Probability overview

Probabilistic Graphical Models

Jerónimo Hernández-González

What does all this have to do with function approximation?

instead of $F : X \to Y$,

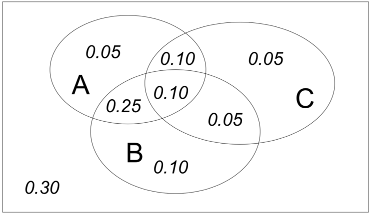
learn $P(Y|X)$

Joint distribution

Recipe for making a joint distribution of M variables:

1. Make a truth table listing all possible combinations of values
   (M variables $\rightarrow 2^M$ combinations)
2. Say how probable each combination is
   Subscribed to the axioms of probability if sum to 1

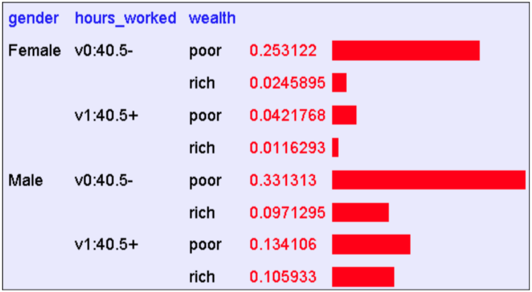
A	B	C	Prob
0	0	0	0.30
0	0	1	0.05
0	1	0	0.10
0	1	1	0.05
1	0	0	0.05
1	0	1	0.10
1	1	0	0.25
1	1	1	0.10



Inference with the joint distribution

You can ask for the probability of any logical expression involving your attribute

$$P(E) = \sum_{\text{r: rows matching } E} P(r)$$



gender	hours_worked	wealth		
Female	v0:40.5-	poor	0.253122	
		rich	0.0245895	
	v1:40.5+	poor	0.0421768	
		rich	0.0116293	
Male	v0:40.5-	poor	0.331313	
		rich	0.0971295	
	v1:40.5+	poor	0.134106	
		rich	0.105933	

Inference with the joint distribution

You can ask for the probability of any logical expression involving a subset of attributes given another expression involving other attributes

$$P(E_1|E_2) = \frac{P(E_1 \wedge E_2)}{P(E_2)}$$

$$= \frac{\sum_{\text{r: rows matching } E_1 \text{ \& } E_2} P(r)}{\sum_{\text{o: rows matching } E_2} P(o)}$$

gender	hours_worked	wealth	
Female	v0:40.5-	poor	0.253122
		rich	0.0245895
	v1:40.5+	poor	0.0421768
		rich	0.0116293
Male	v0:40.5-	poor	0.331313
		rich	0.0971295
	v1:40.5+	poor	0.134106
		rich	0.105933

Learning and the joint distribution

Suppose we want to learn the function $f :< G, H >\rightarrow W$

Equivalently, $P(W|G, H)$

Solution:

- Learn joint distribution from train data
- Calculate $P(W|G, H)$ for test data

  E.g., given a *female* patient of 39 years old:

  $\arg\max_{w \in \{rich, poor\}} P(W = w|G = female, H = 40{,}5-)$

Solution?

$P(Y|X)$ sounds like a nice alternative solution to function $F : X \to Y$

Are we done?

Main problem

Learning $P(Y|X)$ may require more data than we have

E.g., consider learning the joint distribution for 100 binary variables

- # of rows in this table?
- # of data samples to learn faithfully?
- # of rows never observed?

Facing practical problems

What to do?

1. Be smart about how to represent joint distributions
   - Bayesian networks, probabilistic graphical models
2. Be smart about how we estimate probabilities from *sparse* data
   - maximum likelihood estimates
   - maximum a posteriori estimates

Facing practical problems

What to do?

1. **Be smart about how to represent joint distributions**
   - Bayesian networks, probabilistic graphical models
2. Be smart about how we estimate probabilities from *sparse* data
   - maximum likelihood estimates
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Conditional independence

A *qualitative* relationship between random variables

Let A, B, C be disjoint subsets of $V = \{1, ..., v\}$. We say that $\boldsymbol{X}_A$ is independent from $\boldsymbol{X}_B$ given $\boldsymbol{X}_C$ if and only if for all $(\boldsymbol{x}_A, \boldsymbol{x}_B, \boldsymbol{x}_C)$ we have that $p(\boldsymbol{x}_A|\boldsymbol{x}_B, \boldsymbol{x}_C) = p(\boldsymbol{x}_A|\boldsymbol{x}_C)$.

- Denoted by $X_A \perp\!\!\!\perp X_B|X_C$
- $p(\boldsymbol{x}_A|\boldsymbol{x}_B, \boldsymbol{x}_C) = p(\boldsymbol{x}_A|\boldsymbol{x}_C)$: Knowing/observing/fixing $\boldsymbol{x}_C$, the value $\boldsymbol{x}_B$ does not modify the probability of $\boldsymbol{x}_A$
- Exercise: Prove that
  $X_A \perp\!\!\!\perp X_B|X_C \Rightarrow p(\boldsymbol{x}_A, \boldsymbol{x}_B|\boldsymbol{x}_C) = p(\boldsymbol{x}_A|\boldsymbol{x}_C) \cdot p(\boldsymbol{x}_B|\boldsymbol{x}_C)$

Conditional independence

- Allow to simplify the factorization given by the chain rule
- Choose an appropriate ordering that allows to apply the independence over a conditional distribution

Example

- $\boldsymbol{X} = X_1, ..., X_5$
- $3 \perp\!\!\!\perp 4 | 1, 5$
- Ordering: $1, 4, 5, 3, 2$

$$
\begin{aligned}
p(\boldsymbol{X}) &= p(\boldsymbol{X}_{1,4,5})p(X_3|\boldsymbol{X}_{1,4,5})p(X_2|\boldsymbol{X}_{1,3,4,5}) \\
&= p(\boldsymbol{X}_{1,4,5})p(X_3|\boldsymbol{X}_{1,5})p(X_2|\boldsymbol{X}_{1,3,4,5})
\end{aligned}
$$

Counting the parameters

Marginal distribution

Let A and B be a partition of V. Then,

$$\forall \boldsymbol{x}_A, \boldsymbol{x}_B, p(\boldsymbol{x}_A) = \sum_{\boldsymbol{x}_B} p(\boldsymbol{x}_A, \boldsymbol{x}_B)$$

- Number of free parameters: $\prod_{i \in A} r_i - 1 = r_A - 1$

Counting the parameters

Conditional distribution

Let A and B be a partition of V

$$\forall \boldsymbol{x}_A, p(\boldsymbol{x}_A | \boldsymbol{x}_B) = \frac{p(\boldsymbol{x}_A, \boldsymbol{x}_B)}{p(\boldsymbol{x}_B)} = \frac{p(\boldsymbol{x}_A, \boldsymbol{x}_B)}{\sum_{\boldsymbol{x}_A} p(\boldsymbol{x}_A, \boldsymbol{x}_B)}$$

- Family of distribution:
  A marginal distribution for each value assignment $\boldsymbol{X}_B = \boldsymbol{x}_B$
  $\sum_{\boldsymbol{x}_A} p(\boldsymbol{x}_A | \boldsymbol{x}_B) = 1$
- Number of free parameters:
  $(\prod_{i \in A} r_i - 1) \cdot (\prod_{j \in B} r_j) = (r_A - 1) \cdot r_B$

Counting the parameters

Conditional independences reduce the number of (free) parameters

- $X_A \perp\!\!\!\perp X_B$
  - $\forall \boldsymbol{x}$: $p(\boldsymbol{x}_A, \boldsymbol{x}_B) = p(\boldsymbol{x}_A)p(\boldsymbol{x}_B)$
  - From $r_A \cdot r_B - 1$ to $r_A - 1 + r_B - 1$

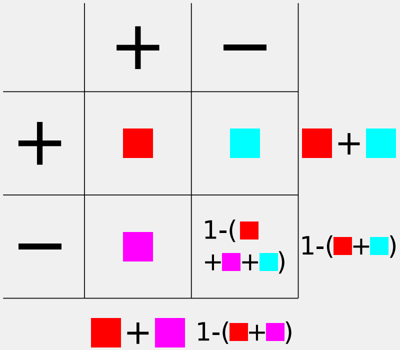
    E.g., $r_A = 3$, $r_B = 4$
    From $3 \times 4 - 1$ to $2 + 3$

- $X_A \perp\!\!\!\perp X_B | X_C$
  - $\forall \boldsymbol{x}$: $p(\boldsymbol{x}_A, \boldsymbol{x}_B | \boldsymbol{x}_C) = p(\boldsymbol{x}_A | \boldsymbol{x}_C)p(\boldsymbol{x}_B | \boldsymbol{x}_C)$
  - From $(r_A \cdot r_B - 1) \cdot r_C$ to $(r_A - 1 + r_B - 1) \cdot r_C$

    E.g., $r_A = 3$, $r_B = 4$, $r_C = 3$
    From $(3 \times 4 - 1) \times 3$ to $(2 + 3) * 3$

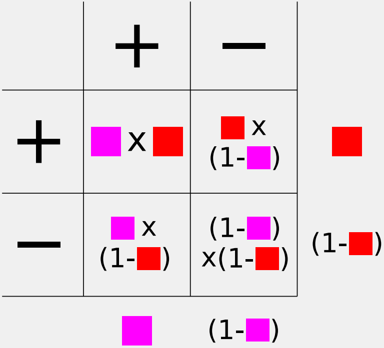
Counting the parameters, without independence

$p(A, B)$: three parameters



Counting the parameters, with independence

$p(A, B)$: two parameters



Conditional independence

Discarding parameters

The complexity of a statistical model can be understood as its flexibility for learning or its requirement of memory

Conditional independences:

- Reduce the complexity (i.e., # parameters) of the statistical model represented by the (simplified) chain rule
- Allow for avoiding to model (irrelevant) parameters associated to soft conditional dependences
- Help to deal with the trade-off between the complexity of the statistical model and amount of train data
  This has crucial implications in statistical models, e.g., overfitting

Exercise

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MENÚ

Actualidad sanitaria, Portada, Salud Pública, Sanidad 5 de noviembre de 2021

El 84 por ciento de la población canaria de más de 12 años está ya inmunizado contra la COVID-19

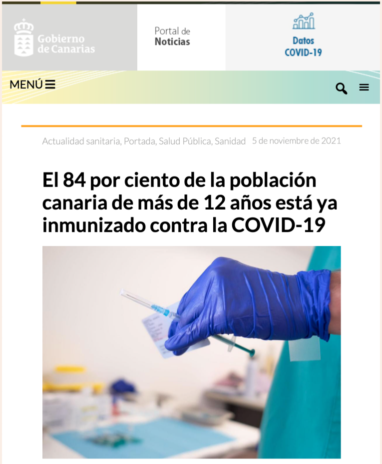
SOCIEDAD ED+

El 57% de los ingresados por coronavirus en Canarias está sin vacunar y no sufre patologías previas

Yanira Martín

- Can you tell which is the probability of ending up in the hospital if you get COVID19 and you are not vaccinated?

Exercise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- Can you tell which is the probability of ending up in the hospital if you get COVID19 and you are not vaccinated?
- Can say anything?

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