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# A new monotonic, clone-independent, reversal symmetric, and condorcet-consistent single-winner election method

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**Abstract** In recent years, the Pirate Party of Sweden, the Wikimedia Foundation, the Debian project, the "Software in the Public Interest" project, the Gentoo project, and many other private organizations adopted a new single-winner election method for internal elections and referendums. In this article, we will introduce this method, demonstrate that it satisfies, e.g., resolvability, Condorcet, Pareto, reversal symmetry, monotonicity, and independence of clones and present an O($C$ˆ3) algorithm to calculate the winner, where $C$ is the number of alternatives.

## 1 Introduction

One important property of a good single-winner election method is that it minimizes the number of "overruled" voters (according to some heuristic). Because of this reason, the Simpson–Kramer method, that always chooses that alternative whose worst pairwise defeat is the weakest, was very popular over a long time. However, in recent years, the Simpson–Kramer method has been criticized by many social choice theorists. Smith (1973) criticizes that this method does not choose from the top-set of alternatives. Tideman (1987) complains that this method is vulnerable to the strategic nomination of a large number of similar alternatives, so-called *clones*. And Saari (1994) rejects this method for violating *reversal symmetry*. A violation of reversal symmetry can lead to strange situations where still the same alternative is chosen when all ballots are reversed, meaning that the same alternative is identified as best one and simultaneously as worst one.

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**Table 1** Simulations by Wright (2009)

| Number of alternatives | A (%) | B (%) | C (%) |
|---|---|---|---|
| 3 | 100.0 | 100.0 | 100.0 |
| 4 | 99.7 | 98.5 | 98.2 |
| 5 | 99.2 | 96.0 | 95.3 |
| 6 | 99.1 | 93.0 | 92.3 |
| 7 | 98.9 | 90.0 | 89.1 |

A: Probability that the Schulze method conforms with the Simpson–Kramer method
B: Probability that the Schulze method conforms with the ranked pairs method
C: Probability that the ranked pairs method conforms with the Simpson–Kramer method

In this article, we will show that only a slight modification (Sect. 4.8) of the Simpson–Kramer method is needed so that the resulting method satisfies the criteria proposed by Smith (Sect. 4.7), Tideman (Sect. 4.6), and Saari (Sect. 4.4). The resulting method will be called *Schulze method*. Random simulations by Wright (2009) confirmed that, in almost 99% of all instances, the Schulze method conforms with the Simpson–Kramer method (Table 1). In this article, we will prove that, nevertheless, the Schulze method still satisfies all important criteria that are also satisfied by the Simpson–Kramer method, like resolvability (Sect. 4.2), Pareto (Sect. 4.3), monotonicity (Sect. 4.5), and prudence (Sect. 4.9). Because of these reasons, already several private organizations have adopted the Schulze method. The Schulze method is currently used by the Wikimedia Foundation (about 26,000 eligible members), the Pirate Party of Sweden (about 50,000 eligible members), and the Pirate Party of Germany (about 12,000 eligible members). It is also used by the Debian project, the "Software in the Public Interest" (SPI) project, and the Gentoo project, three software projects with about 1,000 resp. 400 resp. 300 eligible members.

In Sect. 2 of this article, the Schulze method is defined. In Sect. 3, this method is applied to a concrete example. In Sect. 4, this method is analyzed. Short descriptions of this method can also be found in publications by Tideman (2006, pp. 228–232), Stahl and Johnson (2006, pp. 119–129), Camps et al. (2008), McCaffrey (2008), and Börgers (2009, pp. 37–42). This method is also discussed in articles by Yue et al. (2007), Wright (2009), and Rivest and Shen (2010).

## 2 Definition of the Schulze method

### 2.1 Preliminaries

A *strict partial order* is a transitive and asymmetric relation "$x \succ y$". A *strict weak order* is a strict partial order with the additional property that also the relation "not $x \succ y$" is transitive. A *profile* is a finite list $V$ of $0 < N < \infty$ strict weak orders each on the same finite set $A$ of $1 < C < \infty$ alternatives. "$a \succ_v b$" means "voter $v \in V$ strictly prefers alternative $a \in A$ to alternative $b \in A \backslash \{a\}$". Input of the proposed method is a profile. Output of the proposed method are (1) a strict partial order $\mathcal{O}$ on $A$ and (2) a set $\varnothing \neq \mathcal{S} \subseteq A$ of winners.

Suppose $N[e, f]$ is the number of voters who strictly prefer alternative $e$ to alternative $f$. We presume that the strength of the link $ef$ depends only on $N[e, f]$ and $N[f, e]$. Therefore, the strength of the link $ef$ can be denoted $(N[e, f], N[f, e])$. We presume that a binary relation $\succ_D$ on $\mathbb{N}_0 \times \mathbb{N}_0$ is defined such that the link $ef$ is stronger than the link $gh$ if and only if $(N[e, f], N[f, e]) \succ_D (N[g, h], N[h, g])$. $N[e, f]$ is the *support* for the link $ef$; $N[f, e]$ is its *opposition*.

*Example 1* (*margin*): When the strength of the link $ef$ is measured by *margin*, then its strength is the difference $N[e, f] - N[f, e]$ between its support $N[e, f]$ and its opposition $N[f, e]$.

$(N[e, f], N[f, e]) \succ_{\text{margin}} (N[g, h], N[h, g])$ if and only if $N[e, f] - N[f, e] > N[g, h] - N[h, g]$.

*Example 2* (*ratio*): When the strength of the link $ef$ is measured by *ratio*, then its strength is the ratio $N[e, f]/N[f, e]$ between its support $N[e, f]$ and its opposition $N[f, e]$.

$(N[e, f], N[f, e]) \succ_{\text{ratio}} (N[g, h], N[h, g])$ if and only if at least one of the following conditions is satisfied:

1. $N[e, f] > N[f, e]$ and $N[g, h] \leq N[h, g]$.
2. $N[e, f] \geq N[f, e]$ and $N[g, h] < N[h, g]$.
3. $N[e, f] \cdot N[h, g] > N[f, e] \cdot N[g, h]$.
4. $N[e, f] > N[g, h]$ and $N[f, e] \leq N[h, g]$.
5. $N[e, f] \geq N[g, h]$ and $N[f, e] < N[h, g]$.

*Example 3* (*winning votes*): When the strength of the link $ef$ is measured by *winning votes*, then its strength is measured primarily by its support $N[e, f]$.

$(N[e, f], N[f, e]) \succ_{\text{win}} (N[g, h], N[h, g])$ if and only if at least one of the following conditions is satisfied:

1. $N[e, f] > N[f, e]$ and $N[g, h] \leq N[h, g]$.
2. $N[e, f] \geq N[f, e]$ and $N[g, h] < N[h, g]$.
3. $N[e, f] > N[f, e]$ and $N[g, h] > N[h, g]$ and $N[e, f] > N[g, h]$.
4. $N[e, f] > N[f, e]$ and $N[g, h] > N[h, g]$ and $N[e, f] = N[g, h]$ and $N[f, e] < N[h, g]$.
5. $N[e, f] < N[f, e]$ and $N[g, h] < N[h, g]$ and $N[e, f] > N[g, h]$.
6. $N[e, f] < N[f, e]$ and $N[g, h] < N[h, g]$ and $N[e, f] = N[g, h]$ and $N[f, e] < N[h, g]$.

*Example 4* (*losing votes*): When the strength of the link $ef$ is measured by *losing votes*, then its strength is measured primarily by its opposition $N[f, e]$.

$(N[e, f], N[f, e]) \succ_{\text{los}} (N[g, h], N[h, g])$ if and only if at least one of the following conditions is satisfied:

1. $N[e, f] > N[f, e]$ and $N[g, h] \leq N[h, g]$.
2. $N[e, f] \geq N[f, e]$ and $N[g, h] < N[h, g]$.
3. $N[e, f] > N[f, e]$ and $N[g, h] > N[h, g]$ and $N[f, e] < N[h, g]$.

4. $N[e, f] > N[f, e]$ and $N[g, h] > N[h, g]$ and $N[f, e] = N[h, g]$ and $N[e, f] > N[g, h]$.
5. $N[e, f] < N[f, e]$ and $N[g, h] < N[h, g]$ and $N[f, e] < N[h, g]$.
6. $N[e, f] < N[f, e]$ and $N[g, h] < N[h, g]$ and $N[f, e] = N[h, g]$ and $N[e, f] > N[g, h]$.

The most intuitive definitions for the strength of a link are its *margin* and its *ratio*. However, we only presume that $\succ_D$ is a strict weak order on $\mathbb{N}_0 \times \mathbb{N}_0$ with at least the following properties:

$$\begin{aligned}
&\forall\,(x_1, x_2)\,,\,(y_1, y_2) \in \mathbb{N}_0 \times \mathbb{N}_0 : \\
&((x_1 > y_1 \text{ and } x_2 \leq y_2) \text{ or } (x_1 \geq y_1 \text{ and } x_2 < y_2)) \\
&\quad \Rightarrow (x_1, x_2) \succ_D (y_1, y_2)\,. \qquad (2.1.1) \\
&\forall\,(x_1, x_2)\,,\,(y_1, y_2) \in \mathbb{N}_0 \times \mathbb{N}_0 : \\
&((x_1 > x_2 \text{ and } y_1 \leq y_2) \text{ or } (x_1 \geq x_2 \text{ and } y_1 < y_2)) \\
&\quad \Rightarrow (x_1, x_2) \succ_D (y_1, y_2)\,. \qquad (2.1.2) \\
&\forall\,(x_1, x_2)\,,\,(y_1, y_2) \in \mathbb{N}_0 \times \mathbb{N}_0 \forall c_1, c_2 \in \mathbb{N} : \\
&(c_1 \cdot x_1, c_1 \cdot x_2) \succ_D (c_1 \cdot y_1, c_1 \cdot y_2) \\
&\quad \Rightarrow (c_2 \cdot x_1, c_2 \cdot x_2) \succ_D (c_2 \cdot y_1, c_2 \cdot y_2)\,. \qquad (2.1.3)
\end{aligned}$$

The presumption, that the strength of the link *ef* depends only on $N[e, f]$ and $N[f, e]$, guarantees (1) that the proposed method satisfies anonymity and neutrality, (2) that adding a ballot, on which all alternatives are ranked equally, cannot change the result of the elections, and (3) that the proposed method is a C2 *Condorcet social choice function* (CSCF) according to Fishburn's (1977) terminology.

(2.1.1) says that, when the support of a link increases and its opposition does not increase or when its opposition decreases and its support does not decrease, then the strength of this link increases. So (2.1.1) says that the strength of a link responses to a change of its support or its opposition in the correct manner. (2.1.1) guarantees that the proposed method satisfies resolvability (Sect. 4.2), Pareto (Sect. 4.3), and monotonicity (Sect. 4.5). When each voter $v \in V$ casts a linear order $\succ_v$ on $A$, then all definitions for $\succ_D$, that satisfy (2.1.1), are identical.

(2.1.2) says that every pairwise victory is stronger than every pairwise tie and that every pairwise tie is stronger than every pairwise defeat. (2.1.2) guarantees that the proposed method satisfies the Smith criterion (Sect. 4.7).

*Homogeneity* means that the result depends only on the proportion of ballots of each type, not on their absolute numbers. (2.1.2) guarantees that the proposed method satisfies homogeneity.

Suppose $\varnothing \neq \mathcal{M} \subset \mathbb{N}_0 \times \mathbb{N}_0$ is finite and non-empty. Then "$\max_D \mathcal{M}$", the *set of maximum elements* of $\mathcal{M}$, and "$\min_D \mathcal{M}$", the *set of minimum elements* of $\mathcal{M}$, are defined as follows: $(\beta_1, \beta_2) \in \max_D \mathcal{M}$ if and only if (1) $(\beta_1,\beta_2) \in \mathcal{M}$ and (2) $(\beta_1,\beta_2) \succsim_D (\delta_1,\delta_2) \forall (\delta_1, \delta_2) \in \mathcal{M}$. $(\gamma_1,\gamma_2) \in \min_D \mathcal{M}$ if and only if (1) $(\gamma_1,\gamma_2) \in \mathcal{M}$ and (2) $(\gamma_1,\gamma_2) \precsim_D (\delta_1,\delta_2) \forall (\delta_1,\delta_2) \in \mathcal{M}$.

We write "$(\beta_1,\beta_2) := \max_D \mathcal{M}$" and "$(\gamma_1,\gamma_2) := \min_D \mathcal{M}$" for "$(\beta_1,\beta_2)$ is an arbitrarily chosen element of $\max_D \mathcal{M}$" and "$(\gamma_1,\gamma_2)$ is an arbitrarily chosen element of $\min_D \mathcal{M}$".

## 2.2 Basic definitions

In this section, the Schulze method is defined. A concrete example can be found in Sect. 3.

Basic idea of the Schulze method is that the *strength* of the indirect comparison "alternative $a$ vs. alternative $b$" is the *strength* of the *strongest path* $a \equiv c(1), \ldots, c(n) \equiv b$ from alternative $a \in A$ to alternative $b \in A \backslash \{a\}$ and that the *strength* of a path is the $strength(N[c(i), c(i+1)], N[c(i+1), c(i)])$ of its *weakest link* $c(i), c(i+1)$.

The Schulze method is defined as follows:

> A *path* from alternative $x \in A$ to alternative $y \in A$ is a sequence of alternatives $c(1), \ldots, c(n) \in A$ with the following properties:
>
> 1. $x \equiv c(1)$.
> 2. $y \equiv c(n)$.
> 3. $2 \leq n < \infty$.
> 4. For all $i = 1, \ldots, (n-1) : c(i) \not\equiv c(i+1)$.
>
> The *strength* of the path $c(1), \ldots, c(n)$ is
>
> $$\min_D\{(N[c(i), c(i+1)], N[c(i+1), c(i)])|i = 1, \ldots, (n-1)\}.$$
>
> In other words: The strength of a path is the strength of its weakest link.
>
> $P_D[a, b] := \max_D\{\min_D\{(N[c(i), c(i+1)], N[c(i+1), c(i)])|i = 1, \ldots, (n-1)\} \,|c(1), \ldots, c(n)$ is a path from alternative $a$ to alternative $b$ $\}$.
>
> In other words: $P_D[a, b] \in \mathbb{N}_0 \times \mathbb{N}_0$ is the strength of the strongest path from alternative $a \in A$ to alternative $b \in A \backslash \{a\}$.
>
> The binary relation $\mathcal{O}$ on $A$ is defined as follows:
>
> $$ab \in \mathcal{O} :\Leftrightarrow P_D[a, b] \succ_D P_D[b, a]. \tag{2.2.1}$$
>
> $$\mathcal{S} := \{a \in A | \forall b \in A \backslash \{a\} : ba \notin \mathcal{O}\} \text{ is the } \textit{set of winners}. \tag{2.2.2}$$

As the link $ab$ is already a path from alternative $a$ to alternative $b$ of strength $(N[a, b], N[b, a])$, we get

$$\forall a, b \in A : P_D[a, b] \succsim_D (N[a, b], N[b, a]). \tag{2.2.3}$$

With (2.2.1) and (2.2.3), we get

$$(N[a, b], N[b, a]) \succ_D P_D[b, a] \Rightarrow ab \in \mathcal{O}. \tag{2.2.4}$$

Furthermore, we get

$$\forall a, b, c \in A : \ \min_D \{P_D [a, b], P_D [b, c]\} \precsim_D P_D [a, c] . \qquad (2.2.5)$$

Otherwise, if $\min_D\{P_D[a, b], P_D[b, c]\}$ was strictly larger than $P_D[a, c]$, then this would be a contradiction to the definition of $P_D[a, c]$ since there would be a path from alternative $a$ to alternative $c$ via alternative $b$ with a strength of more than $P_D[a, c]$.

The asymmetry of $\mathcal{O}$ follows directly from (2.2.1) and the asymmetry of $\succ_D$. Furthermore, in Sect. 4.1, we will see that the binary relation $\mathcal{O}$ is transitive. This guarantees that there is always at least one winner.

## 2.3 Implementation

The strength $P_D[i, j]$ of the strongest path from alternative $i \in A$ to alternative $j \in A\backslash\{i\}$ can be calculated with the Floyd (1962) algorithm. The runtime to calculate the strengths of all strongest paths is O($C$ˆ3), where $C$ is the number of alternatives in $A$.

Input: $N[i, j] \in \mathbb{N}_0$ is the number of voters who strictly prefer alternative $i \in A$ to alternative $j \in A\backslash\{i\}$.

Output: $P_D[i, j] \in \mathbb{N}_0 \times \mathbb{N}_0$ is the strength of the strongest path from alternative $i \in A$ to alternative $j \in A\backslash\{i\}$.
$pred[i, j] \in A\backslash\{j\}$ is the predecessor of alternative $j$ in the strongest path from alternative $i \in A$ to alternative $j \in A\backslash\{i\}$.
$\mathcal{O}$ is the binary relation as defined in (2.2.1).
"*winner*[*i*] = *true*" if and only if $i \in \mathcal{S}$.

Stage 1 (initialization):

```
for i := 1 to C
begin
  for j := 1 to C
  begin
    if (i ≠ j) then
    begin
      P_D[i, j] := (N[i, j], N[j, i])
      pred[i, j] := i
    end
  end
end
```

Stage 2 (calculation of the strengths of the strongest paths):

```
for i := 1 to C
begin
  for j := 1 to C
  begin
    if (i ≠ j) then
    begin
      for k := 1 to C
      begin
        if (i ≠ k) then
        begin
          if (j ≠ k) then
          begin
            if (P_D[j, k] ≺_D min_D{P_D[j, i], P_D[i, k]}) then
            begin
              P_D[j, k] := min_D{P_D[j, i], P_D[i, k]}
              pred[j, k] := pred[i, k]
            end
          end
        end
      end
    end
  end
end
```

Stage 3 (calculation of the binary relation $\mathcal{O}$ and the winners):

```
for i := 1 to C
begin
  winner[i] := true
  for j := 1 to C
  begin
    if (i ≠ j) then
    begin
      if (P_D[j, i] ≻_D P_D[i, j]) then
      begin
        ji ∉ O
        winner[i] := false
      end
      if (P_D[j, i] ≾_D P_D[i,j]) then
      begin
        ji ∉ O
      end
    end
  end
end
```

## 3 Example

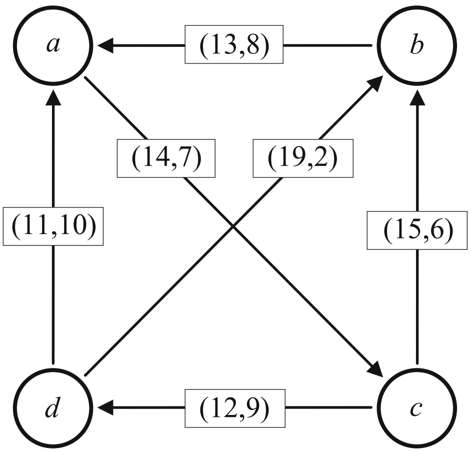
*Example 1*

8 voters $a \succ_v c \succ_v d \succ_v b$
2 voters $b \succ_v a \succ_v d \succ_v c$
4 voters $c \succ_v d \succ_v b \succ_v a$
4 voters $d \succ_v b \succ_v a \succ_v c$
3 voters $d \succ_v c \succ_v b \succ_v a$

$N[i, j] \in \mathbb{N}_0$ is the number of voters who strictly prefer alternative $i \in A$ to alternative $j \in A \setminus \{i\}$. In example 1, the pairwise matrix $N$ looks as follows:

| | $N[*,a]$ | $N[*,b]$ | $N[*,c]$ | $N[*,d]$ |
|---|---|---|---|---|
| $N[a,*]$ | – | 8 | 14 | 10 |
| $N[b,*]$ | 13 | – | 6 | 2 |
| $N[c,*]$ | 7 | 15 | – | 12 |
| $N[d,*]$ | 11 | 19 | 9 | – |

The following digraph illustrates the graph theoretic interpretation of pairwise elections. If $N[i, j] > N[j, i]$, then there is a link from vertex $i$ to vertex $j$ of strength $(N[i, j], N[j, i])$:




The above digraph can be used to determine the strengths of the strongest paths. In the following, "$x$, $(Z_1, Z_2)$, $y$" means "$(N[x, y], N[y, x]) = (Z_1, Z_2)$".

$a \to b$: There are 2 paths from alternative $a$ to alternative $b$.

Path 1: $a$, (14,7), $c$, (15,6), $b$
with a strength of $\min_D$ { (14,7), (15,6) } $\approx_D$ (14,7).

Path 2: $a$, (14,7), $c$, (12,9), $d$, (19,2), $b$
with a strength of $\min_D$ { (14,7), (12,9), (19,2) } $\approx_D$ (12,9).

So the strength of the strongest path from alternative $a$ to alternative $b$ is $\max_D$ { (14,7), (12,9) } $\approx_D$ (14,7).

$a \to c$ : There is only one path from alternative $a$ to alternative $c$.
Path 1: $a$, (14,7), $c$ with a strength of (14,7).

$a \to d$ : There is only one path from alternative $a$ to alternative $d$.
Path 1: $a$, (14,7), $c$, (12,9), $d$
with a strength of $\min_D$ { (14,7), (12,9) } $\approx_D$ (12,9).

$b \to a$ : There is only one path from alternative $b$ to alternative $a$.
Path 1: $b$, (13,8), $a$ with a strength of (13,8).

$b \to c$ : There is only one path from alternative $b$ to alternative $c$.
Path 1: $b$, (13,8), $a$, (14,7), $c$
with a strength of $\min_D$ { (13,8), (14,7) } $\approx_D$ (13,8).

$b \to d$ : There is only one path from alternative $b$ to alternative $d$.
Path 1: $b$, (13,8), $a$, (14,7), $c$, (12,9), $d$
with a strength of $\min_D$ { (13,8), (14,7), (12,9) } $\approx_D$ (12,9).

$c \to a$ : There are 3 paths from alternative $c$ to alternative $a$.
Path 1: $c$, (15,6), $b$, (13,8), $a$
with a strength of $\min_D$ { (15,6), (13,8) } $\approx_D$ (13,8).
Path 2: $c$, (12,9), $d$, (11,10), $a$
with a strength of $\min_D$ { (12,9), (11,10) } $\approx_D$ (11,10).
Path 3: $c$, (12,9), $d$, (19,2), $b$, (13,8), $a$
with a strength of $\min_D$ { (12,9), (19,2), (13,8) } $\approx_D$ (12,9).
So the strength of the strongest path from alternative $c$ to alternative $a$ is $\max_D$ { (13,8), (11,10), (12,9) } $\approx_D$ (13,8).

$c \to b$ : There are 2 paths from alternative $c$ to alternative $b$.
Path 1: $c$, (15,6), $b$ with a strength of (15,6).
Path 2: $c$, (12,9), $d$, (19,2), $b$
with a strength of $\min_D$ { (12,9), (19,2) } $\approx_D$ (12,9).
So the strength of the strongest path from alternative $c$ to alternative $b$ is $\max_D$ { (15,6), (12,9) } $\approx_D$ (15,6).

$c \to d$ : There is only one path from alternative $c$ to alternative $d$.
Path 1: $c$, (12,9), $d$ with a strength of (12,9).

$d \to a$ : There are 2 paths from alternative $d$ to alternative $a$.
Path 1: $d$, (11,10), $a$ with a strength of (11,10).
Path 2: $d$, (19,2), $b$, (13,8), $a$
with a strength of $\min_D$ { (19,2), (13,8) } $\approx_D$ (13,8).
So the strength of the strongest path from alternative $d$ to alternative $a$ is $\max_D${ (11,10), (13,8) } $\approx_D$ (13,8).

$d \to b$ : There are 2 paths from alternative $d$ to alternative $b$.
Path 1: $d$, (11,10), $a$, (14,7), $c$, (15,6), $b$
with a strength of $\min_D$ { (11,10), (14,7), (15,6) } $\approx_D$ (11,10).
Path 2: $d$, (19,2), $b$ with a strength of (19,2).

So the strength of the strongest path from alternative $d$ to alternative $b$ is $\max_D$ { (11,10), (19,2) } $\approx_D$ (19,2).

$d \to c$ : There are 2 paths from alternative $d$ to alternative $c$.
Path 1: $d$, (11,10), $a$, (14,7), $c$
with a strength of $\min_D$ { (11,10), (14,7) } $\approx_D$ (11,10).
Path 2: $d$, (19,2), $b$, (13,8), $a$, (14,7), $c$
with a strength of $\min_D$ { (19,2), (13,8), (14,7) } $\approx_D$ (13,8).
So the strength of the strongest path from alternative $d$ to alternative $c$ is $\max_D$ { (11,10), (13,8) } $\approx_D$ (13,8).

The following table lists the strongest paths. The critical links of the strongest paths are underlined:

| | ... to $a$ | ... to $b$ | ... to $c$ | ... to $d$ |
|---|---|---|---|---|
| from $a$ ... | – | $a$, $\underline{(14,7)}$, $c$,<br>(15,6), $b$ | $a$, $\underline{(14,7)}$, $c$ | $a$, (14,7), $c$,<br>$\underline{(12,9)}$, $d$ |
| from $b$ ... | $b$, $\underline{(13,8)}$, $a$ | – | $b$, $\underline{(13,8)}$, $a$,<br>(14,7), $c$ | $b$, (13,8), $a$,<br>(14,7), $c$,<br>$\underline{(12,9)}$, $d$ |
| from $c$ ... | $c$, (15,6), $b$,<br>$\underline{(13,8)}$, $a$ | $c$, $\underline{(15,6)}$, $b$ | – | $c$, $\underline{(12,9)}$, $d$ |
| from $d$ ... | $d$, (19,2), $b$,<br>$\underline{(13,8)}$, $a$ | $d$, $\underline{(19,2)}$, $b$ | $d$, (19,2), $b$,<br>$\underline{(13,8)}$, $a$,<br>$\underline{(14,7)}$, $c$ | – |

The strengths of the strongest paths are:

| | $P_D[*,a]$ | $P_D[*,b]$ | $P_D[*,c]$ | $P_D[*,d]$ |
|---|---|---|---|---|
| $P_D[a,*]$ | – | (14,7) | (14,7) | (12,9) |
| $P_D[b,*]$ | (13,8) | – | (13,8) | (12,9) |
| $P_D[c,*]$ | (13,8) | (15,6) | – | (12,9) |
| $P_D[d,*]$ | (13,8) | (19,2) | (13,8) | – |

$xy \in \mathcal{O}$ if and only if $P_D[x,y] \succ_D P_D[y,x]$. So in example 1, we get $\mathcal{O} = \{ab, ac, cb, da, db, dc\}$.

$x \in \mathcal{S}$ if and only if $yx \notin \mathcal{O}$ for all $y \in A \backslash \{x\}$. So in example 1, we get $\mathcal{S} = \{d\}$.

## 4 Analysis of the Schulze method

### 4.1 Transitivity

In this section, we will prove that the binary relation $\mathcal{O}$, as defined in (2.2.1), is *transitive*. This means: If $ab \in \mathcal{O}$ and $bc \in \mathcal{O}$, then $ac \in \mathcal{O}$. This guarantees that the set $\mathcal{S}$ of winners, as defined in (2.2.2), is non-empty. When we interpret the Schulze method as a method to find a set $\mathcal{S}$ of winners, rather than a method to generate a binary relation $\mathcal{O}$, then the proof of the transitivity of $\mathcal{O}$ is an essential part of the proof that the Schulze method is well defined.

**Definition** An election method satisfies *transitivity* if the following holds for all $a, b, c \in A$:

Suppose:

$$ab \in \mathcal{O}. \tag{4.1.1}$$
$$bc \in \mathcal{O}. \tag{4.1.2}$$

Then:

$$ac \in \mathcal{O}. \tag{4.1.3}$$

**Claim** The binary relation $\mathcal{O}$, as defined in (2.2.1), is transitive.

*Proof* With (4.1.1), we get

$$P_D[a,b] \succ_D P_D[b,a]. \tag{4.1.4}$$

With (4.1.2), we get

$$P_D[b,c] \succ_D P_D[c,b]. \tag{4.1.5}$$

With (2.2.5), we get

$$\min_D\{P_D[a,b], P_D[b,c]\} \precsim_D P_D[a,c]. \tag{4.1.6}$$
$$\min_D\{P_D[b,c], P_D[c,a]\} \precsim_D P_D[b,a]. \tag{4.1.7}$$
$$\min_D\{P_D[c,a], P_D[a,b]\} \precsim_D P_D[c,b]. \tag{4.1.8}$$

Case 1: Suppose

$$P_D[a,b] \succsim_D P_D[b,c]. \tag{4.1.9a}$$

Combining (4.1.5) and (4.1.9a) gives

$$P_D[a,b] \succ_D P_D[c,b]. \tag{4.1.10a}$$

Combining (4.1.8) and (4.1.10a) gives

$$P_D[c,a] \precsim_D P_D[c,b]. \tag{4.1.11a}$$

Combining (4.1.6) and (4.1.9a) gives

$$P_D[b,c] \precsim_D P_D[a,c]. \tag{4.1.12a}$$

Combining (4.1.11a), (4.1.5), and (4.1.12a) gives

$$P_D[c,a] \precsim_D P_D[c,b] \prec_D P_D[b,c] \precsim_D P_D[a,c]. \tag{4.1.13a}$$

With (4.1.13a), we get (4.1.3).

Case 2: Suppose

$$P_D\,[a, b] \prec_D P_D\,[b, c]\,. \tag{4.1.9b}$$

Combining (4.1.4) and (4.1.9b) gives

$$P_D\,[b, a] \prec_D P_D\,[b, c]\,. \tag{4.1.10b}$$

Combining (4.1.7) and (4.1.10b) gives

$$P_D\,[c, a] \precsim_D P_D\,[b, a]\,. \tag{4.1.11b}$$

Combining (4.1.6) and (4.1.9b) gives

$$P_D\,[a, b] \precsim_D P_D\,[a, c]\,. \tag{4.1.12b}$$

Combining (4.1.11b), (4.1.4), and (4.1.12b) gives

$$P_D\,[c, a] \precsim_D P_D\,[b, a] \prec_D P_D\,[a, b] \precsim_D P_D\,[a, c]\,. \tag{4.1.13b}$$

With (4.1.13b), we get (4.1.3). □

The following corollary says that the set $\mathcal{S}$ of winners, as defined in (2.2.2), is non-empty:

**Corollary** *For the Schulze method, as defined in Sect. 2.2, we get*

$$\forall b \notin \mathcal{S} \exists a \in \mathcal{S} : ab \in \mathcal{O}. \tag{4.1.14}$$

*Proof of the corollary* As $b \notin \mathcal{S}$, there must be a $c(1) \in A$ with $c(1), b \in \mathcal{O}$.

If $c(1) \in \mathcal{S}$, then the corollary is proven. If $c(1) \notin \mathcal{S}$, then there must be a $c(2) \in A$ with $c(2), c(1) \in \mathcal{O}$. With the transitivity and the asymmetry of $\mathcal{O}$, we get $c(2), b \in \mathcal{O}$ and $c(2) \notin \{b, c(1)\}$.

We now proceed as follows: If $c(i) \in \mathcal{S}$, then the corollary is proven. If $c(i) \notin \mathcal{S}$, then there must be a $c(i+1) \in A$ with $c(i+1), c(i) \in \mathcal{O}$. With the transitivity and the asymmetry of $\mathcal{O}$, we get $c(i+1), b \in \mathcal{O}$ and $c(i+1) \notin \{b, c(1), \ldots, c(i)\}$.

We proceed until $c(i) \in \mathcal{S}$ for some $i \in \mathbb{N}$. Such an $i \in \mathbb{N}$ exists because $A$ is finite. □

## 4.2 Resolvability

*Resolvability* basically says that usually there is a unique winner $\mathcal{S} = \{a\}$. There are two different versions of the resolvability criterion. We will prove that the Schulze method, as defined in Sect. 2.2, satisfies both.

### *4.2.1 Formulation #1*

**Definition** An election method satisfies the first version of the *resolvability criterion* if (for every given number of alternatives) the proportion of profiles without a unique winner tends to zero as the number of voters in the profile tends to infinity.

**Claim** If $\succ_D$ satisfies (2.1.1), then the Schulze method, as defined in Sect. 2.2, satisfies the first version of the resolvability criterion.

*Proof* (overview)
Suppose $(x_1, x_2), (y_1, y_2) \in \mathbb{N}_0 \times \mathbb{N}_0$. Then, according to (2.1.1), there is a $v_1 \in \mathbb{N}_0$ such that for all $w_1 \in \mathbb{N}_0$:

1. $w_1 < v_1 \Longrightarrow (x_1, x_2) \succ_D (w_1, y_2)$.
2. $w_1 > v_1 \Longrightarrow (x_1, x_2) \prec_D (w_1, y_2)$.

When the number of voters tends to infinity (i.e., when $x_1, x_2, y_1$, and $y_2$ become large), then the proportion of profiles, where the condition "$y_1 = v_1$" happens to be satisfied, tends to zero. Therefore, when the number of voters tends to infinity, then the proportion of profiles, where two links *ef* and *gh* happen to have equivalent strengths $(N[e, f], N[f, e]) \approx_D (N[g, h], N[h, g])$, tends to zero.

Therefore, we will prove that, unless there are links *ef* and *gh* of equivalent strengths, there is always a unique winner. We will prove this by showing that, when we simultaneously presume (a) that there is more than one winner and (b) that there are no links *ef* and *gh* of equivalent strengths, then we necessarily get to a contradiction.

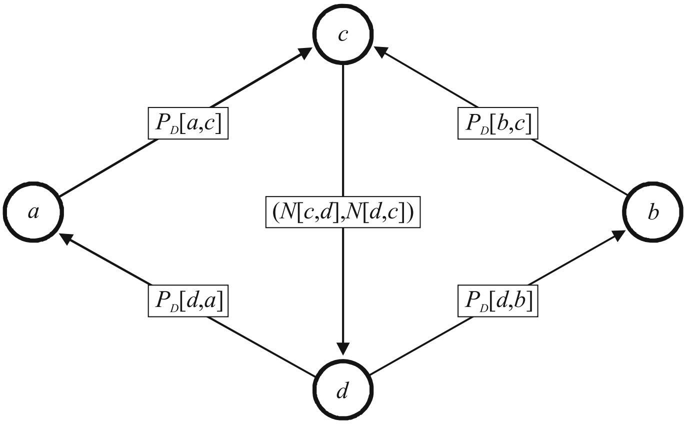
*Proof* (details)
Suppose that there is more than one winner. Suppose alternative $a \in A$ and alternative $b \in A$ are winners. Then

$$\forall i \in A \backslash \{a\} : P_D[a, i] \succsim_D P_D[i, a]. \tag{4.2.1.1}$$

$$\forall j \in A \backslash \{b\} : P_D[b, j] \succsim_D P_D[j, b]. \tag{4.2.1.2}$$

$$P_D[a, b] \approx_D P_D[b, a]. \tag{4.2.1.3}$$

Suppose there are no links *ef* and *gh* of equivalent strengths $(N[e, f], N[f, e]) \approx_D (N[g, h], N[h, g])$. Then $P_D[a, b] \approx_D P_D[b, a]$ means that the weakest link in the strongest path from alternative $a$ to alternative $b$ and the weakest link in the strongest path from alternative $b$ to alternative $a$ must be the same link, say *cd*. Therefore, the strongest paths have the following structure:




As $cd$ is the weakest link in the strongest path from alternative $a$ to alternative $b$, we get

$$P_D[a,d] \approx_D P_D[a,b]. \tag{4.2.1.4}$$

$$P_D[d,b] \succ_D P_D[a,b]. \tag{4.2.1.5}$$

As $cd$ is the weakest link in the strongest path from alternative $b$ to alternative $a$, we get

$$P_D[b,d] \approx_D P_D[b,a]. \tag{4.2.1.6}$$

$$P_D[d,a] \succ_D P_D[b,a]. \tag{4.2.1.7}$$

With (4.2.1.7), (4.2.1.3), and (4.2.1.4), we get

$$P_D[d,a] \succ_D P_D[b,a] \approx_D P_D[a,b] \approx_D P_D[a,d]. \tag{4.2.1.8}$$

But (4.2.1.8) contradicts (4.2.1.1).
Similarly, with (4.2.1.5), (4.2.1.3), and (4.2.1.6), we get

$$P_D[d,b] \succ_D P_D[a,b] \approx_D P_D[b,a] \approx_D P_D[b,d]. \tag{4.2.1.9}$$

But (4.2.1.9) contradicts (4.2.1.2). □

### *4.2.2 Formulation #2*

The second version of the *resolvability criterion* says that, when there is more than one winner, then, for every alternative $a \in \mathcal{S}$, it is sufficient to add a single ballot $w$ so that alternative $a$ becomes the unique winner.

**Definition** An election method satisfies the second version of the *resolvability criterion* if the following holds:
$\forall a \in \mathcal{S}^{\text{old}}$: It is possible to construct a strict weak order $w$ such that $\mathcal{S}^{\text{new}} = \{a\} \text{for} V^{\text{new}} := V^{\text{old}} + \{w\}$.

**Claim** If $\succ_D$ satisfies (2.1.1), then the Schulze method, as defined in Sect. 2.2, satisfies the second version of the resolvability criterion.

*Proof* Suppose $a \in \mathcal{S}^{\text{old}}$. Then we get

$$\forall b \in A \setminus \{a\} : P_D^{\text{old}}[a,b] \succsim_D P_D^{\text{old}}[b,a]. \tag{4.2.2.1}$$

Suppose the strict weak order $w$ is chosen as follows:

$$\forall f \in A \setminus \{a\} : pred^{\text{old}}[a,f] \succ_w f. \tag{4.2.2.2}$$
$$\forall e, f \in A \setminus \{a\} : \left(P_D^{\text{old}}[e,a] \succ_D P_D^{\text{old}}[f,a] \Rightarrow e \succ_w f\right). \tag{4.2.2.3}$$

With (4.2.2.2), we get

$$\forall f \in A \setminus \{a\} : a \succ_w f. \tag{4.2.2.4}$$

We will prove the following three claims:

Claim #1: It is not possible that (4.2.2.2) requires $e \succ_w f$ and that simultaneously (4.2.2.3) requires $f \succ_w e$.
Claim #2: $\forall g \in A \setminus \{a\} : P_D^{\text{new}}[a,g] \succ_D P_D^{\text{old}}[a,g]$.
Claim #3: $\forall g \in A \setminus \{a\} : P_D^{\text{new}}[g,a] \prec_D P_D^{\text{old}}[a,g]$.

With claim #2 and claim #3, we get

$$P_D^{\text{new}}[a,g] \succ_D P_D^{\text{new}}[g,a] \text{ for all } g \in A \setminus \{a\}$$

so that $ag \in \mathcal{O}^{\text{new}}$ for all $g \in A \setminus \{a\}$
so that $\mathcal{S}^{\text{new}} = \{a\}$.

*Proof of claim #1:* Suppose $e, f \in A \setminus \{a\}$. With (2.2.3), we get

$$P_D^{\text{old}}[e,f] \succsim_D \left(N^{\text{old}}[e,f], N^{\text{old}}[f,e]\right). \tag{4.2.2.5}$$

With (2.2.5), we get

$$\min_D \left\{P_D^{\text{old}}[e,f], P_D^{\text{old}}[f,a]\right\} \precsim_D P_D^{\text{old}}[e,a]. \tag{4.2.2.6}$$

With (4.2.2.1), we get

$$P_D^{\text{old}}[a,f] \succsim_D P_D^{\text{old}}[f,a]. \tag{4.2.2.7}$$

Suppose (4.2.2.2) requires $e \succ_w f$. Then $e = pred^{\text{old}}[a,f]$. Therefore, the link $ef$ was in the strongest path from alternative $a$ to alternative $f$. Thus, we get

$$P_D^{\text{old}}[a,f] \precsim_D \left(N^{\text{old}}[e,f], N^{\text{old}}[f,e]\right). \tag{4.2.2.8}$$

Suppose (4.2.2.3) requires $f \succ_w e$. Then

$$P_D^{\text{old}}[f,a] \succ_D P_D^{\text{old}}[e,a]. \tag{4.2.2.9}$$

With (4.2.2.5), (4.2.2.8), (4.2.2.7), and (4.2.2.9), we get

$$\begin{aligned} &P_D^{\text{old}}[e,f] \succsim_D \left(N^{\text{old}}[e,f], N^{\text{old}}[f,e]\right) \succsim_D P_D^{\text{old}}[a,f] \succsim_D \\ &P_D^{\text{old}}[f,a] \succ_D P_D^{\text{old}}[e,a]. \end{aligned} \tag{4.2.2.10}$$

But (4.2.2.9) and (4.2.2.10) together contradict (4.2.2.6).

*Proof of claim #2:* With (2.1.1) and (4.2.2.2), we get: The strength of each link of the strongest paths from alternative $a$ to each other alternative $g \in A\backslash\{a\}$ is increased. Therefore,

$$\forall g \in A\backslash\{a\} : P_D^{\text{new}}[a,g] \succ_D P_D^{\text{old}}[a,g]. \tag{4.2.2.11}$$

*Proof of claim #3:* Suppose $g \in A\backslash\{a\}$. Suppose

$$\mathfrak{T}(g) := \left(\{a\} \cup \left\{h \in A\backslash\{a\} \mid P_D^{\text{old}}[h,a] \succ_D P_D^{\text{old}}[a,g]\right\}\right). \tag{4.2.2.12}$$

With (4.2.2.1) and (4.2.2.12), we get

$$g \notin \mathfrak{T}(g) \quad \text{and} \quad a \in \mathfrak{T}(g) \tag{4.2.2.13}$$

and, therefore, $\varnothing \neq \mathfrak{T}(g) \subsetneq A$. Furthermore, we get

$$\forall i \notin \mathfrak{T}(g)\, \forall j \in \mathfrak{T}(g) : \left(N^{\text{old}}[i,j], N^{\text{old}}[j,i]\right) \precsim_D P_D^{\text{old}}[a,g]. \tag{4.2.2.14}$$

Otherwise, there was a path from alternative $i$ to alternative $a$ via alternative $j$ with a strength of more than $P_D^{\text{old}}[a,g]$. But (as $i \notin \mathfrak{T}(g)$) this would contradict the definition of $\mathfrak{T}(g)$.

With (4.2.2.3), (4.2.2.4), and (4.2.2.12), we get

$$\forall i \notin \mathfrak{T}(g)\, \forall j \in \mathfrak{T}(g) : j \succ_w i. \tag{4.2.2.15}$$

With (2.1.1) and (4.2.2.15), we get

$$\begin{aligned} &\forall i \notin \mathfrak{T}(g)\, \forall j \in \mathfrak{T}(g) : \left(N^{\text{new}}[i,j], N^{\text{new}}[j,i]\right) \prec_D \\ &\quad \left(N^{\text{old}}[i,j], N^{\text{old}}[j,i]\right). \end{aligned} \tag{4.2.2.16}$$

With (4.2.2.14) and (4.2.2.16), we get

$$\forall i \notin \mathfrak{T}(g)\, \forall j \in \mathfrak{T}(g) : \left(N^{\text{new}}[i,j], N^{\text{new}}[j,i]\right) \prec_D P_D^{\text{old}}[a,g]. \tag{4.2.2.17}$$

With (4.2.2.13) and (4.2.2.17), we get

$$P_D^{\text{new}}[g,a] \prec_D P_D^{\text{old}}[a,g]. \tag{4.2.2.18}$$

□

### 4.3 Pareto

The *Pareto criterion* says that the election method must respect unanimous opinions. There are two different versions of the Pareto criterion. The first version addresses situations with "$a \succ_v b$ for all $v \in V$", while the second version addresses situations with "$a \succsim_v b$ for all $v \in V$" (for some pair of alternatives $a, b \in A$). The first version says that, when every voter strictly prefers alternative $a$ to alternative $b$ (i.e., $a \succ_v b$ for all $v \in V$), then alternative $a$ must perform better than alternative $b$. The second version says that, when no voter strictly prefers alternative $b$ to alternative $a$ (i.e., $a \succsim_v b$ for all $v \in V$), then alternative $b$ must not perform better than alternative $a$. We will prove that the Schulze method, as defined in Sect. 2.2, satisfies both versions of the Pareto criterion.

#### *4.3.1 Formulation #1*

**Definition** An election method satisfies the first version of the *Pareto criterion* if the following holds:

Suppose:

$$\forall v \in V : a \succ_v b. \tag{4.3.1.1}$$

Then:

$$ab \in \mathcal{O}. \tag{4.3.1.2}$$

$$b \notin \mathcal{S}. \tag{4.3.1.3}$$

**Claim** If $\succ_D$ satisfies (2.1.1), then the Schulze method, as defined in Sect. 2.2, satisfies the first version of the Pareto criterion.

*Proof* With (2.1.1) and (4.3.1.1), we get

$$\forall e, f \in A : (N[a,b], N[b,a]) \succsim_D (N[e,f], N[f,e]). \tag{4.3.1.4}$$

$$[(N[a,b], N[b,a]) \approx_D (N[e,f], N[f,e])] \Leftrightarrow [\forall v \in V : e \succ_v f]. \tag{4.3.1.5}$$

With (2.2.4), we get: $ab \in \mathcal{O}$, unless the link $ab$ is in a directed cycle that consists of links of which each is at least as strong as the link $ab$.

However, as we presumed that the individual ballots $\succ_v$ are transitive, it is not possible that there is a directed cycle of unanimous opinions. Therefore, it is not possible that the link $ab$ is in a directed cycle that consists of links of which each is at least as strong as the link $ab$. Therefore, with (2.2.4), (4.3.1.4), and (4.3.1.5), we get (4.3.1.2). With (4.3.1.2), we get (4.3.1.3). □

*4.3.2 Formulation #2*

**Definition** An election method satisfies the second version of the *Pareto criterion* if the following holds:

Suppose:

$$\forall v \in V : a \succsim_v b. \tag{4.3.2.1}$$

Then:

$$ba \notin \mathcal{O}. \tag{4.3.2.2}$$

$$\forall f \in A \backslash \{a, b\} : bf \in \mathcal{O} \Rightarrow af \in \mathcal{O}. \tag{4.3.2.3}$$

$$\forall f \in A \backslash \{a, b\} : fa \in \mathcal{O} \Rightarrow fb \in \mathcal{O}. \tag{4.3.2.4}$$

$$b \in \mathcal{S} \Rightarrow a \in \mathcal{S}. \tag{4.3.2.5}$$

**Claim** If $\succ_D$ satisfies (2.1.1), then the Schulze method, as defined in Sect. 2.2, satisfies the second version of the Pareto criterion.

*Proof* With (4.3.2.1), we get

$$\forall e \in A \backslash \{a, b\} : N[a, e] \geq N[b, e]. \tag{4.3.2.6}$$

With (4.3.2.1), we get

$$\forall e \in A \backslash \{a, b\} : N[e, b] \geq N[e, a]. \tag{4.3.2.7}$$

With (2.1.1), (4.3.2.6), and (4.3.2.7), we get

$$\forall e \in A \backslash \{a, b\} : (N[a, e], N[e, a]) \succsim_D (N[b, e], N[e, b]). \tag{4.3.2.8}$$

With (2.1.1), (4.3.2.6), and (4.3.2.7), we get

$$\forall e \in A \backslash \{a, b\} : (N[e, b], N[b, e]) \succsim_D (N[e, a], N[a, e]). \tag{4.3.2.9}$$

Suppose $c(1), \ldots, c(n) \in A$ is the strongest path from alternative $b$ to alternative $a$. With (4.3.2.8) and (4.3.2.9), we get: $a, c(2), \ldots, c(n-1), b$ is a path from alternative $a$ to alternative $b$ with at least the same strength. Therefore,

$$P_D[a, b] \succsim_D P_D[b, a]. \tag{4.3.2.10}$$

With (4.3.2.10), we get (4.3.2.2).

Suppose $c(1), \ldots, c(n) \in A$ is the strongest path from alternative $b$ to alternative $f \in A \backslash \{a, b\}$. With (4.3.2.8), we get:$a, c(m+1), \ldots, c(n)$, where $c(m)$ is the last occurrence of an alternative of the set $\{a, b\}$, is a path from alternative $a$ to alternative $f$ with at least the same strength. Therefore,

$$\forall f \in A \backslash \{a, b\} : P_D[a, f] \succsim_D P_D[b, f]. \tag{4.3.2.11}$$

Suppose $c(1), \ldots, c(n) \in A$ is the strongest path from alternative $f \in A \backslash \{a, b\}$ to alternative $a$. With (4.3.2.9), we get:$c(1), \ldots, c(m-1), b$, where $c(m)$ is the first occurrence of an alternative of the set $\{a, b\}$, is a path from alternative $f$ to alternative $b$ with at least the same strength. Therefore,

$$\forall f \in A \backslash \{a, b\} : P_D[f, b] \succsim_D P_D[f, a]. \tag{4.3.2.12}$$

Part 1: Suppose $f \in A \backslash \{a, b\}$. Suppose

$$bf \in \mathcal{O}. \tag{4.3.2.13a}$$

With (4.3.2.13a), we get

$$P_D[b, f] \succ_D P_D[f, b]. \tag{4.3.2.14a}$$

With (4.3.2.11), (4.3.2.14a), and (4.3.2.12), we get

$$P_D[a, f] \succsim_D P_D[b, f] \succ_D P_D[f, b] \succsim_D P_D[f, a]. \tag{4.3.2.15a}$$

With (4.3.2.15a), we get (4.3.2.3).

Part 2: Suppose $f \in A \backslash \{a, b\}$. Suppose

$$fa \in \mathcal{O}. \tag{4.3.2.13b}$$

With (4.3.2.13b), we get

$$P_D[f, a] \succ_D P_D[a, f]. \tag{4.3.2.14b}$$

With (4.3.2.12), (4.3.2.14b), and (4.3.2.11), we get

$$P_D[f, b] \succsim_D P_D[f, a] \succ_D P_D[a, f] \succsim_D P_D[b, f]. \tag{4.3.2.15b}$$

With (4.3.2.15b), we get (4.3.2.4).

Part 3: Suppose

$$b \in \mathcal{S}. \tag{4.3.2.13c}$$

With (4.3.2.13c), we get

$$\forall f \in A \backslash \{b\} : fb \notin \mathcal{O}. \tag{4.3.2.14c}$$

With (4.3.2.4) and (4.3.2.14c), we get

$$\forall f \in A \backslash \{a, b\} : fa \notin \mathcal{O}. \tag{4.3.2.15c}$$

With (4.3.2.2) and (4.3.2.15c), we get

$$\forall f \in A \backslash \{a\} : fa \notin \mathcal{O}. \tag{4.3.2.16c}$$

With (4.3.2.16c), we get (4.3.2.5). □

## 4.4 Reversal symmetry

*Reversal symmetry* as a criterion for single-winner election methods has been proposed by Saari (1994). This criterion says that, when $\succ_v$ is reversed for all $v \in V$, then also the result of the elections must be reversed; see (4.4.2). When alternative $a \in A$ was the unique winner in the original situation (i.e., $\mathcal{S}^{\text{old}} = \{a\}$), then alternative $a \in A$ should not be a winner in the reversed situation (i.e., $a \notin \mathcal{S}^{\text{new}}$); see (4.4.3). It should not be possible that the same alternatives are elected in the original situation and in the reversed situation, unless all alternatives are tied; see (4.4.4).

Basic idea of this criterion is that, when there is a vote on the best alternatives and then there is a vote on the worst alternatives and when in both cases the same alternatives are chosen, then this questions the logic of the underlying heuristic of the used election method.

**Definition** An election method satisfies *reversal symmetry* if the following holds:
Suppose:

$$\forall e, f \in A \forall v \in V : e \succ_v^{\text{old}} f \Leftrightarrow f \succ_v^{\text{new}} e. \tag{4.4.1}$$

Then:

$$\forall a, b \in A : ab \in \mathcal{O}^{\text{old}} \Leftrightarrow ba \in \mathcal{O}^{\text{new}}. \tag{4.4.2}$$

$$\left(\exists i \in A : i \in \mathcal{S}^{\text{old}} \text{ and } i \notin \mathcal{S}^{\text{new}}\right) \Leftrightarrow$$
$$\left(\exists j \in A : j \notin \mathcal{S}^{\text{old}} \text{ and } j \in \mathcal{S}^{\text{new}}\right). \tag{4.4.3}$$

$$\mathcal{S}^{\text{old}} = \mathcal{S}^{\text{new}} \Leftrightarrow \mathcal{S}^{\text{old}} = A. \tag{4.4.4}$$

**Claim** The Schulze method, as defined in Sect. 2.2, satisfies reversal symmetry.

*Proof* With (4.4.1), we get

$$\forall e, f \in A : N^{\text{old}}[e, f] = N^{\text{new}}[f, e]. \tag{4.4.5}$$

With (4.4.5), we get

$$\forall e, f \in A : \left(N^{\text{old}}[e, f], N^{\text{old}}[f, e]\right) \approx_D \left(N^{\text{new}}[f, e], N^{\text{new}}[e, f]\right). \tag{4.4.6}$$

With (4.4.6), we get: When $c(1), \ldots, c(n) \in A$ was a path from alternative $g \in A$ to alternative $h \in A \backslash \{g\}$, then $c(n), \ldots, c(1)$ is a path from alternative $h$ to alternative $g$ with the same strength. Therefore,

$$\forall g, h \in A : P_D^{\text{old}}[g, h] \approx_D P_D^{\text{new}}[h, g]. \tag{4.4.7}$$

With (4.4.7), we get (4.4.2).

Part 1: Suppose $\exists i \in A : i \in \mathcal{S}^{\text{old}}$ and $i \notin \mathcal{S}^{\text{new}}$. With $i \notin \mathcal{S}^{\text{new}}$ and (4.1.14), we get that there is a $j \in \mathcal{S}^{\text{new}}$ with $ji \in \mathcal{O}^{\text{new}}$. With (4.4.2), we get $ij \in \mathcal{O}^{\text{old}}$ and, therefore, $j \notin \mathcal{S}^{\text{old}}$. With $j \notin \mathcal{S}^{\text{old}}$ and $j \in \mathcal{S}^{\text{new}}$, we get the "$\Longrightarrow$" direction of (4.4.3). The proof for the "$\Longleftarrow$" direction of (4.4.3) is analogous.

Part 2: Suppose $\mathcal{S}^{\text{old}} = A$. Then we get $\mathcal{O}^{\text{old}} = \emptyset$. Otherwise, if there was an $ij \in \mathcal{O}^{\text{old}}$, we would immediately get $j \notin \mathcal{S}^{\text{old}}$ and, therefore, $\mathcal{S}^{\text{old}} \neq A$. With $\mathcal{O}^{\text{old}} = \emptyset$ and (4.4.2), we get $\mathcal{O}^{\text{new}} = \emptyset$ and, therefore, $\mathcal{S}^{\text{new}} = A$. With $\mathcal{S}^{\text{old}} = A$ and $\mathcal{S}^{\text{new}} = A$, we get $\mathcal{S}^{\text{old}} = \mathcal{S}^{\text{new}}$.

Part 3: Suppose $\mathcal{S}^{\text{old}} \neq A$. Then there is a $j \notin \mathcal{S}^{\text{old}}$. With (4.1.14), we get that there is an $i \in \mathcal{S}^{\text{old}}$ with $ij \in \mathcal{O}^{\text{old}}$. With (4.4.2), we get $ji \in \mathcal{O}^{\text{new}}$ and, therefore, $i \notin \mathcal{S}^{\text{new}}$. With $i \in \mathcal{S}^{\text{old}}$ and $i \notin \mathcal{S}^{\text{new}}$, we get $\mathcal{S}^{\text{old}} \neq \mathcal{S}^{\text{new}}$. With part 2 and part 3, we get (4.4.4). □

## 4.5 Monotonicity

*Monotonicity* says that, when some voters rank alternative $a \in A$ higher (see (4.5.1) and (4.5.2)) without changing the order in which they rank the other alternatives relatively to each other (see (4.5.3)), then this must not hurt alternative $a$ (see (4.5.6)). Monotonicity is also known as *mono-raise* and *non-negative responsiveness*.

**Definition** An election method satisfies *monotonicity* if the following holds:

Suppose $a \in A$. Suppose the ballots are modified in such a manner that the following three statements are satisfied:

$$\forall f \in A \backslash \{a\} \, \forall v \in V : a \succ_v^{\text{old}} f \Rightarrow a \succ_v^{\text{new}} f. \tag{4.5.1}$$

$$\forall f \in A \backslash \{a\} \, \forall v \in V : a \succsim_v^{\text{old}} f \Rightarrow a \succsim_v^{\text{new}} f. \tag{4.5.2}$$

$$\forall e, f \in A \backslash \{a\} \, \forall v \in V : e \succ_v^{\text{old}} f \Leftrightarrow e \succ_v^{\text{new}} f. \tag{4.5.3}$$

Then:

$$\forall b \in A \setminus \{a\} : ab \in \mathcal{O}^{\text{old}} \Rightarrow ab \in \mathcal{O}^{\text{new}}. \tag{4.5.4}$$

$$\forall b \in A \setminus \{a\} : ba \notin \mathcal{O}^{\text{old}} \Rightarrow ba \notin \mathcal{O}^{\text{new}}. \tag{4.5.5}$$

$$a \in \mathcal{S}^{\text{old}} \Rightarrow a \in \mathcal{S}^{\text{new}} \subseteq \mathcal{S}^{\text{old}}. \tag{4.5.6}$$

**Claim** If $\succ_D$ satisfies (2.1.1), then the Schulze method, as defined in Sect. 2.2, satisfies monotonicity.

*Proof* Part 1: With (4.5.1), we get

$$\forall f \in A \setminus \{a\} : N^{\text{old}}[a, f] \leq N^{\text{new}}[a, f]. \tag{4.5.7}$$

With (4.5.2), we get

$$\forall f \in A \setminus \{a\} : N^{\text{old}}[f, a] \geq N^{\text{new}}[f, a]. \tag{4.5.8}$$

With (4.5.3), we get

$$\forall e, f \in A \setminus \{a\} : N^{\text{old}}[e, f] = N^{\text{new}}[e, f]. \tag{4.5.9}$$

With (2.1.1), (4.5.7), and (4.5.8), we get

$$\forall f \in A \setminus \{a\} : \left(N^{\text{old}}[a, f], N^{\text{old}}[f, a]\right) \precsim_D \left(N^{\text{new}}[a, f], N^{\text{new}}[f, a]\right). \tag{4.5.10}$$

With (2.1.1), (4.5.7), and (4.5.8), we get

$$\forall f \in A \setminus \{a\} : \left(N^{\text{old}}[f, a], N^{\text{old}}[a, f]\right) \succsim_D \left(N^{\text{new}}[f, a], N^{\text{new}}[a, f]\right). \tag{4.5.11}$$

With (4.5.9), we get

$$\forall e, f \in A \setminus \{a\} : \left(N^{\text{old}}[e, f], N^{\text{old}}[f, e]\right) \approx_D \left(N^{\text{new}}[e, f], N^{\text{new}}[f, e]\right). \tag{4.5.12}$$

Suppose $c(1), \ldots, c(n) \in A$ was the strongest path from alternative $a$ to alternative $b \in A \setminus \{a\}$. Then with (4.5.10) and (4.5.12), we get:$c(1),\ldots,c(n)$ is a path from alternative $a$ to alternative $b$ with at least the same strength. Therefore,

$$\forall b \in A \setminus \{a\} : P_D^{\text{new}}[a, b] \succsim_D P_D^{\text{old}}[a, b]. \tag{4.5.13}$$

Suppose $c(1), \ldots, c(n) \in A$ is the strongest path from alternative $b \in A\backslash\{a\}$ to alternative $a$. Then with (4.5.11) and (4.5.12), we get:$c(1)$,…,$c(n)$ was a path from alternative $b$ to alternative $a$ with at least the same strength. Therefore,

$$\forall b \in A\backslash\{a\} : P_D^{\text{old}}[b, a] \succsim_D P_D^{\text{new}}[b, a]. \tag{4.5.14}$$

With (4.5.13) and (4.5.14), we get (4.5.4) and (4.5.5).

Part 2: It remains to prove (4.5.6). Suppose $a \in \mathcal{S}^{\text{old}}$. Then "$a \in \mathcal{S}^{\text{new}}$" follows directly from (4.5.5). To prove "$\mathcal{S}^{\text{new}} \subseteq \mathcal{S}^{\text{old}}$", we have to prove: $h \notin \mathcal{S}^{\text{old}} \Longrightarrow h \notin \mathcal{S}^{\text{new}}$.
As $a \in \mathcal{S}^{\text{old}}$, we get

$$\forall b \in A\backslash\{a\} : P_D^{\text{old}}[a, b] \succsim_D P_D^{\text{old}}[b, a]. \tag{4.5.15}$$

Suppose $h \notin \mathcal{S}^{\text{old}}$. Then there must have been an alternative $g \in A\backslash\{h\}$ with

$$P_D^{\text{old}}[g, h] \succ_D P_D^{\text{old}}[h, g]. \tag{4.5.16}$$

With (4.5.10–4.5.12) and (4.5.16), we get: $P_D^{\text{new}}[g, h] \succ_D P_D^{\text{new}}[h, g]$, unless at least one of the following two cases occurred.

Case 1: $xa$ was a weakest link in the strongest path from alternative $g$ to alternative $h$.

Case 2: $ay$ was the weakest link in the strongest path from alternative $h$ to alternative $g$.

With (4.5.15), we get: $P_D^{\text{old}}[a, h] \succsim_D P_D^{\text{old}}[h, a]$ .For $P_D^{\text{old}}[a, h] \succ_D P_D^{\text{old}}[h, a]$, we would, with (4.5.4), immediately get $P_D^{\text{new}}[a, h] \succ_D P_D^{\text{new}}[h, a]$, so that alternative $h$ is still not a winner. Therefore, without loss of generality, we can presume $g \in A\backslash\{a, h\}$ and

$$P_D^{\text{old}}[a, h] \approx_D P_D^{\text{old}}[h, a]. \tag{4.5.17}$$

With (4.5.15), we get

$$P_D^{\text{old}}[a, g] \succsim_D P_D^{\text{old}}[g, a]. \tag{4.5.18}$$

With (2.2.5), we get

$$\min_D\left\{P_D^{\text{old}}[g, h], P_D^{\text{old}}[h, a]\right\} \precsim_D P_D^{\text{old}}[g, a]. \tag{4.5.19}$$

$$\min_D\left\{P_D^{\text{old}}[h, a], P_D^{\text{old}}[a, g]\right\} \precsim_D P_D^{\text{old}}[h, g]. \tag{4.5.20}$$

Case 1: Suppose $xa$ was a weakest link in the strongest path from alternative $g$ to alternative $h$. Then

$$P_D^{\text{old}}[g,h] \approx_D P_D^{\text{old}}[g,a] \text{ and} \tag{4.5.21a}$$

$$P_D^{\text{old}}[a,h] \succsim_D P_D^{\text{old}}[g,h]\,. \tag{4.5.22a}$$

Now (4.5.18), (4.5.21a), and (4.5.16) give

$$P_D^{\text{old}}[a,g] \succsim_D P_D^{\text{old}}[g,a] \approx_D P_D^{\text{old}}[g,h] \succ_D P_D^{\text{old}}[h,g]\,, \tag{4.5.23a}$$

while (4.5.17), (4.5.22a), and (4.5.16) give

$$P_D^{\text{old}}[h,a] \approx_D P_D^{\text{old}}[a,h] \succsim_D P_D^{\text{old}}[g,h] \succ_D P_D^{\text{old}}[h,g]\,. \tag{4.5.24a}$$

But (4.5.23a) and (4.5.24a) together contradict (4.5.20).

Case 2: Suppose $ay$ was the weakest link in the strongest path from alternative $h$ to alternative $g$. Then

$$P_D^{\text{old}}[h,g] \approx_D P_D^{\text{old}}[a,g] \text{ and} \tag{4.5.21b}$$

$$P_D^{\text{old}}[h,a] \succ_D P_D^{\text{old}}[h,g]\,. \tag{4.5.22b}$$

Now (4.5.22b), (4.5.21b), and (4.5.18) give

$$P_D^{\text{old}}[h,a] \succ_D P_D^{\text{old}}[h,g] \approx_D P_D^{\text{old}}[a,g] \succsim_D P_D^{\text{old}}[g,a]\,, \tag{4.5.23b}$$

while (4.5.16), (4.5.21b), and (4.5.18) give

$$P_D^{\text{old}}[g,h] \succ_D P_D^{\text{old}}[h,g] \approx_D P_D^{\text{old}}[a,g] \succsim_D P_D^{\text{old}}[g,a]\,. \tag{4.5.24b}$$

But (4.5.23b) and (4.5.24b) together contradict (4.5.19).

We have proven that neither case 1 nor case 2 is possible. Therefore,

$$P_D^{\text{new}}[g,h] \succ_D P_D^{\text{new}}[h,g]\,. \tag{4.5.25}$$

With (4.5.25), we get: $h \notin \mathcal{S}^{\text{new}}$. $\square$

## 4.6 Independence of clones

*Independence of clones* as a criterion for single-winner election methods has been proposed by Tideman (1987). This criterion says that running a large number of similar alternatives, so-called *clones*, must not have any impact on the result of the elections.

The precise definition for a *set of clones* stipulates that every voters ranks all the alternatives of this set in a consecutive manner; see (4.6.1) and (4.6.2). Replacing an alternative $d \in A^{\text{old}}$ by a set of clones $K$ should not change the winner; see (4.6.7) and (4.6.8).

This criterion is very desirable especially for referendums because, while it might be difficult to find several candidates who are simultaneously sufficiently popular to campaign with them and sufficiently similar to misuse them for this strategy, it is usually very simple to formulate a large number of almost identical proposals. For example: In 1969, when the Canadian city that is now known as *Thunder Bay* was amalgamating, there was some controversy over what the name should be. In opinion polls, a majority of the voters preferred the name *The Lakehead* to the name *Thunder Bay*. But when the polls opened, there were three names on the referendum ballot: *Thunder Bay*, *Lakehead*, and *The Lakehead*. As the ballots were counted using *plurality voting*, it was not a surprise when *Thunder Bay* won. The votes were as follows: *Thunder Bay* 15870, *Lakehead* 15302, *The Lakehead* 8377.

**Definition** An election method is *independent of clones* if the following holds:

Suppose $d \in A^{\text{old}}$. Suppose $A^{\text{new}} := (A^{\text{old}} \cup K)\backslash\{d\}$.

Suppose alternative $d$ is replaced by the set of alternatives $K$ in such a manner that the following three statements are satisfied:

$$\forall e \in A^{\text{old}} \backslash \{d\} \forall g \in K \forall v \in V : e \succ_v^{\text{old}} d \Leftrightarrow e \succ_v^{\text{new}} g. \tag{4.6.1}$$

$$\forall f \in A^{\text{old}} \backslash \{d\} \forall g \in K \forall v \in V : d \succ_v^{\text{old}} f \Leftrightarrow g \succ_v^{\text{new}} f. \tag{4.6.2}$$

$$\forall e, f \in A^{\text{old}} \backslash \{d\} \forall v \in V : e \succ_v^{\text{old}} f \Leftrightarrow e \succ_v^{\text{new}} f. \tag{4.6.3}$$

Then the following statements are satisfied:

$$\forall a \in A^{\text{old}} \backslash \{d\} \forall g \in K : ad \in \mathcal{O}^{\text{old}} \Leftrightarrow ag \in \mathcal{O}^{\text{new}}. \tag{4.6.4}$$

$$\forall b \in A^{\text{old}} \backslash \{d\} \forall g \in K : db \in \mathcal{O}^{\text{old}} \Leftrightarrow gb \in \mathcal{O}^{\text{new}}. \tag{4.6.5}$$

$$\forall a, b \in A^{\text{old}} \backslash \{d\} : ab \in \mathcal{O}^{\text{old}} \Leftrightarrow ab \in \mathcal{O}^{\text{new}}. \tag{4.6.6}$$

$$d \in \mathcal{S}^{\text{old}} \Leftrightarrow \mathcal{S}^{\text{new}} \cap K \neq \emptyset. \tag{4.6.7}$$

$$\forall a \in A^{\text{old}} \backslash \{d\} : a \in \mathcal{S}^{\text{old}} \Leftrightarrow a \in \mathcal{S}^{\text{new}}. \tag{4.6.8}$$

**Claim** The Schulze method, as defined in Sect. 2.2, is independent of clones.

*Proof* With (4.6.1), we get

$$\forall e \in A^{\text{old}} \backslash \{d\} \forall g \in K : N^{\text{old}}[e, d] = N^{\text{new}}[e, g]. \tag{4.6.9}$$

With (4.6.2), we get

$$\forall f \in A^{\text{old}} \backslash \{d\} \forall g \in K : N^{\text{old}}[d, f] = N^{\text{new}}[g, f]. \tag{4.6.10}$$

With (4.6.3), we get

$$\forall e, f \in A^{\text{old}} \backslash \{d\} : N^{\text{old}}[e, f] = N^{\text{new}}[e, f]. \tag{4.6.11}$$

With (4.6.9) and (4.6.10), we get

$$\forall e \in A^{\text{old}} \backslash \{d\} \forall g \in K : \left(N^{\text{old}}[e, d], N^{\text{old}}[d, e]\right) \approx_D \left(N^{\text{new}}[e, g], N^{\text{new}}[g, e]\right). \tag{4.6.12}$$

With (4.6.9) and (4.6.10), we get

$$\forall f \in A^{\text{old}} \backslash \{d\} \forall g \in K : \left(N^{\text{old}}[d, f], N^{\text{old}}[f, d]\right) \approx_D \left(N^{\text{new}}[g, f], N^{\text{new}}[f, g]\right). \tag{4.6.13}$$

With (4.6.11), we get

$$\forall e, f \in A^{\text{old}} \backslash \{d\} : \left(N^{\text{old}}[e, f], N^{\text{old}}[f, e]\right) \approx_D \left(N^{\text{new}}[e, f], N^{\text{new}}[f, e]\right). \tag{4.6.14}$$

Suppose $c(1), \ldots, c(n) \in A^{\text{old}}$ was the strongest path from alternative $a \in A^{\text{old}} \backslash \{d\}$ to alternative $d$. Then with (4.6.12) and (4.6.14), we get: $c(1), \ldots, c(n-1), g$ is a path from alternative $a$ to alternative $g \in K$ with the same strength. Therefore,

$$\forall a \in A^{\text{old}} \backslash \{d\} \forall g \in K : P_D^{\text{new}}[a, g] \succsim_D P_D^{\text{old}}[a, d]. \tag{4.6.15}$$

Suppose $c(1), \ldots, c(n) \in A^{\text{new}}$ is the strongest path from alternative $a \in A^{\text{new}} \backslash K$ to alternative $g \in K$. Then with (4.6.12) and (4.6.14), we get: $c(1), \ldots, c(m-1), d$, where $c(m)$ is the first occurrence of an alternative of the set $K$, was a path from alternative $a$ to alternative $d$ with at least the same strength. Therefore,

$$\forall a \in A^{\text{new}} \backslash K \forall g \in K : P_D^{\text{old}}[a, d] \succsim_D P_D^{\text{new}}[a, g]. \tag{4.6.16}$$

Suppose $c(1), \ldots, c(n) \in A^{\text{old}}$ was the strongest path from alternative $d$ to alternative $b \in A^{\text{old}} \backslash \{d\}$. Then with (4.6.13) and (4.6.14), we get: $g, c(2), \ldots, c(n)$ is a path from alternative $g \in K$ to alternative $b$ with the same strength. Therefore,

$$\forall b \in A^{\text{old}} \backslash \{d\} \forall g \in K : P_D^{\text{new}}[g, b] \succsim_D P_D^{\text{old}}[d, b]. \tag{4.6.17}$$

Suppose $c(1), \ldots, c(n) \in A^{\text{new}}$ is the strongest path from alternative $g \in K$ to alternative $b \in A^{\text{new}} \backslash K$. Then with (4.6.13) and (4.6.14), we get: $d, c(m+1), \ldots, c(n)$, where $c(m)$ is the last occurrence of an alternative of the set $K$, was a path from alternative $d$ to alternative $b$ with at least the same strength. Therefore,

$$\forall b \in A^{\text{new}} \backslash K \forall g \in K : P_D^{\text{old}}[d, b] \succsim_D P_D^{\text{new}}[g, b]. \tag{4.6.18}$$

($\alpha$) Suppose the strongest path $c(1), \ldots, c(n) \in A^{\text{old}}$ from alternative $a \in A^{\text{old}} \backslash \{d\}$ to alternative $b \in A^{\text{old}} \backslash \{a, d\}$ did not contain alternative $d$. Then with (4.6.14), we get: $c(1), \ldots, c(n)$ is still a path from alternative $a$ to alternative $b$ with the same strength. Therefore, $P_D^{\text{new}}[a, b] \succsim_D P_D^{\text{old}}[a, b]$.

($\beta$) Suppose the strongest path $c(1), \ldots, c(n) \in A^{\text{old}}$ from alternative $a \in A^{\text{old}} \backslash \{d\}$ to alternative $b \in A^{\text{old}} \backslash \{a, d\}$ contained alternative $d$. Then with (4.6.12), (4.6.13), and (4.6.14), we get: $c(1), \ldots, c(n)$, with alternative $d$ replaced by an arbitrarily chosen alternative $g \in K$, is still a path from alternative $a$ to alternative $b$ with the same strength. Therefore, $P_D^{\text{new}}[a, b] \succsim_D P_D^{\text{old}}[a, b]$.

With ($\alpha$) and ($\beta$), we get

$$\forall a, b \in A^{\text{old}} \backslash \{d\} : P_D^{\text{new}}[a, b] \succsim_D P_D^{\text{old}}[a, b]. \tag{4.6.19}$$

($\gamma$) Suppose the strongest path $c(1), \ldots, c(n) \in A^{\text{new}}$ from alternative $a \in A^{\text{new}} \backslash K$ to alternative $b \in A^{\text{new}} \backslash (K \cup \{a\})$ does not contain alternatives of the set $K$. Then with (4.6.14), we get: $c(1),\ldots,c(n)$ was a path from alternative $a$ to alternative $b$ with the same strength. Therefore, $P_D^{\text{old}}[a,b] \succsim_D P_D^{\text{new}}[a,b]$.

($\delta$) Suppose the strongest path $c(1), \ldots, c(n) \in A^{\text{new}}$ from alternative $a \in A^{\text{new}} \backslash K$ to alternative $b \in A^{\text{new}} \backslash (K \cup \{a\})$ contains some alternatives of the set $K$. Then with (4.6.12), (4.6.13), and (4.6.14), we get: $c(1),\ldots,c(s-1),d,c(t+1),\ldots,c(n)$, where $c(s)$ is the first occurrence of an alternative of the set $K$ and $c(t)$ is the last occurrence of an alternative of the set $K$, was a path from alternative $a$ to alternative $b$ with at least the same strength. Therefore, $P_D^{\text{old}}[a,b] \succsim_D P_D^{\text{new}}[a,b]$.

With ($\gamma$) and ($\delta$), we get

$$\forall a, b \in A^{\text{new}} \backslash K : P_D^{\text{old}}[a, b] \succsim_D P_D^{\text{new}}[a, b]. \tag{4.6.20}$$

Combining (4.6.15) and (4.6.16) gives

$$\forall a \in A^{\text{old}} \backslash \{d\} \forall g \in K : P_D^{\text{old}}[a, d] \approx_D P_D^{\text{new}}[a, g]. \tag{4.6.21}$$

Combining (4.6.17) and (4.6.18) gives

$$\forall b \in A^{\text{old}} \backslash \{d\} \forall g \in K : P_D^{\text{old}}[d, b] \approx_D P_D^{\text{new}}[g, b]. \tag{4.6.22}$$

Combining (4.6.19) and (4.6.20) gives

$$\forall a, b \in A^{\text{old}} \backslash \{d\} : P_D^{\text{old}}[a, b] \approx_D P_D^{\text{new}}[a, b]. \tag{4.6.23}$$

With (4.6.21–4.6.23), we get (4.6.4–4.6.6).

Part 1: Suppose $d \in \mathcal{S}^{\mathrm{old}}$. Then

$$\forall a \in A^{\mathrm{old}} \setminus \{d\} : ad \notin \mathcal{O}^{\mathrm{old}}. \tag{4.6.24}$$

With (4.6.4) and (4.6.24), we get

$$\forall a \in A^{\mathrm{new}} \setminus K \forall g \in K : ag \notin \mathcal{O}^{\mathrm{new}}. \tag{4.6.25}$$

Since the binary relation $\mathcal{O}^{\mathrm{new}}$, as defined in (2.2.1), is asymmetric and transitive, there must be an alternative $k \in K$ with

$$\forall l \in K \setminus \{k\} : lk \notin \mathcal{O}^{\mathrm{new}}. \tag{4.6.26}$$

With (4.6.25) and (4.6.26), we get $k \in \mathcal{S}^{\mathrm{new}} \cap K$ and, therefore, $\mathcal{S}^{\mathrm{new}} \cap K \neq \varnothing$.

Part 2: Suppose $d \notin \mathcal{S}^{\mathrm{old}}$. Then

$$\exists a \in A^{\mathrm{old}} \setminus \{d\} : ad \in \mathcal{O}^{\mathrm{old}}. \tag{4.6.27}$$

With (4.6.4) and (4.6.27), we get

$$\exists a \in A^{\mathrm{new}} \setminus K \forall g \in K : ag \in \mathcal{O}^{\mathrm{new}}. \tag{4.6.28}$$

With (4.6.28), we get: $\mathcal{S}^{\mathrm{new}} \cap K = \varnothing$.
With part 1 and part 2, we get (4.6.7).

Part 3: Suppose $a \in A^{\mathrm{old}} \setminus \{d\}$ and $a \in \mathcal{S}^{\mathrm{old}}$. Then

$$da \notin \mathcal{O}^{\mathrm{old}}. \tag{4.6.29}$$
$$\forall b \in A^{\mathrm{old}} \setminus \{a, d\} : ba \notin \mathcal{O}^{\mathrm{old}}. \tag{4.6.30}$$

With (4.6.5) and (4.6.29), we get

$$\forall g \in K : ga \notin \mathcal{O}^{\mathrm{new}}. \tag{4.6.31}$$

With (4.6.6) and (4.6.30), we get

$$\forall b \in A^{\mathrm{new}} \setminus (K \cup \{a\}) : ba \notin \mathcal{O}^{\mathrm{new}}. \tag{4.6.32}$$

With (4.6.31) and (4.6.32), we get: $a \in \mathcal{S}^{\mathrm{new}}$.

Part 4: Suppose $a \in A^{\mathrm{old}} \setminus \{d\}$ and $a \notin \mathcal{S}^{\mathrm{old}}$. Then at least one of the following two statements must have been valid:

$$da \in \mathcal{O}^{\mathrm{old}}. \tag{4.6.33a}$$
$$\exists b \in A^{\mathrm{old}} \setminus \{a, d\} : ba \in \mathcal{O}^{\mathrm{old}}. \tag{4.6.33b}$$

With (4.6.5), (4.6.6), and (4.6.33), we get that at least one of the following two statements must be valid:

$$\forall g \in K : ga \in \mathcal{O}^{\text{new}}. \tag{4.6.34a}$$
$$\exists b \in A^{\text{new}} \backslash (K \cup \{a\}) : ba \in \mathcal{O}^{\text{new}}. \tag{4.6.34b}$$

With (4.6.34), we get: $a \notin \mathcal{S}^{\text{new}}$.
With part 3 and part 4, we get (4.6.8). □

## 4.7 Smith

The *Smith criterion* and *Smith-IIA* (where IIA means "independence of irrelevant alternatives") say that *weak* alternatives should have no impact on the result of the elections.

Suppose:

$$\varnothing \neq B_1 \subsetneq A, \varnothing \neq B_2 \subsetneq A, B_1 \cup B_2 = A, B_1 \cap B_2 = \emptyset. \tag{4.7.1}$$
$$\forall a \in B_1 \forall b \in B_2 : N[a, b] > N[b, a]. \tag{4.7.2}$$

Then a *weak* alternative in the Smith paradigm is an alternative $b \in B_2$. Adding or removing a weak alternative $b \in B_2$ should have no impact on the set $\mathcal{S}$ of winners.

**Definition** An election method satisfies the *Smith criterion* if the following holds: Suppose (4.7.1) and (4.7.2). Then:

$$\forall a \in B_1 \forall b \in B_2 : ab \in \mathcal{O}. \tag{4.7.3}$$
$$\mathcal{S} \subseteq B_1. \tag{4.7.4}$$

*Remark* If $B_1$ consists of only one alternative $a \in A$, then this is the so-called *Condorcet criterion*. If $B_2$ consists of only one alternative $b \in A$, then this is the so-called *Condorcet loser criterion*.

**Claim** If $\succ_D$ satisfies (2.1.2), then the Schulze method, as defined in Sect. 2.2, satisfies the Smith criterion.

*Proof* The proof is trivial. Presumption (2.1.2) guarantees that any pairwise victory is stronger than any pairwise defeat. If $a \in B_1$ and $b \in B_2$, then already the link $ab$ is a path from alternative $a$ to alternative $b$ that consists only of a pairwise victory. On the other side, (4.7.2) says that there cannot be a path from alternative $b$ to alternative $a$ that contains no pairwise defeat. So already the link $ab$ is stronger than any path from alternative $b$ to alternative $a$. □

**Definition** An election method satisfies *Smith-IIA* if the following holds:
Suppose (4.7.1) and (4.7.2). Then:

$$\begin{array}{l} \text{If } d \in B_2 \text{ is removed, then} \\ (a) \quad \forall e, f \in B_1 : ef \in \mathcal{O}^{\text{old}} \Longleftrightarrow ef \in \mathcal{O}^{\text{new}}. \\ (b) \quad \mathcal{S}^{\text{old}} = \mathcal{S}^{\text{new}}. \end{array} \tag{4.7.5}$$

$$\begin{array}{l} \text{If } d \in B_1 \text{ is removed, then} \\ \forall e, f \in B_2 : ef \in \mathcal{O}^{\text{old}} \Leftrightarrow ef \in \mathcal{O}^{\text{new}}. \end{array} \tag{4.7.6}$$

**Claim** If $\succ_D$ satisfies (2.1.2), then the Schulze method, as defined in Sect. 2.2, satisfies Smith-IIA.

*Proof* We will prove (4.7.5)(a). The proof for (4.7.6) is analogous.
(4.7.5)(b) follows directly from (4.7.4) and (4.7.5)(a).

Part 1: Suppose $e, f \in B_1$. Suppose $ef \in \mathcal{O}^{\text{old}}$. Then

$$P_D^{\text{old}}[e, f] \succ_D P_D^{\text{old}}[f, e]. \tag{4.7.7}$$

With (2.2.3), we get

$$P_D^{\text{old}}[e, f] \succsim_D (N[e, f], N[f, e]). \tag{4.7.8}$$

With (4.7.7) and (2.2.3), we get

$$P_D^{\text{old}}[e, f] \succ_D P_D^{\text{old}}[f, e] \succsim_D (N[f, e], N[e, f]). \tag{4.7.9}$$

With (4.7.8) and (4.7.9), we get

$$P_D^{\text{old}}[e, f] \succsim_D \max_D \{(N[e, f], N[f, e]), (N[f, e], N[e, f])\}. \tag{4.7.10}$$

With (4.7.2), we get: Any path from alternative $e \in B_1$ to alternative $f \in B_1$ that contained alternative $d \in B_2$ necessarily contained a pairwise defeat. As it is not possible that the link *ef* is a pairwise defeat and that simultaneously the link *fe* is a pairwise defeat, $\max_D$ { ($N[e, f]$,$N[f, e]$), ($N[f, e]$, $N[e, f]$) } is stronger than any pairwise defeat [ because of (2.1.2) ]. Therefore, with (4.7.2) and (4.7.10), we get: The strongest path from alternative $e \in B_1$ to alternative $f \in B_1$ did not contain alternative $d \in B_2$. Therefore,

$$P_D^{\text{new}}[e, f] \approx_D P_D^{\text{old}}[e, f]. \tag{4.7.11}$$

As the elimination of alternative $d \in B_2$ only removes paths, we get

$$P_D^{\text{new}}[f, e] \precsim_D P_D^{\text{old}}[f, e]. \tag{4.7.12}$$

With (4.7.11), (4.7.7), and (4.7.12), we get

$$P_D^{\text{new}}[e,f] \approx_D P_D^{\text{old}}[e,f] \succ_D P_D^{\text{old}}[f,e] \succsim_D P_D^{\text{new}}[f,e]. \quad (4.7.13)$$

With (4.7.13), we get: $ef \in \mathcal{O}^{\text{new}}$.

Part 2: The proof "$ef \notin \mathcal{O}^{\text{old}} \Longrightarrow ef \notin \mathcal{O}^{\text{new}}$" is analogous. □

The *majority criterion for solid coalitions* says that, when a majority of the voters strictly prefers every alternative of a given set of alternatives to every alternative outside this set of alternatives, then the winner must be chosen from this set. In short, an election method satisfies the *majority criterion for solid coalitions* if the following holds:

Suppose (4.7.1).
Suppose $\|\{v \in V \mid \forall a \in B_1 \forall b \in B_2 : a \succ_v b\}\| > N/2$.
Then $\mathcal{S} \subseteq B_1$.

If $B_1$ consists of only one alternative $a \in A$, then this is the so-called *majority criterion*. If $B_2$ consists of only one alternative $b \in A$, then this is the so-called *majority loser criterion*.

*Participation* says that adding a list $W$ of ballots, on which every alternative of a given set of alternatives is strictly preferred to every alternative outside this set, must not hurt the alternatives of this set. In short, an election method satisfies *participation* if the following holds:

Suppose (4.7.1).
Suppose $\forall a \in B_1 \forall b \in B_2 \forall w \in W : a \succ_w b$.
Suppose $V^{\text{new}} := V^{\text{old}} + W$.
Then

$$\mathcal{S}^{\text{old}} \cap B_1 \neq \emptyset \Rightarrow \mathcal{S}^{\text{new}} \cap B_1 \neq \emptyset. \quad (4.7.14)$$

$$\mathcal{S}^{\text{old}} \cap B_2 = \emptyset \Rightarrow \mathcal{S}^{\text{new}} \cap B_2 = \emptyset. \quad (4.7.15)$$

The Smith criterion implies the majority criterion for solid coalitions, the Condorcet criterion, and the Condorcet loser criterion. The majority criterion for solid coalitions implies the majority criterion and the majority loser criterion. The Condorcet criterion implies the majority criterion. The Condorcet loser criterion implies the majority loser criterion. Unfortunately, the Condorcet criterion is incompatible with the participation criterion (Moulin 1988).

## 4.8 MinMax set

For all $\varnothing \neq B \subsetneq A$, we define

$$\Gamma_D(B) := \max_D\{(N[y,x], N[x,y]) \mid y \notin B, x \in B\}.$$

Suppose $\beta_D := \min_D\{\Gamma_D(B) \mid \emptyset \neq B \subsetneq A\}$.

Suppose $\mathfrak{B}_D := \cup\{\varnothing \neq B \subsetneq A | \Gamma_D(B) \approx_D \beta_D\}$ is the *MinMax set*. Then $\mathfrak{B}_D$ has the following properties:

1. $\mathfrak{B}_D \neq \varnothing$.
2. If $\mathfrak{B}_D$ consists of only one alternative $a \in A$, then alternative $a$ is the unique Simpson–Kramer winner (i.e., that alternative $a \in A$ with minimum $\max_D\{(N[b, a], N[a, b]) | b \in A \backslash \{a\}\}$).
3. If $d \in \mathfrak{B}_D$ is replaced by a set of alternatives $K$ as described in (4.6.1–4.6.3), then $\mathfrak{B}_D^{\text{new}} = (\mathfrak{B}_D \cup K) \backslash \{d\}$.
4. If $d \notin \mathfrak{B}_D$ is replaced by a set of alternatives $K$ as described in (4.6.1–4.6.3), then $\mathfrak{B}_D^{\text{new}} = \mathfrak{B}_D$.

So, in some sense, the MinMax set $\mathfrak{B}_D$ is a clone-proof generalization of the Simpson–Kramer winner.

When we want primarily that the used election method is independent of clones and secondarily that the strongest link *ef*, that is overruled when determining the winner, is minimized, then we have to demand that the winner is always chosen from the MinMax set $\mathfrak{B}_D$.

**Claim** The Schulze method, as defined in Sect. 2.2, has the following properties:

$$\forall a \in \mathfrak{B}_D \forall b \notin \mathfrak{B}_D : ab \in \mathcal{O}. \tag{4.8.1}$$

$$\mathcal{S} \subseteq \mathfrak{B}_D. \tag{4.8.2}$$

*Proof* Suppose $a \in \mathfrak{B}_D$. Then we get

$$\exists \emptyset \neq B \subsetneq A : \Gamma_D(B) \approx_D \beta_D \text{ and } a \in B. \tag{4.8.3}$$

Suppose $b \notin \mathfrak{B}_D$. Then we get

$$\gamma_D := \min_D\{\Gamma_D(B) | \emptyset \neq B \subsetneq A \text{ and } b \in B\} \succ_D \beta_D. \tag{4.8.4}$$

We will prove the following claims:

Claim #1: $P_D[b, a] \precsim_D \beta_D$.
Claim #2: $P_D[a, b] \succsim_D \gamma_D$.

With claim #1, claim #2, and (4.8.4), we get

$$P_D[a, b] \succsim_D \gamma_D \succ_D \beta_D \succsim_D P_D[b, a]. \tag{4.8.5}$$

With (4.8.5), we get (4.8.1). With (4.8.1), we get (4.8.2).

*Proof of claim #1:* With (4.8.3) and (4.8.4), we get

$$\exists \emptyset \neq B \subsetneq A : \Gamma_D(B) \approx_D \beta_D \text{ and } a \in B \text{ and } b \notin B. \tag{4.8.6}$$

Suppose $c(1), \ldots, c(n) \in A$ is the strongest path from alternative $b$ to alternative $a$. Suppose $c(i)$ is the last alternative with $c(i) \notin B$. Then we get $(N[c(i), c(i+1)], N[c(i+1), c(i)]) \precsim_D \beta_D$. Therefore, we get

$$P_D[b, a] \precsim_D \beta_D. \tag{4.8.7}$$

*Proof of claim #2:* We can construct a path from alternative $a$ to alternative $b$ with a strength of at least $\gamma_D$ as follows:

(1) We start with $E_1 := \{a\}$ and $i := 1$. Trivially, we get $b \notin E_1$ and $P_D[a, h] \succsim_D \gamma_D$ forall $h \in E_1 \backslash \{a\}$.

(2) At each stage, we consider the set $B_i := A \setminus E_i$.
With $b \in B_i$ and with (4.8.4), we get

$$\Gamma_D(B_i) \approx_D \max_D\{(N[y, x], N[x, y]) \mid y \notin B_i, x \in B_i\} \succsim_D \gamma_D. \tag{4.8.8}$$

We choose $f \in E_i$ and $g \in B_i$ with

$$(N[f, g], N[g, f]) \approx_D \max_D\{(N[y, x], N[x, y]) \mid y \notin B_i, x \in B_i\} \succsim_D \gamma_D. \tag{4.8.9}$$

We define $E_{i+1} := E_i \cup \{g\}$.
With $f \in E_i$, with $P_D[a, h] \succsim_D \gamma_D$ for all $h \in E_i \backslash \{a\}$, with $(N[f, g], N[g, f]) \succsim_D \gamma_D$, and with $E_{i+1} := E_i \cup \{g\}$, we get

$$P_D[a, h] \succsim_D \gamma_D \text{ for all } h \in E_{i+1} \backslash \{a\}. \tag{4.8.10}$$

(3) We repeat stage 2 with $i \to i+1$, until $g \equiv b$.

Therefore, we get

$$P_D[a, b] \succsim_D \gamma_D. \tag{4.8.11}$$

□

## 4.9 Prudence

*Prudence* as a criterion for single-winner election methods has been popularized mainly by Arrow and Raynaud (1986). This criterion says that the strength $\lambda_D$ of the strongest link *ef*, that is not supported by the binary relation $\mathcal{O}$, should be as small as possible. So $\lambda_D := \max_D\{(N[e, f], N[f, e]) \mid ef \notin \mathcal{O}\}$ should be minimized.

When there is a directed cycle $c(1), \ldots, c(n) \in A$ with $c(1) \equiv c(n)$, then it is obvious that the strongest link, that is not supported by the binary relation $\mathcal{O}$, is at least as strong as the weakest link$c(i)$,$c(i+1)$ of this directed cycle. So we get:

$$\lambda_D \succsim_D \min_D\{(N[c(i), c(i+1)], N[c(i+1), c(i)]) \mid i = 1, \ldots, (n-1)\}. \tag{4.9.1}$$

As we have to make this consideration for all directed cycles, the maximum, that we can ask for, is the following criterion.

**Definition** Suppose $\lambda_D \in \mathbb{N}_0 \times \mathbb{N}_0$ is the strength of the strongest directed cycle.

$$\begin{aligned}\lambda_D := \; & \max_D \{ \min_D \{ (N[c(i), c(i+1)], N[c(i+1), c(i)]) \mid i = 1, \ldots, (n-1) \} \\ & \mid c(1), \ldots, c(n) \text{ is a path with } c(1) \equiv c(n) \}.\end{aligned} \tag{4.9.2}$$

Then an election method is *prudent* if the following holds:

$$\forall a, b \in A : (N[a, b], N[b, a]) \succ_D \lambda_D \Rightarrow ab \in \mathcal{O}. \tag{4.9.3}$$

$$\forall a, b \in A : (N[a, b], N[b, a]) \succ_D \lambda_D \Rightarrow b \notin \mathcal{S}. \tag{4.9.4}$$

**Claim** The Schulze method, as defined in Sect. 2.2, is prudent.

*Proof* The proof is trivial. With (2.2.4), we get: $ab \in \mathcal{O}$, unless the link $ab$ is in a directed cycle that consists of links of which each is at least as strong as the link $ab$. □

## 5 Comparison with other methods

Table 2 compares the Schulze method with its main contenders. Extensive descriptions of the different methods can be found in publications by Fishburn (1977), Nurmi (1987), Kopfermann (1991), Levin and Nalebuff (1995), and Tideman (2006). As most of these methods only generate a set $\mathcal{S}$ of winners and don't generate a binary relation $\mathcal{O}$, only that part of the different criteria is considered that refers to the set $\mathcal{S}$ of winners.

In terms of satisfied and violated criteria, that election method, that comes closest to the Schulze method, is Tideman's ranked pairs method (Tideman 1987). The only difference is that the ranked pairs method doesn't choose from the MinMax set $\mathfrak{B}_D$.

The ranked pairs method works from the strongest to the weakest link. The link $xy$ is locked if and only if it doesn't create a directed cycle with already locked links. Otherwise, this link is locked in its opposite direction.

In Example 1, the ranked pairs method locks $db$. Then it locks $cb$. Then it locks $ac$. Then it locks $ab$, since locking $ba$ in its original direction would create a directed cycle with the already locked links $ac$ and $cb$. Then it locks $cd$. Then it locks $ad$, since locking $da$ in its original direction would create a directed cycle with the already locked links $ac$ and $cd$.

The winner of the ranked pairs method is alternative $a \notin \mathfrak{B}_D = \{d\}$, because there is no locked link that ends in alternative $a$.

Although Tideman's ranked pairs method is that election method that comes closest to the Schulze method in terms of satisfied and violated criteria, random simulations by Wright (2009) showed that that election method, that agrees the most frequently with the Schulze method, is the Simpson–Kramer method (Table 1).

**Table 2** Comparison of Election Methods

| | Resolvability | Pareto | Reversal symmetry | Monotonicity | Independence of clones | Smith | Smith-IIA | Condorcet | Condorcet loser | Majority for solid coalitions | Majority | Majority loser | Participation | MinMax set | Prudence | Polynomial runtime |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Baldwin | Y | Y | N | N | N | Y | N | Y | Y | Y | Y | Y | N | N | N | Y |
| Black | Y | Y | Y | Y | N | N | N | Y | Y | N | Y | Y | N | N | N | Y |
| Borda | Y | Y | Y | Y | N | N | N | N | Y | N | N | Y | Y | N | N | Y |
| Bucklin | Y | Y | N | Y | N | N | N | N | N | Y | Y | Y | N | N | N | Y |
| Copeland | N | Y | Y | Y | N | Y | Y | Y | Y | Y | Y | Y | N | N | N | Y |
| Dodgson | Y | Y | N | N | N | N | N | Y | N | N | Y | N | N | N | N | N |
| Instant runoff | Y | Y | N | N | Y | N | N | N | Y | Y | Y | Y | N | N | N | Y |
| Kemeny–Young | Y | Y | Y | Y | N | Y | Y | Y | Y | Y | Y | Y | N | N | N | N |
| Nanson | Y | Y | Y | N | N | Y | N | Y | Y | Y | Y | Y | N | N | N | Y |
| Plurality | Y | Y | N | Y | N | N | N | N | N | N | Y | N | Y | N | N | Y |
| Ranked pairs | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | N | N | Y | Y |
| Schulze | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | N | Y | Y | Y |
| Simpson–Kramer | Y | Y | N | Y | N | N | N | Y | N | N | Y | N | N | N | Y | Y |
| Slater | N | Y | Y | Y | N | Y | Y | Y | Y | Y | Y | Y | N | N | N | N |
| Young | Y | Y | N | Y | N | N | N | Y | N | N | Y | N | N | N | N | N |

Y = compliance
N = violation

## 6 Discussion

Suppose $\Lambda_D(a) := \max_D\{(N[b,a], N[a,b]) | b \in A\backslash\{a\}\}$ is the Simpson–Kramer score of alternative $a \in A$. Then the Simpson–Kramer method is defined as follows:

$$a \in \mathcal{S}_{\text{SK}} :\Leftrightarrow \Lambda_D(a) \precsim_D \Lambda_D(b) \quad \text{for all } b \in A\backslash\{a\}. \tag{6.1}$$

Over a long period of time, this method was the most popular election method among Condorcet activists, because this method minimizes the number of overruled voters. However, a very serious problem of this method is that it is not independent of clones, because it can happen that, when alternative $a \in A$ is replaced by a set of clones $K$ as described in (4.6.1–4.6.3), then the alternatives of the set $K$ disqualify each other in such a manner that for some alternative $b \in A\backslash\{a\}$:

$$\Lambda_D^{\text{old}}(a) \prec_D \Lambda_D^{\text{old}}(b) \text{ and } \Lambda_D^{\text{new}}(b) \prec_D \Lambda_D^{\text{new}}(g) \, \forall g \in K. \tag{6.2}$$

To make the Simpson–Kramer method clone-proof, the concept of Simpson–Kramer scores has to be generalized from individual alternatives $a \in A$ to sets of alternatives $\varnothing \neq B \subsetneq A$ :

$$\Gamma_D(B) := \max_D\{(N[b,a], N[a,b]) | b \notin B, a \in B\}. \tag{6.3}$$

We get

$$\forall a \in A : \Lambda_D(a) \approx_D \Gamma_D(\{a\}). \tag{6.4}$$

The $\Gamma_D$ scores are clone-proof because, when alternative $a \in A$ is replaced by a set of clones $K$, then we get for all $\emptyset \neq B \subsetneq A$ :

$$a \in B \Rightarrow \Gamma_D^{\text{new}}((B \cup K)\backslash\{a\}) \approx_D \Gamma_D^{\text{old}}(B). \tag{6.5a}$$

$$a \notin B \Rightarrow \Gamma_D^{\text{new}}(B) \approx_D \Gamma_D^{\text{old}}(B). \tag{6.5b}$$

Suppose $\beta_D := \min_D\{\Gamma_D(B) | \emptyset \neq B \subsetneq A\}$ and $\mathfrak{B}_D := \cup\{\emptyset \neq B \subsetneq A | \Gamma_D(B) \approx_D \beta_D\}$. Then when we want primarily that the used election method is clone-proof and secondarily that it minimizes the number of overruled voters, then the maximum, that we can ask for, is

$$\mathcal{S} \subseteq \mathfrak{B}_D. \tag{6.6}$$

In this article, we propose a new single-winner election method (*Schulze method*) that is clone-proof (Sect. 4.6) and that always chooses from the MinMax set $\mathfrak{B}_D$ (Sect. 4.8). The latter property is the most characteristic property of the Schulze method, since this is the first time that an election method with this property is proposed.

The Schulze method also satisfies many other criteria; some of them are also satisfied by the Simpson–Kramer method, like the Pareto criterion (Sect. 4.3), resolvability

(Sect. 4.2), monotonicity (Sect. 4.5), and prudence (Sect. 4.9); some of them are violated by the Simpson–Kramer method, like the Smith criterion (Sect. 4.7) and reversal symmetry (Sect. 4.4). Because of this large number of satisfied criteria, we consider the Schulze method to be a promising alternative to the Simpson–Kramer method for actual implementations, especially when manipulation through clones or weak alternatives is an issue.

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## References

Arrow KJ, Raynaud H (1986) Social choice and multicriterion decision-making. MIT Press, Cambridge
Börgers C (2009) Mathematics of social choice: voting, compensation, and division. SIAM, Philadelphia
Camps R, Mora X, Saumell L (2008) A continuous rating method for preferential voting. Working paper
Fishburn PC (1977) Condorcet social choice functions. SIAM J Appl Math 33:469–489
Floyd RW (1962) Algorithm 97 (Shortest Path). Commun ACM 5:345
Kopfermann K (1991) Mathematische Aspekte der Wahlverfahren. BI-Verlag, Mannheim
Levin J, Nalebuff B (1995) An introduction to vote-counting schemes. J Econ Perspect 9:3–26
McCaffrey JD (2008) Test run: group determination in software testing. MSDN Magazine, Redmond, Washington
Moulin H (1988) Condorcet's principle implies the no show paradox. J Econ Theory 45:53–64
Nurmi HJ (1987) Comparing voting systems. Springer-Verlag, Berlin
Rivest RL, Shen E (2010) An optimal single-winner preferential voting system based on game theory. Working paper
Saari DG (1994) Geometry of voting. Springer-Verlag, Berlin
Smith JH (1973) Aggregation of preferences with variable electorate. Econometrica 41:1027–1041
Stahl S, Johnson PE (2006) Understanding modern mathematics. Jones & Bartlett Publishing, Boston
Tideman TN (1987) Independence of clones as a criterion for voting rules. Soc Choice Welf 4:185–206
Tideman TN (2006) Collective decisions and voting: the potential for public choice. Ashgate Publishing, Burlington
Wright B (2009) Objective measures of preferential ballot voting systems. Doctoral dissertation, Duke University, Durham, North Carolina
Yue A, Liu W, Hunter A (2007) Approaches to constructing a stratified merged knowledge base. Symbolic and quantitative approaches to reasoning with uncertainty, 9th European Conference, ECSQARU 2007