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Master's Thesis in Finance

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Risk and Return: Stochastic Dynamics in Portfolio Optimization

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Abstract

This thesis analyzes three distinct utility functions across two market settings: one with constant investment opportunities and one with a stochastic market price of risk. Six closed-form solutions are derived using the Martingale method, revealing that a stochastic market price of risk introduces a hedging component to the optimal portfolio strategy that accounts for expected future changes in the market price of risk. Notably, models with a consumption benchmark lead to lower allocations to risky assets over time, which goes against the typical advice of increasing risk with a longer investment horizon.

To link theory with practice, the relative risk aversion coefficient is estimated via a two-state Markov-switching model combined with a Taylor expansion of expected utility, yielding estimates between 20 and 30, consistent with the litterature while also highlighting the equity premium puzzle. Finally, the theoretical results are discussed with an experienced investment manager, which inspired a small microeconomic game-theoretic model to represent real-world asset allocation.

All Python code related to this paper can be found here.

Acknowledgements

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"It ain't what you don't know that gets you into trouble.
It's what you know for sure that just ain't so."

— Mark Twain

Contents

1 Introduction

Financial markets are constantly evolving. Interest rates fluctuate, risk premia change, and uncertainty remains a persistent feature of the markets. For investors, this creates a fundamental challenge: how to allocate their wealth not only in response to current market conditions, but also in anticipation of how these conditions might change in the future. While classical portfolio theory provides a solid foundation, suggesting that rational investors maximize expected utility based on risk-return tradeoffs, it often relies on simplifying assumptions that may not fully capture the complexities of real-world investment decisions.

This thesis explores how optimal portfolio strategies are affected when the investor faces stochastic investment opportunities. In particular, it examines how a stochastic market price of risk influences decisions under different utility frameworks. Rather than assuming a fixed investment environment, the models includes uncertainty in future returns, which creates new reasons to hedge beyond the standard allocation. By comparing models with constant and stochastic dynamics, the thesis highlights how investor behavior shifts when market conditions become more realistic and less predictable.

A central part of the analysis involves solving the portfolio optimization problem under three different utility functions: standard CRRA utility, a benchmark-adjusted CRRA utility, and habit formation utility. These preferences capture increasingly sophisticated aspects of investor behavior, such as heightened risk aversion near a benchmark level or sensitivity to past consumption. The portfolio problems are solved using the Martingale method, an elegant alternative to dynamic programming that essentially rewrites the dynamic optimization problem as a static one, using pricing kernels.

In addition to the theoretical analysis, the thesis sets out to estimate relative risk aversion based on past stock market returns on `SPY`. By combining a two state Markov-switching regime model with a moment-based Taylor expansion of expected utility, the thesis estimates time-varying risk aversion coefficients that are consistent with full investment in the broad stock market.

Finally, the numerical results are discussed in light of real-world investment practice. In collaboration with an experienced investment manager, the results are evaluated from a professional perspective. This adds a practical dimension to the thesis and helps assess which insights from the models are most relevant in an applied portfolio management context.

In short, the thesis aims to shed light on how stochastic features of the market interact with investor preferences, and what this means for portfolio optimization in theory and practice. By combining rigorous financial modeling with empirical estimation and practical insights, the thesis offers a deeper understanding of dynamic investment behavior.

1.1 Problem statement

The main problem statement considered in this thesis is:

How does a stochastic market price of risk influence optimal portfolio strategies under different investor preferences, and how do these theoretical insights compare to real-world portfolio management?

To address this question, the thesis sets out to:

- introduce three utility functions and their preferences toward risk,
- formulate a dynamic optimization problem for an investor with frictionless access to financial markets and a continous budget constraint,
- explain how such problems can be solved, formulate a general solution using the Martingale method, and derive optimal solutions,
- estimate the CRRA risk aversion parameter using real-world return data, by combining a two state Markov-switching model and a Taylor-expanded expected utility function,
- present numerical results under two different market environments: one with constant investment opportunities and one with a stochastic market price of risk,
- relate and discuss the results in light of insights from an experienced investment manager.

1.2 Delimitation

Asset allocation and portfolio optimization are among the most widely studied topics in finance. As a result, the literature is extensive, and for nearly every approach or result, there are alternative methods, extensions, or counterarguments. To ensure a focused and meaningful answer to the main research question, the thesis is delimited in several areas. This helps keep the analysis both relevant and manageable, while also aligning with the formal requirements. Thus, the thesis does not:

- consider alternative utility specifications beyond CRRA, benchmark-adjusted CRRA, and habit formation utility,
- assess whether the use of dynamic programming or the Martingale method is the most appropriate method to solve investor problems,
- explain general undergraduate finance concepts such as CAPM, the basic risk-return tradeoff, etc.,
- provide a full empirical test of the investment models using historical data beyond the estimation of the risk aversion parameter.

Furthermore, the thesis builds extensively on the framework and results presented in Munk (2011), and unless otherwise stated, most theoretical derivations in continuous time are inspired by this reference. Any additional contributions or modifications from other sources are explicitly referenced.

1.3 Thesis structure

The thesis opens by introducing foundational concepts related to risk preferences and utility theory, followed by a formal definition of the investor's portfolio optimization problem. Chapter 4 outlines two solution methods, and the thesis applies the Martingale method throughout. Chapter 5 introduces two market settings, one with constant investment opportunities and another with a stochastic market price of risk. Chapters 6 and 7 derive optimal portfolio strategies under the three distinct utility functions and two markets. Chapter 8 turns to the empirical estimation of risk aversion. The final chapters present numerical results, offer new insights contributed by an experienced investment manager and reflect on their broader implications.

1.4 Litterature review

The foundation of modern portfolio theory was laid by Markowitz (1952), where single-period mean-variance analysis dominated early thinking about optimal portfolio choice. A major development came with Merton (1971), who introduced the continuous-time consumption-investment problem and provided explicit solutions under specific assumptions. In Merton (1973), he extended this framework by distinguishing between two components in the optimal portfolio: one capturing speculative demand for risky assets, and the other serving as a hedge against stochastic changes in the investment opportunities.

Cox and Huang (1989) later expanded the model by introducing non-negativity constraints on terminal wealth, opening the door to a stream of literature on constrained dynamic portfolio optimization. Since then, a number of papers have refined the framework by incorporating stochastic state variables, such as time-varying interest rates, inflation, or the market price of risk, to better reflect real world uncertainty.

Wachter (2002) studies optimal terminal wealth when the market price of risk follows a mean-reverting Ornstein-Uhlenbeck process, showing that risk premia fluctuations alone can generate a non-myopic hedging demand. Kim and Omberg (1996) offer similar insights in a comparable setting, demonstrating that stochastic variation in expected returns directly affects portfolio composition. These results generalize Merton's original two-fund separation and have been extended in models that incorporate more complex investor preferences. For instance, Munk (2008) explores how habit formation preferences interact with stochastic investment opportunities, amplifying the hedging motive. Brennan and Xia (2002) also contribute to this literature by analyzing optimal consumption and investment decisions in an environment with stochastic inflation and price levels. Finally, Munk et al. (2004) present a comprehensive framework with mean-reverting returns, stochastic interest rates, and inflation uncertainty, unifying this extended literature.

A related branch of the literature focuses specifically on interest rate risk hedging within dynamic portfolio models. When the problem is formulated in nominal terms, the short-term interest rate often becomes the key state variable. Studies such as Jaredsen (1999), Brennan and Xia (2000), Deelstra et al. (2000), and Bajeux-Besnainou et al. (2003) show that while the models vary in how they specify interest rate dynamics, their conclusions are broadly similar. In particular, the interest rate hedging component of the optimal portfolio typically consists of bonds maturing at the investment horizon, while the equity allocation is affected by the myopic demand, regardless of whether stock returns are correlated with interest rates.

During the late 1980s and into the early 1990s, the Martingale method was introduced as an alternative to the dynamic programming framework based on the Hamilton-Jacobi-Bellman equation. Key contributions include Karatzas, Lehoczky, and Shreve (1987) and Cox and Huang (1991), who demonstrated how this method could be applied to both unconstrained and constrained optimization problems under general conditions.

2 Risk and utility

2.1 Introduction

To analyze optimal portfolio allocation, lets consider three types of investors, each characterized by a different utility function: Constant Relative Risk Aversion (CRRA), benchmark-adjusted CRRA, and habit formation utility. These functions capture distinct features of investor preferences and allow for a comparison of how different attitudes toward risk shape investment choices.

The goal is not to dive into the theoretical foundations of these utility functions, but to explore their practical implications for portfolio decisions. CRRA and habit utility are commonly used in the literature, see Liu (2007) and Munk (2002), while the benchmark-adjusted CRRA utility is a unique version of the HARA utility function that introduces a tolerance band around a benchmark level.

Studying these specifications provides a way to understand how preference structures, especially those that deviate from standard models, interact with market dynamics. This becomes particularly relevant when different markets are introduced, as it affects how investors adjust their optimal strategies.

2.2 Risk aversion

To understand how risk is incorporated into the utility function, the following example provides valuable insights. Consider an individual whose preferences are represented by an utility function $U(\cdot)$. The individual derives marginal utility from each additional unit of consumption, i.e., higher consumption is always preferred to lower consumption. Suppose a fixed consumption level $C \in Z$, where $Z = \mathbb{R}_+ \equiv [0, \infty)$, is given. Further, let ε be a random variable with $E[\varepsilon] = 0$, representing a stochastic shock to consumption. The term $C + \varepsilon$ can be interpreted as a random consumption plan, where the realized consumption is $C + \varepsilon(\omega)$ if state ω occurs.

An individual is classified as risk-averse if, for all $C \in Z$ and all fair gambles ε, they strictly prefer the certain consumption level C over the stochastic alternative $C + \varepsilon$. Conversely, an individual is risk-loving if, for all $C \in Z$ and all fair gambles

ε, they prefer the risky alternative $C + \varepsilon$ to the certain level C. A risk-neutral individual is indifferent between the two.

A fundamental concept in risk analysis is the certainty equivalent, defined as the consumption level $C^\star \in Z$ that satisfies

$$U(C^\star) = E[U(C)] \tag{1}$$

This implies that the individual is indifferent between receiving $C^\star$ with certainty and facing the uncertain consumption C. For a risk-averse individual, it always holds that $E[C] \geq C^\star$, reflecting a preference for certainty over risk. The risk premium is then given by

$$\lambda(C) = E[C] - C^\star \tag{2}$$

The risk premium represents the amount the individual is willing to forgo to eliminate uncertainty. Rewriting the certainty equivalent condition, we obtain

$$E[U(C)] = U(C^\star) = U(E[C] - \lambda(C)) \tag{3}$$

A robust measure of risk aversion should remain invariant to strictly positive affine transformations. This is satisfied by the Arrow-Pratt measures of risk aversion. The absolute risk aversion (ARA) is given by

$$ARA(C) = -\frac{U''(C)}{U'(C)} \tag{4}$$

While the relative risk aversion (RRA) is given by

$$RRA(C) = -C \cdot \frac{U''(C)}{U'(C)} = C \cdot ARA(C) \tag{5}$$

These measures quantify risk preferences by capturing the concavity of the utility function relative to its slope. Since the second derivative of a concave function is negative for a risk-averse investor, the negative sign ensures that risk aversion is expressed as a positive quantity. The Arrow-Pratt measures are fundamental in portfolio- and expected utility theory, as they allow for a consistent comparison of risk preferences across different utility functions and consumption levels.

2.3 Utility functions

Before presenting the individual utility functions, it is meaningful to outline a few general assumptions that apply across all specifications. These include the standard microeconomic axioms, completeness, reflexivity, and transitivity, which ensure well-behaved preferences. The utility functions are assumed to be increasing and concave, satisfying $U'(\cdot) > 0$ and $U''(\cdot) \leq 0$, and to be continuously differentiable, i.e., $U(\cdot) \in C^2(\mathbb{R}^+)$.

The CRRA utility function is given by

$$U(C_t) = \frac{C_t^{1-\gamma} - 1}{1-\gamma}, \quad \gamma \neq 1 \tag{6}$$

Where γ represents the coefficient of relative risk aversion. A higher γ corresponds to greater risk aversion, and, by construction, the CRRA utility function ensures that relative risk aversion remains constant across different levels of wealth. It follows directly that the function satisfies monotonicity and concavity. The CRRA utility implies homothetic preferences, meaning that optimal portfolio allocation depends only on relative wealth rather than absolute wealth. In the special case where $\gamma = 1$, it can be shown that $U(C_t) = \ln(C_t)$ by L'Hôpital's rule. For the CRRA utility function, the corresponding risk aversion measures are given by

$$ARA(C_t)_{\text{CRRA}} = \frac{\gamma}{C_t} \tag{7}$$

$$RRA(C_t)_{\text{CRRA}} = \gamma \tag{8}$$

These expressions illustrate a central feature of CRRA utility: while absolute risk aversion diminishes with higher consumption, relative risk aversion remains fixed at the level of the parameter γ.

The standard CRRA utility function assumes symmetric preferences around all levels of consumption, with a constant coefficient of relative risk aversion. However, such symmetry may fail to capture important behavioral aspects of investor decision-making, particularly when consumption nears a critical reference point. To address this, the utility function can be modified so that the agent becomes more risk-averse when consumption is close to a fixed benchmark level $\bar{C}$.

The benchmark-adjusted CRRA utility function is given by

$$U(C_t) = \frac{(C_t - \bar{C})^{1-\gamma(C_t)} - 1}{1 - \gamma(C_t)}, \tag{9}$$

where

$$\gamma(C_t) = \begin{cases} \gamma_H, & \text{if } |C_t - \bar{C}| < \varepsilon \quad \text{(i.e., near subsistence)} \\ \gamma, & \text{otherwise} \end{cases}$$

The threshold $\varepsilon > 0$ defines a region around $\bar{C}$ in which the investor's risk aversion increases to γ_H. This captures the idea that the agent becomes more sensitive to fluctuations in consumption when it approaches a critical subsistence level. Outside this interval, the investor reverts to a lower risk preference γ. The choice of ε is not a structural parameter but rather a modeling device that determines when the heightened risk aversion is activated. This structure keeps the simplicity of CRRA in functional form, while capturing asymmetric risk attitudes through a locally adjusted risk aversion coefficient. The corresponding absolute and relative risk aversion measures are

$$ARA(C_t) = \frac{\gamma(C_t)}{C_t - \bar{C}}, \qquad RRA(C_t) = \frac{C_t \cdot \gamma(C_t)}{C_t - \bar{C}}. \tag{10}$$

As consumption approaches $\bar{C}$, both $ARA(C_t)$ and $RRA(C_t)$ increase sharply, reflecting the investor's rising caution. This increase in risk aversion results in more conservative portfolio choices near the benchmark.

The benchmark-adjusted CRRA utility function can be motivated by comparing it to the Constant Proportion Portfolio Insurance (CPPI) strategy. In CPPI, the investor allocates a fixed proportion of their cushion, wealth above a floor, to risky assets, and reduces risk when wealth gets close to the floor. The benchmark-adjusted CRRA model follows a similar idea: it increases risk aversion when consumption approaches a benchmark level, thereby limiting downside risk.

The internal habit utility function is given by

$$U(C_t, h_t) = \frac{(C_t - h_t)^{1-\gamma} - 1}{1 - \gamma}, \quad \gamma \neq 1 \tag{11}$$

$$h_t = h_0 e^{-\beta t} + \alpha \int_0^t e^{-\beta(t-s)} C_s ds \tag{12}$$

Where h_0, α, and β are non-negative constants. This definiftion follows directly from Munk (2002). Habit utility functions can be specified in various ways, as illustrated in Wachter (2002). The key property, however, is that $U(\cdot)$ is decreasing in h, $\frac{\partial U(\cdot)}{\partial h} < 0$. For the habit utility function, the risk aversion measures are given by

$$ARA(C_t)_{\text{Habit}} = \frac{(C_t - h_t)^{-1-\gamma}\gamma}{(C_t - h_t)^{-\gamma}} = \frac{\gamma}{C_t - h_t} \tag{13}$$

$$RRA(C_t)_{\text{Habit}} = \frac{C_t\gamma}{C_t - h_t} \tag{14}$$

These expressions show that habit formation makes risk preferences depend on both current and past consumption. Unlike standard CRRA utility, where preferences are constant over time, habit utility reflects how past consumption affects today's utility. This leads to smoother consumption patterns, as investors try to avoid drops below their usual standard of living.

As consumption approaches the internal habit level h_t, both absolute and relative risk aversion increase. This means investors become more cautious when their current consumption is near their usual standard of living. The parameter α controls how strongly past consumption affects today's utility, while β determines how quickly that influence fades over time.

Although the benchmark-adjusted CRRA utility function also reflects increased caution near a reference level, it does so in a simpler and static way. Habit formation, in contrast, introduces a dynamic feedback mechanism: higher consumption today increases the habit level tomorrow, which raises future consumption needs and affects portfolio choices over time.

Empirical evidence on the validity of habit formation remains mixed. Brunnermeier and Nagel (2006) find no significant support for habit utility, showing that households do not adjust their relative asset allocation in response to changes in wealth, as habit models would suggest. Wachter (2002) presents evidence in favor of habit formation, demonstrating its ability to explain differences in stock and bond returns over time. Also, Ravina (2007) reports strong support for habit formation utility.

3 The investor problem

3.1 Introduction

Independent of the choice of utility function, lets consider a price-taking investor with a fixed horizon $[t, T]$. At the current time $t = 0$, the investor seeks to maximize expected lifetime utility over the time horizon. A simple version of the investor's optimization problem is given by

$$\max_{C_t} U(C_t) = E_0 \left[\int_0^T e^{-\delta t} U(C_t, t) dt + e^{-\delta T} \bar{U}(W_T) \right] \tag{15}$$

However, this maximization problem alone has limited economic relevance, as real-world investors face constraints in their consumption choices. In particular, consumption cannot exceed available wealth, making it necessary to introduce a budget constraint. Moreover, economic agents typically have access to financial markets, allowing them to allocate their wealth across risky and risk-free assets in pursuit of higher expected returns. To fully capture these considerations, the investor's problem must be formulated within a dynamic framework that accounts for both consumption and investment decisions over time. This naturally motivates the introduction of a stochastic wealth process, which formalizes the budget constraint and links preferences to market opportunities.

3.2 The budget constraint

Consider a financial market with $d + 1$ assets, where one asset, P_t^0, represents a bank account earning a risk-free return of $r_t \Delta t$. The remaining d assets have traded prices denoted by $P_t = (P_t^1, \ldots, P_t^d)$. Let M_t^i represent the number of units of asset i held in the investor's portfolio during the period $[t, t + \Delta t)$. At time t, the total value of the investor's portfolio is given by

$$W_t = \sum_{i=0}^{d} M_{t-\Delta t}^i P_t^i \tag{16}$$

At time t, the investor determines consumption $c_t \Delta t$, which is financed by liquidating a portion of the portfolio. This can be expressed as

$$c_t \Delta t = \sum_{i=0}^{d} M_{t-\Delta t}^i P_t^i - \sum_{i=0}^{d} M_t^i P_t^i \tag{17}$$

The corresponding budget constraint is given by

$$\sum_{i=0}^{d} M_{t-\Delta t}^i P_t^i - c_t \Delta t = \sum_{i=0}^{d} M_t^i P_t^i \tag{18}$$

This equation follows directly from the interpretation that the wealth held in the portfolio, net of consumption in the current period, determines the remaining wealth available in the next period. In other words, consumption is financed through the sale of assets. Munk (2011) extends this framework by incorporating labor income y_t. For simplicity, labor income is omitted.

The change in the value of the portfolio from t to $t + \Delta t$ is given by

$$W_{t+\Delta t} - W_t = \sum_{i=0}^{d} M_t^i (P_{t+\Delta t}^i - P_t^i) - c_t \Delta t \tag{19}$$

Define $\theta_t^i = M_t^i P_t^i$ as the amount invested in asset i at time t, and let $\boldsymbol{\theta}_t = (\theta_t^1, \ldots, \theta_t^d)^\top$. Using this notation, the change in portfolio value can be rewritten as

$$W_{t+\Delta t} - W_t = \theta_t^0 r_t \Delta t + \boldsymbol{\theta}_t^\top \boldsymbol{R}_{t+\Delta t} - c_t \Delta t \tag{20}$$

Here, $\boldsymbol{R}_{t+\Delta t}$ represents the return on the risky assets, and consists of both an expected and an unexpected component

$$\boldsymbol{R}_{t+\Delta t} = \boldsymbol{\mu}_t \Delta t + \underline{\underline{\sigma_t}} \varepsilon_{t+\Delta t} \sqrt{\Delta t} \tag{21}$$

The amount invested in the risk-free asset can be expressed as

$$\theta_t^0 = W_t - \boldsymbol{\theta}_t^\top \mathbf{1} \tag{22}$$

Substituting this into Equation (18), the portfolio dynamics can be rewritten as

$$W_{t+\Delta t} - W_t = \left(r_t W_t + \boldsymbol{\theta}_t^\top (\boldsymbol{\mu}_t - r_t \mathbf{1}) - c_t \right) \Delta t + \boldsymbol{\theta}_t^\top \underline{\underline{\sigma_t}} \varepsilon_{t+\Delta t} \sqrt{\Delta t} \tag{23}$$

Taking the limit as $\Delta t \to 0$, transitioning from discrete to continuous time, yields the wealth dynamics

$$dW_t = \left(r_t W_t + \boldsymbol{\theta}_t^\top (\boldsymbol{\mu}_t - r_t \mathbf{1}) - c_t\right) dt + \boldsymbol{\theta}_t^\top \underline{\underline{\sigma_t}} d\boldsymbol{Z_t} \tag{24}$$

For a more intuitive interpretation, the wealth dynamics can also be expressed in terms of relative portfolio weights $\boldsymbol{\pi}_t$ instead of the nominal amounts $\boldsymbol{\theta}_t$. Assuming the denominator is nonzero, the portfolio weight of asset i at time t is defined as

$$\pi_t^i = \frac{\theta_t^i}{W_t - c_t dt} \tag{25}$$

Equation (24) can now be rewritten as

$$dW_t = \left[(r_t + \boldsymbol{\pi}_t^\top (\boldsymbol{\mu}_t - r_t \mathbf{1}))W_t - c_t\right] dt + W_t \boldsymbol{\pi}_t^\top \underline{\underline{\sigma_t}} d\boldsymbol{Z}_t \tag{26}$$

Since $\underline{\underline{\sigma_t}}$ is assumed to be a non-singular $(d \times d)$ matrix, we define the d-dimensional Sharpe ratio process $\boldsymbol{\lambda}_t$ as

$$\boldsymbol{\lambda}_t = \underline{\underline{\sigma_t}}^{-1} (\boldsymbol{\mu}_t - r_t \mathbf{1}) \tag{27}$$

Which summarizes the risk-return tradeoff of all risk, and so that

$$\boldsymbol{\mu}_t = r_t \mathbf{1} + \underline{\underline{\sigma_t}} \boldsymbol{\lambda}_t \tag{28}$$

This expression aligns closely with standard asset pricing theory: the expected return on the risky assets equals the risk-free rate plus a risk adjustment term, given by the product of the volatility matrix and the market price of risk.

Substituting this into Equation (26), we obtain the final formulation of the wealth dynamics in continous time

$$dW_t = \left[(r_t + \boldsymbol{\pi}_t^\top \underline{\underline{\sigma_t}} \boldsymbol{\lambda}_t) W_t - c_t\right] dt + W_t \boldsymbol{\pi}_t^\top \underline{\underline{\sigma_t}} d\boldsymbol{Z}_t \tag{29}$$

The deterministic component represents the change in wealth due to the risk-free return r_t, the excess return from exposure to risk (captured by $\boldsymbol{\pi}_t^\top \underline{\underline{\sigma_t}} \boldsymbol{\lambda}_t$), and the outflow due to consumption c_t. The stochastic component, captures random fluctuations in wealth due to exposure to the underlying sources of risk $\boldsymbol{Z}_t$. The portfolio weights $\boldsymbol{\pi}_t$ and the volatility matrix $\underline{\underline{\sigma_t}}$ determine how sensitive the wealth process is to these shocks.

3.3 The optimization problem

Having established the investor's constraint, the objective remains to maximize expected lifetime utility, now explicitly incorporating the wealth dynamics. Define the indirect utility process $J = (J_t)$, and formulate the optimization problem as

$$\begin{aligned} J_t &= \sup_{(C_t, \boldsymbol{\pi}_t) \in \mathcal{A}_t} \mathbb{E}_0 \left[\int_0^T e^{-\delta t} U(C_t, t) dt + e^{-\delta T} \bar{U}(W_T) \right] \\ \text{w.r.t.} \quad dW_t &= \left[(r_t + \boldsymbol{\pi}_t^\top \underline{\underline{\sigma_t}} \boldsymbol{\lambda}_t) W_t - c_t \right] dt + W_t \boldsymbol{\pi}_t^\top \underline{\underline{\sigma_t}} d\boldsymbol{Z}_t \end{aligned} \tag{30}$$

The agent must determine a continuous-time consumption process $(C_t)_{t \in [0,T]}$ and a continuous-time portfolio process $(\boldsymbol{\pi}_t)_{t \in [0,T]}$. The vector $\boldsymbol{\pi}_t$ represents the d-dimensional allocation of wealth across the d risky assets at time t, while the remaining wealth is invested in the locally risk-free asset, $\theta_t^0 = W_t - \boldsymbol{\theta}_t^\top \mathbf{1}$. Any admissible consumption and investment strategy $(C_t, \boldsymbol{\pi}_t)$ must satisfy the wealth dynamics. The set of all such feasible strategies over the interval $[t, T]$ is denoted by $\mathcal{A}_t$. Assume that $P(C_t \geq 0) = 1$, implying that consumption cannot be negative. The investor is unconstrained in the sense that any portfolio strategy $\boldsymbol{\pi}_t \in \mathbb{R}^d$ is admissible, thereby allowing for short-selling and other unrestricted investment strategies. Moreover, an admissible strategy pair $(C_t, \boldsymbol{\pi}_t)$ must satisfy a lower bound constraint $K \in \mathbb{R}$. In this setting, it is required that wealth remains non-negative at all times, corresponding to the special case $K = 0$. This assumption is particularly relevant in models without labor income, such as the one considered in this thesis.

At time $t = 0$, the investor must select an admissible strategy based on a set of variables that evolve stochastically over time. These could be:

- the risk-free rate r_t (i.e., the short-term interest rate),
- asset prices, the expected returns on the considered assets, and the variance-covariance matrix of returns on the risky assets,
- covariances or correlations among these variables.

4 Solutions to the investor problem

4.1 Introduction to dynamic programming

The optimization problem presented in Chapter 3 differs significantly from the standard introductory microeconomics setup, where first-order conditions can be derived directly from a static Lagrangian formulation to obtain rather simple expressions for the optimal strategy. In contrast, this setting is formulated in continuous time and involves multiple stochastic elements that influence the investor's optimal decision. Consequently, a central question arises: how can the optimal strategies and the associated indirect utility function be computed?

In the literature, two primary approaches are typically employed to solve this class of dynamic optimization problems: i) the dynamic programming approach, and ii) the Martingale approach.

The dynamic programming approach relies on the existence of a state variable that follows a Markov process, such that all relevant quantities can be expressed as functions of this state variable and time. Specifically, consider a process $\boldsymbol{x} = (\boldsymbol{x}_t)$ that captures the variation in r, $\boldsymbol{\mu}$, and $\underline{\underline{\sigma}}$ over time, i.e.,

$$r_t = r(\boldsymbol{x}_t), \quad \boldsymbol{\mu}_t = \boldsymbol{\mu}(\boldsymbol{x}_t, t), \quad \underline{\underline{\sigma}}_t = \underline{\underline{\sigma}}(\boldsymbol{x}_t, t) \tag{31}$$

Although the theory of dynamic programming provides results on the existence of optimal strategies, these results often rely on restrictive assumptions, such as requiring admissible strategies to lie within a compact set, an assumption that is generally unsuitable for portfolio problems. For this reason, verification theorems are commonly applied. These involve solving the associated Hamilton-Jacobi-Bellman (HJB) equation, and under appropriate technical conditions, the solution to the HJB equation yields both the optimal strategy and the value function.

Dynamic programming is widely considered the standard method in the literature. However, as this thesis will not pursue that approach further, only this brief overview is provided.

4.2 The Martingale method

The Martingale method serves as the primary solution approach in this thesis and will be explained in detail to the extent necessary for understanding the models and their connection to the focus of this thesis. The Martingale method builds on intermediate concepts from modern asset pricing theory. While the key ideas will be introduced and discussed as needed, readers are encouraged to consult Björk (2009) or Duffie (2001) for background on topics such as state-price deflators, market prices of risk, and other related modern asset pricing concepts.

This is not to suggest that these topics are unimportant, on the contrary, they form the theoretical backbone of the models analyzed. However, they are not treated in detail here, and it is assumed that the reader possesses a basic familiarity with them.

As previously mentioned, the dynamic programming approach requires the existence of a finite-dimensional Markov process $\boldsymbol{x} = (\boldsymbol{x}_t)$, such that all relevant quantities, including the indirect utility function, can be expressed as functions of $\boldsymbol{x}_t$. However, this method often yields limited conclusions when the associated partial differential equation cannot be solved explicitly, and it is generally difficult to verify whether an optimal strategy actually exists. In contrast, the Martingale method imposes no additional structural assumptions beyond those already introduced in Chapter 3.

4.2.1 Establishing the pricing framework

Based on the framework introduced in Chapter 3, the analysis will proceed under the assumption of a complete market. This implies that variations in the risk-free rate r_t, expected returns $\boldsymbol{\mu}_t$, and the variance-covariance matrix $\underline{\underline{\sigma}}_t$ are all driven by the same d-dimensional Brownian motion $\boldsymbol{Z}$. Before solving the continuous-time portfolio optimization problem, i.e., choosing C_t and $\boldsymbol{\pi}_t$ to maximize expected utility, it is necessary to establish the following:

- a method for pricing future consumption and terminal wealth,
- a probability measure under which expectations are easier to compute,
- a connection between investor preferences and market dynamics.

This is achieved by introducing the state-price deflator and the risk-neutral measure $\mathbb{Q}$. In a complete market, there exists a unique state-price deflator process (also referred to as the pricing kernel) $\zeta = (\zeta_t)$, defined as

$$\zeta_t = \exp\left\{-\int_0^t r_s\,ds - \int_0^t \boldsymbol{\lambda}_s^\top\,d\boldsymbol{Z}_s - \frac{1}{2}\int_0^t \|\boldsymbol{\lambda}_s\|^2\,ds\right\} \tag{32}$$

Applying Itô's Lemma yields the following dynamics (see Appendix)

$$d\zeta_t = -\zeta_t r_t\,dt - \boldsymbol{\lambda}_t^\top \zeta_t\,d\boldsymbol{Z}_t \tag{33}$$

The deflator ζ_t allows for pricing of any asset or consumption stream by discounting its payoff in present-value terms, adjusted for both time and risk.

There also exists a unique equivalent martingale measure (also known as the risk-neutral probability measure) $\mathbb{Q}$, defined through the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left\{\int_0^T r_s\,ds\right\}\zeta_T \tag{34}$$

The measure $\mathbb{Q}$ is mathematically convenient, as all discounted asset prices become martingales under it, hence the name of the method.

Let $\boldsymbol{\lambda}$ be an $\mathcal{L}^2[0,T]$ process[1]. The time-zero price of a stochastic payoff X_T at time T is given by

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_s\,ds}X_T\right] = \mathbb{E}\left[\zeta_T X_T\right] \tag{35}$$

Similarly, the time-t price is given by

$$\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s\,ds}X_T\right] = \mathbb{E}_t\left[\frac{\zeta_T}{\zeta_t}X_T\right] \tag{36}$$

The motivation for introducing the state-price deflator is not simply to price arbitrary future payoffs, but to restate the dynamic optimization problem in terms of a present value budget constraint. This transformation is what allows the Martingale method to bypass the need for solving a HJB equation.

[1] This assumption on $\boldsymbol{\lambda}$ implies that $\mathbb{E}\left[\int_0^T \|\boldsymbol{\lambda}\|^2 dt\right] < \infty$, ensuring that the state-price deflator is well-defined and has a meaningful interpretation.

4.2.2 General solution with the Martingale method

Having briefly established the pricing framework, the next step is to proceed with solving the investor's problem, as initially proposed in Equation (30). Recall that the wealth dynamics are given by

$$dW_t = \left[(r_t + \boldsymbol{\pi}_t^\top \underline{\underline{\sigma_t}} \boldsymbol{\lambda}_t)W_t - c_t\right] dt + W_t \boldsymbol{\pi}_t^\top \underline{\underline{\sigma_t}} d\boldsymbol{Z}_t \tag{37}$$

With the dynamics of W_t known, Itô's product rule (see Appendix) is applied to find the dynamics of $d\zeta_t W_t$

$$d\zeta_t W_t = -\zeta_t c_t \, dt + \zeta_t W_t \left(\boldsymbol{\pi}_t^\top \underline{\underline{\sigma}}_t - \boldsymbol{\lambda}_t^\top\right) d\boldsymbol{Z}_t \tag{38}$$

Or equivalently

$$\zeta_t W_t + \int_0^t \zeta_s c_s \, ds = W_0 + \int_0^t \zeta_s W_s \left(\boldsymbol{\pi}_s^\top \underline{\underline{\sigma}}_t - \boldsymbol{\lambda}_s^\top\right) d\boldsymbol{Z}_s \tag{39}$$

Next, the stopping times $\tau_{n\in\mathbb{N}}$ are defined by

$$\tau_n = T \wedge \inf \left\{ t \in [0,T] \mid \int_0^t \|\zeta_s W_s \left[\boldsymbol{\pi}_s^\top \underline{\underline{\sigma}}_t - \boldsymbol{\lambda}_s\right] \|^2 ds \geq n \right\} \tag{40}$$

The stopping times τ_n represent the first moment when the cumulative portfolio risk, defined by the squared difference between portfolio returns and the market price of risk, exceeds a threshold n.

Once the stopping times are defined, the integral on the right-hand side of Equation (39) is a martingale[2] on $[0, \tau_n]$. Taking expectations of Equation (39)

$$\mathbb{E}[\zeta_{\tau_n} W_{\tau_n}] + \mathbb{E}\left[\int_0^{\tau_n} \zeta_t c_t \, dt\right] = W_0 \tag{41}$$

Since the stochastic integral is a martingale, its time dependence disappears when taking expectations. This follows from the property of martingales, where the expected value at any time t, conditioned on the present, equals its current value, leading to the simplification of the process.

[2] A martingale is a stochastic process where the future value, conditional on the present, is equal to the current value.

To proceed, Lebesgue's monotone convergence theorem is applied, ensuring the limit of the integral is well-behaved. The theorem states that if (f_n) is a sequence of measurable functions such that $f_n \to f$ pointwise and $f_n \geq 0$, then

$$\lim_{n\to\infty} \int_0^T f_n \, d\mu = \int_0^T \lim_{n\to\infty} f \, d\mu \tag{42}$$

The sequence of integrals of the consumption process $\zeta_t c_t$ over the stopping times τ_n converges as $\tau_n \to T$. Therefore, by applying the monotone convergence theorem

$$\mathbb{E}\left[\int_0^{\tau_n} \zeta_t c_t \, dt\right] \to \mathbb{E}\left[\int_0^T \zeta_t c_t \, dt\right] \tag{43}$$

Furthermore, Fatou's lemma can be applied to show that

$$\liminf_{n\to\infty} \mathbb{E}[\zeta_{\tau_n} W_{\tau_n}] \geq \mathbb{E}[\zeta_T W_T] \tag{44}$$

Fatou's lemma ensures that the expected wealth at time T is at least as large as the limit of expected wealth at the stopping times τ_n as they approach T. It guarantees that the optimal strategy remains consistent and does not degrade as the terminal time is approached.

Thus, a natural constraint for any admissible strategy pair $(C_t, \boldsymbol{\pi}_t)$ at time $t = 0$ is

$$\mathbb{E}\left[\int_0^T \zeta_t C_t dt + \zeta_T W_T\right] \leq W_0 \tag{45}$$

where W_T is the terminal wealth generated by the strategy $(C_t, \boldsymbol{\pi}_t)$, and W_0 is the investor's initial wealth. This condition implies that the value or "price" of the chosen strategy at time $t = 0$ cannot exceed the available initial wealth.

Instead of the optimization problem presented in Chapter 3, the central idea of the Martingale method is to now focus on a static formulation of the problem

$$\begin{aligned} \sup_{(C_t, W_t)} \mathbb{E}_0 &\left[\int_0^T e^{-\delta t} U(C_t, t) dt + e^{-\delta T} \bar{U}(W_T)\right] \\ \text{w.r.t.} \quad \mathbb{E} &\left[\int_0^T \zeta_t C_t dt + \zeta_T W_T\right] \leq W_0 \end{aligned} \tag{46}$$

The investor selects a terminal wealth variable W from the set of non-negative, integrable random variables that are measurable with respect to the final information set $\mathcal{F}_T$, which essentially means that the chosen value of W must be based only on information available at the final time T and must have finite expected value under the probability measure.

The Lagrangian for the constrained optimization problem from Equation (46) is given by

$$\begin{aligned}\mathcal{L} = \mathbb{E}_0 \left[\int_0^T e^{-\delta t} U(C_t, t)\, dt + e^{-\delta T} \bar{U}(W_T) \right] \\ + \psi \left(W_0 - \mathbb{E}_0 \left[\int_0^T \zeta_t C_t \, dt + \zeta_T W_T \right] \right)\end{aligned} \tag{47}$$

Or equivalently

$$\mathcal{L} = \psi W_0 + \mathbb{E} \left[\int_0^T \left(e^{-\delta t} U(C_t, t) - \psi \zeta_t C_t \right) dt + \left(e^{-\delta T} \bar{U}(W_T) - \psi \zeta_T W \right) \right] \tag{48}$$

Where ψ is the Lagrange multiplier. The entire Lagrangian problem can be maximized under the expectation operator. This involves first maximizing $e^{-\delta t} U(C_t, t) - \psi \zeta_t C_t$ with respect to C_t for all possible values of ζ_t, and second, maximizing $e^{-\delta T} \bar{U}(W_T) - \psi \zeta_T W_T$ with respect to W_T for all possible values of ζ_T for all t.

Simple differentiation results in the following first-order conditoins

$$e^{-\delta t} U'_{C_t}(C_t, t) = \psi \zeta_t, \quad e^{-\delta t} \bar{U}'_{W_t}(W_t) = \psi \zeta_T \tag{49}$$

The FOC show that the marginal utility of consumption at time t must align with the cost of consumption, adjusted for time value. Similarly, the marginal utility of wealth at time T must correspond to the wealth the agent wants to leave behind. Thus, the optimal consumption at time T is directly related to the terminal wealth.

Let $I_U(\cdot)$ represent the inverse of $U'_{C_t}(\cdot)$, and $I_{\bar{U}}(\cdot)$ represent the inverse of $\bar{U}'_{W_t}(\cdot)$. The candidates for the optimal consumption and terminal wealth can then be expressed as

$$C_t = I_U(e^{\delta t} \psi \zeta_t), \quad W_T = I_{\bar{U}}(e^{\delta T} \psi \zeta_T) \tag{50}$$

The optimal choices depend on the Lagrange multiplier ψ, and the present value of these choices can be expressed as a function of ψ

$$\mathcal{D}(\psi) = \mathbb{E} \left[\int_0^T \zeta_t I_U \left(e^{\delta t} \psi \zeta_t \right) dt + \zeta_T I_{\bar{U}} \left(e^{\delta T} \psi \zeta_T \right) \right] \tag{51}$$

It is natural to choose ψ such that $\mathcal{D}(\psi) = W_0$, ensuring that the entire budget is spent. Furthermore, assuming that $\mathcal{D}(\psi) < \infty$ for all $\psi > 0$, the function $\mathcal{D}$ has an inverse, $\mathcal{Y}$, where it holds that $\psi = \mathcal{Y}(W_0)$. Given this, it can be shown that a reasonable solution to the static problem in Equation (46) will also be the optimal solution for the dynamic problem in Equation (30). The optimal solutions are denoted with stars

$$C_t^* = I_U\left(e^{\delta t}\mathcal{Y}(W_0)\zeta_t\right) \tag{52}$$

$$W^* = I_{\bar{U}}\left(e^{\delta T}\mathcal{Y}(W_0)\zeta_T\right) \tag{53}$$

With the corresponding wealth process, given the optimal strategy for $(C_t, \boldsymbol{\pi}_t)$

$$W_t^* = \frac{1}{\zeta_t}\mathbb{E}_t\left[\int_t^T \zeta_s C_s^*\, ds + \zeta_T W^*\right] \tag{54}$$

Define $W^\star$ from Equation (54), and it is trivial that

$$\zeta_t W_t^* + \int_0^t \zeta_s C_s^*\, ds = \mathbb{E}_t^P\left[\int_t^T \zeta_s C_s^*\, ds + \zeta_T W^*\right] \tag{55}$$

The left-hand side of Equation (55) represents the wealth at time t plus the accumulated consumption from 0 to t. The equation implies that this must equal the expectation, at time t, of the future evolution of consumption and terminal wealth over the interval $[t, T]$, which is consistent with the definition of a martingale.

By the martingale representation theorem, there exists an adapted $\mathcal{L}^2[0, T]$ process η such that

$$\zeta_t W_t^* + \int_0^t \zeta_s C_s^*\, ds = W_0 + \int_0^t \boldsymbol{\eta}_s^\top\, d\boldsymbol{Z}_s \tag{56}$$

Thus, the optimal portfolio strategy can be found by solving the following equation for $\boldsymbol{\pi_t}$

$$\begin{aligned} &\int_0^t \zeta_s W_s(\boldsymbol{\pi}_s^\top \underline{\underline{\sigma_s}} - \boldsymbol{\lambda}_s^\top)\, d\boldsymbol{Z}_s = \int_0^t \boldsymbol{\eta}_s^\top\, d\boldsymbol{Z}_s \Leftrightarrow \\ &\boldsymbol{\pi}_t = \left(\underline{\underline{\sigma_t}}^\top\right)^{-1}\left(\frac{\boldsymbol{\eta}_t}{W_t^*\zeta_t} + \boldsymbol{\lambda}_t\right) \end{aligned} \tag{57}$$

With the remainder of the wealth, $W_t^\star(1 - \boldsymbol{\pi_t}^\top \mathbf{1})$, invested in the risk-free asset. Finally, by comparing Equations (37) and (55), it can be seen that the portfolio strategy above, together with the optimal consumption strategy from Equation (52), exactly gives the optimal wealth process in Equation (54).

5 Market models

5.1 Introduction

This section introduces the two types of market models. The purpose is not to cover all possible ways of modelling financial markets, but rather to focus on two commonly used approaches that differ in how investment opportunities evolve over time.

The first model assumes that key financial variables are constant. This makes the model relatively simple and allows for clear insights into how a risk-averse investor allocates wealth across assets. However, this setup also ignores several important features observed in financial markets, such as time variation in the compensation investors require for taking on risk.

To address this, the second model introduces uncertainty about future investment conditions. This is done by allowing the market price of risk to change over time, through a mean-reverting process. Although this adds complexity, it also captures more realistic investor behaviour. In particular, the investor may want to hedge against future changes in expected returns, something that does not occur in the simpler model.

The comparison between the two models helps to clarify when and why it is important to account for such changes, and how they influence the optimal investment strategy.

While the constant model serves as a useful starting point, its implications are largely in line with standard intuition. For instance, as we will see, increasing risk aversion simply shifts the portfolio weight toward the safe asset. While this thesis focuses exclusively on a setting with a time-varying market price of risk, there are several alternative ways to introduce dynamic investment opportunities. These include stochastic interest rates, endogenous labor income, and borrowing constraints, all of which can significantly influence investor behavior and optimal portfolio choice. Each approach introduces different economic mechanisms and trade-offs, depending on the context being studied.

5.2 Constant investment opportunities

The model with constant investment opportunities is the simpler of the two considered. This version excludes any stochastic elements, except for the risk associated with the risky asset in the market, as briefly discussed in Chapter 3 when deriving the wealth dynamics constraint. In this model, the behavior of the risky assets and the single risk-free asset is formally defined as

$$\begin{aligned} dP_t &= (r + \sigma\lambda)P_t \, dt + \sigma P_t \, dZ_t \\ dB_t &= rB_t \, dt \end{aligned} \tag{58}$$

Here, P_t represents the price of the risky asset, with its dynamics driven by the risk-free rate r, the volatility σ, and the market price of risk λ. The process dZ_t denotes the Brownian motion. The equation dB_t describes the evolution of the risk-free asset, which grows deterministically at the risk-free rate r.

In this framework, the drift of the risky asset is $r + \sigma\lambda$, where λ represents the market price of risk, and σ is the volatility of the asset. The volatility component determines the degree of fluctuation in the asset's price, while the market price of risk adjusts the drift of the risky asset to account for the additional risk premium demanded by investors.

5.3 Stochastic investment opportunities

In the stochastic model, the market price of risk evolves over time according to an Ornstein–Uhlenbeck process, mean-reverting toward a long-term average level. This framework is inspired by Kim and Omberg (1996) and Wachter (2002), who derive closed-form solutions for optimal investment strategies under a constant risk-free rate r and a single risky asset, interpreted as a stock market index. In their models, the price of the risky asset follows a geometric Brownian motion with constant volatility σ, while the market price of risk λ_t evolves as a mean-reverting process. Following this line of modeling, the financial market with stochastic investment opportunities is specified as

$$\begin{aligned} dP_t &= (r + \sigma\lambda_t)P_t \, dt + \sigma P_t \, dZ_t \\ d\lambda_t &= \kappa(\bar{\lambda} - \lambda_t) \, dt + \rho\sigma_\lambda \, dZ_t + \sqrt{1 - \rho^2}\sigma_\lambda \, dZ_{\lambda,t} \\ dB_t &= rB_t \, dt \end{aligned} \tag{59}$$

Here, κ, $\bar{\lambda}$, and σ_λ are positive constants, and $\rho \in [-1, 1]$ denotes the correlation between the Brownian motions driving the stock return and the market price of risk. The parameter ρ plays a central role in shaping the dynamics of the model. In particular, a negative value of ρ captures the idea that a positive shock dZ_t affects the current stock return

$$\frac{dP_t}{P_t} = (r + \sigma\lambda_t)\, dt + \sigma\, dZ_t$$

positively, while simultaneously reducing the market price of risk at the next instant

$$\lambda_{t+dt} = \lambda_t + d\lambda_t.$$

Hence, expected returns over the short horizon $[t, t + dt]$ are lower than before the shock, reflecting a mean-reverting structure in the Sharpe ratio. This mechanism implies that high realized returns today are typically followed by periods of lower expected returns, and vice versa. Applying Itô's Lemma, the future value of the Sharpe ratio λ_s for $s > t$ is normally distributed with conditional mean and variance

$$\mathbb{E}(\lambda_s) = \lambda_t e^{-\kappa(s-t)} + \bar{\lambda}\left(1 - e^{-\kappa(s-t)}\right)$$

$$\mathrm{Var}(\lambda_s) = \frac{\sigma_\lambda^2}{2\kappa}\left(1 - e^{-2\kappa(s-t)}\right)$$

As t increases, the mean converges to $\bar{\lambda}$ and the variance to $\sigma_\lambda^2/(2\kappa)$. When $\kappa \to \infty$, mean reversion is immediate, while $\kappa \to 0$ implies no mean reversion and variance growing linearly over time.

Wachter (2002) argues that a correlation of $\rho = -1$ is empirically reasonable. This assumption will be applied in this framework, which removes the last term in the $d\lambda_t$ equation and links shocks to returns directly with changes in the market price of risk. This dynamic implies that fluctuations in risk premia are entirely driven by the same shocks affecting current returns. Empirical support for this mechanism is found in Fama and French (1989), who document that periods of high realized returns are typically followed by lower expected returns.

6 Solving the model with constant investment opportunities

6.1 Introduction

Chapter 2 introduced the utility functions employed to represent investor preferences. Chapters 3 and 4 established the dynamic budget constraint, formulated the corresponding maximization problem, and presented the Martingale method as the primary solution technique. Chapter 5 outlined the two market environments under consideration: one characterized by constant investment opportunities and the other by stochastic dynamics.

The objective is now to determine the optimal consumption and portfolio strategies, $(C_t, \boldsymbol{\pi}_t)$, across both market specifications and for each type of investor, using the Martingale method. The derivation begins with the case of constant investment opportunities and a CRRA utility function. This choice reflects both the natural starting point in terms of model complexity and a desire to build the solution method step by step. The following chapter extend the framework to settings involving stochastic market dynamics, which, as will be seen, introduce additional analytical challenges.

6.2 CRRA utility

Recall that the CRRA utility functions for consumption C_t and terminal wealth W_T are defined as

$$U(C_t) = \varepsilon_1 \frac{C_t^{1-\gamma} - 1}{1-\gamma}, \qquad \bar{U}(W_t) = \varepsilon_2 \frac{W_T^{1-\gamma} - 1}{1-\gamma} \tag{60}$$

Here, $\varepsilon_1, \varepsilon_2 > 0$, ensuring that the investor derives utility from both consumption and terminal wealth. The first-order derivatives are computed as

$$U'(C_t) = \varepsilon_1 C_t^{-\gamma}, \qquad \bar{U}'(W_T) = \varepsilon_2 W_T^{-\gamma} \tag{61}$$

With inverse functions

$$I_U(z) = \varepsilon_1^{\frac{1}{\gamma}} z^{-\frac{1}{\gamma}}, \qquad I_{\bar{U}}(z) = \varepsilon_2^{\frac{1}{\gamma}} z^{-\frac{1}{\gamma}} \tag{62}$$

Inserting into the definition of $\mathcal{D}(\psi)$

$$\begin{aligned}\mathcal{D}(\psi) &= \mathbb{E}\left[\int_0^T \zeta_t \varepsilon_1^{\frac{1}{\gamma}} e^{-\frac{\delta}{\gamma}t} \psi^{-\frac{1}{\gamma}} \zeta_t^{-\frac{1}{\gamma}} dt + \zeta_T \varepsilon_2^{\frac{1}{\gamma}} e^{-\frac{\delta}{\gamma}T} \psi^{-\frac{1}{\gamma}} \zeta_T^{-\frac{1}{\gamma}}\right] \\ &= \psi^{-\frac{1}{\gamma}}\, \mathbb{E}\left[\int_0^T \varepsilon_1^{\frac{1}{\gamma}} e^{-\frac{\delta}{\gamma}t} \zeta_t^{1-\frac{1}{\gamma}} dt + \varepsilon_2^{\frac{1}{\gamma}} e^{-\frac{\delta}{\gamma}T} \zeta_T^{1-\frac{1}{\gamma}}\right] \\ &= \psi^{-\frac{1}{\gamma}} g_0\end{aligned} \tag{63}$$

This result follows by applying the definition

$$g_t = \mathbb{E}_t\left[\int_t^T \varepsilon_1^{\frac{1}{\gamma}} e^{-\frac{\delta}{\gamma}(s-t)} \left(\frac{\zeta_s}{\zeta_t}\right)^{1-\frac{1}{\gamma}} ds + \varepsilon_2^{\frac{1}{\gamma}} e^{-\frac{\delta}{\gamma}(T-t)} \left(\frac{\zeta_T}{\zeta_t}\right)^{1-\frac{1}{\gamma}}\right] \tag{64}$$

This definition will be helpful in the derivations ahead. For g_0, $\zeta_0 = 1$ holds, which makes the expression in Equation (63) slightly more simple.

To proceed with the Martingale method, recall that the inverse of $\mathcal{D}(\psi)$ with respect to W_0 is denoted as $\mathcal{Y}(W_0)$, and is given by

$$\mathcal{Y}(W_0) = W_0^{-\gamma} g_0^{\gamma} \tag{65}$$

Inserting this result into Equations (52) and (53) yields the optimal consumption policy and optimal terminal wealth

$$\begin{aligned}C_t^* &= \varepsilon_1^{\frac{1}{\gamma}} \left(W_0^{-\gamma} g_0^{\gamma} e^{\delta t} \zeta_t\right)^{-\frac{1}{\gamma}} = \varepsilon_1^{\frac{1}{\gamma}} \frac{W_0}{g_0} e^{-\frac{\delta}{\gamma}t} \zeta_t^{-\frac{1}{\gamma}} \\ &= e^{-\frac{\delta}{\gamma}t} \zeta_t^{-\frac{1}{\gamma}} W_0 \left(\mathbb{E}\left[\int_0^T e^{-\frac{\delta}{\gamma}t} \zeta_t^{1-\frac{1}{\gamma}} dt + \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^{\frac{1}{\gamma}} e^{-\frac{\delta}{\gamma}T} \zeta_T^{1-\frac{1}{\gamma}}\right]\right)^{-1}\end{aligned} \tag{66}$$

$$\begin{aligned}W^* &= \varepsilon_2^{\frac{1}{\gamma}} \left(W_0^{-\gamma} g_0^{\gamma} e^{\delta T} \zeta_T\right)^{-\frac{1}{\gamma}} = \varepsilon_2^{\frac{1}{\gamma}} \frac{W_0}{g_0} e^{-\frac{\delta}{\gamma}T} \zeta_T^{-\frac{1}{\gamma}} \\ &= e^{-\frac{\delta}{\gamma}T} \zeta_T^{-\frac{1}{\gamma}} W_0 \left(\mathbb{E}\left[\int_0^T \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^{\frac{1}{\gamma}} e^{-\frac{\delta}{\gamma}t} \zeta_t^{1-\frac{1}{\gamma}} dt + e^{-\frac{\delta}{\gamma}T} \zeta_T^{1-\frac{1}{\gamma}}\right]\right)^{-1}\end{aligned} \tag{67}$$

From Equation (54), the wealth process under the optimal policy is given by

$$
\begin{aligned}
W_t^* &= \frac{1}{\zeta_t}\,\mathbb{E}_t\left[\int_t^T \zeta_s C_s^* ds + \zeta_T W^*\right] \\
&= \frac{W_0}{g_0}\frac{1}{\zeta_t}\,\mathbb{E}_t\left[\int_t^T \varepsilon_1^{\frac{1}{\gamma}} e^{-\frac{\delta}{\gamma}s}\zeta_s^{1-\frac{1}{\gamma}} ds + \varepsilon_2^{\frac{1}{\gamma}} e^{-\frac{\delta}{\gamma}T}\zeta_T^{1-\frac{1}{\gamma}}\right] \\
&= \frac{W_0}{g_0} e^{-\frac{\delta}{\gamma}t}\zeta_t^{-\frac{1}{\gamma}}\,\mathbb{E}_t\left[\int_t^T \varepsilon_1^{\frac{1}{\gamma}} e^{-\frac{\delta}{\gamma}(s-t)}\left(\frac{\zeta_s}{\zeta_t}\right)^{1-\frac{1}{\gamma}} ds + \varepsilon_2^{\frac{1}{\gamma}} e^{-\frac{\delta}{\gamma}(T-t)}\left(\frac{\zeta_T}{\zeta_t}\right)^{1-\frac{1}{\gamma}}\right] \\
&= \frac{W_0}{g_0} e^{-\frac{\delta}{\gamma}t}\zeta_t^{-\frac{1}{\gamma}} g_t
\end{aligned}
\tag{68}
$$

Consequently

$$
\frac{W_t^*}{g_t} = \frac{W_0}{g_0} e^{-\frac{\delta}{\gamma}t}\zeta_t^{-\frac{1}{\gamma}} \tag{69}
$$

The LHS of Equation (69) can be inserted into the optimal consumption policy such that

$$
C_t^* = \varepsilon_1^{\frac{1}{\gamma}}\frac{W_t^*}{g_t} \quad \Leftrightarrow \quad \frac{C_t^*}{W_t^*} = \frac{\varepsilon_1^{\frac{1}{\gamma}}}{g_t} \tag{70}
$$

The consumption-to-wealth ratio is inversely proportional to g_t. As $t \to T$, the ratio $\frac{C_t^*}{W_t^*}$ converges to $\left(\frac{\varepsilon_1}{\varepsilon_2}\right)^{\frac{1}{\gamma}}$ when $\varepsilon_1, \varepsilon_2 > 0$. In contrast, if $\varepsilon_2 = 0$, then $\frac{C_t^*}{W_t^*} \to \infty$, which is consistent with the interpretation that an investor without utility from terminal wealth will fully deplete their wealth.

For any future time point s such that $t < s < T$, Equation (66) directly yields the expression for the uncertain consumption rate and optimal terminal wealth at time s, conditional on the information available at time t

$$
\begin{aligned}
C_s^* &= \epsilon_1^{\frac{1}{\gamma}}\frac{W_0}{g_0} e^{-\frac{\delta}{\gamma}s}\zeta_s^{-\frac{1}{\gamma}} = \epsilon_1^{\frac{1}{\gamma}}\frac{W_0}{g_0} e^{-\frac{\delta}{\gamma}t}\zeta_t^{-\frac{1}{\gamma}} e^{-\frac{\delta}{\gamma}(s-t)}\left(\frac{\zeta_s}{\zeta_t}\right)^{-\frac{1}{\gamma}} \\
&= \frac{W_t^*}{g_t}\epsilon_1^{\frac{1}{\gamma}} e^{-\frac{\delta}{\gamma}(s-t)}\left(\frac{\zeta_s}{\zeta_t}\right)^{-\frac{1}{\gamma}}
\end{aligned}
\tag{71}
$$

And

$$
W^* = \frac{W_t^*}{g_t}\epsilon_2^{\frac{1}{\gamma}} e^{-\frac{\delta}{\gamma}(T-t)}\left(\frac{\zeta_T}{\zeta_t}\right)^{-\frac{1}{\gamma}} \tag{72}
$$

Following Equation (30), (71) and (72) the indirect utility at time t is

$$\begin{aligned}
J_t &= \mathbb{E}_t\left[\int_t^T e^{-\delta(s-t)}U(C_s^*)ds + e^{-\delta(T-t)}\bar{U}(W^*)\right] \\
&= \frac{1}{1-\gamma}\mathbb{E}_t\left[\int_t^T e^{-\delta(s-t)}\epsilon_1(C_s^*)^{1-\gamma}ds + e^{-\delta(T-t)}\epsilon_2(W^*)^{1-\gamma}\right] \\
&= \frac{1}{1-\gamma}\left(\frac{W_t^*}{g_t}\right)^{1-\gamma}\mathbb{E}_t\left[\int_t^T e^{-\frac{\delta}{\gamma}(s-t)}\frac{1}{\epsilon_1}\left(\frac{\zeta_s}{\zeta_t}\right)^{-\frac{1}{\gamma}}ds + e^{-\frac{\delta}{\gamma}(T-t)}\frac{1}{\epsilon_2}\left(\frac{\zeta_T}{\zeta_t}\right)^{-\frac{1}{\gamma}}\right] \\
&= \frac{1}{1-\gamma}g_t^{\gamma}(W_t^*)^{1-\gamma}
\end{aligned} \tag{73}$$

Recall that the state-price deflator is given by

$$\zeta_t = \exp\left\{-\int_0^t r_s\,ds - \int_0^t \boldsymbol{\lambda}_s^\top\,d\boldsymbol{Z}_s - \frac{1}{2}\int_0^t \|\boldsymbol{\lambda}_s\|^2\,ds\right\} \tag{74}$$

To proceed, consider the future values of the state-price deflator for any $s > t$

$$\begin{aligned}
\mathbb{E}_t\left[\int_t^T e^{-\delta(s-t)}\left(\frac{\zeta_s}{\zeta_t}\right)^{1-\frac{1}{\gamma}}ds\right] &= \mathbb{E}_t\left[\int_t^T e^{-\delta(s-t)}\left(e^{-r(s-t)-\boldsymbol{\lambda}^\top(\boldsymbol{Z}_s-\boldsymbol{Z}_t)-\frac{1}{2}\|\boldsymbol{\lambda}\|^2(s-t)}\right)^{1-\frac{1}{\gamma}}ds\right] \\
&= e^{-\frac{\delta}{\gamma}(s-t)}e^{-(1-\frac{1}{\gamma})r(s-t)-\frac{1}{2}(1-\frac{1}{\gamma})\|\boldsymbol{\lambda}\|^2(s-t)}\mathbb{E}_t\left[e^{-(1-\frac{1}{\gamma})\boldsymbol{\lambda}^\top(\boldsymbol{Z}_s-\boldsymbol{Z}_t)}\right] \\
&= e^{-\frac{\delta}{\gamma}(s-t)}e^{-(1-\frac{1}{\gamma})r(s-t)-\frac{1}{2}(1-\frac{1}{\gamma})\|\boldsymbol{\lambda}\|^2(s-t)}e^{\frac{1}{2}(1-\frac{1}{\gamma})^2\|\boldsymbol{\lambda}\|^2(s-t)} \\
&= e^{-\left(\frac{\delta-r(1-\gamma)}{\gamma}-\frac{1}{2}\frac{1-\gamma}{\gamma^2}\|\boldsymbol{\lambda}\|^2\right)(s-t)} \\
&= e^{-A(s-t)}
\end{aligned} \tag{75}$$

Where $A = \frac{\delta+r(\gamma-1)}{\gamma} + \frac{1}{2}\frac{\gamma-1}{\gamma^2}\|\boldsymbol{\lambda}\|^2$. The expectation in the second equation can be evaluated explicitly, as future values of the state-price deflator follow a lognormal distribution: If $x \sim \mathcal{N}(m, s^2)$, then $\mathbb{E}[e^{-ax}] = e^{-am+\frac{1}{2}a^2s^2}$. In the present setting, $a = 1 - \frac{1}{\gamma}$, and $x = \boldsymbol{\lambda}^\top(\mathbf{Z}_s - \mathbf{Z}_t) = \sum_{i=1}^d \lambda_i(z_{is} - z_{it})$, which follows a normal distribution with mean zero and variance $\sum_{i=1}^d \lambda_i^2(s-t) = \|\boldsymbol{\lambda}\|^2(s-t)$.

The close-form expression for g_t is given by

$$
\begin{aligned}
g_t &= \mathbb{E}_t\left[\int_t^T \epsilon_1^{\frac{1}{\gamma}} e^{-\frac{\delta}{\gamma}(s-t)}\left(\frac{\zeta_s}{\zeta_t}\right)^{1-\frac{1}{\gamma}} ds + \epsilon_2^{\frac{1}{\gamma}} e^{-\frac{\delta}{\gamma}(T-t)}\left(\frac{\zeta_T}{\zeta_t}\right)^{1-\frac{1}{\gamma}}\right] \\
&= \int_t^T \epsilon_1^{\frac{1}{\gamma}} \mathbb{E}_t\left[e^{-\frac{\delta}{\gamma}(s-t)}\left(\frac{\zeta_s}{\zeta_t}\right)^{1-\frac{1}{\gamma}}\right] ds + \epsilon_2^{\frac{1}{\gamma}} \mathbb{E}_t\left[e^{-\frac{\delta}{\gamma}(T-t)}\left(\frac{\zeta_T}{\zeta_t}\right)^{1-\frac{1}{\gamma}}\right] \\
&= \int_t^T \epsilon_1^{\frac{1}{\gamma}} e^{-A(s-t)} ds + \epsilon_2^{\frac{1}{\gamma}} e^{-A(T-t)} \\
&= \frac{1}{A}\left(\epsilon_1^{\frac{1}{\gamma}} + \left(A\epsilon_2^{\frac{1}{\gamma}} - \epsilon_1^{\frac{1}{\gamma}}\right) e^{-A(T-t)}\right)
\end{aligned}
\tag{76}
$$

Applying Itô's Lemma on the optimal wealth process (see Appendix)

$$
dW_t^* = \left(\frac{r-\delta}{\gamma} + \frac{g_t'}{g_t} + \frac{1}{2}\left(\frac{1}{\gamma^2} + \frac{1}{\gamma}\right)\|\boldsymbol{\lambda}\|^2\right) W_t^*\, dt + W_t^* \frac{1}{\gamma}\boldsymbol{\lambda}^\top d\mathbf{Z}_t \tag{77}
$$

This result follows directly from the closed-form expression for g_t, the optimal wealth process in Equation (68), and the dynamics of the state-price deflator presented in Equation (31).

Using the primary result from the Martingale method presented in Equation (57), together with the dynamics of the optimal wealth process, the optimal portfolio strategy is obtained by equating the diffusion terms, since both must be the same along the optimal path

$$
W_t^* \boldsymbol{\pi}_t^\top \underline{\underline{\sigma}}\, d\mathbf{Z}_t = W_t^* \frac{1}{\gamma}\boldsymbol{\lambda}^\top\, d\mathbf{Z}_t \tag{78}
$$

Since both expressions represent the same stochastic component (complete market), the optimal portfolio strategy is given by

$$
\boldsymbol{\pi}_t^* = \frac{1}{\gamma}(\underline{\underline{\sigma}}^\top)^{-1}\boldsymbol{\lambda}. \tag{79}
$$

The optimal portfolio strategy increases with the market price of risk $\boldsymbol{\lambda}$, meaning the investor allocates more to assets offering higher excess returns per unit of risk. The allocation is scaled by $\frac{1}{\gamma}$, reflecting that more risk-averse investors (higher γ) take smaller positions in risky assets. The term $(\boldsymbol{\sigma}^\top)^{-1}$ adjusts the risky position, assigning a smaller weight when the asset is more volatile, since the investor is risk-averse.

6.3 Benchmark-adjusted CRRA utility

Recall that the benchmark-adjusted CRRA utility functions for consumption C_t and terminal wealth W_T are defined as follows

$$U(C_t) = \varepsilon_1 \frac{(C_t - \bar{C})^{1-\gamma(C_t)} - 1}{1 - \gamma(C_t)}, \qquad \bar{U}(W_t) = \varepsilon_2 \frac{(W_T - \bar{W})^{1-\gamma(C_t)} - 1}{1 - \gamma(C_t)} \tag{80}$$

The first-order derivatives are computed as

$$U'(C_t) = \varepsilon_1 (C_t - \bar{C})^{-\gamma(C_t)}, \qquad \bar{U}'(W_T) = \varepsilon_2 (W_T - \bar{W})^{-\gamma(C_t)} \tag{81}$$

With inverse functions

$$I_U(z) = \bar{C} + \varepsilon_1^{\frac{1}{\gamma(C_t)}} z^{-\frac{1}{\gamma(C_t)}}, \qquad I_{\bar{U}}(z) = \bar{W} + \varepsilon_2^{\frac{1}{\gamma(C_t)}} z^{-\frac{1}{\gamma(C_t)}} \tag{82}$$

Inserting into the definition of $\mathcal{D}(\psi)$

$$\begin{aligned} \mathcal{D}(\psi) &= \mathbb{E}\left[\int_0^T \left(\zeta_t \bar{C} + \epsilon_1 \zeta_t^{1-\frac{1}{\gamma(C_t)}} e^{-\frac{\delta}{\gamma(C_t)} t} \psi^{-\frac{1}{\gamma(C_t)}}\right) dt + \epsilon_2 e^{-\frac{\delta}{\gamma(C_t)} T} \psi^{-\frac{1}{\gamma(C_t)}} \zeta_T^{1-\frac{1}{\gamma(C_t)}} + \zeta_T \bar{W}\right] \\ &= \psi^{-\frac{1}{\gamma(C_t)}} g_0 + \bar{C}\mathbb{E}\left[\int_0^T \zeta_t dt + \zeta_T \bar{W}\right] \\ &= \psi^{-\frac{1}{\gamma(C_t)}} g_0 + \bar{C} F(0) \end{aligned} \tag{83}$$

Where, for now, $F(t)$ is a pricing formula such that

$$F(t) = \mathbb{E}\left[\int_t^T \frac{\zeta_s}{\zeta_t} ds + \frac{\zeta_T}{\zeta_t} \bar{W}\right] \tag{84}$$

The inverse of $\mathcal{D}(\psi)$ with respect to W_0 is denoted as $\mathcal{Y}(W_0)$, and is given by

$$\mathcal{Y}(W_0) = \left(\frac{W_0 - \bar{C}F(0)}{g_0}\right)^{-\gamma(C_t)} \tag{85}$$

Inserting this into Equation (52) and (53) yields the optimal consumption policy and terminal wealth

$$\begin{aligned} C_t^* &= \bar{C} + \epsilon_1^{\frac{1}{\gamma(C_t)}} (W_0 - \bar{C}F(0))^{-\gamma(C_t)} g_0^{\gamma(C_t)} e^{\delta t} \zeta_t^{-\frac{1}{\gamma(C_t)}} \\ &= \bar{C} + \epsilon_1^{\frac{1}{\gamma(C_t)}} \frac{W_t^* - \bar{C}F(t)}{g_t} \end{aligned} \tag{86}$$

$$\begin{aligned} W^* &= \bar{W} + \epsilon_2^{\frac{1}{\gamma(C_t)}} (W_0 - \bar{C}F(0))^{-\gamma(C_t)} g_0^{\gamma(C_t)} e^{\delta T} \zeta_T^{-\frac{1}{\gamma(C_t)}} \\ &= \bar{W} + \epsilon_2^{\frac{1}{\gamma(C_t)}} \frac{W_T - \bar{C}F(T)}{g_T} \end{aligned} \tag{87}$$

From Equation (54), the wealth process under the optimal policy is given by

$$\begin{aligned} W_t^* &= \frac{1}{\zeta_t}\mathbb{E}_t\bigg[\int_t^T \left(\zeta_s \bar{C} + \epsilon_1^{\frac{1}{\gamma(C_t)}} \frac{W_0 - F(0)}{g_0} e^{-\frac{\delta}{\gamma(C_t)} s} \zeta_s^{1-\frac{1}{\gamma(C_t)}} \right) ds \\ &\quad + \epsilon_2^{\frac{1}{\gamma(C_t)}} \frac{W_0 - F(0)}{g_0} e^{-\frac{\delta}{\gamma(C_t)} T} \zeta_T^{1-\frac{1}{\gamma(C_t)}} + \zeta_T \bar{W} \bigg] \\ &= \frac{W_0 - F(0)}{g_0} \frac{1}{\zeta_t} \mathbb{E}_t \left[\int_t^T \epsilon_1^{\frac{1}{\gamma(C_t)}} e^{-\frac{\delta}{\gamma(C_t)} s} \zeta_s^{1-\frac{1}{\gamma(C_t)}} ds + \epsilon_2^{\frac{1}{\gamma(C_t)}} e^{-\frac{\delta}{\gamma(C_t)} T} \zeta_T^{1-\frac{1}{\gamma(C_t)}} \right] \\ &\quad + \frac{1}{\zeta_t} \mathbb{E}_t \left[\int_t^T \zeta_s ds + \zeta_T \bar{W} \right] \\ &= \frac{W_0 - F(0)}{g_0} e^{-\frac{\delta}{\gamma(C_t)} t} \zeta_t^{-\frac{1}{\gamma(C_t)}} g_t + \bar{C}F(t) \end{aligned} \tag{88}$$

Applying Itô's Lemma on the optimal wealth process in Equation (see Appendix)

$$\begin{aligned} dW_t^* &= \cdots dt + \left(-\frac{1}{\gamma(C_t)} \frac{W_t^* - \bar{C}F(t)}{\zeta_t^*} \right) \left(-\zeta_t r dt - \zeta_t \boldsymbol{\lambda}^\top d\boldsymbol{Z}_t \right) \\ &= \cdots dt + \frac{1}{\gamma(C_t)} (W_t^* - \bar{C}F(t)) \boldsymbol{\lambda}_t^\top d\boldsymbol{Z}_t \end{aligned} \tag{89}$$

The optimal portfolio strategy is again, obtained by equating the diffusion terms

$$\begin{aligned} W_t^* \boldsymbol{\pi}_t \underline{\underline{\sigma}} d\boldsymbol{Z}_t &= \frac{1}{\gamma(C_t)} (W_t^* - \bar{C}F(t)) \boldsymbol{\lambda}^\top d\boldsymbol{Z}_t \;\Leftrightarrow \\ \boldsymbol{\pi}_t^* &= \frac{1}{\gamma(C_t)} (\underline{\underline{\sigma}}^\top)^{-1} \boldsymbol{\lambda} \frac{W_t^* - \bar{C}F(t)}{W_t^*} \end{aligned} \tag{90}$$

The optimal strategy suggests that the investor ensures a minimum level of consumption by investing a fraction $\frac{F(t)}{W_t^*}$ of total wealth in the risk-free asset. To implement this strategy, it is necessary that the investor's initial wealth satisfies $W_0 \geq F_0$.

In the numerical results, the behavior of $\gamma(C_t)$ is implemented by an `if` statement in the Python code, which selects between the high and low risk aversion parameters depending on whether consumption is sufficiently close to the benchmark level $\bar{C}$. This is determined by checking if the absolute difference $|C_t - \bar{C}|$ falls within a specified tolerance ε.

6.4 Habit utility

To derive the optimal strategy under habit utility, it is convenient to build on the solution obtained under stochastic investment opportunities, setting the terms that arise from market stochasticity to zero ($\boldsymbol{\sigma}_{gt} = 0$). Accordingly, the optimal solution $(C_t, \boldsymbol{\pi}_t)$ is stated here and formally explained in Section 7.4. Note that it is assumed that $\epsilon_1 = 1$ and $\epsilon_2 = 0$, in clear contrast to the two other cases. While this may initially seem problematic, it allows for a more elegant solution. Moreover, including ϵ_2 would not provide additional economic insight, as it would only imply that the investor allocates more wealth to the risk-free asset maturing at time T. The optimal consumption strategy is

$$C_t^* = h_t^* + (1 + \alpha F(t))^{-\frac{1}{\gamma}} \frac{W_t^* - h_t^* F(t)}{g_t} \tag{91}$$

And the optimal portfolio strategy is

$$\boldsymbol{\pi}_t^* = \frac{1}{\gamma} (\underline{\underline{\sigma}}^\top)^{-1} \boldsymbol{\lambda} \frac{W_t^* - h_t^* F(t)}{W_t^*} \tag{92}$$

The optimal portfolio under habit utility depends on surplus wealth, reflected by the term $W_t^* - h_t^* F(t)$. This introduces state-dependence in the strategy, as the allocation to risky assets decreases when wealth W_t^* approaches the habit-adjusted floor $h_t^* F(t)$.

$$F(t) = \int_t^T e^{-(r+\beta-\alpha)(s-t)} ds = \frac{1}{r + \beta - \alpha} \left(1 - e^{-(r+\beta-\alpha)(T-t)}\right) \tag{93}$$

The function $F(t)$ represents the present value of future habit-adjusted consumption, discounted at the effective rate $r + \beta - \alpha$. It determines the cost, in today's terms, of sustaining the habit level h_t^* until terminal time T.

7 Solving the model with stochastic investment opportunities

7.1 Introduction

Solving the model with stochastic investment opportunities is naturally more complex than in the case with constant parameters. In the previous setting, g_t was a deterministic function since both the interest rate r and the market price of risk $\boldsymbol{\lambda}$ were constant. When either of these components becomes stochastic, g_t evolves as a stochastic process. g_t captures the value of future investment opportunities as perceived at time t. If investment opportunities evolve stochastically, through changes in r, $\boldsymbol{\lambda}$, or both, then g_t must reflect this uncertainty. In general, the dynamics of g_t can be written as

$$dg_t = g_t \left[\mu_{gt}\, dt + \boldsymbol{\sigma}_{gt}^{\top} d\boldsymbol{Z}_t\right] \tag{94}$$

For some drift process $\mu_g = (\mu_{gt})$ and sensitivity process $\boldsymbol{\sigma}_g = (\boldsymbol{\sigma}_{gt})$.

It is useful to briefly recall the CRRA utility function solution and study how it shapes the optimal portfolio choice. Recall that the wealth process under the optimal policy is given by Equation (68)

$$W_t^{\star} = \frac{W_0}{g_0} e^{-\frac{\delta}{\gamma}t} \zeta_t^{-\frac{1}{\gamma}} g_t \tag{95}$$

As shown repeatedly in earlier sections, Itô's Lemma can be applied to Equation (95), since the dynamics of ζ_t and g_t are known (see Appendix)

$$\begin{aligned} dW_t^* &= \dots\, dt - \frac{1}{\gamma} \frac{W_t^*}{\zeta_t}\, d\zeta_t + \frac{W_t^*}{g_t}\, dg_t && (96) \\ &= \dots\, dt + W_t^* \left(\frac{1}{\gamma}\boldsymbol{\lambda}_t + \boldsymbol{\sigma}_{gt}\right)^{\top} d\boldsymbol{Z}_t && (97) \end{aligned}$$

Comparing with the dynamics of wealth for a general portfolio, it follows that an optimal investment strategy satsifies

$$\boldsymbol{\pi}_t^* = \underbrace{\frac{1}{\gamma}\left(\underline{\underline{\sigma_t}}^{\top}\right)^{-1} \boldsymbol{\lambda}_t}_{\text{Myopic term}} + \underbrace{\left(\underline{\underline{\sigma_t}}^{\top}\right)^{-1} \boldsymbol{\sigma}_{gt}}_{\text{Hedging term}} \tag{98}$$

The optimal portfolio under stochastic investment opportunities closely resembles the solution in the constant case, but includes an additional hedging component. This term, involving $\boldsymbol{\sigma}_{gt}$, reflects the investor's desire to hedge against changes in future investment opportunities. These arise through fluctuations in the stochastic discount factor, driven by time variation in the market price of risk. Since g_t enters the optimal consumption and wealth process, any uncertainty in its evolution creates an incentive to hedge, resulting in the additional portfolio term.

At first glance, Equation (98) appears rather harmless. An optimal portfolio strategy has been derived, so initially it might seem easy to just substitute into the expression to obtain the portfolio weights and the allocation to the risk-free asset. However, implementing this solution requires an understanding of the element $\boldsymbol{\sigma}_{gt}$. The following sections explore this component in detail.

Under stochastic investment opportunities, the analysis will assume $\varepsilon_1 = 1$ and $\varepsilon_2 = 0$. This choice is primarily motivated by the aim to obtain clearer results. Including utility from terminal wealth adds considerable analytical complexity without offering substantial additional insight into the asset allocation problem. Furthermore, this assumption makes it easier to follow the approach applied in Munk (2002).

Also, it is worth noting that the case with $\varepsilon_2 = 1$ has already been treated in the setting with constant investment opportunities, demonstrating how it can be incorporated into the framework.

7.2 CRRA utility

Instead of repeating the same derivations from earlier, simply recall that

$$g_t = \mathbb{E}_t\left[\int_t^T \varepsilon_1^{\frac{1}{\gamma}} e^{-\frac{\delta}{\gamma}(s-t)}\left(\frac{\zeta_s}{\zeta_t}\right)^{1-\frac{1}{\gamma}} ds + \varepsilon_2^{\frac{1}{\gamma}} e^{-\frac{\delta}{\gamma}(T-t)}\left(\frac{\zeta_T}{\zeta_t}\right)^{1-\frac{1}{\gamma}}\right] \tag{99}$$

For $\varepsilon_1 = 1$ and $\varepsilon_2 = 0$ this shortens to

$$\begin{aligned} g_t &= \mathbb{E}_t\left[\int_t^T e^{-\frac{\delta}{\gamma}(s-t)}\left(\frac{\zeta_s}{\zeta_t}\right)^{1-\frac{1}{\gamma}} ds\right] \\ &= \int_t^T e^{-\frac{\delta}{\gamma}(s-t)}\mathbb{E}_t\left[\left(\frac{\zeta_s}{\zeta_t}\right)^{1-\frac{1}{\gamma}}\right] ds \end{aligned} \tag{100}$$

Rewriting the expectation as

$$\begin{aligned}
\mathbb{E}_t\left[\left(\frac{\zeta_s}{\zeta_t}\right)^{1-\frac{1}{\gamma}}\right] &= e^{-r(1-\frac{1}{\gamma})(s-t)}\,\mathbb{E}_t\left[e^{-\left(1-\frac{1}{\gamma}\right)\int_t^s \boldsymbol{\lambda}_u\, d\boldsymbol{Z}_u - \frac{1}{2}\left(1-\frac{1}{\gamma}\right)\int_t^s \boldsymbol{\lambda}_u^2\, du}\right] \\
&= e^{-r(1-\frac{1}{\gamma})(s-t)}\,\mathbb{E}_t\left[e^{-\left(1-\frac{1}{\gamma}\right)\int_t^s \boldsymbol{\lambda}_u\, d\boldsymbol{Z}_u - \frac{1}{2}\left(1-\frac{1}{\gamma}\right)^2\int_t^s \boldsymbol{\lambda}_u^2\, du} \cdot e^{-\frac{1}{2\gamma}(1-\frac{1}{\gamma})\int_t^s \boldsymbol{\lambda}_u^2\, du}\right] \\
&= e^{-r(1-\frac{1}{\gamma})(s-t)}\,\mathbb{E}_t^{\mathbb{Q}^\gamma}\left[e^{\frac{1}{2\gamma}(1-\frac{1}{\gamma})\int_t^s \boldsymbol{\lambda}_u^2\, du}\right]
\end{aligned} \tag{101}$$

Here, we simply insert the definition of the state-price deflator from Equation (32). To change the measure, the second equation simply places the Radon-Nikodym derivative in front, allowing the expectation to be taken under the measure $\mathbb{Q}^\gamma$. This measure is defined via Girsanov's theorem, where the Brownian motion under $\mathbb{Q}^\gamma$ is given by

$$\boldsymbol{Z}_t^{\mathbb{Q}^\gamma} = \boldsymbol{Z}_t + \left(1-\frac{1}{\gamma}\right)\int_0^t \boldsymbol{\lambda}_u\, du \tag{102}$$

Intuitively, $\mathbb{Q}^\gamma$ reflects the perspective of an investor with CRRA preferences and risk aversion parameter γ. Under this measure, the investor perceives adjusted dynamics for the investment opportunities.

Since the Sharpe ratio λ_t follows an Ornstein–Uhlenbeck process under the physical measure, then under $\mathbb{Q}^\gamma$ its dynamics become

$$d\boldsymbol{\lambda}_t = \left(\kappa(\bar{\lambda}-\boldsymbol{\lambda}_t) + \left(1-\frac{1}{\gamma}\right)\sigma_\lambda \boldsymbol{\lambda}_t\right)dt - \sigma_\lambda\, d\boldsymbol{Z}_t^{\mathbb{Q}^\gamma} \tag{103}$$

From the Feynman-Kac formula it follows that the function V defined by

$$V(\lambda,t) = \mathbb{E}_t^{\mathbb{Q}^\gamma}\left[e^{-\frac{1}{2\gamma}\left(1-\frac{1}{\gamma}\right)\int_t^s \boldsymbol{\lambda}_u^2\, du}\right] \tag{104}$$

Solves the partial differential equation

$$\frac{\partial V}{\partial t} + \left(\kappa(\bar{\lambda}-\lambda_t) + \left(1-\frac{1}{\gamma}\right)\sigma_\lambda\lambda_t\right)\frac{\partial V}{\partial \lambda} + \frac{1}{2}\sigma_\lambda^2\frac{\partial^2 V}{\partial \lambda^2} = \frac{1}{2\gamma}\left(1-\frac{1}{\gamma}\right)\lambda^2 V \tag{105}$$

where the terminal condition $V(\lambda,s)=1$ follows immediately from the structure of the exponential function. The solution to this partial differential equation is given by

$$V(\lambda,t) = e^{\hat{g}_0(s-t)+g_1(s-t)\lambda+\frac{1}{2}g_2(s-t)\lambda^2} \tag{106}$$

The functions g_0, g_1, and g_2 solve a system of ordinary differential equations, with the following solutions, see Munk (2002)

$$\begin{aligned} g_0(\tau) = & -\frac{\delta}{\gamma}\tau - r\left(1-\frac{1}{\gamma}\right)\tau - \frac{1}{2}\ln\left(\frac{2\theta-(\theta-b)(1-e^{-2\theta\tau})}{2\theta}\right) \\ & + \frac{1-\gamma}{2\gamma^2}\left[\left(\frac{\kappa^2\bar{\lambda}^2}{\theta^2}+\frac{\sigma_\lambda^2}{\theta+b}\right)\tau + \frac{\kappa^2\bar{\lambda}^2}{\theta^3}\frac{(\theta-2b)e^{-2\theta\tau}+4be^{-\theta\tau}-\theta-2b}{2\theta-(\theta-b)(1-e^{-2\theta\tau})}\right] \end{aligned} \tag{107}$$

$$g_1(\tau) = \frac{1-\gamma}{\gamma^2}\frac{\kappa\bar{\lambda}}{\theta}\frac{(1-e^{-\theta\tau})^2}{2\theta-(\theta-b)(1-e^{-2\theta\tau})} \tag{108}$$

$$g_2(\tau) = \frac{1-\gamma}{\gamma^2}\frac{1-e^{-2\theta\tau}}{2\theta-(\theta-b)(1-e^{-2\theta\tau})} \tag{109}$$

Where $b = \kappa - (1-\frac{1}{\gamma})\sigma_\lambda$ and $\theta = \sqrt{b^2 - \sigma_\lambda^2(1-\lambda)/\gamma^2}$. With $V(\lambda, t)$, this yields the solution for $g(\lambda, t)$

$$g(\lambda, t) = \int_t^T e^{g_0(s-t)+g_1(s-t)\lambda+\frac{1}{2}g_2(s-t)\lambda^2}\, ds \tag{110}$$

By applying Itô's Lemma, the dynamics of $g(\lambda, t)$ can be derived as

$$\begin{aligned} dg(\lambda, t) &= \ldots\, dt + \frac{\partial g(\lambda,t)}{\partial \lambda}\left(\left(\kappa(\bar{\lambda}-\lambda_t)+\left(1-\frac{1}{\gamma}\right)\sigma_\lambda\lambda_t\right)dt - \sigma_\lambda\, dZ_t^{\mathbb{Q}^\gamma}\right) \\ &= \ldots\, dt - \frac{\partial g(\lambda,t)}{\partial \lambda}\,\sigma_\lambda\, dZ_t^{\mathbb{Q}^\gamma} \end{aligned} \tag{111}$$

Equating diffusion terms with Equation (94)

$$\boldsymbol{\sigma}_{gt}g_t = -\frac{\partial g(\lambda,t)}{\partial \lambda}\,\sigma_\lambda \quad \Leftrightarrow \quad \boldsymbol{\sigma}_{gt} = -\frac{1}{g(\lambda,t)}\frac{\partial g(\lambda,t)}{\partial \lambda}\,\sigma_\lambda \tag{112}$$

Inserting this expression into Equation (99), and defining $D(\lambda, t) = -\frac{\partial g(\lambda,t)}{\partial \lambda}/g(\lambda, t)$, the optimal portfolio strategy becomes

$$\boldsymbol{\pi}_t^\star = \frac{\lambda}{\gamma\sigma} - \frac{\sigma_\lambda}{\sigma}D(\lambda, t) \tag{113}$$

Where $D(\lambda, t)$ is given by

$$D(\lambda, t) = \frac{\int_t^T (g_1(s-t)+g_2(s-t)\lambda)\, e^{g_0(s-t)+g_1(s-t)\lambda+\frac{1}{2}g_2(s-t)\lambda^2}\, ds}{\int_t^T e^{g_0(s-t)+g_1(s-t)\lambda+\frac{1}{2}g_2(s-t)\lambda^2}\, ds}. \tag{114}$$

7.3 Benchmark-adjusted CRRA utility

The solution for the benchmark-adjusted CRRA utility is naturally closely related to the standard CRRA utility. We can build on the results already derived under the assumption of constant investment opportunities. Recall the optimal wealth process from Equation (88)

$$W_t^\star = \frac{W_0 - F(0)}{g_0} e^{-\frac{\delta}{\gamma(C_t)}t} \zeta_t^{-\frac{1}{\gamma(C_t)}} g_t + \bar{C}F(t) \tag{115}$$

Previously, when g_t was deterministic, we derived the dynamics using Itô's Lemma, relying only on the dynamics of ζ. However, recall that under stochastic investment opportunities, we must also incorporate the dynamics of g_t

$$\begin{aligned} dW_t^* &= dt + \left(-\frac{1}{\gamma(C_t)} \frac{W_t^* - \bar{C}F(t)}{\zeta_t}\right)(-\zeta_t r\, dt - \zeta_t \boldsymbol{\lambda}\, d\boldsymbol{Z}_t) + \frac{W_t^* - \bar{C}F(t)}{g_t}(\mu_{gt} g_t\, dt + \boldsymbol{\sigma}_{gt}\, g_t d\boldsymbol{Z}_t) \\ &= dt + \frac{1}{\gamma(C_t)}(W_t^* - \bar{C}F(t))\boldsymbol{\lambda}\, d\boldsymbol{Z}_t + (W_t^* - \bar{C}F(t))\boldsymbol{\sigma}_{gt}\, d\boldsymbol{Z}_t \end{aligned} \tag{116}$$

Equating the diffusion terms with the wealth dynamics for any given portfolio

$$\begin{aligned} W_t^* \boldsymbol{\pi}_t^* \sigma d\boldsymbol{Z}_t &= \frac{1}{\gamma(C_t)}(W_t^* - Cf_t)\boldsymbol{\lambda} d\boldsymbol{Z}_t + (W_t^* - Cf_t)\boldsymbol{\sigma}_{gt} d\boldsymbol{Z}_t \\ \boldsymbol{\pi}_t^* &= \frac{\lambda}{\gamma(C_t)\sigma} \frac{W_t^* - \bar{C}F(t)}{W_t^*} + \frac{\boldsymbol{\sigma}_{gt}}{\sigma} \frac{W_t^* - \bar{C}F(t)}{W_t^*} \end{aligned} \tag{117}$$

Recall that

$$\boldsymbol{\sigma}_{gt} = -\frac{1}{g(\lambda,t)} \frac{\partial g(\lambda,t)}{\partial \lambda}\, \sigma_\lambda \tag{118}$$

Defining $D(\lambda,t) = -\frac{\partial g(\lambda,t)}{\partial \lambda}/g(\lambda,t)$, the optimal portfolio strategy becomes

$$\pi_t^* = \frac{\lambda}{\gamma(C_t)\sigma} \frac{W_t^* - \bar{C}F(t)}{W_t^*} - \frac{\sigma_\lambda}{\sigma} D(\lambda,t) \frac{W_t^* - \bar{C}F(t)}{W_t^*} \tag{119}$$

Compared to the case of constant investment opportunities, the optimal consumption remains almost identical

$$C_t^\star = \bar{C} + \frac{W_t^* - \bar{C}F(t)}{g(\lambda,t)} \tag{120}$$

7.4 Habit utility

Recall the derivation presented in Section 7.2 for the CRRA utility case. This represents a simplified version of the habit utility framework. Rather than repeating the steps involving the Girsanov theorem, the dynamics under $\mathbb{Q}^{\gamma}$, and the solution to the associated system of ODEs, only the key results are stated here. The difference between the CRRA and habit utility cases stems from the factor $(1+\alpha F(s))^{1-\frac{1}{\gamma}}$, which appears repeatedly in the latter. Under CRRA preferences, $\alpha = 0$, and this term reduces to 1. The derivation under CRRA utility follows closely from Munk (2002), and is sufficient to illustrate the solution structure. The complete solution for habit utillity can be seen in Theorem 3 in Munk (2002); it has been left out here to keep things short, as it doesn't add anything beyond what is already shown. The optimal consumption strategy is

$$C_t^\star = h_t^\star + (1+\alpha F(t))^{-\frac{1}{\gamma}} \frac{W_t^\star - h_t^\star F(t)}{g(\lambda, t)} \tag{121}$$

And the optimal portfolio strategy is

$$\boldsymbol{\pi}_t^\star = \frac{\lambda}{\gamma\sigma} \frac{W_t^\star - h_t^\star F(t)}{W_t^\star} - \frac{\sigma_\lambda}{\sigma} D(\lambda, t) \frac{W_t^\star - h_t^\star F(t)}{W_t^\star} \tag{122}$$

Where

$$F(t) = \int_t^T e^{-(r+\beta-\alpha)(s-t)} ds = \frac{1}{r+\beta-\alpha} \left(1 - e^{-(r+\beta-\alpha)(T-t)}\right) \tag{123}$$

What differs here, as previously noted, is the term $(1+\alpha F(s))^{1-\frac{1}{\gamma}}$, which now appears in the expressions below, as it no longer equals one

$$D(\lambda, t) = \frac{\int_t^T \left(g_1(s-t) + g_2(s-t)\lambda\right) (1+\alpha F(s))^{1-\frac{1}{\gamma}} e^{g_0(s-t)+g_1(s-t)\lambda+\frac{1}{2}g_2(s-t)\lambda^2} ds}{\int_t^T (1+\alpha F(s))^{1-\frac{1}{\gamma}} e^{g_0(s-t)+g_1(s-t)\lambda+\frac{1}{2}g_2(s-t)\lambda^2} ds} \tag{124}$$

$$g(\lambda, t) = \int_t^T (1+\alpha F(s))^{1-\frac{1}{\gamma}} e^{g_0(s-t)+g_1(s-t)\lambda+\frac{1}{2}g_2(s-t)\lambda^2} ds \tag{125}$$

8 Estimating risk aversion γ

8.1 Introduction

Understanding how investors respond to risk is a central concern in both financial theory and practical asset management. In modern portfolio theory, risk preferences guide how capital is allocated across assets with different risk and return profiles. In classic finance theory, these preferences are often summarized using the coefficient of relative risk aversion, γ, which plays a key role in shaping long-term investment strategies. Identifying this parameter is essential, since portfolios that are misaligned with investors' tolerance for risk, particularly during periods of heightened market volatility, can lead to suboptimal decisions such as panic selling or abrupt shifts in the strategic asset allocation. Recent market events, such as the sharp market declines observed in April 2025, highlight how important it is to base portfolio decisions on a clear understanding of investor behavior under uncertainty.

While γ is theoretically a preference parameter, its empirical estimation has proven to be complex. Elminejad et al. (2023) review 92 studies and find that estimated values of relative risk aversion vary widely depending on model choice, data frequency, region, and estimation method. In general economics contexts, estimates tend to cluster around 1, whereas studies within financial economics often report values between 2 and 7. However, unadjusted results from finance-specific studies show average estimates as high as 45, emphasizing the difficulty of explaining observed asset returns using standard utility-based models, a problem often referred to as the equity premium puzzle, see Constantinides (1990).

Another area of debate concerns whether γ is constant over time. Early studies, such as Friend and Blume (1975), find support for time-invariant risk aversion, a result reaffirmed by Paiella (2011). In contrast, more recent contributions including Das and Sarkar (2010), using a GARCH-M framework, provide strong evidence that risk aversion is dynamic and varies with market conditions. Empirical support for higher and potentially time-varying values of γ also comes from Pindyck (1988), Stambaugh (1987), and Tödter (2008).

The dominant empirical strategy for estimating γ relies on the Euler equation and is often implemented using Generalized Method of Moments (GMM), see Chernozhukov and Fernandez-Val (2017). However, this approach requires detailed consumption or income data and relies on strong structural assumptions. This thesis instead implements a method that avoids the need for consumption data by combining a Markov-switching model of returns with a Taylor expansion of expected utility. The Markov-switching framework captures regime-dependent return dynamics, interpreted as persistent bull and bear states, with distinct means and variances. These are used to generate time-varying conditional moments based on rolling regime probabilities. By feeding these moments into a fourth-order Taylor expansion of the CRRA utility function, the model produces a sequence of implied γ values that reflect how attractive the equity market is to a risk-averse investor at each point in time.

This approach uses only return data and provides a straightforward way to study how investor behavior changes with market conditions. At the same time, it stays consistent with the core principles of expected utility theory.

Beyond its practical use, the method also relates to existing research on time-varying volatility and structural changes in financial markets. Volatility clustering is often modeled using ARCH or GARCH frameworks, where current volatility depends on past shocks, as in Engle (1982) and Bollerslev (1986). In contrast, the Markov-switching model assumes that markets shift between different states, such as bull and bear regimes, each with its own level of return and risk, see for instance Hamilton (1989). By combining this with a utility-based approach, the model links changes in market conditions to investor behavior. In particular, it shows how shifts in regime probabilities affect the level of risk aversion needed to justify full investment in equities.

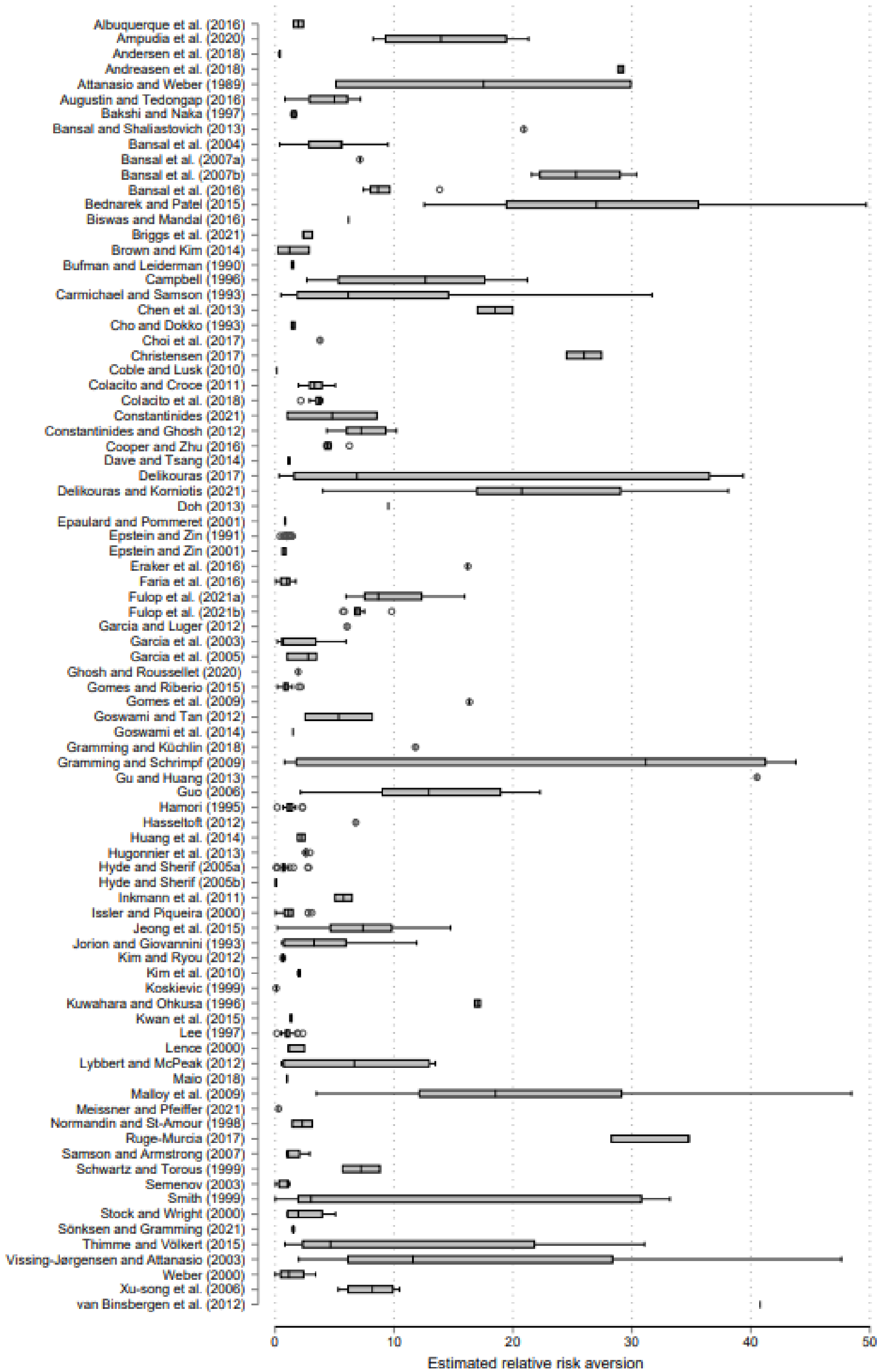


Figure 1: Overview over risk aversion. Taken from Elminejad et al. (2023).

8.2 Methodology

In contrast to the structure of typical econometric papers, this section presents the methodology before the data. The reasoning is that it is more natural to first derive the model framework and estimation procedure, thereby clarifying the data requirements for estimating γ.

8.2.1 Markov-switching regime framework

To capture structural shifts in financial markets, particularly the alternation between persistent high-return/low-volatility and low-return/high-volatility periods, we model log returns using a Markov-switching framework. This class of models allows both the conditional mean and variance of returns to depend on an unobserved state variable that evolves stochastically over time.

Let $\{R_t\}_{t=1}^{T}$ denote the log return series. The process is managed by a discrete latent state variable $s_t \in \{1, 2\}$, which follows a first-order Markov chain with transition probabilities

$$P(s_t = j \mid s_{t-1} = i) = p_{ij}, \quad i, j \in \{1, 2\} \tag{126}$$

The observed return R_t is modeled as

$$R_t = \mu_{s_t} + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma^2_{s_t}) \tag{127}$$

Where μ_{s_t} and $\sigma^2_{s_t}$ denote the regime-specific mean and variance. These parameters are assumed constant within each regime, and are typically interpreted as representing "bull" (μ_{s_t} high, σ_{s_t} low) and "bear" (μ_{s_t} low, σ_{s_t} high) market states.

Since the regime s_t is unobserved, the model is estimated via maximum likelihood using the Expectation-Maximization (EM) algorithm. Instead of detailing the EM algorithm, this thesis applies it in practice. In short, it iteratively fills in missing data and updates estimates until convergence, see Dempster et al. (1977). This yields smoothed regime probabilities

$$\xi_t^{(k)} = P(s_t = k \mid \mathcal{F}_T), \quad k \in \{1, 2\} \tag{128}$$

Where $\mathcal{F}_T$ denotes the full information set available at time T. These probabilities represent the model-implied likelihood that each observation belongs to regime k.

The analysis begins by estimating a single coefficient of relative risk aversion γ using the full return sample. In this setting, the regime-specific first and second moments μ_k and σ_k^2 are held constant, while regime probabilities are averaged across the full sample. Letting $\xi_t^{(k)}$ denote the smoothed probability that observation t belongs to regime k, the average regime probability is defined as

$$\bar{\pi}^{(k)} = \frac{1}{T}\sum_{t=1}^{T} \xi_t^{(k)} \tag{129}$$

The regime-weighted expected return is then given by

$$E[R] = \sum_{k=1}^{2} \bar{\pi}^{(k)} \mu_k \tag{130}$$

and the corresponding regime-weighted variance is

$$\text{Var}(R) = \sum_{k=1}^{2} \bar{\pi}^{(k)} \left(\sigma_k^2 + \mu_k^2\right) - \left(E[R]\right)^2 \tag{131}$$

where $\bar{\pi}^{(k)}$ denotes the average smoothed probability of regime k across the full sample. Skewness and kurtosis are computed once from the full distribution of log returns

$$\text{Skew}_{\text{full}} = \text{skew}(R_{1:T}) \tag{132}$$

$$\text{Kurt}_{\text{full}} = \text{kurt}(R_{1:T}) \tag{133}$$

This yields a static benchmark estimate of γ under average regime conditions. To examine how this risk aversion measure evolves with changing market dynamics, the approach is extended by estimating γ_t using rolling windows. In the rolling procedure, regime probabilities $\pi_t^{(k)}$ are updated in each window based on smoothed values, while the regime-specific means and volatilities, as well as skewness θ_1 and kurtosis θ_2, remain fixed. This isolates the effect of time-varying regime exposure on the level of risk aversion consistent with full investment in the risky asset. For further properties on Markov chains, see Rahbek and Pedersen (2020).

8.2.2 Moment-based CRRA estimation via Taylor expansion

As usual, consider a rational economic agent who maximizes expected utility, $E[U(W)]$, where $U(\cdot)$ is assumed to be the standard CRRA utility function. The agent begins with initial wealth W_0 and faces a simple investment opportunity set $\mathcal{K}$, consisting of a risk-free asset with gross return R_f and a risky asset with stochastic return $\tilde{R}$. For generality, these can be interpreted as a short-term government bond (e.g., T-bill) and a broad market equity index, respectively. The agent allocates a fraction $\alpha \in [0, 1]$ of wealth to the risky asset, and seeks to choose α to maximize expected utility over final wealth. The problem is formally defined as

$$\alpha^* \in \arg\max_{\alpha} E(U) \quad \text{where} \quad U = \frac{\left(W_0\left(1 + R_f + \alpha\left(\tilde{R} - R_f\right)\right)\right)^{1-\gamma}}{1-\gamma} \tag{134}$$

Next, we perform a Taylor series expansion of Equation (126) around expected wealth, $E[\tilde{W}]$

$$\begin{aligned} E(U) = E\Big[& U(E[\tilde{W}]) + (\tilde{W} - E[\tilde{W}])U'(E[\tilde{W}]) \\ & + \frac{(\tilde{W} - E[\tilde{W}])^2 U''(E[\tilde{W}])}{2} \\ & + \frac{(\tilde{W} - E[\tilde{W}])^3 U'''(E[\tilde{W}])}{2 \times 3} \\ & + \frac{(\tilde{W} - E[\tilde{W}])^4 U''''(E[\tilde{W}])}{2 \times 3 \times 4} \Big] \end{aligned} \tag{135}$$

Here, U', U'', U''', and U'''' denote the first through fourth derivatives of the utility function with respect to α. The wealth expressions are given by

$$\tilde{W} = W_0 \left(1 + R_f + \alpha(\tilde{R} - R_f)\right) \tag{136}$$

$$E(\tilde{W}) = W_0 \left(1 + R_f + \alpha\left(E[\tilde{R}] - R_f\right)\right) \tag{137}$$

This leads to the following identities

$$\tilde{W} - E[\tilde{W}] = W_0\alpha(\tilde{R} - E[\tilde{R}]) \tag{138}$$

$$E[(\tilde{W} - E[\tilde{W}])^2] = W_0^2\alpha^2 E[(\tilde{R} - E[\tilde{R}])^2] = W_0^2\alpha^2\sigma^2 \tag{139}$$

$$E[(\tilde{W}-E[\tilde{W}])^3] = W_0^3\alpha^3E[(\tilde{R}-E[\tilde{R}])^3] = W_0^3\alpha^3\sigma^3\theta_1 \tag{140}$$

$$E[(\tilde{W}-E[\tilde{W}])^4] = W_0^4\alpha^4E[(\tilde{R}-E[\tilde{R}])^4] = W_0^4\alpha^4\sigma^4\theta_2 \tag{141}$$

Here, θ_1 and θ_2 represent the skewness and kurtosis, respectively, and are defined as follows, see Hill et al. (2012)

$$\sigma^2 = E[(\tilde{R}-E[\tilde{R}])^2], \quad \theta_1 = \frac{E[(\tilde{R}-E[\tilde{R}])^3]}{\sigma^3}, \quad \theta_2 = \frac{E[(\tilde{R}-E[\tilde{R}])^4]}{\sigma^4} \tag{142}$$

Replacing Equation (130) to (133) into Equation (127) gives

$$\begin{aligned} E(U) =& \frac{\left(1+R_f+\alpha\left(E(\tilde{R})-R_f\right)\right)^{1-\gamma}}{1-\gamma} \\ &- \frac{\gamma\sigma^2\alpha^2W_0^2W_0^{-\gamma-1}\left(1+R_f+\alpha\left(E(\tilde{R})-R_f\right)\right)^{-\gamma-1}}{2} \\ &+ \frac{\gamma(1+\gamma)\sigma^3\alpha^3W_0^3W_0^{-\gamma-2}\left(1+R_f+\alpha\left(E(\tilde{R})-R_f\right)\right)^{-\gamma-2}\theta_1}{6} \\ &- \frac{\gamma(1+\gamma)(2+\gamma)\sigma^4\alpha^4W_0^4W_0^{-\gamma-3}\left(1+\alpha\left(E(\tilde{R})-R_f\right)\right)^{-\gamma-3}\theta_2}{24} \end{aligned} \tag{143}$$

As Equation (135) can be factored, the W_0 term merely represents a scaling factor that does not influence the optimization outcome. We can therefore normalize Equation (135) by $W_0 = 1$ without affecting the maximization results

$$\begin{aligned} E(U) =& \frac{\left(1+R_f+\alpha\left(E(\tilde{R})-R_f\right)\right)^{1-\gamma}}{1-\gamma} \\ &- \frac{\gamma\sigma^2\alpha^2\left(1+R_f+\alpha\left(E(\tilde{R})-R_f\right)\right)^{-\gamma-1}}{2} \\ &+ \frac{\gamma(1+\gamma)\sigma^3\alpha^3\left(1+R_f+\alpha\left(E(\tilde{R})-R_f\right)\right)^{-\gamma-2}\theta_1}{6} \\ &- \frac{\gamma(1+\gamma)(2+\gamma)\sigma^4\alpha^4\left(1+R_f+\alpha\left(E(\tilde{R})-R_f\right)\right)^{-\gamma-3}\theta_2}{24} \end{aligned} \tag{144}$$

The initial optimization problem involves maximizing Equation (136) with respect to the allocation parameter α. However, the current analysis instead fixes $\alpha =$

1, representing full investment in the risky asset (the broad stock index). The relative risk aversion coefficient γ can then be estimated numerically using fourth-order moment conditions of the stock index returns. The optimization problem is formally stated as

$$\gamma^* \in \arg\min_{\gamma} \mathcal{J} \tag{145}$$

This approach implements Equation (144) within Python, where the objective function $\mathcal{J} = -E(U)$, represents negative expected utility.

The moment inputs used in Equation (144) are not computed directly from raw return data, but are estimated using the Markov-switching framework. For the baseline estimate of γ, regime probabilities are averaged across the full sample, resulting in a single, static measure of risk aversion. To examine how this measure evolves, rolling estimates of γ_t are computed using 2- and 10-year windows. These horizons are chosen to capture both shorter-term shifts and longer-term structural changes in regime dynamics, providing a clearer view of how risk preferences evolve over time.

8.3 Empirical data

The analysis requires data for two assets: (i) a risk-free security and (ii) a broad stock market index to serve as a proxy for the market portfolio. Brealey et al. (2014) estimate the risk-free rate at 3.9%, which is applied here for simplicity. As an alternative, one could use historical returns on short-term government securities, such as 3-month Treasury bills, from well-established economies like the United States, Germany, or Denmark.

Regarding the market portfolio proxy, the S&P 500 is commonly used in the literature. This naturally leads to a discussion of Roll's critique, which argues that the CAPM is inherently untestable because the true market portfolio is unobservable. Consequently, the estimation of γ is dependent on the chosen proxy. According to Roll (1977), a true market portfolio must: (i) be complete and include exposure to all sources of risk in the economy (i.e., all tangible assets), (ii) be value-weighted, (iii) reflect the total wealth of the economy, and (iv) be observable and accessible. These conditions are clearly not fully satisfied by the index, yet it still represent large, diversified sets of companies for the U.S. market, which remains the world's largest economy. Thus, it seems reasonable to include the index. Monthly data is retrieved from Yahoo Finance for the ticker `SPY`, representing the SPDR S&P 500 ETF Trust. The ETF exhibits low tracking error and high liquidity, making it a reasonable proxy for the index it tracks (see factsheet here). Data is available from 1993-01-31, providing 384 monthly observations. Monthly log returns are computed as

$$R_t = \ln\left(\frac{P_t}{P_{t-1}}\right), \tag{146}$$

Where P_t denotes the closing price of the SPDR S&P 500 ETF at time t. The use of continuously compounded log returns is standard in the asset pricing literature due to their desirable statistical properties: they are time-additive across periods and symmetric in gains and losses, which simplifies both estimation and interpretation in dynamic models. Log returns are also well-suited for the Markov-switching framework, which assumes that returns are conditionally normally distributed within each regime. While real-world returns often exhibit deviations from normality, such as fat tails and skewness, the regime-dependent normal distribution provides an approximation of changing market dynamics.

It is worth noting that the empirical analysis is based on the ETF rather than the underlying S&P 500 Index. As a result, the return data reflect net performance after management fees and tracking costs associated with holding the ETF. While this might initially appear problematic from a theoretical perspective, it arguably provides a more realistic representation of investor experience. In practice, real-life investors access broad equity exposure through exchange-traded funds such as `SPY`, not the index itself. Uusing ETF returns aligns more closely with the actual investment outcomes faced by investors and therefore supports the empirical relevance of the analysis.

8.4 Results

Figure 2 presents the monthly log returns of `SPY` along with the smoothed probability of being in the bear regime. The model identifies several distinct high-probability bear episodes, most notably during the early 2000s (corresponding to the dot-com bust), the period from 2008 to 2009 (global financial crisis), and the sharp volatility spike in early 2020 (the COVID-19 pandemic). These results support a structural interpretation of regime dynamics, in which the bear state is associated with persistently lower expected returns and elevated volatility, in contrast to a more stable bull state.

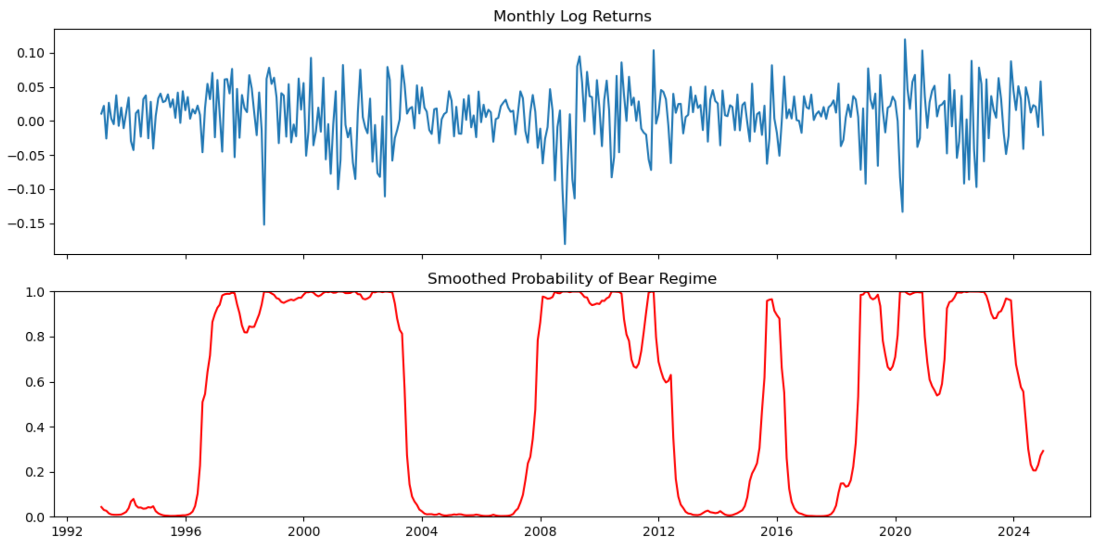


Figure 2: Monthly log returns and smoothed probabilities

Table 1 shows the estimated regime-specific moments. The bull regime is characterized by a higher mean return (1.32%) and lower variance (0.0006), while the bear regime has a lower mean (0.40%) and higher variance (0.0029). Using these moments (together with 3rd and 4th order moments) and empirical regime probabilities, the implied CRRA coefficient $\hat{\gamma}$ is estimated to be 24.12. This reflects the level of risk aversion required to justify full exposure to `SPY`, given the underlying return dynamics and regime mix.

When expressed in annual terms, the bull regime yields an average return of approximately 15.8% with a volatility of 7.7%, while the bear regime exhibits a markedly lower average return of 4.8% and significantly higher volatility of 18.8%.

It is worth noting, that the estimated mean return in the bear regime is not statistically significant (p = 0.324), suggesting that average returns during these episodes are not reliably different from zero. This aligns with the empirical observation that crisis periods are characterized more by elevated uncertainty than by sustained directional trends. The key feature of the bear regime is instead its elevated volatility, which is highly significant and consistent with the intuition that investors face greater risk rather than systematically negative returns during such periods.

Parameter	**Estimate**	**Std. Error**	**Z**	**P-value**
Regime 1 (Bull)				
Mean (μ_1)	0.0132	0.0020	6.510	0.000
Variance (σ_1^2)	0.0006	7.97e–05	7.015	0.000
Sharpe Ratio (λ, annualized)	1.4669	–	–	–
Regime 2 (Bear)				
Mean (μ_2)	0.0040	0.0040	0.986	0.324
Variance (σ_2^2)	0.0029	0.0000	8.651	0.000
Sharpe Ratio (λ, annualized)	0.0529	–	–	–

Table 1: Markov-Switching Model Results

The estimated transition probabilities suggest that the market exhibits strong persistence within each regime. Specifically, the probability of remaining in the bull regime is $p_{11} = 0.9641$, which corresponds to an expected duration of approximately $\frac{1}{1-p_{11}} \approx 27.9$ months. Similarly, the probability of switching from the bear to the bull regime is $p_{21} = 0.0344$, implying $p_{22} = 0.9656$ and an expected duration of about $\frac{1}{1-p_{22}} \approx 29.0$ months in the bear state. These durations are somewhat in line with empirical observations from financial markets, where bull and bear phases often last over multiple years. Gonzalez et al. (2006) analyze over two centuries of data and find that bull markets typically last around 50 months, while bear markets average approximately 15 months in duration. Although the results differ, most notably in the significantly higher presence of the bear regime, it is clear that the model effectively captures the cyclical nature of asset returns in some way.

The return distribution shows clear signs of non-normality, with skewness of -0.75 and kurtosis of 4.37. The Jarque-Bera test confirms this with a test statistic of 64.70 and a p-value below 0.001. Although the Markov model assumes normal returns within each regime, the unconditional distribution becomes non-normal due to regime shifts. This justifies both the use of regime switching and the inclusion of skewness and kurtosis in the risk aversion estimation.

Figures 3 and 4 show the estimated CRRA coefficient $\hat{\gamma}_t$ based on rolling windows of 2 and 10 years, respectively.

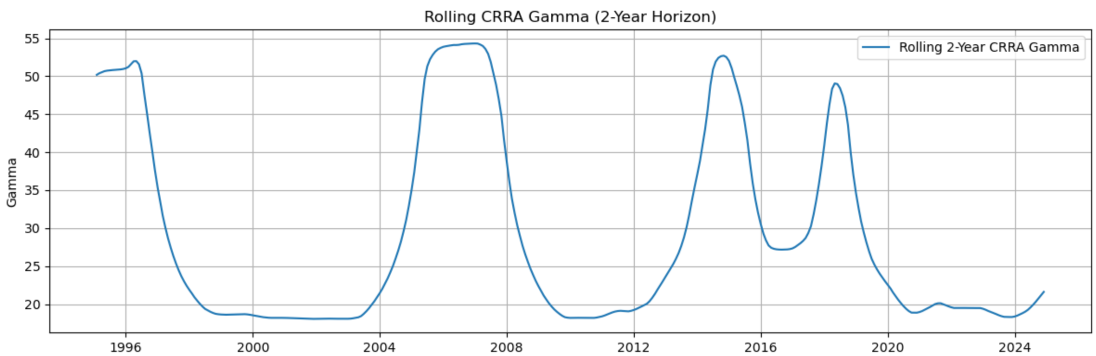


Figure 3: Rolling CRRA Gamma (2Y)

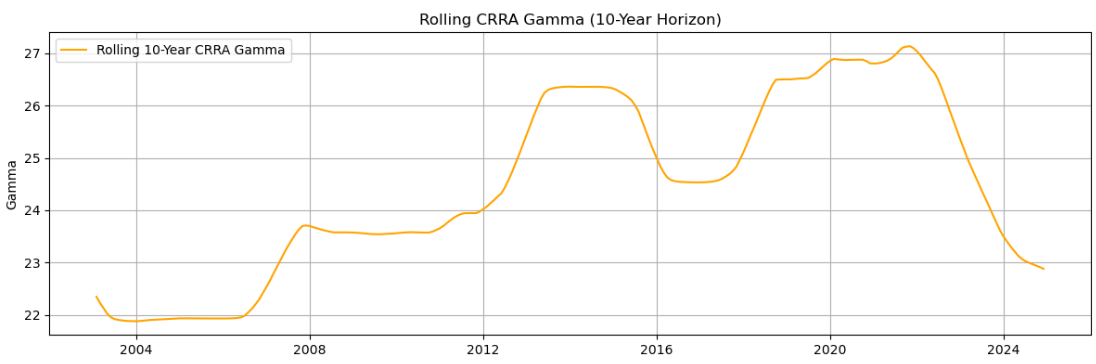


Figure 4: Rolling CRRA Gamma (10Y)

The 2-year estimate shows substantial variation, with $\hat{\gamma}_t$ reaching values around 50 during periods such as the mid-2000s and the market recovery following the financial crisis in the mid-2010s. These high values suggest that the return distribution during these times was particularly favorable, characterized by strong excess returns relative to volatility, making it optimal even for highly risk-averse investors to fully invest in `SPY`. Low values of γ_t appear during and shortly after major downturns, such as the dot-com crash, the Global Financial Crisis, and the COVID-19 shock, when weaker returns reduce the attractiveness `SPY`.

The 10-year rolling estimates are more stable, capturing longer-term patterns in the market. Unlike the 2-year estimates, which are more sensitive to temporary shocks and regime fluctuations, the 10-year measure reflects more persistent dynamics.

These estimates of γ_t should not be interpreted as changes in investor preferences. Rather, they reflect the level of risk aversion required to rationalize full investment in `SPY`, conditional on the past observed return distribution and the probability of being in each regime. High values signal periods in which the market environment was particularly attractive.

As a concluding remark, this section has shown that when using the full sample with fixed regime-specific moments and no rolling structure, the estimated relative risk aversion coefficient is $\hat{\gamma} = 24.12$. In contrast, the 2-year rolling estimation produce highly volatile results, reflecting the sensitivity of changes in regime probabilities. The 10-year rolling estimates are more stable, generally fluctuating between values of 20 and 30, indicating a more persistent level of risk aversion over longer horizons. While these values may appear high, they are not unrealistic in the context of the equity premium puzzle and are consistent with earlier findings in the finance literature, such as Elminejad et al. (2023), as well as the results presented in Figure 1.

9 Numerical results

This chapter presents the numerical results based on simulations of the portfolio optimization models developed in earlier sections. While multiple structures could be used to present the results, such as grouping by market setting or by utility specification, the results are organized by utility function. This choice reflects the central aim of examining how changes in the investment environment, particularly through a stochastic market price of risk, affect optimal strategies across the different utility functions. Presenting the results this way also allows for a natural comparison, while preserving the emphasis on how market dynamics shape optimal behavior.

Before proceeding, the choice of parameter values must be discussed. In the constant market model, the general parameter set for all three utility functions is

$$\theta_1 = \{W, T, r, \delta, \sigma, \lambda, \gamma, \bar{C}, h, \alpha, \beta\}$$

In the stochastic market model, the parameter set extends to

$$\theta_2 = \theta_1 \cup \{\sigma_\lambda, \kappa, \bar{\lambda}\}$$

Some parameters are held fixed throughout the analysis, while others are systematically varied to conduct a comparative analysis of how optimal strategies respond to changes in, for instance, γ or σ. The constant parameters are set as $t = 0$, $W_0 = 1000$ and $\delta = 0.02$. These parameters are not central to addressing the problem formulation and are therefore kept fixed to ensure meaningful comparisons.

The remaining parameters are varied systematically to assess their influence on the optimal portfolio strategies. Specifically, for each utility function and each model, the analysis investigates: (i) changes in the subjective elements of the model, represented by variations in γ, and (ii) changes in the objective elements of the model, captured through variations in r, σ, λ, σ_λ, and $\bar{\lambda}$. The baseline parameter values, based on Munk (2002), are set to $\gamma = 5$, $\lambda = \bar{\lambda} = 0.3$, $\sigma = \sigma_\lambda = 0.2$, $r = 0.02$, $\bar{C} = h_0 = 50$, $\kappa = 0.3$, $T = 5$ and are used throughout the analysis unless stated otherwise.

9.1 Results for CRRA utility

9.1.1 Constant market

The results are presented in Table 2 and 3.

γ	Stock (%)	Cash (%)	Stock/Cash
0.1	1500.0	-1400.0	-1.1
1	150.0	-50.0	-3.0
5	30.00	70.0	0.4
10	15.0	85.0	0.2
20	7.5	92.5	0.1
30	5.0	95.0	0.1
∞	0	100.0	0.0

Table 2: CRRA w. constant investment opportunities (1)

RRA	λ	σ	r	Description	Stock (%)	Cash (%)	Stock/Cash
$\gamma = 5$	0.3	0.2	0.02	Baseline	30.0	70.0	0.43
	0.6	0.2	0.02	Increase λ	60.0	40.0	1.5
	0.3	0.4	0.02	Increase σ	15.0	85.0	0.18
	0.6	0.4	0.02	Increase λ, σ	30.0	70.0	0.43
	0.3	0.2	0.04	Increase r	30.0	70.0	0.43
$\gamma = 10$	0.3	0.2	0.02	Baseline	15.0	85.0	0.18
	0.6	0.2	0.02	Increase λ	30.0	70.0	0.43
	0.3	0.4	0.02	Increase σ	7.5	92.5	0.08
	0.6	0.4	0.02	Increase λ, σ	15.0	85.0	0.18
	0.3	0.2	0.04	Increase r	15.0	85.0	0.18

Table 3: CRRA w. constant investment opportunities (2)

The optimal stock weight increases with the market price of risk λ and decreases with volatility σ, while proportional changes to λ and σ leave the strategy unchanged. Higher risk aversion γ shifts the portfolio towards safer assets, whereas low γ leads to aggressive leveraging. The optimal allocation is independent of the investment horizon T, and since λ is held constant, changes in the risk-free rate r have no effect on the stock weight. This reflects the static nature of the model, where future investment opportunities are constant and intertemporal discounting is not present. These results highlight the simplicity of the CRRA model under constant investment opportunities and are consistent with theoretical expectations.

9.1.2 Stochastic market

The results are presented in Table 4 and 5.

Horizon	γ	Myopic (%)	Hedging (%)	Total (%)	Cash (%)	Hedging/Myopic
$T = 5$	0.1	1500.0	-1175.1	324.9	-224.9	-0.78
	1	150.0	0.0	150.0	-50.0	0.00
	5	30.0	11.0	41.0	59.0	0.37
	10	15.0	6.7	21.7	78.3	0.44
	20	7.5	3.6	11.1	88.9	0.49
	30	5.0	2.5	7.5	92.5	0.50
$T = 10$	0.1	1500.0	-1199.6	300.4	-200.4	-0.80
	1	150.0	0.0	150.0	-50.0	0.00
	5	30.0	18.7	48.7	51.3	0.62
	10	15.0	12.0	27.0	73.0	0.80
	20	7.5	6.8	14.3	85.7	0.91
	30	5.0	4.7	9.7	90.3	0.95

Table 4: CRRA w. stochastic investment opportunities (1)

Introducing a stochastic market price of risk fundamentally changes the structure of optimal portfolio choice compared to the constant market setting. The presence of stochastic investment opportunities activates a non-zero hedging demand. This hedging component can be substantial, particularly for low values of γ, where it serves to offset the excessive leverage implied by the myopic position, rather than reflecting speculative intent. As a result, the total risky share is no longer a simple inverse function of risk aversion, and investment horizon becomes meaningful in the strategic allocation. Longer horizons amplify the role of hedging, especially for more risk-averse investors. Additionally, the investor's reaction to the Sharpe ratio depends on more than just current market conditions, it also reflects expectations about future returns. This shows how the model accounts for intertemporal trade-offs under stochastic investment opportunities, something the constant model does not consider.

Horizon	r	σ	σ_λ	λ	$\bar{\lambda}$	Description	Myopic (%)	Hedging (%)	Total (%)	Cash (%)	Hedging/Myopic
$T = 5$	0.02	0.2	0.2	0.3	0.3	Baseline	30.0	11.0	41.0	59.0	0.37
	0.02	0.2	0.2	0.6	0.3	Increase λ	60.0	18.4	78.4	21.6	0.31
	0.02	0.2	0.2	0.3	0.6	Increase $\bar{\lambda}$	30.0	14.3	44.3	55.7	0.48
	0.02	0.2	0.2	0.3	-0.6	Negative $\bar{\lambda}$	30.0	0.7	30.7	69.3	0.02
	0.02	0.4	0.4	0.3	0.3	Increase σ, σ_λ	15.0	16.0	31.0	69.0	1.07
	0.04	0.2	0.2	0.3	0.3	Increase r	30.0	10.9	40.9	59.1	0.36
$T = 10$	0.02	0.2	0.2	0.3	0.3	Baseline	30.0	18.7	48.7	51.3	0.62
	0.02	0.2	0.2	0.6	0.3	Increase λ	60.0	28.8	88.8	11.2	0.48
	0.02	0.2	0.2	0.3	0.6	Increase $\bar{\lambda}$	30.0	25.7	55.7	44.3	0.86
	0.02	0.2	0.2	0.3	-0.6	Negative $\bar{\lambda}$	30.0	-5.7	24.3	75.7	-0.19
	0.02	0.4	0.4	0.3	0.3	Increase σ, σ_λ	15.0	29.9	44.9	55.1	2.00
	0.04	0.2	0.2	0.3	0.3	Increase r	30.0	18.3	48.3	51.7	0.61

Table 5: CRRA w. stochastic investment opportunities (2)

Table 5 presents results under varying market parameters. A longer investment horizon generally increases exposure to the risky asset, driven by a higher hedging demand. This pattern holds across all specifications, except when the long-run market price of risk $\bar{\lambda}$ is negative, where total exposure instead declines with horizon. Increasing today's market price of risk λ lowers the hedging-to-myopic ratio, as less future improvement is expected. In contrast, raising $\bar{\lambda}$ amplifies the hedging demand, reflecting more favorable long-term expectations. Higher return volatility σ reduces the myopic allocation, while increased Sharpe ratio volatility σ_λ strengthens hedging incentives. Changes in the risk-free rate r have only a minor effect, slightly lowering the hedging demand due to stronger discounting.

In summary, the stochastic setting introduces a forward-looking hedging component that dynamically adjusts today's allocation in response to expected future changes in investment opportunities, an effect entirely absent in the constant market model.

9.2 Results for Benchmark-adjusted CRRA utillity

To simulate results for the benchmark-adjusted CRRA utility function, the code structure is slightly different due to the nature of $\gamma(C_t)$. In general, all utility specifications could be implemented within the habit formation framework by adjusting α, β, and h accordingly. However, because the benchmark model requires calculating $C_0^\star$ and verifying its proximity to $\bar{C}$, an `if`-statement is needed to switch between regions based on a tolerance level. To implement the benchmark adjustment, the model first computes a candidate consumption $C_0^{\text{candidate}}$ using the initial value of γ

$$C_0^{\text{candidate}} = \bar{C} + \frac{W_0 - \bar{C}F(0)}{g(\lambda, 0)}$$

The candidate consumption is then compared to the benchmark $\bar{C}$. If $|C_0^{\text{candidate}} - \bar{C}| < \epsilon$, the investor is considered sufficiently close to the benchmark, and the model switches to the "high" region. In this case, the investor's effective risk aversion doubles, $\gamma_{\text{used}} = 2\gamma$.

In the following analysis, initially $\epsilon = 0$, thereby disabling the switching mechanism, and later allow switching to observe how impactful it is. Finally, it should be noted that $\varepsilon_1 = 1$ is assumed for simplicity when computing $C_0^\star$.

9.2.1 Constant market

The results are presented in Table 6, 7 and 8. Table 6 presents results for varying risk aversion γ, investment horizon T, and benchmark level $\bar{C}$. As expected, higher values of $\bar{C}$ lead to more conservative portfolios, since investors seek to avoid deviations from the benchmark. The transition from aggressive to safe positioning is sharper compared to the standard CRRA model, with risky exposure dropping off quickly as γ increases. The model also captures highly leveraged behavior at low γ, but adjusts this depending on the benchmark. These effects highlight the benchmark-adjusted model's ability to reflect both defensive and return-seeking investment behavior depending on the investor's consumption benchmark.

Table 7 presents results for varying market parameters. Compared to the standard CRRA case, most qualitative effects remain intact: increasing λ raises the risky share, while higher volatility σ reduces it. However, unlike in the standard

Horizon	γ	$\bar{C} = 25$			$\bar{C} = 50$		
		Stock (%)	Cash (%)	Stock/Cash	Stock (%)	Cash (%)	Stock/Cash
$T = 5$	0.1	1321.6	-1221.6	-1.1	1143.1	-1043.1	-1.1
	1	132.2	-32.2	-4.1	114.3	-14.3	-8.0
	5	26.4	73.6	0.4	22.9	77.1	0.3
	10	13.2	86.8	0.2	11.4	88.6	0.1
	20	6.6	93.4	0.1	5.7	94.3	0.1
	30	4.4	95.6	0.0	3.8	96.2	0.0
$T = 10$	0.1	1160.1	-1060.1	-1.1	820.2	-720.2	-1.1
	1	116.0	-16.0	-7.2	82.0	18.0	4.6
	5	23.2	76.8	0.3	16.4	83.6	0.2
	10	11.6	88.4	0.1	8.2	91.8	0.1
	20	5.8	94.2	0.1	4.1	95.9	0.0
	30	3.9	96.1	0.0	2.7	97.3	0.0

Table 6: Benchmark-adjusted CRRA w. constant investment opportunities (1)

CRRA model, changes in the risk-free rate r now affect portfolio weights. A higher r increases the stock allocation across all scenarios, as it enters the benchmark component through the function $F(t)$. Intuitively, a higher risk-free rate reduces the need for cash to satisfy the benchmark, allowing for greater risk-taking. Despite these effects, the benchmark level $\bar{C}$ continues to have a central influence, with higher benchmarks consistently leading to more conservative portfolios.

Horizon	λ	σ	r	Description	$\bar{C} = 25$			$\bar{C} = 50$		
					Stock (%)	Cash (%)	Stock/Cash	Stock (%)	Cash (%)	Stock/Cash
$T = 5$	0.3	0.2	0.02	Baseline	26.4	73.6	0.4	22.9	77.1	0.3
	0.6	0.2	0.02	Increase λ	52.9	47.1	1.1	45.7	5.3	0.8
	0.3	0.4	0.02	Increase σ	13.2	86.8	0.2	11.4	88.6	0.1
	0.6	0.4	0.02	Increase λ, σ	26.4	73.6	0.4	22.9	77.1	0.3
	0.3	0.2	0.04	Increase r	26.6	73.4	0.4	23.2	76.8	0.3
$T = 10$	0.3	0.2	0.02	Baseline	23.2	76.8	0.3	16.4	83.6	0.2
	0.6	0.2	0.02	Increase λ	46.4	53.6	0.9	32.8	67.2	0.5
	0.3	0.4	0.02	Increase σ	11.6	88.4	0.1	8.2	91.8	0.1
	0.6	0.4	0.02	Increase λ, σ	23.2	76.8	0.3	16.4	83.6	0.2
	0.3	0.2	0.04	Increase r	23.8	76.2	0.3	17.6	82.4	0.2

Table 7: Benchmark-adjusted CRRA w. constant investment opportunities (2)

Table 8 presents results for varying the tolerance threshold ε. To activate meaningful differences in behavior across ε, a high market price of risk $\lambda = 1.5$ is used. This ensures a sufficiently large spread in consumption, making it possible for the model to switch between regions within economically plausible tolerance bands. The results reveal that the switching mechanism is highly non-linear: for extremely low risk aversion (e.g., $\gamma = 0.1$), the defensive regime is activated immediately across all ε. In contrast, moderately risk-averse investors (e.g., $\gamma = 1$) may only switch at intermediate thresholds, while highly risk-averse investors (e.g., $\gamma = 30$) also switch early. This pattern arises because the candidate consumption depends non-linearly on γ, due to the structure of the consumption function. As a result, the activation of the defensive mechanism reflects a trade-off between risk preferences and proximity to the benchmark, not a simple increasing function of γ.

$\boldsymbol{\lambda = 1.5}$, $T = 10$						
Initial γ	$\varepsilon = 50$		$\varepsilon = 75$		$\varepsilon = 100$	
	Region	Used γ	Region	Used γ	Region	Used γ
0.1	high	0.2	high	0.2	high	0.2
1	low	1	high	2	high	2
5	low	5	low	5	low	5
10	low	10	low	10	high	20
20	low	20	low	20	high	40
30	low	30	high	60	high	60

Table 8: Benchmark-adjusted CRRA with constant investment opportunities (3)

9.2.2 Stochastic market

The results are presented in Table 9 and 10. Table 9 presents results for varying risk aversion γ, investment horizon T and benchmark level $\bar{C}$. While the nominal allocations shift with the benchmark level $\bar{C}$, with higher benchmarks leading to lower risky exposure, the relative allocation of the portfolio remains stable. In particular, the hedging-to-myopic ratio is identical across benchmark levels for all combinations of T and γ. This consistency is both intuitive and appealing: while the model adjusts cash positions to meet the benchmark, the relative structure of the risky allocation remains unchanged. Beyond this, the overall qualitative patterns align with those previously observed in the simpler models.

Horizon	γ	$\bar{C} = 25$			$\bar{C} = 50$		
		Myopic (%)	Hedging (%)	Hedging/Myopic	Myopic (%)	Hedging (%)	Hedging/Myopic
$T = 5$	0.1	1321.6	-1035.3	-0.8	1143.1	-895.5	-0.8
	1	132.2	0.0	0.0	114.3	0.0	0.0
	5	26.4	9.7	0.4	22.9	8.4	0.4
	10	13.2	5.9	0.4	11.4	5.1	0.4
	20	6.6	3.2	0.5	5.7	2.8	0.5
	30	4.4	2.2	0.5	3.8	1.9	0.5
$T = 10$	0.1	1160.1	-927.8	-0.8	820.2	-656.0	-0.8
	1	116.0	0.0	0.0	82.0	0.0	0.0
	5	23.2	14.5	0.6	16.4	10.2	0.6
	10	11.6	9.2	0.8	8.2	6.5	0.8
	20	5.8	5.3	0.9	4.1	3.7	0.9
	30	3.9	3.7	0.9	2.7	2.6	0.9

Table 9: Benchmark-adjusted CRRA w. stochastic investment opportunities (1)

Tables 9 and 10 present results for varying market parameters. To avoid repetition, it is worth noting that the model largely reflects qualitative dynamics already observed: the standard CRRA framework exhibited similar qualitative responses to changes in market parameters, and Table 9 has already shown how increasing $\bar{C}$ reduces absolute allocations while preserving the relative portfolio structure. The most noteworthy, and perhaps counterintuitive, finding is that the total allocation to risky assets tends to decrease with the investment horizon T. This pattern, also visible under constant investment opportunities, becomes more evident in models that incorporate a benchmark or bequest motive. In both the benchmark-adjusted and habit utility models, long-term investors allocate less to risky assets, which contrasts with the traditional view that longer horizons justify greater risk-taking.

The ε sensitivity analysis under stochastic investment opportunities largely mirrors the constant market results and is therefore omitted. The focus now shifts to the habit formation model, which extends the benchmark-adjusted framework by introducing a time-varying reference level h, an adjustment speed α, and a weighting parameter β.

$\bar{C} = 25$											
Horizon	r	σ	σ_λ	λ	$\bar{\lambda}$	Description	Myopic (%)	Hedging (%)	Total (%)	Cash (%)	Hedging/Myopic
$T = 5$	0.02	0.2	0.2	0.3	0.3	Baseline	26.4	9.7	36.2	63.8	0.4
	0.02	0.2	0.2	0.6	0.3	Increase λ	52.9	16.2	69.1	30.9	0.3
	0.02	0.2	0.2	0.3	0.6	Increase $\bar{\lambda}$	26.4	12.6	39.0	61.0	0.5
	0.02	0.2	0.2	0.3	-0.6	Negative $\bar{\lambda}$	26.4	0.7	27.1	72.9	0.0
	0.02	0.4	0.4	0.3	0.3	Increase σ, σ_λ	13.2	14.1	27.3	72.7	1.1
	0.04	0.2	0.2	0.3	0.3	Increase r	26.6	9.7	36.3	63.7	0.4
$T = 10$	0.02	0.2	0.2	0.3	0.3	Baseline	23.2	14.5	37.7	62.3	0.6
	0.02	0.2	0.2	0.6	0.3	Increase λ	46.4	22.2	68.7	31.3	0.5
	0.02	0.2	0.2	0.3	0.6	Increase $\bar{\lambda}$	23.2	19.8	43.0	57.0	0.9
	0.02	0.2	0.2	0.3	-0.6	Negative $\bar{\lambda}$	23.2	-4.4	18.8	81.2	-0.2
	0.02	0.4	0.4	0.3	0.3	Increase σ, σ_λ	11.6	23.2	34.8	65.2	2.0
	0.04	0.2	0.2	0.3	0.3	Increase r	23.8	14.5	38.3	61.7	0.6

Table 10: Benchmark-adjusted CRRA w. stochastic investment opportunities (2)

$\bar{C} = 50$											
Horizon	r	σ	σ_λ	λ	$\bar{\lambda}$	Description	Myopic (%)	Hedging (%)	Total (%)	Cash (%)	Hedging/Myopic
$T = 5$	0.02	0.2	0.2	0.3	0.3	Baseline	22.9	8.4	31.3	68.7	0.4
	0.02	0.2	0.2	0.6	0.3	Increase λ	45.7	14.0	59.8	40.2	0.3
	0.02	0.2	0.2	0.3	0.6	Increase $\bar{\lambda}$	22.9	10.9	33.7	66.3	0.5
	0.02	0.2	0.2	0.3	-0.6	Negative $\bar{\lambda}$	22.9	0.6	23.4	76.6	0.0
	0.02	0.4	0.4	0.3	0.3	Increase σ, σ_λ	11.4	12.2	23.6	76.4	1.1
	0.04	0.2	0.2	0.3	0.3	Increase r	23.2	8.4	31.6	68.4	0.4
$T = 10$	0.02	0.2	0.2	0.3	0.3	Baseline	16.4	10.2	26.7	73.3	0.6
	0.02	0.2	0.2	0.6	0.3	Increase λ	32.8	15.7	48.5	51.5	0.5
	0.02	0.2	0.2	0.3	0.6	Increase $\bar{\lambda}$	16.4	14.0	30.4	69.6	0.9
	0.02	0.2	0.2	0.3	-0.6	Negative $\bar{\lambda}$	16.4	-3.1	13.3	86.7	-0.2
	0.02	0.4	0.4	0.3	0.3	Increase σ, σ_λ	8.2	16.4	24.6	75.4	2.0
	0.04	0.2	0.2	0.3	0.3	Increase r	17.6	10.7	28.4	71.6	0.6

Table 11: Benchmark-adjusted CRRA w. stochastic investment opportunities (3)

9.3 Results for habit utility

Having analyzed portfolio allocation dynamics under both constant and stochastic investment opportunities, both with and without a consumption benchmark, the focus now shifts to the habit formation model. To avoid repeating previously established results, this part of the analysis emphasizes how habit formation differs from the benchmark-adjusted CRRA utility.

This approach is taken because it offers the most potential for new and meaningful insights. As the habit model builds on the benchmark framework by introducing additional dynamic elements, the earlier models can be seen as special cases of it. Therefore, the qualitative behaviors will remain similar, and the key interest lies in understanding the impact of the added flexibility through the parameters α and β. When both are set to zero, the benchmark-adjusted CRRA and habit model are equivalent.

9.3.1 Constant market

The results are presened in Table 12.

Horizon	γ	$h_0 = 25,\ \alpha = 0.1,\ \beta = 0.2$			$h_0 = 50,\ \alpha = 0.1,\ \beta = 0.2$		
		Stock (%)	Cash (%)	Stock/Cash	Stock (%)	Cash (%)	Stock/Cash
$T = 5$	0.1	1359.0	-1259.0	-1.1	1218.0	-1118.0	-1.1
	1	135.9	-35.9	-3.8	121.8	-21.8	-5.6
	5	27.2	72.8	0.4	24.4	75.6	0.3
	10	13.6	86.4	0.2	12.2	87.8	0.1
	20	6.8	93.2	0.1	6.1	93.9	0.1
	30	4.5	95.5	0.0	4.1	95.9	0.0
$T = 10$	0.1	1281.6	-1181.6	-1.1	1063.3	-963.3	-1.1
	1	128.2	-28.2	-4.6	106.3	-6.3	-16.8
	5	25.6	74.4	0.3	21.3	78.7	0.3
	10	12.8	87.2	0.1	10.6	89.4	0.1
	20	6.4	93.6	0.1	5.3	94.7	0.1
	30	4.3	95.7	0.0	3.5	96.5	0.0

Table 12: Habit utility w. constant investment opportunities (1)

Since the weights are determined from $t = 0$, the habit level $h = h_0$ is fixed and not updated over time. As a result, the parameters α and β influence the allocation indirectly through the adjustment term $h \cdot F(T)$. A higher α increases the investor's sensitivity to past consumption, raising $F(T)$ and thereby lowering the risky allocation. Conversely, a higher β, which affects the speed at which the habit decays, reduces $F(T)$ and leads to a higher risky share. Figure 5 illustrates how the stock-to-cash ratio varies across combinations of α and β. The figure illustrates a non-linear relationship.

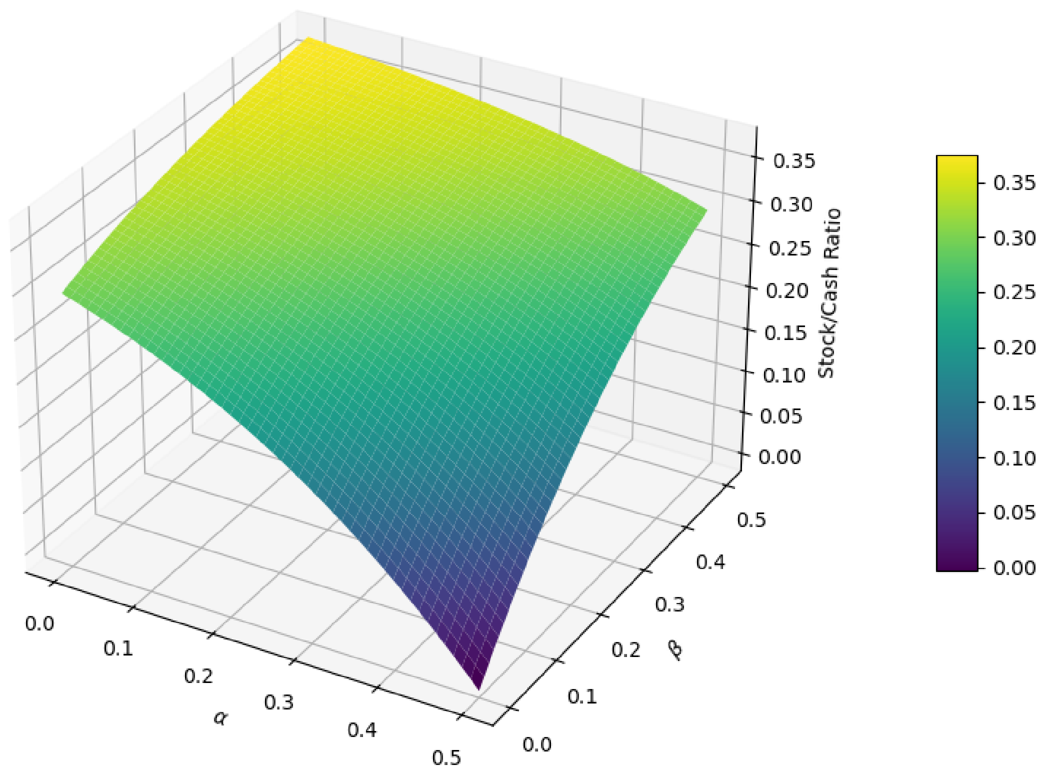


Figure 5: The effect of α and β on the Stock/Cash ratio

9.3.2 Stochastic market

The results are presented in Table 13 and Figure 6.

Horizon	γ	$h_0 = 25,\ \alpha = 0.1,\ \beta = 0.2$			$h_0 = 50,\ \alpha = 0.1,\ \beta = 0.2$		
		Myopic (%)	Hedging (%)	Hedging/Myopic	Myopic (%)	Hedging (%)	Hedging/Myopic
$T = 5$	0.1	1359.0	-1074.6	-0.7	1218.0	-963.1	-0.8
	1	135.9	0.0	0.0	121.8	0.0	0.0
	5	27.2	9.6	0.4	24.4	8.6	0.4
	10	13.6	5.8	0.4	12.2	5.2	0.4
	20	6.8	3.1	0.5	6.1	2.8	0.5
	30	4.5	2.2	0.5	4.1	1.9	0.5
$T = 10$	0.1	1281.6	-1025.2	-0.8	1063.2	-850.5	-0.8
	1	128.2	0.0	0.0	106.3	0.0	0.0
	5	25.6	15.2	0.6	21.3	12.6	0.6
	10	12.8	9.6	0.8	10.6	8.0	0.8
	20	6.4	5.4	0.8	5.3	4.5	0.8
	30	4.3	3.8	0.9	3.5	3.1	0.9

Table 13: Habit utility w. stochastic investment opportunities (1)

Table 13 does not provide any new insights into the quantitative relationships between the variables. Its primary purpose is to allow a direct comparison of the exact portfolio weights with those observed in the other models. Overall, the results resemble those from the benchmark-adjusted CRRA model in terms of portfolio composition. What sets the habit model apart is the role of the habit parameters α and β, which influence the hedging term and the hedging-to-myopic ratio. For fixed values of α and β, changes in h_0 affect only the absolute size of the risky position, while the hedging-to-myopic ratio remains constant, just as in the benchmark-adjusted case.

The interesting part lies in how α and β shape this ratio, as illustrated in Figure 6. Increasing habit sensitivity via α raises the hedging demand, whereas increasing the decay rate β reduces it. Importantly, the interaction between the two is nonlinear. The marginal effect of one parameter depends on the level of the other, and their joint influence cannot be captured by a simple linear relationship. This is clearly visible in Figure 6, where the surface is curved rather than flat, indicating that the marginal effect of each parameter varies across the domain.

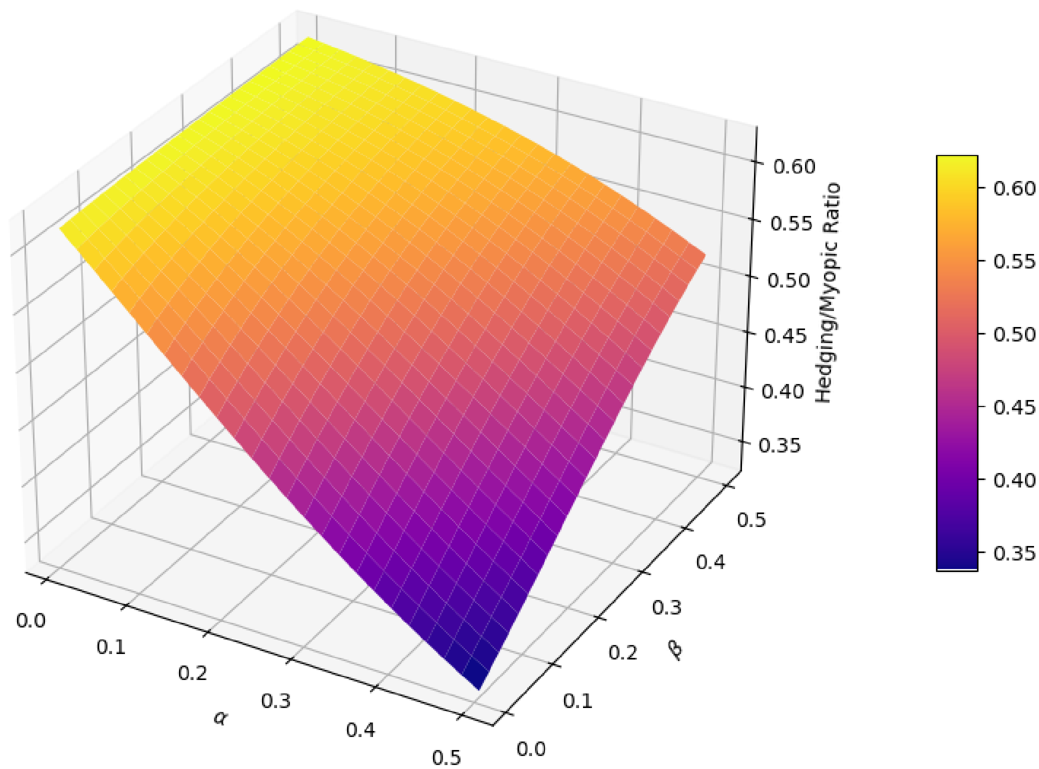


Figure 6: The effect of α and β on the Hedging/Myopic ratio

10 Discussion

This section of the thesis begins with a brief summary of the key findings from the empirical estimation of relative risk aversion and highlights the most notable results from the numerical analysis of the six portfolio strategies.

The next part presents the discussion with an experienced investment manager and aims to bridge the central theme of this thesis: risk and return, from a theoretical perspective to its application in real-world investment practice.

Finally, based on the insights from this dialogue, concluding reflections are presented. To round off, the thesis returns to the theoretical domain by modelling the investment manager's decision-making through a microeconomic, game-theory-inspired framework.

10.1 Quick recap

The goal of this thesis was to explore how a stochastic market price of risk affects the optimal portfolio strategies for different investor preferences, and to connect the resulting theoretical insights to real-world asset allocation, both through an empirical estimation of relative risk aversion and through dialogue with a professional investor.

The empirical estimation yielded a value of $\hat{\gamma} = 24.12$, with rolling 10-year estimates fluctuating between 20 and 30. These values are unusually high within the CRRA framework and reflect a well-known phenomenon in financial economics: the equity premium puzzle. Historically, equity returns have been so attractive relative to safer assets that extreme levels of risk aversion are required to justify a lack of full equity exposure.

When inserting this empirically derived range of $\hat{\gamma}$ into the numerical simulations, the resulting portfolios are far from realistic. They imply extremely conservative allocations, dominated by the risk-free asset. As such, the numerical results should not be interpreted as portfolio recommendations. Rather, they serve as a way to examine how portfolio weights shift in response to changes in underlying parameters, and whether those patterns align with practical asset allocation principles.

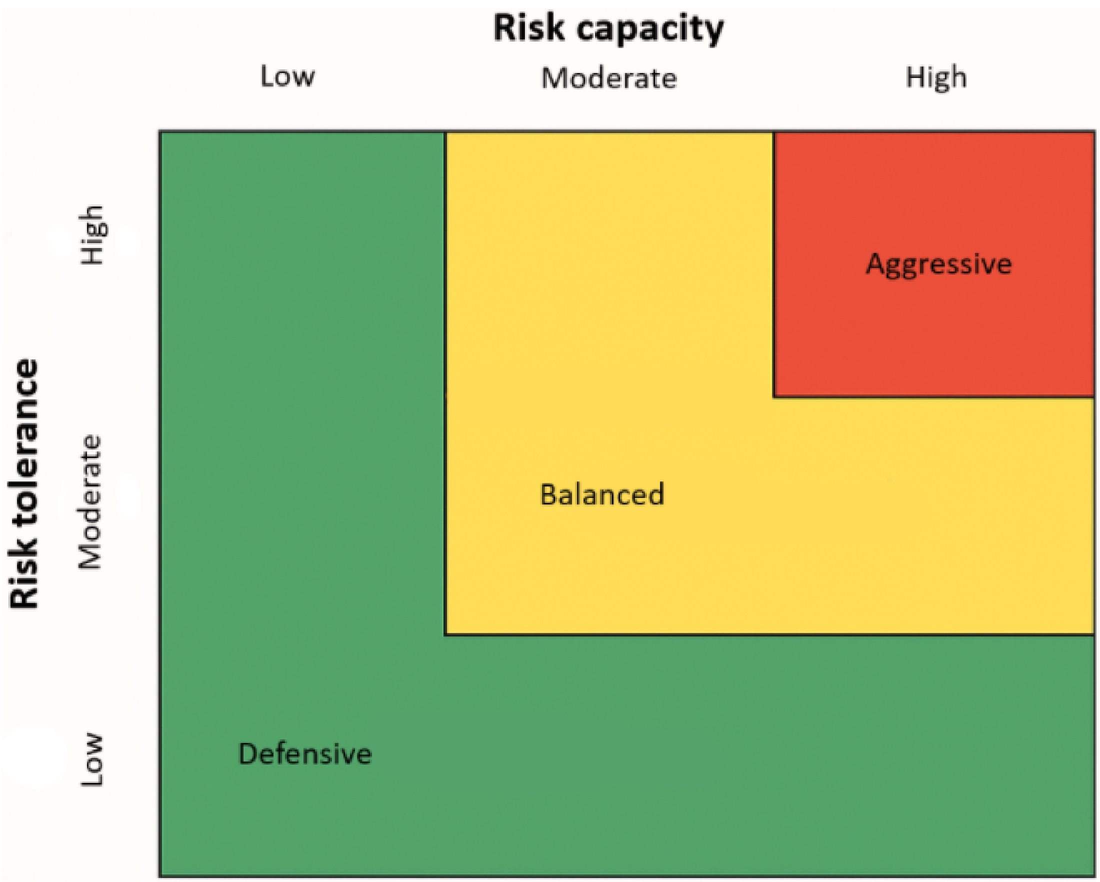
In general, the portfolio behavior aligns with theoretical expectations: higher γ leads to greater allocation to the risk-free asset, higher λ increases risky exposure, and higher volatility σ reduces it. However, an interesting result appears when a consumption benchmark is introduced, as in the benchmark-adjusted CRRA and habit model. In these cases, longer investment horizons tend to result in more conservative portfolios. This outcome holds both in constant and stochastic investment environments, and it contrasts with the common view in practice that longer horizons justify increased risk-taking.

10.2 Insights from an investment manager

This subsection builds on a reflective dialogue with an experienced investment manager. For professional reasons, the individual's identity has been anonymized and will from now on, be referred to as Jared. His background includes an MSc in Economics, a CFA certificate, and over 10 years of experience in various senior positions within strategic asset allocation. During our conversation, I had the opportunity to present the core ideas and findings of this thesis, which led to an engaging discussion centered on its two main themes: risk and return. What follows is a synthesis of the insights gained from our meeting, with an emphasis on relating the theoretical framework to real-world portfolio management practices.

Risk is central: When onboarding new clients or initiating investment mandates, the first and most essential step is to establish a clear understanding of risk in collaboration with the client. Within the utility framework, this is reduced to a single parameter, γ, which may appear simple. However, in practice, determining an investor's true risk profile is far more complex. This challenge is well recognized in the literature and has given rise to an entire subfield within finance: behavioral finance, which seeks to understand how investors actually behave.

As Jared pointed out, it is quite common for individuals new to investing to initially express preferences such as: "I want the highest possible return with the lowest possible risk." Interestingly, this aligns quite well with the intuition behind standard CAPM theory. Yet, once discussions move into evaluating specific outcome scenarios and potential losses, it quickly becomes evident that investors differ significantly in how they perceive and accept risk. To address this, potential clients need to be categorized more precisely. Jared emphasized the importance of distinguishing between two fundamental concepts: risk capacity and risk tolerance. The former refers to the investor's financial ability to bear risk, while the latter captures their psychological comfort with uncertainty and potential loss. The relationship between these two concepts is determined by a principle of the lowest common denominator, meaning that the more conservative of the two determines the appropriate risk profile for the client. This relationship is illustrated in the figure below.



An intuitive way to illustrate this line of thinking is through the fictional character Scrooge McDuck. Although he is extremely wealthy (high risk capacity), he is notoriously distressed by even marginal losses (low risk tolerance). As a result, despite his financial strength, his risk profile would place him in a defensive portfolio, since the principle of the lowest common denominator dictates that the more conservative of the two dimensions defines the risk profile.

Determning risk tolerance can be complicated: Risk capacity is a more objective measure, which Jared and his team are typically able to assess through a straightforward evaluation of the client's financial situation, future plans, and current life circumstances. Based on their experience, they can reliably categorize clients into one of the three risk capacity levels.

Risk tolerance, on the other hand, is a more subjective and qualitative dimension. It must be quantified in order to support a concrete classification, but unlike observable variables such as temperature, it is not directly measurable. Different firms adopt different methods for assessing risk tolerance, but these approaches are generally built around similar principles. Jared provided an overview of how this is typically done in practice.

The client is initially presented with a preliminary test designed to confirm their familiarity with the types of investments that may be included in the portfolio. This step is essential, as any assessment of the client's risk profile is based on an understanding of the underlying instruments.

In the next stage, the client is presented with a series of questions, such as[3]

- What is your overall goal with the investment?
- When do you expect to use the invested capital?
- How do you view fluctuations in returns over time?
- How would you prefer your investment to evolve?

Each question offers three answer options, reflecting increasing willingness to take risk. The most conservative response receives the fewest points, while the most risk-seeking earns the most. The total score is then calculated by summing all responses, and the client is categorized into one of three risk tolerance levels based on the final score.

[3]These are not exact questions, but questions that Jared approved of

Risk is more than a standard deviation: In this thesis, the primary measure of volatility is standard deviation. While Jared acknowledged that this metric is analytically convenient, he emphasized that it is a difficult quantity to use as the sole basis for portfolio construction. In client-facing situations, it is nearly impossible to make meaningful assessments such as: "You appear to be an 8% standard deviation type of investor, whereas you seem more like a 14% one." In contrast, forming an impression of an investor's desired expected return is often more intuitive.

This led to a broader discussion on the trade-off between complex theoretical models and simple, practical solutions. Jared noted that if a straightforward approach captures the client's needs, it is often preferred over abstract, model-driven strategies. Nonetheless, an important takeaway was that he prefers to incorporate higher-order moments, such as skewness and kurtosis, as well as alternative risk measures like Value at Risk (VaR) and, in particular, Conditional Value at Risk (CVaR), which he finds especially useful.

Jared highlighted a key shortcoming of the standard VaR measure. He noted that in extreme market scenarios, quantitative analysts may respond with statements such as: "Our model did not anticipate this level of loss." This reaction stems from the fact that VaR is essentially a quantile-based measure, which only indicates a threshold that losses will not exceed with a given confidence level. It does not provide any information about the magnitude of losses beyond that point.

For this reason, CVaR is preferred, which addresses this limitation. CVaR estimates the expected loss given that the portfolio ends up in the worst-case tail of the distribution, typically the worst 5%. This measure tends to be more intuitive for clients, as it offers a clearer picture of potential losses in adverse scenarios and implicitly incorporates the expected return, which, as previously mentioned, is often easier for clients to relate to.

Portfolio construction is nuanced in practice: In this thesis, we derived optimal portfolio solutions for three distinct investor preferences under two different market settings. The resulting framework led to a closed-form equation that could be used to directly compute the optimal portfolio weights. In practice, portfolio construction is far less straightforward. Jared demonstrated a simple method for constructing portfolios using Morningstar, which in many ways parallels the structure of this thesis. Still, this model is only one of several components that contribute to the design of a final portfolio for a given investor type.

The first step in this process is to define the investment landscape. In this thesis, we considered a market consisting of a risk-free bank account and a risky asset, interpreted as a broad stock index. Jared pointed out that this setup represents the bare minimum for modeling investment decisions. While adding additional asset classes may not always offer further insights in theoretical comparisons, in practice it is essential to include a broader range of assets with different higher-order moments and correlation structures.

One of the primary differences, is the size and complexity of the investment universe. In the model Jared showcased, the landscape includes a complete money market, short- and long-term government bonds, investment-grade and high-yield corporate bonds, hedge funds, microfinance products, equities from different geographical regions, emerging market equities, and even proxies for private equity investments.

Once the investment universe is defined, the next step involves assigning expected returns to the various asset classes. For this, Jared refers to data provided by Rådet for Afkastforventninger, which publishes semiannually forecasts for returns across asset classes. Based on this input and using Morningstar's platform, he constructs an efficient frontier that resembles the classical CAPM mean-variance frontier. However, instead of using standard deviation as the risk measure, the model is built around CVaR.

Given an investor's CVaR preference and the defined investment landscape, it becomes possible to identify an optimal portfolio allocation. Although the approach shares similarities with the classical portfolio theory presented in this thesis, it differs due to the incorporation of CVaR and a broader, more realistic set of asset classes with varying correlations and higher order moments.

There is no perfect model: CVaR optimization is just one of several models used in practice. When determining a final portfolio, Jared applies multiple models and forms an average, by qualitatively weighing the outcomes of each. For example, he compares results across a bootstrap model and a Johnson distribution model, and uses their combined insights to guide the overall direction of the portfolio. This stands in clear contrast to the framework applied in this thesis, where a given preference structure and market setting lead to a single, closed-form solution.

While neither model is perfect, they generally outperform the Markowitz framework in real-world applications. Standard deviation remains a relevant metric, but it is used alongside CVaR to perform a more nuanced and robust assessment of risk.

Modelling key figures is reasonable: I was particularly interested in hearing Jared's perspective on the idea of modelling key quantities such as the market price of risk directly, as done in this thesis. While he was generally receptive to the idea, he noted that he would approach the problem quite differently in practice. In academic literature, especially within continuous-time models, quantities are often modeled to create a mathematical environment that captures key features of the real world. Jared acknowledged the theoretical elegance of this approach but emphasized that, at this level of abstraction, the models become extremely complex and detached from practical solutions.

Instead, he expressed a strong preference for using factor models. As an example, he described a model he had encountered while working with Morgan Stanley, which consisted of over fifty factors used to estimate expected returns. These factors spanned a wide range, from firm-specific metrics such as EBITDA and revenue to broader macroeconomic indicators like employment and GDP growth. Each factor could be linked through its own sub-equation, creating a highly interconnected web of dynamic relationships.

What is particularly valuable about these models is their applications for scenario analysis. For instance, one could investigate the implications of a rise in unemployment over the next few years, and analyze how this assumption would affect the expected returns.

Real-life investors are not always rational: In this thesis, it is assumed that investors behave rationally, expressed through standard utility functions based on principles, such as the idea that more consumption is always preferred to less. However, based on Jared's experience, this assumption often fails to hold in practice. He specifically pointed out that investor behavior tends to shift depending on market conditions: when markets perform well, clients are typically focused on relative performance versus benchmarks. Yet in downturns, benchmark comparisons tend to lose relevance, and clients become primarily concerned with the absolute losses in their portfolios. This shift in focus is highly inconsistent with rational behavior.

We also discussed the idea that real-life utility appears to be path-dependent, which motivates the use of models such as Markov chains. For example, investor behavior during periods of "buying the dip" often reflects a backward-looking evaluation of portfolio performance. A rational investor should, in theory, only care about future prospects, not how a position has evolved. However, as Jared noted, many investors express dissatisfaction when a position that previously yielded a 50% gain now stands at 20%, even though 20% is objectively a strong return. This effect is another reminder that behavioral biases play a significant role in real-world investment decisions.

Relating this phenomenon to the models presented in this thesis, it can be argued that habit formation preferences implicitly capture the path-dependence of real-life utility, as current utility depends on past consumption through the parameters α and β. This may further motivate the use of such models.

This final point captures a broader theme that has emerged throughout the discussion: while theoretical models offer structure and insight, real-life portfolio construction must always account for the human element. Investors are not perfectly rational or purely utility-maximizing. They are people with emotions, experiences, and changing perspectives. And that, ultimately, makes all the difference.

10.3 Modelling the investment manager

The following model is a microeconomic, game-theory-inspired framework developed to illustrate how Jared's work with asset allocation and client interaction can be represented in a structured micro setting. While the earlier discussion moved from theory to practice, this model takes the reverse approach. It starts from practical observations from our discussion, and tries to translate them into a theoretical model.

Let R_C denote an investor's risk capacity and R_T denote their risk tolerance, both defined on the ordinal scale

$$\{\text{Low},\ \text{Moderate},\ \text{High}\}$$

Define a total order such that

$$\text{Low} < \text{Moderate} < \text{High}$$

Then, the effective risk profile R^* that determines the investor's portfolio type can be expressed as

$$R^* = \min(R_C,\ R_T)$$

This implies that the final risk category is constrained by the more conservative of the two inputs. For example, an investor with $R_C = \text{High}$ but $R_T = \text{Moderate}$ would be assigned $R^* = \text{Moderate}$, corresponding to a balanced portfolio. Let the set of portfolio types be defined as

$$\mathcal{P} = \{\text{Defensive},\ \text{Balanced},\ \text{Aggressive}\}$$

Define a mapping from the effective risk profile $R^* \in \{\text{Low},\ \text{Moderate},\ \text{High}\}$ to portfolio type $P \in \mathcal{P}$ as follows

$$P(R^*) = \begin{cases} \text{Defensive} & \text{if } R^* = \text{Low} \\ \text{Balanced} & \text{if } R^* = \text{Moderate} \\ \text{Aggressive} & \text{if } R^* = \text{High} \end{cases}$$

Since $R^* = \min(R_C,\ R_T)$, the assigned portfolio is always the one corresponding to the more conservative of the investor's financial ability and psychological preference.

The investment manager's objective is now to maximize the client's utility based on the available information set $\mathcal{I}$, by selecting an optimal portfolio within the category determined by the investor's effective risk profile R^*

$$\max_{P \in \mathcal{P}_{R^*}} \quad U(P \mid \mathcal{I})$$

Where $\mathcal{P}_{R^*} \subseteq \mathcal{P}$ represents the set of feasible portfolios classified as Defensive, Balanced, or Aggressive, corresponding respectively to low, moderate, and high effective risk profiles. This means the manager first identifies the appropriate risk category R^* for the client and then optimizes within that portfolio class to maximize utility, leveraging professional experience, market data, and risk measures such as CVaR.

11 Conclusion

This thesis set out to investigate how a stochastic market price of risk influences optimal portfolio strategies under different investor preferences, and how these theoretical insights compare to real-world portfolio management. To address this, mathematical solutions were derived for three distinct utility functions across two market settings, yielding six different portfolio strategies. These solutions were obtained using the Martingale method and analyzed through both their analytical form and numerical simulations.

Simultaneously, the study sought to bridge the gap between theory and practice by empirically estimating the relative risk aversion coefficient using historical market data, and by having a discussion with a professional investment manager. This allowed for an evaluation of how well the theoretical models capture the complexities of real-world portfolio construction.

A key theoretical insight was the appearance of a hedging demand in the portfolio when accounting for a stochastic market price of risk, reflecting investors' attempts to mitigate future changes in the market price of risk. Interestingly, models incorporating benchmark or habit preferences showed that longer investment horizons led to more conservative allocations, contrary to traditional advice favoring increased risk-taking over time.

The empirical estimation revealed high relative risk aversion values, ranging from 20 to 30, consistent with the well-documented equity premium puzzle. When applied to the numerical models, these values resulted in unrealistically conservative portfolios.

The discussion with the investment manager showcased that real-world asset allocation is far more nuanced than what a single risk parameter can capture. Risk is multifaceted, and incorporating additional measures such as CVaR is important.

Ultimately, this thesis demonstrates that while mathematical models provide valuable frameworks for understanding portfolio optimization, real-world investing involves layers of complexity, judgment, and human behavior that extend beyond formal theory. The art of portfolio management lies in integrating these elements, reminding us that the most elegant models must still confront the messy realities of the market and its participants.

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Appendix

Itô's Lemma: Pricing kernel

The pricing kernel ζ_t is given by

$$\zeta_t = \exp\left\{-\int_0^t r_s\,ds - \int_0^t \boldsymbol{\lambda}_s^\top\,d\boldsymbol{Z}_s - \frac{1}{2}\int_0^t \|\boldsymbol{\lambda}_s\|^2\,ds\right\}$$

Let the exponent be denoted by X_t, so that

$$X_t = \int_0^t r_s\,ds + \int_0^t \boldsymbol{\lambda}_s^\top d\boldsymbol{Z}_s + \frac{1}{2}\int_0^t \|\boldsymbol{\lambda}_s\|^2 ds, \quad \text{and} \quad \zeta_t = e^{-X_t}$$

Apply Itô's Lemma to $f(X_t) = e^{-X_t}$

$$d\zeta_t = -\zeta_t\,dX_t + \frac{1}{2}\zeta_t\,d[X]_t$$

Define dX_t

$$dX_t = r_t\,dt + \boldsymbol{\lambda}_t^\top d\boldsymbol{Z}_t + \frac{1}{2}\|\boldsymbol{\lambda}_t\|^2 dt$$

And the quadratic variation

$$d[X]_t = \|\boldsymbol{\lambda}_t\|^2 dt$$

Substituting into Itô's Lemma

$$\begin{aligned} d\zeta_t &= -\zeta_t\left(r_t\,dt + \boldsymbol{\lambda}_t^\top d\boldsymbol{Z}_t + \frac{1}{2}\|\boldsymbol{\lambda}_t\|^2 dt\right) + \frac{1}{2}\zeta_t\|\boldsymbol{\lambda}_t\|^2 dt \\ &= -\zeta_t r_t\,dt - \zeta_t\boldsymbol{\lambda}_t^\top d\boldsymbol{Z}_t \end{aligned}$$

The dynamics of ζ_t are given by

$$d\zeta_t = -\zeta_t r_t\,dt - \boldsymbol{\lambda}_t^\top\zeta_t\,d\boldsymbol{Z}_t$$

Itô's product rule: Equation (38)

Itô's product rule, for two Itô processes X_t and Y_t, is given by

$$d(X_t Y_t) = X_t\, dY_t + Y_t\, dX_t + dX_t\, dY_t$$

Let $X_t = \zeta_t$ and $Y_t = W_t$ with known dynamics

$$dW_t = \left[\left(r_t + \boldsymbol{\pi}_t^\top \underline{\underline{\sigma}}_t \boldsymbol{\lambda}_t\right) W_t - c_t\right] dt + W_t\, \boldsymbol{\pi}_t^\top \underline{\underline{\sigma}}_t\, d\boldsymbol{Z}_t$$

$$d\zeta_t = -\zeta_t r_t\, dt - \zeta_t \boldsymbol{\lambda}_t^\top d\boldsymbol{Z}_t$$

Inserting

$$d(\zeta_t W_t) = \zeta_t\, dW_t + W_t\, d\zeta_t + d\zeta_t\, dW_t$$

Computing each term

$$\begin{aligned}
\zeta_t\, dW_t &= \zeta_t \left[\left(r_t + \boldsymbol{\pi}_t^\top \underline{\underline{\sigma}}_t \boldsymbol{\lambda}_t\right) W_t - c_t\right] dt + \zeta_t W_t\, \boldsymbol{\pi}_t^\top \underline{\underline{\sigma}}_t\, d\boldsymbol{Z}_t \\
W_t\, d\zeta_t &= -\zeta_t W_t r_t\, dt - \zeta_t W_t\, \boldsymbol{\lambda}_t^\top d\boldsymbol{Z}_t \\
d\zeta_t\, dW_t &= (-\zeta_t\, \boldsymbol{\lambda}_t^\top d\boldsymbol{Z}_t)(W_t\, \boldsymbol{\pi}_t^\top \underline{\underline{\sigma}}_t\, d\boldsymbol{Z}_t) = -\zeta_t W_t\, \boldsymbol{\lambda}_t^\top\, \boldsymbol{\pi}_t^\top \underline{\underline{\sigma}}_t\, dt
\end{aligned}$$

Combining all drift terms

$$\begin{aligned}
&\zeta_t r_t W_t dt + \zeta_t W_t \boldsymbol{\pi}_t^\top \underline{\underline{\sigma}}_t \boldsymbol{\lambda}_t dt - \zeta_t c_t dt - \zeta_t W_t r_t dt - \zeta_t W_t \boldsymbol{\lambda}_t^\top \boldsymbol{\pi}_t^\top \underline{\underline{\sigma}}_t dt \\
&= -\zeta_t c_t dt
\end{aligned}$$

Combining all diffusion terms

$$\zeta_t W_t\, \boldsymbol{\pi}_t^\top \underline{\underline{\sigma}}_t\, d\boldsymbol{Z}_t - \zeta_t W_t\, \boldsymbol{\lambda}_t^\top\, d\boldsymbol{Z}_t = \zeta_t W_t \left(\boldsymbol{\pi}_t^\top \underline{\underline{\sigma}}_t - \boldsymbol{\lambda}_t^\top\right) d\boldsymbol{Z}_t$$

Thus, the dynamics of $\zeta_t W_t$ are given by

$$d(\zeta_t W_t) = -\zeta_t c_t\, dt + \zeta_t W_t \left(\boldsymbol{\pi}_t^\top \underline{\underline{\sigma}}_t - \boldsymbol{\lambda}_t^\top\right) d\boldsymbol{Z}_t$$

Itô's Lemma: CRRA with constant investment opportunities

The optimal wealth dynamics are given by

$$W_t^* = \frac{W_0}{g_0} e^{-\frac{\delta}{\gamma}t} \zeta_t^{-\frac{1}{\gamma}} g_t$$

The dynamics of ζ_t are given by

$$d\zeta_t = -\zeta_t r_t \, dt - \boldsymbol{\lambda}_t^\top \zeta_t \, d\boldsymbol{Z}_t$$

Calculating all the needed components

$$\frac{\partial W_t^*}{\partial t} = \frac{W_0}{g_0} \zeta_t^{-\frac{1}{\gamma}} \left(-\frac{\delta}{\gamma} e^{-\frac{\delta}{\gamma}t} g_t + e^{-\frac{\delta}{\gamma}t} g_t' \right) = W_t^* \left(-\frac{\delta}{\gamma} + \frac{g_t'}{g_t} \right)$$

$$\frac{\partial W_t^*}{\partial \zeta_t} = \frac{W_0}{g_0} e^{-\frac{\delta}{\gamma}t} g_t \cdot \left(-\frac{1}{\gamma} \right) \zeta_t^{-\frac{1}{\gamma}-1} = -\frac{1}{\gamma} \frac{W_t^*}{\zeta_t}$$

$$\frac{\partial^2 W_t^*}{\partial \zeta_t^2} = \frac{W_0}{g_0} e^{-\frac{\delta}{\gamma}t} g_t \cdot \left(-\frac{1}{\gamma} \left(\frac{1}{\gamma} + 1 \right) \zeta_t^{-\frac{1}{\gamma}-2} \right) = -\frac{1}{\gamma} \left(\frac{1}{\gamma} + 1 \right) \frac{W_t^*}{\zeta_t^2}$$

$$d(\zeta_t)^2 = \zeta_t^2 \|\boldsymbol{\lambda}_t\|^2 dt$$

Itô's Lemma is given by

$$dW_t^* = \frac{\partial W_t^*}{\partial t} dt + \frac{\partial W_t^*}{\partial \zeta_t} d\zeta_t + \frac{1}{2} \frac{\partial^2 W_t^*}{\partial \zeta_t^2} (d\zeta_t)^2$$

Inserting into Itô's Lemma

$$dW_t^* = W_t^* \left(-\frac{\delta}{\gamma} + \frac{g_t'}{g_t} \right) dt - \frac{1}{\gamma} \frac{W_t^*}{\zeta_t} (-\zeta_t r_t dt - \boldsymbol{\lambda}_t^\top \zeta_t d\boldsymbol{Z}_t) - \frac{1}{2} \left(-\frac{1}{\gamma} \left(\frac{1}{\gamma} + 1 \right) \right) \frac{1}{\zeta_t^2} W_t^* \zeta_t^2 \|\boldsymbol{\lambda}_t\|^2 dt$$

Collecting dt and $d\boldsymbol{Z}_t$ terms

$$dW_t^* = -\frac{\delta}{\gamma} W_t^* dt + \frac{g_t'}{g_t} W_t^* dt + \frac{r_t}{\gamma} W_t^* dt + \frac{1}{\gamma} W_t^* \boldsymbol{\lambda}_t^\top d\boldsymbol{Z}_t + \frac{1}{2} \left(\frac{1}{\gamma^2} + \frac{1}{\gamma} \right) W_t^* \|\boldsymbol{\lambda}_t\|^2 dt$$

$$dW_t^* = \left(\frac{g_t'}{g_t} + \frac{r_t - \delta}{\gamma} + \frac{1}{2} \left(\frac{1}{\gamma^2} + \frac{1}{\gamma} \right) \|\boldsymbol{\lambda}_t\|^2 \right) W_t^* dt + \frac{1}{\gamma} W_t^* \boldsymbol{\lambda}_t^\top d\boldsymbol{Z}_t$$

Itô's Lemma: Benchmark-adjusted CRRA with constant investment opportunities

The optimal wealth dynamics are given by

$$W_t^* = \frac{W_0 - F(0)}{g_0} e^{-\frac{\delta}{\gamma}t} \zeta_t^{-\frac{1}{\gamma}} g_t + \bar{C}F(t)$$

The dynamics of ζ_t are given by

$$d\zeta_t = -\zeta_t r_t \, dt - \boldsymbol{\lambda}_t^\top \zeta_t \, d\boldsymbol{Z}_t$$

Applying Itô's Lemma on the diffusion-relevant term (as shown earlier, the drift is redundant for the method)

$$\frac{\partial W_t^*}{\partial \zeta_t} = \frac{W_0 - F(0)}{g_0} e^{-\frac{\delta}{\gamma}t} \left(-\frac{1}{\gamma} \zeta_t^{-\frac{1}{\gamma}-1} \right) g_t = -\left(\frac{W_t^* - \bar{C}F(t)}{\gamma \zeta_t} \right)$$

Itô's Lemma is given as (red part is diffusion)

$$dW_t^* = \frac{\partial W_t^*}{\partial t} dt + \frac{\partial W_t^*}{\partial \zeta_t} d\zeta_t + \frac{1}{2} \frac{\partial^2 W_t^*}{\partial \zeta_t^2} (d\zeta_t)^2$$

Inserting

$$dW_t^* = \ldots dt + \left(-\frac{W_t^* - \bar{C}F(t)}{\gamma \zeta_t} \right) (-\zeta_t r_t dt - \boldsymbol{\lambda}_t^\top \zeta_t d\boldsymbol{Z}_t)$$

The part with $-\zeta_t r_t dt$ just goes into $\ldots$ for simplicity

$$dW_t^* = \ldots dt + \left(\frac{W_t^* - \bar{C}F(t)}{\gamma} \right) (\boldsymbol{\lambda}_t^\top d\boldsymbol{Z}_t)$$

$$dW_t^* = \ldots dt + \frac{1}{\gamma} (W_t^* - \bar{C}F(t)) \boldsymbol{\lambda}_t^\top d\boldsymbol{Z}_t$$

Itô's Lemma: General solution with stochastic investment opportunities

The optimal wealth dynamics are given by

$$W_t^* = \frac{W_0}{g_0} e^{-\frac{\delta}{\gamma} t} \zeta_t^{-\frac{1}{\gamma}} g_t$$

The dynamics of ζ are given by

$$d\zeta_t = -\zeta_t r_t \, dt - \boldsymbol{\lambda}_t^\top \zeta_t \, d\boldsymbol{Z}_t$$

Now g has dynamics, which are given by

$$dg_t = g_t \left[\mu_{gt} \, dt + \boldsymbol{\sigma}_{gt}^\top d\boldsymbol{Z}_t\right]$$

Since we have two dynamics, Ito's Lemma is now defined as

$$\begin{aligned}
dW_t^* = {} & \underbrace{\frac{\partial W_t^*}{\partial t}}_{\text{time drift}} dt + \underbrace{\frac{\partial W_t^*}{\partial \zeta_t}}_{\text{effect of } \zeta_t} d\zeta_t + \underbrace{\frac{\partial W_t^*}{\partial g_t}}_{\text{effect of } g_t} dg_t \\
& + \underbrace{\frac{1}{2}\frac{\partial^2 W_t^*}{\partial \zeta_t^2}}_{\text{volatility from } \zeta_t} (d\zeta_t)^2 + \underbrace{\frac{1}{2}\frac{\partial^2 W_t^*}{\partial g_t^2}}_{\text{volatility from } g_t} (dg_t)^2 + \underbrace{\frac{\partial^2 W_t^*}{\partial \zeta_t \partial g_t}}_{\text{covariation term}} d\zeta_t \, dg_t
\end{aligned}$$

Applying Ito's Lemma we obtain

$$\begin{aligned}
dW_t^* = {} & -\frac{\delta}{\gamma} W_t^* dt - \frac{1}{\gamma}\frac{W_t^*}{\zeta_t}\big(-\zeta_t r_t dt - \zeta_t \boldsymbol{\lambda}_t^\top d\boldsymbol{Z}_t\big) + \frac{W_t^*}{g_t}\big(\mu_{gt} g_t dt + \boldsymbol{\sigma}_{gt}^\top g_t d\boldsymbol{Z}_t\big) \\
& + \frac{1}{2}\Bigg[-\left(\frac{1}{\gamma} - 1\right)\frac{1}{\gamma}\frac{W_t^*}{\zeta_t}\big(\zeta_t^2 \|\boldsymbol{\lambda}_t\|^2 dt\big) - \frac{W_t^*}{g_t^2} g_t^2 \|\boldsymbol{\sigma}_{gt}\|^2 dt \\
& + 2\Bigg(-\frac{1}{\gamma}\frac{W_t^*}{\zeta_t g_t}\big(-\zeta_t r dt - \zeta_t \boldsymbol{\lambda}_t^\top d\boldsymbol{Z}_t\big)\big(\mu_{gt} g_t dt + \boldsymbol{\sigma}_{gt}^\top g_t d\boldsymbol{Z}_t\big)\Bigg)\Bigg]
\end{aligned}$$

Expanding terms

$$\begin{aligned}
dW_t^* = {} & -\frac{\delta}{\gamma} W_t^* dt + \frac{r_t}{\gamma} W_t^* dt + \frac{1}{\gamma} W_t^* \boldsymbol{\lambda}_t^\top d\boldsymbol{Z}_t + W_t^* \mu_{gt} dt + W_t^* \boldsymbol{\sigma}_{gt}^\top d\boldsymbol{Z}_t \\
& + \frac{1}{2}\left(\left(\frac{1}{\gamma^2} - \frac{1}{\gamma}\right) W_t^* \|\boldsymbol{\lambda}_t\|^2 dt - W_t^* \|\boldsymbol{\sigma}_{gt}\|^2 dt + \frac{2}{\gamma} W_t^* \boldsymbol{\lambda}_t^\top \boldsymbol{\sigma}_{gt} dt\right)
\end{aligned}$$

Collecting dt amd dZ_t terms

$$dW_t^* = \cdots dt + W_t^* \left(\frac{1}{\gamma}\boldsymbol{\lambda}_t + \boldsymbol{\sigma}_{gt}\right)^\top d\boldsymbol{Z}_t$$