Exercise 6. (Karush–Kuhn–Tucker conditions)

Given a vector $\boldsymbol{y}$, consider the problem

$$\max_{\boldsymbol{x}} \ \boldsymbol{y}^T \boldsymbol{x}$$

$$\text{subject to:} \quad \boldsymbol{x}^T Q \boldsymbol{x} \leq 1$$

where Q is a positive definite symmetric matrix. Show that the optimal value is $\sqrt{\boldsymbol{y}^T Q^{-1} \boldsymbol{y}}$, and use this fact to establish the inequality

$$(\boldsymbol{x}^T \boldsymbol{y})^2 \leq (\boldsymbol{x}^T Q \, \boldsymbol{x})(\boldsymbol{y}^T Q^{-1} \boldsymbol{y})$$

Solution

We rewrite the problem as

$$\min_{\boldsymbol{x}} \ -\boldsymbol{y}^T \boldsymbol{x}$$

$$\text{subject to:} \quad g(\boldsymbol{x}) = 1 - \boldsymbol{x}^T Q \boldsymbol{x} \geq 0$$

The Lagrange function is

$$L(\boldsymbol{x}, \lambda) = -\boldsymbol{y}^T \boldsymbol{x} - \lambda(1 - \boldsymbol{x}^T Q \boldsymbol{x}).$$

The K-K-T conditions are

$$\begin{aligned} \nabla_{\boldsymbol{x}} L(\boldsymbol{x}, \lambda) &= 0 \\ \lambda(1 - \boldsymbol{x}^T Q \boldsymbol{x}) &= 0 \\ \lambda &\geq 0 \end{aligned}$$

where, using that Q is symmetric, we have

$$\nabla_{\boldsymbol{x}} L(\boldsymbol{x}, \lambda) = -\boldsymbol{y}^T + \lambda(\boldsymbol{x}^T Q + Q\boldsymbol{x}) = -\boldsymbol{y}^T + 2\lambda \boldsymbol{x}^T Q.$$

- If $\lambda^* = 0$, we have $\nabla_{\boldsymbol{x}} L(\boldsymbol{x}, \lambda^*) = -\boldsymbol{y}^T$, which is equal to 0 only for the trivial case $\boldsymbol{y} = 0$, with $\boldsymbol{y}^T \boldsymbol{x} = 0$ for all $\boldsymbol{x}$.
- If $\lambda^* \neq 0$, we compute $\boldsymbol{x}^*$ from $\nabla_{\boldsymbol{x}} L(\boldsymbol{x}^*, \lambda^*) = 0$ and get

  $$\boldsymbol{x}^{*T} = \frac{1}{2\lambda^*} \boldsymbol{y}^T Q^{-1}, \quad \Rightarrow \quad \boldsymbol{x}^* = \frac{1}{2\lambda^*} Q^{-1} \boldsymbol{y}.$$

  From the second K-K-T condition, when $\lambda^* \neq 0$, we can write

  $$1 = \boldsymbol{x}^{*T} Q \boldsymbol{x}^* = \frac{1}{4\lambda^{*2}} \boldsymbol{y}^T Q^{-1} Q Q^{-1} \boldsymbol{y} \quad \Rightarrow \quad \lambda^{*2} = \frac{1}{4} \boldsymbol{y}^T Q^{-1} \boldsymbol{y} \quad \Rightarrow \quad \lambda^* = \frac{1}{2} \sqrt{\boldsymbol{y}^T Q^{-1} \boldsymbol{y}}.$$

Since Q is positive defined Q^{-1} is also positive defined, so $\sqrt{\boldsymbol{y}^T Q^{-1} \boldsymbol{y}}$ is well defined, $\lambda^* \geq 0$, and

$$\boldsymbol{x}^{*T} = \frac{1}{2\lambda^*} \boldsymbol{y}^T Q^{-1} = \frac{\boldsymbol{y}^T Q^{-1}}{\sqrt{\boldsymbol{y}^T Q^{-1} \boldsymbol{y}}}.$$

The optimal value is

$$-\boldsymbol{x}^{*T} \boldsymbol{y} = -\frac{\boldsymbol{y}^T Q^{-1} \boldsymbol{y}}{\sqrt{\boldsymbol{y}^T Q^{-1} \boldsymbol{y}}} = -\sqrt{\boldsymbol{y}^T Q^{-1} \boldsymbol{y}}.$$

SInce the above value minimizes $-\boldsymbol{x}^T \boldsymbol{y}$, we can write that for any $\boldsymbol{x} \in \mathbb{R}^n$

$$\boldsymbol{x}^T \boldsymbol{y} \leq \sqrt{\boldsymbol{y}^T Q^{-1} \boldsymbol{y}} \quad \Rightarrow \quad (\boldsymbol{x}^T \boldsymbol{y})^2 \leq \boldsymbol{y}^T Q^{-1} \boldsymbol{y} \leq \left(\boldsymbol{x}^T Q \boldsymbol{x}\right) \left(\boldsymbol{y}^T Q^{-1} \boldsymbol{y}\right),$$

where we have used that $\boldsymbol{x}^T Q \boldsymbol{x} = 1$.