Predicting Network Latency with Graph-Inspired Models

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1 Introduction

This work addresses the problem of predicting the communication latency between two network locations, or addresses, using machine learning techniques. This task is an essential component of matchmaking applications with minimal latency. Unlike existing methods that only generalize to unseen pairs from a pool of known addresses, typically by positioning each address in a network map, we also wish to generalize to *unseen addresses*. This task is inherently very noisy and while the obtained relative errors can be as high as 50%, our objective for matchmaking purposes is twofold: (1) to correctly predict the order of magnitude of a given connection latency, and (2) to correctly determine the fastest of two candidate connections.

Our basic approach is to model the Internet network as an undirected graph where the shortest distance between two nodes is an estimate of the latency between those locations. While we could in principle learn both graph connectivity and edge lengths, in this work we will derive connectivity from available geographical information and learn only the edge lengths.

This report extends a previous version from April 22nd, 2013, by Nicolas Boulanger-Lewandowski.

2 Model

2.1 Tree model

Our basic assumption is that the network topology corresponds to a graph with several interconnected hubs (highly connected nodes) and that the latency between two network locations can be approximated by the distance of the shortest path between two nodes. Intuitively, the edges will have non-negative weights but this constraint will be lifted later.

In the basic model (Figure 1), we further assume that graph connectivity is known and corresponds to a hierarchy of geographical entities (country, region and city). Since the resulting graph is a tree, there is a unique path between two nodes and it is easy to learn the edge lengths to minimize the squared error C by solving a non-negative linear system:

$$C \equiv \sum_{j=1}^{N} (t_j - y_j)^2 \tag{1}$$

for N training examples with targets t_j, $1 \leq j \leq N$ with

$$y = M \cdot l \tag{2}$$

the predictions of the model and $M_{ji} = 1$ if edge i is part of the path linking location pair j and 0 otherwise.

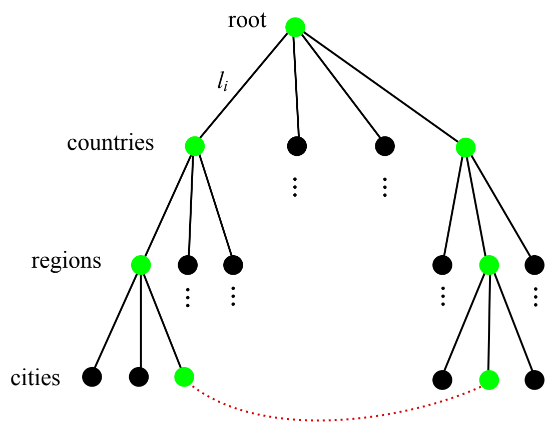


Figure 1: The basic graphical model is a tree of depth 3 with edges $l_i \geq 0$ representing the latency between network hubs. The overall latency between two addresses is the distance between two terminal nodes (path in green). Shortcut edges (red) are added in the extended version.

2.2 Extended model

We now extend the basic tree model to have the following properties:

1. It should be possible to add direct edges, or shortcuts, between important hubs to bypass the default hierarchy (e.g., red edge in Figure 1).

2. The prediction should degrade gracefully with sparse data, e.g., a city unseen during training should inherit the average behavior of its parent region or country.

3. The model should also incorporate additional information such as physical distance, connection medium or full IP address.

4. The model should be robust to outliers, in particular unusually long delays.

The strategy is to express the predictions as a regularized sum of K terms with weight decay coefficients λ_k and each term will be chosen to represent an aspect (feature) of the input address pair. To limit the weight of outliers, we minimize the *absolute error*, instead of the squared error used in Equation 2[1].

Equations (1-2) become:

$$C \equiv \sum_{j=1}^{N} |t_j - y_j| + \sum_{k=1}^{K} \lambda_k \left\| l^{(k)} \right\|^2 \tag{3}$$

[1]The model in the previous report used the squared error, or L_2 norm.

$$y = \sum_{k=1}^{K} M^{(k)} \cdot l^{(k)} \tag{4}$$

where $M^{(k)}_{j:}$ represents the k-th feature vector of the j-th example, e.g., a one-hot vector representing the city of the host address, a one-hot vector representing the country combination of both addresses, or a real value giving the physical distance between the two locations. It is easy to see that the basic tree model of Figure 1 can be recovered by this framework simply by using $K = 7$ terms: `city1`, `region1`, `country1`, `country2`, `region2`, `city2` and a constant. It is also possible to use second-order terms such as `(country1, country2)` or `(city1, city2)` to add direct edges in the graph, satisfying requirement 1 above, or other miscellaneous information fulfilling requirement 3 above.

The regularization coefficient λ_k will effectively diminish the salience of a feature rarely present in the data (that could be spurious) by demanding that each $l_i^{(k)}$ affects a large number of training examples (equation 3). For example, the correction due to a `(city1, city2)` term will be weighted by

$$(1 + \lambda_k/n)^{-1} \tag{5}$$

for n occurences of a given city pair during training due to the regularization. Since corrections associated with sparse data will tend to be small, the model will naturally fall back to features with a lot of training data (e.g., first-order terms), satisfying requirement 2 above. It is important not to constrain the $l_i^{(k)}$ to be non-negative in this context because corrections to a coarse predictor may go in either direction.

Minimizing equation (3) involves solving a huge sparse linear system with $N \cdot n_f$ entries, where the number of training examples $N \simeq 10^6$ to 10^8 and the number of input features $n_f \simeq 10^{10}$ in our experiments.

A more efficient approach is to optimize each successive term $l^{(k)}$ in a greedy fashion by keeping the previous ones fixed, which makes each $M^{(k)}$ in reduced row echelon form and readily invertible. This greedy approximation becomes exact as $\lambda_k \to 0$, eliminates the underdetermination associated with subpopulation terms (e.g. `city1` $\subset$ `country1`) and helps generalization. This approach also makes it easier to select the hyperparameters λ_k and to adaptively learn the parameters $l^{(k)}$ in an online setting.

3 Experiments

3.1 Error metrics

In the following experiments, we will report results using the following metrics, where, for example j, t_j is the target (actual measured delay), and y_j is the predicted delay.

We report L_1 *and* L_2 *errors:*

$$L_p = \sqrt[p]{\frac{1}{N}\sum_{j=1}^{N} |t_j - y_j|^p}, \tag{6}$$

the relative error:

$$RE = \frac{1}{N}\sum_{j=1}^{N} \frac{|t_j - y_j|}{t_j}, \tag{7}$$

the *confusion matrix* C:

$$C_{ab} = \sum_{j=1}^{N} \begin{cases} 1 & \text{if } a = c(t_j), b = c(y_j) \\ 0 & \text{otherwise} \end{cases}, \tag{8}$$

where $c(\cdot)$ denotes the class label of a given latency value as defined in Table 5, or the row-normalized version:

$$C'_{ab} = \frac{C_{ab}}{\sum_{b'} C_{ab'}}, \tag{9}$$

the *classification accuracy*:

$$CA = \frac{\mathrm{Tr}(C)}{N}. \tag{10}$$

We also report a few measurements of *ordering accuracy*, where we define the ordering accuracy for two examples j and j' as:

$$OA(j, j') = \begin{cases} 1 & \text{if } t_j > t_{j'} \text{ and } y_j > y_{j'} \\ 1 & \text{if } t_j < t_{j'} \text{ and } y_j < y_{j'} \\ 1 & \text{if } t_j = t_{j'} \\ \frac{1}{2} & \text{if } t_j \neq t_{j'} \text{ and } y_j = y_{j'} \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

The ordering accuracy corresponds to the probability of choosing successfully, between two address pairs, which has the smallest latency, and is an important indicator of the usefulness of a model in matchmaking applications. When both pairs have exactly the same latency (third case), we consider both predictions as successes. When both predictions are the same (fourth case), we use a score of $\frac{1}{2}$, so the measure is symmetrical[2].

In particular, we will report the ordering accuracy averaged over all pairs of data:

$$OA_{\text{Pairs}} = \frac{2}{N(N-1)} \sum_{j=1}^{N} \sum_{j'>j}^{N} OA(j, j'), \tag{12}$$

which can also be defined class-wise, i.e., the sum for OA_{ab} being only over j, j' indices for which $a = c(t_j), b = c(t_{j'})$, and properly normalized. In practice, to avoid considering all possible pairs, we estimate this cost by sampling 500,000 pairs at random.

In order to reflect the need for a matchmaking application to choose the fastest destination IP for a given source IP, we will also compute the ordering accuracy of all (j, j') pairs sharing the same source IP, and report their mean (where each origin IP has the same weight, denoted "ordering accuracy normalized by source IP", or "OA/IP").

[2] The previous report used a slightly different definition of the ordering accuracy.

3.2 Data Sets

3.2.1 iOS Games

The iOS Games data set (or "iOS") consists of ICMP round-trip times between wireless peers (Wi-Fi, 3G or other) collected during iOS games, from May 2012 to June 2013.

The original data was filtered to remove unsuccessful ping attempts, duplicates (examples where the origin IP, the destination IP and the measured delay were exactly the same) and remove examples with delays greater than 5000 ms, giving $N = 4,146,598$ examples. The data, still ordered chronologically, was then split into training, validation and test sets using a 8:1:1 ratio[3].

3.2.2 Assassin's Creed

The "Assassin's Creed" (or "AC") data set is composed of data collected from two games: *Assassin's Creed Revelations* (2,086,908 samples collected between November 2011 and March 2013) and *Assassin's Creed III* (1,439,458 samples collected between October 2012 and March 2013).

Each of these samples contain ping measurements from both ends of the connection. After creating two training samples for each of the original samples (one for each direction) and removing duplicates, the data from each source was split independently into training, validation and test using the same 8:1:1 ratio. These sets are then combined to form the final training, validation and test set, for a total of $N = 6,533,214$ examples.

3.2.3 Internet Census 2012 ICMP

This data set was extracted from the Internet Census 2012 ICMP Ping data[4], collected from compromised embedded systems on the Internet (for instance, routers).

Only the successful attempts with a delay of 5000 ms or less were kept. The $N = 74,679,517$ samples were ordered chronologically, and split between training, validation, and test sets with a 8:1:1 ratio.

3.3 Baseline Models

We compare our graphical model to other baselines:

- a constant prediction,
- a linear regression on the physical distance obtained via IP2Location, and
- predicting the average latency between the two communicating countries.

4 Results

4.1 Absolute and Relative Errors

Table 1 shows the $K = 21$ terms we have retained in our extended model, and their contribution to the test L_1 error, on the different data sets. For each term, λ_k is chosen so as to optimize the validation L_1 error using the downhill simplex algorithm with initial value $\lambda_{k0} = 10$.

Due to the large size of the Internet Census 2012 data set, it was not possible to perform the regression using the `(ip1, ip2)` pairs.

Tables 2 to 4 present the performance of the evaluated models in the latency prediction task on the different data sets described in Section 3.2. The proposed model clearly outperforms other baselines on all data sets.

While the obtained $L_{1,2}$ and relative errors remain relatively high, the classification accuracy (53.5% to 61.2%) and ordering accuracy between all pairs (around 75%) indicate that the graphical model predictions lie in a reasonable range and suffice to determine the fastest of two addresses in the majority of cases.

The ordering accuracy normalized by IP is lower than when it is computed among all pairs, but still reasonable on iOS (69.3%) and Assassin's Creed (68.2%), but surprisingly low (50.4%, with no discriminative power) on IC2012. Further measurements and analysis of ordering accuracy are presented in Section 4.3.

4.2 Latency Classes and Confusion Matrix

Table 5 indicates the ranges used to define the latency classes. Surprisingly, the L_1 and L_2 errors do not vary much with the magnitude of the target. This effect is consistent across all three data sets. This could indicate an intrinsic noise to the measurement process that is added to a more deterministic network-specific baseline latency.

The ability of the graphical model to correctly identify the order of magnitude of latency can also be measured in the confusion matrix, displayed in Figure 2. The model is able to capture the correlation, especially for classes 2 to 4, on all data sets. Really small values (class 1, less than 100 ms), however, are usually predicted as class 2. The same phenomenon arises for larger values: classes 5 and 6 are often mapped to class 4. However, these classes represent a small proportion of the data, and it is possible that most of the large values are outlier values for two addresses that usually have a more reasonable delay. This would also explain why the confusion matrix is worse on the Assassin's Creed data set, as most of its data consists of class 2, and even classes 3 and 4 have a really small number of examples.

4.3 Ordering Accuracy

In a matchmaking scenario, it is more important to be able to discriminate between latency pairs when the difference between both is bigger: a difference of a few milliseconds does not matter as much. Figures 3 to 5 show the ordering accuracy when we only consider pairs of test examples where the difference in targets is bigger than some threshold. The different thresholds used are used as the x axis. Figures on the left show the ordering accuracy between random pairs, figures on the right show the ordering accuracy normalized by source IP.

[3]The previous report used data collected until August 2012 only, without filtering out duplicates, and used a random (rather than chronological) split.

[4]`http://internetcensus2012.bitbucket.org/download.html`

L_1 (ms)			Added Term
iOS	AC	IC2012	
169.4	71.3	120.0	constant
155.4	55.3	120.0	distance
155.4	55.3	120.0	type1
155.4	55.3	120.0	type2
155.4	55.3	120.0	(type1, type2)
141.4	53.8	91.9	country1
133.5	53.5	91.8	country2
133.1	53.0	91.8	same_country
127.1	50.4	91.5	(country1, country2)
127.1	50.4	91.5	(country1, country2, type1, type2)
126.4	50.1	90.1	region1
126.2	50.1	89.6	region2
126.1	49.8	89.6	same_region
125.3	49.8	89.3	(region1, region2)
125.2	49.7	88.8	city1
125.0	49.7	88.7	city2
125.0	49.3	88.7	same_city
125.0	49.4	88.3	(city1, city2)
124.9	49.4	84.7	ip1
124.7	49.4	84.7	ip2
124.7	49.4	-	(ip1, ip2)

Table 1: Evolution of the test L_1 error with each term added to our graphical model, for the different data sets used.

Model	L_1 (ms)	L_2 (ms)	Rel Err	Class Acc	OA/Pairs	OA/IP
Constant prediction	169.4	283.7	68.3%	31.3%	50.1%	50.1%
IP2Location	155.4	283.4	62.8%	33.1%	65.4%	62.0%
Country pairs	127.9	268.5	51.7%	52.0%	74.4%	68.5%
Graphical model	**124.7**	**264.8**	**50.5%**	**54.0%**	**75.7%**	**69.3%**

Table 2: Prediction performance on the iOS data set, compared with the baseline models.

Model	L_1 (ms)	L_2 (ms)	Rel Err	Class Acc	OA/Pairs	OA/IP
Constant prediction	71.3	133.3	59.4%	49.0%	50.2%	50.4%
IP2Location	55.3	117.2	46.0%	53.0%	73.7%	66.9%
Country pairs	54.2	117.2	45.2%	54.4%	69.2%	64.0%
Graphical model	**49.4**	**112.9**	**41.1%**	**61.2%**	**74.8%**	**68.2%**

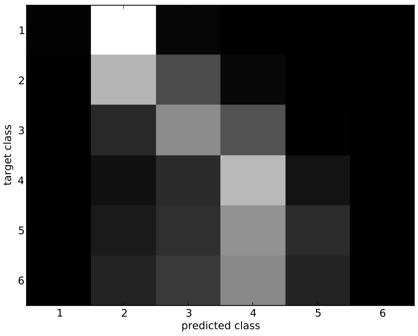
Table 3: Prediction performance on the Assassin's Creed data set, compared with the baseline models.

Model	L_1 (ms)	L_2 (ms)	Rel Err	Class Acc	OA/Pairs	OA/IP
Constant prediction	120.1	172.5	67.2%	25.6%	50.1%	50.4%
IP2Location	120.1	172.5	67.1%	25.6%	51.3%	50.4%
Country pairs	91.4	151.8	51.1%	51.6%	72.1%	50.4%
Graphical model	**84.7**	**143.8**	**47.4%**	**53.5%**	**74.7%**	50.4%

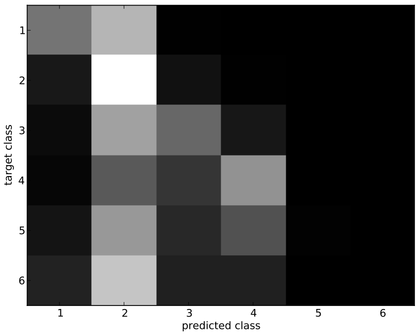
Table 4: Prediction performance on the Internet Census 2012 data set, compared with the baseline models.

Class	Range (ms)	iOS L_1 (ms)	iOS L_2 (ms)	AC L_1 (ms)	AC L_2 (ms)	IC2012 L_1 (ms)	IC2012 L_2 (ms)
1	0–99	124.4	266.8	49.4	114.5	84.7	144.4
2	100–199	125.1	266.7	49.5	114.1	84.5	143.5
3	200–299	125.0	266.2	49.3	106.9	84.7	143.7
4	300–499	124.7	264.6	48.7	110.5	84.7	144.0
5	500–999	123.9	260.1	50.5	103.8	84.9	143.4
6	1000–5000	121.8	259.4	49.5	110.7	84.9	141.9

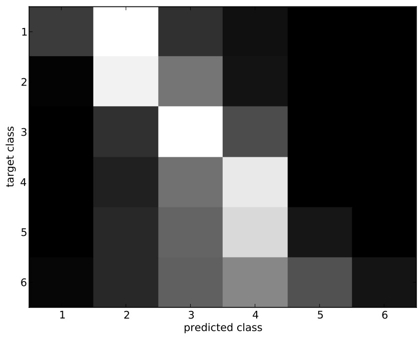
Table 5: Definition of the latency classes and their associated L_1 and L_2 errors.



(a) iOS Games



(b) Assassin's Creed



(c) Internet Census 2012

Figure 2: Row-normalized confusion matrices C'_{ab} (row: target class $c(t_j)$, column: predicted class $c(y_j)$).

The peak for a threshold of 0, observed in particular for the constant predictor (bringing their accuracy to more than 50%) is due to fact that we consider both predictions correct when the targets are equal (case 4 of Equation 11), which cannot happen when there is a minimal delay.

As we already saw in Tables 2 to 4, the ordering accuracy normalized by source IP is lower than when considering all pairs. One explanation for that is that this task is harder, because for each comparison, the origin IP is the same, so none of the factors related to the source only (`type1`, `country1`, etc.) are actually useful. This is consistent with the fact that, on IC2012, where this effect is the strongest, these are the only factors that contribute significantly to the minimization of the L_1 error (see Table 1).

4.4 Qualitative Assessments

A qualitative assessment of the model performance and predictions can be obtained by plotting a few predictions randomly chosen from the test set (Figure 6). Again, the model correctly predicts the order of magnitude of the latency most of the time. We also see, on the right side of these plots, the distribution of higher-latency samples, that seem to be distributed randomly, regardless of the prediction.

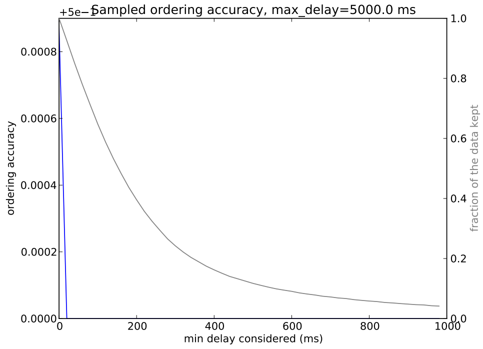
Finally, plots showing the correlation of the targets with the predictions of different models can be found in Figures 7 to 9.

4.5 Other Considerations

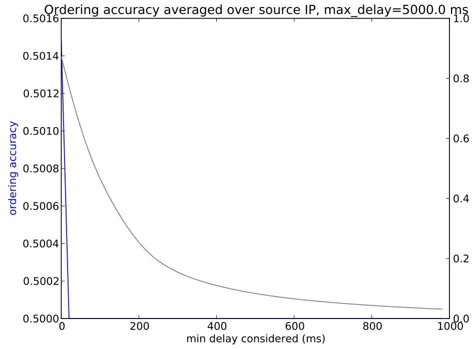
When performing preliminary experiments, which are not thoroughly reported here, we came across the following conclusions, that can be important to keep in mind when collecting data and using this model.

The GeoIP data associated to an IP can change over time. For instance, the GeoLite Database[5] reports a loss

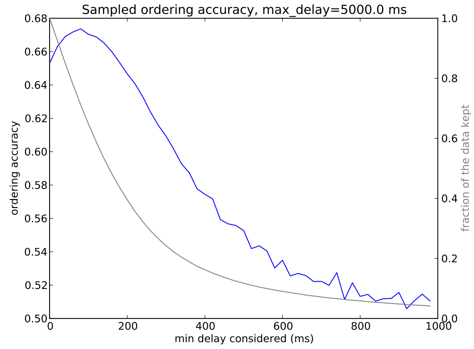
[5] `http://dev.maxmind.com/geoip/legacy/geolite/`



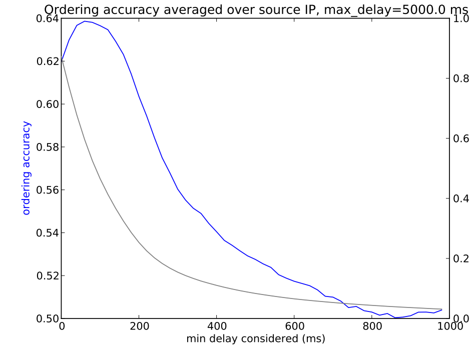
(a) Constant prediction, OA



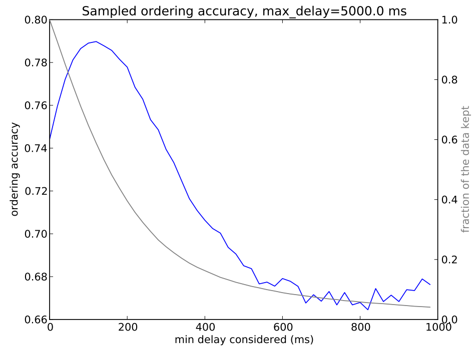
(b) Constant prediction, OA/IP



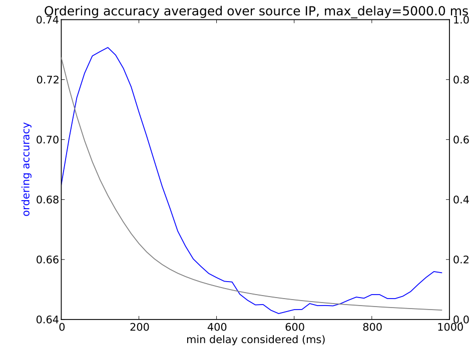
(c) IP2Location, OA



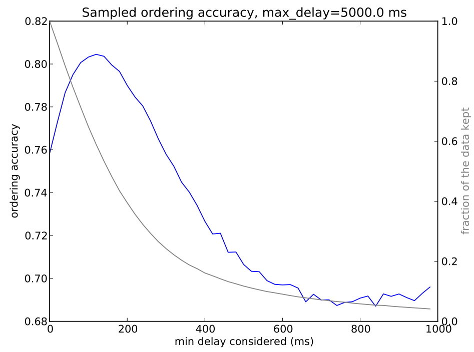
(d) IP2Location, OA/IP



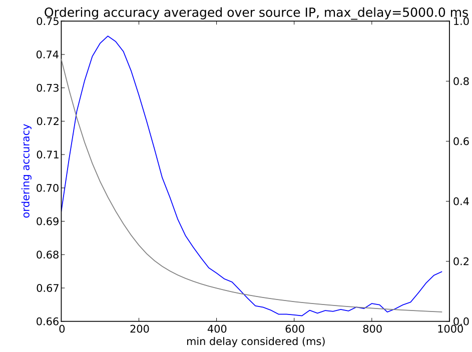
(e) Country pairs, OA



(f) Country pairs, OA/IP

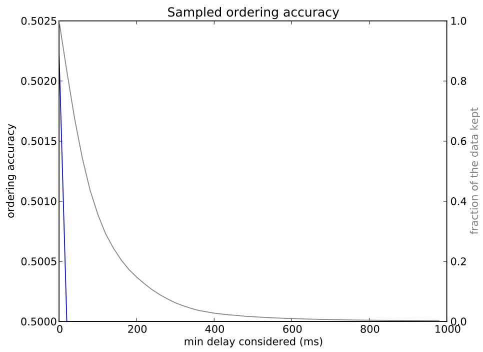


(g) Graphical model, OA

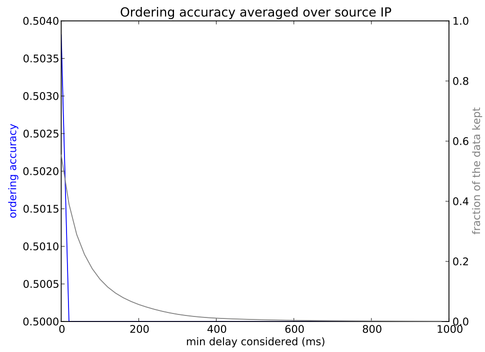


(h) Graphical model, OA/IP

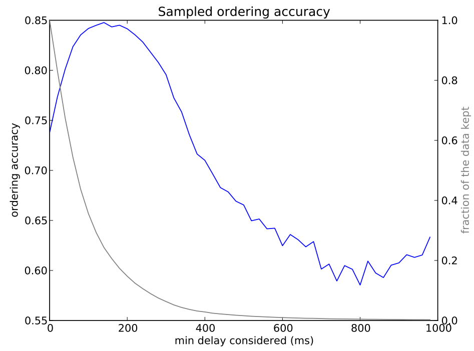
Figure 3: Ordering accuracy (blue curve) between pairs of examples from the **iOS Games** test set, considering only pairs for which the difference in delays is over some threshold. That threshold corresponds to the x axis. The gray curve shows the proportion of data actually considered.



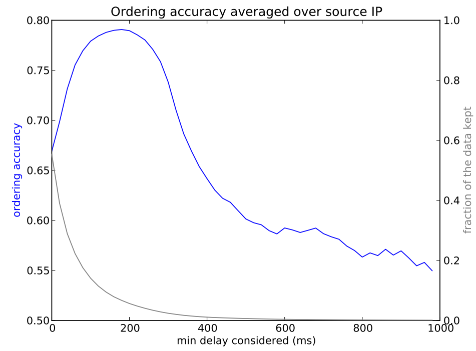
(a) Constant prediction, OA



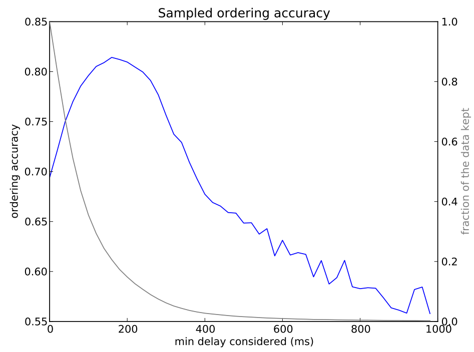
(b) Constant prediction, OA/IP



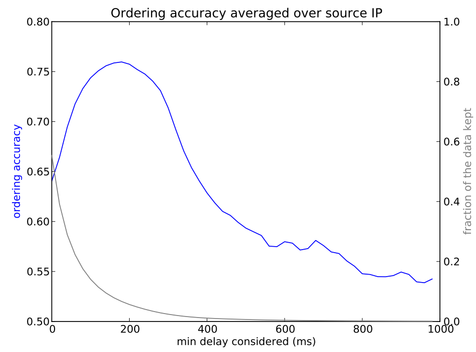
(c) IP2Location, OA



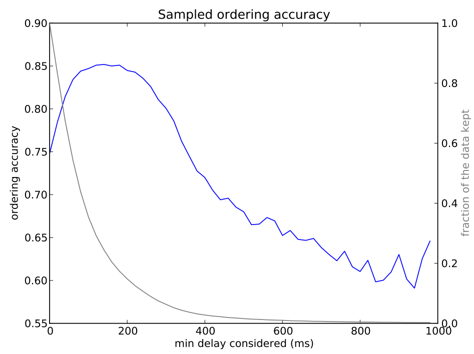
(d) IP2Location, OA/IP



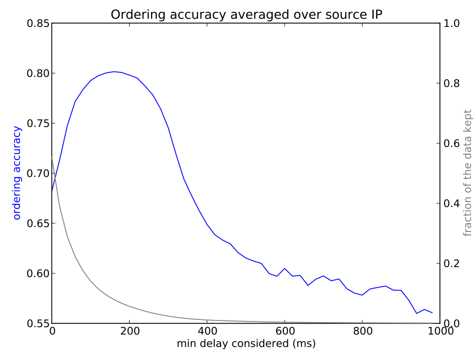
(e) Country pairs, OA



(f) Country pairs, OA/IP

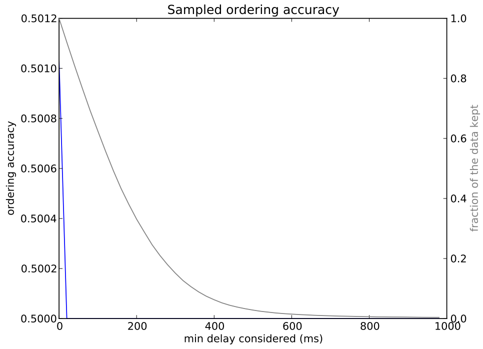


(g) Graphical model, OA

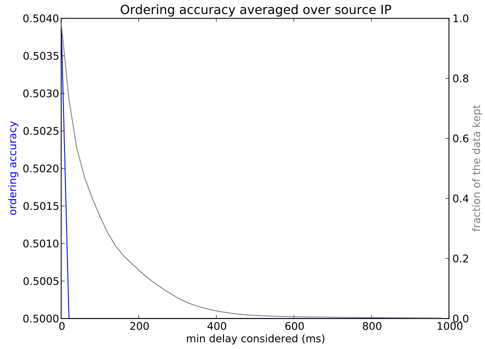


(h) Graphical model, OA/IP

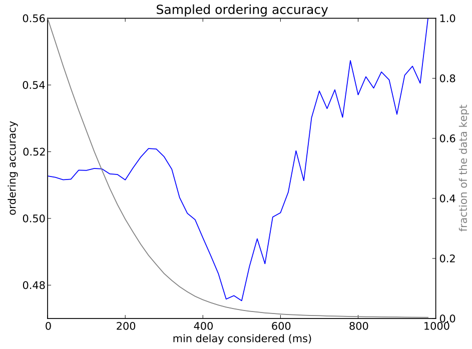
Figure 4: Ordering accuracy (blue curve) between pairs of examples from the **Assassin's Creed** test set, considering only pairs for which the difference in delays is over some threshold. That threshold corresponds to the x axis. The gray curve shows the proportion of data actually considered.



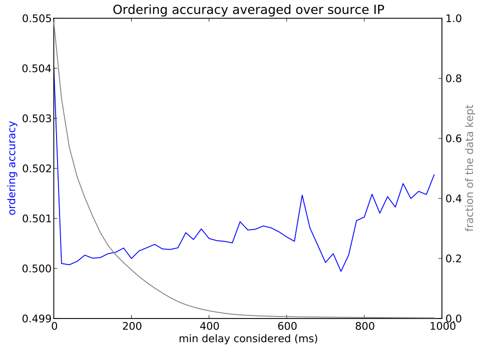
(a) Constant prediction, OA



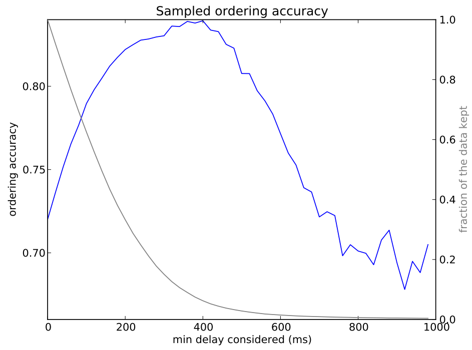
(b) Constant prediction, OA/IP



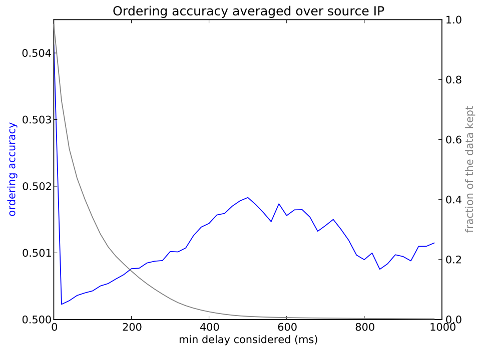
(c) IP2Location, OA



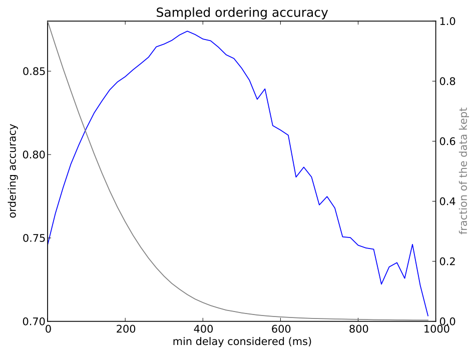
(d) IP2Location, OA/IP



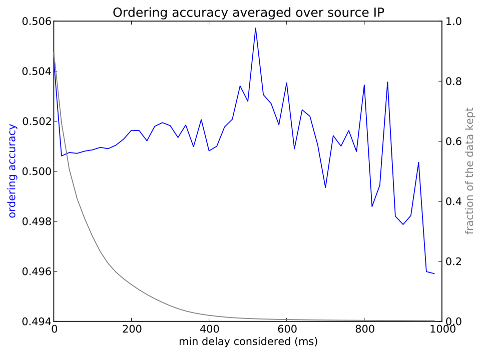
(e) Country pairs, OA



(f) Country pairs, OA/IP

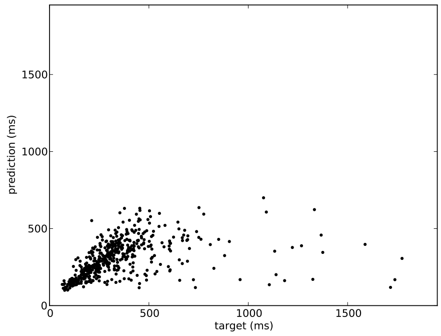


(g) Graphical model, OA

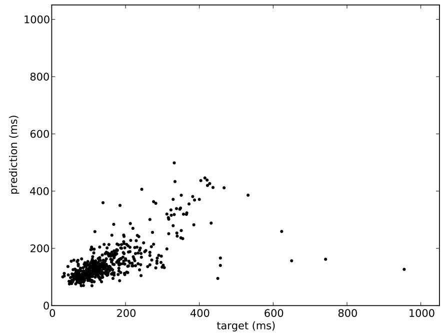


(h) Graphical model, OA/IP

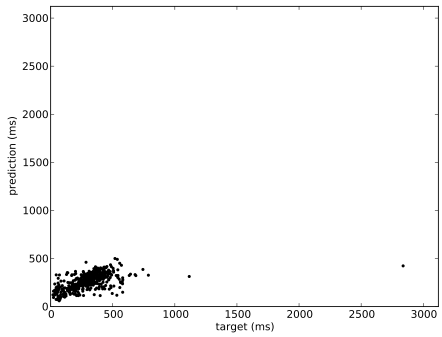
Figure 5: Ordering accuracy (blue curve) between pairs of examples from the **Internet Census 2012** test set, considering only pairs for which the difference in delays is over some threshold. That threshold corresponds to the x axis. The gray curve shows the proportion of data actually considered.



(a) iOS Games



(b) Assassin's Creed



(c) Internet Census 2012

Figure 6: A few targets and predictions by the graphical model randomly chosen from the different test sets.

of accuracy of 1.5% each month. Therefore, it is important to extract the GeoIP data (coordinates, country, region, and city) using a recent version of the data base, as soon as the data is collected. For the experiments presented here, the GeoIP data was not collected along with the IPs and delays, so we used a recent version of the GeoIP data, even for older data.

To investigate the bad performance of the IP2Location distance as a predictor on the iOS data set, we trained the different models on a subset of data, containing only Wi-Fi to Wi-Fi connection (excluding cellular connections at source and destination). The results were not significantly different from the ones using the whole data set, and were not reported here.

To try and explain why the ordering accuracy by IP gives bad performances (50.4%, the same as the constant predictor) on the Internet Census 2012 data, we hypothesized that difference in the relative weights of examples played a role. For instance, if an IP is present as a source IP in a data set $10,000$ times more often than another one, then it will have a relative weight $10,000$ higher in the training phase, but the OA/IP will give the same weight to both. Instead of the simple average, we also computed a weighted average where each source IP is given a weight relative to its frequency in the test test. These weighted average have values similar to the un-weighted ones, and are not reported here.

Among the data sets used, iOS is the one wich resembles most the context in which a matchmaking predictor would take place, for two reasons. First, all the device participating in the data collection (source and destination of the ping) are devices used for gaming at that time, in contrast to IC2012, where source devices are compromised system all over the Internet, and destinations are random IPs. Second, these devices are not necessarily playing with each other, whereas Assassin's Creed data is biased by the fact that the two players are actually connected to each other, which may not have been the case if the latency was too high.

5 Conclusion

We have proposed a graph-inspired model to predict the communication latency between pairs of unseen network addresses. The model obtained a test classification accuracy over 50% on all data sets, and the L_1 error showing low variability across all ranges indicates an accurate prediction of the latency order of magnitude.

This ability translates into a high ordering accuracy (around 75% on all data sets). The model is also able to discriminate fairly well between two candidate pairs with the same origin IP on data coming from games (OA/IP around 70% for iOS and Assassin's Creed), which is sufficiently useful to discern between candidate connections in matchmaking applications.

The proposed model outperforms IP2Location on all tested data sets, for a variety of metrics. On the iOS Games data set, which seems most relevant for the task of matchmaking, the global ordering accuracy is largely better (75.7% compared to 65.4%), as well as the ordering accuracy by origin IP (69.3% vs. 62.0%).

On the Assassin's Creed data set, for which the players already have a low latency, the gain of the graphical model compared to IP2Location is lower (74.8% vs. 73.7% in global ordering accuracy, 68.2% vs. 66.9% normalized by source IP), but still consistent across measures.

For the Internet Census 2012 data set, the graph-inspired model also outperforms IP2Location for all metrics (74.7% in ordering accuracy vs. 51.3%), except for the discriminating ordering accuracy by origin IP, where no model was able to get better than a random prediction, probably because the distribution of training examples was quite different from the distribution we expect in a matchmaking task.

A Code and Saved Model

The code to train and load the graphical model is available in the "pings" repository on GitHub[6]. The code for training the model and retrieving its predictions is in `models/work_in_progress/src/graph.py`. It was tested with the following version of its dependencies:

- Python 2.7.2
- NumPy 1.6.1
- SciPy 0.9.0.

The GeoLite GeoIP data base was retrieved in May 2013.

After training, the model parameters are saved in a pickle file with the following structure. The file contains a pair `(saved_params, terms)`, where `terms` is a list of the terms used in the model, as described in Table 1. For each term in that list, the last value represents the regularization penalty used; it will be 0 for the constant term or bias, and for the GeoIP distance, and -1 for all other terms, meaning it should be optimized on the validation set.

`saved_params` is a list of the same length, each element of that list is a triplet containing the data necessary to compute the contribution of that term. There are 3 possible cases:

- If the term is a real value, which is only the case for the distance, the triplet contains `(coef, bias, reg)`, where `coef` is the coefficient of the linear regression, `bias` is the bias, and `reg` is the regularization coefficient (similar to the one described in Equations 3 and 5, except no division by n would occur). so the final contribution of that term in the prediction is:

$$M_{j,:}^{(k)} \cdot \frac{\texttt{coeff}}{1 + \texttt{reg}} + \texttt{bias}. \tag{13}$$

- If the term is a one-hot vector with fewer than 2000 possible classes (for instance, the connection type `type1`), the triplet contains `(params, number, reg)`, where `params` and `numbers` are NumPy `ndarray` vectors, with their length equal to the number of possible classes, containing respectively the regression coefficients and the training example count for each of the possible input classes. `reg` is the coefficient λ_k, as described in Equations 3 and 5, so the final contribution of that term in the prediction is:

$$M_{j,:}^{(k)} \cdot \frac{\texttt{params}}{1 + \texttt{reg}/\texttt{numbers}}, \tag{14}$$

  where all divisions are done element-wise. Note that, if $M_{j,:}^{(k)}$ is a one-hot vector where $M_{j,i}^{(k)} = 1$, this is equivalent to the scalar:

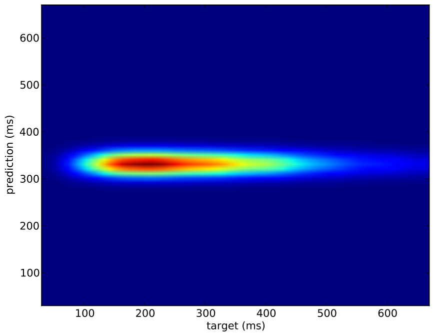
$$\frac{\texttt{params}_i}{1 + \texttt{reg}/\texttt{numbers}_i}. \tag{15}$$

- If the term would be represented by a one-hot vector with 2000 classes or more (for instance, the source IP `ip1`), the triplet is also `(params, number, reg)`, but a dictionary structure (more precisely, a Python `collections.defaultdict`) is used to store the coefficients (`params`) and counts (`numbers`) instead. If the k-th feature of example j is the class i, the contribution of that term to the prediction is:

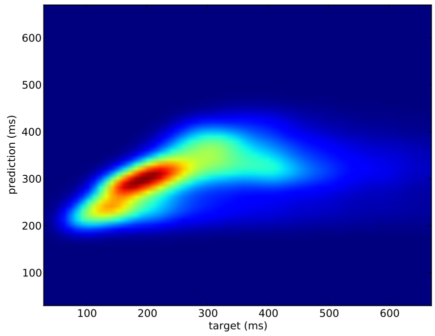
$$\frac{\texttt{params}[\texttt{i}]}{1 + \texttt{reg}/\texttt{numbers}[\texttt{i}]}, \tag{16}$$

  where `d[i]` denotes the value of dictionary `d` associated to key `i`, or 0 if `i` is not a key of `d`.

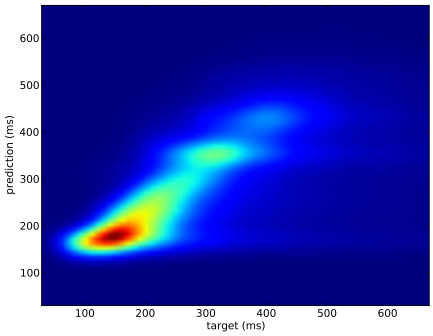
[6] https://github.com/lisa-lab/pings



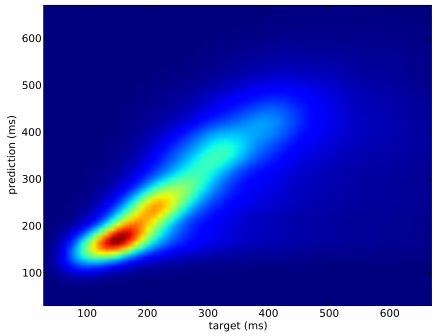
(a) Constant prediction



(b) IP2Location

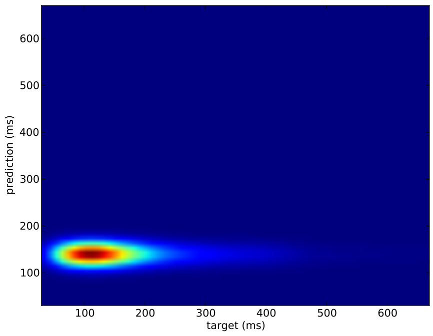


(c) Country pairs

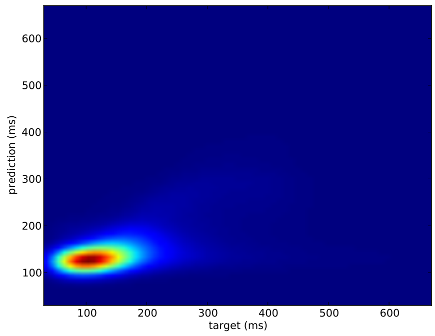


(d) Graphical model

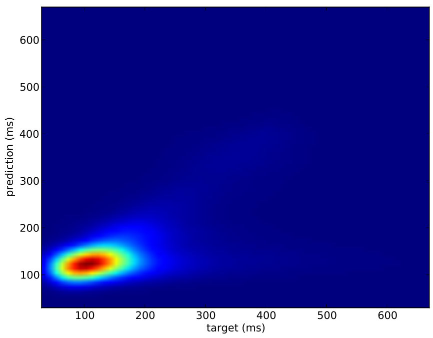
Figure 7: Heat map of the test joint distribution $P(t_j, y_j)$ estimated with a Gaussian kernel ($\sigma = 18$ ms) on the **iOS games** data set.



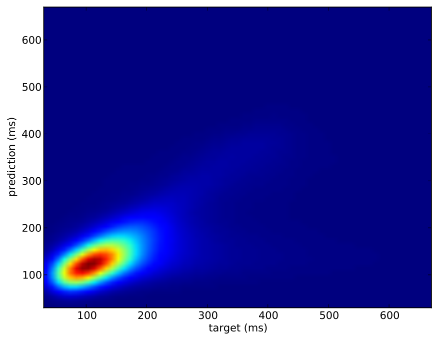
(a) Constant prediction



(b) IP2Location

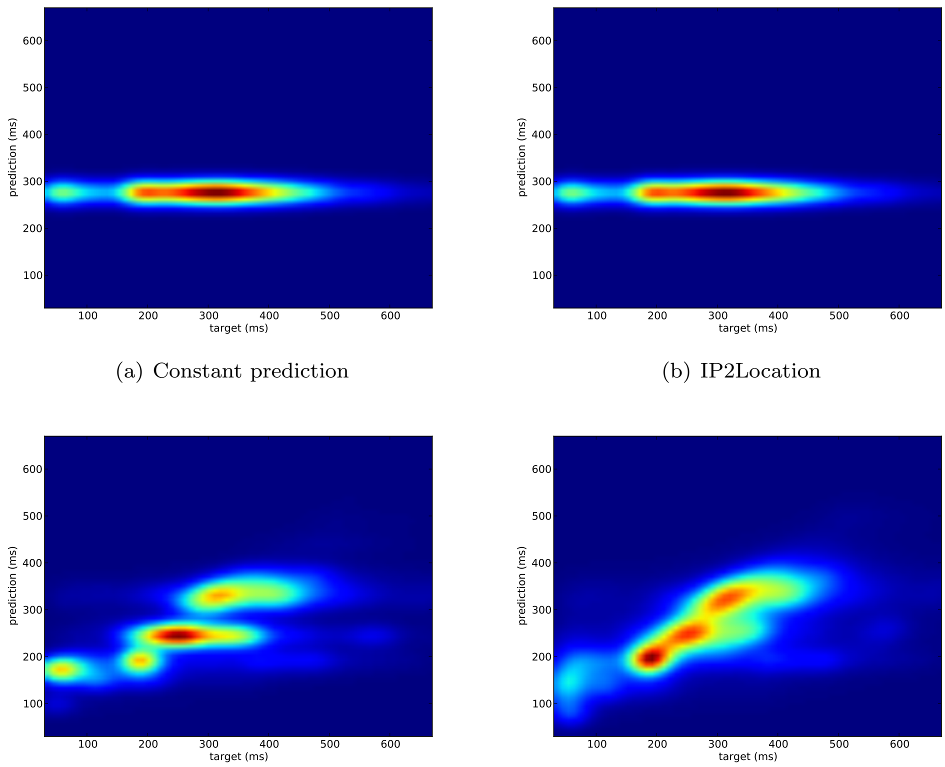


(c) Country pairs



(d) Graphical model

Figure 8: Heat map of the test joint distribution $P(t_j, y_j)$ estimated with a Gaussian kernel ($\sigma = 18$ ms) on the **Assassin's Creed** data set.



(a) Constant prediction (b) IP2Location

(c) Country pairs (d) Graphical model

Figure 9: Heat map of the test joint distribution $P(t_j, y_j)$ estimated with a Gaussian kernel ($\sigma = 18$ ms) on the **Internet Census 2012** data set.