L1.1 (Teacher) Let ε be a small positive real number and consider $A = \begin{pmatrix} \varepsilon & 1 \\ 1 & 1 \end{pmatrix}$ $b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

1) Compute the condition number with respect to the ∞-norm of A and of the matrices in its LU Factorization.

Change notation: $\varepsilon \longrightarrow \eta$ now ε is machine epsilon. $\{ \varepsilon > \eta > 0 \rightarrow \begin{cases} 1 \oplus \eta = 1 \\ 1 \oplus \varepsilon > 1 \end{cases}$

$$A^{-1} = \begin{pmatrix} \frac{1}{\eta - 1} & -\frac{1}{\eta-1} \\ -\frac{1}{\eta-1} & \frac{\eta}{\eta-1} \end{pmatrix} \approx \begin{pmatrix} -1 & +1 \\ +1 & 0 \end{pmatrix} \Rightarrow \left.\begin{array}{l} \|A^{-1}\|_\infty = 2 \\ \|A\|_\infty = 2 \end{array}\right\} K(A) = 4 \quad \text{well conditioned}$$

= LU factorization = (pag 2)

We need to use pivoting, otherwise it would be unstable. $\underbrace{\left|\tfrac{1}{\eta}\right|}_{\text{my guide}} + |1| \approx \eta^{-1}$

$$A = LU \text{ with } \begin{cases} L = \begin{pmatrix} 1 & 0 \\ 1/\eta & 1 \end{pmatrix} \Rightarrow L^{-1} = \begin{pmatrix} 1 & 0 \\ -1/\eta & 1 \end{pmatrix} \Rightarrow K(L) = \|L\|_\infty \|L^{-1}\|_\infty \approx \eta^{-1} \cdot \eta^{-1} = \eta^{-2} \\ U = \begin{pmatrix} \eta & 1 \\ 0 & 1-\eta^{-1} \end{pmatrix} \Rightarrow U^{-1} = \begin{pmatrix} \eta^{-1} & \frac{1}{1-\eta} \\ 0 & \frac{1}{1-\eta^{-1}} \end{pmatrix} \Rightarrow K(U) = \|U\|_\infty \|U^{-1}\|_\infty \approx \eta^{-2} \end{cases} \text{ ill cond.}$$

$K(U)K(L) >> K(A)$ (problematic)

2) Suppose that η $\overbrace{\text{is smaller than machine precision}}^{\text{rounding}}$, so that $1 \oplus \eta = 1$. Show that Gaussian elimination without pivoting is numerically unstable algorithm for solving the linear equation $Ax=b$. (As seen in pg 7)

GEWP gives us $\tilde{L} = \begin{pmatrix} 1 & 0 \\ 1/\eta & 1 \end{pmatrix}$

$\tilde{U} = \begin{pmatrix} \eta & 1 \\ 0 & \underbrace{-\eta^{-1}}_{\text{rounded}^{*}} \end{pmatrix}$

Solve $\tilde{L}y = b$: $\tilde{y} = \begin{pmatrix} 1 \\ -\eta^{-1} \end{pmatrix}$; comes from rounding $fl(2-\eta^{-1})$

$\tilde{U}x = \tilde{y}$: $\tilde{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 1 \end{pmatrix} = x$ $\Rightarrow$ not numerically stable

* Assume that $\eta = \beta^e$ (power of the base)

$fl(1-\eta^{-1}) = fl(\eta^{-1}(\eta - 1)) = -\eta^{-1}$

3) Show that solving $Ax=b$ using Gaussian elimination with partial pivoting is numerically stable.

$P^{-1}A$ should be equal to $\begin{pmatrix} 1 & 1 \\ \varepsilon & 1 \end{pmatrix}$ (according to page 1)

Then $\begin{cases} L = \begin{pmatrix} 1 & 0 \\ \eta & 1 \end{pmatrix} \xrightarrow{\text{round}} \tilde{L} = \begin{pmatrix} 1 & 0 \\ \eta & 1 \end{pmatrix} \\ U = \begin{pmatrix} 1 & 1 \\ 0 & 1-\eta \end{pmatrix} \xrightarrow{\text{round}} \tilde{U} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{cases}$ $\tilde{A} = PLU = \begin{pmatrix} \eta & \eta+1 \\ 1 & 1 \end{pmatrix} \approx \begin{pmatrix} \eta & 1 \\ 1 & 1 \end{pmatrix}$

Following pg 7:

GEPP stable $\Leftrightarrow$ $\dfrac{\|\tilde{A}-A\|_\infty}{\|A\|_\infty} \leq C\varepsilon$ with small C. ■ because $\tilde{A} = A$.

Analysis can be done through solutions too: $\dfrac{\|\tilde{x}-x\|}{\|x\|} \leq C K(A)\varepsilon$ with small C $\Leftrightarrow$ GEPP stable.

L1.2 Given a symmetric and positive definite matrix $A \in \mathbb{R}^{n\times n}$, Cholesky's algorithm computes a factorization

$$A = L \cdot L^T$$

where L is a lower triangular matrix with positive diagonal entries.

1) Write down Cholesky's algorithm in pseudocode notation and describe how it works.

(Copy page 13)

2) Using this pseudocode, derive a bound for the complexity of this algorithm in terms of flops. (student)

$$\sum_{j=1}^{n}\left[\sum_{k=1}^{j-1} 1 + \sum_{k=1}^{j-1} 1 + 1 + \sum_{i=j+1}^{n}\left(1+\sum_{k=1}^{j-1} 2\right)\right] = \sum_{j=1}^{n}\left[2j-2+1+\sum_{i=j+1}^{n}(1+2j-2)\right] =$$

$$\sum_{j=1}^{n}\left[2j-1+\sum_{i=j+1}^{n}(2j-1)\right] = 2\underbrace{\sum_{j=1}^{n} j}_{} - \sum_{j=1}^{n} 1 + \sum_{j=1}^{n}\sum_{i=j+1}^{n}(2j-1) = \cancel{2}\cdot\frac{n^2}{\cancel{2}} + O(n) - n + \sum_{j=1}^{n}\left[(n-j)+2\sum_{i=j+1}^{n} j\right]$$

$$\sum_{i=1}^{n} i^k = \frac{n^{k+1}}{k+1} + O(n^k)$$

$$= n^2 - n + O(n) + \sum_{j=1}^{n} n - \sum_{j=1}^{n} j + 2\sum_{j=1}^{n} j(n-j) = n^2 - n + O(n) + n^2 - \frac{n^2}{2} + \cancel{O(n)} + 2\left(\sum_{j=1}^{n} nj - \sum_{j=1}^{n} j^2\right)$$

$$= 2n^2 - \frac{n^2}{2} - n + O(n) + \cancel{2}n\frac{n^2}{\cancel{2}} + \cancel{O(n)} - 2\frac{n^3}{3} + O(n^2) = \frac{n^3}{3} + \underbrace{\frac{3}{2}n^2 - n + O(n^2)}_{\text{redundant}} \text{ Flops}$$

3) Describe how you proceed to solve $Ax=b$, once computed $A = L\cdot L^T$ (student)

$$Ax=b \Rightarrow L\cdot L^T x = b \quad \begin{cases} \overset{\text{upper triang}}{L^T} x = y & \text{(Forward)} \\ \underset{\text{lower triang.}}{L} y = b & \text{(backward)} \end{cases}$$

4) Compute the Cholesky's factorization of $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 10 & 3 \\ 0 & 3 & 5 \end{pmatrix} = \begin{pmatrix} g_{11} & & \\ g_{21} & g_{22} & \\ g_{31} & g_{32} & g_{33} \end{pmatrix}\begin{pmatrix} g_{11} & g_{21} & g_{31} \\ & g_{22} & g_{32} \\ & & g_{33} \end{pmatrix}$

Por sistema de ecuaciones: $g_{11}^2 = a_{11} \Rightarrow g_{11} = \pm 1$ → must be positive

$g_{21} = -1$

$g_{31} = 0$

$10 = (-1)^2 + g_{22}^2 \Rightarrow g_{22} = 3$

$g_{32} = 0$

$g_{33} = 2$

L1.3 Let $A \in \mathbb{R}^{m\times n}$ be a matrix of rank n and $b \in \mathbb{R}^m$.

1) Explain how to use the QR factorization of A to solve the least square problem that asks to find the vector $x_{min} \in \mathbb{R}^n$ minimizing the quantity $\|Ax-b\|_2$ for $x \in \mathbb{R}^n$, and give the expression for the residual error $\|Ax_{min}-b\|_2$ (Student)

Describe steps in page 20

$$\|Ax_{min}-b\|_2 \underset{\text{pg 20}}{=} \|A(R^{-1}Q^Tb)-b\|_2 \underset{A=QR}{=} \|QRR^{-1}Q^Tb-b\|_2 = 0 \quad \circledast$$

2) Find the affine function $l(x)=\alpha x+\beta$ whose graph fits better the points $(-1,1)$, $(0,0)$ and $(1,1)$, in the sense that the euclidean norm of the vector

$$(l(-1)-1,\ l(0)-0,\ l(1)-1) \in \mathbb{R}^3$$

is minimal along all possible choices of $\alpha, \beta \in \mathbb{R}^3$. (Teacher)

$$\underbrace{\begin{pmatrix}-1 & 1\\ 0 & 1\\ 1 & 1\end{pmatrix}}_{A}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \underbrace{\begin{pmatrix}l(-1)\\ l(0)\\ l(1)\end{pmatrix}}_{b} \xrightarrow[\text{minimal norm}]{} b \approx \begin{pmatrix}1\\0\\1\end{pmatrix}$$

A is orthogonal $\Rightarrow$ you just need to normalize

$$A = QR = \begin{pmatrix}-1/\sqrt{2} & 1/\sqrt{3}\\ 0 & 1/\sqrt{3}\\ 1/\sqrt{2} & 1/\sqrt{3}\end{pmatrix}\begin{pmatrix}\sqrt{2} & 0\\ 0 & \sqrt{3}\end{pmatrix}$$

$$x_{min} = R^{-1}Q^Tb = \begin{pmatrix}0\\ 2/3\end{pmatrix} = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

Teacher solution of 1) Explain that

$$\boxed{A} = \underset{\text{orthogonal}}{\boxed{Q}}\ \underset{\nabla \text{ with pos. diagonal}}{\boxed{R}} \underset{\text{full}}{=} \boxed{Q \mid \tilde{Q}}\ \boxed{\begin{matrix}R\\ 0\end{matrix}}$$

We need to compute the residual of full QR! (Follow page 26)

$$\|Ax-b\|_2^2 = \|(Q|\tilde{Q})\begin{pmatrix}R\\0\end{pmatrix}x-b\|_2^2 = \|\underbrace{(Q|\tilde{Q})}_{\text{orthogonal (can be deleted in norms)}}\left(\begin{pmatrix}R\\0\end{pmatrix}x-\begin{pmatrix}Q\\ \tilde{Q}^T\end{pmatrix}^T b\right)\|_2^2 = \|\begin{pmatrix}R\\0\end{pmatrix}x-\begin{pmatrix}Q^T\\ \tilde{Q}^T\end{pmatrix}b\|_2^2 = \underbrace{\|Rx-Q^Tb\|_2^2}_{\text{will be 0 with } x_{min}} + \underbrace{\|\tilde{Q}^Tb\|_2^2}_{\text{residual}}$$

(because $x_{min} = R^{-1}Q^Tb$

L1.4 Consider the matrix $A = \begin{pmatrix} 4 & 1 \\ -2 & -1 \end{pmatrix}$

1) Compute its QR factorization using Householder reflexions. (Student)

$$\tilde{v}_1 = y + \text{sign}(y_1)\|y\|_2 e_1 = \begin{pmatrix} 4 \\ -2 \end{pmatrix} + \begin{pmatrix} \sqrt{20} \\ 0 \end{pmatrix} = \begin{pmatrix} 4+\sqrt{20} \\ -2 \end{pmatrix} \quad ; \quad v_1 = \text{House}\begin{bmatrix} 4 \\ -2 \end{bmatrix} = \frac{\tilde{v}_1}{\|\tilde{v}_1\|} = \begin{pmatrix} 0.97 \\ -0.23 \end{pmatrix}$$

$$P_1 = \mathbb{1}_2 - 2v_1v_1^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 2\begin{pmatrix} 0.97 \\ -0.23 \end{pmatrix}(0.97 \quad -0.23) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 2\cdot\begin{pmatrix} 0.94 & -0.22 \\ -0.22 & 0.05 \end{pmatrix} = \underbrace{\begin{pmatrix} -0.88 & 0.44 \\ 0.44 & 0.9 \end{pmatrix}}_{\text{orthogonal } (M^T = M^{-1})}$$

quick check: $\det M = \pm 1$

$$A_1 = P_1 A = \begin{pmatrix} -0.88 & 0.44 \\ 0.44 & 0.9 \end{pmatrix} \cdot \begin{pmatrix} 4 & 1 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} -4.4 & -1.32 \\ 0 & -0.46 \end{pmatrix} = R$$

$$Q = P_1^{-1} = P_1^T = \begin{pmatrix} -0.88 & 0.44 \\ 0.44 & 0.9 \end{pmatrix}$$

2) Compute the same factorization, but this time using Givens rotation instead of reflexions.

$4 = x_i > x_j = -2$

$$R_1 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 4/\sqrt{20} & -2/\sqrt{20} \\ 2/\sqrt{20} & 4/\sqrt{20} \end{pmatrix}$$

$$\cos\theta = \frac{x_i}{(x_i^2 + x_j^2)^{1/2}} = \frac{4}{\sqrt{20}}$$

$$\sin\theta = \frac{-x_j}{(x_i^2 + x_j^2)^{1/2}} = \frac{2}{\sqrt{20}}$$

$$A_1 = R_1 A = \begin{pmatrix} 0.89 & -0.44 \\ 0.44 & 0.89 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} 4.44 & 1.33 \\ 0 & -0.45 \end{pmatrix} = \tilde{R}$$

$$Q = R_1^{-1} = R_1^T = \begin{pmatrix} 0.89 & 0.44 \\ -0.44 & 0.89 \end{pmatrix}$$

Not considering signs and errors due to rounding, both methods give similar values.

L15 Consider the singular value decomposition (SVD):

$$A = \begin{pmatrix} 1 & 1 & 0.41 \\ -1 & 0 & 0.41 \\ 0 & 1 & -0.41 \end{pmatrix} = \begin{pmatrix} -0.82 & 0 & -0.58 \\ 0.41 & -0.71 & -0.58 \\ -0.41 & -0.71 & 0.58 \end{pmatrix} \begin{pmatrix} 1.73 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.71 \end{pmatrix} \begin{pmatrix} -0.71 & -0.71 & 0 \\ 0.71 & -0.71 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

1) Compute the condition number of the matrix A with respect to the 2-norm (Student)

$$K_{\|\cdot\|_2} := \|A\|_2 \|A^{-1}\|_2$$

We know that $(A^{-1}) = (USV^T)^{-1} = (V^T)^{-1} S^{-1} U^{-1} = \begin{pmatrix} -0.71 & 0.71 & 0 \\ -0.71 & -0.71 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1/1.73 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/0.71 \end{pmatrix} \begin{pmatrix} -0.82 & 0.41 & -0.41 \\ 0 & -0.71 & -0.71 \\ -0.58 & -0.58 & 0.58 \end{pmatrix}$

$\|\cdot\|_2$ = max singular value ⊛ $\Rightarrow K_{\|\cdot\|_2} = \|A\|_2 \|A^{-1}\|_2 = \dfrac{1.73}{0.71} \approx 2.44$

2) Compute the best rank 1 and rank 2 approximations of A with respect to the same norm, and determine the distance to A of these approximations. (Student)

Using Young Eckart Theorem (page 30)

$\boxed{k=1}$

$$A_1 = \sum_{i=1}^{1} \sigma_i u_i v_i^T = \sigma_1 \cdot \underset{\text{column}}{u_1} \cdot \underset{\text{row (incl. T)}}{v_1^T} = 1.73 \cdot \begin{pmatrix} -0.82 \\ 0.41 \\ -0.41 \end{pmatrix} \begin{pmatrix} -0.71 & -0.71 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ -0.5 & -0.5 & 0 \\ 0.5 & 0.5 & 0 \end{pmatrix}$$

$\|A - A_1\|_2 = \sigma_2 = 1$ Any quick way to compute it?

$\boxed{k=2}$

$$A_2 = \sum_{i=1}^{2} \sigma_i u_i v_i^T = \sigma_2 u_2 v_2^T + A_1 = \begin{pmatrix} 1 & 1 & 0 \\ -0.5 & -0.5 & 0 \\ 0.5 & 0.5 & 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ -0.71 \\ -0.71 \end{pmatrix} \begin{pmatrix} 0.71 & -0.71 & 0 \end{pmatrix}$$

$\|A - A_2\|_2 = \sigma_3 = 0.71$

⊛ $\|\cdot\|_2 = \sqrt{\lambda_{max}(\bar{A}^T A)} = \sqrt{\lambda_{max}(A^T A)} = \sqrt{\lambda_*^2} = \lambda_*$

the eigvals of $A^T A$ are λ_i^2 for λ_i eigvals of A $\longrightarrow$

Let λ_* be the highest value eigvals of A

L1.6 Let $A \in \mathbb{R}^{2\times 2}$ such that the eigenvalues of $A \cdot A^T$ are 9 and $\frac{1}{4}$ with respective eigenvectors:

$$\begin{pmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{pmatrix} \approx \begin{pmatrix} 0.71 \\ -0.71 \end{pmatrix} \qquad \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix} \approx \begin{pmatrix} 0.71 \\ 0.71 \end{pmatrix}$$

and the eigenvalues of A^TA are 9 and $\frac{1}{4}$ with respective eigenvectors

$$\begin{pmatrix} 1/2 \\ -\sqrt{3}/2 \end{pmatrix} \approx \begin{pmatrix} 0.5 \\ -0.87 \end{pmatrix} \qquad \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \approx \begin{pmatrix} 0.87 \\ 0.5 \end{pmatrix}$$

1) Compute a SVD of A. (Teacher) (Development by student)

SVD of A would be $A = U\Sigma V^T$

$$A^TA = (U\Sigma V^T)^T (U\Sigma V^T) = V\Sigma \underbrace{U^TU}_{\text{orthogonal}}\Sigma V^T = V\Sigma^2V^T \Rightarrow V^T = \begin{pmatrix} 0.5 & -0.87 \\ 0.87 & 0.5 \end{pmatrix} \begin{matrix} \to 9 \\ \to 1/4 \end{matrix}$$

eigvecs →

$$AA^T = U\Sigma^2U^T \text{ (similarly)} \Rightarrow U = \begin{array}{cc} \begin{pmatrix} 0.71 & 0.71 \\ -0.71 & 0.71 \end{pmatrix} \\ \begin{matrix} 9 & \quad 1/4 \end{matrix} \end{array}$$

eigvecs ↓

$$\Sigma^2 = \begin{pmatrix} 9 & 0 \\ 0 & 1/4 \end{pmatrix} \Rightarrow \Sigma = \begin{pmatrix} 3 & 0 \\ 0 & 1/2 \end{pmatrix}$$

más grande primero

2) Using this SVD, compute the condition number of A respect to the 2-norm. (Student)

$\|A\|_2 = 3$ (highest e-val)

$\|A^{-1}\|_2 = 2$ (inverse of lowest e-val)

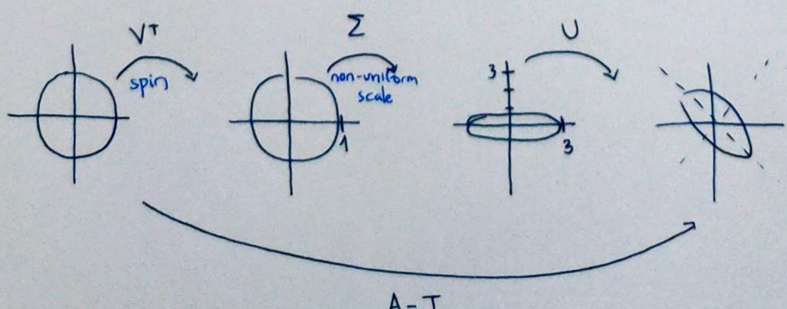
$$\Rightarrow K_{\|\cdot\|_2}(A) = \|A\|_2 \|A^{-1}\|_2 = \frac{3}{1/2} = 6$$

3) Determine the image of the unit disk of $\mathbb{R}^2$ under the linear map defined by A. (Teacher)

Interpret $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$x \longmapsto T(x) = USV^Tx$

$\mathbb{D} = \{(x_1, x_2) \in \mathbb{R} \mid x_1^2 + x_2^2 = 1\}$



L2.1 Let $A \in \mathbb{C}^{n \times n}$ be a matrix whose eigenvalues satisfy $|\lambda_1| > \dots > |\lambda_n|$

1) Which iterative algorithm would you apply to compute the eigenvalue λ_i of largest absolute value? Give an estimate of its rate of convergence. (Teacher)

Power method:
- start $x_0 \in \mathbb{C}^n \,/\, \|x_0\| = 1$ (following page 49)
- $x_{k+1} = \dfrac{Ax_k}{\|Ax_k\|} = \dfrac{A^k x_0}{\|A^k x_0\|}$

Relative error:

$$\frac{\|x - x_k\|}{\|x\|} \leq C \left(\frac{|\lambda_2|}{|\lambda_1|} \right)^k$$

Precision:

$$-\log_\beta \frac{\|x - x_k\|}{\|x\|} \geq k \log \frac{|\lambda_1|}{|\lambda_2|} + O(1)$$

similar to page 50

2) How would you proceed to compute the eigenvalue λ_n of smallest absolute value? Similarly as before, give an estimate for the rate of convergence of such an iterative method. (Teacher)

Set $B = A^{-1} \Rightarrow$ Eigenvalues of B: $|\lambda_n^{-1}| > \dots > |\lambda_1^{-1}|$

Apply power method to B:
- $x_{k+1} = \dfrac{Bx_k}{\|Bx_k\|} = \dfrac{B^k x_0}{\|B^k x_0\|} = \dfrac{A^{-k} x_0}{\|A^{-k} x_0\|}$

$$-\log_\beta \frac{\|x - x_k\|}{\|x\|} \geq k \log \frac{|\lambda_{n-1}|}{|\lambda_n|} + O(1)$$

Teacher defines $M_k = y_k^T B y_k$ and states $M_k \longrightarrow \lambda_n^{-1}$

Then

$$-\log_\beta \left| \frac{\lambda_n^{-1} - M_k}{\lambda_n} \right| \geq k \log \frac{|\lambda_{n-1}|}{|\lambda_n|} + O(1)$$

the other alternative in page 50

L2.2 Consider the matrix

$$A = \begin{pmatrix} 6.45 & -4.17 \\ 1.79 & -0.76 \end{pmatrix}$$

1. For an starting vector $x_0 \in \mathbb{R}^2$, compute the iterative scheme for the power method, aimed to approximate the largest eigenvalue and its corresponding eigenvector (Student)

Following pages 49, 50

Given x_0: $\quad x_{k+1} = \dfrac{Ax_k}{\|Ax_k\|} = \dfrac{A^k x_0}{\|A^k x_0\|}$

If we stop at iteration l, then our approximations are: $\quad \tilde{x} = x_{l+1}$

$$\tilde{\lambda} = \tilde{x}^* A \tilde{x}$$

2. Choose the starting vector $x_0 = (1,0)^T$. Knowing that the largest eigenvalue of A is about 10 times bigger than the other one, how many iterations of this scheme are necessary to compute 10 decimal digits of the coordinates of the solution? (Student)

We will asume that "10 times bigger than the other one" means $\left|\frac{\lambda_1}{\lambda_2}\right| = 10$ (we are in $\mathbb{R}^2$)

Then we need, with K the number of iterations:

$$K \underbrace{\log_{10}}_{\text{base}} \underbrace{(10)}_{\left|\frac{\lambda_1}{\lambda_2}\right|} \geq 10 \rightarrow \text{number of wanted decimals.}$$

Then,

$$K \cdot 1 \geq 10 \quad \longrightarrow \quad \text{we need at least 10 iterations.}$$

And how many if we want to compute 30 decimal digits? (Student)

$$K \cdot 1 \geq 30 \quad \Rightarrow \quad 30 \text{ iterations.}$$

We have chosen C=0 for this exersise, as it returns the minimum amount of needed iterations.

L23 The Schur decomposition of a matrix $A \in \mathbb{C}^{n \times n}$ is its factorization as:

$$A = QRQ^T$$

where Q is a unitary matrix and R is upper triangular

1) Given A, which algorithm would you apply to compute its Schur decompositions? (Student)
   QR iteration

2) How can you the compute the eigenvalues and eigenvectors of A from its Schur decomposition.
   eigvals included in T
   eigvecs (pages 46-47) using Q

L24 Same but real QR it.

L2.4 A matrix $H = (h_{ij})_{ij} \in \mathbb{R}^{n\times n}$ is upper Hessenberg if all its coefficients below the lower secondary diagonal are zero, that is, if $h_{ij} = 0$ whenever $i \geq j+2$

(1) Explain how a matrix $A \in \mathbb{R}^{n\times n}$ can be reduced to upper Hessenberg form via an orthogonal similarity, that is, how to compute an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ such that QAQ^T is upper Hessenberg. (Student)

Follow page 53.

(2) Show that the QR iteration preserves the Hessenberg form. (Student)

Follow page 54.

L25 Let

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

1) For this matrix and vector, compute the iterative scheme given by (Student)

a) Jacobi method: for $j = 1, \dots, n$:

$$x_{l+1,j} \leftarrow \frac{1}{a_{jj}} \Big(b_j - \sum_{k \neq j} a_{j,k} x_k \Big)$$

b) Gauss-Seidel: for $j = 1, \dots, n$

$$x_{l+1,j} \leftarrow \frac{1}{a_{jj}} \Big(b_j - \sum_{k<j} a_{jk} x_{l+1,k} - \sum_{k>j} a_{jk} x_{l,k} \Big)$$

c) SOR(ω) with $\omega \in \mathbb{R}$ $\quad x_{l+1,j} = (1-\omega) x_l^{GS} + \omega x_{l+1}^{GS}$

2) Using the criterium based on the spectral radius, check if you can guarantee if these Jacobi and Gauss-Seidel schemes converge for any choice of initial vector $x_0 \in \mathbb{R}^2$. (Student)

Page 60: We need that $\rho(R) < 1$

For Jacobi: $R = L + U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with $A = \overbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}^{D} \left[\overbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}^{I} - \overbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}^{L} - \overbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}^{U} \right]$

$$\chi_R = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 \Rightarrow \lambda = \pm\sqrt{1} \Rightarrow \rho(R) = 1 \quad \text{no guaranteed convergence}$$

For Gauss-Seidel:

$$R = (\mathbb{1}_n - L)^{-1} U = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$

$$\chi_{GS} = \begin{vmatrix} -\lambda & 1 \\ 0 & -1-\lambda \end{vmatrix} = \lambda^2 + \lambda - 1 \Rightarrow \lambda = \frac{-1 \pm \sqrt{1 - 4 \cdot 1 \cdot -1}}{2} = \frac{-1 \pm \sqrt{5}}{2} \Rightarrow \rho(R) > 1 \quad \text{no conv.}$$

3) Choosing the starting vector $x_0 = (0,0)^T$, how many iterations of each of these schemes are necessary to compute 30 decimal digits of the solution? And how many if we want to compute 100 decimal digits?

Pg 60 con $\rho(R) \geq 1$ $\quad x - x_{l+1} = \lambda^{l+1} (x - x_0)$

1. Consider the matrix $A=\begin{pmatrix}1 & 1 & -2\\ 1 & 2 & -1\\ -2 & -1 & 9\end{pmatrix}$

1) Compute the PLU factorization of A given by the GEPP algorithm.

$$A=\underbrace{\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}}_{\text{swap rows 1 and 3}}\begin{pmatrix}-2&-1&9\\1&2&-1\\1&1&-2\end{pmatrix} \longrightarrow \text{compute LU of new A}$$

$$\begin{pmatrix}-2 & \boxed{-1 \quad 9}^{A_{1,2}}\\ \boxed{1} & 2 \quad -1\\ \boxed{1} & 1 \quad -2\end{pmatrix}_{A_{2,1}} = \begin{pmatrix}1&0&0\\-1/2&1&0\\-1/2&0&1\end{pmatrix}_{a_{11}^{-1}A_{2,1}}\begin{pmatrix}\underset{a_{1,1}}{-2} & \overset{A_{,2}}{\boxed{-1 \quad 9}}\\ 0 & \boxed{3/2 \quad 7/2}\\ 0 & \boxed{1/2 \quad 5/2}\end{pmatrix}$$

$$\longrightarrow = \begin{pmatrix}2&-1\\1&-2\end{pmatrix}-\begin{pmatrix}-1/2\\-1/2\end{pmatrix}(-1,\ 9)=\begin{pmatrix}2&-1\\1&-2\end{pmatrix}-\begin{pmatrix}1/2&-9/2\\1/2&-9/2\end{pmatrix}$$

We should apply LU to $\square$ (no pivoting needed) → 3/2 already biggest

$$A=\underbrace{\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}}_{P}\underbrace{\begin{pmatrix}1&0&0\\-1/2&1&0\\-1/2&1/3&1\end{pmatrix}}_{L}\underbrace{\begin{pmatrix}-2&-1&9\\0&3/2&7/2\\0&0&4/3\end{pmatrix}}_{U}$$

$\frac{1}{2}\cdot\left(\frac{3}{2}\right)^{-1}$ $\qquad \frac{5}{2}-\frac{1}{3}\cdot\frac{7}{2}=$

2) Compute the cholesky factorization of A.

$$G=\begin{pmatrix}1&?&?\\1&?&?\\-2&?&?\end{pmatrix}=\begin{pmatrix}1&0&0\\1&1&0\\-2&1&?\end{pmatrix}=\begin{pmatrix}1&0&0\\1&1&0\\-2&1&2\end{pmatrix}$$

copy column

$g_{2,2}=(\underset{a_{22}}{2}-(\underset{g_{2,1}}{1})^2)^{1/2}=1$ $\qquad g_{33}=(9-(-2)^2-1^2)^{1/2}=$

$g_{3,2}=\frac{1}{\underset{g_{22}}{(1)}}(-1-\underset{g_{3,1}}{(-2)}\cdot\underset{g_{2,1}}{(1)})$

3) Explain how you would solve the equation $Ax=b$ for $b=\begin{pmatrix}1\\0\\-1\end{pmatrix}$ using each one of these factorizations, and compute the solution vector x using one of them

PLU → page 1
Cholesky → similar

} Follow one to get $x=\begin{pmatrix}2\\1\\0\end{pmatrix}$

$G\underbrace{G^Tx}_{y}=b \rightarrow \cdot\ Gy=b$
$\qquad G^Tx=y$

4. Given a symmetric matrix $B \in \mathbb{R}^{n \times n}$, explain a procedure to obtain a Schur normal form of it in a computational efficient way. Give details about the algorithms for computing the involved factorizations.

Remark 3 pg 45 $\Rightarrow$ B admits a Schur decomposition $A = QTQ^*$ with Q orthogonal, T diagonal

From LA. pg 12 $\rightarrow$ B admits $B = Q^T \Lambda Q$ similarity with Q orthogonal, diagonalizable

LDL^T