PHYS 414 Term Project

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PART A

$\frac{dm(r)}{dr} = 4\pi r^2 \rho(r)$ and $\frac{dp(r)}{dr} = -\frac{Gm(r)\rho(r)}{r^2}$ After rearranging terms of equation (2) such that only $m(r)$ appear at right-handside as a function of r,

$$\frac{r^2}{\rho(r)}\frac{dp(r)}{dr} = -Gm(r)$$

Taking derivatives of both sides, and using equation (1) yield

$$\frac{d}{dr}(\frac{r^2}{\rho(r)}\frac{dp(r)}{dr}) = -G\frac{dm(r)}{dr} = -4\pi G r^2 \rho(r)$$

Thus, the equation is free of $m(r)$. Dividing both sides by r^2,

$$\frac{1}{r^2}\frac{d}{dr}(\frac{r^2}{\rho(r)}\frac{dp(r)}{dr}) = -4\pi G\rho(r)$$

Using polytropic EOS $p = K\rho^{1+\frac{1}{n}}$, along with factorization $\rho(r) = \rho_c\theta^n(r)$ such that $p(r) = K\rho_c^{1+\frac{1}{n}}\theta^{n+1}(r)$,

$$\frac{K\rho_c^{\frac{1}{n}}}{r^2}\frac{d}{dr}(\frac{r^2}{\theta^n(r)}\frac{d}{dr}\theta^{n+1}(r)) = -4\pi G\rho_c\theta^n(r)$$

$$\rightarrow \frac{(n+1)K\rho_c^{\frac{1}{n}-1}}{4\pi G}\frac{1}{r^2}\frac{d}{dr}(r^2\frac{d\theta(r)}{dr}) = -\theta^n(r)$$

If we use scaled radius such that $r = \alpha\xi$ where $\alpha^2 = \frac{(n+1)K\rho_c^{\frac{1}{n}-1}}{4\pi G}$, we obtain the final form

$$\frac{1}{\xi^2}\frac{d}{d\xi}(\xi^2\frac{d\theta(\xi)}{d\xi}) + \theta^n(\xi) = 0$$

Since $M = \int_0^R 4\pi r^2\rho(r)dr = \frac{4\pi\rho_c}{\alpha^3}\int_0^{\xi_n}\xi^2\theta^n(\xi)d\xi$, substituting θ^n from Lane-Emden equation yields,

$$M = 4\pi\rho_c\alpha^3\int_0^{\xi_n} -\frac{d}{d\xi}(\xi^2\frac{d\theta(\xi)}{d\xi})d\xi$$

Evaluating the integral, and substituting $R = \alpha\xi_n$,

$$M = 4\pi\rho_c\alpha^3[-\xi^2\frac{d\theta(\xi)}{d\xi}]\Big|_0^{\xi_n} = 4\pi\rho_c\alpha^3\xi_n^3[-\frac{\theta'(\xi_n)}{\xi_n}] \quad (1)$$

$$= 4\pi\rho_c R^3[-\frac{\theta'(\xi_n)}{\xi_n}] \quad (2)$$

For the stars sharing same polytropic EOS, $\frac{R}{\alpha\xi_n} = 1$, then

$$\frac{4\pi G}{(n+1)K\xi_n^2}\rho_c^{\frac{n-1}{n}}R^2 = 1$$

Since K, n, ξ_n and others are constant for same polytropic equation except ρ_c, $\rho_c \propto R^{\frac{-2n}{n-1}}$.

$$M \propto \rho_c R^3 \rightarrow M \propto R^{3-\frac{2n}{n-1}} = R^{\frac{3-n}{1-n}}$$