Exercise 3. (Convex functions)

Proof that for any $a \in \mathbb{R}$, the function $f(x) = e^{ax}$, $x \in \mathbb{R}$ is convex whithout using the using the following characterization of convex functions:

$$f \text{ is convex} \quad \Leftrightarrow \quad \nabla f(x)^T (y-x) \le f(y) - f(x) \le \nabla f(y)^T (y-x), \quad \forall x, y \in \mathbb{R}.$$

Solution

The domain of definition of f is $\mathbb{R}$, which is convex, so we only need to prove that

$$f(\lambda x + (1-\lambda) y) \le \lambda f(x) + (1-\lambda) f(y), \quad \forall x, y \in \mathbb{R}, \quad \forall \lambda \in [0,1],$$

with $f(x) = e^{ax}$, this is

$$e^{a(\lambda x + (1-\lambda)y)} \le \lambda e^{ax} + (1-\lambda) e^{ay} \quad \forall x, y \in \mathbb{R}, \quad \forall \lambda \in [0,1].$$

Assume that $ay > ax$, and divide both sides of the above inequality by e^{ax} (if $ay < ax$ divide e^{ay}, and if $ay = ax$ the equality holds). In this way we get

$$e^{(\lambda-1)ax + (1-\lambda)ay} \le \lambda + (1-\lambda) e^{ay - ax} \quad \Leftrightarrow \quad e^{(1-\lambda)a(y-x)} \le \lambda + (1-\lambda) e^{a(y-x)}.$$

Define $h = a(y-x) > 0$ (since we have assumed $ay > ax$), then, the above inequality becomes

$$e^{(1-\lambda)h} \le \lambda + (1-\lambda) e^h.$$

Computing the Taylor expansions of both sides we get

$$\begin{aligned}
e^{(1-\lambda)h} &= 1 + (1-\lambda)h + \frac{1}{2}(1-\lambda)^2 h^2 + \frac{1}{6}(1-\lambda)^3 h^3 + \dots \\
\lambda + (1-\lambda)e^h &= \lambda + (1-\lambda)\left(1 + h + \frac{1}{2}h^2 + \frac{1}{6}h^3 + \dots\right) \\
&= 1 + (1-\lambda)h + \frac{1}{2}(1-\lambda)h^2 + \frac{1}{6}(1-\lambda)h^3 + \dots
\end{aligned}$$

Since $\lambda \in [0,1]$, then $1-\lambda \in [0,1]$, and since $h > 0$ it follows that for $n \ge 0$

$$(1-\lambda)^n h^n \le 1 - \lambda) h^n,$$

so

$$1 + (1-\lambda)h + \frac{1}{2}(1-\lambda)^2 h^2 + \frac{1}{6}(1-\lambda)^3 h^3 + \dots \le 1 + (1-\lambda)h + \frac{1}{2}(1-\lambda)h^2 + \frac{1}{6}(1-\lambda)h^3 + \dots$$

which is the inequality that we want to proof.