Exercise 5. (Karush–Kuhn–Tucker conditions)

Solve the two-dimensional problem

$$\text{minimize} \quad (x-a)^2 + (y-b)^2 + xy$$

$$\text{subject to} \quad 0 \le x \le 1, \quad 0 \le y \le 1$$

for all possible values of the scalars a and b

Solution

Rewrite the constraints in the standard form as

$$\begin{aligned} x &\ge 0, \\ 1-x &\ge 0, \\ y &\ge 0, \\ 1-y &\ge 0. \end{aligned}$$

In this way, the Lagrangian can be writen as

$$L(x,y,\lambda_1,\lambda_2,\lambda_3,\lambda_4) = (x-a)^2+(y-b)^2+xy+\lambda_1 x+\lambda_2(1-x)+\lambda_3 y+\lambda_4(1-y),$$

and the K-K-T conditions become

$$\begin{aligned} 2(x-a)+y+\lambda_1-\lambda_2 &= 0, \\ 2(y-b)+x+\lambda_3-\lambda_4 &= 0, \\ \lambda_1 x &= 0, \\ \lambda_2(1-x) &= 0, \\ \lambda_3 y &= 0, \\ \lambda_4(1-y) &= 0, \\ \lambda_1,\lambda_2,\lambda_3,\lambda_4 &\ge 0. \end{aligned}$$

Some of the last five conditions are incompatible:

$$\begin{array}{rrrr} \lambda_1 = 0, & \lambda_2 = 0, & y = 0, & y = 1 \\ \lambda_1 = 0, & x = 1, & y = 0, & y = 1 \\ x = 0, & \lambda_2 = 0, & y = 0, & y = 1 \\ x = 0, & x = 1, & \lambda_3 = 0, & y = 1 \\ x = 0, & x = 1, & \lambda_3 = 0, & \lambda_4 = 0 \\ x = 0, & x = 1, & y = 0, & \lambda_4 = 0 \\ x = 0, & x = 1, & y = 0, & y = 1 \end{array}$$

In the compatible cases, and introducing $\boldsymbol{\lambda} = (\lambda_1,\lambda_2,\lambda_3,\lambda_4)$, we have the following compatible solutions:

1. $\boldsymbol{\lambda} = (0,0,0,0)$

$$\begin{cases} 2(x-a)+y=0 \\ 2(y-b)+x=0 \end{cases} \Leftrightarrow \begin{cases} x = 2b-2y \\ 2(2b-2y)+y=2a \end{cases} \Leftrightarrow \begin{cases} x = 2b-2y \\ y = \dfrac{4b-2a}{3} \end{cases} \Leftrightarrow \begin{cases} x = \dfrac{4a-2b}{3} \\ y = \dfrac{4b-2a}{3} \end{cases}$$

a and b must satisfy

$$\begin{cases} 0 \le \dfrac{4a-2b}{3} \le 1 \\ 0 \le \dfrac{4b-2a}{3} \le 1 \end{cases} \Leftrightarrow \begin{cases} 0 \le a \le \dfrac{1}{2} \\ \dfrac{a}{2} \le b \le 2a \end{cases}$$

The solution is

$$\boxed{(x,y) = \left(\frac{4a-2b}{3}, \frac{4b-2a}{3}\right),\ 0 \le a \le \frac{1}{2},\ \frac{a}{2} \le b \le 2a}$$

2. $\boldsymbol{\lambda} = (\lambda_1,0,0,0)$

$$\begin{cases} x = 0 \\ 2(-a)+y+\lambda_1 = 0 \\ 2(y-b)=0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ \lambda_1 = 2a-b \\ y = b \end{cases}$$

a and b must satisfy

$$\begin{cases} 0 \le b \le 1 \\ \lambda_1 = 2a-b \ge 0 \end{cases} \Leftrightarrow \begin{cases} 0 \le a \le 1 \\ 2a \ge b \end{cases}$$

The solution is

$$\boxed{(x,y) = (0,b)\,,\ \boldsymbol{\lambda} = (2a-b,0,0,0),\ 0 \le b \le 1,\ 2a \ge b}$$

3. $\boldsymbol{\lambda} = (0,\lambda_2,0,0)$

$$\begin{cases} x = 1 \\ 2(1-a)+y-\lambda_2 = 0 \\ 2(y-b)+1=0 \end{cases} \Leftrightarrow \begin{cases} x = 1 \\ y = \dfrac{2b-1}{2} \\ \lambda_2 = 2-2a+\dfrac{2b-1}{2} = \dfrac{3-4a+2b}{2} \end{cases}$$

a and b must satisfy

$$\begin{cases} \lambda_2 = \dfrac{3-4a+2b}{2} \ge 0 \\ 0 \le \dfrac{2b-1}{2} \le 1 \end{cases} \Leftrightarrow \begin{cases} 3-4a+2b \ge 0 \\ 1 \le 2b \le 3 \end{cases} \Leftrightarrow \begin{cases} 2b-4a \ge 3 \\ \dfrac{1}{2} \le b \le \dfrac{3}{2} \end{cases}$$

The solution is

$$\boxed{(x,y)=\left(1,\frac{2b-1}{2}\right),\ \boldsymbol{\lambda}=\left(0,\frac{3-4a+2b}{2},0,0\right),\ 2b-4a\geq 3,\ \frac{1}{2}\leq b\leq\frac{3}{2}}$$

4. $\boldsymbol{\lambda}=(0,0,\lambda_3,0)$

$$\begin{cases} y=0 \\ 2(x-a)=0 \\ 2(-b)+x+\lambda_3=0 \end{cases} \Leftrightarrow \begin{cases} y=0 \\ x=a \\ \lambda_3=2b-a \end{cases}$$

a and b must satisfy

$$\begin{cases} 0\leq a\leq 1 \\ \lambda_3=2b-a\geq 0 \end{cases} \Leftrightarrow \begin{cases} 0\leq a\leq 1 \\ 2b\geq a \end{cases}$$

The solution is

$$\boxed{(x,y)=(a,0)\,,\ \boldsymbol{\lambda}=(0,0,2b-a,0),\ 0\leq a\leq 1,\ 2b\geq a}$$

5. $\boldsymbol{\lambda}=(0,0,0,\lambda_4)$

$$\begin{cases} y=1 \\ 2(x-a)+1=0 \\ 2(1-b)+x-\lambda_4=0 \end{cases} \Leftrightarrow \begin{cases} y=1 \\ x=a-\dfrac{1}{2} \\ \lambda_4=a-\dfrac{1}{2}+2-2b=a+\dfrac{3}{2}-2b \end{cases}$$

a and b must satisfy

$$\begin{cases} 0\leq a-\dfrac{1}{2}\leq 1 \\ \lambda_4=a+\dfrac{3}{2}-2b\geq 0 \end{cases} \Leftrightarrow \begin{cases} \dfrac{1}{2}\leq a\leq\dfrac{3}{2} \\ \dfrac{a}{2}+\dfrac{3}{4}\geq b \end{cases}$$

The solution is

$$\boxed{(x,y)=\left(a-\frac{1}{2},1\right),\ \boldsymbol{\lambda}=(0,0,0,a+\frac{3}{2}-2b),\ \frac{1}{2}\leq a\leq\frac{3}{2},\ \frac{a}{2}+\frac{3}{4}\geq b}$$

6. $\boldsymbol{\lambda}=(\lambda_1,0,\lambda_3,0)$

$$\begin{cases} x=0 \\ y=0 \\ -2a+\lambda_1=0 \\ -2b+\lambda_3=0 \end{cases} \Leftrightarrow \begin{cases} x=0 \\ y=0 \\ \lambda_1=2a \\ \lambda_3=2b \end{cases}$$

a and b must satisfy

$$\begin{cases} \lambda_1 = 2a \geq 0 \\ \lambda_3 = 2b \geq 0 \end{cases} \Leftrightarrow \begin{cases} a \geq 0 \\ b \geq 0 \end{cases}$$

The solution is

$$\boxed{(x, y) = (0, 0),\ \boldsymbol{\lambda} = (2a, 0, 2b, 0),\ a \geq 0,\ b \geq 0}$$

7. $\boldsymbol{\lambda} = (\lambda_1, 0, 0, \lambda_4)$

$$\begin{cases} x = 0 \\ y = 1 \\ -2a + 1 + \lambda_1 = 0 \\ 2 - 2b - \lambda_4 = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 1 \\ \lambda_1 = 2a - 1 \\ \lambda_4 = 2 - 2b \end{cases}$$

a and b must satisfy

$$\begin{cases} \lambda_1 = 2a - 1 \geq 0 \\ \lambda_4 = 2 - 2b \geq 0 \end{cases} \Leftrightarrow \begin{cases} a \geq \dfrac{1}{2} \\ b \leq 1 \end{cases}$$

The solution is

$$\boxed{(x, y) = (0, 1),\ \boldsymbol{\lambda} = (2a - 1, 0, 0, 2 - 2b),\ a \geq \frac{1}{2},\ b \leq 1}$$

8. $\boldsymbol{\lambda} = (0, \lambda_2, \lambda_3, 0)$

$$\begin{cases} x = 1 \\ y = 0 \\ 2 - 2a - \lambda_2 = 0 \\ -2b + 1 + \lambda_3 = 0 \end{cases} \Leftrightarrow \begin{cases} x = 1 \\ y = 0 \\ \lambda_2 = 2 - 2a \\ \lambda_3 = 2b - 1 \end{cases}$$

a and b must satisfy

$$\begin{cases} \lambda_2 = 2 - 2a \geq 0 \\ \lambda_3 = 2b - 1 \geq 0 \end{cases} \Leftrightarrow \begin{cases} a \leq 1 \\ b \geq \dfrac{1}{2} \end{cases}$$

The solution is

$$\boxed{(x, y) = (1, 0),\ \boldsymbol{\lambda} = (0, 2 - 2a, 2b - 1, 0),\ a \leq 1,\ b \geq \frac{1}{2}}$$

9. $\boldsymbol{\lambda} = (0, \lambda_2, 0, \lambda_4)$

$$\begin{cases} x = 1 \\ y = 1 \\ 2 - 2a + 1 - \lambda_2 = 0 \\ 2 - 2b + 1 - \lambda_4 = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} x = 1 \\ y = 1 \\ \lambda_2 = 3 - 2a \\ \lambda_4 = 3 - 2b \end{cases}$$

a and b must satisfy

$$\begin{cases} \lambda_2 = 3 - 2a \geq 0 \\ \lambda_4 = 3 - 2b \geq 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} a \leq \dfrac{3}{2} \\ b \leq \dfrac{3}{2} \end{cases}$$

The solution is

$$\boxed{(x, y) = (1, 1),\ \boldsymbol{\lambda} = (0, 3 - 2a, 0, 3 - 2b - 1),\ a \leq \frac{3}{2},\ b \leq \frac{3}{2}}$$