RESUMEN

Análisis Numérico I [75.12]

Exámenes **PARCIAL** e **INTEGRADOR**

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Errores

- ★ Redondeo → el redondeo de la computadora
- ★ Inherente → error del ser humano
- ★ Truncamiento → discretización o aproximación

Tipos de Errores

- ➢ Absoluto → $e_{ax} = x - \bar{x}$
  - ○ Cota → $|e_{ax}| \leq \Delta x$
- ➢ Relativo → $e_{rx} = \frac{x-\bar{x}}{x}$
  - ○ Cota → $|e_{rx}| = \frac{\Delta x}{\bar{x}}$

Convención

$x = \bar{x} \pm \Delta x$

$\Delta x = 0, d_1 * 10^{-t}$ → se mayora a 1 dígito

Propagación de Errores

$z = f(x, y, t, ..., q)$

$\Delta z \leq \left|\frac{dz}{dx}\right|\Delta x + \left|\frac{dz}{dy}\right|\Delta y + \left|\frac{dz}{dt}\right|\Delta t + ... + \left|\frac{dz}{dq}\right|\Delta q,\ \Delta z \neq 0$

Búsqueda de Raíces

Criterios de Paro

- ★ Error Absoluto: $\left|p_n - p_{n-1}\right| < \frac{\varepsilon}{10^{-2}}$
- ★ Error Relativo: $\frac{\left|p_n - p_{n-1}\right|}{\left|p_n\right|} < \varepsilon$
- ★ N° Iteraciones: $n \leq N$
- ★ Resultado: $\left|f\left(p_n\right)\right| < \frac{\varepsilon}{10^{-5}}$

Bisección

Con $f \in C[a; b],\ f(a) < 0 \wedge f(b) > 0 \Rightarrow por\ Bolzano\ \exists p \in [a; b] \,/\, f(p) = 0$

$p_n = \frac{a_n + b_n}{2},\ n \geq 1,\ a_1 = a,\ b_1 = b$

n	a_n	b_n	p_n	$\left\|p_n - p_{n-1}\right\| < \varepsilon$

Cota

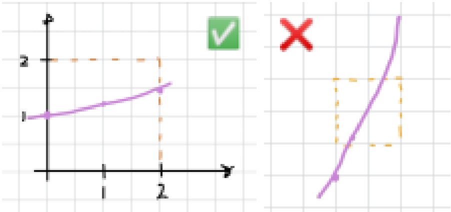
$\frac{\log(b-a) - \log(\varepsilon)}{\log(2)} < n$

Punto Fijo

Con $f(p) = p \Rightarrow p \in punto\, fijo$

Buscamos $g(x)$ admisible: $g(x) = x - \varphi(x) \cdot f(x)$

1. Existencia:



2. Unicidad:

   $\exists g'(x) \forall x \in (a; b) \wedge |g'(x)| \leq k < 1$

Si $\varphi(x) = 1$ no existe, probamos con $\varphi(x) = \frac{1}{f'(x)} \to \varphi(x_0) \to \varphi\left(\frac{b-a}{2}\right) = \varphi_0 \to g_{\varphi_0}(x) = x - \varphi_0 \cdot f(x)$

n	p_n	$g(p_n)$	$\lvert p_n - p_{n-1} \rvert < \varepsilon$

$p_0 \in semilla$

$p_{n+1} = g(p_n),\ n \geq 0$

→ convergencia lineal

Newton Raphson

Con $f \in C^2[a; b],\ f(p) = 0 \wedge p_0\, cercano\, a\, p$:

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$$

n	p_n	p_{n+1}	$\lvert p_n - p_{n+1} \rvert < \varepsilon$

→ convergencia cuadrática

Secante

$$p_n = p_{n-1} - \frac{f(p_{n-1}) * (p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}$$

n	p_n	p_{n+1}	$\lvert p_n - p_{n+1} \rvert < \varepsilon$

→ convergencia supralineal

Newton Raphson Modificado (Raíces Múltiples)

$$p_{n+1} = p_n - \frac{f(p_n) * f'(p_n)}{(f'(p_n))^2 - f(p_n) * f''(p_n)}$$

Ajuste Cuadrados Mínimos: $Ax = b \to A^T A\hat{x} = A^T b$

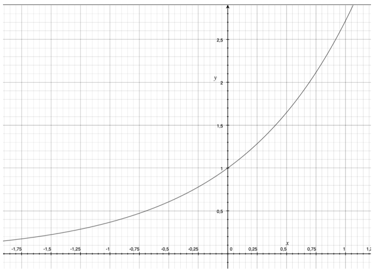
Distintos modelos…

Modelo Lineal: $y = a_0 + a_1 x + e$

$$\begin{vmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{vmatrix} \begin{vmatrix} a_0 \\ a_1 \end{vmatrix} = \begin{vmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} y_i * x_i \end{vmatrix}$$

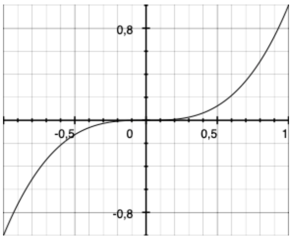
Modelo Exponencial: $y = a * e^{bx} \rightarrow \ln(y) = \ln(a) + bx$

$$\begin{vmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{vmatrix} \begin{vmatrix} \ln(a) \\ b \end{vmatrix} = \begin{vmatrix} \sum_{i=1}^{n} \ln(y_i) \\ \sum_{i=1}^{n} x_i * \ln(y_i) \end{vmatrix}$$



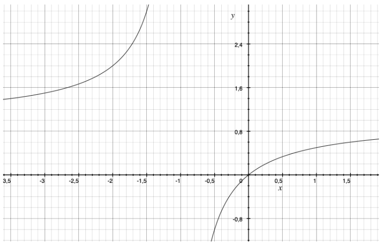
Modelo Potencial: $y = a * x^b \rightarrow \ln(y) = \ln(a) + b\ln(x)$

$$\begin{vmatrix} n & \sum_{i=1}^{n} \ln(x_i) \\ \sum_{i=1}^{n} \ln(x_i) & \sum_{i=1}^{n} \ln^2(x_i) \end{vmatrix} \begin{vmatrix} \ln(a) \\ b \end{vmatrix} = \begin{vmatrix} \sum_{i=1}^{n} \ln(y_i) \\ \sum_{i=1}^{n} \ln(x_i) * \ln(y_i) \end{vmatrix}$$



Modelo Racional: $y = \frac{ax}{b+x} \rightarrow \frac{1}{y} = \frac{b}{ax} + \frac{1}{a}$

$$\begin{vmatrix} \sum_{i=1}^{n} \frac{1}{x_i^2} & \sum_{i=1}^{n} \frac{1}{x_i} \\ \sum_{i=1}^{n} \frac{1}{x_i} & n \end{vmatrix} \begin{vmatrix} \frac{b}{a} \\ \frac{1}{a} \end{vmatrix} = \begin{vmatrix} \sum_{i=1}^{n} \frac{1}{x_i * y_i} \\ \sum_{i=1}^{n} \frac{1}{y_i} \end{vmatrix}$$



Modelo Polinomial: $y = a_0 + a_1 x + a_2 x^2 + e$

$$\begin{vmatrix} n & \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i^3 \\ \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i^3 & \sum_{i=1}^{n} x_i^4 \end{vmatrix} \begin{vmatrix} a_0 \\ a_1 \\ a_2 \end{vmatrix} = \begin{vmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} y_i * x_i \\ \sum_{i=1}^{n} y_i * x_i^2 \end{vmatrix}$$

Sistemas Lineales

Norma Infinito

$||A||_\infty = max_{1 \le i \le n} \left\{ \sum_{i=1}^{n} \left| a_{i,j} \right| \right\}$ (máximo de la suma de las filas)

Radio Espectral

$\rho(A) = max\left\{ \left| \lambda_i \right| \right\}$, $\lambda_i \in AVA$ de A

$\rho(A) < ||A||_\infty$

Métodos Directos

Descomposición LU

Sea $A \in R^{m \times n}$, necesitaremos $n - 1$ iteraciones:

1) $f_2 \leftarrow f_2 - m_{21} f_1,\ m_{21} = \frac{a_{21}}{a_{11}}$ ... $f_n \leftarrow f_n - m_{n1} f_1,\ m_{n1} = \frac{a_{n1}}{a_{11}}$

n-1) $f_n \leftarrow f_n - m_{n\,n-1} f_{n-1},\ m_{n\,n-1} = \frac{a_{n\,n-1}}{a_{n-1\,n-1}}$

Luego:

U = A (triangular superior)

L =

1	0	0	...	0
m_{21}	1	0	...	0
$\vdots$	$\vdots$	$\ddots$	...	$\vdots$
$m_{n-1\,1}$	$m_{n-1\,2}$	...	1	0
m_{n1}	m_{n2}	...	$m_{n\,n-1}$	1

Y resuelvo el sistema: $L\overline{y} = b,\ U\overline{x} = y$

Cholesky

$A \cdot x = b,\ A = L \cdot L^T \Rightarrow L^T\overline{x} = \overline{y},\ L\overline{y} = b$

Métodos Iterativos

$A = D - L - U \rightarrow \overline{x}_{k+1} = T \cdot \overline{x}_k + c$

Jacobi

$T_J = D^{-1} \cdot (L + U),\ c_J = D^{-1} \cdot b$

Gauss Seidel

$T_{GS} = (D - L)^{-1} \cdot U,\ c_{GS} = (D - L)^{-1} \cdot b$

Refinamiento Iterativo

Vector Residual $\Rightarrow\ r = b - A \cdot \bar{x},\ A \cdot \bar{y} = r$

Número de Condición de A $\Rightarrow\ K(A) \approx 10^t \cdot \frac{||\bar{y}||_\infty}{||\bar{x}||_\infty}$ (t: #dígitos de la aritmética)

Refinamiento $\Rightarrow\ \bar{x}_2 = \bar{x}_1 + \bar{y}_1$

Sistemas No Lineales

Método de Newton

$$\bar{x}_k = \bar{x}_{k-1} + \bar{y}_{k-1} \rightarrow JF\left(\bar{x}_{k-1}\right) \cdot \bar{y}_{k-1} = - F\left(\bar{x}_{k-1}\right)$$

Interpolación Polinomial

Polinomio de Lagrange

$P_L(x) = \sum_{k=0}^{n} L_k(x) * f(x_k),\ \ L_k(x) = \prod_{i=0,\, i\neq k}^{n} \frac{x-x_i}{x_k-x_i}$

Polinomio de Newton

una **diferencia dividida** es -aproximadamente- la derivada de una función en un punto y podemos reemplazar en caso de ser necesario

$P_N(x) = f[x_0] + \sum_{k=1}^{n} f[x_0,..., x_k](x - x_0)...(x - x_{k-1})$

$\varepsilon = f[x_0,..., x_n] = \left|P_N(x_0) - P_{N-1}(x_0)\right|$

x_i	$f\left(x_i\right)$	$f\left[x_i, x_{i+1}\right]$	$f\left[x_i, x_{i+2}\right]$
x_1	$f\left(x_1\right)$		
		$\frac{f\left(x_2\right)-f\left(x_1\right)}{x_2-x_1}$	
x_2	$f\left(x_2\right)$		
			$\frac{f\left[x_3,x_4\right]-f\left[x_1,x_2\right]}{x_4-x_1}$
x_3	$f\left(x_3\right)$		
		$\frac{f\left(x_4\right)-f\left(x_3\right)}{x_4-x_3}$	
x_4	$f\left(x_4\right)$		

Hermite

$$H_{2N+1}(x) = f[z_0] + \sum_{k=1}^{2n+1} f[z_0,..., z_k](x - z_0)...(x - z_{k-1})$$

z_i	x_i	$f(x_i)$	$f[z_i, z_{i+1}]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+2}]$
z_1	x_1	$f(x_1)$			
			$f'(x_1)$		
z_2	x_1	$f(x_1)$		$\frac{f[z_3,z_2]-f[z_2,z_1]}{z_3-z_1}$	
			$\frac{f(x_2)-f(x_1)}{z_3-z_2}$		$\frac{f[z_4,z_2]-f[z_3,z_1]}{z_4-z_1}$
z_3	x_2	$f(x_2)$		$\frac{f[z_4,z_3]-f[z_3,z_2]}{z_4-z_2}$	
			$f'(x_2)$		
z_4	x_2	$f(x_2)$			

Spline Cúbico

$$S(x) = \begin{cases} S_0(x) = a_0 + b_0(x - x_0) + c_0(x - x_0)^2 + d_0(x - x_0)^3,\ x \in [x_0, x_1] \\ S_1(x) = a_1 + b_1(x - x_1) + c_n(x - x_1)^2 + d_n(x - x_1)^3,\ x \in [x_1, x_2] \\ \vdots \\ S_n(x) = a_n + b_n(x - x_n) + c_n(x - x_n)^2 + d_n(x - x_n)^3,\ x \in [x_{n-1}, x_n] \end{cases}$$

$S_j(x_j) = f(x_j) = a_j$

$S_j(x_{j+1}) = S_{j+1}(x_{j+1}) = f(x_{j+1})$ (empalme)

$S'_j(x_{j+1}) = S'_{j+1}(x_{j+1})$ (las derivadas coinciden en el punto intermedio)

$S''_j(x_{j+1}) = S''_{j+1}(x_{j+1})$

- Spline Natural: $S''_0(x_{inicio}) = S''_n(x_{fin}) = 0$

Diferenciación Numérica

Hacia Atrás

$$f'\left(x_i\right) \approx \frac{f\left(x_i\right)-f\left(x_{i-1}\right)}{h}$$

Hacia Adelante

$$f'\left(x_i\right) \approx \frac{f\left(x_{i+1}\right)-f\left(x_i\right)}{h}$$

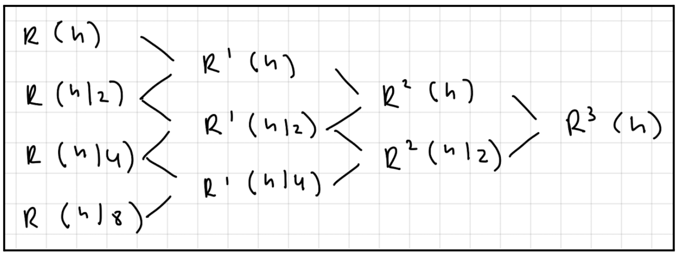
Centrada

$$f'\left(x_i\right) \approx \frac{f\left(x_{i+1}\right)-f\left(x_{i-1}\right)}{2h}$$

Extrapolación de Richardson

$$R^0(h) = \frac{f(x+h)-f(x-h)}{2h}$$

$$R^k(h) = \frac{4^k \cdot R^{k-1}\left(\frac{h}{2}\right)-R^{k-1}(h)}{4^k-1}$$



Integración Numérica

Regla de los Trapecios Compuesta

$$\int_a^b f(x)dx \approx \frac{h}{2} \cdot \left[f(a) + f(b) + 2 \cdot \sum_{k=1}^{N-1} f\left(x_k\right)\right],\ E_T = O\left(h^2\right)$$

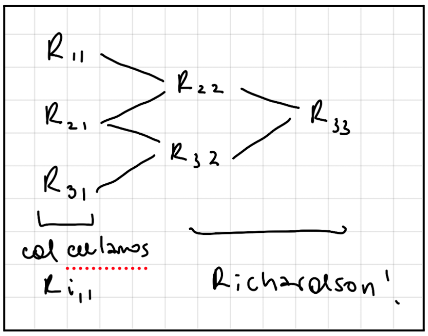
Regla de Simpson ⅓

$$\int_a^b f(x)dx \approx \frac{h}{3} \cdot \left[f(a) + f(b) + 4 \cdot \sum_{k=0}^{\frac{N-2}{2}} f\left(x_{2k+1}\right) + 2 \cdot \sum_{k=1}^{\frac{N-2}{2}} f\left(x_{2k}\right)\right],\ E_T = O\left(h^4\right)$$

Método de Romberg

$$R_{i,1} = \frac{1}{2} \cdot \left[R_{i-1,1} + h_{i-1} \cdot \sum_{k=1}^{2^{i-2}} f\left(a + \frac{2k-1}{2} \cdot h_{i-1}\right)\right]$$

$$R_{i,j} = \frac{4^{j-1} \cdot R_{i,j-1} - R_{i-1,j-1}}{4^{j-1}-1},\ E_T = O\left(h^6\right)$$



Ecuaciones Diferenciales

Ecuaciones Diferenciales Ordinarias

$\frac{dy}{dt} = f(t, y), \;\; y(a) = y_0$

Euler

$y_{i+1} = y_i + h \cdot f(t_i, y_i)$

Runge Kutta

Punto Medio

$y_{i+1} = y_i + h \cdot k_2$

$k_1 = f(t_i, y_i)$

$k_2 = f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2} \cdot k_1\right)$

Ralston

$y_{i+1} = y_i + \frac{h}{3} \cdot k_1 + \frac{2h}{3} \cdot k_2$

$k_1 = f(t_i, y_i)$

$k_2 = f\left(t_i + \frac{3h}{4}, y_i + \frac{3h}{4} \cdot k_1\right)$

Sistema de Ecuaciones Diferenciales

$\frac{dx}{dt} = f(t, x, y)$

$\frac{dy}{dt} = g(t, x, y)$

$x(a) = x_0$

$y(b) = y_0$

Euler

$$\begin{vmatrix} x_{i+1} \\ y_{i+1} \end{vmatrix} = \begin{vmatrix} x_i \\ y_i \end{vmatrix} + h \begin{vmatrix} f(t_i, x_i, y_i) \\ g(t_i, x_i, y_i) \end{vmatrix}$$

Runge Kutta Punto Medio

$$\begin{vmatrix} x_{i+1} \\ y_{i+1} \end{vmatrix} = \begin{vmatrix} x_i \\ y_i \end{vmatrix} + h \begin{vmatrix} m_2 \\ k_2 \end{vmatrix}$$

$m_1 = f(t_i, x_i, y_i)$

$k_1 = g(t_i, x_i, y_i)$

$m_2 = f\left(t_i + \frac{h}{2}, x_i + \frac{h}{2} \cdot m_1, y_i + \frac{h}{2} \cdot k_1\right)$

$k_2 = g\left(t_i + \frac{h}{2}, x_i + \frac{h}{2} \cdot m_1, y_i + \frac{h}{2} \cdot k_1\right)$

Ecuaciones Diferenciales de 2do Orden

$$a \cdot \frac{d^2y}{dt^2} + b \cdot \frac{dy}{dt} + c \cdot y = f(t)$$

Despejo:

$\frac{d^2y}{dt^2} = f(t, y, y')$

$y(t_0) = y_0$

$y'(t_0) = u_0$

Luego:

$\frac{dy}{dt} = u$

$\frac{du}{dt} = \frac{d^2y}{dt^2} = f(t, y, u)$

$y(t_0) = y_0$

$u(t_0) = u_0$

Euler

$$\begin{vmatrix} y_{i+1} \\ u_{i+1} \end{vmatrix} = \begin{vmatrix} y_i \\ u_i \end{vmatrix} + h \begin{vmatrix} u_i \\ f(t_i, y_i, u_i) \end{vmatrix}$$

Runge Kutta Punto Medio

$$\begin{vmatrix} y_{i+1} \\ u_{i+1} \end{vmatrix} = \begin{vmatrix} y_i \\ u_i \end{vmatrix} + h \begin{vmatrix} m_2 \\ k_2 \end{vmatrix}$$

$m_1 = u_i$

$k_1 = f(t_i, y_i, u_i)$

$m_2 = u_i + \frac{h}{2} \cdot k_1$

$k_2 = f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2} \cdot m_1, u_i + \frac{h}{2} \cdot k_1\right)$

Problema de Valores en la Frontera

En Diferencias Finitas

$\frac{d^2y}{dx^2} + P(x) \cdot \frac{dy}{dx} + Q(x) \cdot y = f(x)$

$y(a) = \alpha$

$y(b) = \beta$

Resolvemos:

$$\left(1 + \frac{h}{2} \cdot P_i\right) \cdot y_{i+1} + \left(-2 + h^2 \cdot Q_i\right) \cdot y_i + \left(1 - \frac{h}{2} \cdot P_i\right) \cdot y_{i-1} = h^2 \cdot f_i$$