PHYS 414 Final Project

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NEWTON

(a) Analytical Derivations of Lane-Emden Equation

Using Newtonian mechanics, two hydrostatic equilibrium equations are provided as

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho(r) \tag{1}$$

$$\frac{dp(r)}{dr} = -\frac{Gm(r)\rho(r)}{r^2} \tag{2}$$

After rearranging terms of equation 2 such that only $m(r)$ appears at right-hand side as a function of r, we obtain

$$\frac{r^2}{\rho(r)}\frac{dp(r)}{dr} = -Gm(r) \tag{3}$$

Taking derivatives of both sides, and using equation 1 yield

$$\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dp(r)}{dr}\right) = -G\frac{dm(r)}{dr} = -4\pi G r^2 \rho(r) \tag{4}$$

Thus, the equation is free of $m(r)$. Dividing both sides by r^2,

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dp(r)}{dr}\right) = -4\pi G\rho(r) \tag{5}$$

Using polytropic EOS $p = K\rho^{1+\frac{1}{n}}$, along with factorization $\rho(r) = \rho_c\theta^n(r)$ such that $p(r) = K\rho_c^{1+\frac{1}{n}}\theta^{n+1}(r)$,

$$\frac{K\rho_c^{\frac{1}{n}}}{r^2}\frac{d}{dr}\left(\frac{r^2}{\theta^n(r)}\frac{d}{dr}\theta^{n+1}(r)\right) = -4\pi G\rho_c\theta^n(r) \tag{6}$$

$$\frac{(n+1)K\rho_c^{\frac{1}{n}-1}}{4\pi G}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta(r)}{dr}\right) = -\theta^n(r) \tag{7}$$

If we use scaled radius such that $r = \alpha\xi$ where $\alpha^2 = \frac{(n+1)K\rho_c^{\frac{1}{n}-1}}{4\pi G}$, we obtain the final form

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta(\xi)}{d\xi}\right) + \theta^n(\xi) = 0 \tag{8}$$

Using `Mathematica`'s asymptotic solve function `AsymptoticDSolveValue` to obtain a series solution of Lane-Emden equation, we obtain

$$\theta(\xi) \approx 1 - \frac{\xi^2}{6} + \frac{n\xi^4}{120} - \frac{(8n^2-5n)\xi^6}{1520} + \dots \tag{9}$$

After this step, we can find closed-form analytical solutions of Lane-Emden equation. Evaluating Lane-Emden equation with `Mathematica`'s `DSolve` function for $n = 1$ yields

$$\theta(\xi) = \frac{sin(\xi)}{\xi} \tag{10}$$

Since $M = \int_0^R 4\pi r^2\rho(r)dr = \frac{4\pi\rho_c}{\alpha^3}\int_0^{\xi_n}\xi^2\theta^n(\xi)d\xi$, substituting θ^n from Lane-Emden equation yields,

$$M = 4\pi\rho_c\alpha^3\int_0^{\xi_n} -\frac{d}{d\xi}\left(\xi^2\frac{d\theta(\xi)}{d\xi}\right)d\xi \tag{11}$$

Evaluating the integral, and substituting $R = \alpha\xi_n$,

$$M = 4\pi\rho_c\alpha^3\left[-\xi^2\frac{d\theta(\xi)}{d\xi}\right]\Bigg|_0^{\xi_n} = 4\pi\rho_c\alpha^3\xi_n^3\left[-\frac{\theta'(\xi_n)}{\xi_n}\right] \tag{12}$$

$$= 4\pi\rho_c R^3\left[-\frac{\theta'(\xi_n)}{\xi_n}\right] \tag{13}$$

For the stars sharing same polytropic EOS, $\frac{R}{\alpha\xi_n} = 1$, then

$$\frac{4\pi G}{(n+1)K\xi_n^2}\rho_c^{\frac{n-1}{n}}R^2 = 1 \tag{14}$$

Since K, n, ξ_n and others are constant for same polytropic equation except ρ_c, $\rho_c \propto R^{\frac{-2n}{n-1}}$.

$$M \propto \rho_c R^3 \rightarrow M \propto R^{3-\frac{2n}{n-1}} = R^{\frac{3-n}{1-n}} \tag{15}$$

(b) Mass versus Radius Distributions of White Dwarfs

Using `Python` for reading and plotting the given white dwarf, we obtain mass distributions as a function of radii. After calculating corresponding radii `R` for data, the distribution is plotted.

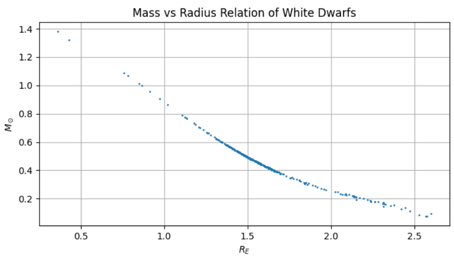

FIG. 1.

(c) Obtaining Fitting Parameters for White Dwarf Data

Given pressure equation for the white dwarfs,

$$C(x(2x^2-3)(x^2+1)^{-1/2}+3sinh^{-1}(x)) \tag{16}$$

where x is the scaled density parameter $(\frac{\rho}{D})^{\frac{1}{q}}$. After obtaining series expansion for the pressure as $x \to 0$ using `Mathematica`'s `Series` function, we get the leading parameter to approximate pressure for small x

$$P \approx \frac{8C}{5}(\frac{\rho}{D})^{\frac{5}{q}} \tag{17}$$

After arranging the leading term in the form $P \approx K_*\rho^{1+\frac{1}{n_*}}$, we obtain parameters K_* and n_* as,

$$n_* = \frac{q}{5-q} \qquad \& \qquad K_*\frac{8C}{5D^{\frac{5}{q}}} \tag{18}$$

After fitting the white dwarf data in `Python` for integer q, we obtain $q = 3$ and using fitting parameters, we can plot the curve for small M

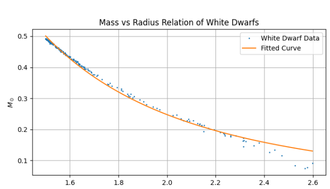

FIG. 2.

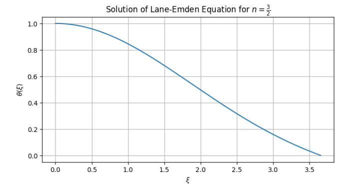

FIG. 3.

Numerically solving Lane-Emden equation in `Python`, we obtain the solution for $n = \frac{3}{2}$ given below

By calculating the central densities of white dwarfs using **??** and numerical solutions of Lane-Emden equation, we obtain the plot

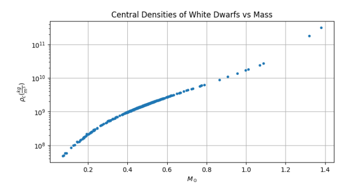

FIG. 4.

After fitting the data for parameter K, we obtain $K = 2774995.74$ and using the parameters, we plot the fitting curve

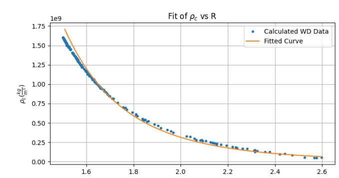

FIG. 5.

(d) Obtaining Parameter D by Interpolation

Using interpolation and solving IVPs using differential equations $\frac{dm}{dr} = 4\pi r^2\rho$ and,

$$\frac{d\rho}{dr} = -G\frac{\sqrt{x^2+1}}{8Cx^5}\frac{qm\rho^2}{r^2} \tag{19}$$

we obtained that $D = 3022830886$ and C = 10960543496614858194944.

(e)

Using parameters, numerical solution of white dwarf mass-radius relation is plotted as

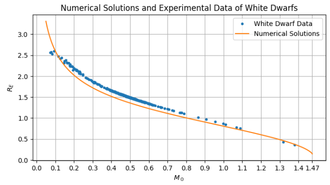

FIG. 6.

EINSTEIN

(a) Numerical Solutions of Tolman–Oppenheimer–Volkoff Equations

Using given polytropic definition of pressure $P = K\rho^2$, TOV equations converted into:

$$\frac{dm}{dr} = 4\pi r^2 \rho \tag{20}$$

$$\frac{d\nu}{dr} = 2\frac{m + 4\pi K\rho^2 r^3}{r(r-2m)} \tag{21}$$

$$\frac{d\rho}{dr} = -\frac{m + 4\pi K\rho^2 r^3}{r(r-2m)}\frac{1+K\rho}{2K} = -\frac{1}{2}\frac{1+K\rho}{2K}\nu' \tag{22}$$

Such that there is not explicit P-dependence. Instead of pressure, we can directly solve the TOV equations for density. Given initial conditions $\rho(0) = \rho_c$, $m(0) = 0$, and $\nu(0) = 0$, the coupled equations are integrated with `SciPy`'s `solve_ivp` function. For provided central density values, mass and radii of neutron stars are obtained. The curve of mass-radius relation of neutron stars is given below.

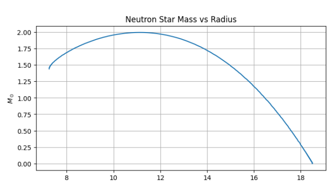

FIG. 7.

(b) Baryonic Mass and Fractional Binding Energy

Baryonic mass of the neutron stars are calculated by appending the equation 23 to TOV equations,

$$\frac{dm_P}{dr} = 4\pi(1 - \frac{2m}{r})^{-1/2} r^2 \rho \tag{23}$$

Fractional binding energy of the neutron stars is also given as $\Delta = \frac{M_P - M}{M}$. The plot of fractional binding energy for varying radii is provided below.

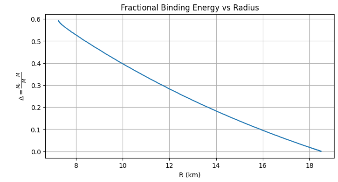

FIG. 8.

(c) Stability of the Neutron Stars

The stability condition is given as $\frac{dM}{d\rho_c} > 0$. On our specific case, that ρ_c is always increasing, thus $d\rho_c$ is always greater than zero. Then, reduced stability condition is $dM > 0$. Although the derivative can be calculated as an irregular derivative, in our specific case, the condition is simplified. ΔM is calculated using numerical differencing. The obtained stability curve of the neutron stars is provided below.

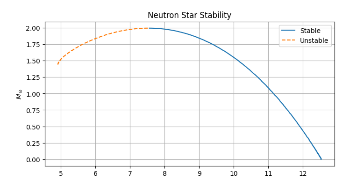

FIG. 9.

(d) Maximum Neutron Star Masses

Maximal mass of neutron stars described by polytropic equation depends on the value of constant K. For $K = 100$, $M_{max} \approx 2$. Although maximal masses increase with increasing K, maximum observed neutron star mass $2.14 M_{\odot}$ limits K. Calculations yield that maximum of allowed value of K is 114. Above this value, neutron star masses exceed the maximum observed value. Maximal mass depending on varying K values are provided below.

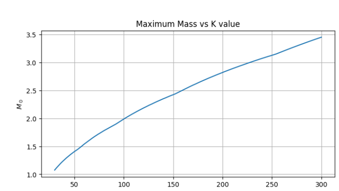

FIG. 10.

(e) TOV Equation outside of Star

The differential equation of parameter ν outside of the star is reduced to the form

$$\frac{d\nu}{dr} = \frac{2M}{r(r-2M)} \tag{24}$$

Since we can factorize right-hand side as $\frac{1}{r-2M} - \frac{1}{r}$, we collect variables and differentials side by side, and put under integral sign as

$$\int_R^r d\nu = \int_R^r \frac{dr'}{r'-2M} - \int_R^r \frac{dr'}{r'} \tag{25}$$

After integrating, we obtain

$$\nu(r) - \nu(R) = ln(r-2M) - ln(R-2M) - (ln(r) - ln(R)) \tag{26}$$

Then simplifying it yields the final form for $r > R$

$$\nu(r > R) = ln(1 - \frac{2M}{r}) - ln(1 - \frac{2M}{R}) + \nu(R) \tag{27}$$