THE LIKELIHOOD OF CHOOSING THE BORDA-WINNER WITH PARTIAL PREFERENCE RANKINGS OF THE ELECTORATE*

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ABSTRACT

This study aims to empirically investigate the likelihood of choosing the Borda winner when n voters report only the first r ranks of their *linear* preference rankings over m alternatives. The voters' preferences are generated through Impartial Anonymous and Neutral Culture model where both the names of the alternatives and voters are immaterial. Monte-Carlo experiments are run to quantify the content of the information contained in the first r ranks of the voters' preference rankings from the perspective of predicting the Borda outcome with respect to changes in m, n and r. For a given r, the likelihood of choosing the Borda winner approaches to zero independent of n as m increases within the computed range of parameter values, $1 \leq m,n \leq 30$. We show empirically that this probability is inversely proportional to m, and determine the constant of proportionality for two different types of likelihood that we consider.

1. INTRODUCTION

A voting rule solves the collective decision problem, where the voters must jointly choose one among the possible outcomes (alternatives), on the basis of the reported ordinal preferences. The choice of a voting rule has been a major ethical question ever since the political philosophy of the Enlightenment. When only two alternatives are at stake, the ordinary majority voting, whose axiomatic formulation is due to May(1952) is unambiguously regarded as the 'fairest' method. The theory of voting deals with the ways of extending majority voting among pairs when there are three or more alternatives.

Plurality voting, where each voter reports the name of (exactly) one alternative on his ballot and the alternative receiving the most votes wins, has been historically the most popular voting rule. The two celebrated critiques of Plurality voting, Borda(1781) and Condorcet(1785) noted that Plurality voting may elect a poor candidate, namely, one that would lose in a simple pair-wise majority comparison to every other candidate. Borda and Condorcet individually devised different rules to replace Plurality voting. Condorcet suggested electing the alternative defeating every other alternative in pair-wise majority

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comparisons; whereas Borda assigned points to each candidate, linearly increasing with the candidate's ranking in a voter's opinion, and proposed electing the alternative with the highest total score over all alternatives. These two approaches have generated most of the modern scholarly research on inventing and analyzing voting rules that aggregate individual preferences into social decisions in a manner compatible with the fulfillment of a variety of normative criteria. However, the fact that plurality voting, despite its well-known deficiencies in fulfilling most of these criteria, is still by far the most widely used method gives an indication of the speed at which all those theoretical inputs find their way into the real world. The major reason for this is surely that, unlike Plurality rule, most of the 'ethically superior' rules require the voters to report their full rankings over the possible alternatives. In reality, this requirement is very difficult to implement due to associated practical complications both on the side of the voters and of the administrators who are to collect this information.

Let us consider a social planner who believes that Borda Rule is the 'fairest' voting rule to be adopted for an election, but is unable to ask the voters to report their full preference rankings over m alternatives because of the practical complications alluded to above. As it is generally practiced, he can adopt Plurality voting, and ask the electorate to report only their first-best alternatives. Or, he can require the voters to state the first r ($1 < r < m-1$) ranks of their linear preference rankings in the hope of increasing the probability of choosing the Borda winner that would be elected if the full linear preference rankings could be collected. Given that the costs of implementation associated with this requirement is surely higher than the cost of asking only the top choices, the social planner has then to decide whether this would be a worthwhile attempt.

This study aims to empirically investigate the likelihood of choosing the Borda winner when n voters report their first r ($1 \leq r < m$) ranks of their *linear* (i.e. full or total) preference rankings over m alternatives. Monte-Carlo experiments are run to ascertain the information content of the first r ranks of the electorate's preferences from the perspective of implementing the Borda outcome, and the changes in this value is observed as a function of m, n and r by considering all possible values of these parameters in an appropriate range.

In the present study, the voters' preferences are generated through Impartial Anonymous and Neutral Culture (IANC) model. As introduced by Eğecioğlu and Giritligil (2005), IANC model treats the voters' preferences through a class of preference profiles, namely *root profiles*, where the names of both voters and alternatives are immaterial.

For a root profile generated for m alternatives and n voters, first, the set of Borda winners, namely 'original' (or m-) Borda winners for the profile is detected and recorded. Then, the first r ranks of the voters' preferences are considered in the root profile, which is assumed to be the reported part. A scoring method is adopted for aggregating the incomplete preference rankings in this partial profile. This a là Borda method assigns strictly positive points to each alternative that appear in the first r-ranks of a voter's actual preference, linearly increasing with its rank, and gives zero points to the ones that are not among the top r-ranks in the voter's decision. Then, the alternative(s) that receive(s) the highest score over the entire electorate is elected as the *r-Borda winner(s)*. It should be noted that, when $r = 1$, the set of first-rank Borda winners corresponds to the set of Plurality winners, and the set of *(m-1)*-Borda winners is necessarily equal to the set of original Borda winners (i.e. the set of m-Borda winners).

For triples of m, n and r, two types of probabilities are computed as the likelihood of choosing the Borda winner with partially reported linear preference rankings. The first type, $\boldsymbol{Pr_1}$, refers to the likelihood of choosing *exactly* the same set of Borda winners of the original profile by considering only the reported partial profile, i.e. the probability of the set of m-Borda winners and r-Borda winners to be equal. The second type of probability, $\boldsymbol{Pr_2}$, considers the likelihood of the r-Borda winners to also be m-Borda winners. For a given root profile, this is the probability that an r-Borda winner selected to also be an actual Borda winner. Through Monte-Carlo methods, the values of the both probability types are computed for each given triple of m, n and r.

2. PRELIMINARIES

2. 1. Preference Profiles and the Borda Rule

By a *preference* on a set A we mean any function p: $A \rightarrow 2^A$ which assigns to every $a \in A$ a subset ("lower contour set") $p(A) \subset A$ such that, at all a, $b \in A$ we have

(1) $b \in p(a)$ or $a \in p(b)$ [completeness]

(2) $p(b) \subset p(a)$ whenever $b \in p(a)$ [transitivity]

(3) $b \in p(a)$ and $a \in p(b)$ only if $a = b$ [antisymmetry]

Such a preference clearly corresponds to a linear (or "total") order on A.

We denote by $\boldsymbol{p}(A)$ the set of all preferences on a set A. For any positive integer n we write $[n] = \{1,2,..,n\}$, and by a *preference profile* for a society of n voters on a set A we mean any family $P_{m,n} = (p_i)_{i\in[n]} \in \boldsymbol{p}(A)^{[n]}$ of preferences p_i on A indexed by "voters" $i \in [n]$.

Let $card(p_i(\mathrm{a}))$ be the cardinality of the lower counter set of $a \in A$ for $i \in [n]$. Note that the cardinalities of the top- and bottom- ranked alternatives are m and 1, respectively.

The *Borda score* of $a \in A$ for $i \in [n]$ is defined as,

$$B_i^a = card(p_i(\mathrm{a})),$$

and the set of m-Borda winners at each $P_{m,n} \in \boldsymbol{p}(A)^{[n]}$ is determined by setting

$$\mathrm{B}(P_{m,n}) = \underset{a\in A}{\arg\max} \sum_{i\in[n]} B_i^a .$$

Thus, the Borda Rule chooses the candidates who maximize the total Borda score aggregated over the set of all n voters.

Let $P^r_{m,n}$ denote the portion of a preference profile $P_{m,n}$ where only the first r ranks of the voters' preferences can be observed. When we view the profile $P_{m,n}$ as an $m \times n$ matrix with m rows and n columns, then $P^r_{m,n}$ corresponds to the $r \times n$ sub matrix of $P_{m,n}$ consisting of the first r rows. Note that, when $r = m$ or $r = m$-1, the Borda outcome of the entire preference profile is detectable. However when $r < m$-1, the observable preference p^r_i of voter i corresponds to a partial strict ordering on A which is transitive and antisymmetric, however incomplete. Let $A^r_i \subseteq A$ be the set of alternatives that appear at p^r_i .

Let $card(p^r_i(\text{a}))$ be the cardinality of the *observable* lower counter set of $a \in A^r_i$ for $i \in [n]$. Note that $1 \leq card(p^r_i(\text{a})) \leq r$. We now re-define the Borda score of $a \in A$ for $i \in [n]$ as,

$$B^a_i = \begin{cases} card(p^r_i(\text{a})), \text{ if } a \in A^r_i \\ 0, \text{ otherwise,} \end{cases}$$

and the set of r-Borda winners at any $P^r_{m,n}$ is given by

$$\mathrm{B}(P^r_{m,n}) = \underset{a \in A}{\arg\max} \sum_{i \in [n]} B^a_i .$$

In other words, if an alternative is among the first r-ranks of a voter's original preference ranking, then its Borda score is equal to the number of the alternatives it beats at the first r-ranks of this preference. If it is not one of the top r-ranked alternatives, it receives a Borda score of zero. Then, the 'modified' Borda rule chooses the alternatives with the highest Borda scores aggregated over all voters as the set of r-Borda winners.

When the preference profile and the dependence on m and n is clear from the context, we denote $\mathrm{B}(P^r_{m,n})$ simply by B_r, for $1 \leq r \leq m$. Note that $\mathrm{B}_m = \mathrm{B}_{m-1}$ is the set of Borda winners of the full profile, and B_1 is the set of plurality winners.

2.2 Root Profiles and IANC

Let $\Omega\ (m,n)$ denote the set of all preference profiles that can be generated for m alternatives and n voters. As shown in Eğecioğlu and Giritligil (2005), a product permutation group on the names of alternatives and of voters *acts* on $\Omega\ (m,n)$, and splits it up into a disjoint union of subsets called *orbits*, i.e.

$$\Omega\ (m,n) = \theta_1 + \theta_2 + \ldots.. + \theta_\omega ,$$

where each θ_i is an *anonymous and neutral equivalence class*. All the preference profiles within an anonymous and neutral equivalence class can be generated from each other through re-labeling the alternatives and/or the voters. The preference profiles of any such class are 'equal' in the sense that any anonymous and neutral social choice rule, such as Borda Rule, yields the same alternative as the winner (however, under different names) for all of them.

A *root profile* is any preference profile that represents an anonymous and neutral equivalence class. That is, all the other preference profiles within the same equivalence class can be generated from this root profile via permuting the names of the *m* alternatives and simultaneously those of the *n* voters. We denote by $R = R(m,n)$ the collection of root profiles for *m* alternatives and *n* voters. Each element of this set represents an equivalence class of preference profiles from Ω (m,n).

IANC model as introduced by Eğecioğlu and Giritligil (2005), uses root profiles to generate voters' preferences through an application of the Dixon Wilf algorithm. The roots can then be generated from the uniform distribution for *m* alternatives and *n* for large values of these parameters. Each root profile is generated uniformly with probability $1/card(R(m,n))$. This allows for Monte-Carlo algorithms for the empirical analysis of the behaviors and applications of anonymous and neutral social choice rules even for large electorate size and high number of alternatives as we describe below in the analysis of selecting the Borda winner, when only partial preference rankings of the voters are available.

3. LIKELIHOOD MEASURES

3.1 Types of Likelihood:

Two types of probabilities are considered for measure of the likelihood of implementing the Borda outcome with partially observed linear preference rankings of the voters.

- $\boldsymbol{Pr_1} = \boldsymbol{Pr_1(m,n,r)}$ refers to the likelihood of choosing exactly the set of Borda winners themselves by considering only the first *r* rows of a preference profile, $P^r_{m,n}$. In other words $\boldsymbol{Pr_1}$ is the probability that $B_r = B_m$. For a given preference profile $P_{m,n}$ we consider the random variable

$$\boldsymbol{f_1}(P^r_{m,n}) = \begin{cases} 1, \text{ if } B(P^r_{m,n}) = B(P_{m,n}) \\ 0, \text{ otherwise.} \end{cases}$$

Then,

$$\boldsymbol{Pr_1} = \frac{1}{card(R(m,n))} \sum_{P_{m,n} \in R(m,n)} f_1(P^r_{m,n})$$

- $\boldsymbol{Pr_2} = \boldsymbol{Pr_2}(m,n,r)$ is the likelihood that an r-Borda winner (selected on the basis of the first r rows of the profile) is one of the Borda winners of the full profile. Thus it is the likelihood that an element of B_r is actually an element of B_m. For a given $P_{m,n}$, consider the random variable

$$\boldsymbol{f_2}(P^r_{m,n}) = \frac{card(B(P_{m,n}) \cap B(P^r_{m,n}))}{card(B(P^r_{m,n}))} \qquad (1)$$

Then,

$$\boldsymbol{Pr_2} = \frac{1}{card(R(m,n))} \sum_{P_{m,n} \in R(m,n)} f_2(P^r_{m,n})$$

Note that the above definition of $\boldsymbol{f_2}(P^r_{m,n})$ is intuitive: If $B(P^r_{m,n}) \subseteq B(P_{m,n})$, then $\boldsymbol{f_2}(P^r_{m,n}) = 1$, since any r-Borda winner selected on the basis of the first r ranks is automatically a Borda winner; if $B(P^r_{m,n})$ and $B(P_{m,n})$ are disjoint, then $\boldsymbol{f_2}(P^r_{m,n}) = 0$, as in this case no r-Borda winner can possibly be a Borda winner; and if $B(P^r_{m,n})$ contains, say, two elements of $B(P_{m,n})$, then $\boldsymbol{f_2}(P^r_{m,n}) = \frac{2}{card(B(P^r_{m,n}))}$, i.e. only two elements out of a total of $B(P^r_{m,n})$ would be Borda winners for the complete profile.

Given the distribution of profiles to be generated for a given m and n, and a given r, the approximate probability (of both types) is computed through Monte-Carlo integration by making use of the law of large numbers. The law of large numbers implies that the average of a random sample from a large population is likely to be close to the mean of the whole population. We apply this to the averages that define $\boldsymbol{Pr_1}$ and $\boldsymbol{Pr_2}$ through the random variables $\boldsymbol{f_1}$ and $\boldsymbol{f_2}$.

The tools provided by Eğecioğlu and Giritligil (2005) allows for the generation of roots profiles from $R(m,n)$ with probability $1/card(R(m,n))$. Then we can approximate the actual probability as follows: we generate a large number of root profiles from $R(m,n)$ with uniform probability $1/card(R(m,n))$, where each selection is done independently of the others. Let $S(m,n)$ denote the set of these generated profiles. Then the law of large numbers implies that

$$\boldsymbol{Pr_1} = \frac{1}{card(R(m,n))} \sum_{P_{m,n} \in R(m,n)} f_1(P^r_{m,n}) \approx \frac{1}{card(S(m,n))} \sum_{P_{m,n} \in S(m,n)} f_1(P^r_{m,n}) \qquad (2)$$

Hence, when only the first r-ranks of the n voters' preferences over m available alternatives are reported, the probability of implementing the Borda outcome by using only these partial preferences equals to the computed probabilities aggregated over all generated profiles in $S(m,n)$, divided by the total number generated. Note that for the formulation (2) to result in a valid Monte-Carlo algorithm for the computation of $\boldsymbol{Pr_1}$, it is essential that each $P_{m,n}$ in *S(m,n)* be drawn from the uniform probability on *R(m,n)*.

The computation of $\boldsymbol{Pr_2}$ by a Monte-Carlo method is similar. We generate a set *S(m,n)* of root profiles of *m* alternatives and *n* voters, where each profile in *S(m,n)* is generated uniformly and independently. Then

$$\boldsymbol{Pr_2} \approx \frac{1}{card(S(m,n))} \sum_{P_{m,n} \in S(m,n)} f_2(P^r_{m,n}) \qquad (3)$$

Again, for the formulation (3) to result in a valid Monte-Carlo algorithm for the computation of $\boldsymbol{Pr_2}$, each $P_{m,n}$ in *S(m,n)* must be drawn from the uniform probability on *R(m,n)*.

3.2. Examples:

Below we present a preference profile as a matrix of size m$\times$n where the voters correspond to the columns and the alternatives correspond to the rows.

Example 1: Let us consider the following profile:

a	a	b	c	d
b	b	d	d	a
c	c	c	b	b
d	d	a	a	c

The original Borda winner for the above profile is 'b', i.e. B($P_{4,5}$) = {b}. If only the first ranked alternatives are reported, then 'a' is chosen as the 1-Borda (or Plurality) winner, i.e. B($P^1_{4,5}$) = {a}. Since {a} $\neq$ {b}, $\boldsymbol{f_1}$($P^1_{4,5}$) = 0. On the other hand, $\boldsymbol{f_2}$($P^1_{4,5}$) = 0 because {a} $\cap$ {b}= $\varnothing$. If the first two ranks of the voters' preferences are reported, i.e. if we pick r = 2, then B($P^2_{4,5}$) = {a}. Since B($P_{4,5}$) = {b}, we again have $\boldsymbol{f_1}$($P^2_{4,5}$) = 0 and $\boldsymbol{f_2}$($P^2_{4,5}$) = 0.

Example 2: Now let us consider the profile below:

a	a	d	d	b
b	d	c	c	c
c	c	b	a	a
d	b	a	b	d

The set of Borda winners is B($P_{4,5}$) = {a,c,d}and the set of 1-Borda winners is B($P^1_{4,5}$) = {a,d}. Then, for this profile $\boldsymbol{f_1}$($P^1_{4,5}$) = 0 and $\boldsymbol{f_2}$($P^1_{4,5}$) = 1. For r=2, we find that B($P^2_{m,n}$) = {d}, and therefore the profile yields $\boldsymbol{f_1}$($P^2_{4,5}$) = 0 and $\boldsymbol{f_2}$($P^2_{4,5}$) = 1.

Example 3:

a	a	b	b	c
b	c	c	c	a
c	b	a	d	b
d	d	d	a	d

For the above profile, the set of Borda winners is B($P_{4,5}$) = {b,c}, and the set of 1-Borda (Plurality) winners is B($P^1_{4,5}$) = {a,b}. Then, for this profile $\boldsymbol{f_1}(P^1_{4,5}) = 0$ and $\boldsymbol{f_2}(P^1_{4,5}) = 1/2$. We find that B($P^2_{m,n}$) = {a,b,c}, and hence the profile yields $\boldsymbol{f_1}(P^2_{4,5}) = 0$ and $\boldsymbol{f_2}(P^2_{4,5}) = 2/3$.

We observe that when $\boldsymbol{f_1}(P^r_{m,n}) = 1$, then B($P^r_{m,n}$) = B($P_{m,n}$), and consequently

$$\boldsymbol{f_2}(P^r_{m,n}) = \frac{card(B(P_{m,n}) \cap B(P^r_{m,n}))}{card(B(P^r_{m,n}))} = \frac{card(B(P^r_{m,n}))}{card(B(P^r_{m,n}))} = 1.$$

On the other hand when $\boldsymbol{f_1}(P^r_{m,n}) = 0$, then B($P^r_{m,n}$) $\neq$ B($P_{m,n}$), but it is still possible that these two sets are not disjoint. In other words when $\boldsymbol{f_1}(P^r_{m,n}) = 0$, $\boldsymbol{f_2}(P^r_{m,n}) \geq 0$. It follows that $\boldsymbol{f_1}(P^r_{m,n}) \leq \boldsymbol{f_2}(P^r_{m,n})$. As a result of this observation, for any selection of *m, n, r*, we always have $\boldsymbol{Pr_1} \leq \boldsymbol{Pr_2}$.

4. MONTE-CARLO EXPERIMENTS

The IANC model as introduced by Eğecioğlu and Giritligil (2005) uses root profiles to generate voters' preferences through an application of the Dixon-Wilf algorithm. The roots can be generated from the uniform distribution for *m* alternatives and *n* voters for even large values of these parameters. At the heart of the Monte-Carlo experiments given here is the Mathematica program GenerateRoot[m,n] (the Mathematica notebook containing this function can be accessed online for experimentation: see Eğecioğlu (2004)). The program GenerateRoot[m,n] takes two integers m and n as input parameters and generates an IANC preference profile as an m$\times$ n matrix as its output. The preference profile returned each time by GenerateRoot[m,n] is guaranteed to be distributed over the *R(m,n)* roots uniformly. As we remarked before, to be able to estimate the probabilities $\boldsymbol{Pr_1}$ and $\boldsymbol{Pr_2}$ through the formulations (2) and (3) by using the law of large numbers, we need the preference profiles generated be uniform over the set of roots *R(m,n)*. GenerateRoot[m,n] does exactly that.

The design of the Monte-Carlo experiments was as follows. We have generated 1000 root profiles for each value of the parameters *m, n* under consideration. Thus we took *card(S(m,n))* = 1000. Taking larger sample sizes is a matter of computational resources available, as this particular algorithm is of the type referred to as "embarrassingly parallel". The ranges of the parameters were taken to be $1 \leq m, n \leq 30$ for most Monte-Carlo experiments carried out. Below are the basic steps followed for the computation of both $\boldsymbol{Pr_1}$ and $\boldsymbol{Pr_2}$ type probabilities in the symbolic algebra package Mathematica:

Step 1: We have generated the values of m and n themselves, $1 \leq m, n \leq 30$ iteratively by means of two nested loops.

Step 2: For the given values of m and n, we have invoked the function GenerateRoot[m,n], which returned a preference profile $P_{m,n}$ from the uniform distribution on the set of root profiles *R(m,n)*.

Step 3: We have computed the set of Borda winners B($P_{m,n}$) for the complete profile $P_{m,n}$ returned.

Step 4: In addition, for every value of r in the range $1 \leq r < m$, at this point we also computed the set of Borda winners B($P^r_{m,n}$) by considering only the first r rows of the profile $P_{m,n}$.

Step 5: Using the sets B($P_{m,n}$) and B($P^r_{m,n}$) available, we evaluated the random variables $\boldsymbol{f_1}$ and $\boldsymbol{f_2}$ corresponding to the triple m, n, r.

Step 2 through Step 5 were executed *card(S(m,n))* times, and the approximations to $\boldsymbol{Pr_1}$ and $\boldsymbol{Pr_2}$ for given m, n and r were calculated afterwards by dividing the sum of the computed values of $\boldsymbol{f_1}$ and $\boldsymbol{f_2}$ Step 5 by *card(S(m,n))*.

4.1 Experimental results on $\boldsymbol{Pr_1}$ type probabilities

Recall that $\boldsymbol{Pr_1} = \boldsymbol{Pr_1}(m,n,r)$ is the probability of choosing exactly the set of Borda winners themselves by considering only the first r rows of a preference profile. Picking r =1 results in the Plurality winners for the profile. We first determine the extent to which the Plurality winners determine the Borda winners.

4.1.1 Plurality vs. Borda winners for $\boldsymbol{Pr_1}$

Table 1 shows the $\boldsymbol{Pr_1}$ type probabilities of the set of Plurality winners to be equal to the set of Borda winners, i.e. the set of 1-Borda winners to be exactly the same as the set of m-Borda winners. The probabilities were computed for m and n both varying from 1 to 30. The columns are indexed by the number of voters n, varying from 1 to 30, and the rows are indexed by the number of alternatives m, also varying from 1 to 30. For the calculation of the entries, *card(S(m,n))* = 1000 sample profiles for each value of n and m were generated from the uniform distribution. The values given are the fraction of the number out of this *card(S(m,n))* in which the Plurality winners and the Borda winners for the profile were identical. As an example, the entry 0.331 in row 8 and column 22 in Table 1 indicates that for n = 22 voters and m = 8 alternatives, the computed probability that a random profile has identical Plurality and Borda winners is about 1/3.

Note that in Table I, for $n = 1$ and $1 \leq m \leq 30$, and for $m = 1,2$ and $1 \leq n \leq 30$, $\boldsymbol{Pr_1} = 1$. A three-dimensional plot of the computed probabilities with these boundary cases discarded is presented in Figure 1.

It is interesting that for the values of the parameters shown, $\boldsymbol{Pr_1}$ appears to be independent of n especially as the value of n increases. We observe that $\boldsymbol{Pr_1} \to 0$ as m gets large, and the behavior is roughly as $1/m$. We have made a plot of the sections of this surface across different n, $3 \leq n \leq 30$. By considering the family of functions $f(n,m) = c/m$ that are constant multiples of $1/m$, we computed the function which best approximates these sections in the least-squares sense. We have found that the probabilities are very well approximated in this range by

$$\boldsymbol{Pr_1} \sim f(n,m) = \frac{2.4}{m} \tag{4}$$

In fact, a slightly tighter approximation obtained by considering a larger class of functions is given by

$$\boldsymbol{Pr_1} \sim f(n,m) = \frac{2.5}{m} - \frac{0.5}{m^2}. \tag{5}$$

These two plots are given in Figures 2 and 3 respectively. Comparing these plots with the plot of actual value of $\boldsymbol{Pr_1}$ given in Figure 1, we see that the analytic expression in (4) does extremely well in approximating the probability of the Plurality winner and the Borda winners to be identical.

4.1.2 *r* -Borda winners vs. *m*-Borda winners for $\boldsymbol{Pr_1}$

In Table 2, we present the computed $\boldsymbol{Pr_1}$ type probabilities of the set of r-Borda winners to be exactly equal to the set of m-Borda winners. To be able to present these results in a comprehensible fashion, we have used a fixed value $n = 30$ for the tabulation given. The range of m is $1 \leq m \leq 30$. For m alternatives, the value of r changes in the range$1 \leq r \leq m$. The rows of Table 2 are indexed by m, and the columns are indexed by r. For the computation of the values of this table, we consider a special case: suppose there are m = 5 alternatives. We generated 1000 IANC profiles from the uniform distribution, and computed the fraction of the cases for which B_r and B were identical, where B_r is the set of Borda winners obtained by scoring the first r rows of the profile, and $B = B_m$ is the set of Borda winners for the profile. These are the computed probabilities listed in the table. Let B_1, B_2, B_3, B_4, B_5 denote the r-Borda winners obtained by considering the first 1, 2, 3, 4, and 5 rows respectively. Thus B_1 is the set of plurality winners, $B = B_5$ the set of Borda winners. Then

$Pr[B_1=B] = 0.51$,

$Pr[B_2=B] = 0.671$,

$Pr[B_3=B] = 0.818$,

$Pr[B_4=B] = Pr[B_5=B] = 1$.

Figure 4 is a three-dimensional plot of the computed $\boldsymbol{Pr_1}$ type probabilities. A close observation shows that for n fixed at 30, $\boldsymbol{Pr_1} \to 0$ as m gets large for fixed r, and the behavior is roughly as $(1+r)/m$. We have made a least squares fit of the sections of this surface at the values of m, $1 \leq m \leq 30$, by considering the family of functions of the form $f(m,r) = c(1+r)/m$ (c constant). The best approximating function in the least-squares sense was found to be

$$Pr_1 \sim f(m,r) = r/m + 1.4/m \qquad (6)$$

Note that for $r = 1$, the above expression reduces to $2.4/m$. This is consistent with the value obtained in section 4.1.1 for likelihood of the set of Plurality winners to be equal to the set of m-Borda winners.

Figure 5 is a three-dimensional plot of the values of the function $f(m,r) = r/m + 1.4/m$. Comparing with the plot of the actual probabilities given in Figure 4, we see that the probability that the set of r-Borda winners to be exactly equal to the set of Borda winners very well approximated by the expression for Pr_1 in (6).

4.2 Experimental results on Pr_2 type probabilities

Recall that $Pr_2 = Pr_2(m,n,r)$ is the probability that an r-Borda winner (i.e. a Borda winner selected on the basis of the first r rows) is in fact one of the Borda winners of the full profile. Thus Pr_2 is the likelihood that an element of B_r is actually an element of B_m. As in the case of the Pr_1 type probabilities, we first determine the extent to which the Plurality winners determine the Borda winners. This is followed by the experiments for the computation of Pr_2 type probability for capturing the Borda winners by r-Borda winners.

4.2.1 Plurality vs. Borda winners for Pr_2

Table 3 shows the computed probabilities of the Pr_2 type that a Plurality winner is a Borda winner in the IANC model. The columns are indexed by the number of voters n, varying from 1 to 30, and the rows are indexed by the number of alternatives m, also varying from 1 to 30. For the calculation of the entries, $card(S(m,n)) = 1000$ sample profiles for each value of n and m were generated from the uniform distribution. For each profile generated, the set of Plurality winners, and the set of Borda winners for the profile were computed. The values for Pr_2 given in the table are the average values of quantity (1) for $r = 1|$ over the 1000 profiles generated. As an example, the entry 0.75 in row 4 and column 7 in Table 3 indicates that for $n = 7$ voters and $m = 4$ alternatives, the computed probability that a random IANC profile has a Plurality winner which also happens to be a Borda winner is about 3/4.

A three-dimensional plot of the computed Pr_2 probabilities that a Plurality winner is a Borda winner is presented in Figure 6. We have again eliminated the boundary cases and plotted the values of the parameters in the range $3 \leq m \leq 30$ and $3 \leq n \leq 30$.

As in the case of the Pr_1 type probabilities, we can approximate the probabilities analytically. We compute a that a first order least-squares analytic approximation to Pr_2 is given by

$$Pr_2 \sim f(n,m) = \frac{3.1}{m} \qquad (7)$$

As expected from the inequality $Pr_1 \leq Pr_2$ discussed before, the asymptotic approximation to Pr_2 in (7) is larger than its counterpart to Pr_1 in (4). A tighter analytic approximation to the values of Pr_2 can be obtained by considering a larger class of functions. We have

$$\boldsymbol{Pr_2} \sim f(n,m) = \frac{4.2}{m} - \frac{4.9}{m^2}. \tag{8}$$

The three-dimensional plot of this latter approximation is given in Figure 7. Comparing with the plot of the actual values of $\boldsymbol{Pr_2}$ in Figure 5 , we see that in this range the expression in (8) is a good approximation to the actual values.

4.2.2 *r* -Borda winners vs. *m*-Borda winners for $\boldsymbol{Pr_2}$

In Table 4, we present the computed $\boldsymbol{Pr_2}$ type probabilities of the set of *r*-Borda winners to contain an element which is also a *m*-Borda winner. We present these results for the fixed value $n = 30$. The range of m is $1 \leq m \leq 30$. For m alternatives, the value of r changes in the range$1 \leq r \leq m$. The rows of Table 4 are indexed by *m*, and the columns are indexed by *r*. If we take $m = 5$ and let B_1, B_2, B_3, B_4, B_5 denote the *r*-Borda winners obtained by considering the first 1, 2, 3, 4, and 5 rows respectively, then we can read off the . Thus B_1 is the set of plurality winners, $B = B_5$ the set of Borda winners. Then we can read off the $\boldsymbol{Pr_2}$ type probabilities from Table 4 as

$Pr[B_1=B] = 0.607$,

$Pr[B_2=B] = 0.759$,

$Pr[B_3=B] = 0.895$,

$Pr[B_4=B] = Pr[B_5=B] = 1$.

Figure 8 is a three-dimensional plot of the computed $\boldsymbol{Pr_2}$ type probabilities. We observe that as in the case of the $\boldsymbol{Pr_1}$ type probabilities, and for n fixed at 30, $\boldsymbol{Pr_2} \to 0$ as m gets large for fixed r, and the behavior is roughly as $(1+r)/m$. We can make use of the properties of the analytic approximations for the $\boldsymbol{Pr_2}$ type probabilities and surmise that $\boldsymbol{Pr_2}$ probabilities should be well approximated by

$$\boldsymbol{Pr_2} \sim f(m,r) = r/m + 2.1/m \tag{9}$$

Indeed, the plot of this function in Figure 9 is a good approximation to the actual values in Figure 8.

5. CONCLUDING REMARKS

Voting rules pertain to democratic procedures according to which a social will is extracted out of individual wills. Each voting rule is a particular method to aggregate profiles of individual preferences into a social preference or to arrive at a choice based on such profiles. Being concerned with 'fairness' in the matter of how individual preferences should be mapped into a social choice, most of the voting rules of modern era are defined on the complete information in these profiles concerning the individual preferences of the voters over the set of available alternatives. The original contributions by Borda and Condorcet, and most of the Condorcet-consistent and positional rules are such methods. However, the practical complications associated with the requirement of complete information block the all these theoretical contributions on their way to be used in the real world. The fact that, despite its well-known insufficiencies in terms of 'fairness', plurality voting which aggregates the first-best choices of the electorate, is still by far the most widely used method is an indication of this problem.

This study is an attempt to empirically investigate the value of information revealed in the partial preference rankings from the perspective of predicting the Borda outcome. For the voters' preferences generated through IANC model, Monte-Carlo experiments are run to detect the likelihood of choosing the Borda winner when only the first *r* ranks of the linear preference rankings of n voters over m alternatives are reported. The computed values of this likelihood are presented as tables for varying triples of *m, n* and *r*. In order to detect the change in this likelihood as a function of m, n and r, least squares fit method is adopted. Through this method, it is shown that for fixed *n* and *r* within the computed range of parameter values, the likelihood of choosing the Borda winner approaches zero as *m* increases.

As an immediate direction for future research, empirical studies can be designed to investigate the likelihood of choosing the winner of the other well-known rules, with partial preference rankings of the electorate. However, in the case of pair-wise majority rules, it should be noted that the method to be used for aggregating partial preference rankings through such a rule (i.e. Condorcet's rule) is not as straightforward or intuitive as it is through a scoring method, such as a lá Borda. The fact that the lower-counter set cardinalities of the alternatives does not constitute a sufficient information for determining the winner of such a rule calls for the design of rules of a pair-wise majority fashion for aggregating the partial preference rankings.

As another and more general direction for future research, more studies should be run to guide the theoretical inputs to find their way into the real world. The mature theoretical literature on social choice should be accompanied by empirical studies, not only to analyze the properties and behaviors of voting rules, but also to address to the nature and the extent of problems associated with the applications of the theory through using sophisticated Monte-Carlo models and algorithms to generate voters' preferences.

REFERENCES

Borda, J.C. de., 1781. Mémoire sur les Élections au Scrutin. *Histoire de l'Académie Royale des Sciences*, Paris.

Condorcet, M. de., 1785. Essai sur l'Application de l'Analyse à la Probabilité des Décisions Rendues à la Pluralité des Voix, Paris.

Eğecioğlu Ö., 2004. "GenerateRoot[m,n]", Mathematica function: available online as a Mathematica notebook http://www.cs.ucsb.edu/~omer/randomvote.nb.

Eğecioğlu Ö. and A. E. Giritligil, 2005. "Public Preference Structures with Impartial Anonymous and Neutral Culture Model", mimeo.

May, K., 1952. "A Set of Independent Necessary and Sufficient Conditions for Simple Majority Decision", *Econometrica* **20**:680-684.

1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.
1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.
1.	0.602	0.788	0.703	0.706	0.762	0.762	0.71	0.768	0.746	0.749	0.781	0.755	0.771	0.775	0.771	0.774	0.781	0.796	0.775	0.796	0.769	0.797	0.813	0.804	0.785	0.79	0.812	0.791	0.799
1.	0.456	0.559	0.586	0.531	0.567	0.598	0.572	0.595	0.578	0.591	0.601	0.622	0.584	0.591	0.617	0.596	0.616	0.623	0.585	0.633	0.58	0.573	0.614	0.624	0.607	0.619	0.615	0.617	0.595
1.	0.353	0.446	0.487	0.469	0.408	0.418	0.473	0.437	0.434	0.457	0.469	0.464	0.468	0.46	0.479	0.452	0.474	0.507	0.491	0.462	0.511	0.496	0.468	0.502	0.491	0.512	0.48	0.46	0.51
1.	0.263	0.369	0.432	0.395	0.354	0.354	0.366	0.428	0.382	0.367	0.385	0.382	0.391	0.42	0.376	0.393	0.405	0.387	0.412	0.399	0.398	0.407	0.415	0.401	0.405	0.419	0.397	0.44	0.444
1.	0.223	0.287	0.35	0.36	0.342	0.338	0.343	0.359	0.346	0.359	0.36	0.348	0.329	0.341	0.37	0.338	0.367	0.338	0.358	0.347	0.366	0.347	0.336	0.367	0.376	0.354	0.343	0.362	0.363
1.	0.184	0.248	0.316	0.336	0.327	0.266	0.297	0.314	0.318	0.297	0.31	0.322	0.287	0.293	0.32	0.31	0.327	0.312	0.319	0.32	0.331	0.301	0.304	0.312	0.304	0.316	0.303	0.321	0.344
1.	0.181	0.232	0.268	0.296	0.27	0.239	0.236	0.238	0.311	0.266	0.298	0.285	0.294	0.252	0.27	0.277	0.278	0.269	0.299	0.271	0.271	0.289	0.293	0.286	0.29	0.297	0.295	0.282	0.274
1.	0.146	0.184	0.22	0.28	0.271	0.25	0.252	0.241	0.246	0.264	0.259	0.272	0.242	0.234	0.236	0.25	0.241	0.252	0.258	0.255	0.268	0.24	0.248	0.247	0.272	0.248	0.25	0.257	0.259
1.	0.115	0.171	0.239	0.252	0.249	0.226	0.202	0.189	0.22	0.223	0.224	0.227	0.27	0.209	0.243	0.203	0.237	0.232	0.238	0.233	0.213	0.244	0.23	0.248	0.227	0.253	0.259	0.241	0.226
1.	0.12	0.144	0.181	0.218	0.236	0.202	0.216	0.195	0.178	0.221	0.214	0.227	0.225	0.195	0.228	0.208	0.215	0.208	0.216	0.204	0.213	0.242	0.209	0.205	0.202	0.229	0.208	0.249	0.202
1.	0.104	0.139	0.179	0.226	0.233	0.202	0.184	0.178	0.193	0.185	0.201	0.216	0.185	0.171	0.207	0.186	0.197	0.185	0.174	0.204	0.194	0.198	0.223	0.19	0.218	0.194	0.182	0.184	0.198
1.	0.103	0.139	0.175	0.191	0.226	0.207	0.179	0.155	0.15	0.171	0.186	0.162	0.179	0.172	0.171	0.17	0.187	0.178	0.178	0.187	0.185	0.2	0.176	0.196	0.173	0.169	0.202	0.175	0.208
1.	0.078	0.13	0.151	0.204	0.186	0.177	0.168	0.158	0.16	0.147	0.153	0.162	0.172	0.157	0.17	0.181	0.183	0.162	0.181	0.166	0.158	0.185	0.172	0.188	0.176	0.198	0.16	0.169	0.184
1.	0.06	0.109	0.153	0.166	0.199	0.164	0.149	0.148	0.139	0.157	0.167	0.165	0.158	0.143	0.155	0.165	0.151	0.167	0.148	0.159	0.145	0.168	0.154	0.187	0.174	0.158	0.153	0.145	0.16
1.	0.07	0.099	0.148	0.157	0.167	0.162	0.16	0.138	0.134	0.126	0.129	0.151	0.132	0.155	0.13	0.162	0.164	0.153	0.131	0.142	0.143	0.156	0.153	0.149	0.153	0.149	0.147	0.146	0.175
1.	0.062	0.097	0.121	0.16	0.159	0.155	0.151	0.141	0.111	0.128	0.122	0.133	0.137	0.155	0.145	0.153	0.143	0.129	0.141	0.151	0.138	0.136	0.136	0.122	0.145	0.153	0.155	0.168	0.131
1.	0.067	0.082	0.117	0.115	0.168	0.15	0.136	0.117	0.131	0.127	0.115	0.111	0.14	0.146	0.124	0.153	0.105	0.143	0.137	0.141	0.118	0.124	0.125	0.128	0.128	0.146	0.145	0.124	0.133
1.	0.057	0.071	0.109	0.127	0.142	0.151	0.155	0.129	0.13	0.11	0.108	0.115	0.106	0.109	0.12	0.116	0.125	0.133	0.122	0.124	0.112	0.132	0.114	0.12	0.118	0.106	0.125	0.12	0.117
1.	0.059	0.068	0.1	0.121	0.133	0.147	0.132	0.131	0.132	0.108	0.1	0.094	0.106	0.121	0.102	0.114	0.13	0.116	0.106	0.121	0.107	0.122	0.126	0.125	0.113	0.109	0.106	0.123	0.124
1.	0.042	0.081	0.104	0.11	0.123	0.124	0.128	0.111	0.115	0.08	0.106	0.096	0.098	0.116	0.126	0.104	0.118	0.113	0.114	0.119	0.11	0.129	0.103	0.093	0.127	0.111	0.105	0.105	0.116
1.	0.048	0.076	0.084	0.124	0.129	0.129	0.138	0.109	0.108	0.094	0.111	0.087	0.099	0.107	0.115	0.117	0.119	0.108	0.101	0.104	0.097	0.114	0.125	0.107	0.1	0.112	0.114	0.09	0.089
1.	0.044	0.068	0.106	0.105	0.1	0.137	0.117	0.117	0.091	0.098	0.098	0.095	0.106	0.102	0.104	0.101	0.12	0.112	0.111	0.103	0.106	0.108	0.099	0.093	0.11	0.099	0.131	0.112	0.096
1.	0.047	0.058	0.092	0.099	0.119	0.111	0.12	0.116	0.12	0.116	0.086	0.084	0.078	0.089	0.106	0.101	0.101	0.081	0.094	0.087	0.073	0.091	0.103	0.092	0.099	0.092	0.1	0.091	0.107
1.	0.054	0.055	0.093	0.086	0.123	0.109	0.129	0.081	0.097	0.089	0.085	0.076	0.094	0.069	0.077	0.079	0.098	0.091	0.097	0.108	0.098	0.106	0.095	0.088	0.078	0.087	0.1	0.103	0.079
1.	0.054	0.06	0.075	0.113	0.096	0.107	0.109	0.098	0.096	0.077	0.083	0.075	0.089	0.085	0.085	0.104	0.1	0.09	0.091	0.091	0.104	0.092	0.09	0.096	0.076	0.097	0.101	0.086	0.086
1.	0.055	0.04	0.063	0.094	0.124	0.092	0.099	0.1	0.109	0.065	0.094	0.07	0.09	0.087	0.093	0.078	0.082	0.103	0.102	0.111	0.089	0.083	0.091	0.061	0.077	0.083	0.086	0.086	0.078
1.	0.049	0.048	0.072	0.072	0.109	0.087	0.108	0.092	0.093	0.09	0.084	0.094	0.065	0.065	0.08	0.062	0.089	0.094	0.099	0.077	0.085	0.077	0.075	0.096	0.056	0.083	0.086	0.078	0.076
1.	0.041	0.052	0.077	0.088	0.099	0.101	0.101	0.092	0.088	0.086	0.098	0.071	0.085	0.065	0.076	0.071	0.092	0.085	0.089	0.077	0.085	0.078	0.075	0.086	0.103	0.081	0.083	0.082	0.07

Table 1: For $1 \leq m \leq 30$ and $1 \leq n \leq 30$, the computed probabilities that the set of Plurality winners and the set of Borda winners are equal.

1.																													
1.	1.																												
0.799	1.	1.																											
0.595	0.8	1.	1.																										
0.51	0.671	0.818	1.	1.																									
0.444	0.582	0.704	0.832	1.	1.																								
0.363	0.521	0.622	0.738	0.843	1.	1.																							
0.344	0.478	0.579	0.691	0.775	0.873	1.	1.																						
0.274	0.404	0.512	0.603	0.7	0.79	0.878	1.	1.																					
0.259	0.38	0.471	0.576	0.664	0.741	0.823	0.909	1.	1.																				
0.226	0.362	0.459	0.548	0.629	0.695	0.774	0.838	0.91	1.	1.																			
0.202	0.308	0.386	0.489	0.558	0.629	0.701	0.772	0.849	0.904	1.	1.																		
0.198	0.312	0.387	0.461	0.531	0.598	0.661	0.721	0.786	0.845	0.925	1.	1.																	
0.208	0.296	0.358	0.44	0.512	0.581	0.632	0.689	0.769	0.824	0.885	0.93	1.	1.																
0.184	0.269	0.352	0.413	0.476	0.546	0.606	0.671	0.734	0.782	0.821	0.867	0.922	1.	1.															
0.16	0.245	0.338	0.387	0.437	0.498	0.572	0.626	0.701	0.737	0.794	0.861	0.903	0.954	1.	1.														
0.175	0.259	0.33	0.387	0.443	0.498	0.551	0.599	0.643	0.691	0.752	0.805	0.851	0.893	0.946	1.	1.													
0.131	0.224	0.286	0.334	0.394	0.449	0.504	0.559	0.617	0.671	0.716	0.762	0.822	0.866	0.901	0.943	1.	1.												
0.133	0.209	0.271	0.332	0.389	0.449	0.51	0.551	0.607	0.655	0.709	0.754	0.791	0.837	0.878	0.914	0.949	1.	1.											
0.117	0.204	0.259	0.326	0.388	0.436	0.475	0.513	0.556	0.613	0.648	0.7	0.748	0.785	0.832	0.875	0.911	0.956	1.	1.										
0.124	0.198	0.259	0.309	0.365	0.416	0.456	0.5	0.543	0.596	0.631	0.662	0.714	0.754	0.801	0.838	0.873	0.914	0.955	1.	1.									
0.116	0.188	0.247	0.298	0.332	0.378	0.435	0.471	0.508	0.562	0.604	0.655	0.691	0.73	0.77	0.809	0.85	0.891	0.922	0.955	1.	1.								
0.089	0.17	0.222	0.269	0.319	0.361	0.39	0.448	0.495	0.532	0.573	0.611	0.657	0.709	0.753	0.779	0.821	0.872	0.904	0.93	0.959	1.	1.							
0.096	0.181	0.211	0.251	0.302	0.341	0.387	0.437	0.48	0.517	0.554	0.601	0.644	0.688	0.733	0.77	0.804	0.85	0.883	0.912	0.934	0.969	1.	1.						
0.107	0.17	0.216	0.263	0.302	0.329	0.38	0.422	0.459	0.506	0.549	0.596	0.635	0.669	0.704	0.738	0.771	0.811	0.835	0.874	0.908	0.94	0.969	1.	1.					
0.079	0.138	0.183	0.22	0.271	0.305	0.346	0.38	0.423	0.454	0.497	0.543	0.573	0.602	0.635	0.676	0.715	0.756	0.796	0.83	0.866	0.892	0.925	0.962	1.	1.				
0.086	0.159	0.203	0.248	0.283	0.312	0.353	0.383	0.419	0.455	0.497	0.534	0.566	0.598	0.63	0.663	0.703	0.746	0.781	0.804	0.853	0.886	0.911	0.936	0.964	1.	1.			
0.078	0.141	0.186	0.232	0.274	0.299	0.337	0.369	0.402	0.438	0.465	0.509	0.543	0.574	0.609	0.644	0.675	0.706	0.733	0.76	0.799	0.838	0.88	0.91	0.947	0.97	1.	1.		
0.076	0.145	0.182	0.229	0.27	0.303	0.339	0.383	0.416	0.444	0.468	0.507	0.54	0.583	0.603	0.633	0.672	0.694	0.726	0.758	0.795	0.818	0.851	0.881	0.909	0.943	0.969	1.	1.	
0.07	0.124	0.173	0.214	0.246	0.291	0.319	0.34	0.374	0.418	0.451	0.491	0.518	0.547	0.572	0.619	0.648	0.69	0.725	0.765	0.799	0.83	0.849	0.878	0.903	0.917	0.941	0.971	1.	1.

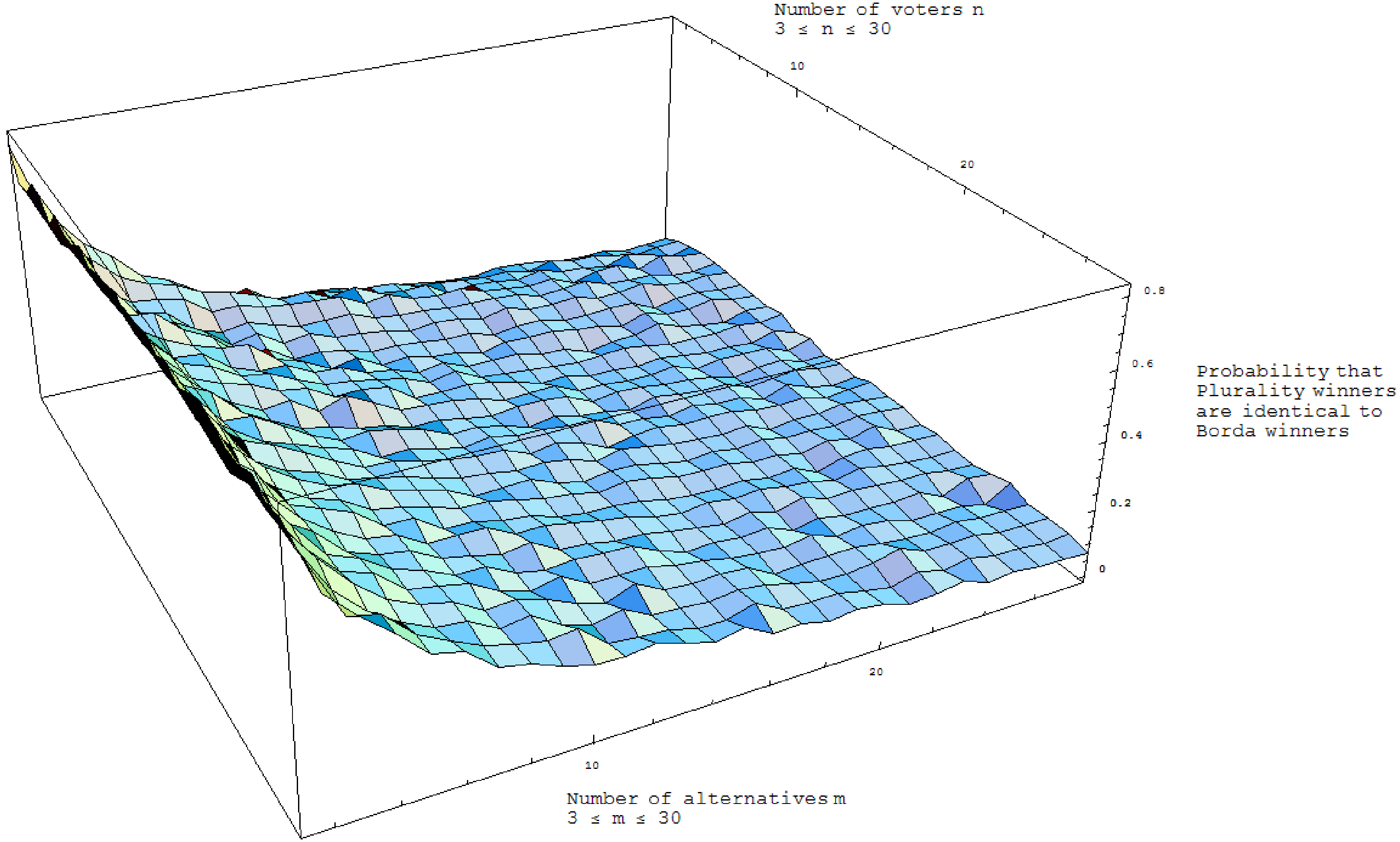
Table 2: For 1≤ m≤ 30, n = 30 and 1≤ r≤ m, the behavior of the probability that the set of r-Borda winners is equal to the set of actual Borda winners.

1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.
1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.
1.	0.893	0.927	0.896	0.876	0.887	0.89	0.854	0.876	0.86	0.854	0.865	0.844	0.86	0.85	0.855	0.857	0.852	0.861	0.843	0.847	0.824	0.846	0.872	0.846	0.84	0.844	0.857	0.841	0.844
1.	0.758	0.773	0.77	0.74	0.75	0.75	0.736	0.739	0.726	0.726	0.731	0.739	0.706	0.71	0.73	0.711	0.717	0.721	0.701	0.712	0.683	0.675	0.691	0.711	0.695	0.708	0.698	0.707	0.685
1.	0.648	0.675	0.668	0.673	0.618	0.608	0.635	0.596	0.592	0.617	0.612	0.611	0.618	0.598	0.608	0.588	0.596	0.622	0.602	0.576	0.623	0.595	0.58	0.597	0.597	0.615	0.577	0.552	0.607
1.	0.568	0.594	0.61	0.579	0.559	0.552	0.527	0.565	0.537	0.532	0.535	0.521	0.521	0.562	0.511	0.521	0.519	0.515	0.528	0.515	0.523	0.519	0.525	0.514	0.514	0.522	0.504	0.534	0.537
1.	0.528	0.515	0.516	0.519	0.524	0.501	0.505	0.502	0.473	0.496	0.503	0.485	0.473	0.482	0.489	0.472	0.481	0.459	0.473	0.462	0.479	0.476	0.441	0.47	0.48	0.452	0.456	0.462	0.474
1.	0.49	0.482	0.474	0.48	0.475	0.439	0.451	0.462	0.452	0.437	0.441	0.447	0.424	0.418	0.426	0.424	0.435	0.428	0.436	0.424	0.436	0.415	0.411	0.412	0.408	0.406	0.403	0.416	0.429
1.	0.464	0.451	0.42	0.435	0.419	0.405	0.396	0.389	0.438	0.398	0.418	0.408	0.417	0.377	0.399	0.388	0.379	0.39	0.402	0.37	0.39	0.389	0.387	0.377	0.373	0.386	0.398	0.372	0.371
1.	0.431	0.398	0.387	0.412	0.404	0.395	0.41	0.396	0.381	0.39	0.378	0.381	0.353	0.342	0.352	0.362	0.351	0.351	0.349	0.339	0.359	0.329	0.338	0.354	0.369	0.351	0.33	0.335	0.352
1.	0.404	0.37	0.38	0.369	0.366	0.348	0.339	0.336	0.337	0.339	0.348	0.328	0.382	0.332	0.348	0.298	0.333	0.334	0.338	0.33	0.296	0.325	0.313	0.331	0.318	0.333	0.334	0.312	0.313
1.	0.397	0.336	0.32	0.336	0.348	0.333	0.332	0.335	0.317	0.341	0.32	0.331	0.315	0.302	0.326	0.304	0.326	0.305	0.302	0.295	0.306	0.333	0.282	0.281	0.286	0.31	0.295	0.325	0.285
1.	0.375	0.334	0.319	0.329	0.346	0.324	0.31	0.307	0.317	0.306	0.312	0.312	0.286	0.257	0.303	0.292	0.291	0.275	0.26	0.286	0.283	0.282	0.301	0.266	0.289	0.277	0.272	0.268	0.278
1.	0.354	0.323	0.317	0.297	0.334	0.317	0.292	0.276	0.259	0.277	0.286	0.251	0.263	0.255	0.266	0.257	0.288	0.268	0.266	0.277	0.274	0.271	0.264	0.267	0.253	0.251	0.278	0.262	0.278
1.	0.333	0.309	0.287	0.303	0.294	0.27	0.27	0.271	0.28	0.255	0.259	0.248	0.269	0.245	0.256	0.267	0.267	0.249	0.276	0.244	0.237	0.263	0.257	0.259	0.25	0.274	0.232	0.242	0.252
1.	0.324	0.282	0.28	0.272	0.286	0.258	0.245	0.246	0.239	0.265	0.263	0.25	0.25	0.223	0.24	0.245	0.239	0.254	0.244	0.242	0.224	0.246	0.233	0.252	0.245	0.231	0.232	0.227	0.225
1.	0.311	0.261	0.265	0.256	0.257	0.25	0.258	0.235	0.237	0.231	0.224	0.243	0.22	0.231	0.225	0.232	0.237	0.226	0.21	0.218	0.22	0.233	0.234	0.218	0.221	0.215	0.217	0.214	0.239
1.	0.31	0.265	0.231	0.252	0.238	0.244	0.236	0.243	0.206	0.222	0.218	0.213	0.218	0.24	0.219	0.223	0.223	0.205	0.221	0.224	0.219	0.211	0.214	0.198	0.206	0.222	0.219	0.226	0.187
1.	0.31	0.243	0.235	0.213	0.242	0.227	0.219	0.203	0.222	0.232	0.209	0.206	0.227	0.216	0.202	0.225	0.171	0.213	0.207	0.216	0.182	0.181	0.189	0.189	0.186	0.211	0.202	0.192	0.196
1.	0.28	0.225	0.23	0.214	0.215	0.227	0.24	0.209	0.215	0.201	0.194	0.202	0.188	0.181	0.191	0.182	0.192	0.194	0.19	0.212	0.18	0.202	0.186	0.187	0.188	0.171	0.18	0.182	0.18
1.	0.278	0.219	0.212	0.215	0.202	0.224	0.208	0.218	0.22	0.197	0.182	0.185	0.186	0.193	0.17	0.185	0.198	0.186	0.173	0.191	0.178	0.187	0.181	0.195	0.185	0.163	0.178	0.181	0.192
1.	0.27	0.231	0.214	0.192	0.19	0.19	0.206	0.194	0.188	0.17	0.194	0.185	0.17	0.193	0.186	0.172	0.19	0.184	0.178	0.188	0.178	0.181	0.175	0.154	0.187	0.177	0.164	0.168	0.173
1.	0.267	0.233	0.191	0.212	0.203	0.19	0.206	0.18	0.192	0.178	0.184	0.164	0.178	0.176	0.185	0.195	0.177	0.175	0.158	0.157	0.159	0.181	0.189	0.166	0.162	0.176	0.175	0.148	0.151
1.	0.257	0.209	0.208	0.191	0.173	0.194	0.181	0.19	0.176	0.174	0.179	0.177	0.178	0.169	0.176	0.163	0.193	0.175	0.173	0.157	0.162	0.161	0.167	0.152	0.164	0.151	0.193	0.16	0.154
1.	0.266	0.187	0.198	0.18	0.183	0.18	0.183	0.188	0.188	0.19	0.152	0.156	0.155	0.161	0.173	0.163	0.162	0.135	0.151	0.144	0.136	0.142	0.158	0.145	0.158	0.151	0.156	0.145	0.154
1.	0.258	0.193	0.187	0.161	0.199	0.167	0.19	0.148	0.17	0.162	0.157	0.158	0.173	0.136	0.141	0.14	0.157	0.152	0.146	0.159	0.154	0.164	0.146	0.148	0.133	0.139	0.154	0.162	0.127
1.	0.27	0.185	0.173	0.186	0.163	0.179	0.161	0.168	0.168	0.151	0.154	0.142	0.16	0.148	0.147	0.164	0.156	0.144	0.154	0.152	0.161	0.142	0.139	0.146	0.132	0.147	0.157	0.14	0.135
1.	0.252	0.169	0.169	0.176	0.184	0.153	0.15	0.166	0.174	0.139	0.166	0.141	0.162	0.149	0.156	0.138	0.137	0.163	0.157	0.163	0.137	0.137	0.145	0.119	0.135	0.137	0.129	0.14	0.129
1.	0.243	0.171	0.164	0.147	0.168	0.148	0.162	0.149	0.151	0.158	0.151	0.173	0.131	0.136	0.14	0.133	0.151	0.148	0.152	0.128	0.141	0.125	0.128	0.139	0.108	0.135	0.132	0.131	0.132
1.	0.246	0.179	0.174	0.155	0.159	0.162	0.157	0.155	0.158	0.148	0.166	0.138	0.156	0.129	0.139	0.126	0.145	0.137	0.134	0.129	0.137	0.126	0.117	0.134	0.155	0.131	0.133	0.133	0.127

Table 3: For $1 \leq m \leq 30$ and $1 \leq n \leq 30$, the computed probabilities that a Plurality winner is also a Borda winner.

1.																													
1.	1.																												
0.844	1.	1.																											
0.685	0.87	1.	1.																										
0.607	0.759	0.895	1.	1.																									
0.537	0.663	0.788	0.907	1.	1.																								
0.474	0.618	0.706	0.827	0.919	1.	1.																							
0.429	0.545	0.643	0.754	0.837	0.928	1.	1.																						
0.371	0.475	0.582	0.666	0.761	0.849	0.936	1.	1.																					
0.352	0.444	0.533	0.63	0.715	0.796	0.877	0.954	1.	1.																				
0.313	0.428	0.519	0.602	0.681	0.754	0.827	0.893	0.96	1.	1.																			
0.285	0.362	0.44	0.54	0.604	0.678	0.749	0.82	0.893	0.954	1.	1.																		
0.278	0.372	0.445	0.512	0.579	0.647	0.712	0.774	0.84	0.906	0.971	1.	1.																	
0.278	0.36	0.422	0.489	0.563	0.626	0.678	0.743	0.812	0.871	0.932	0.971	1.	1.																
0.252	0.318	0.395	0.462	0.522	0.592	0.655	0.716	0.773	0.82	0.863	0.913	0.967	1.	1.															
0.225	0.301	0.386	0.437	0.483	0.541	0.616	0.676	0.74	0.784	0.839	0.904	0.944	0.982	1.	1.														
0.239	0.301	0.37	0.418	0.476	0.529	0.582	0.631	0.681	0.73	0.785	0.842	0.891	0.931	0.976	1.	1.													
0.187	0.264	0.328	0.371	0.428	0.483	0.535	0.586	0.644	0.698	0.745	0.794	0.854	0.898	0.928	0.97	1.	1.												
0.196	0.257	0.309	0.365	0.423	0.48	0.542	0.584	0.635	0.684	0.736	0.779	0.822	0.875	0.909	0.946	0.979	1.	1.											
0.18	0.247	0.299	0.36	0.42	0.469	0.506	0.547	0.588	0.639	0.679	0.728	0.776	0.819	0.862	0.908	0.947	0.983	1.	1.										
0.192	0.25	0.303	0.355	0.406	0.446	0.485	0.529	0.576	0.627	0.669	0.699	0.75	0.797	0.839	0.88	0.914	0.954	0.984	1.	1.									
0.173	0.231	0.281	0.323	0.36	0.409	0.465	0.503	0.541	0.591	0.634	0.683	0.724	0.766	0.807	0.844	0.883	0.928	0.956	0.984	1.	1.								
0.151	0.211	0.259	0.302	0.347	0.385	0.423	0.475	0.522	0.558	0.601	0.645	0.687	0.745	0.782	0.813	0.857	0.908	0.934	0.965	0.987	1.	1.							
0.154	0.222	0.252	0.291	0.334	0.371	0.424	0.47	0.51	0.551	0.593	0.635	0.681	0.719	0.765	0.802	0.838	0.883	0.915	0.942	0.964	0.99	1.	1.						
0.154	0.212	0.247	0.293	0.325	0.354	0.4	0.446	0.48	0.531	0.573	0.617	0.66	0.692	0.728	0.76	0.797	0.837	0.869	0.902	0.938	0.966	0.989	1.	1.					
0.127	0.172	0.213	0.256	0.301	0.333	0.377	0.409	0.447	0.486	0.53	0.576	0.604	0.64	0.673	0.714	0.751	0.793	0.837	0.87	0.902	0.93	0.962	0.988	1.	1.				
0.135	0.195	0.233	0.271	0.305	0.338	0.38	0.408	0.45	0.484	0.523	0.557	0.593	0.623	0.656	0.692	0.735	0.774	0.81	0.838	0.886	0.917	0.94	0.967	0.989	1.	1.			
0.129	0.179	0.22	0.261	0.295	0.32	0.362	0.393	0.423	0.459	0.49	0.534	0.57	0.602	0.631	0.667	0.704	0.73	0.758	0.792	0.829	0.868	0.911	0.94	0.97	0.99	1.	1.		
0.132	0.184	0.219	0.259	0.298	0.335	0.371	0.411	0.443	0.472	0.499	0.539	0.579	0.616	0.638	0.668	0.7	0.727	0.765	0.796	0.832	0.858	0.89	0.918	0.946	0.972	0.99	1.	1.	
0.127	0.168	0.207	0.251	0.279	0.322	0.349	0.368	0.406	0.448	0.481	0.52	0.549	0.576	0.603	0.648	0.678	0.724	0.758	0.795	0.832	0.861	0.882	0.914	0.936	0.952	0.972	0.992	1.	1.

Table 4: For 1 ≤ m ≤ 30, n = 30 and 1 ≤ r ≤ m, the behavior of the probability of the set of r-Borda winners to be a sub-set of the set of m-Borda winners.



Number of voters n
3 ≤ n ≤ 30
10
20
0.8
0.6
0.4
0.2
0
Probability that
Plurality winners
are identical to
Borda winners
20
10
Number of alternatives m
3 ≤ m ≤ 30

Figure 1

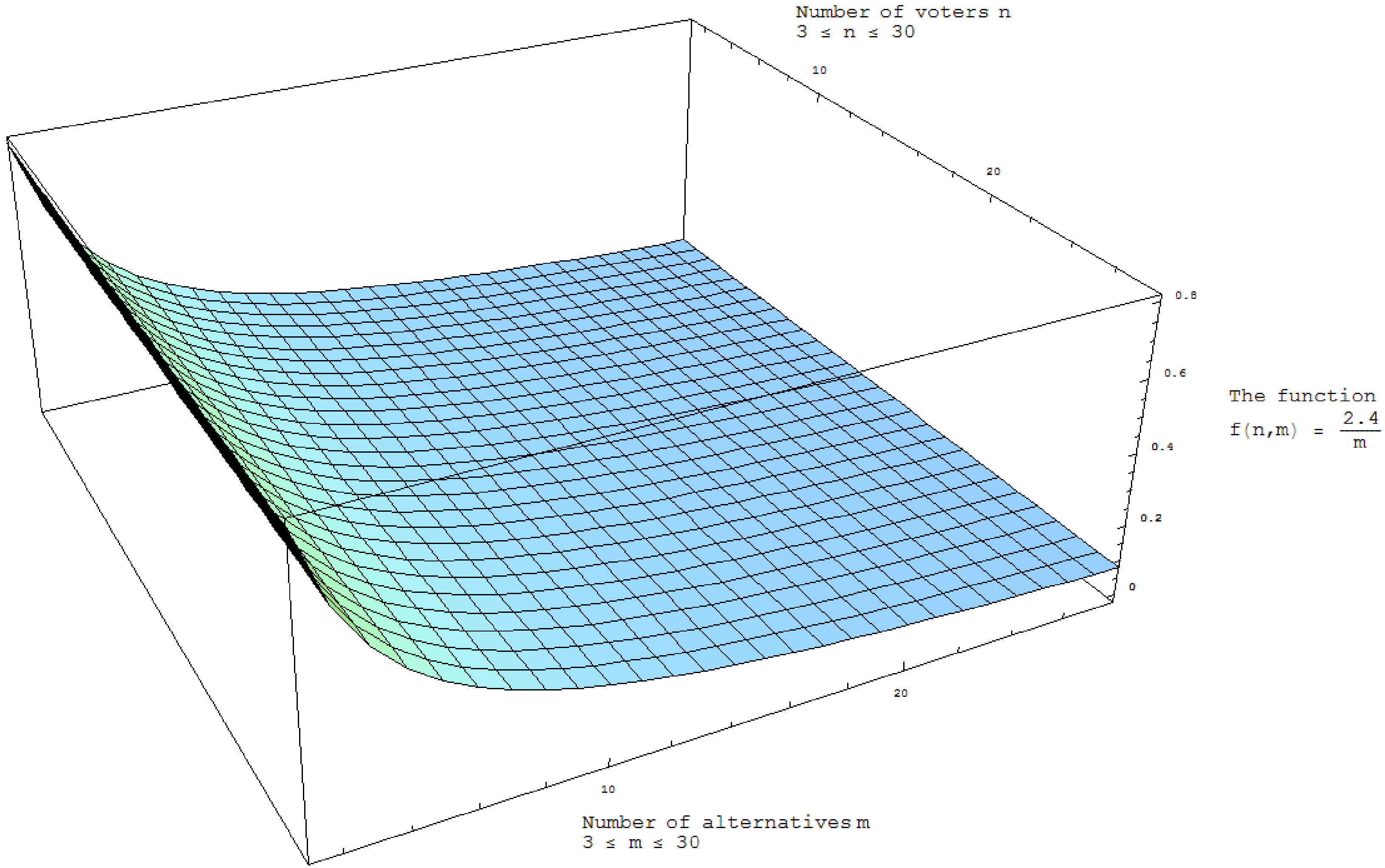

Number of voters n
3 ≤ n ≤ 30
10
20
0.8
0.6
0.4
0.2
0
The function
$f(n,m) = \frac{2.4}{m}$
20
10
Number of alternatives m
3 ≤ m ≤ 30

Figure 2

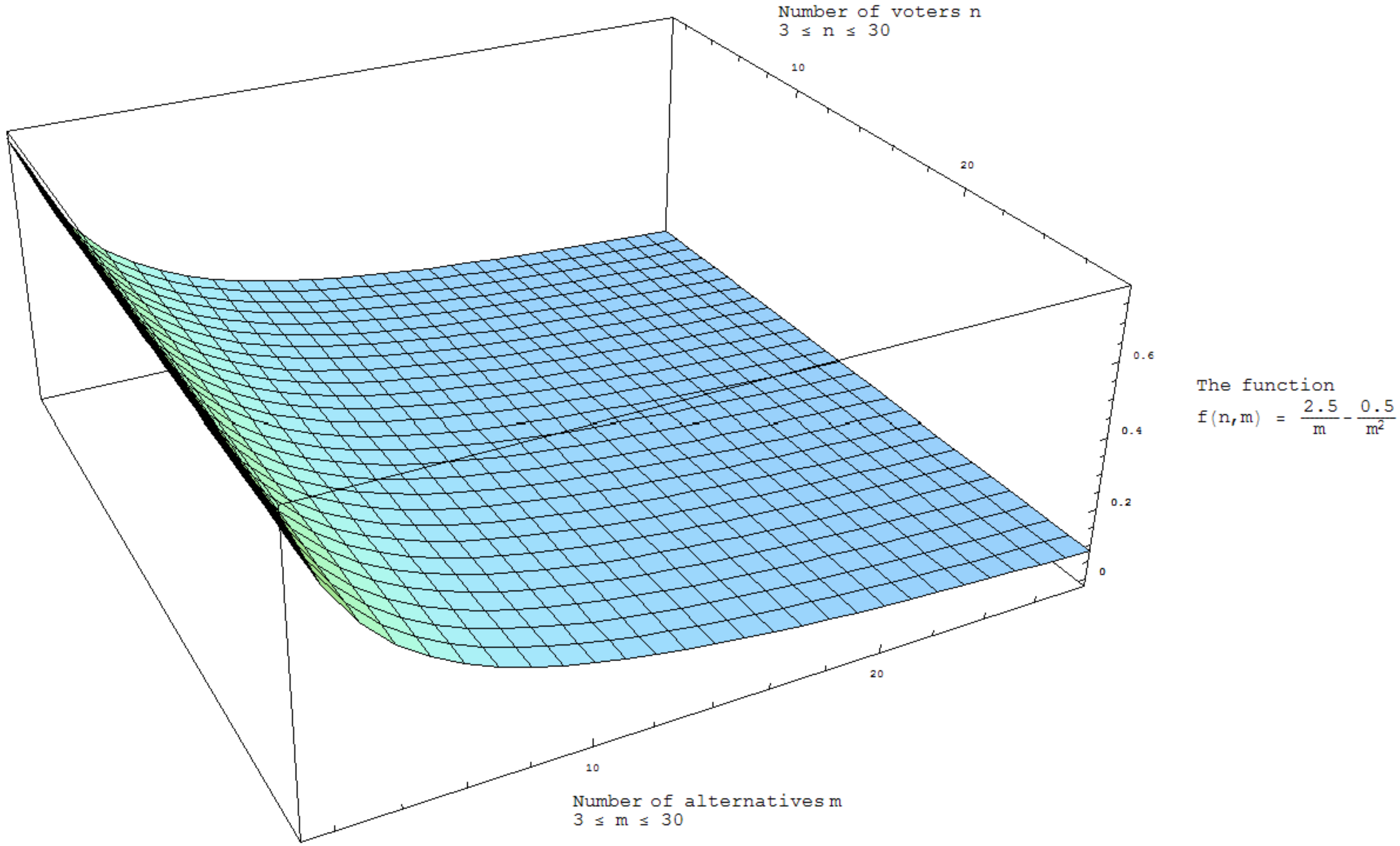

Number of voters n
3 ≤ n ≤ 30
10
20
0.6
0.4
0.2
0
The function
$f(n,m) = \frac{2.5}{m} - \frac{0.5}{m^2}$
20
10
Number of alternatives m
3 ≤ m ≤ 30

Figure 3

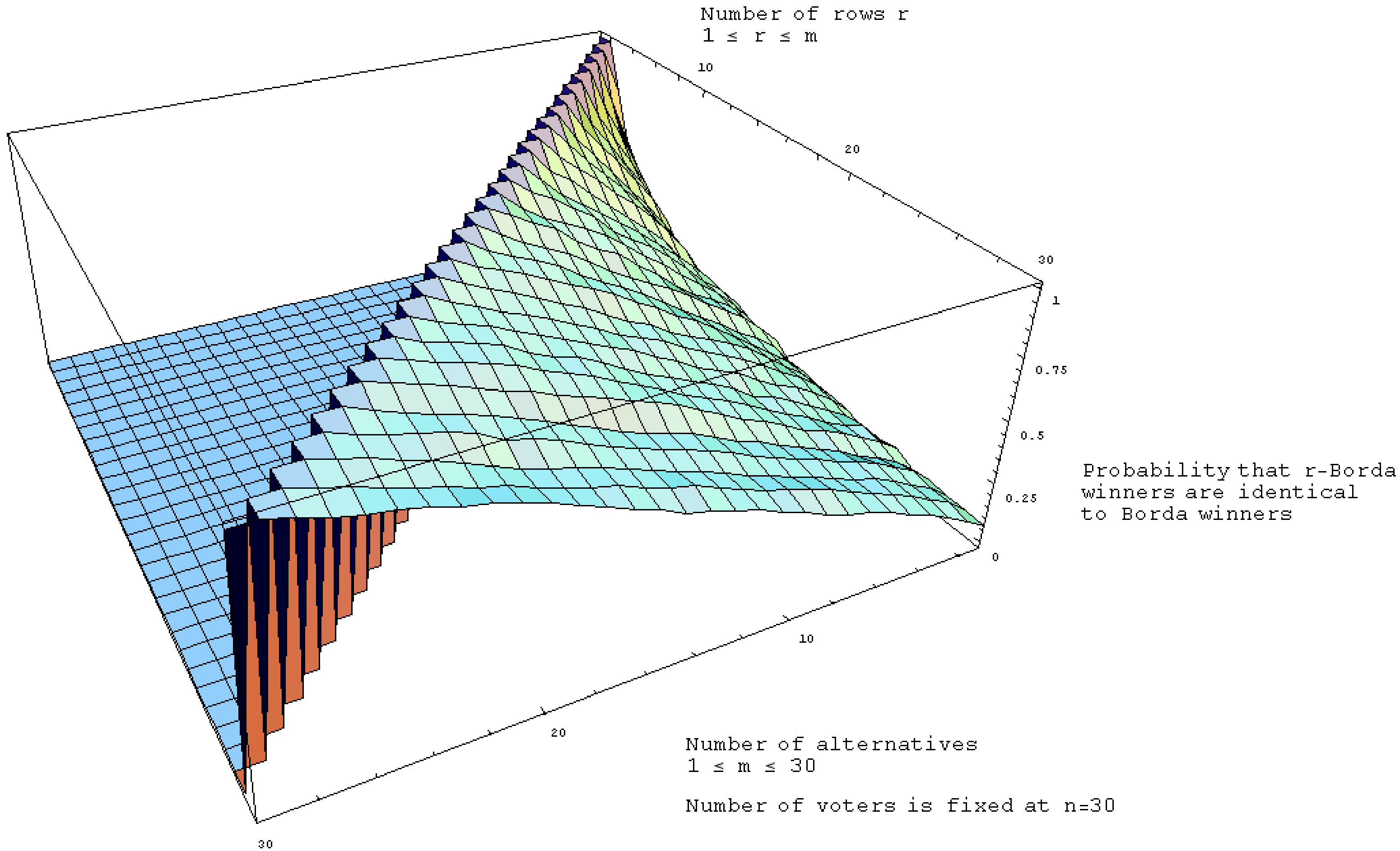

Number of rows r
1 ≤ r ≤ m
10
20
30
1
0.75
0.5
0.25
0
Probability that r-Borda
winners are identical
to Borda winners
10
20
30
Number of alternatives
1 ≤ m ≤ 30
Number of voters is fixed at n=30

Figure 4

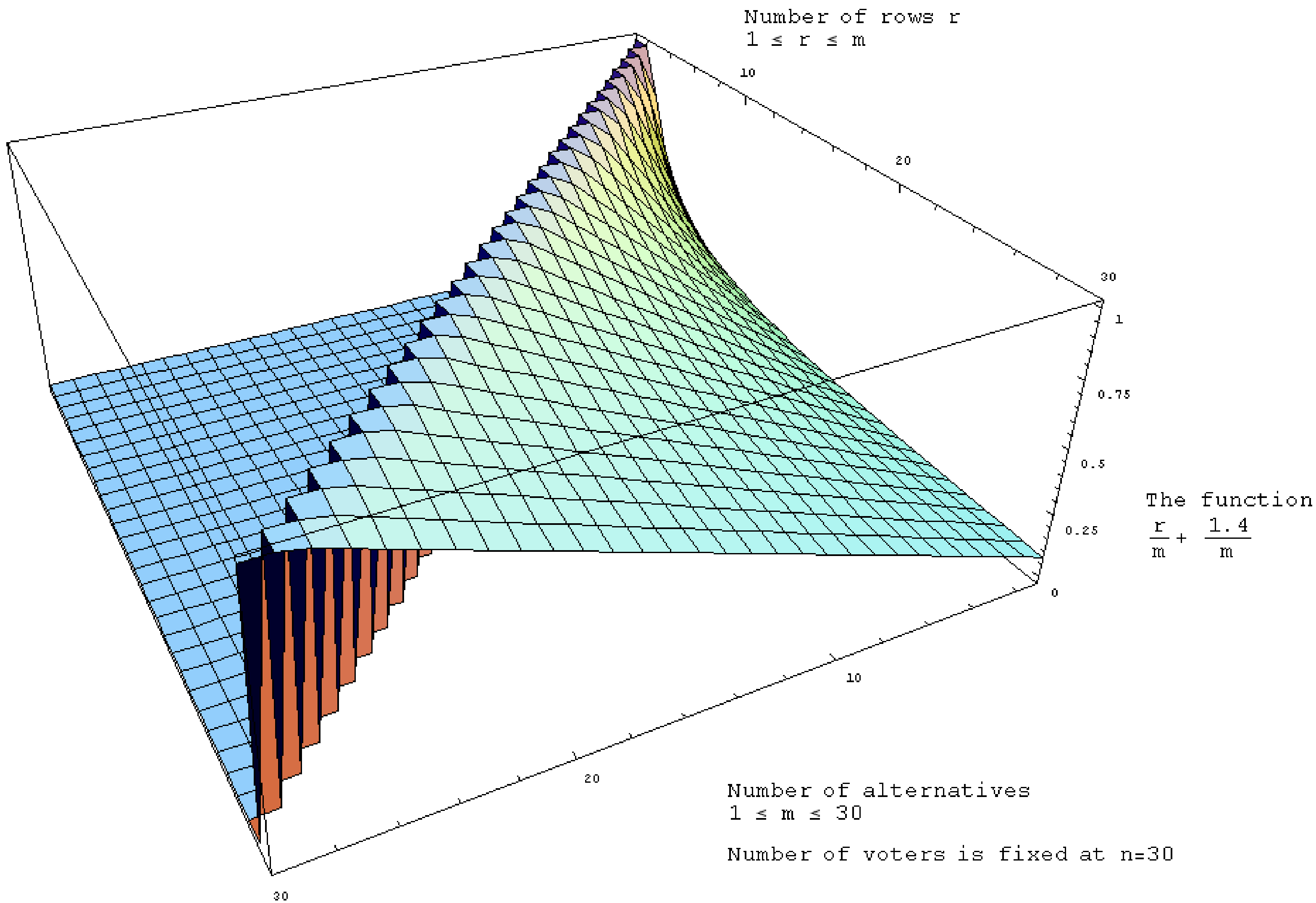

Number of rows r
1 ≤ r ≤ m
10
20
30
1
0.75
0.5
0.25
0
The function
$\frac{r}{m} + \frac{1.4}{m}$
10
20
30
Number of alternatives
1 ≤ m ≤ 30
Number of voters is fixed at n=30

Figure 5

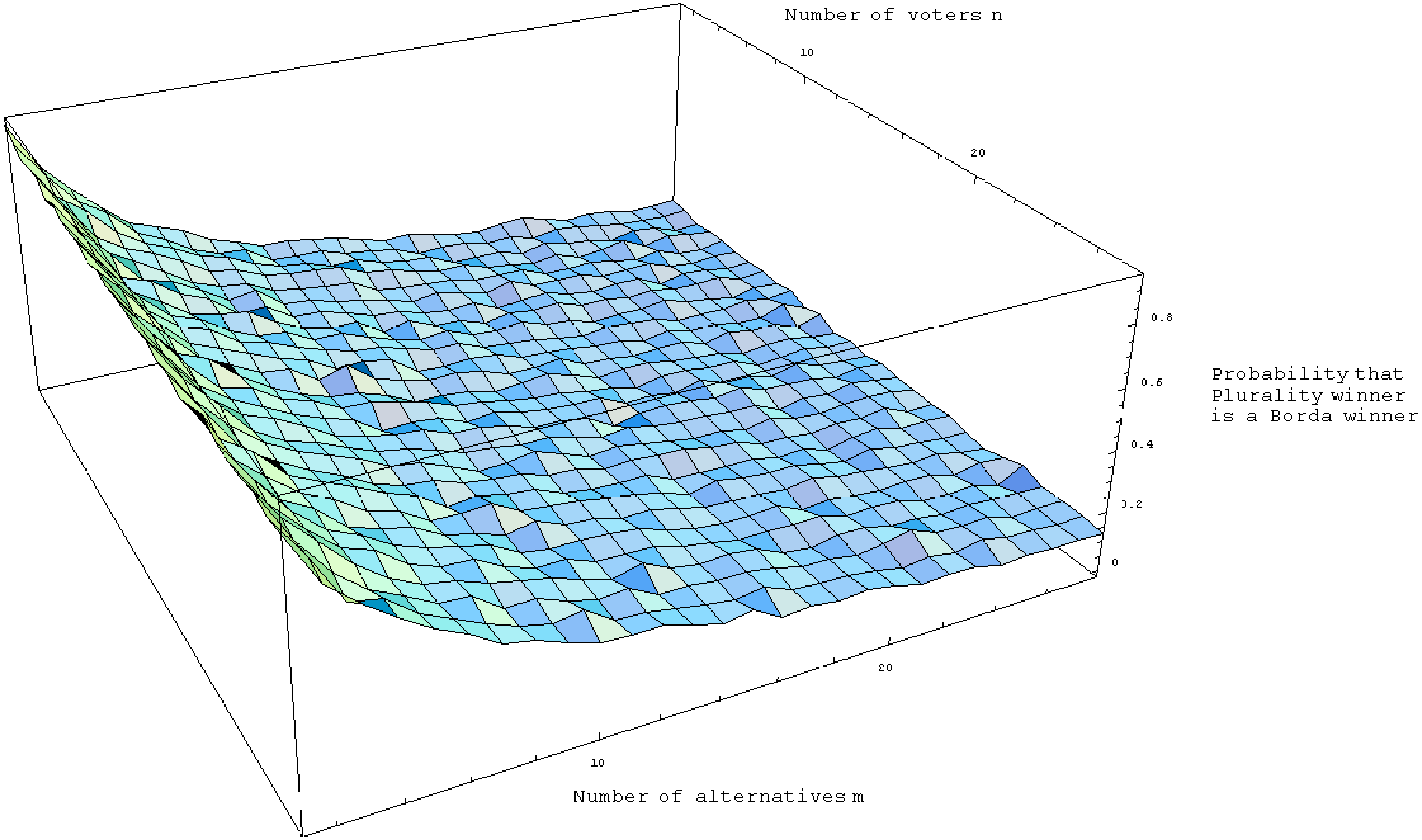
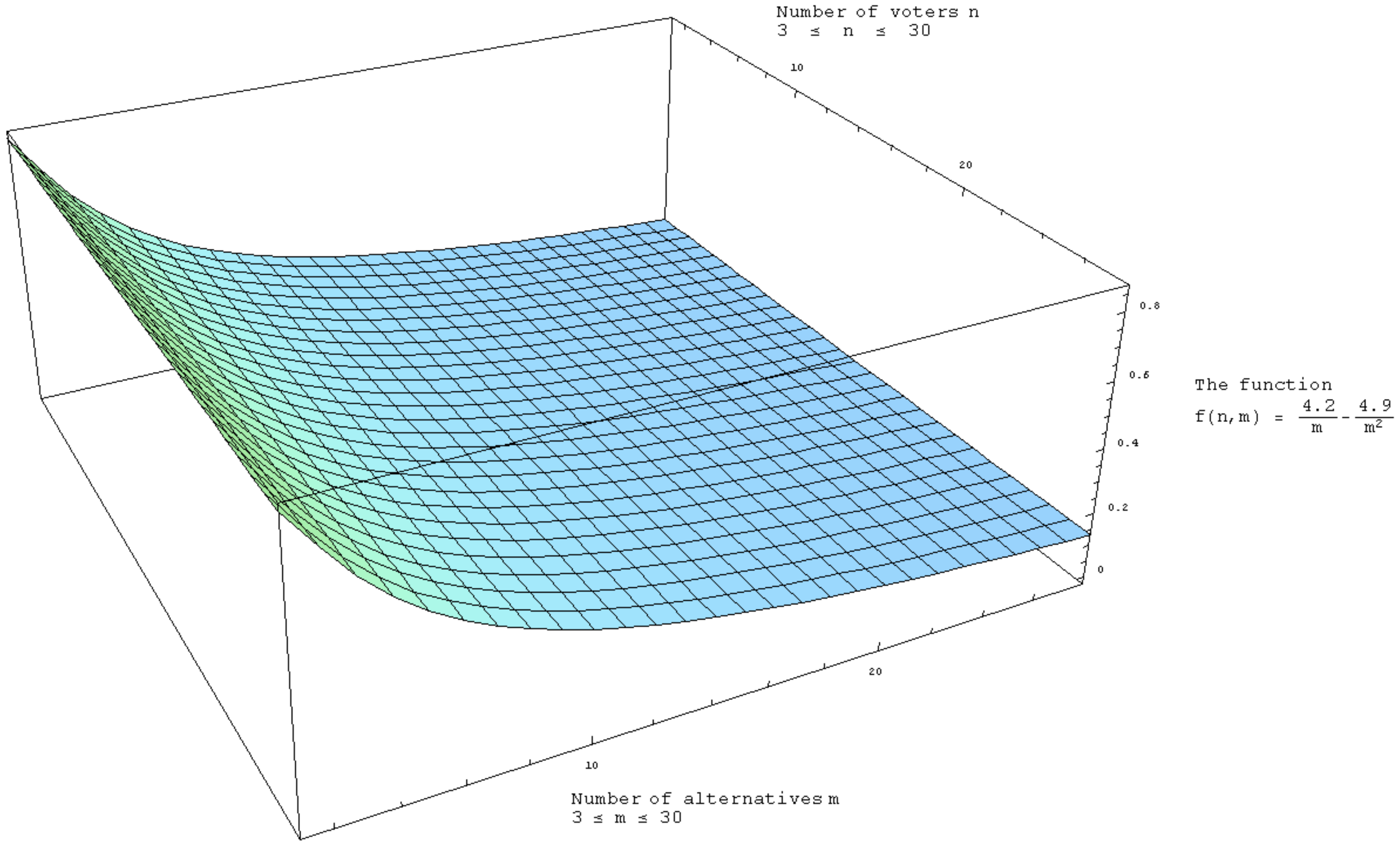

Number of voters n
10
20
0.8
0.6
0.4
0.2
0
Probability that
Plurality winner
is a Borda winner
20
10
Number of alternatives m

Figure 6



Number of voters n
3 ≤ n ≤ 30
10
20
0.8
0.6
0.4
0.2
0
The function
$f(n,m) = \frac{4.2}{m} - \frac{4.9}{m^2}$
20
10
Number of alternatives m
3 ≤ m ≤ 30

Figure 7

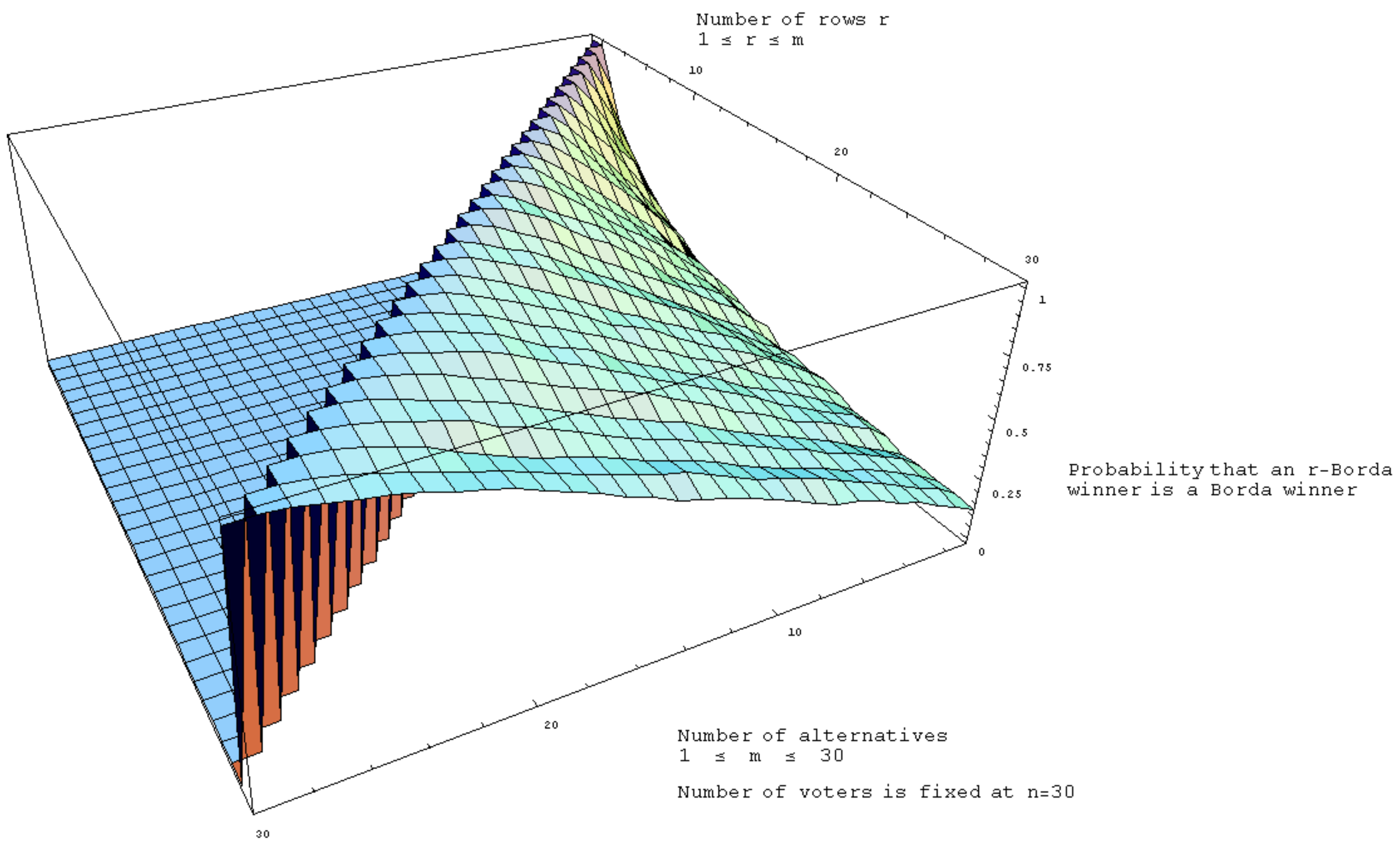
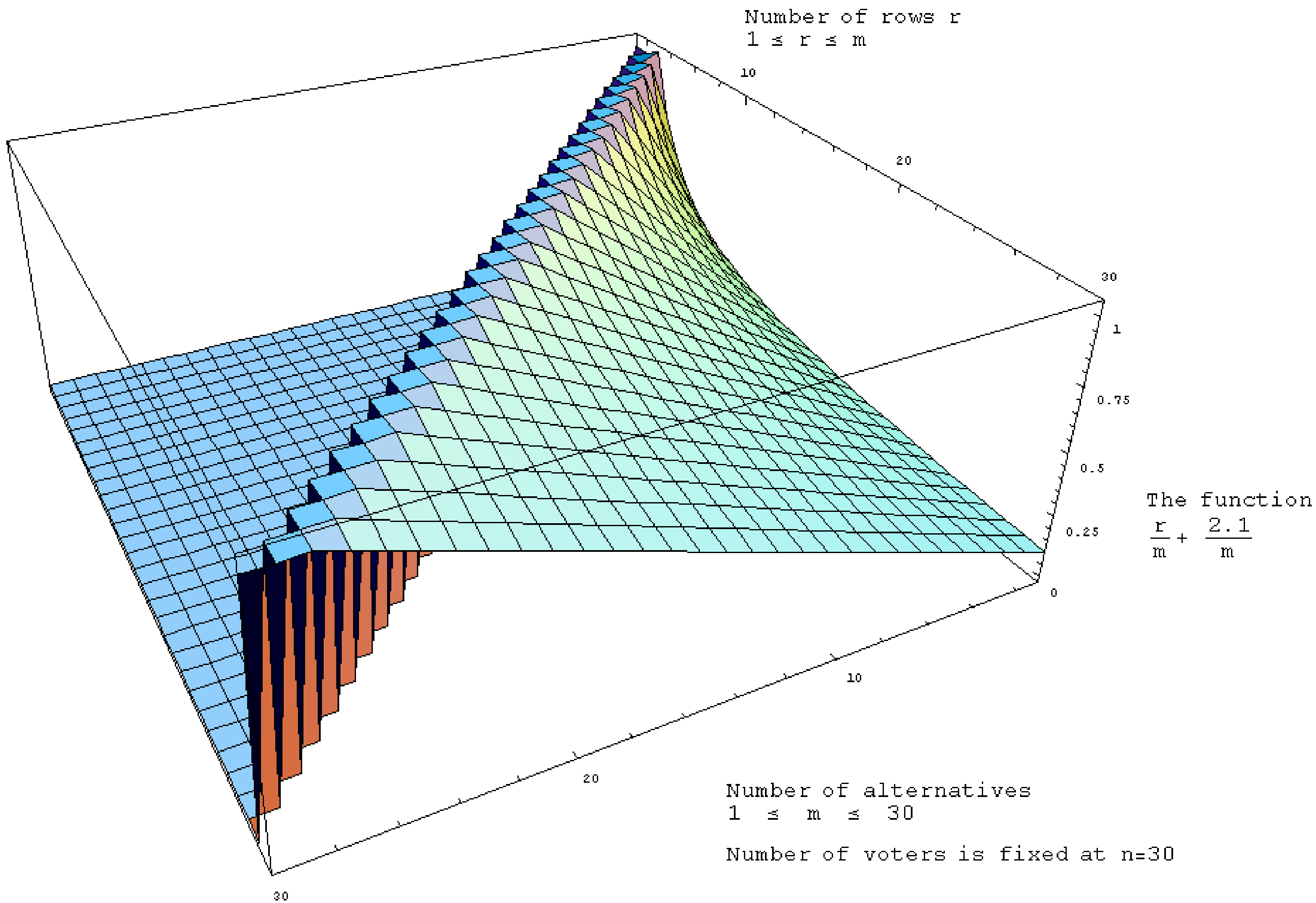

Number of rows r
1 ≤ r ≤ m
10
20
30
1
0.75
0.5
0.25
0
Probability that an r-Borda
winner is a Borda winner
10
20
30
Number of alternatives
1 ≤ m ≤ 30
Number of voters is fixed at n=30

Figure 8



Number of rows r
1 ≤ r ≤ m
10
20
30
1
0.75
0.5
0.25
0
The function
r/m + 2.1/m
10
20
30
Number of alternatives
1 ≤ m ≤ 30
Number of voters is fixed at n=30

Figure 9