Exercise 6. (Conjugate gradient method)

Consider the conjugate gradient method applied to the minimization of

$$f(x) = \frac{1}{2}\boldsymbol{x}^T A\boldsymbol{x} - \boldsymbol{b}^T\boldsymbol{x}$$

where A is a positive definite and symmetric matrix.

Show that the iterate $\boldsymbol{x}^k$ minimizes f over

$$\boldsymbol{x}^0 + < \boldsymbol{v}^0, A\boldsymbol{v}^0, ..., A^{k-1}\boldsymbol{v}^0 >$$

where $\boldsymbol{v}^0 = \nabla f(\boldsymbol{x}^0)$, and $< v^0, A\boldsymbol{v}^0, ..., A^{k-1}\boldsymbol{v}^0 >$ is the subspace generated by $v^0, A\boldsymbol{v}^0, ..., A^{k-1}\boldsymbol{v}^0$

Solution

According to the conjugate gradient method's theorem, x^k minimizes f in the subspace $x^0 + < z^1, ..., z^k >$, so we only need to prove that

$$V^k := < v^0, Av^0, ..., A^{k-1}v^0 > = < z^1, ..., z^k > := Z^k.$$

Since the z^j are linearly independent $\dim(Z^k) = k$, and since $\dim(V^k) \leq k$ it follows that $V^k \subset Z^k$. Thus, it is sufficient to prove that $Z^k \subset V^k$. This will be proved by induction on k.

- Case $k = 1$.

  Since $z^1 = -\nabla f(x^0)$, we have

  $$Z^1 = < z^1 > = < -\nabla f(x^0) > = < v^0 > = V^1.$$

- Induction hypothesis. $Z^p \subset V^p$ for any $p < k$.

- Final statement.

  Note that $z^{p+1} = -\nabla f(x^p) + \sum_{j=1}^{p} \beta_{kj} z^j$, and that the sumation of the right hand side is in V^p (using the induction hypothesis). From the other hand

  $$\nabla f(x^p) = Ax^p - b = A(x^{p-1} + \alpha^p z^p) - b = Ax^{p-1} - b + \alpha^p Az^p = \nabla f(x^{p-1}) + \alpha^p Az^p.$$

  By the induction hypothesis, it follows that $\alpha^p Az^p \in V^{p+1}$. Iterating the process (expanding $\nabla f(x^{p-1})$,...) we finally get

  $$\nabla f(x^p) = \nabla f(x^0) + p, \quad \text{with} \quad p \in V^{p+1} \quad \Rightarrow \quad \nabla f(x^p) \in V^{p+1}.$$

  So $z^{p+1} \in V^{p+1}$ and, consequently, $Z^{p+1} \subset V^{p+1}$.