$

$$e^{ipa/\hbar} f(x) e^{-ipa/\hbar} = f(x) + a \cdot f'(x) + \frac{1}{2}a^2 f''(x) + \cdots$$

$$\boxed{e^{ipa/\hbar} f(x) e^{-ipa/\hbar} = f(x+a)}$$