Exercise 0

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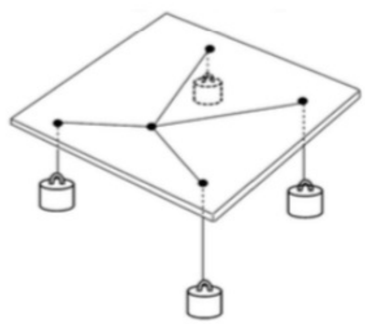
Optimization: 30^{th} December 2021

Exercise 0. (The Fermat point of a set of points) To be delivered before the end of the course as: `Ex00-YourSurname.pdf`.
Given set of points $y_1,\dots,y_m$ in the plane, find a point x^* whose sum of weighted distances to the given set of points is minimized.
Mathematically, the problem is

$$\min \sum_{i=1}^{m} w_i \|x^* - y_i\|, \quad \textit{subject to } x^* \in \mathbb{R}^2,$$

where $w_1,\dots,w_m$ are given positive real numbers.

1. *Show that there exists a global minimum for this problem (that it can be realized by means of the mechanical model shown in the figure).*



2. *Is the optimal solution always unique?*
3. *Show that an optimal solution minimizes the potential energy of the mechanical model defined as $\sum_{i=1}^{m} w_i h_i$, where h_i is the height of the ith weight measureed from some reference level.*

First of all, we will define the function that expresses our problem mathematically:

$$f(x) = \sum_{i=1}^{m} w_i ||x^* - y_i||$$

In order to claim that this function has a minimum we will show a simplistic approach using to our benefit that this is a problem that can be based in physics. Due to the constraints on w_i and the properties of norms, we know that $f(x) \geq 0$ for all x.

Since f is continuous, the only way for it not to have a minimum is for it to be horizontally asymptotic towards infinite values of x. There are many ways to figure out that this may not happen. One we may consider is the general behavior of norms, as they tend to explode with big values.

A second approach is taking into account that this problem needs to find an equilibrium point within the system, which can not be a point in infinity. For that reason, we are sure we may define a sufficiently big compact set within the domain of f where this equilibrium point will be included. Continuous functions always have a minimum within a closed and bounded set.

We may assume now that f has at least one global minimum. To show that the function has a unique minimum, we will first prove that norms are convex functions:

Proposition 1. *Let $g(\cdot) = ||\cdot||$ be a norm in the vector space V, then g is a convex function.*

Proof. In order to prove this, we need to show that for any $x, y \in V$ and any $t \in [0, 1]$ it follows that:

$$||tx + (1-t)y|| \leq t||x|| + (1-t)||y||$$

This is easily provable due to the properties of norms, the triangular inequality and the fact that $1 - t => 0$:

$$||tx + (1-t)y|| \leq ||tx|| + ||(1-t)y|| = |t|||x|| + |1-t|||y|| = t||x|| + (1-t)||y||$$

$\square$

Now, we need to show that the positive weighted sum of convex functions is convex. To achieve this, we need the following two propositions:

Proposition 2. *If g is a convex function and $\alpha > 0$, then ag is a convex function.*

Proof. Let $t \in [0, 1]$. Due to the convexity of g:

$$\begin{aligned}(\alpha g)(tx + (1-t)y) = \alpha(g(tx + (1-t)y)) &\leq \alpha(tg(x) + (1-t)g(y)) \\ &= t(\alpha g)(x) + (1-t)(\alpha g)(y)\end{aligned}$$

$\square$

Proposition 3. *If g and h are convex functions, then $g+h$ is a convex function.*

Proof. Let $t \in [0,1]$. Due to the convexity of both g and h:

$$\begin{aligned}(g+h)(tx+(1-t)y) &= g(tx+(1-t)y) + h(tx+(1-t)y) \\ &\leq tg(x) + (1-t)g(y) + th(x) + (1-t)h(y) \\ &= t(g(x)+h(x)) + (1-t)(g(y)+h(y)) \\ &= t(g+h)(x) + (1-t)(g+h)(y)\end{aligned}$$

□

Applying the three propositions at once, we can see that our function f is convex. Therefore, the given problem is trying to minimize a convex function. We will use this to prove that the minimum is unique.

Theorem 0.1 (Global Minimum Unicity in Convex Functions defined in Convex Sets)**.** *Let $\mathcal{X} \subset \mathbb{R}^n$ be a convex set for any n and f be a convex function with $dom(f) = \mathcal{X}$. If f has a global minimum, it is unique.*

Proof. Let's suppose that our function f has two global minima x_1, x_2. Then, $x_1 < x_2$, $f(x_1) = f(x_2)$ and $f(x) > f(x_1) = f(x_2)$ for any other x in the domain of the function.

We need to define $\alpha(x_1, x_2)$ as the shortest path connecting both points. Taking $t \in \alpha(x_1, x_2)$ we know that $f(t) > f(x_1) = f(x_2)$. Since we are working in a convex set $\mathcal{X}$, there exists $s \in (0,1)$ such that $t = sx_1 + (1-s)x_2$. From the convexity of f, we may reach the following contradiction:

$$f(t) = f(sx_1+(1-s)x_2) \leq sf(x_1)+(1-s)f(x_2) = sf(x_1)+(1-s)f(x_1) = f(x_1)$$

□

At this point, we have proved that the global minimum of our problem exists and is unique.

Referring to the physical notion of this problem, it is generally known as the **Fermat-Torricelli Problem**. It has been physically modelled by the Hungarian mathematician **George Pólya**, even though he modelled it only with 3 weights forming a triangle.

The equilibrium point in this physical model is represented by the point where all ropes are knotted together. This point will move until it reaches the equilibrium. Of course, the higher weights will pull the knot closer to their holes. This pull effect minimizes both the potential energy (as the weight will be closer to the floor since now it has more rope available to hang from the table with) and the distance between the equilibrium point and the knot (which is what is modelled in our mathematical problem: the weighted sum of the distances between the equilibrium point and the holes). The weights in both formulas represent the notion of higher weights pulling the knot closer to their holes more than the rest (they have priority when it comes to minimizing the distance).