Exercise 7

Lorenzo Vigo

Optimization: 8^{th} November 2021

Exercise 7. To be delivered before 9-XI-2021 as: `Ex07-YourSurname.pdf`

Solve the two-dimensional problem

$$\text{minimize} \quad (x-a)^2 + (y-b)^2 + xy$$

$$\text{subject to} \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1$$

for all possible values of the scalars a and b

We are presented with a constrained optimization problem characterized by:

$$\begin{aligned} \min\ f(x) &= (x-a)^2 + (y-b)^2 + xy \\ \text{subject to } g_1(x) &= x \geq 0 \\ g_2(x) &= 1 - x \geq 0 \\ g_3(y) &= y \geq 0 \\ g_4(y) &= 1 - y \geq 0 \end{aligned}$$

We could solve this normally by minimizing the function in the open unit square. That point would be a candidate to be the desired solution, but we should also consider that the edges and the minimum in each one of the sides of the square also may be.

This approach gives us the following candidates: $(2(b-y), 2(a-x))$ (if it is inside the square), $(0, b)$, $(1, b-\frac{1}{2})$, $(a, 0)$, $(a-\frac{1}{2}, 1)$ and all the four edges $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$. For any a, b, the minimum is among these candidates.

Let's try to get the same result with the Theorems and properties seen in class:

$$L(x, y, \lambda) = f(x, y) - \sum_{i=1}^{4} \lambda_i g_i(x, y)$$

$$X = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq 1\}$$

We will use the Theorem of sufficient conditions seen in class. The Karush-Kuhn-Tucker conditions generate the following equations, with all $\lambda_i \geq 0$:

$$(2(x-a)+y, 2(y-b)+x) - \lambda_1(1,0) - \lambda_2(-1,0) - \lambda_3(0,1) - \lambda_4(0,-1) = (0,0)$$

$$\begin{aligned} \lambda_1 x &= 0 \\ \lambda_2(1-x) &= 0 \\ \lambda_3 y &= 0 \\ \lambda_4(1-y) &= 0 \end{aligned}$$

Solving this system of 6 equations with 6 variables will lead you to the candidates stated before. This is possible because the second needed condition is fulfilled. As all the second derivatives of the functions $g_i(x)$ are 0, we just need to check that:

$$z^T \nabla^2 f(x) z > 0$$

For all the candidate points. But since the Hessian matrix of f is positive definite, it is trivial:

$$\nabla^2 f(x) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$