Hidden Markov Models (I)

Natural Language Processing

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Sequential Labelling

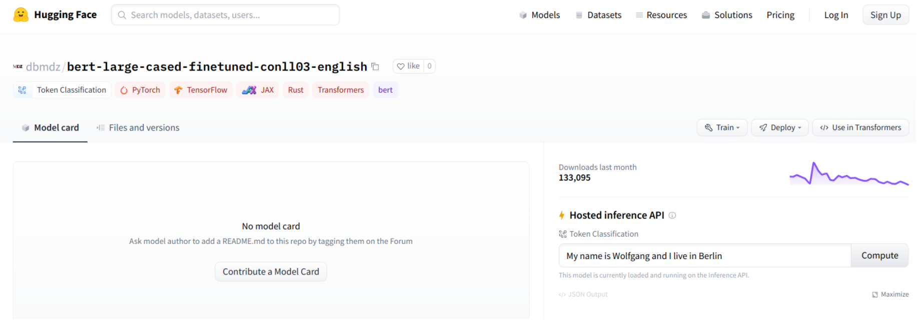
- Many problems in NLP require text to be labelled:
  - Part-of-Speech Tagging (POS): Find lexical terms in a sentence.
    - Input: 'red', 'flower'
    - Output: 'adjective', 'noun'
  - Named Entity Recognition (NER): Find 'Entities' in a sentence.
    - Input: 'Bill', 'Gates', 'founded', 'Microsoft', 'decades', 'ago'
    - Output: 'Person', 'Person', '-', 'Company', 'O', 'O'

Sequential Labelling

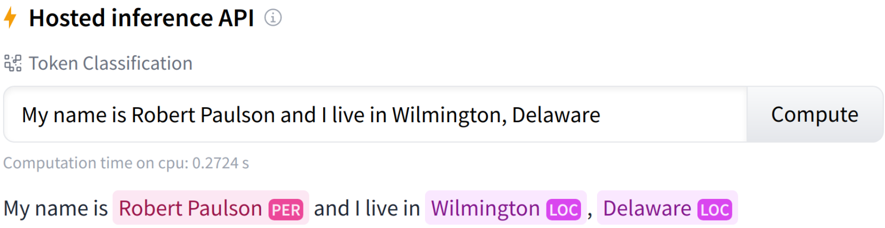
- Example: 'Bill', 'Gates', 'founded', 'Microsoft', 'decades', 'ago'
- We would like to know that 'Bill Gates' is a person, not consider 'Bill' and 'Gates' as two different people
- **Inside-Outside-Beginning Tagging (IOB Tagging)** uses B (Beginning) and I (Inside) tags to change the class label of a word to allow multiple words to belong to a single concept
  - Input: 'Bill', 'Gates', 'founded', 'Microsoft', 'decades', 'ago'
  - Output: 'B-Per', 'I-Per', 'O', 'B-COMP', 'O', 'O'
  - Input: Angelina Jolie was born in Los Angeles.
  - Output: 'B-PER', 'I-PER', 'O', 'O', 'O', 'B-LOC', 'I-LOC'

Sequential Labelling is Hard

https://huggingface.co/dbmdz/bert-large-cased-finetuned-conll03-english



Sequential Labelling is Hard



Sequential Labelling is Hard

Hosted inference API

Token Classification

Jesus of nazareth was born in Bethlehem — Compute

Computation time on cpu: 0.2496 s

Jesus PER of nazareth LOC was born in Bethlehem LOC

Sequential Labelling is Hard

Hosted inference API

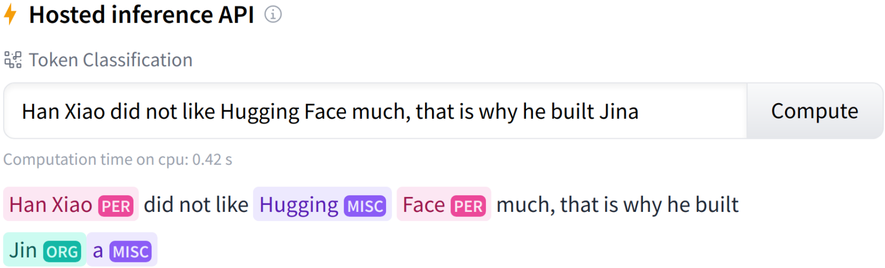
Token Classification

Han Xiao thought Hugging Face was the competition of Jina — Compute

Computation time on cpu: 0.2628 s

Han Xiao PER thought Hugging Face MISC was the competition of Jina PER

Sequential Labelling is Hard



Sequential Labelling is Hard

Hosted inference API

Token Classification

I love that Julia replaces standard classes by multiple dispatch | Compute

Computation time on cpu: 0.21359999999999998 s

I love that Julia PER replaces standard classes by multiple dispatch

Sequential Labelling: Notation

- Let $\Sigma := w_1, ..., w_J$ be a set of words
- Let $\Lambda := c_1, ..., c_K$ be a set of labels
- A sentence is an element from Σ^* which is the Kleene closure of Σ
  - The Kleene closure of a set Σ is defined as the set containing all possible strings of arbitrary length made up with elements in Σ, ϵ is used for the empty string
  - $\Sigma = a, b$ then $\Sigma^* = \epsilon, a, b, aa, ab, ba, bb, aaa, aab, bab, bba, ...$
  - $\Sigma = ab, c$ then $\Sigma^* = \epsilon, ab, c, abab, abc, cab, cc, ababab, ababc, ...$

Sequential Labelling: Notation

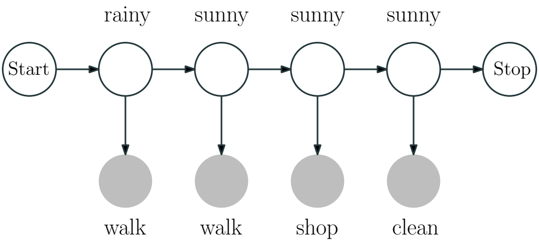
- A Hidden Markov Model (HMM) is a probabilistic model that defines a probability distribution over input/output pairs as follows:

$$
\begin{aligned}
P(X_1 = x_1, \dots, X_N = x_N, Y_1 = y_1, \dots, Y_N = y_N) = & P_{\mathrm{init}}(y_1|\mathrm{start}) \cdot \prod_{i=1}^{N-1} P_{\mathrm{trans}}(y_{i+1}|y_i) \cdot \\
& P_{\mathrm{final}}(\mathrm{stop}|y_N) \cdot \prod_{i=1}^{N} P_{\mathrm{emiss}}(x_i|y_i)
\end{aligned}
$$

- The previous formula tells us how to compute the probability of a pair if we have the initial, transition, final and emission probabilities, but how do we find such probabilities?

HMM Example

$$(x, y) = ([\text{walk}, \text{walk}, \text{shop}, \text{clean}], [\text{rainy}, \text{sunny}, \text{sunny}, \text{sunny}])$$



$$P([\text{walk}, \text{walk}, \text{shop}, \text{clean}], [\text{rainy}, \text{sunny}, \text{sunny}, \text{sunny}]) =$$
$$P_{\text{init}}(\text{rainy}|\text{start}) \cdot P_{\text{trans}}(\text{sunny}|\text{rainy}) \cdot P_{\text{trans}}(\text{sunny}|\text{sunny}) \cdot P_{\text{trans}}(\text{sunny}|\text{sunny}) \cdot P_{\text{final}}(\text{stop}|\text{sunny}) \cdot$$
$$P_{\text{emiss}}(\text{walk}|\text{rainy}) \cdot P_{\text{emiss}}(\text{walk}|\text{sunny}) \cdot P_{\text{emiss}}(\text{shop}|\text{sunny}) \cdot P_{\text{emiss}}(\text{clean}|\text{sunny})$$

HMM Parameters in memory

Parameter	#Floats
$P_{\mathrm{init}}(c\vert\mathrm{start})$	$\vert\Lambda\vert$
$P_{\mathrm{trans}}(c\vert\hat{c})$	$\vert\Lambda\vert^2$
$P_{\mathrm{final}}(\mathrm{stop}\vert c)$	$\vert\Lambda\vert$
$P_{\mathrm{emiss}}(w\vert c)$	$\vert\Sigma\vert \cdot \vert\Lambda\vert$

HMM Properties

- **Independence of Previous States**: the probability of being in a given state at position i only depends on the state of the previous position $i-1$ (this defines a first order Markov chain)

$$P(Y_i = y_i | Y_{i-1} = y_{i-1}, Y_{i-2} = y_{i-2}, ..., Y_1 = y_1) = P(Y_i = y_i | Y_{i-1} = y_{i-1})$$

- **Homogeneus Transition**: the probability of making a transition from state c_l to state c_k is independent of the particular sequence position

$$P(Y_i = c_k | Y_{i-1} = c_l) = P(Y_t = c_k | Y_{t-1} = c_l)$$

- **Observation Independence**: the probability of observing $X_i = x_i$ at position i is fully determined by the state Y_i at that position. The probability is also independent of the particular position

$$P(X_i = x_i | Y_1 = y_1, ..., Y_i = y_i, ..., Y_N = y_N) = P(X_i = x_i | Y_i = y_i)$$

$$P(X_i = w_j | Y_i = c_k) = P(X_t = w_j | Y_t = c_k)$$

Decoding Problems in HMM

- Posterior decoding

$$y_i^* = \arg\max_{y_i \in \Lambda} P(Y_i = y_i | X_1 = x_1, ..., X_N = x_N)$$

- Viterbi decoding

$$y^* = \arg\max_{y \in \Lambda^N} P(Y_1 = y_1, ..., Y_N = y_N | X_1 = x_1, ..., X_N = x_N)$$

Learning the Parameters of an HMM

- HMM parameters can be learnt by counting the different *events* we encounter in the training data*

Parameter	**Value**
$c_{\text{init}}(c_k)$	$\sum_{m=1}^{M} 1_{(y_1^m = c_k)}$
$P_{\text{init}}(c_k \mid \text{start})$	$\frac{c_{\text{init}}(c_k)}{\sum_{l=1}^{L} c_{\text{init}}(c_l)}$
$c_{\text{trans}}(c_k, c_l)$	$\sum_{m=1}^{M} \sum_{i=1}^{N} 1_{(y_i^m = c_k \wedge y_{i-1}^m = c_l)}$
$P_{\text{trans}}(c_l \mid c_k)$	$\frac{c_{\text{trans}}(c_k, c_l)}{\sum_{p=1}^{P} c_{\text{trans}}(c_k, c_p) + c_{\text{final}}(c_k)}$
$c_{\text{final}}(c_k)$	$\sum_{m=1}^{M} 1_{(y_N^m = c_k)}$
$P_{\text{final}}(\text{stop} \mid c_l)$	$\frac{c_{\text{final}}(c_l)}{\sum_{p=1}^{P} c_{\text{trans}}(c_l, c_p) + c_{\text{final}}(c_l)}$
$c_{\text{emiss}}(w_j, c_k)$	$\sum_{m=1}^{M} \sum_{i=1}^{N} 1_{(x_i^m = w_j \wedge y_i^m = c_k)}$
$P_{\text{emiss}}(w_j \mid c_k)$	$\frac{c_{\text{emiss}}(w_j, c_k)}{\sum_{q=1}^{J} c_{\text{emiss}}(w_q, c_k)}$

* M: number of samples

Forward Backward Algorithm

- Forward quantity

$$f(i, x, c) := P(X_1 = x_1, ..., X_i = x_i, Y_i = c)$$

- Backward quantity

$$b(i, x, c) := P(X_{i+1} = x_{i+1}, ..., X_N = x_N | Y_i = c)$$

Efficient Forward Quantity Computations

- If we define $f(i, x, c) := P(X_1 = x_1, ..., X_i = x_i, Y_i = c)$ then, the forward quantity at position i can be written as:

$$f(i, x, c) = P_{\text{emiss}}(x_i|c) \cdot \left(\sum_{\tilde{c} \in \Lambda} P_{\text{trans}}(c|\tilde{c}) \cdot f(i-1, x, \tilde{c}) \right)$$

Recurrence Rule for the Forward Quantities

$$f(i,x,c) := P(X_1 = x_1, ..., X_i = x_i, Y_i = c)$$

$$\begin{aligned}
P(y_i, x_{1:i}) = \sum_{y_{i-1} \in \Lambda} & P(y_i, y_{i-1}, x_{1:i}) = \\
\sum_{y_{i-1} \in \Lambda} & P(y_i, y_{i-1}, x_{1:i-1}, x_i) = \\
\sum_{y_{i-1} \in \Lambda} & P(x_i | y_i, y_{i-1}, x_{1:i-1}) \cdot P(y_i, y_{i-1}, x_{1:i-1}) = \\
\sum_{y_{i-1} \in \Lambda} & P(x_i | y_i, y_{i-1}, x_{1:i-1}) \cdot P(y_i | y_{i-1}, x_{1:i-1}) \cdot P(y_{i-1}, x_{1:i-1}) \approx \\
\sum_{y_{i-1} \in \Lambda} & P(x_i | y_i) \cdot P(y_i | y_{i-1}) \cdot P(y_{i-1}, x_{1:i-1}) = \\
P(x_i | y_i) \cdot & \sum_{y_{i-1} \in \Lambda} P(y_i | y_{i-1}) \cdot P(y_{i-1}, x_{1:i-1})
\end{aligned}$$

Forward Quantities Definition

- Let us consider a sequence x with N words
- For every state c let us define:

$$f(1, x, c) = P_{\text{init}}(c_k|\text{start}) \cdot P_{\text{emiss}}(x_1|c_k)$$

- For every position $i = 2$ up to N

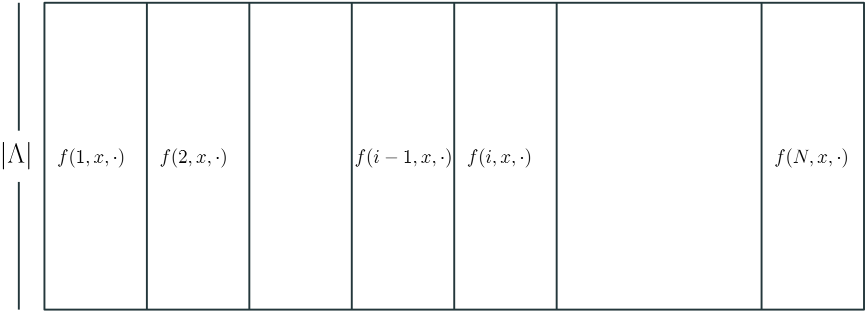
$$f(i, x, c) = P_{\text{emiss}}(x_i|c) \cdot \left(\sum_{\tilde{c} \in \Lambda} P_{\text{trans}}(c|\tilde{c}) \cdot f(i-1, x, \tilde{c}) \right)$$

- For position $N+1$

$$f(N+1, \text{stop}) = \sum_{c_l \in \Lambda} P_{\text{final}}(\text{stop}|c) \cdot f(N, x, c)$$

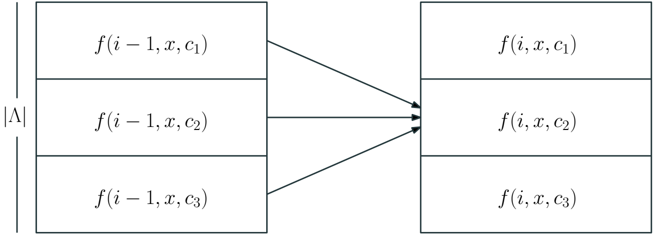
Forward Quantities

$$f(i, x, c) = P_{\text{emiss}}(x_i | c) \cdot \left(\sum_{\tilde{c} \in \Lambda} P_{\text{trans}}(c | \tilde{c}) \cdot f(i-1, x, \tilde{c}) \right)$$



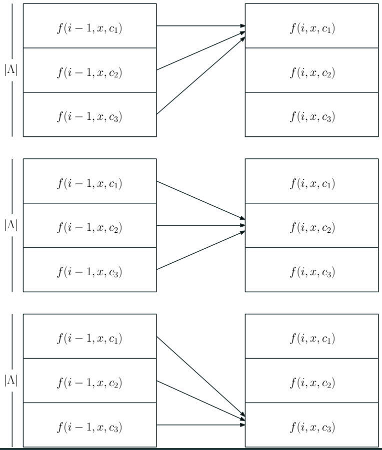
Forward Quantities

$$f(i, x, c) = P_{\text{emiss}}(x_i | c) \cdot \left(\sum_{\tilde{c} \in \Lambda} P_{\text{trans}}(c | \tilde{c}) \cdot f(i-1, x, \tilde{c}) \right)$$



Forward Quantity Given the Previous Forward Quantity

$$f(i, x, c) = P_{\text{emiss}}(x_i|c) \cdot \left(\sum_{\tilde{c} \in \Lambda} P_{\text{trans}}(c|\tilde{c}) \cdot f(i-1, x, \tilde{c}) \right)$$



Recurrence Rule for the Backward Quantities

$$b(i, x, c) := P(X_{i+1} = x_{i+1}, ..., X_N = x_N | Y_i = c)$$

Exercise!

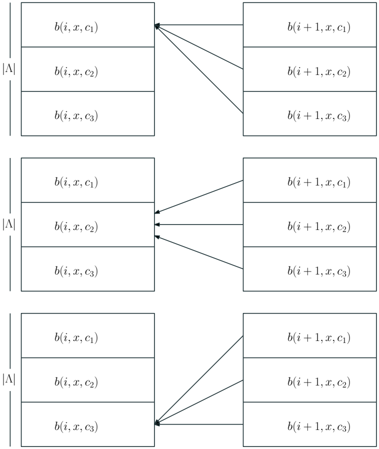
Backward Quantities

$$b(i, x, c) := \sum_{\tilde{c} \in \Lambda} P_{\mathrm{trans}}(\tilde{c}|c) \cdot P_{\mathrm{emiss}}(x_{i+1}|\tilde{c}) \cdot b(i+1, x, \tilde{c})$$

$\lvert\Lambda\rvert$	$b(1, x, \cdot)$	$b(2, x, \cdot)$		$b(i-1, x, \cdot)$	$b(i, x, \cdot)$		$b(N, x, \cdot)$

Backward Quantities

$$b(i, x, c) := \sum_{\tilde{c} \in \Lambda} P_{\text{trans}}(\tilde{c}|c) \cdot P_{\text{emiss}}(x_{i+1}|\tilde{c}) \cdot b(i+1, x, \tilde{c})$$



Forward Backward Algorithm

- Proposition I: $P(X = x_{1:N}, Y_i = c) = f(i,x,c) \cdot b(i,x,c)$
- Proposition II: $P(X = x_{1:N}) = \sum_{c \in \Lambda} f(i,x,c) \cdot b(i,x,c)$
- Proposition III: $P(Y_i = c | X = x_{1:N}) = \frac{f(i,x,c) \cdot b(i,x,c)}{P(X=x)}$
- Proposition IV (Posterior decoding):

$$y^* = \arg\max_{c \in \Lambda} P(Y_i = c | X_1 = x_1, ..., X_N = x_N) = \arg\max_{c \in \Lambda} f(i,x,c) \cdot b(i,x,c)$$

Forward Backward Algorithm

```
forward_backward(x,Λ,P_init,P_trans,P_final,P_emiss)
    # Forward pass
    for c_k ∈ Λ:
        forward(1, x, c_k) = P_init(c_k|start) · P_emiss(x_1|c_k)
    for i = 1 to N:
        for c̃ ∈ Λ:
            forward(i, x, c̃) = (Σ_{c_k∈Λ} P_trans(c̃|c_k) · forward(i − 1, x, c_k)) P_emiss(x_i|c̃)

    # Backward pass
    for c_k ∈ Λ:
        backward(N, x, c_k) = P_final(stop|c_k)
    for i = N − 1 to 1:
        for c̃ ∈ Λ:
            backward(i, x, c̃) = Σ_{c_k∈Λ} P_trans(c_k|c̃) · P_emiss(x_{i+1}|c_k) · backward(i + 1, x, c_k)
```