Exercise 8

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Exercise 8. To be delivered before 9-XI-2021 as: `Ex08-YourSurname.pdf`

Given a vector $\boldsymbol{y}$, consider the problem

$$\text{maximize} \quad \boldsymbol{y}^T\boldsymbol{x}$$

$$\text{subject to:} \quad \boldsymbol{x}^T Q \boldsymbol{x} \leq 1$$

where Q is a positive definite symmetric matrix. Show that the optimal value is $\sqrt{\boldsymbol{y}^T Q^{-1} \boldsymbol{y}}$, and use this fact to establish the inequality

$$(\boldsymbol{x}^T\boldsymbol{y})^2 \leq (\boldsymbol{x}^T Q\, \boldsymbol{x})(\boldsymbol{y}^T Q^{-1} \boldsymbol{y})$$

We are presented with a problem that is equivalent to the following constrained optimization problem:

$$\min f(x) = -y^T x$$
$$\text{subject to } g(x) = 1 - x^T Q x \geq 0$$

We will follow the same steps as in Exercise 7 to begin with:

$$L(x, \lambda) = f(x) - \lambda g(x)$$
$$X = \{x | 1 - x^T Q x \geq 0\}$$

We will use the Theorem of sufficient conditions seen in class. The Karush-Kuhn-Tucker conditions generate the following equations, with $\lambda > 0$:

$$\nabla L(x, \lambda) = -y + 2\lambda Q x = 0$$
$$\lambda(1 - x^T Q x) = 0$$

We are using that Q is symmetrical and positive definite. So now, we get that:

$$x = \frac{1}{2\lambda} Q^{-1} y$$

From the second condition, knowing that $\lambda > 0$:

$$\begin{aligned}
1 &= x^T Q x \\
1 &= \frac{1}{2\lambda}^2 (y^T Q^{-1}) Q (Q^{-1} y) \\
1 &= \frac{1}{2\lambda}^2 y^T Q^{-1} y \\
2\lambda &= \sqrt{y^T Q^{-1} y}
\end{aligned}$$

Therefore:

$$x = \frac{1}{\sqrt{y^T Q^{-1} y}} Q^{-1} y$$

And in the original function, as we wanted to prove:

$$y^T x = \frac{y^T}{\sqrt{y^T Q^{-1} y}} Q^{-1} y = \sqrt{y^T Q^{-1} y}$$

To prove the last inequality, we will consider the case where $(x^T y)^2 = ((y^T x)^T)^2$ is maximized, that is when the base is maximized. In that case we already know that:

$$x = \frac{1}{2\lambda} Q^{-1} y$$

We will be using that $y^* = 2\lambda Q x^*$. We are changing notations in order not to confound generic x, y and their values when the function is maximized.

$$\begin{aligned}
(x^T y)^2 &= (2\lambda x^T Q x^*)^2 \\
(x^T y)^2 &= (2\lambda)^2 (x^T Q x^*)^T (x^T Q x^*)
\end{aligned}$$

Now, applying the Cauchy-Schwartz inequality:

$$(x^T y)^2 < (x^T Q x)(y^T Q^{-1} y)$$

As for the second condition of the Theorem of sufficient conditions we used during this exercise, we know that $\nabla^2 f(x) = 0$ and that $\nabla^2 g(x) = -2Q$. Therefore, for any z:

$$z^T [2Q] z = 2(z^T Q z) > 0$$

Which is true because Q is positive definite.