PHYS 414 Term Project

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NEWTON

(a) Analytical Derivations of Lane-Emden Equation

Using Newtonian mechanics, two hydrostatic equilibrium equations are provided as

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho(r) \tag{1}$$

$$\frac{dp(r)}{dr} = -\frac{Gm(r)\rho(r)}{r^2} \tag{2}$$

After rearranging terms of equation 2 such that only $m(r)$ appears at right-hand side as a function of r, we obtain

$$\frac{r^2}{\rho(r)}\frac{dp(r)}{dr} = -Gm(r) \tag{3}$$

Taking derivatives of both sides, and using equation 1 yield

$$\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dp(r)}{dr}\right) = -G\frac{dm(r)}{dr} = -4\pi G r^2 \rho(r) \tag{4}$$

Thus, the equation is free of $m(r)$. Dividing both sides by r^2,

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dp(r)}{dr}\right) = -4\pi G\rho(r) \tag{5}$$

Using polytropic EOS $p = K\rho^{1+\frac{1}{n}}$, along with factorization $\rho(r) = \rho_c\theta^n(r)$ such that $p(r) = K\rho_c^{1+\frac{1}{n}}\theta^{n+1}(r)$,

$$\frac{K\rho_c^{\frac{1}{n}}}{r^2}\frac{d}{dr}\left(\frac{r^2}{\theta^n(r)}\frac{d}{dr}\theta^{n+1}(r)\right) = -4\pi G\rho_c\theta^n(r) \tag{6}$$

$$\frac{(n+1)K\rho_c^{\frac{1}{n}-1}}{4\pi G}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta(r)}{dr}\right) = -\theta^n(r) \tag{7}$$

If we use scaled radius such that $r = \alpha\xi$ where $\alpha^2 = \frac{(n+1)K\rho_c^{\frac{1}{n}-1}}{4\pi G}$, we obtain the final form

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta(\xi)}{d\xi}\right) + \theta^n(\xi) = 0 \tag{8}$$

Using `Mathematica`'s asymptotic solve function `AsymptoticDSolveValue` to obtain a series solution of Lane-Emden equation, we obtain

$$\theta(\xi) \approx 1 - \frac{\xi^2}{6} + \frac{n\xi^4}{120} - \frac{(8n^2-5n)\xi^6}{1520} + \dots \tag{9}$$

After this step, we can find closed-form analytical solutions of Lane-Emden equation. Evaluating Lane-Emden equation with `Mathematica`'s `DSolve` function for $n = 1$ yields

$$\theta(\xi) = \frac{sin(\xi)}{\xi} \tag{10}$$

Since $M = \int_0^R 4\pi r^2\rho(r)dr = \frac{4\pi\rho_c}{\alpha^3}\int_0^{\xi_n}\xi^2\theta^n(\xi)d\xi$, substituting θ^n from Lane-Emden equation yields,

$$M = 4\pi\rho_c\alpha^3\int_0^{\xi_n} -\frac{d}{d\xi}\left(\xi^2\frac{d\theta(\xi)}{d\xi}\right)d\xi \tag{11}$$

Evaluating the integral, and substituting $R = \alpha\xi_n$,

$$M = 4\pi\rho_c\alpha^3\left[-\xi^2\frac{d\theta(\xi)}{d\xi}\right]\Bigg|_0^{\xi_n} = 4\pi\rho_c\alpha^3\xi_n^3\left[-\frac{\theta'(\xi_n)}{\xi_n}\right] \tag{12}$$

$$= 4\pi\rho_c R^3\left[-\frac{\theta'(\xi_n)}{\xi_n}\right] \tag{13}$$

For the stars sharing same polytropic EOS, $\frac{R}{\alpha\xi_n} = 1$, then

$$\frac{4\pi G}{(n+1)K\xi_n^2}\rho_c^{\frac{n-1}{n}}R^2 = 1 \tag{14}$$

Since K, n, ξ_n and others are constant for same polytropic equation except ρ_c, $\rho_c \propto R^{\frac{-2n}{n-1}}$.

$$M \propto \rho_c R^3 \rightarrow M \propto R^{3-\frac{2n}{n-1}} = R^{\frac{3-n}{1-n}} \tag{15}$$