My NIUB is 20194344, thus $r' = 1$. The exercises to do are 2,8 and 13.

Exercise 2.

Proof. Let us start by finding points satisfying necessary conditions for extrema of the function f. Notice that the function belongs to C^∞ class, thus we can apply the necessary and sufficient conditions for minima seen in theory. First, we want to find (x_1, x_2) such that $\nabla f(x_1, x_2) = (0, 0)$. Observe that

$$\nabla f(x_1, x_2) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right) = \left(\frac{3 - x_1^2 - 2x_1x_2}{(3 + x_1^2 + x_2^2 + x_1x_2)^2}, \frac{3 - x_2^2 - 2x_1x_2}{(3 + x_1^2 + x_2^2 + x_1x_2)^2}\right).$$

Then, $\nabla f(x_1, x_2) = (0, 0)$ if and only if $(3 - x_1^2 - 2x_1x_2, 3 - x_2^2 - 2x_1x_2) = (0, 0)$, thus, solving this equations, we can find two points satisfying necessary conditions for extrema. This points are $(1, 1)$ and $(-1, -1)$. Now, we are going to check the sufficient conditions to decide whether they are extrema or not. In order to do that, let us compute the Hessian matrix of the function. Remind that in this case, the Hessian matrix is as follows:

$$\begin{pmatrix} \frac{(-2x_1-2x_2)h(x_1,x_2)^2-2(3-x_1^2-2x_1x_2)h(x_1,x_2)(2x_1+x_2)}{h(x_1,x_2)^4} & \frac{-2x_1h(x_1,x_2)^2-2(3-x_1^2-2x_1x_2)h(x_1,x_2)(2x_2+x_1)}{h(x_1,x_2)^4} \\ \frac{-2x_1h(x_1,x_2)^2-2(3-x_1^2-2x_1x_2)h(x_1,x_2)(2x_2+x_1)}{h(x_1,x_2)^4} & \frac{(-2x_2-2x_1)h(x_1,x_2)^2-2(3-x_2^2-2x_1x_2)h(x_1,x_2)(2x_2+x_1)}{h(x_1,x_2)^4} \end{pmatrix},$$

where $h(x_1, x_2) = 3 + x_1^2 + x_2^2 + x_1x_2$. Since $h(1, 1) = h(-1, -1) = 6$, we have that

$$H(f)(1,1) = \begin{pmatrix} \frac{-1}{9} & \frac{-1}{18} \\ \frac{-1}{18} & \frac{-1}{9} \end{pmatrix}, \quad H(f)(-1,-1) = \begin{pmatrix} \frac{1}{9} & \frac{1}{18} \\ \frac{1}{18} & \frac{1}{9} \end{pmatrix}.$$

Notice that $H(f)(1, 1)$ is a negative definite matrix, hence, $(1, 1)$ is a local maximum. Observe also that $H(f)(-1, -1)$, is a positive definite matrix, hence, $(-1, -1)$ is a local minimum. To know when the matrices are positive definite or negative definite, we can use Sylvester criterion or we can compute the eigenvalues of the matrices. In the second case, if both eigenvalues are strictly positive, the matrix is positive definite and if both eigenvalues are strictly negative, the matrix is negative definite. □

Exercise 8.

Proof. Notice that $x = 1 - 2y$, thus, the problem to solve is the following:

$$\max_{y\in\mathbb{R}} \min\{1 - 2y, y\}.$$

Let us define $f(y) := \min\{1 - 2y, y\}$. We can describe this function as follows:

$$f(y) = \begin{cases} y & \text{if } y \leqslant \frac{1}{3} \\ 1 - 2y & \text{if } y > \frac{1}{3} \end{cases}$$

Since $\phi(y) := y$ is a linear function with positive slope and $\Phi(y) := 1 - 2y$ is also a linear function with negative slope, it is easy to see that the function f, increase from $\{-\infty\}$ to $\{\frac{1}{3}\}$ and then decreases. Hence, the maximum of this function is in $y = \frac{1}{3}$. Since $x = 1 - 2y$, the solution of our problem is $(\frac{1}{3}, \frac{1}{3})$. □

Exercise 13.

(a) Let us prove that there exist a global minimum. In order to do that, we define the function $f(x) := \sum_{j=1}^m w_k||x - y_j||$. Since norms are continuous and the sum of continuous functions are

continuous, f is continuous. Let us prove that there exist a global minimum. Let x_1 be any point in $\mathbb{R}^2$. As $f(x) \to +\infty$ when $\|x\| \to +\infty$, there exists $M > 0$ such that

$$\|x\| \geq \text{M} \Longrightarrow f(x) \geq f(x_1).$$

Therefore, the problem to find the minimum in $\mathbb{R}^2$ reduces to find the minimum in the closed ball $B(0, M) := \{x \in \mathbb{R}^n : \|x\| \leqslant M\}$ which is compact and Weierstrass theorem applies to conclude that there exist a global minimum for this function. The physics approach turns out of the definition of center of mass. The center of mass of this problem is the unique point at the center of a distribution mass in space that has the property that the weighted position vectors relative to this point sum zero, that is exactly our problem.

(b) Let us prove that the global minimum is unique. For that, we may assume that the function f has at least one global minimum. If we are able to prove that f is a convex function we are done, since a global minimum of a convex function in a convex set must be unique. First of all, we are going to prove that f is a convex function. In order to see that, let us prove that the function $g : \mathbb{R}^2 \to \mathbb{R}$ given by $g(x) := ||x||$ is convex. We have to see that for all $x, y \in \mathbb{R}^2$ and for all $\lambda \in [0, 1]$ satisfies $g(\lambda x + (1 - \lambda)y) \leqslant \lambda g(x) + (1 - \lambda)g(y)$. This comes from the triangle inequality of the norms. More precisely,

$$g(\lambda x + (1 - \lambda)y) = ||\lambda x + (1 - \lambda)y|| \leqslant ||\lambda x|| + ||(1 - \lambda)y|| = \lambda||x|| + (1 - \lambda)||y||,$$

since $1 - \lambda \geq 0$. Then, the norm is a convex function and since the finite sum of convex functions is also a convex function, we have that f is a convex function[1]. To see that, assume f, g convex functions in $\mathbb{R}^2$, let us prove that $f + g$ is also a convex function. For all $x, y \in \mathbb{R}^2$ and for all $\lambda \in [0, 1]$ we have

$$\begin{aligned}
(f + g)(\lambda x + (1 - \lambda)y) &= f(\lambda x + (1 - \lambda)y) + g(\lambda x + (1 - \lambda)y) \\
&\leqslant \lambda f(x) + (1 - \lambda)f(x) + \lambda g(x) + (1 - \lambda)g(x) \\
&= \lambda(f + g)(x) + (1 - \lambda)(f + g)(x).
\end{aligned}$$

Finally we are going to prove that if we have a convex function defined in a convex set, if it has a global minimum, it is unique. Notice that this concludes the prove, since our function is convex and the ball $B(0, M)$, the domain where the minimum is found (see *(a)*), is a convex set. To prove that fact, we may assume that we have two global minimum, x_1, x_2. Without loss of generality, assume $x_1 < x_2$, $f(x_1) = f(x_2)$ and $f(x) > f(x_1) = f(x_2)$ for all x in the domain. Since the domain is convex, for all $\lambda \in [0, 1]$, $\lambda x_1 + (1 - \lambda)x_2$ belongs to the domain and

$$f(\lambda x_1 + (1 - \lambda)x_2) \leqslant \lambda f(x_1) + (1 - \lambda)f(x_2) = \lambda f(x_1) + (1 - \lambda)f(x_1) = f(x_1),$$

which is a contradiction with $f(x) > f(x_1) = f(x_2)$ for all x in the domain.

(c) Let us consider the model from the point of view of a mechanical system, as the shown figure. Notice that if we let the system achieve the equilibrium only with the gravitational force, the system will achieve a state of minimum energy, and remain at equilibrium with x being the minimal solution of f.

[1] Notice that f is not exactly the sum of norms, which are convex functions, f is the sum of norms times positive constants. An easy computation, as the ones we have done, shows that a function that comes from multiply a positive constant by a convex function is still a convex function. Thus, at this point, we only have to proof that the finite sum of convex functions is a convex function