$\boxed{\text{Dirac Equation}}$ ⇝ improvement on the Schrodinger equation

$$H_0|\psi\rangle = i\hbar\frac{\partial}{\partial t}|\psi\rangle$$

$$\hookrightarrow H_0 = c\vec{\alpha}\cdot(\vec{p}-q\vec{A}) + \overbrace{q\Phi}^{V} + \beta mc^2$$

$$\left.\begin{array}{l}
\alpha_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix} \\[2ex]
\alpha_y = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix} \\[2ex]
\alpha_z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix}
\end{array}\right\} \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$$

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

<u>With $\vec{A}=0$ and $V=V(r)$:</u> $H_0 = c\vec{\alpha}\cdot\vec{p} + q\Phi + \beta mc^2$

can show
$$\left\{\begin{array}{ll} \vec{L}=\vec{r}\times\vec{p}: & [\vec{L}, H_0] = i\hbar c\vec{\alpha}\times\vec{p} \\ \vec{S}=\frac{1}{2}\hbar\vec{\sigma}: & [\vec{S}, H_0] = -i\hbar c\vec{\alpha}\times\vec{p} \end{array}\right\} \therefore \begin{array}{l} l\ m_l \\ s\ m_s \end{array}$$
not good q.n.

But: $[\vec{L}+\vec{S}, H_0] = 0 \rightsquigarrow \vec{J}=\vec{L}+\vec{S} \rightarrow j, m_j$ are good q.n.

Stationary Solutions

$\vec{A}, \phi$ independent of time $\Longrightarrow \Psi(\vec{r},t) = \chi(\vec{r})\, e^{-iE_{tot}t/\hbar}$

$$i\hbar\frac{\partial}{\partial t}\Psi = -\frac{iE}{\cancel{\hbar}}(i\cancel{\hbar})\Psi = E_{tot}\Psi$$

$$\therefore H_D\,\chi = E_{tot}\,\chi$$

Four-component spinor $\Longrightarrow$ two, two-component spinors

$$\chi(\vec{r}) = \begin{pmatrix} \psi(\vec{r}) \\ \eta(\vec{r}) \end{pmatrix}$$

$$H_D\,\chi = E\,\chi$$

$$c\begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}\cdot\left(\vec{p} - q\vec{A}\right)\begin{pmatrix} \psi \\ \eta \end{pmatrix} + q\phi\begin{pmatrix} \psi \\ \eta \end{pmatrix} + mc^2\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} \psi \\ \eta \end{pmatrix} = E_{tot}\begin{pmatrix} \psi \\ \eta \end{pmatrix}$$

$$c\begin{pmatrix} 0 & \vec{\sigma}\cdot(\vec{p}-q\vec{A}) \\ \vec{\sigma}\cdot(\vec{p}-q\vec{A}) & 0 \end{pmatrix}\begin{pmatrix} \psi \\ \eta \end{pmatrix} + q\phi\begin{pmatrix} \psi \\ \eta \end{pmatrix} + mc^2\begin{pmatrix} \psi \\ -\eta \end{pmatrix} = E_{tot}\begin{pmatrix} \psi \\ \eta \end{pmatrix}$$

$$\begin{pmatrix} c\vec{\sigma}\cdot(\vec{p}-q\vec{A})\eta \\ c\vec{\sigma}\cdot(\vec{p}-q\vec{A})\psi \end{pmatrix} + q\phi\begin{pmatrix} \psi \\ \eta \end{pmatrix} + mc^2\begin{pmatrix} \psi \\ -\eta \end{pmatrix} = E_{tot}\begin{pmatrix} \psi \\ \eta \end{pmatrix}$$

$$\boxed{\begin{aligned} &(1)\quad c(\vec{p}-q\vec{A})\cdot\vec{\sigma}\,\eta + q\phi\psi + mc^2\psi = E_{tot}\psi \\ &(2)\quad c(\vec{p}-q\vec{A})\cdot\vec{\sigma}\,\psi + q\phi\eta - mc^2\eta = E_{tot}\eta \end{aligned}}$$

coupled

Nonrelativistic Limit

Let $E_{tot} = E + mc^2$ $\qquad E << mc^2$

① $c(\vec{p}-q\vec{A})\cdot\vec{\sigma}\,\eta(\vec{r}) + q\phi\psi(\vec{r}) + \cancel{mc^2}\psi = (E+\cancel{mc^2})\psi$

$$c(\vec{p}-q\vec{A})\cdot\vec{\sigma}\,\eta + q\phi\psi = E\psi$$

② $c(\vec{p}-q\vec{A})\cdot\vec{\sigma}\,\psi + q\phi\eta - mc^2\eta = (E+mc^2)\eta$

$$c(\vec{p}-q\vec{A})\cdot\vec{\sigma}\,\psi + q\phi\eta = (E+2mc^2)\eta$$

$$\eta = \frac{c(\vec{p}-q\vec{A})\cdot\vec{\sigma}}{E+\underbrace{2mc^2}_{\text{bigger}}-\underbrace{q\phi}_{\text{smaller}}}\psi \longrightarrow \text{use again later}$$

solve approximately in non-rel. limit $E-q\phi << 2mc^2$

$$\underbrace{\eta}_{\text{"small" component}} \approx \frac{c(\vec{p}-q\vec{A})\cdot\vec{\sigma}}{2mc^2}\underbrace{\psi}_{\text{"large" component}}$$

Substitute back into ① $\longrightarrow$ to uncouple

$$\cancel{c}(\vec{p}-q\vec{A})\cdot\vec{\sigma}\left[\frac{\cancel{c}(\vec{p}-q\vec{A})\cdot\vec{\sigma}}{2m\cancel{c^2}}\psi\right] + q\phi\psi = E\psi$$

$$\frac{\left[(\vec{p}-q\vec{A})\cdot\vec{\sigma}\right]\left[(\vec{p}-q\vec{A})\cdot\sigma\right]}{2m}\psi + q\phi\psi = E\psi$$

Use identity: $(\vec{\sigma}\cdot\vec{V})(\vec{\sigma}\cdot\vec{W})\psi = \vec{V}\cdot\vec{W}\psi + i\vec{\sigma}\cdot(\vec{V}\times\vec{W})\psi$

vector operators that commute with $\vec{\sigma}$

Let $\vec{V} = \vec{W} = \vec{p} - q\vec{A}$

$$\vec{V}\cdot\vec{W} = (\vec{p}-q\vec{A})^2$$

$$\vec{V}\times\vec{W} = (\vec{p}-q\vec{A})\times(\vec{p}-q\vec{A})$$ ← $\vec{p}$ and $\vec{A}$ do not commute

$\vec{p} = -i\hbar\vec{\nabla}$

$$= \cancel{\vec{p}\times\vec{p}} - q\vec{p}\times\vec{A} - q\vec{A}\times p + q^2\cancel{\vec{A}\times\vec{A}}$$

$$= i\hbar q\,\underbrace{(\vec{\nabla}\times\vec{A} + \vec{A}\times\vec{\nabla})}_{\vec{\nabla}\times\vec{A}\psi + \vec{A}\times\vec{\nabla}\psi}$$

z component:

$$\left(\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right)\psi + \left(A_x\frac{\partial}{\partial y} - A_y\frac{\partial}{\partial x}\right)\psi$$

$$\Big[\underbrace{\frac{\partial A_y}{\partial x} + \cancel{A_y\frac{\partial}{\partial x}} - \frac{\partial A_x}{\partial y}}_{(\vec{\nabla}\times\vec{A})_z\psi} - \cancel{A_x\frac{\partial}{\partial y}} + \cancel{A_x\frac{\partial}{\partial y}} - \cancel{A_y\frac{\partial}{\partial x}}\Big]\psi$$

$B_z\psi \rightsquigarrow$ similarly for $\hat{y}, \hat{z}$

$$= i\hbar q\vec{B}$$

In identity: $[\vec{\sigma}\cdot(\vec{p}-q\vec{A})][\vec{\sigma}\cdot(\vec{p}-q\vec{A})] = (\vec{p}-q\vec{A})^2 + i\vec{\sigma}\cdot(i\hbar q\vec{B})$

$$[(\vec{p}-q\vec{A})\cdot\vec{\sigma}][(\vec{p}-q\vec{A})\cdot\vec{\sigma}] = (\vec{p}-q\vec{A})^2 - q\hbar\vec{\sigma}\cdot\vec{B}$$

Back in Dirac eq. in non-relativistic limit

$$\frac{[(\vec{p}-q\vec{A})\cdot\vec{\sigma}][(\vec{p}-q\vec{A})\cdot\vec{\sigma}]}{2m}\psi + q\Phi\psi = E\psi$$

$$\left[\frac{(\vec{p}-q\vec{A})^2}{2m} - \frac{q\hbar}{2m}\underbrace{\vec{\sigma}}_{\vec{S}/(\frac{1}{2}\hbar)}\cdot\vec{B} + q\Phi\right]\psi = E\psi$$

$$\left[\frac{(\vec{p}-q\vec{A})^2}{2m} - \underbrace{\frac{q\hbar}{2m}\frac{\vec{S}}{\frac{1}{2}\hbar}}\cdot\vec{B} + q\Phi\right]\psi = E\psi$$

$$\vec{\mu} = \underbrace{\frac{q\hbar}{2m}}\frac{\vec{S}}{\frac{1}{2}\hbar} = \frac{q}{m}\vec{S}$$

Spin magnetic moment comes out naturally

For electron and positron

$$|\vec{\mu}| = \underbrace{\frac{e\hbar}{2m}}_{\mu_B}\frac{|\vec{S}|}{\frac{1}{2}\hbar}$$

$\mu_B \Leftarrow$ Bohr magneton

In Coulomb gauge: $\vec{A} = \frac{1}{2}\vec{B}\times\vec{r}$ for uniform magnetic field

$$H = \frac{\vec{p}^{\,2}}{2m} - \frac{q}{2m}\vec{L}\cdot\vec{B} - \frac{q}{m}\vec{S}\cdot\vec{B} + \underbrace{\frac{q^2B^2(x^2+y^2)}{8m}}_{\text{for } \vec{B}=B\hat{z}}$$

$$\boxed{H = \frac{\vec{p}^{\,2}}{2m} - \frac{q}{2m}(\vec{L}+2\vec{S})\cdot\vec{B} + q\Phi + \frac{q^2B^2}{8m}(x^2+y^2)}$$

Schrodinger Hamiltonian from Dirac

$$H_0 = \frac{\vec{p}^{\,2}}{2m} + q\Phi$$

$$V = -\frac{q}{2m}(\vec{L}+2\vec{S})\cdot\vec{B} + \frac{q^2B^2(x^2+y^2)}{8m} \quad \text{for } \vec{B}=B\hat{z}$$

Fine Structure

Higher Order Corrections

$\vec{A}=0$

Nuclear potential energy

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{-Ze^2}{r}$$

$q\Phi = V$

$$② \rightarrow \eta = \frac{1}{\underbrace{E+2mc^2-V}}\, c(\vec{p}\cdot\vec{\sigma})\psi$$

$$\frac{1}{2mc^2}\frac{1}{1+\frac{E-V}{2mc^2}} \simeq \frac{1}{2mc^2}\left[1-\frac{E-V}{2mc^2}+\cancel{\cdots}\right]$$

$$① \rightarrow (E-V)\psi = c\vec{p}\cdot\vec{\sigma}\,\eta$$

$$= \cancel{c}\,\vec{p}\cdot\vec{\sigma}\,\frac{1}{2m\cancel{c^2}}\left[1-\frac{E-V}{2mc^2}\right]\cancel{c}(\vec{p}\cdot\vec{\sigma})\psi$$

does not commute with V

$$= \frac{1}{2m}\left[1-\frac{E-V}{2mc^2}\right](\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{p})\psi + \frac{1}{4m^2c^2}(-i\hbar[\vec{\nabla}V]\cdot\vec{\sigma})(\vec{p}\cdot\vec{\sigma})\psi$$

$\frac{p^2}{2m}\phi + V\phi = E\phi$ to lowest order $\therefore E-V \rightarrow \frac{p^2}{2m}$

$$= \frac{1}{2m}\left[1-\frac{p^2}{4m^2c^2}\right]\underbrace{(\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{p})}_{\boxed{A}}\psi \underbrace{- \frac{i\hbar}{4m^2c^2}\left[\left(\frac{\hat{r}\,\partial V}{\partial r}\right)\cdot\vec{\sigma}\right](\vec{p}\cdot\vec{\sigma})\psi}_{\text{Not Hermitian}}$$

Darwin related this problem to normalization difficulties. The fact that $1 = \int d^3r\,[|\psi|^2 + |\eta|^2]$ does not mean that $1 = \int d^3r\,|\psi|^2$.

This normalization is established by making the term Hermitian

$$\frac{1}{2}(\;)+\frac{1}{2}(\;)^{+} = \frac{-i\hbar}{8m^2c^2}\left[\underbrace{\left(\frac{\hat{r}\,\partial V}{\partial r}\cdot\vec{\sigma}\right)(\vec{p}\cdot\vec{\sigma})}_{\boxed{B}} - \underbrace{(\vec{p}\cdot\vec{\sigma})^{+}\left(\frac{\hat{r}\,\partial V}{\partial r}\cdot\vec{\sigma}\right)^{+}}_{\boxed{C}}\right]$$

2nd use

Apply the identity: $(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b})\psi = \vec{a}\cdot\vec{b}\,\psi + i\vec{\sigma}\cdot(\vec{a}\times\vec{b})\psi$ for $\vec{a},\vec{b}$ that commute with $\vec{\sigma}$

$$\boxed{A}\;(\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{p})\psi = \vec{p}^{\,2}\psi + i\vec{\sigma}\cdot\cancel{(\vec{p}\times\vec{p})\psi} = \vec{p}^{\,2}\psi$$

$\boxed{B}$ $\left(\hat{r}\frac{\partial V}{\partial r}\cdot\vec{\sigma}\right)(\vec{p}\cdot\vec{\sigma})\psi = \hat{r}\frac{\partial V}{\partial r}\cdot\vec{p}\,\psi + i\vec{\sigma}\cdot\left(\hat{r}\frac{\partial V}{\partial r}\times\vec{p}\right)\psi$

$\boxed{C}$ $(\vec{p}\cdot\vec{\sigma})\left(\hat{r}\frac{\partial V}{\partial r}\cdot\vec{\sigma}\right)\psi = \underbrace{\vec{p}\cdot\hat{r}\frac{\partial V}{\partial r}\psi} + i\vec{\sigma}\cdot\overbrace{\underbrace{\left(\vec{p}\times\hat{r}\frac{\partial V}{\partial r}\right)}}^{\vec\nabla\times\vec\nabla}\psi$

X $\left\{-i\hbar\vec{\nabla}\cdot\{\vec{\nabla}V\}\right\}\psi + \hat{r}\frac{\partial V}{\partial r}\cdot\vec{p}\,\psi \quad \cancel{-i\hbar\vec{\nabla}\times\hat{r}\frac{\partial V}{\partial r}} - \hat{r}\frac{\partial V}{\partial r}\times\vec{p}\,\psi$

X $= \hat{r}\frac{\partial V}{\partial r}\cdot\vec{p}\,\psi - i\hbar\{\nabla^2 V\}\psi - i\vec{\sigma}\cdot\left(\hat{r}\frac{\partial V}{\partial r}\times\vec{p}\right)\psi$

$$\boxed{B}-\boxed{C} = \left[\cancel{\hat{r}\frac{\partial V}{\partial r}\cdot\vec{p}} + i\vec{\sigma}\cdot\left(\hat{r}\frac{\partial V}{\partial r}\times\vec{p}\right)\right] - \left[\cancel{\hat{r}\frac{\partial V}{\partial r}\cdot\vec{p}} - i\hbar\frac{\partial^2 V}{\partial r^2} - i\vec{\sigma}\cdot\left(\hat{r}\frac{\partial V}{\partial r}\times\vec{p}\right)\right]$$

$$= 2i\vec{\sigma}\cdot\left(\frac{\partial V}{\partial r}\frac{\vec{r}}{r}\times\vec{p}\right) + i\hbar\nabla^2 V$$

$$(E-V)\psi = \frac{1}{2m}\left[1-\frac{\vec{p}^2}{4m^2c^2}\right]\vec{p}^2 - \frac{i\hbar}{8m^2c^2}\left[2i\vec{\sigma}\cdot\frac{\partial V}{\partial r}\frac{\vec{r}\times\vec{p}}{r} + i\hbar\frac{\partial^2 V}{\partial r^2}\right]$$

$$E\psi = \left[\frac{p^2}{2m} + V - \frac{p^4}{8m^3c^2} + \frac{\cancel{\hbar}}{4m^2c^2}\frac{2\vec{S}}{\cancel{\hbar}}\cdot\frac{1}{r}\frac{\partial V}{\partial r}\vec{L} + \frac{\hbar^2}{8m^2c^2}\{\nabla^2 V\}\right]\psi$$

$\vec{S} = \frac{1}{2}\hbar\vec{\sigma}$ $$= \left[\frac{\vec{p}^2}{2m} + V - \frac{p^4}{8m^3c^2} + \frac{1}{2m^2c^2}\frac{1}{r}\frac{\partial V}{\partial r}\vec{L}\cdot\vec{S} + \frac{\hbar^2}{8m^2c^2}(\nabla^2 V)\right]\psi$$

<u>A little E&M:</u> $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \rightarrow \rho = q\,\delta(\vec{r})$

for a charge q $\quad \int\rho\, d^3r = q\int d^3r\,\delta^3(\vec{r}) = q$

$$\vec{\nabla}\cdot\left(-\vec{\nabla}\frac{e}{4\pi\epsilon_0 r}\right) = \frac{e\,\delta(\vec{r})}{\epsilon_0}$$

mult. by Ze $\quad \vec{\nabla}\cdot\vec{\nabla}\frac{-Ze^2}{4\pi\epsilon_0 r} = \frac{Ze^2\delta(\vec{r})}{\epsilon_0}$

$$\nabla^2 V = \frac{Ze^2\delta^3(\vec{r})}{\epsilon_0} \rightarrow \frac{\hbar^2 Ze^2}{8m^2c^2\epsilon_0}\delta^3(\vec{r})$$

$$\boxed{E\psi = \left[\frac{\vec{p}^2}{2m} + V \underbrace{- \frac{p^4}{8m^3c^2}}_{H_1} + \underbrace{\frac{1}{2m^2c^2}\frac{1}{r}\frac{\partial V}{\partial r}\vec{L}\cdot\vec{S}}_{H_2} + \underbrace{\frac{\hbar^2 Ze^2}{8m^2c^2\epsilon_0}\delta^3(\vec{r})}_{H_3}\right]\psi}$$

rel. kinetic energy $\quad$ spin orbit $\quad$ Darwin

Laplacian:

$$\nabla^2 f=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 f}{\partial\phi^2}+\frac{\partial^2 f}{\partial z^2}.$$

Laplacian of vector:

$$(\nabla^2\mathbf{A})_r=\nabla^2 A_r-\frac{2}{r^2}\frac{\partial A_\phi}{\partial\phi}-\frac{A_r}{r^2},$$

$$(\nabla^2\mathbf{A})_\phi=\nabla^2 A_\phi+\frac{2}{r^2}\frac{\partial A_r}{\partial\phi}-\frac{A_\phi}{r^2},$$

$$(\nabla^2\mathbf{A})_z=\nabla^2 A_z.$$

Components of $(\mathbf{A}\cdot\nabla)\mathbf{B}$:

$$(\mathbf{A}\cdot\nabla\mathbf{B})_r=A_r\frac{\partial B_r}{\partial r}+\frac{A_\phi}{r}\frac{\partial B_r}{\partial\phi}+A_z\frac{\partial B_r}{\partial z}-\frac{A_\phi B_\phi}{r},$$

$$(\mathbf{A}\cdot\nabla\mathbf{B})_\phi=A_r\frac{\partial B_\phi}{\partial r}+\frac{A_\phi}{r}\frac{\partial B_\phi}{\partial\phi}+A_z\frac{\partial B_\phi}{\partial z}+\frac{A_\phi B_r}{r},$$

$$(\mathbf{A}\cdot\nabla\mathbf{B})_z=A_r\frac{\partial B_z}{\partial r}+\frac{A_\phi}{r}\frac{\partial B_z}{\partial\phi}+A_z\frac{\partial B_z}{\partial z}.$$

Divergence of tensor:

$$(\nabla\cdot\mathbf{T})_r=\frac{1}{r}\frac{\partial}{\partial r}(rT_{rr})+\frac{1}{r}\frac{\partial}{\partial\phi}(T_{\phi r})+\frac{\partial T_{zr}}{\partial z}-\frac{1}{r}T_{\phi\phi},$$

$$(\nabla\cdot\mathbf{T})_\phi=\frac{1}{r}\frac{\partial}{\partial r}(rT_{r\phi})+\frac{1}{r}\frac{\partial T_{\phi\phi}}{\partial\phi}+\frac{\partial T_{z\phi}}{\partial z}+\frac{1}{r}T_{\phi r},$$

$$(\nabla\cdot\mathbf{T})_z=\frac{1}{r}\frac{\partial}{\partial r}(rT_{rz})+\frac{1}{r}\frac{T_{\phi z}}{\partial\phi}+\frac{\partial T_{zz}}{\partial z}.$$

b. Spherical coordinates. Divergence:

$$\nabla\cdot\mathbf{A}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}.$$

Gradient:

$$(\nabla f)_r=\frac{\partial f}{\partial r},\quad(\nabla f)_\theta=\frac{1}{r}\frac{\partial f}{\partial\theta},\quad(\nabla f)_\phi=\frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}.$$

Curl:

$$(\nabla\times\mathbf{A})_r=\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\phi\sin\theta)-\frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi},$$

$$(\nabla\times\mathbf{A})_\theta=\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi}-\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi),$$

$$(\nabla\times\mathbf{A})_\phi=\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta)-\frac{1}{r}\frac{\partial A_r}{\partial\theta}.$$

Laplacian:

$$\nabla^2 f=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}.$$

Laplacian of vector:

$$(\nabla^2\mathbf{A})_r=\nabla^2 A_r-\frac{2A_r}{r^2}-\frac{2}{r^2}\frac{\partial A_\theta}{\partial\theta}-\frac{2}{r^2\sin\theta}\frac{\partial A_\phi}{\partial\phi},$$

$$(\nabla^2\mathbf{A})_\theta=\nabla^2 A_\theta+\frac{2}{r^2}\frac{\partial A_r}{\partial\theta}-\frac{A_\theta}{r^2\sin^2\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\phi}{\partial\phi},$$

$$(\nabla^2\mathbf{A})_\phi=\nabla^2 A_\phi-\frac{A_\phi}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\frac{\partial A_r}{\partial\phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\theta}{\partial\phi}.$$

Components of $(\mathbf{A}\cdot\nabla)\mathbf{B}$:

$$(\mathbf{A}\cdot\nabla\mathbf{B})_r=A_r\frac{\partial B_r}{\partial r}+\frac{A_\theta}{r}\frac{\partial B_r}{\partial\theta}+\frac{A_\phi}{r\sin\theta}\frac{\partial B_r}{\partial\phi}-\frac{A_\theta B_\theta+A_\phi B_\phi}{r},$$

$$(\mathbf{A}\cdot\nabla\mathbf{B})_\theta=A_r\frac{\partial B_\theta}{\partial r}+\frac{A_\theta}{r}\frac{\partial B_\theta}{\partial\theta}+\frac{A_\phi}{r\sin\theta}\frac{\partial B_\theta}{\partial\phi}+\frac{A_\theta B_r}{r}-\frac{A_\phi B_\phi\cot\theta}{r},$$

$$(\mathbf{A}\cdot\nabla\mathbf{B})_\phi=A_r\frac{\partial B_\phi}{\partial r}+\frac{A_\theta}{r}\frac{\partial B_\phi}{\partial\theta}+\frac{A_\phi}{r\sin\theta}\frac{\partial B_\phi}{\partial\phi}+\frac{A_\phi B_r}{r}+\frac{A_\phi B_\theta\cot\theta}{r}.$$

Divergence of tensor:

$$(\nabla\cdot\mathbf{T})_r=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2T_{rr})+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(T_{\theta r}\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial T_{\phi r}}{\partial\phi}-\frac{T_{\theta\theta}+T_{\phi\phi}}{r},$$

$$(\nabla\cdot\mathbf{T})_\theta=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2T_{r\theta})+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(T_{\theta\theta}\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial T_{\phi\theta}}{\partial\phi}+\frac{T_{\theta r}}{r}-\frac{\cot\theta}{r}T_{\phi\phi},$$

$$(\nabla\cdot\mathbf{T})_\phi=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2T_{r\phi})+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(T_{\theta\phi}\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi}}{\partial\phi}+\frac{T_{\phi r}}{r}+\frac{\cot\theta}{r}T_{\phi\theta}.$$

L. Fourier series[4]

A function $f(x)$ in the interval $-a/2\leqslant x\leqslant a/2$ may be expanded in the series

$$f(x)=\tfrac{1}{2}A_0+\sum_{m=1}^{\infty}\left[A_m\cos\left(\frac{2\pi mx}{a}\right)+B_m\sin\left(\frac{2\pi mx}{a}\right)\right],$$