Exercise 4. (Quadratic approximation)

Let f be a real function on $\mathbb{R}^n$. Also let $\boldsymbol{x}_0 \in \mathbb{R}^n$, $\boldsymbol{z} \in \mathbb{R}^n$, and $\theta \in \mathbb{R}$. Define

$$F(\theta) = f(\boldsymbol{x}_0 + \theta \boldsymbol{z})$$

and suppose that we are looking for the minimum of F (that is, for the minimum of f in the direction $\boldsymbol{z}$ through the point $\boldsymbol{x}_0$). Let $\boldsymbol{x}_0+\theta_1\boldsymbol{z}$, $\boldsymbol{x}_0+\theta_2\boldsymbol{z}$ and $\boldsymbol{x}_0+\theta_3\boldsymbol{z}$ be three points where f is evaluated. Show that the minimum predicted by applying the quadratic approximation method is $\boldsymbol{x}_0 + \theta^*\boldsymbol{z}$, where

$$\theta^* = \frac{[\theta_2^2 - \theta_3^2]F(\theta_1) + [\theta_3^2 - \theta_1^2]F(\theta_2) + [\theta_1^2 - \theta_2^2]F(\theta_3)}{2[(\theta_2 - \theta_3)F(\theta_1) + (\theta_3 - \theta_1)F(\theta_2) + (\theta_1 - \theta_2)F(\theta_3)]}$$

and it is indeed the minimum of the parabola passing through the above three points if

$$\frac{(\theta_2 - \theta_3)F(\theta_1) + (\theta_3 - \theta_1)F(\theta_2) + (\theta_1 - \theta_2)F(\theta_3)}{(\theta_2 - \theta_3)(\theta_3 - \theta_1)(\theta_1 - \theta_2)} < 0$$

Solution

The minimum of the quadratic real valued function

$$F(\theta) = f(\boldsymbol{x}_0 + \theta \boldsymbol{z}) = a + b\theta + c\theta^2,$$

will be achieved at a value θ^* such that $F'(\theta^*) = 0$, so $\theta^* = -b/(2c)$, and $F''(\theta^*) = c > 0$.

Using three different points θ_1, θ_2 and θ_1 we can compute the values of the coefficients a, b and c of (θ) solving the linear system

$$\begin{pmatrix} 1 & \theta_1 & \theta_1^2 \\ 1 & \theta_2 & \theta_2^2 \\ 1 & \theta_3 & \theta_3^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} F(\theta_1) \\ F(\theta_2) \\ F(\theta_3) \end{pmatrix},$$

whose solution is given by

$$\begin{aligned}
a &= \frac{(\theta_2^2\theta_3 - \theta_2\theta_3^2)F(\theta_1) + (\theta_1\theta_3^2 - \theta_1^2\theta_3)F(\theta_2) + (\theta_1^2\theta_2 - \theta_1\theta_2^2)F(\theta_3)}{(\theta_2 - \theta_3)(\theta_3 - \theta_1)(\theta_1 - \theta_2)}, \\
b &= \frac{(\theta_2^2 - \theta_3^2)F(\theta_1) + (\theta_3^2 - \theta_1^2)F(\theta_2) + (\theta_1^2 - \theta_2^2)F(\theta_3)}{(\theta_2 - \theta_3)(\theta_3 - \theta_1)(\theta_1 - \theta_2)}, \\
c &= -\frac{(\theta_2 - \theta_3)F(\theta_1) + (\theta_3 - \theta_1)F(\theta_2) + (\theta_1 - \theta_2)F(\theta_3)}{(\theta_2 - \theta_3)(\theta_3 - \theta_1)(\theta_1 - \theta_2)},
\end{aligned}$$

The minimum of $F(\theta)$ is achieved at

$$\theta^* = -\frac{b}{2c} = \frac{1}{2}\frac{(\theta_2^2 - \theta_3^2)F(\theta_1) + (\theta_3^2 - \theta_1^2)F(\theta_2) + (\theta_1^2 - \theta_2^2)F(\theta_3)}{(\theta_2 - \theta_3)F(\theta_1) + (\theta_3 - \theta_1)F(\theta_2) + (\theta_1 - \theta_2)F(\theta_3)},$$

and the $c > 0$ condition becomes

$$\frac{(\theta_2 - \theta_3)F(\theta_1) + (\theta_3 - \theta_1)F(\theta_2) + (\theta_1 - \theta_2)F(\theta_3)}{(\theta_2 - \theta_3)(\theta_3 - \theta_1)(\theta_1 - \theta_2)} < 0.$$