Chaotic Elections:

Constructing and decomposing an election profile using various voting procedures

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Abstract

While elections and voting methods mainly seem to be political matters, they are closely related to mathematics. Based on *Chaotic Elections!: A Mathematician Looks at Voting,* by Donald G. Sarri (2001), this projects looks at various voting procedures, mainly focusing on positional methods (including plurality, anti-plurality, and Borda Count) and pairwise comparisons. Due to their different mathematical properties, the voting procedures often lead to different election outcomes, which become chaotic. The goal of this project is to explore some of the properties of certain voting procedures and develop ways to represent voter profiles and different outcomes in geometric ways. Therefore, this study shows that an election outcome reflects which procedure is used, rather than what voters *really* want.
The second half of this study looks at different ways to decompose a profile, using reversal, Condorcet, basic, and kernel effects. The decomposition, thus, shows how a single profile can lead to different election rankings depending on the voting procedure. Extending Saari's study with three-candidate elections, which can be modeled with a triangle, we attempt to construct an analogous decomposition for four-candidate elections using a tetrahedron. Therefore, we are able to decompose any four-candidate election profile and explain its characteristics.

1. Election X

To demonstrate how various voting procedures can yield different outcomes for a single profile, a simple example, named Election X, is constructed. This election consists of 18 voters and four candidates: A, B, C, and D. When each voter has a ranking of preference, let's say there are:

- four people who prefer A >B>C>D
- four people who prefer A>D>C>B
- four people who prefer C>B>D>A
- six people who prefer D>B>C>A

The term *profile* is indicated as "a listing of each voter's ranking of the candidates," (Saari), the profile for this election can be laid out as the table below:

Table 1: Election with 4 candidates and 18 voters

Number	Preference	Number	Preference
4	**A >B>C>D**	**4**	**C>B>D>A**
4	**A>D>C>B**	**6**	**D>B>C>A**

2. Voting Procedures

2.1 Plurality Voting

Plurality is a procedure commonly used, including in most of the US elections. In plurality voting, each voter votes for one favorite candidate and the candidate receiving the largest number of votes wins.

While this method seems reasonable and fair, it is problematic since plurality voting completely ignores voters' 2^{nd}, 3^{rd}, … and last choices.

As an example, in the 1996 Republican Presidential primaries in New Hampshire, candidate Pat Buchanan won the primary with nearly 30% of support, while the polls indicated that nearly half of the voters ranked him near the bottom. The other candidate, Robert Dole, who was highly regarded by most voters indeed lost because the number of plurality votes was less than Buchanan's.

In the example of Election X, candidate A is the winner as a total of eight voters choose A as their first choice. Note that A is elected without majority support in this particular election, as only eight out of eighteen voters choose as 1^{st} rank.

2.2 Anti-plurality Voting

Anti-plurality asks each voter to vote *against* one candidate and that candidate is assigned -1 point. A similar approach is to vote for k-candidates in n-candidate election, where $k = n - 1$.

This method can elect a candidate who has the fewest votes against, while it still has the risk that the elected candidate is just beneath mediocrity.

In the example X, candidate C wins the election because no one ranks C the last, while B,D, and A receive four, four, and ten votes against, respectively. Thus, A who was the winner with plurality procedure, now is the least-favored candidate in anti-plurality voting; in fact, it is noted that A wins with eight votes with plurality procedure while more people, 10 voters, consider A to be their least preferred candidate.

2.3 Approval Voting

This procedure allows voters to pick as many candidates as they approve to be elected.

Approval voting prevents a minor-party candidate from being a "spoiler" as the votes don't have to be split between the major and minor parties that hold similar platforms. Thus, it also provides more opportunities for the minor party, by reducing the likelihood of "strategic voting" – which will be explained more in the later section. This procedure is also seen to be more expressive than plurality vote, which limits a voter to pick just one candidate.

However, this method is criticized for its potential problems regarding emotional factors including: 1) voting for a candidate as encouragement rather than as approval, or 2) voting for everyone but one strongly opposed candidate to give humiliation.

Nonetheless, this method is implemented in various organization and professional societies, including American Mathematical Society, the Mathematical Association of America, and the Society for Social Choice & Welfare (SC&W).

It is not clear to see who would actually win in the Election X because the meaning of “approval” differs for each individual. A voter could approve just one candidate who is the first choice, while the other could approve three or four candidates.

2.4 Cumulative Voting

In cumulative voting, a voter is given a specific number of points to distribute among voters.

While a voter could spread out the points among many candidates, a voter could also dedicate all the points for one particular candidate, which is called *plumping.* Plumping could be seen both positively and negatively. A minority group can give all their points to elect one favored candidate, which offers the minority group a chance for political inclusion; however, this could also mean that the elected candidate need not be supported broadly among the community members.

Thus, the result of the Election X is not clearly reflected in using cumulative voting procedure because the voter profile doesn’t tell the level of support each voter gives different candidates.

2.5 Borda Count (BC)

The Borda Count was presented in the 1770s by the French mathematician Jean Charles de Borda, who strongly opposed Plurality voting. Borda Count assigns different ‘points’ for each rank: for n candidate election, this method assigns $n – j$ points to a voter’s j th-ranked candidate.

The Borda count is used in many educational institutions in the United States. For example, the University of Michigan College of Literature, Science, and the Arts uses the Borda Count to elect the Student Government and to elect the Michigan Student Assembly for the university at large, and in the Civil Liberties Union of Harvard University to elect its officers, and at Arizona State University to elect officers to the Department of Mathematics and Statistics assembly.

In the Election X with four candidates, the first-choice candidate of each voter gets 3 points, while second, third, and last get 2, 1, and 0, respectively. Using the vector form, the voting vector, which shows how many points are assigned to each candidate, can be represented as (3, 2, 1, 0). Thus, candidate D becomes the winner in the Borda Count election with 30 points.

2.6 Positional Methods

Related to Borda Count, Positional Method also assigns different points for each rank. While the BC specifically gives n-j points to j-th ranked candidates, the Positional Method include different ways to assign ($w1, w2, w3, \ldots, wn$) points while $w1 \geq w2 \geq w3 \geq \ldots \geq wn$.

The Positional Method is popularly used to grant awards for sports in the United States. For example, it is used to determine the Most Valuable Player in Major League Baseball. At the end of the baseball season, the Baseball Writers Association of America uses the voting vector of (14, 9, 8, 7, 6, 5, 4, 3, 2, 1) to choose a winner for the most valuable player (MVP) awards for each league. In this case, a writer only gives positive scores to his or her top-ten candidates using the voting vector.

In case of Election X, the result will be differentiated depends on what voting vector is used. For example, if we use the voting vector $(5, 2, 1, 0)$ then D will be the winner with 42 points, and A, B, and C have 40, 28, and 34 points, respectively. On the other hand, if the positional method uses voting vector of $(7, 2, 0, 0)$, then A is elected with 56 points, beating B,C, and D who receive 28, 28, and 50 points. Therefore, even within the positional method, many different outcomes can be resulted from a single voter profile.

2.7 Pairwise Comparison

This procedure compares and ranks a candidate relative to other candidate. A candidate who wins when matched head-to-head with another candidate gets one point and half point for a tie. The candidate who receives most points is the winner.

When a candidate beats all others, then the candidate is called *Condorcet winner;* and a candidate hat loses to other candidates is called *Condorcet loser.*

In Election X, the result shows:

- A and B: B wins with 8<10: B gets 1 point
- A and C: C wins with 8 <10: C gets 1 point
- A and D: D wins with 8<10: D gets 1 point
- B and C: B wins with 10>8 : B gets 1 point
- B and D: D wins with 8<10 : D gets 1 point
- C and D: D wins with 8<10 :D gets 1 point

At the end of pairwise comparison, D wins with 3 points, while A, B, and C are only receiving 0, 2, and 1 point, respectively. Additionally, candidate D beats all A, B, and C in pairwise comparison and becomes Condorcet winner.

2.8 Recalling Election X

Based on all the information we have, we can describe the result in a more systematic way using the voting vector.

Table 1: Election with 4 candidates and 18 voters

Number	Preference	Number	Preference
4	**A >B>C>D**	**4**	**C>B>D>A**
4	**A>D>C>B**	**6**	**D>B>C>A**

In this profile,

- A is a winner using Plurality procedure: (1,0,0,0)
    - Plurality only counts how many voters pick a candidate as the first choice.
- B wins when voting for two candidates: (1,1,0,0)
    - First two top-ranked candidates both receive one point each in voting for 2-candidate, as it does not differentiate the first and second preference.
- C wins with Anti-Plurality (when three candidates are voted): (1,1,1,0) or (0,0,0,-1)
    - First three top-ranked candidates get one point each – which is the same logic as the least favored candidate gets -1 points.
- D wins with the Borda Count Procedure (3,2,1,0). And each candidate's Borda Count score is the sum of their points for Plurality, voting for 2, and Anti-plurality.

Therefore, even this simple example demonstrates how complicated and chaotic an election can be, depending on different voting procedures; the election outcome indeed reflects the choice of voting procedure rather than the voter's preference. This discovery leads to the theorem 1 and 2 whose proofs would be well beyond the scope of this paper.

Theorem 1 *(Saari). For $N \geq 3$ candidates $\{c_1, c_2, \ldots, c_N\}$, there exist profiles – choice of how the voters rank the candidates – so that c_j wins when the voters vote for j candidates, j = 1, …, N-1, and then c_N is the Borda Count winner.*

Theorem 2 *(Saari). For $N \geq 4$ candidates, there exist a profile where, with appropriate choice of positional election procedures, each candidate is listed in first, second,…, and in last place.*

3. Representation triangle and profiles

3.1 Election Y

So far we have looked at the different voting procedures. And in this section, we will see how the voters' preference could be represented geometrically.

While counting and tallying the ballot could be extremely tedious, the geometric representation can be used to simplify that process. Saari presents a way to use an equilateral simplex for the geometric construction; for an election of three candidates, an equilateral triangle is used.

Each vertex of a triangle is assigned to a different candidate, called A, B, and C. A point in the triangle can indicate a voter's preference by its distance to different vertices. The closer a point is to a particular vertex, the more preferred a candidate - corresponding to that vertex – is by the voter.

To demonstrate the geometric representation, an example is constructed and is called Election Y.

The voter profile is:

Number	Preference	Number	Preference
2	A>B>C	4	B>C>A
6	A>C>B	0	C>A>B
3	B>A>C	5	C>B>A

Thus, in Figure 3.1, six different regions on the triangle indicate six different ranking, where no tie is allowed.

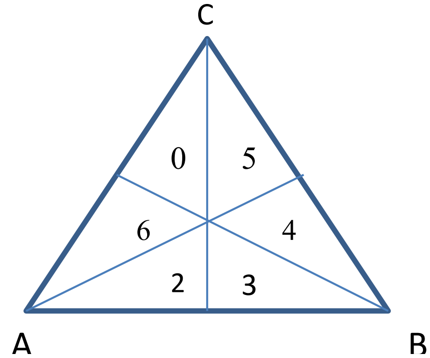


Fig. 3.1

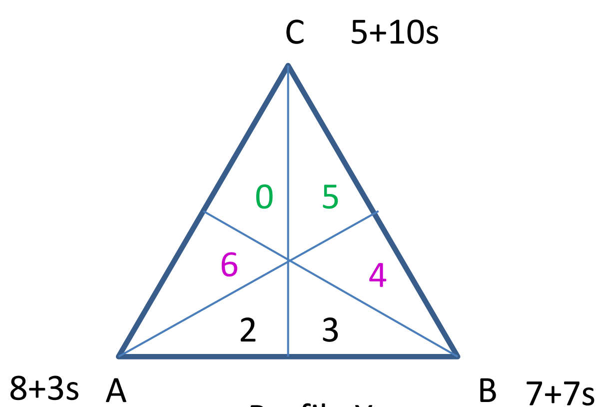
'5' in the upper right corner shows that five voters prefer the ranking C > B > A because the region is closest to C, then to B, then last to A.

3.2 Computing election tallies

In Fig 3.1, the plurality vote can be tallied by counting the votes in each candidate's corner of the triangle. By adding two numbers that are located the closest to each vertex, plurality vote in this profile indicates that A wins with 8 votes, while B gets 7 and C receives 5 votes.

Pairwise comparison is clear in this profile's geometric representation. To compare candidate A and candidate B, one has to add all the votes on the left side of the vertical line in the triangle for A and all on the right to count B. Thus, B wins in pairwise comparison with 12 votes over A with 8 votes. Comparison of B and C also indicates that C wins with 11 votes over B with 9 votes. Also, pairwise comparison between A and C results winning of A with 11 votes, while C has 9 votes.

Election tallies for positional methods can also be computed through geometric representation. Positional methods assign points of $\left(w_1, w_2, w_{3,} \ldots 0\right)$ to each rank and the same election ranking is maintained with assigning$(\boldsymbol{\lambda} w_1, \boldsymbol{\lambda} w_2, \ldots 0)$, when $\lambda > 0$ is a fixed scalar. The vector $(w_1, w_2, \ldots 0)$ becomes $\left(\frac{w_1}{w_1}, \frac{w_2}{w_1}, \ldots 0\right)$ (using $\boldsymbol{\lambda} = 1/w_1$. To apply to our example of Election Y, we can assign points of (*w1, w2, 0*) to the respective rank, and the vector $\left(\frac{w_1}{w_1}, \frac{w_2}{w_1}, 0\right)$ presents the same election rank. This vector form, which indeed becomes $w\_s = (1, s, 0)$ is called *normalized voting vector*, where $s = \frac{w_2}{w_1}$. Thus, in the Election Y, candidate A receives *(2+6)*1 + (3+0)* s* points – which is *8+3s.*



Profile Y

In this normalized voting vector w_s = (1,s,0),

- s = 0 in the case of plurality vote.
- s = 1 in the case of choosing 2 candidates or anti-plurality vote.
- s= ½ using Borda Count.

This allows us to geometrically represent the "winning values of s" for candidate A and B.

When s=0, A wins with plurality and B wins when s=1. Thus, by solving the inequality $\{8 + 3s > 7 + 7s\}$, we know what value of s guarantees A's victory: s > $\frac{1}{4}$.

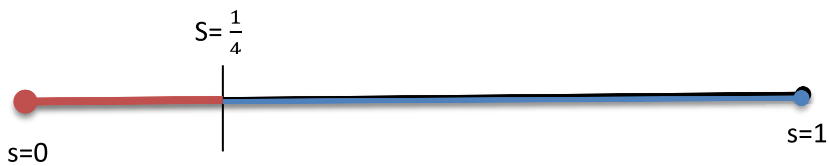


Fig. 3.3: S value

By carefully choosing the value of s, one can strategically make the election outcome to be according to one's own desire.

Using similar ideas, we can discover what happens with all possible positional methods for all election with three candidates.

3.3 Procedure lines

Using the geometric way, one can also visualize all possible arrangement of election rankings. Let q_j : fraction of the total tally won by each candidate. Then $\{q = (q_A, q_B, q_C) \mid q_j \geq 0,\ q_A + q_B + q_C = 1\}$. In the case of Figure. 3.1 profile, q_S = $\frac{1}{20(1+s)}$ (8+3s, 7+7s, 5+10s); dividing each voter's point by the total point.

With algebraic manipulation, Saari presents the formula for procedure line:

$$\boldsymbol{q_S} = (1\text{-}2t)\ \boldsymbol{q_0} + 2t\boldsymbol{q_1},\quad t = \frac{\boldsymbol{s}}{\boldsymbol{1+s}}$$

which shows the line of all possible election outcomes.

Thus, the procedure line connects the end points, q_0 and q_1, where:

- q_0 indicates the election outcome using plurality voting and thus voting vector (1,0,0)
- q_1 indicates the outcome using anti-plurality procedure with voting vector (1,1,0)

And any point on the stretched band connecting the two points is an election outcome for some positional method.

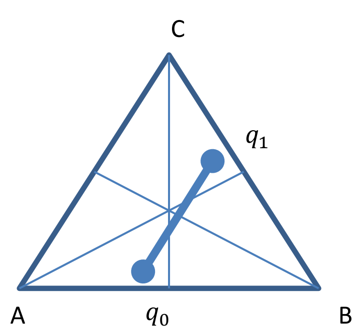


Fig. 3.3

While six regions within the triangle indicate the election ranking without any tie, when the procedure line crosses a line, that crossing point represents a situation where there is a tie between candidates. Additionally, the midpoint, which is equidistant to all A, B, and C, represents a point where all three candidates are in a tie. Thus, in Fig. 3.3, the procedure line indicates that there are seven different ranking outcomes, as it lies on four different regions and crosses three lines.

4. Geometric representation for more candidates

4.1 Four candidates

An election of four candidates can no longer be represented in a triangle; it will require an equilateral tetrahedron that has four vertices.
Now, four vertices are assigned four candidates: A, B, C, and D.
In Fig. 3.1, the flattened tetrahedron shows there are 24 ranking regions (without tie) and these 24 regions still indicate the rankings depending on the distance to each vertex.

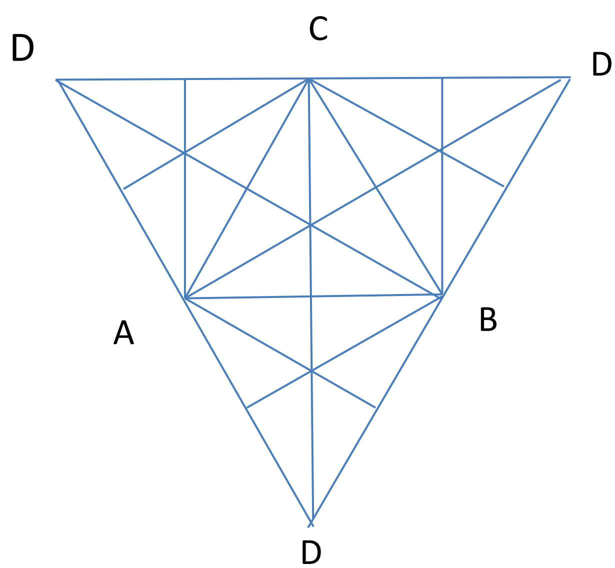
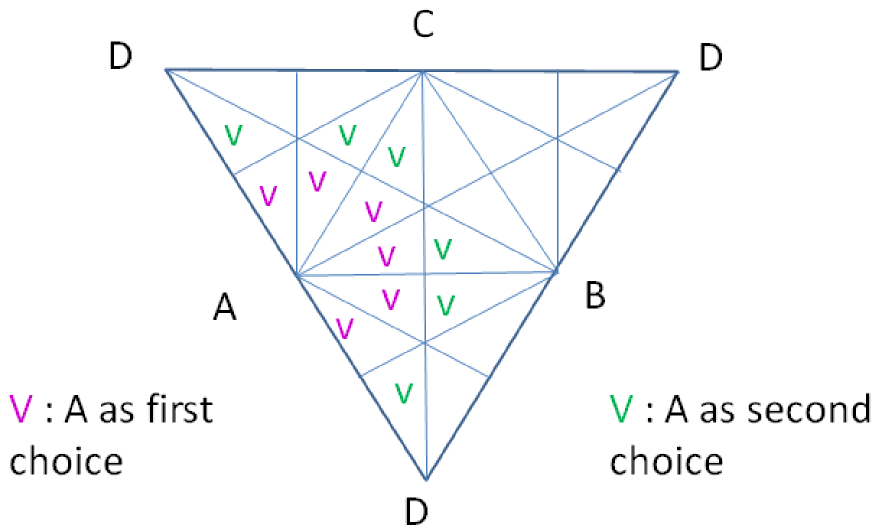
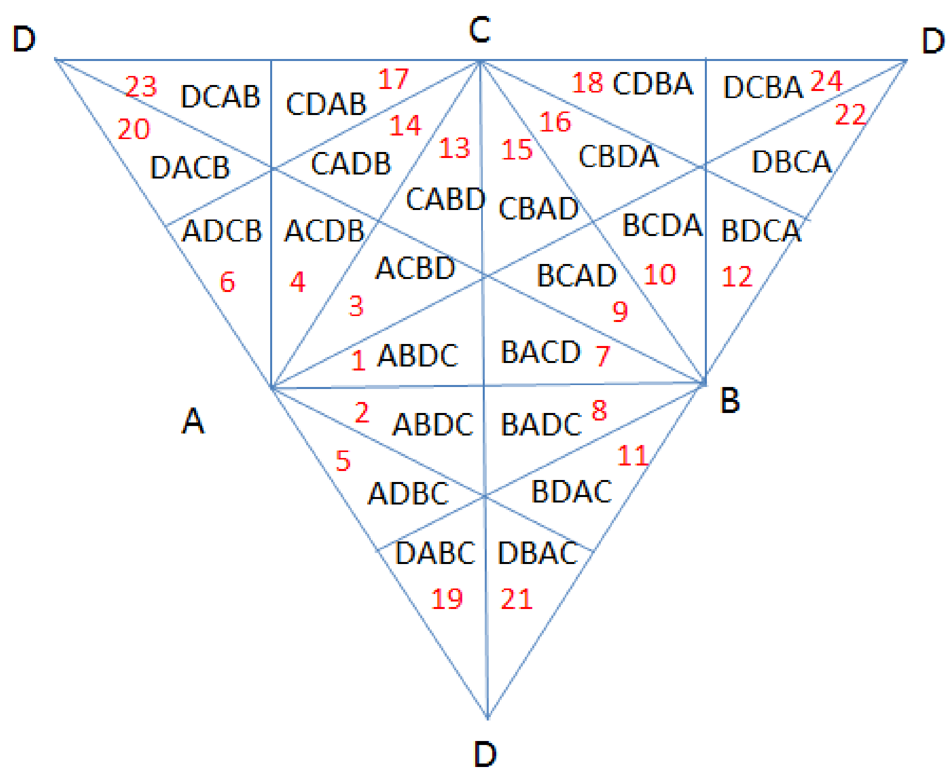


Fig. 4.1 Flattened Tetrahedron



With the same idea used in the triangle, “the closer the better” idea works in a tetrahedron, too. Here, all the pink checks indicate the regions that rank A as the first choice and in the same manner, the green checks indicate the regions that rank A as the second.

Thus, I present another tetrahedron – with all the rankings indicated as well as numbering of the triangle. The numbers are in alphabetical orders of the rankings.



4.2 Procedure triangle

Now that the four candidate situation involves a tetrahedron, the procedure line becomes a procedure triangle. Three vertices of that procedure triangle will be

- An outcome using plurality (1, 0, 0,0): q_0
- An outcome when voting for two candidates (1, 1, 0, 0): q_1
- An outcome with anti-plurality (1, 1, 1, 0): q_2

Note that the normalized vector for four candidate case is thus (1, s_1, s_2, 0)
where 1 $\geq s_1 \geq s_2 \geq$0.

Connecting these three points within the tetrahedron will be a triangle. Any point on the triangle or in the interior of the triangle is an election outcome for appropriate choices of s1 and s2.

For N ≥ 5, the appropriate object is an equilateral simplex which is in an (N-1)-dimensional space.

4.3 Election X and Procedure triangle

I bring the voter profile of election X back to demonstrate the procedure triangle. First, I located four types of voter preferences in the flattened tetrahedron. For example, the red 4 represents the four voters who ranked C > D> B> A.

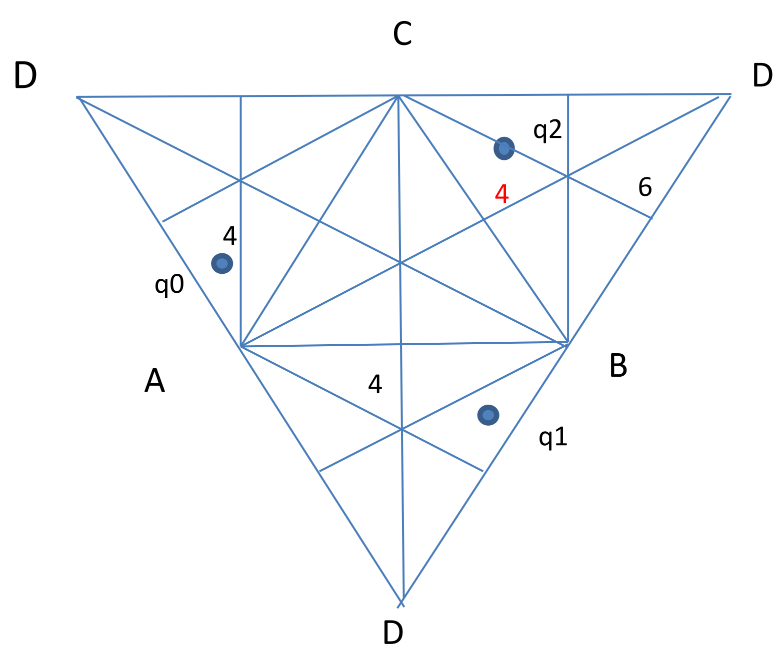


Fig. 4.3 Election X – vertices for procedure triangle

Using Fig. 4.1, we can also construct a procedure triangle in the 3-D tetrahedron.

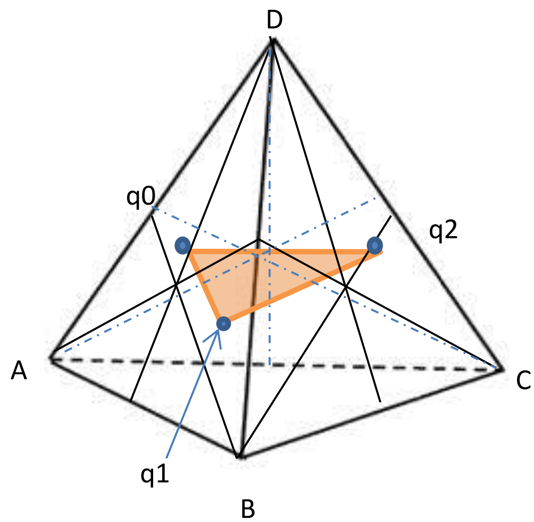


Fig. 4.4 procedure Triangle for Election X

As Fig 4.4 illustrates, we see that there are many possible elections outcomes for Election X's voter profile.

Analogous to Fig. 3.3 where there was one parameter, s, for three candidates, the four-candidate election contains two parameters, s1 and s2, and we can illustrate outcomes with a two-dimensional graph. Based on voter profile of Election X, I have created a graph whose x and y-axis indicate values of s1 and s2, respectively. Finding values of s1 and s2 where any two candidates become tie, I have plotted six functions in total - functions indicating A=B, A=C, A=D, B=C, B=D, and C=D.

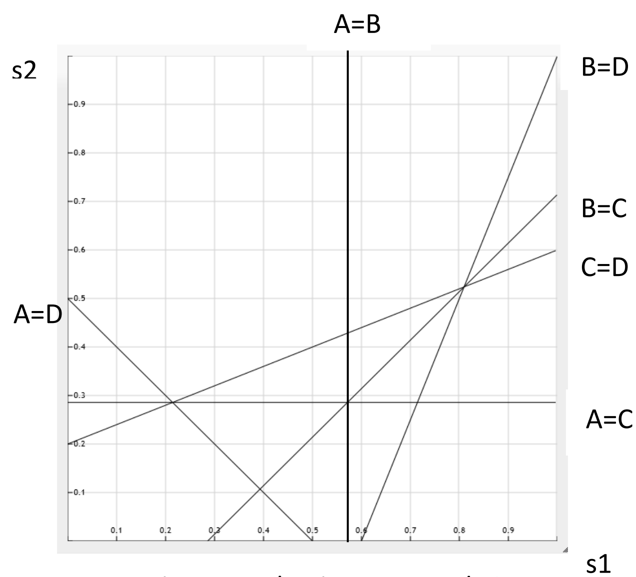


Fig. 4.4.1 Election X; s1 and s2

However, the value of s2 is always less than or equal to that of s1, we have to disregard the upper left region beyond s1=s2 line.

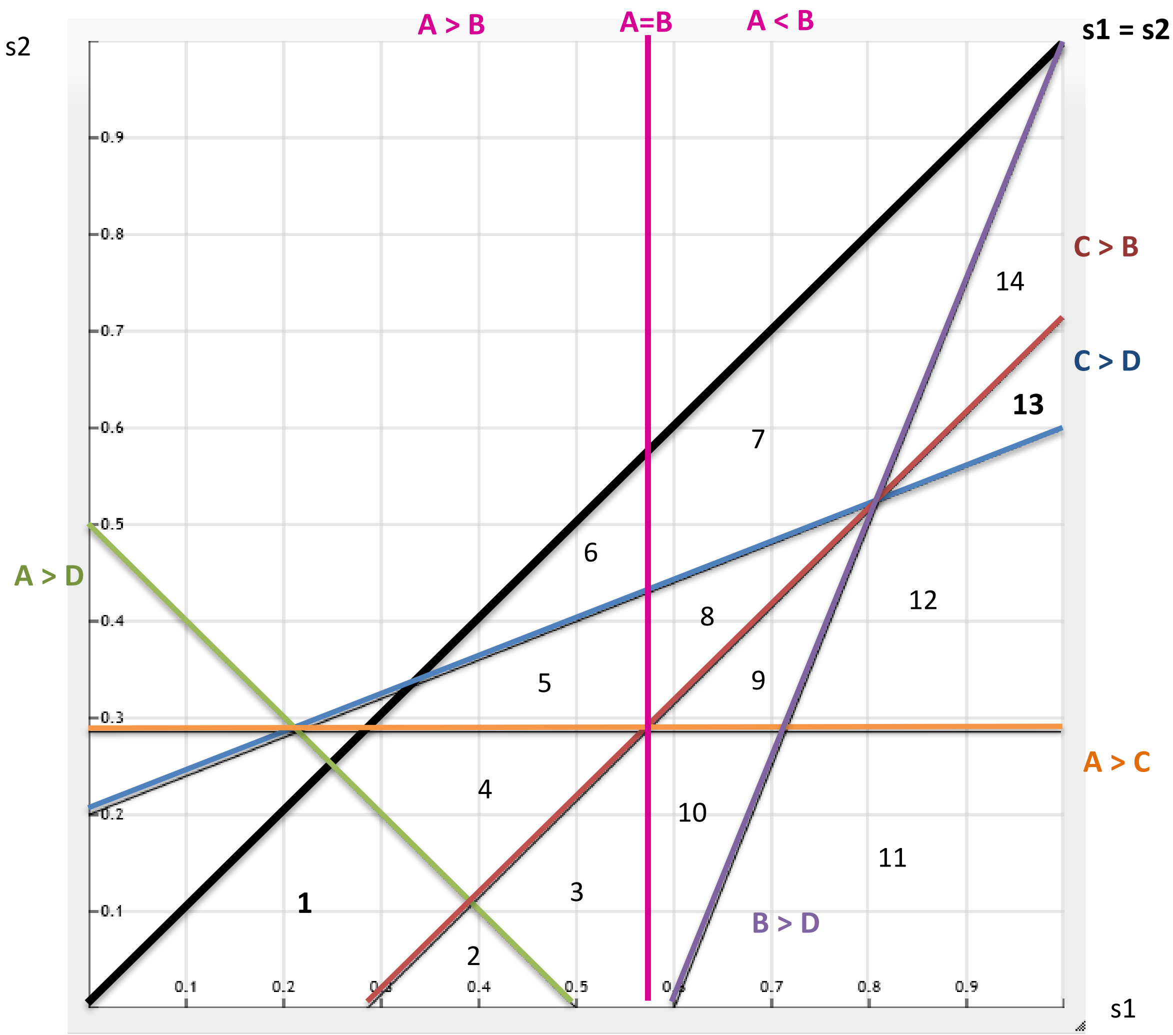


Fig. 4.4.2 Election X; s1 and s2

In this graph there are 14 regions (in the lower right part), which indicate the total number of untied possible outcomes for election X. Remember a tie line indeed divides the regions where each candidate wins. For example, left region of the tie line of A=B is when A beats B and the right region is when B beats A. Thus, we can find what particular ranking of candidates is represented in a region. We also notice that region 1 indicates ranking “A > D > C > B” while region 13 shows the reversal “B > C > D > A.”

5. More Chaotic Consequences

5.1 Elimination/ Withdrawal of a candidate

The selection of a site for the 2000 Olympic Games showed how an elimination of one candidate could affect the election outcome tremendously.

The selection was held in 1993, and the voting procedure was a plurality runoff where the bottom-ranked city is dropped at each stage, and at every round there is a complete revote. (Saari) The competition was amongst Beijing, Sydney, Manchester, Berlin, and Istanbul. Beijing won the first round with 32 votes, closely followed by Sydney with 30 votes, while Istanbul who received the lowest vote, 7, was eliminated. The second round scored Beijing with 37 votes and Sydney with 30 votes, while eliminating Berlin with 9 votes. In the third next round, with three remaining candidates, Beijing was still supported with 40 votes, while Sydney and Manchester received 37 and 11 votes, respectively. As the first three rounds had shown, most people were expecting that Beijing would be the host of 2000 Olympic Games. However, it came as a shock to most people when Beijing lost with 43 votes, to Sydney with 45 votes in the final round. Indeed, we now remember 2000 Sydney Olympics, even though Beijing was chosen for the 2008 Olympics.

Here, even this brief example shows how election rankings can change by an elimination of a candidate. And in similar manner, withdrawal of a candidate could also reverse the election outcome.

5.1.1 Example

In the election of three candidate – A,B, and C - the Committee of 17 people is using plurality procedure to select one committee chair. In the original profile, Alice would win with 8 votes. However, let's say Caren, who realized that she is not going to win, withdraws from the election. Then all of a sudden, Betty wins Alice with 9:8.

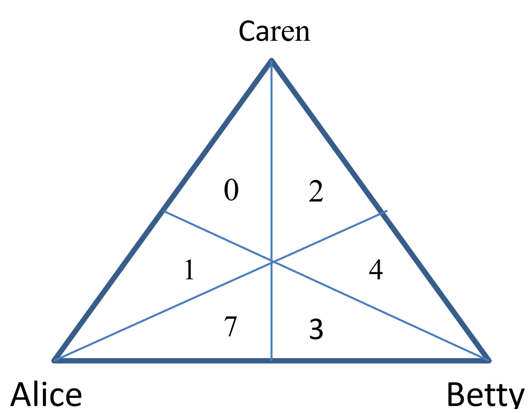


Fig 4.1

Thus, Saari introduced another theorem that summarizes the consequences of withdrawal/ elimination and the possibility of manipulation.

Theorem 3 *(Saari) Let N represent the number of candidates, and suppose there are at least three candidates.*

- *Rank the candidates in any desired transitive manner. Next, select the positional election method to tally ballots.*
- *There are N ways to drop one of the candidates. For each of these ways, rank the remaining (N-1) candidates in any desired transitive manner. These rankings need not have anything to do with one another or with the original ranking. Next, for each subset of candidates, select the positional election method to tally ballots.*
- *Continue this process, namely, for each possible subset of three or more candidates, assign a transitive ranking and a positional voting procedure. The ranking need not be related, in any manner, to any of the other ranking.*
- *For each pair, assign a ranking; the voting procedure is the majority vote.*

For almost all choices of positional voting method, there exists a profile so that when the voters vote on any of the above sets of candidates where ballots are tallied in the indicated manner, the election outcome is the specified one.
A voting method which does not allow this highly chaotic state of affairs is the Borda Count.

The mathematics of proof of this theorem would be beyond the scope of this paper.

5.2 Strategic Voting

In 2000 Presidential election, George W. Bush, who was eventually elected as the president, received 47.9% of popular votes, while Al Gore got 48.4% and the most prominent third-party leader, Ralph Nader from the Green Party got 2.74%. While this result reflects some controversy about Electoral College system, it also invites the idea of strategic voting. Right after the defeat of Gore, a common claim was made that Nader's candidacy acted as a spoiler that split the potential votes for Gore – because Green party holds more common ground with the Democratic Party, than the Republican Party. Even though Gore still lost to Bush, it is impossible to say that voters whose first choice was Nader would have voted for Gore. Some people would have thought about voting for Gore, in case their support for Nader could strengthen Bush's position. In this case like the 2000 election, it is possible that people vote strategically when knowing their first choice candidate is not likely to win and they want to avoid the victory of their least favorite candidate. Thus, voters might vote strategically, rather than sincerely. Further, different voting procedures offer different strategic opportunities.

5.2.1 Being strategic in Approval voting

Approval Voting Procedure is currently used in the Mathematics Association of American and the American Mathematical Society. However, as it has been said before, any voting procedure holds both positive and negative effects, and Approval Voting also offers strategic voting opportunities.

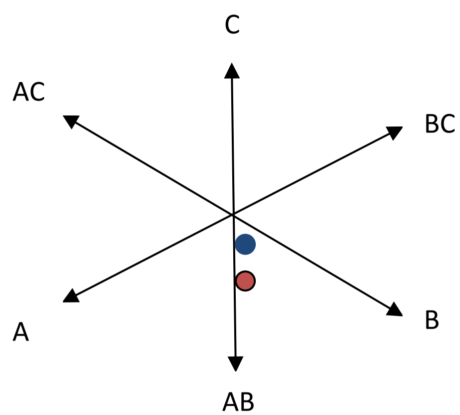


Fig 5.2

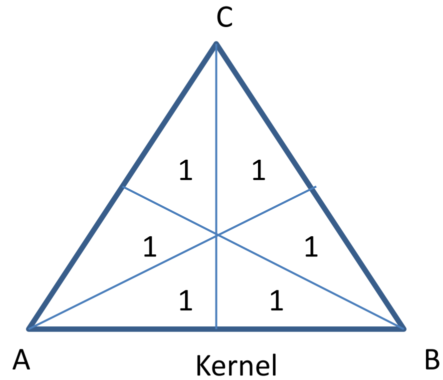
If it is indicated that the current voter preference falls as the red point on the diagram, where the red point is located in the 'B-preferred' region and very close to the tie line of AB. If you prefer C>A>B, and you only vote for C, then your vote might just change the voter preference to the blue point – which still elects B as the winner. Then, you might want to strategic and vote for both A and C, which will pull the election outcome towards the 'AB' line. Thus, even though you are not sincerely voting but still avoid electing your least-preferred candidate B.

6. Profile Component for three candidates

We want to look more closely to understand why a profile could lead to different outcomes by examining profile components whose effect is easy to understand.

6.1 Kernel Component

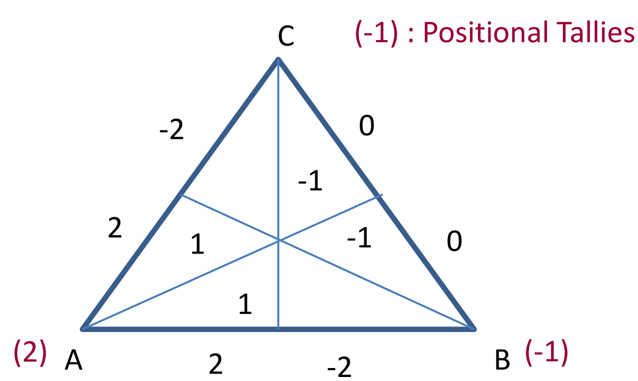
As the name says, the kernel component does not affect the election outcomes, neither the pairwise comparison nor positional method. For three candidates, the kernel is where one vote is added to every single ranking; every triangle gets one more vote. To express this in more systematic way, we use the vector. In this case, k = (1,1,1,1,1,1)



Kernel

6.2 Basic Component

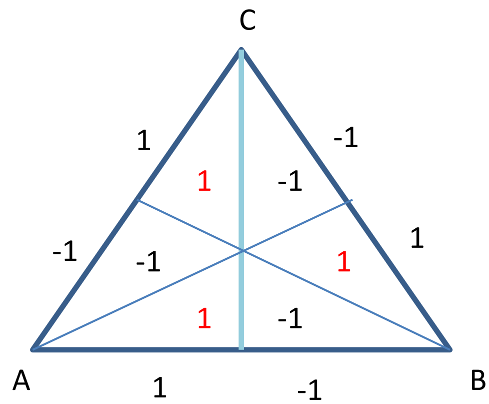
Basic component affect both positional and pairwise comparison. For example, A-basic favors A in both of the method. Here, the regions where A is the first rank get one vote, while all the regions where A is the bottom rank lose one vote. Thus, the pairwise comparison indicates that A wins over both B and C. In counting positional tallies: A =2 and B, C = -1. Since the tallies do not include any s value, the positional outcomes are consistent: A wins. In the same manner, there are B-basic and C-basic but C-basic can be written in terms of A-basic and B-basic: $B_C = -(B_A + B_B)$. Therefore, there are two independent components.



A-Basic

6.3 Condorcet Component

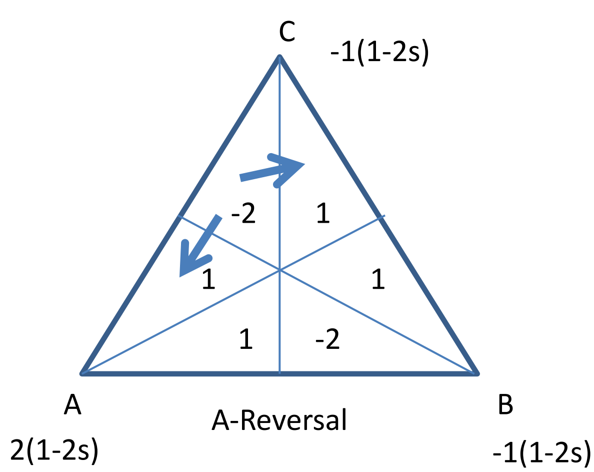
Condorcet component only affect the pairwise comparison outcome but not positional method. The total number of votes does not change here. In this example of ABC Condorcet, the ranking regions of A>B>C, B>C>A, and C>A>B gets one vote each where the other three ranking loses one. Thus, when you compare them in pairwise, the cycle is created where A beats B, B beats C, and C beats A. Thus, when a profile is constructed with many Condorcet components, it is more likely to have a cyclic pairwise comparison outcome. However, in this component, each candidate's positional tallies do not change. Every candidate gains and loses one vote where they are the first, second, and last rank. There is just one independent component and the vector is C= (1,-1,1,-1,1,-1). And here, negative voter might sound strange; however, we use the negative values for calculation purposes and eventually adding kernel components eliminate negative numbers.



ABC Condorcet

6.4 Reversal Component

This component only affects positional method (except for Borda Count) and not pairwise comparison. This component could be explained as: two voters whose preferred rank was C > A > B (where A is in the middle) changed their mind but one moved to ranking A as the first choice while the other moved to ranking A at the bottom. And the same thing happen to the voters whose preferred rank was B > A > C. I call this particular example A-reversal. In the A-reversal, therefore, A gains two first choice votes while losing 4 votes in the second choice, and gaining 2 votes in the bottom rank. Thus, A's positional tally becomes 2-4s while B and C gets -1+2s. Thus, all the positional method with the value of s < $\frac{1}{2}$, A is favored and otherwise, B and C are favored. The vector is R_A = (1,1,-2,1,1,-2). Similarly, we can have B-reversal and C-reversal. However, C reversal could be written in terms of A-reversal and B-reversal: R_C = -(R_A + R_B). Thus, there are only two independent components for three candidate case.



A-Reversal

6.5 Constructing three candidate election profile

Now that we have the four profile component: Kernel, Condorcet, Reversal and Basic, we can construct an election profile by assigning any set of coefficients.

$$P = a_B B_A + b_B B_B + a_R R_A + b_R R_B + \gamma C + kK$$

They can be also written in matrix form.

$$\vec{p} = A\,\vec{v}$$

$$A = \begin{pmatrix} B_A & B_B & R_A & R_B & C & K \end{pmatrix}$$

Where $\vec{p}$ is the profilem, A is the 6 x 6 bases matrix, and $\vec{v}$ is the vector of coefficients.

Thus, in the same manner, we can also decompose a profile using the components (specifically, using the inverse of A matrix)

$$A^{-1} = \frac{1}{6}\begin{pmatrix} 2 & 1 & -1 & -2 & -1 & 1 \\ 1 & -1 & -2 & -1 & 1 & 2 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ -1 & 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Using the matrix, I decomposed the profile Y and got:

$$p = \mathbf{7}R_A + 3R_B - 1B_A + 1\,B_B + \mathbf{-8}C + 20K$$

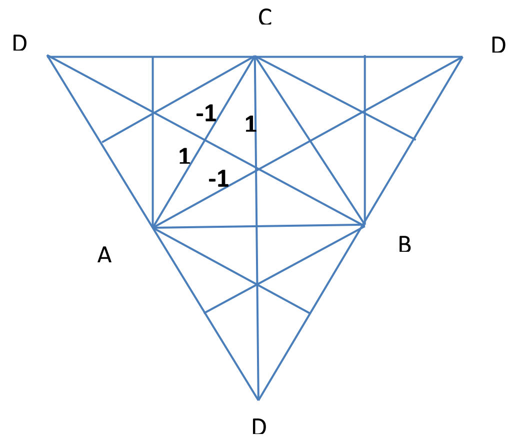
For example, the coefficient of A-reversal, 7, explains how A is the plurality winner and the coefficient of Condorcet, -8, explains the cyclic outcome for the pairwise comparison (A>C, C>B, B>A).

7. Profile Components for Four candidates

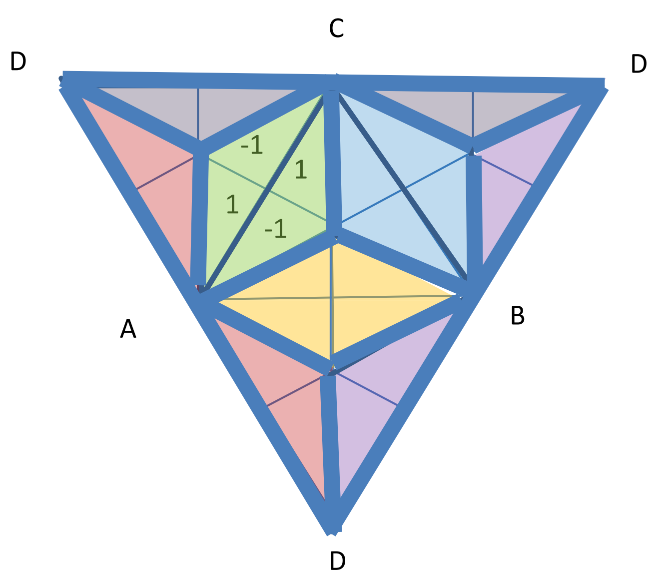
For four candidate case, we use tetrahedron and thus becomes 24 dimensional. We will see seven Kernel, three Basic, three Condorcet, and eleven Reversal components.

7.1 Kernel Components

For four candidate case, there isn't just one kernel component but seven of them. One of the example is shown below:



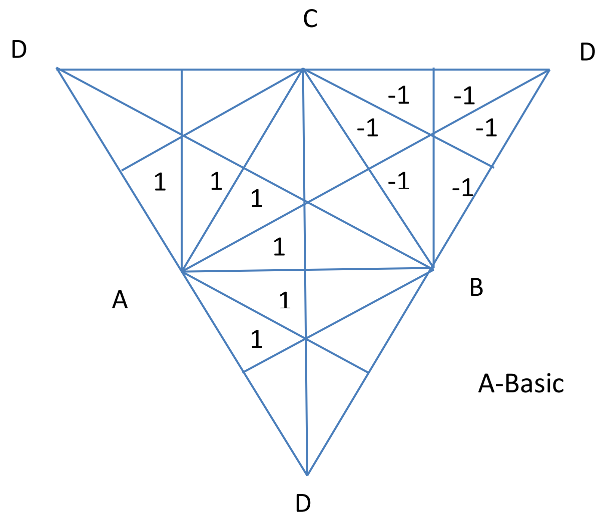
In this case we have A > C > D > B, -(A > C > B > D), C > A > B > D, and -(C > A > D > B). For example, A's first choice vote gets cancelled out as well as its second choice vote. Thus, A's positional tally does not change – and as for every other candidate. It also does not affect the pairwise comparison, either.



Each of 6 colored region generate one independent kernel vector. Thus, adding one kernel component that adds one to each of 24 triangles, we have seven kernel components.

7.2 Basic Components

As we have done in the three candidate case, we add one vote to every region whose first choice is A and take away one vote from every region where A is the bottom rank.



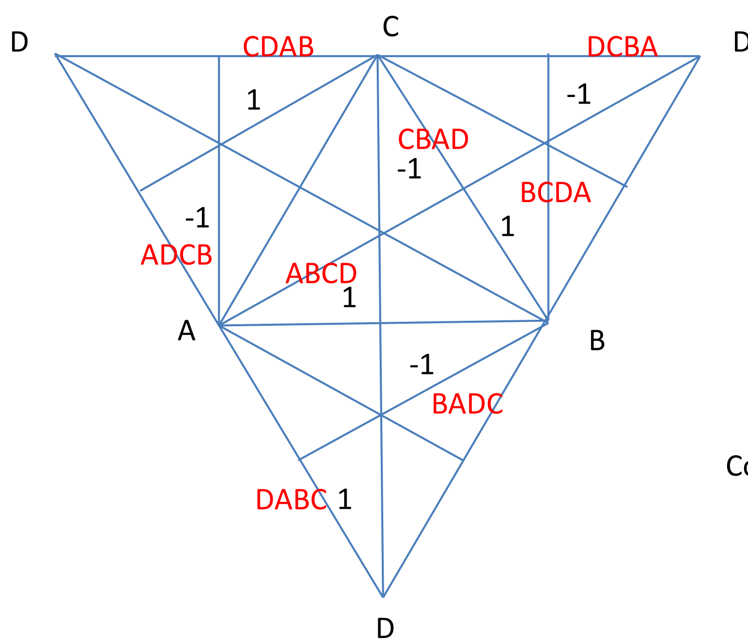
A-Basic

This components make the positional tallies: A: 6 while all B,C, and D have -2. Therefore, A is favored. In Pairwise, A beats all B, C, and D. We can still see B-basic, C-basic, and D-basic but D-basic can be written in terms of the other three: $B_D = -(B_A + B_B + B_C)$. Therefore, there are three basic components.

7.3 Condorcet Components

There are three different components: A>B>C>D, A>B>D>C and A>C>B>D. This example shows the Condorcet component for A>B>C>D.

This ranking can generate 8 different ranking: ABCD, BCDA, CDAB, DABC, -(DCBA),-(ADCB),-(BADC),-(CBAD). As one component can generate eight ranks, it makes sense that it is three dimensional: 8 x 3 = 24. And this will favor A over B with pairwise comparison; there is no effect with A vs. C; A is hurt when vs. D.



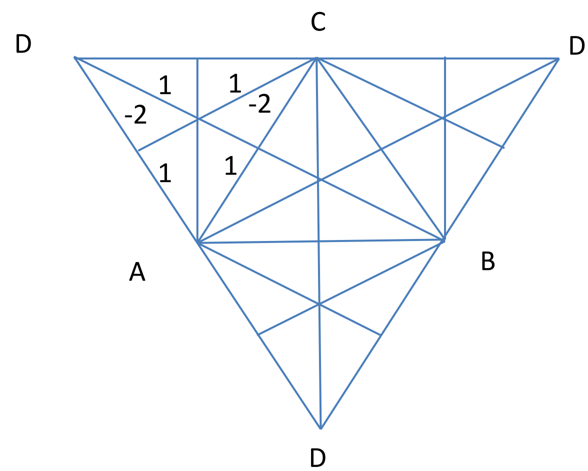
Condorcet
ABCD

7.4 Reversal Components

Four candidates reversals are a little different from the three candidate case. While the voters who voted A as the middle choice still change their minds, it does not involve the votes that rank A as the last. Thus, two voters who rank A as second change their mind: one put A as the first choice and the other put A as the bottom rank.
The positional tallies become:

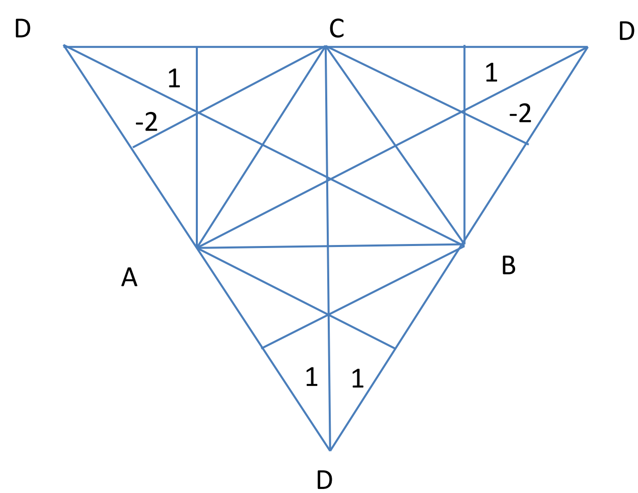
A: 2 – $4s_1$ + $2s_2$, B: 0, while C and D each has -1 +$2s_1$ $- s_2$.

Here, note that there are three different tallies, not two.



In this component, for example, A is favored using plurality method, while C and D are favored when using 'vote for 2 candidate' method. As we have already seen in the three candidate case, each medium sized triangle (i.e. Triangle ABC) have two independent components. Thus, we have eight components (2x4=8) in similar manner.

One of the other three reversal components is shown below. It is using all the votes that rank one particular candidate (in this example, D) as first choice. Here, two voters who prefer ranking of D>A>C>B change their mind: one now prefer D>C>A>B and the other prefer D>A>B>C. Also, two voterS whose original preference was D>B>C>A now vote for D>C>B>A and C>B>A>C.



In this example, the positional tallies are: A and B: $-s_1 + 2s_2$, C: $2s_1 - 4s_2$, and D: 0.

Thus, as the reversal component is supposed to be, this component does not affect pairwise comparison outcome but affects the positional outcome. In this particular one, Plurality helps the candidate C and Antiplurality helps A and B. In same manner, we can have the component using the regions that rank A, B, or C as the first choice. However, only three of those components are linearly independent. Thus, we have shown that there are eleven reversal components (8+3=11).

7.5 Decomposition Matrix

Recall $\vec{p} = A\,\vec{v}$. While $\vec{p}$ is still a profile and $\vec{v}$ is the vector of coefficients, the matrix A is now 24 by 24, putting all the components together.

```
BCols // MatrixForm
```

$$
\left(\begin{array}{rrrrrrrrrrrrrrrrrrrrrrrr}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & -2 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & -2 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\
1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
$$

For decomposition, we need A^{-1} which looks like:

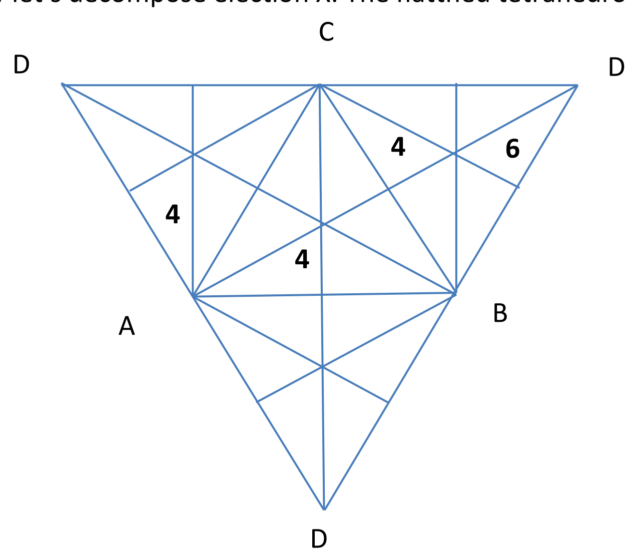
```
24*MatrixPower[BCols, -1] // MatrixForm
```

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & -12 & -9 & 0 & -15 & 0 & 0 & 12 & 9 & 0 & 15 & 0 & 0 & 9 & 0 & -9 & 12 & -12 & -12 & 3 & 12 \\
-9 & 0 & 0 & -12 & 0 & -15 & 0 & 9 & 0 & -9 & 12 & -12 & 0 & 12 & 9 & 0 & 15 & 0 & 3 & -12 & 12 \\
6 & -3 & -6 & 3 & -6 & 6 & 15 & 6 & 6 & 6 & -6 & 9 & -15 & -6 & -6 & -6 & 6 & -9 & -18 & 18 & -9 \\
6 & 6 & 15 & 6 & 9 & -6 & -6 & -6 & -15 & -6 & -9 & 6 & 6 & -3 & -6 & 3 & -6 & 6 & 6 & -9 & -6 \\
0 & 9 & 0 & -9 & 12 & -12 & -9 & 0 & 0 & -12 & 0 & -15 & 9 & 0 & 0 & 12 & 0 & 15 & 12 & -12 & 3 \\
15 & 6 & 6 & 6 & -6 & 9 & 6 & -3 & -6 & 3 & -6 & 6 & -6 & -6 & -15 & -6 & -9 & 6 & -9 & 6 & -18 \\
3 & 0 & 0 & 0 & 0 & -3 & 0 & -3 & 0 & 3 & 0 & 0 & 0 & 0 & -3 & 0 & 3 & 0 & 3 & 0 & 0 \\
0 & 0 & 3 & 0 & -3 & 0 & 0 & 0 & -3 & 0 & 3 & 0 & 0 & -3 & 0 & 3 & 0 & 0 & 0 & 3 & 0 \\
0 & 3 & 0 & -3 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 3 & 3 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & -3 \\
3 & 2 & 3 & 2 & 1 & 1 & 2 & 1 & 1 & -1 & -1 & -2 & 2 & 1 & 1 & -1 & -1 & -2 & -1 & -1 & -2 \\
2 & 1 & 1 & -1 & -1 & -2 & 3 & 2 & 3 & 2 & 1 & 1 & 1 & -1 & 2 & 1 & -2 & -1 & -2 & -3 & -1 \\
1 & -1 & 2 & 1 & -2 & -1 & 1 & -1 & 2 & 1 & -2 & -1 & 3 & 2 & 3 & 2 & 1 & 1 & -3 & -2 & -3 \\
9 & 5 & 5 & 5 & 1 & 5 & 1 & -3 & -7 & 1 & -7 & 9 & -3 & -3 & -11 & -3 & -7 & 9 & -3 & 1 & -11 \\
5 & 5 & 9 & 5 & 5 & 1 & -3 & -3 & -11 & -3 & -7 & 9 & 1 & -3 & -7 & 1 & -7 & 9 & 1 & -3 & -7 \\
1 & -3 & -7 & 1 & -7 & 9 & 9 & 5 & 5 & 5 & 1 & 5 & -11 & -3 & -3 & -3 & 9 & -7 & -11 & 5 & -3 \\
-3 & -3 & -11 & -3 & -7 & 9 & 5 & 5 & 9 & 5 & 5 & 1 & -7 & 1 & 1 & -3 & 9 & -7 & -7 & 9 & 1 \\
-7 & 1 & 1 & -3 & 9 & -7 & -11 & -3 & -3 & -3 & 9 & -7 & 9 & 5 & 5 & 5 & 1 & 5 & 5 & -11 & 9 \\
-11 & -3 & -3 & -3 & 9 & -7 & -7 & 1 & 1 & -3 & 9 & -7 & 5 & 5 & 9 & 5 & 5 & 1 & 9 & -7 & 5 \\
0 & 0 & 4 & 4 & 0 & 4 & -4 & -4 & 4 & 4 & -4 & 4 & -4 & -4 & 0 & 0 & -4 & 0 & 0 & 4 & -4 \\
-4 & -4 & 4 & 4 & -4 & 4 & 0 & 0 & 4 & 4 & 0 & 4 & 0 & 0 & -4 & -4 & 0 & -4 & -4 & 4 & 0 \\
9 & 1 & 9 & 1 & 5 & 5 & 1 & -7 & -7 & 1 & -11 & 5 & 1 & -7 & -7 & 1 & -11 & 5 & 1 & 1 & -7 \\
1 & -7 & -7 & 1 & -11 & 5 & 9 & 1 & 9 & 1 & 5 & 5 & -7 & 1 & 1 & -7 & 5 & -11 & -7 & 9 & 1 \\
-7 & 1 & 1 & -7 & 5 & -11 & -7 & 1 & 1 & -7 & 5 & -11 & 9 & 1 & 9 & 1 & 5 & 5 & 9 & -7 & 9
\end{pmatrix}
$$

Here, we multiplied by 24 just to make it a whole number.

7.6 Decomposing Election X

With the matrix, now let's decompose election X. The flattned tetrahedron representation of this election looks like



And thus, p = (4,0,0,0,0,4,0,0,0,0,0,0,0,0,0,4,0,0,0,0,0,6,0,0)

When we decompose this profile using mathematica, we got the coefficients:

$$V = \{9,\ -27,\ -84,\ 30,\ 33,\ -36,\ 90,\ 0,\ -3,\ 0,\ -3,\ -1,\ -2,\ 37,\ 41,\ 17,\ -3,\ -39,\ -59,\ 20,\ 4,\ 57,\ 1,\ -55\}$$

And this represents, {Kernel, Condorcet, Basic, Reversal}

i) Condorcet component
All three coefficients are small and that explains why Election X does not have cyclic outcome but a strict pairwise outcome: D > B > C > A.

ii) Basic Component
Even though basic components carry more weights (point difference), the coefficients are relatively smaller than those of Reversal components.

iii) Reversal Component
Huge coefficients can explain the various positional outcomes.

Conclusion

This project explored how one election profile could lead to many different outcomes, when using various voting methods; elections can be quite chaotic. Thus, an election outcome is decided by which voting procedure we use and not about what voters *really* want. By studying the profile components for three candidate election and extending this idea to create the basis for the four candidate election, we showed that we can construct and decompose an election profile. Eventually, the decomposition helps us understand the characteristics of the various outcomes.

Reference

Saari, Donald G. *Chaotic Elections! A mathematician Looks at Voting*. United States: American Mathematical Society, 2001. Print.