My NIUB is 20194344, thus $r' = 8$ and $r = 9$. The exercises to do are 3, 9, 14.

Exercise 3

Proof. Assume that $(\hat{x}, \hat{\lambda}, \hat{\mu})$ is a solution of $(\mathcal{S})$. This implies that for every $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^m$ and $\mu \in \mathbb{R}^p$, $\mu \geq 0$, we have $L(\hat{x}, \lambda, \mu) \leqslant L(\hat{x}, \hat{\lambda}, \hat{\mu}) \leqslant L(x, \hat{\lambda}, \hat{\mu})$ where L is the Lagrangian function. Writing explicit formulas for the first inequality, we can see the following:

$$\sum_{j=1}^{m} (\hat{\lambda}_j - \lambda_j) g_j(\hat{x}) + \sum_{j=1}^{p} (\hat{\mu}_j - \mu_j) h_j(\hat{x}) \leqslant 0. \tag{0.1}$$

First of all, we claim that $g_j(\hat{x}) = 0$ for every $j \in \{1, \ldots, m\}$. To prove this, assume $g_i(\hat{x}) \neq 0$ for some i. In this case, $g_i(\hat{x})$ is either strictly positive or strictly negative and in consequence, we can choose a suitable vector $\lambda \in \mathbb{R}^m$ such that $\sum_{j=1}^{m} (\hat{\lambda}_j - \lambda_j) g_j(\hat{x}) > 0$. For example we can choose $\lambda_i = \hat{\lambda}_i - 1$ if $g_i(\hat{x}) > 0$ or $\lambda_i = \hat{\lambda}_i + 1$ if $g_i(\hat{x}) < 0$ and $\lambda_j = \hat{\lambda}_j$ for every $j \neq i$. Moreover, choosing $\mu = \hat{\mu}$, we have $\sum_{j=1}^{p} (\hat{\mu}_j - \mu_j) h_j(\hat{x}) = 0$ and we reach a contradiction with the inequality sign on (0.1), thus $g_j(\hat{x}) = 0$ for every $j \in \{1, \ldots, m\}$.

Since $g_j(\hat{x}) = 0$ for every $j \in \{1, \ldots, m\}$, the inequality (0.1) becomes

$$\sum_{j=1}^{p} (\hat{\mu}_j - \mu_j) h_j(\hat{x}) \leqslant 0. \tag{0.2}$$

First of all notice that $h_j(\hat{x}) \geq 0$ for all j, otherwise, if for some subscript i, $h_i(\hat{x}) < 0$ we always can choose $\mu_i > 0$ sufficiently large, so that (0.2) does not hold. Moreover, observe that for $\mu = 0$, we have that $\sum_{j=1}^{p} \hat{\mu}_j h_j(\hat{x}) \leqslant 0$ and on the other hand, since $\hat{\mu} \geq 0$ and $h_j(\hat{x}) \geq 0$ for all j, $\sum_{j=1}^{p} \hat{\mu}_j h_j(\hat{x}) \geq 0$. This implies that $\sum_{j=1}^{p} \hat{\mu}_j h_j(\hat{x}) = 0$, hence $\hat{\mu}_j h_j(\hat{x}) = 0$ for every $j \in \{1, \ldots, p\}$.

In summary, we have obtain that $g_j(\hat{x}) = 0$ for every $j \in \{1, \ldots, m\}$ and $\hat{\mu}_j h_j(\hat{x}) = 0$ for every $j \in \{1, \ldots, p\}$. Now, if we write explicit formulas for the second inequality of the Lagrangian functions, we obtain

$$f(\hat{x}) \leqslant f(x) + \sum_{j=1}^{m} \hat{\lambda}_j (g_j(\hat{x}) - g_j(x)) + \sum_{j=1}^{p} \hat{\mu}_j (h_j(\hat{x}) - h_j(x)).$$

Adding the conditions we have just obtain, we conclude that

$$f(\hat{x}) \leqslant f(x) - \sum_{j=1}^{m} \hat{\lambda}_j g_j(x) - \sum_{j=1}^{p} \hat{\mu}_j h_j(x).$$

Since $\hat{\mu} \geq 0$, we have $f(\hat{x}) \leqslant f(x)$ for all x such that $g_j(x) = 0$ and $h_j(x) \geq 0$ for all j. This implies that $\hat{x}$ is a solution of $(\mathcal{P})$. □

Exercise 9

Proof. Assume $\Phi(\theta) = a + b\theta + c\theta^2$. If we evaluate this function on the points $\theta_1, \theta_2, \theta_3$, we obtain the following equation system:

$$\begin{pmatrix} 1 & \theta_1 & \theta_1^2 \\ 1 & \theta_2 & \theta_2^2 \\ 1 & \theta_3 & \theta_3^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \Phi(\theta_1) \\ \Phi(\theta_2) \\ \Phi(\theta_3) \end{pmatrix}$$

Notice that the minimum of the quadratic function will be achieved at the point θ^* such that $\Phi'(\theta^*) = 0$, i.e., $\theta^* = -\frac{b}{2c}$ as long as $\Phi''(\theta^*) > 0$, i.e., $c > 0$. Therefore, if we are able to find a, b, c, i.e., solve the previous linear system, we are done. After some computations, assuming that the previous 3×3 matrix is non-singular, one can see that the solution of the linear system is

$$a = \frac{(\theta_2^2\theta_3 - \theta_2\theta_3^2)\Phi(\theta_1) + (\theta_1\theta_3^2 - \theta_1^2\theta_3)\Phi(\theta_2) + (\theta_1^2\theta_2 - \theta_1\theta_2^2)\Phi(\theta_3)}{(\theta_2-\theta_3)(\theta_3-\theta_1)(\theta_1-\theta_2)},$$

$$b = \frac{(\theta_2^2 - \theta_3^2)\Phi(\theta_1) + (\theta_3^2 - \theta_1^2)\Phi(\theta_2) + (\theta_1^2 - \theta_2^2)\Phi(\theta_3)}{(\theta_2-\theta_3)(\theta_3-\theta_1)(\theta_1-\theta_2)},$$

$$c = -\frac{(\theta_2 - \theta_3)\Phi(\theta_1) + (\theta_3 - \theta_1)\Phi(\theta_2) + (\theta_1 - \theta_2)\Phi(\theta_3)}{(\theta_2-\theta_3)(\theta_3-\theta_1)(\theta_1-\theta_2)}.$$

Therefore, the minimum of Φ is achieved at

$$\theta^* = -\frac{b}{2c} = \frac{(\theta_2^2 - \theta_3^2)\Phi(\theta_1) + (\theta_3^2 - \theta_1^2)\Phi(\theta_2) + (\theta_1^2 - \theta_2^2)\Phi(\theta_3)}{2[(\theta_2 - \theta_3)\Phi(\theta_1) + (\theta_3 - \theta_1)\Phi(\theta_2) + (\theta_1 - \theta_2)\Phi(\theta_3)]},$$

as long as $c > 0$, i.e.,

$$\frac{(\theta_2 - \theta_3)\Phi(\theta_1) + (\theta_3 - \theta_1)\Phi(\theta_2) + (\theta_1 - \theta_2)\Phi(\theta_3)}{(\theta_2-\theta_3)(\theta_3-\theta_1)(\theta_1-\theta_2)} < 0.$$

□

Exercise 14

Proof. Let us start solving *(a)*. Remind that a sequence $\{x_k\}_k$, $x_k \in \mathbb{R}^n$, converge superlinearly to x^* if

$$\lim_{k\to\infty} \frac{||x_{k+1} - x^*||}{||x_k - x^*||} = 0.$$

Assume $\{x_k = \frac{1}{k!}\}$. Let us see if the sequence converge superlinearly to $x^* = 0$. Notice that

$$\lim_{k\to\infty} \frac{||x_{k+1} - 0||}{||x_k - 0||} = \lim_{k\to\infty} \frac{k!}{(k+1)!} = \lim_{k\to\infty} \frac{1}{k+1} = 0.$$

Then, the sequence of real number $\{x_k = \frac{1}{k!}\}$ converge superlinearly to $x^* = 0$.

Let us continue with *(b)*. We want to see that the sequence $\{x_k = 1 + (\frac{1}{2})^{2^k}\}$ converge quadratically to 1. Remind that a sequence $\{\alpha_k\}_k$ converge quadratically to α^* if

$$\lim_{k\to\infty} \frac{||\alpha_{k+1} - \alpha^*||}{||\alpha_k - \alpha^*||^2} = C > 0.$$

Notice that

$$\lim_{k\to\infty} \frac{||x_{k+1} - 1||}{||x_k - 1||^2} = \lim_{k\to\infty} \frac{\left(\frac{1}{2}\right)^{2^{k+1}}}{[\left(\frac{1}{2}\right)^{2^k}]^2} = \lim_{k\to\infty} \frac{\left(\frac{1}{2}\right)^{2^{k+1}}}{\left(\frac{1}{2}\right)^{2^{k+1}}} = \lim_{k\to\infty} 1 = 1 > 0.$$

Therefore, the sequence $\{x_k = 1 + (\frac{1}{2})^{2^k}\}$ converge quadratically to 1. □