# Physics 414-2 Problem Set 8 (Last Problem Set)

May 20, 2022

**Due: Friday, June 3 at 4 pm**

**1. The Hall Effect and Helicon Waves in a Conductor.** We consider a similar setup as in problem 6 of homework 7, but now we look at a material that is a conductor rather than a dielectric. Mathematically, this amounts to setting $\omega_0 = 0$, since the electrons are no longer bound.

For a current density $\vec{j}$ of electrons flowing in a conductor with an applied magnetic field $\vec{B}$, recall from undergraduate electrodynamics that a transverse electric field

$$\vec{E}_t = \frac{\vec{j} \times \vec{B}}{n_0 e c} \tag{1}$$

is generated via the Hall effect. This transverse electric field counteracts the transverse force on the electrons from the Lorentz force. **If the Hall effect was not discussed in your undergraduate electrodynamics course, please let me know, and I will be happy to discuss it with you.**

(a) For an electric field with time dependence $e^{-i\omega t}$, show that the resistivity tensor $\rho_{ik}$, defined in terms of the relation $E_i = \rho_{ik} j_k$ (summation over k is assumed) between the electric field $\vec{E}$ and the current density $\vec{j}$, has the following nonzero components in the case of an applied uniform magnetic field $\vec{B} = B_0 \hat{z}$:

$$\rho_{xx} = \rho_{yy} = \rho_{zz} = \frac{1 - i\omega\tau}{\sigma_0} \tag{2}$$

$$\rho_{xy} = -\rho_{yx} = \frac{B_0}{n_0 e c}, \tag{3}$$

where $\sigma_0 = (n_0 e^2/m)\tau$ is the conductivity at zero frequency and $\tau$ is the damping time.

(b) Using your result from part (a), find the conductivity tensor $\sigma_{ik}$ so that $j_i = \sigma_{ik} E_k$. In the high magnetic field, low frequency limit–which we mathematically represent as $\omega\tau \ll 1 \ll \omega_c\tau$, with $\omega_c$ as defined in homework 7–simplify your expressions for the elements of $\sigma_{ik}$ in terms of $m$, $n_0$, $e$, $c$, $\tau$, and $B_0$. As $B_0 \to \infty$, what happens to $\sigma_{ik}$?

(c) For the high magnetic field, low frequency limit, and assuming also that $\tau$ is very large as in homework 7 problem 6, show the circularly polarized waves can propagate in the $\hat{z}$ direction with the dispersion relation

$$\omega = \pm \frac{c^2 k^2 \omega_c}{4\pi n_0 e^2/m} \tag{4}$$

and without attenuation, for sufficiently low frequency $\omega$. These are called helicon waves. Would these waves be able to propagate without attenuation in the absence of the magnetic field? Why or why not?

**2. Group and phase velocity in the Drude-Lorentz Model.** In class, we considered the complex dielectric constant $\epsilon(\omega) = \epsilon_1(\omega) + i\epsilon_2(\omega)$, where $\epsilon_1(\omega)$ is the real part and $\epsilon_2(\omega)$ is the imaginary part. This leads to a complex index of refraction $n(w) = \sqrt{\epsilon(\omega)}$.

(a) In class, we showed that the phase velocity as a function of frequency is given by

$$v_p \equiv \frac{\omega}{\mathrm{Re}(k)} = \frac{c}{\mathrm{Re}(n(\omega))} \tag{5}$$

Show that the group velocity is given by

$$v_g = \frac{c}{\mathrm{Re}\left[\frac{d}{d\omega}\left(\omega n(\omega)\right)\right]} \tag{6}$$

(b) Using Mathematica, make plots of $\epsilon_1$, $\epsilon_2$, $v_p$, and $v_g$ as a function of frequency for the parameters $\omega_0 = 2 \times 10^{16}$ rad/s, $\Omega_p = 5 \times 10^{15}$ rad/s, and $\tau = 15 \times 10^{-15}$ s. The frequency $\Omega_p$, called the plasma frequency, is defined so that $\Omega_p^2 = (4\pi n_0 e^2/m)$.

Focus on the features of the plotted quantities for $\omega$ near the resonance frequency $\omega_0$. You should see that the group velocity becomes negative or exceeds the speed of light at certain points. This, however, does not violate causality, as the notion of a group velocity is only well-defined when the index of refraction varies slowly with $\omega$ and when absorption is small–conditions which are violated near these points. See chapter 20, section 134 in your textbook for further discussion of why causality is not violated.

**3. Slow light in an atomic gas.** In many dilute atomic gases, the low-lying atomic levels can be split by applied laser fields. As an approximate classical model for this behavior, we assume that the resonance at $\omega_0$ is symmetrically split so that the new resonances are at $\omega_0 \pm \delta$. We will then model the dielectric constant as

$$\bar{\epsilon}(\omega) = \frac{1}{2}\left[\epsilon(\omega, \omega_0 + \delta) + \epsilon(\omega, \omega_0 - \delta)\right], \tag{7}$$

where $\epsilon(\omega, \omega_0+\delta)$ is the complex dielectric constant for the Drude-Lorentz model with resonance frequency shifted to $\omega_0+\delta$, and $\epsilon(\omega, \omega_0-\delta)$ is the complex index of refraction for the Drude-Lorentz model with resonance frequency shifted to $\omega_0 - \delta$. Light with frequency $\omega$ near the center $\omega_0$ of this split resonance is dramatically slowed down. A rigorous treatment of this effect requires a full quantum mechanical calculation, but our classical model can qualitatively capture the feature that the light is significantly slowed. An interesting note is that besides being fundamentally interesting, slow light can find applications in the creation of a quantum memory for quantum networks and quantum computation.

For the split resonance, plot the real and imaginary parts of the dielectric constant and index of refraction versus $\omega$. Also, plot the phase velocity and the group velocity versus $\omega$. Focus on the behavior near the split resonance. Use the same parameters as in problem 3 and set $\delta = 0.09 \times 10^{15}$ rad/s. You should see that near $\omega_0$, the group velocity is slowed to about $0.003c$. You should also see that the absorption (which corresponds to the imaginary part of the index of refraction) near $\omega_0$ is reduced in the case of the split resonance as compared to the case of the unsplit resonance considered in problem 2.

**4. Properties of $\epsilon(\omega)$ in the Drude-Lorentz model and the energy-time uncertainty principle.** In class, we derived expressions for the real ($\epsilon_1(\omega)$) and imaginary ($\epsilon_2(\omega)$) parts of $\epsilon(\omega)$ for the Drude-Lorentz model in terms of the following parameters: the resonance frequency $\omega_0$, the damping time $\tau$, the electron number density $n$, the electron mass $m$, and the electron charge $e$.

(a) Sketch $\epsilon_1(\omega)$ and $\epsilon_2(\omega)$ near the resonance $\omega = \omega_0$, and qualitatively describe the features of your sketches. Assume that $\tau^{-1} \ll \omega_0$, which is the typical situation. How do the width of the features near resonance depend on the parameters listed above?

(b) $\epsilon_2(\omega)$, which corresponds to the existence of absorption, peaks within a region of a certain characteristic width surrounding the resonance at $\omega_0$. Using the energy-time uncertainty principle (and assuming that the product of the energy and time uncertainties is near the lower bound allowed by the energy-time uncertainty principle), make a qualitative argument for why the width of the peak in $\epsilon_2(\omega)$ has the value you found in part (a). (Hint: Consider an atom with a transition frequency frequency $\omega_0$ that is excited by absorbing a photon, and the subsequent spontaneous decay of the atom back into the ground state after a characteristic damping time $\tau$).

**5. Generalization of the Drude-Lorentz model for many resonances: sum rule for resonance strengths.** So far, we've considered the Drude-Lorentz model for materials that have a single resonance at frequency $\omega_0$. Typically, real materials will have more than one resonance. Let us generalize the Drude-Lorentz model to include multiples resonances at frequencies $\omega_i$, with the strengths of the respective resonances characterized by parameters $f_i$ and their respective damping times given by $\tau_i$:

$$\epsilon(\omega) = 1 + \frac{4\pi n e^2}{m} \sum_i \frac{f_i}{\omega_i^2 - \omega^2 - i\omega/\tau_i} \tag{8}$$

Use the asymptotic behavior of $\epsilon(\omega)$ in the limit of $\omega \gg \omega_i, \tau_i^{-1}$ to prove that $\sum_i f_i = 1$.

**6. The f-sum rule.** In this problem, we will use the Kramer's-Kronig relations to derive a particular integral relation for $\epsilon_2(\omega)$ called the f-sum rule.

(a) Show that the Kramers-Kronig relation for $\epsilon_1(\omega)$ can be rewritten as

$$\epsilon_1(\omega) - 1 = \frac{2}{\pi} P \int_0^\infty d\omega' \frac{\omega' \epsilon_2(\omega')}{(\omega')^2 - \omega^2} \tag{9}$$

(b) At very high frequencies $\omega$, the form for $\epsilon(\omega)$ given by the Drude-Lorentz model is valid for most materials. Use the asymptotic form of $\epsilon(\omega)$ for $\omega \to \infty$, as well as your result from part (a), to prove the f-sum rule, which states that

$$\int_0^\infty d\omega' \omega' \epsilon_2(\omega') = \frac{2\pi^2 e^2 n}{m}. \tag{10}$$

(c) Using the fact that

$$\int_0^\infty d\omega' \frac{(\omega')^2/\tau_i}{((\omega')^2 - \omega_i^2)^2 + (\omega')^2/\tau_i^2} = \frac{\pi}{2} \tag{11}$$

for general $\tau_i$ and $\omega_i$ (note: Mathematica seems to have trouble evaluating this integral in the general case, but it evaluates the integral correctly for specific values of $\omega_i$ and $\tau_i$ plugged in), check that the f-sum rule is consistent with your result from problem 5–namely, that $\sum_i f_i = 1$.

**7. Free electron laser.** This problem describes some of the basic features of a free electron x-ray laser, although the usual arrangement involves transverse acceleration, in contrast to the longitudinal acceleration considered here. A particle with charge $q$ moves along the $z$ axis with velocity (in units of $c$)

$$\bar{\beta} = \begin{cases} \bar{\beta}_0 & |\bar{t}| > \frac{N\pi}{\omega_0} \\ \bar{\beta}_0 + \frac{a\omega_0}{c}\sin(\omega_0 \bar{t}) & |\bar{t}| < \frac{N\pi}{\omega_0}. \end{cases} \tag{12}$$

As in class, the bar indicates retarded quantities. We assume that $\frac{a\omega_0}{c} \ll \bar{\beta}_0$ and that $N$ is a positive integer.

(a) With the assumptions stated above, find an expression for $\frac{d^2E}{d\omega d\Omega}$, the double differential energy distribution (with respect to frequency and angle) of the emitted radiation. Simplify the form of your expression by making use of the function $f_N$, defined so that

$$f_N(x) = \frac{2}{N\pi}\left[\frac{x\sin(N\pi x)}{x^2-1}\right]^2. \tag{13}$$

(b) Sketch or plot $f_N(x)$ for $N = 1$ and for $N \gg 1$. In both cases, discuss the qualitative features of this function.

(c) Derive a simplified expression for $\frac{d^2E}{d\omega d\Omega}$ in the case that $\bar{\beta}_0 \ll 1$. Qualitatively describe the frequency distribution as a function of angle.

(d) Repeat part (c), but this time in the ultrarelativistic limit that $\bar{\beta}_0 \to 1$, which corresponds to $\bar{\gamma} \gg 1$.