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CONDORCET'S PARADOX AND THE LIKELIHOOD OF ITS OCCURRENCE: DIFFERENT PERSPECTIVES ON BALANCED PREFERENCES*

ABSTRACT. Many studies have considered the probability that a pairwise majority rule (PMR) winner exists for three candidate elections. The absence of a PMR winner indicates an occurrence of Condorcet's Paradox for three candidate elections. This paper summarizes work that has been done in this area with the assumptions of: Impartial Culture, Impartial Anonymous Culture, Maximal Culture, Dual Culture and Uniform Culture. Results are included for the likelihood that there is a strong winner by PMR, a weak winner by PMR, and the probability that a specific candidate is among the winners by PMR. Closed form representations are developed for some of these probabilities for Impartial Anonymous Culture and for Maximal Culture. Consistent results are obtained for all cultures. In particular, very different behaviors are observed for odd and even numbers of voters. The limiting probabilities as the number of voters increases are reached very quickly for odd numbers of voters, and quite slowly for even numbers of voters. The greatest likelihood of observing Condorcet's Paradox typically occurs for small numbers of voters. Results suggest that while examples of Condorcet's Paradox are observed, one should not expect to observe them with great frequency in three candidate elections.

We wish to consider elections that select a single candidate when there are three candidates {A,B,C} available for consideration. The winner of an election should be the candidate that somehow represents the 'most favored' candidate with regard to the aggregated preferences of voters. In an attempt to determine the 'most favored' candidate, societies typically apply plurality rule to elections. With plurality rule, each voter casts a vote for their most favored candidate, and the candidate who receives the most votes is selected as the winner. We begin by showing that some very disturbing outcomes can result in this situation, even when individual voters have preferences that are completely rational. In defining a rational voter, we let

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A$\succ$B denote that a particular voter prefers Candidate A to Candidate B.

A common requirement in defining rational voter behavior is that each voter must have transitive preferences. That is, if a given voter has A$\succ$B and B$\succ$C, then they must also have A$\succ$C. This prevents situations in which voters might respond in a cyclic fashion, such that A$\succ$B, B$\succ$C and C$\succ$A. The usual argument for transitivity of preference for individual voters falls back on some form of the notion of being able to use a person as a 'money pump' if they have cyclic preferences. Suppose, for example, that Candidate A will be the winner of an election. The voter in question could be asked if they would make some small payment to have Candidate C be the winner instead of A. The voter would agree since they have C$\succ$A. Now, we ask the voter to make a small payment to have Candidate B be the winner instead of C. The voter would agree since they have B$\succ$C. The voter is then asked to make a small payment to have Candidate A be the winner instead of B. The voter would agree since they have A$\succ$B. The voter would then have made a series of payments, only to return back to the original situations with Candidate A being the winner.

We are also going to assume that each voter has complete preferences, so that either A$\succ$B or B$\succ$A for all A and B. The notion of complete preferences simply eliminates voter indifference between candidates. When we assume that each voter's preferences on candidates are complete and transitive, there are only six possible preference rankings for three candidate elections.

$$\begin{array}{cccccc} A & A & B & C & B & C \\ B & C & A & A & C & B \\ C & B & C & B & A & A \\ n_1 & n_2 & n_3 & n_4 & n_5 & n_6 \end{array}$$

Here, n_i denotes the number of voters with the associated complete and transitive preference ranking on the candidates. That is, n_1 voters have individual preferences with A$\succ$B$\succ$C. Of course, we also have A$\succ$C for these voters, with the assumption of transitivity. If we let n define the number of voters in the population, $n = \sum_{i=1}^{6} n_i$. Any particular combination of n_i's that sum to n will be referred to as a voter preference profile, or simply as a profile.

1. BORDA'S PARADOX

Borda (1784) made a very interesting observation regarding a possible outcome of election procedures that are based on plurality. Using the notation for a voter profile from above, Candidate A will be the plurality winner over Candidate B, denoted as A***P***B, if $n_1 + n_2 > n_3 + n_5$. Similarly, A***P***C if $n_1 + n_2 > n_4 + n_6$. In examining plurality rule, Borda introduced the notion of looking at pairwise majority rule (PMR) relationships on candidates. Let A***M***B denote the situation in which more voters have A≻B than have B≻A in their preference rankings, regardless of the relative position of Candidate C in the preference ranking. Thus, A***M***C in a profile if $n_1+n_2+n_3 > n_4+n_5+n_6$ and A***M***B if $n_1+n_2+n_4 > n_3+n_5+n_6$. In this case, Candidate A would be the pairwise majority rule winner (PMRW) for the three candidates.

The original example that Borda considered has a profile for 21 voters with complete preferences on three candidates:

$$\begin{array}{cccc} A & A & B & C \\ B & C & C & B \\ C & B & A & A \\ 1\ Voter & 7\ Voters & 7\ Voters & 6\ Voters. \end{array}$$

The concern expressed by Borda is related to the outcome of the election when plurality rule is used, versus the outcome when PMR is used. Using plurality rule A***P***B (8–7), A***P***C (8–6) and B***P***C (7–6) to give a complete and transitive rank by plurality rule, with A***P***B***P***C. A very different result is observed using PMR. Here, B***M***A (13–8), C***M***A (13–8) and C***M***B (13–8) to give a complete and transitive PMR rank, with C***M***B***M***A. With this particular profile, plurality rule and PMR reverse the ranks on the three candidates. The phenomenon is known as Borda's Paradox. Borda's major concern in this example seemed to be that the loser by PMR is elected by plurality rule. Borda strongly endorsed the notion of selecting the PMRW, and proposed a voting rule to be assured of selecting that candidate. Unfortunately, this rule was later shown to not necessarily elect the PMRW in three candidate elections. Borda's introduction of the notion of the impact of using PMR in elections has led to numerous studies that consider possible outcomes of using such an approach.

2. CONDORCET'S PARADOX

The Marquis de Condorcet wrote a series of papers that extended some of the ideas in Borda (1784). Condorcet particularly stressed Borda's notion that the winner of an election should be the PMRW, and as a result, the PMRW is frequently referred to as the Condorcet Winner in the literature. However, a major obstacle to this line of reasoning is given in Condorcet (1785a), which contains a famous example of a profile with 60 voters on three candidates:

$$\begin{array}{ccccc} A & B & B & C & C \\ B & C & A & A & B \\ C & A & C & B & A \\ 23\ Voters & 17\ Voters & 2\ Voters & 10\ Voters & 8\ Voters \end{array}$$

Here, Condorcet notes that we have a 'contradictory system' that represents what has come to be known as Condorcet's Paradox. In particular, we find that under comparison by PMR: A***M***B (33–27), B***M***C (42–18), and C***M***A (35–25). There is a cycle in the PMR relation on the three candidates, so that no candidate emerges as being superior to each of the remaining candidates.

Condorcet was quite adamant that a lack of transitivity of preference for individual voters was irrational. Condorcet (1788) makes the point perfectly clear by stating "Clearly, if anyone's vote was self-contradictory (having cyclic preferences), it would have to be discounted, and we should therefore establish a form of voting which makes such absurdities impossible." However, after eliminating intransitivity from the preferences of individual voters, we find that PMR still might produce intransitive results. This result is quite disconcerting, given Condorcet's strong argument for the notion that the PMRW should be found and selected as the winner of the election. Despite widespread acceptance of the notion that a PMRW as somehow representing the 'most favored' candidate, this notion is not universally accepted. For example, Saari (1994) argues against the use of any voting rule that is based on pairwise majority voting.

Clearly one of the problems of having a profile exhibit an example of Condorcet's Paradox is that it is not easy to determine who the winner should be. A second problem emerges if we have a series of pairwise elimination votes by majority rule, where the loser in each stage is eliminated from further consideration, while the winner goes on to the next pairwise election. In particular, any candidate might emerge as the winner, depending on the order in which candidates are presented to the sequential elimination election process. In such a situation, the person in charge of setting the order of pairwise votes can manipulate the procedure to get whatever outcome they desire as the winner.

The view as to whether an occurrence of Condorcet's Paradox might occur is a good thing or a bad thing, seems quite evident to this point. We have suggested that it will leave an election as being difficult to decide a winner, or that the chair of a committee might be able to manipulate the outcome of an election by altering the order in which candidates are brought forward for consideration. And, this has been the typical view. However, there are opinions to the contrary.

Miller (1983) takes exception to the notion that the existence of majority cycles is necessarily a bad phenomenon. A number of historical quotes are given to suggest that conditions leading to majority cycles are likely to result from the electorate having opinions that are 'crosscut' in many different ways. This situation results in the electorate routinely forming different factions on many different issues over time, to obtain desired outcomes. The end result of this routine change in factions is argued to lead to political stability and viability, without having long-term total domination of minorities.

Rae (1980) criticizes some work of Riker (1980), regarding the suggestion that PMR cycles reflect 'incoherence' within group decision making. A number of quotes, particularly from Dahl (1956), are presented to argue that intransitive majority rule is not a rational restriction on group preferences. Rae (1980) concludes with the statement: "An understanding of majority rule, of democracy, of liberalism which does without utilitarianism, and which does more than assert that rights are right, must travel a more mysterious space, must walk up odder stairs, and must employ a more intricate altimeter than transitive consistency."

Even in Arrow's famous work (Arrow 1963) itself, the discussion of collective rationality addresses the beliefs of researchers who consider that collective transitivity might well be sacrificed in social decision processes in order to satisfy other conditions of collective rationality.

3. ACTUAL OCCURRENCES OF CONDORCET'S PARADOX

An obvious point of interest is whether or not preferences involving Condorcet's Paradox ever actually occur in practice. Many studies have been conducted to try to find these occurrences. The process is complicated by the fact that voting situations seldom require voters to report a complete preference ranking on all of the options that are available. Various techniques have been used to try to reconstruct the complete preference rankings of voters in given situations.

Riker (1958) presents a truly classic example of research in this area. A detailed description is given of the procedure by which amendments can be proposed for addition to various bills before the US House of Representatives, and the sequential elimination procedure by PMR that is used to determine the winning outcome. A detailed examination is then made of the Committee of the Whole vote on a very interesting case of the Agricultural Appropriation Act of 1953. The original bill provided a \$250 000 000 appropriation to the Soil Conservation Service. Four different amendments were put forth by members of the House to modify the appropriation to \$142 410 000 (Javits Amendment), \$100 000 000 (O'Toole Amendment), \$200 000 000 (Andersen Amendment), and \$225 000 000 (Whitten Amendment). The nature of the process that is used to vote on amendments makes it impossible to precisely obtain the preference rankings of the voters on the amendments. However, Riker (1958) argues very convincingly to show that some of the budgeted amendment amounts were involved in a PMR cycle. That cycle very likely contained the winning option, which was the original bill.

Riker (1958) then goes on to argue that the Whitten Amendment was intentionally introduced in order to create that cycle. Representative Whitten was in fact managing the original bill through the House, and he was not certain that it would obtain majority support. By introducing the Whitten Amendment, a cycle was created, and he was thereby able to take advantage of the sequence of votes on amendments to be assured that the original bill passed. Riker (1958) estimated that the House of Representatives and the Senate might have voting results that appear to have cycles in more than 10 percent of cases when two or more amendments are considered with an original bill. However, some of these cycles are likely to have been contrived as a result of strategic manipulation, as in the case discussed above.

More recently, Bjurulf and Niemi (1978) found situations in the Swedish Parliament in which similar strategic cycles were found. Stensholt (1999a) presents an example of a PMR voting cycle in a situation that was before the Norwegian National Assembly. Kurrild-Klitgaard (2001) finds an example of a PMR voting cycle for voters' preferences on presidential candidates in Denmark. Tideman (1992) examined the results of 84 elections that were overseen by the Electoral Reform Society of Great Britain and Ireland to find the existence of PMR cycles in voters' preferences over candidates in a number of elections. Truchon (1998) considers the rankings that were revealed by judges in evaluating ice skaters in 24 different competitions in Olympic Games, according to the rules of the International Skating Union, to find PMR's. In summary, there are definitely cases in which Condorcet's Paradox has been observed in actual voting situations, and interest in finding examples of the paradox remains strong. However, it would not seem that one should expect them to occur with great frequency (Van Deemen and Vergunst 1998). This leads to obvious questions related to the likelihood that Condorcet's Paradox might be observed.

4. THE LIKELIHOOD OF OBSERVING CONDORCET'S PARADOX

Given the generally very negative perspective of possible results when Condorcet's Paradox occurs, we wish to develop some procedures to estimate the likelihood that it might be observed. In doing this, a number of models have been developed which attempt to determine these probability estimates for situations which are contrived to make the outcome most likely to occur. In general, intuition suggests that we would be most likely to observe this paradox on the pairwise majority rule relations when voters' preferences are closest to being balanced between all pairs of candidates. This suggestion is not absolutely true, as indicated by the work of Gillett (1979, 1980) and Buckley (1975), but it does seem to be a generally valid claim, as discussed in Stensholt (1999b) and other studies. The assumption of balanced preferences will also be seen to be consistent with the notion of the 'principle of insufficient reason', given that nothing is known *a priori* about voters' preferences in any particular voting situation.

A number of studies have been conducted to evaluate these probabilities with different perspectives on defining 'balanced preferences'. Interestingly, the first attempt was by Condorcet (1785b) himself, assuming that there is a balance in preferences, as related to possible social outcomes on pairs of candidates.

4.1. *Balance relative to social outcomes*

Condorcet (1785b) starts to discuss the probability that a PMRW might exist for the general case of m candidates. The basis of the analysis that is presented is the assumption that there is an equal likelihood of social outcomes on pairs from voting by PMR. To be more specific, this assumption suggests that there is an equal likelihood that any pair of candidates, A and B, will have B$\boldsymbol{M}$A or A$\boldsymbol{M}$B by pairwise majority rule, regardless of the pairwise majority voting results on any other pairs of candidates.

We follow Condorcet's rather indirect derivation, and suppose that there are m candidates and that Candidate A is the PMRW. Any outcomes of voting are allowed on the social relations on the pairs of voters among the remaining m−1 candidates. There are $\frac{(m-1)(m-2)}{2}$ pairwise comparisons on the remaining candidates, and there are two possible outcomes on each comparison. The total number of social relations in which any one of the m candidates is given to be the PMRW over the remaining candidates is then given by

$$m2^{\frac{(m-1)(m-2)}{2}}$$

Each of these social relations is equally likely to be obtained.

The total number of possible social outcomes by PMR on all m candidates is $2^{\frac{m(m-1)}{2}}$, and each of these is assumed to be equally likely to be obtained. Given all of the above, the probability that there is a PMRW is given by the ratio

$$\frac{m2^{\frac{(m-1)(m-2)}{2}}}{2^{\frac{m(m-1)}{2}}} = \frac{m}{2^{m-1}}$$

Unfortunately, this relationship is stated incorrectly in the original paper by Condorcet (1785b), and Sommerlad and McLean (1989) therefore state it incorrectly in their translation. Riker (1961) develops a representation for this probability that is identical to the one given above.

May (1971) develops an identical representation for the probability that a PMRW exists, while considering societies in which voters have absolutely no requirements for consistency on their individual preferences for candidates. Clearly, May's assumptions are in sharp contrast to the notions in the comments above from Condorcet (1788) regarding requirements for transitivity of individual preference. However, we find that both studies result in the development the same representation for the probability that a PMRW exists.

If we further require that the social relation has completely transitive PMR relations, then similar arguments to those used above can be used to show that this probability is given as

$$\frac{m!}{2^{\frac{m(m-1)}{2}}}$$

Both of these probabilities become quite small as m gets at all large, as seen from computed values in Table 1, which led Condorcet to argue in several articles that some type of elimination procedure must be implemented to remove candidates that are not serious contenders from consideration. If we were to believe that these probabilities represented a realistic situation, we would then expect to be observing majority rule cycles on a regular basis, which is not the case.

Condorcet's underlying notion of considering a balance between social outcomes ignores the assumption that voters will have some type of cohesive preferences, as indicated in the interpretation of May (1971). In this situation, consider an example in an m-candidate election. Suppose that Candidate A has sequentially defeated the first m-2 of the m-1 candidates that it must beat by PMR to become the PMRW. This would seem to suggest that Candidate A is a highly preferred candidate in the preference rankings of the voters. It does not seem plausible to then assume that Candidate A has only a 50–50 chance of beating the one remaining candidate by majority rule, as is assumed in Condorcet's model.

As a result, it would seem that the balance of preference that we seek should not be applied to social outcomes on pairs of candidates, but to some measure of voters' ranked preferences on can-

TABLE I

Probabilities with equally likely social outcomes

Candidates (m)	PMRW exists	Complete transitivity
3	0.7500	0.7500
4	0.5000	0.3750
5	0.3125	0.1172
6	0.1875	0.0220
7	0.1094	0.0024

didates within a profile. The social outcomes on pairs will then be determined from the observed preferences of voters within the given profile.

4.2. *Expected overall balance across voter profiles: fixed number of voters*

Kuga and Nagatani (1974) and Gehrlein and Fishburn (1976a) developed the notion of Impartial Anonymous Culture (IAC). With IAC, it is assumed that all preference profiles for n voters are equally likely to be observed. This process is 'anonymous' in the sense that we only know the values on the n_i terms in a profile, and have no connection to the particular voters who have any particular preferences. IAC produces a balance in expected preferences on pairs of candidates. This follows from matching the set of all possible profiles into pairs. To do this, we match every profile to the profile that interchanges rankings according to: $n_1 \leftrightarrow n_6$, $n_2 \leftrightarrow n_5$, $n_3 \leftrightarrow n_4$. This transformation pairs every profile with its dual profile, in which all preference rankings are reversed. Thus, for every pair of candidates A and B, the number of voters with A$\succ$B in one of the profiles will have B$\succ$A in the matching profile. Since both profiles are equally likely under IAC, there is an expected balance between the number of voters with A$\succ$B and with B$\succ$A when profiles are selected from a uniformly random distribution over possible profiles.

We begin by developing a representation for the probability, P^{S} (3, n, IAC), that a strict PMRW exists for three candidates under IAC, following the derivation in Gehrlein and Fishburn (1976a). When a PMRW is strict, there are no ties allowed in the PMR relation. We begin by noting the conditions on the n_i terms in profiles that have Candidate A as the strict PMRW for the case of odd n:

$$
\begin{aligned}
&0 \le n_6 \le (n-1)/2 \\
&0 \le n_5 \le (n-1)/2 - n_6 \\
&0 \le n_4 \le (n-1)/2 - n_6 - n_5 \\
&0 \le n_3 \le (n-1)/2 - n_6 - n_5 \\
&0 \le n_2 \le n - n_6 - n_5 - n_4 - n_3 \\
&n_1 = n - n_6 - n_5 - n_4 - n_3 - n_2
\end{aligned}
$$

Computing probabilities with IAC becomes a simple process of counting numbers of profiles, since all profiles are equally likely. In doing this, the notions that Condorcet (1785b) used are applied to counting profiles, instead of counting the number of possible social outcomes. The number of profiles meeting the restrictions on n_i terms above can be computed as N^s (A,IAC), with

$$
N^S(A, n, IAC) = \sum_{n_6=0}^{\frac{n-1}{2}} \sum_{n_5=0}^{\frac{n-1}{2}-n_6} \sum_{n_4=0}^{\frac{n_1}{2}-n_6-n_5} \sum_{n_3=0}^{\frac{n-1}{2}-n_6-n_5} \sum_{n_2=0}^{n-n_6-n_5-n_4-n_3},
$$

for odd n.

Gehrlein and Fishburn (1976a) algebraically reduce this representation for N^{S} (A, n, IAC) to obtain

$$
N^S(A, n, IAC) = \frac{(n+1)(n+3)^3(n+5)}{384}, \text{ for odd } n.
$$

It follows from Feller (1957) that the total number of possible profiles, $K(n)$, for n voters on three candidates is given by

$$
K(n) = \frac{\prod_{i=1}^{5}(n+i)}{120}
$$

Using the definition of IAC and its symmetry with respect to candidates, we then find P^{S} (3,n,IAC) as the ratio 3*N^{S} (A,n,IAC)/$K(n)$

with

$$P^{\mathrm{S}}(3, n, \mathrm{IAC}) = \frac{15(n+3)^2}{16(n+2)(n+4)}, \text{ for odd } n.$$

When n is even, we compute N^{S} (A,n,IAC) from

$$N^{\mathrm{S}}(\mathrm{A}, n, \mathrm{IAC}) = \sum_{n_6=0}^{\frac{n-2}{2}} \sum_{n_5=0}^{\frac{n-2}{2}-n_6} \sum_{n_4=0}^{\frac{n-2}{2}-n_6-n_5} \sum_{n_3=0}^{\frac{n-2}{2}-n_6-n_5} \sum_{n_2=0}^{n-n_6-n_5-n_4-n_3},$$

for even n.

Lepelley (1989) uses this to develop a representation for $P^{\mathrm{S}}(3, n,$ IAC) with even n

$$P^S(3, n, IAC) = \frac{15n(n+2)(n+4)}{16(n+1)(n+3)(n+5)}, \text{ for even } n.$$

Fishburn et al. (1979a,b) introduced a different notion related to the probability that a PMRW exists. In particular, they discuss something that suggests the notion of the probability $P^{\{X\}}$ (m,n,IAC) that a given particular set, $\{X\}$, of candidates is included in the set of PMRW's, for n voters with m candidates under IAC. Here, each pair of candidates in $\{X\}$ is tied by majority rule, and each candidate in $\{X\}$ beats or ties all candidates not included in $\{X\}$.

Let $P^{\#i}$(m,n,IAC) denote the value of $P^{\{X\}}$(m,n,IAC) when the cardinality of $\{X\}$ is equal to i. When n is odd, there can be no ties in the majority rule relation and $3P^{\#1}$(3,n,IAC) = P^{S}(3,n,IAC). We develop a representation for $P^{\#1}$(3,n,IAC) as the probability that Candidate A beats or ties all other candidates by pairwise majority rule with an even number of voters. The number of profiles meeting this condition is denoted by $N^{\{A\}}$(n,IAC) with

$$N^{\{A\}}(n, IAC) = \sum_{n_6=1}^{\frac{n}{2}} \sum_{n_5=0}^{\frac{n}{2}-n_6} \sum_{n_4=0}^{\frac{n}{2}-n_6-n_5} \sum_{n_3=0}^{\frac{n}{2}-n_6-n_5} \sum_{n_2=0}^{n-n_6-n_5-n_4-n_3},$$

for even n.

This representation can be algebraically reduced by any of a number of available computer packages to obtain

$$N^{\{A\}}(n, IAC) = \frac{(n+2)^2(n+4)^2(n+6)}{384}, \text{ for even } n,$$

Following earlier arguments related to IAC, we then find $P^{\#1}$ (3,n, IAC) as the ratio $N^{\{A\}}(n, \text{IAC})/K(n)$ with

$$P^{\{1\}}(3, n, \text{IAC}) = \frac{5(n+2)(n+4)(n+6)}{16(n+1)(n+3)(n+5)}, \text{ for even } n.$$

Kelly (1974) introduced the notion of a weak PMRW. A given profile has a weak PMRW if some candidate beats or ties all other candidates under pairwise majority rule. Let P^W (3,n,IAC) denote the probability that a weak PMRW exists for n voters with three candidates under IAC. If n is odd, there can be no ties by majority rule, so $P^W(3, n, \text{IAC}) = 3\,P^{\#1}(3, n, \text{IAC}) = P^S(3, n, \text{IAC})$.

Our next step is to develop a representation for $P^W(3, n, \text{IAC})$, for even n. To do this we use a relationship that follows from our definitions

$$\begin{aligned}
&P^{\mathrm{W}}(3, n, \text{IAC}) = P^{\{A\}}(3, n, \text{IAC}) + \\
&\quad P^{\{B\}}(3, n, \text{IAC}) + P^{\{C\}}(3, n, \text{IAC}) - \\
&[P^{\{A,B\}}(3, n, \text{IAC}) + P^{\{A,C\}}(3, n, \text{IAC}) + \\
&P^{\{B,C\}}(3, n, \text{IAC})] + P^{\{A,B,C\}}(3, n, \text{IAC}).
\end{aligned}$$

Due to the symmetry of IAC with respect to candidates

$$\begin{aligned}
&P^{\mathrm{W}}(3, n, \text{IAC}) = 3P^{\#1}(3, n, \text{IAC}) - 3P^{\#2}(3, n, \text{IAC}) \\
&\quad + P^{\#3}(3, n, \text{IAC})
\end{aligned} \tag{1}$$

First, consider $P^{\#3}$(3,n,IAC), which requires that:

$$\begin{aligned}
&n_1 + n_2 + n_3 = n_4 + n_5 + n_6 \text{ (A tied with C)} \\
&n_1 + n_2 + n_4 = n_3 + n_5 + n_6 \text{ (A tied with B)} \\
&n_1 + n_3 + n_5 = n_2 + n_4 + n_6 \text{ (B tied with C).}
\end{aligned}$$

It is easily shown that this leads to the restriction $n_3 = n_4$, $n_1 = n_6$, $n_2 = n_5$. The profiles which permit this configuration of n_i's are bounded by:

$$\begin{aligned}0 \leqslant n_6 &\leqslant n/2\\ 0 \leqslant n_5 &\leqslant n/2 - n_6\\ n_4 &= n/2 - n_6 - n_5\\ n_3 = n_4 &= n/2 - n_6 - n_5\\ n_2 &= n_5\\ n_1 &= n_6\end{aligned}$$

The number of possible profiles meeting these conditions is $N^{\{A,B,C\}}$ (n,IAC) with

$$N^{\{A,B,C\}}(n, \text{IAC}) = \sum_{n_6=0}^{\frac{n}{2}} \sum_{n_5=0}^{\frac{n}{2}-n_6}$$

which can be reduced to

$$N^{\{A,B,C,\}}(n, \text{IAC}) = \frac{(n+2)(n+4)}{8}$$

Following the discussion above, $P^{\#3}(3, n, \text{IAC}) = N^{\{A,B,C\}}(n, \text{IAC})/K(n)$.

To develop a representation for $P^{\#2}(3, n, \text{IAC})$, we start by obtaining the number, $N^{\{A,B\}}(n, \text{IAC})$, of profiles in which A and B are included in the set of weak PMRW's. The restrictions on the n_i's to produce this event are

$$\begin{aligned}n_1 + n_2 + n_3 &\geqslant n_4 + n_5 + n_6 (A\ beats\ or\ ties\ C)\\ n_1 + n_3 + n_5 &\geqslant n_2 + n_4 + n_6 (B\ beats\ or\ ties\ C)\\ n_1 + n_2 + n_4 &= n_3 + n_5 + n_6 (A\ ties\ B).\end{aligned}$$

$N^{\{A,B\}}(n, \text{IAC})$ is obtained as

$$N^{\{A,B\}}(n, \text{IAC}) = \sum_{n_6=0}^{\frac{n}{2}} \sum_{n_4=0}^{\frac{n}{2}-n_6} \sum_{n_5=0}^{\frac{n}{2}-n_6-n_4} \sum_{n_2=0}^{\frac{n}{2}-n_6-n_4}$$

with $n_3 = \frac{n}{2} - n_5 - n_6$ and $n_1 = \frac{n}{2} - n_2 - n_4$. This can be reduced to

$$N^{\{A,B\}}(n, \text{IAC}) = \frac{(n+2)(n+4)^2(n+6)}{192}$$

Then, $P^{\#2}(3, n, \text{IAC})$ is obtained as $N^{\{A,B\}}(n, \text{IAC})/K(n)$.

Using all of the above with (1), we find:

$$P^{\text{W}}(3, n, \text{IAC}) = \frac{15(n+2)(n^2+8n+8)}{16(n+1)(n+3)(n+5)}, \text{ for even } n.$$

This representation has been verified by computer enumeration, and it is not in agreement with a representation for $P^{\text{W}}(3, n, \text{IAC})$ with even n in Berg and Bjurulf (1983).

Given all of the representations above, the following results follow directly from taking derivatives:

THEOREM 1 (IAC). *$P^S(3, n, IAC) > P^S(3, n+2, IAC)$, for all odd $n \geqslant 1$.*

THEOREM 2 (IAC). *$P^S(3, n, IAC) < P^S(3, n+2, IAC)$, for all even$\geqslant 1$.*

THEOREM 3 (IAC). *$P^S(3, n, IAC) = 3P^{\#1}(3, n, IAC) = P^W(3, n, IAC)$ for all odd $n \geqslant 1$.*

THEOREM 4 (IAC). *$P^{\#1}(3, n, IAC) > P^{\#1}(3, n+2, IAC)$ for all even $n \geqslant 2$.*

THEOREM 5 (IAC). *$P^W(3, n, IAC) > P^W(3, n+2, IAC)$, for all even $n \geqslant 2$.*

Table 2 shows computed values of $P^{\text{S}}(3, n, \text{IAC})$, $P^{\#1}(3, n, \text{IAC})$ and $P^{\text{W}}(3, n, \text{IAC})$ for various values of n. The results show very different behaviors for odd and even n. With odd n, each of the probabilities approaches its limiting values quickly for small values of n. For even n, we find a much slower convergence to the limiting probabilities as n increases, with a rather small probability for $P^{\text{S}}(3, n, \text{IAC})$ when n is small. The additional fact that

TABLE II

Probabilities with impartial anonymous culture condition (IAC)

Voters (n)	$P^{\mathrm{S}}(3, n, \mathrm{IAC})$	$P^{\#1}(3, n, \mathrm{IAC})$	$P^{\mathrm{W}}(3, n, \mathrm{IAC})$
3	0.9643	0.3214	0.9643
4	0.5714	0.4762	1.0000
5	0.9524	0.3175	0.9524
6	0.6494	0.4329	0.9957
7	0.9470	0.3157	0.9470
8	0.6993	0.4079	0.9907
9	0.9441	0.3147	0.9441
10	0.7343	0.3916	0.9860
11	0.9423	0.3141	0.9423
20	0.8199	0.3553	0.9702
21	0.9391	0.3130	0.9391
40	0.8735	0.3348	0.9569
41	0.9380	0.3127	0.9380
100	0.9105	0.3217	0.9462
101	0.9376	0.3125	0.9376
∞	0.9375	0.3125	0.9375

$P^{\mathrm{W}}(3, n, \mathrm{IAC})$ values are near 1.00 for small even n indicate that there is a significant probability for PMR ties in these cases, which might explain the slower rate of convergence for even n.

4.3. *Expected overall balance across voter profiles: variable number of voters*

Another set of assumptions has been developed that is similar to the notion of IAC, in which each of the possible profiles is considered to be equally likely to be observed. However, the Maximal Culture Condition (MC) does not require that there is a fixed number of voters in the population. MC is defined by the situation in which each profile which has $0 \leqslant n_i \leqslant L$, for $i = 1, 2, 3, 4, 5, 6$, for some positive integer L is equally likely to be observed. With the assumption of MC, there are a total of $(L + 1)^6$ different possible profiles that might be observed. The expected total number of voters in a profile, $E(n)$, with MC is given by $E(n) = 6^*(L/2) = 3L$.

Gehrlein and Lepelley (1997) follow the development of the representation for N^{S} (A,n,IAC) given above to find the bounds on n_i's for N^{S} (A,L,MC), the number of MC profiles for which Candidate A is the PMRW with MC, with

$$
\begin{gathered}
0 \leqslant n_3 \leqslant L \\
0 \leqslant n_4 \leqslant L \\
0 \leqslant n_1 \leqslant L \\
\text{Max}\left\{\begin{matrix} 0 \\ n_4 - n_3 - n_1 + 1 \\ n_3 - n_4 - n_1 + 1 \end{matrix}\right\} \leqslant n_2 \leqslant L \\
0 \leqslant n_5 \leqslant \text{Min}\left\{\begin{matrix} L \\ n_1 + n_2 + n_3 - n_4 - 1 \\ n_1 + n_2 + n_4 - n_3 - 1 \end{matrix}\right\} \\
0 \leqslant n_6 \leqslant \text{Min}\left\{\begin{matrix} L \\ n_1 + n_2 + n_3 - n_4 - n_5 - 1 \\ n_1 + n_2 + n_4 - n_3 - n_5 - 1 \end{matrix}\right\}
\end{gathered}
$$

Here, $\text{Min}\{{a \atop b}\}$ and $\text{Max}\{{a \atop b}\}$ denote the minimum and maximum respectively of arguments a and b. The Min and Max functions in the summation limits significantly complicate the problem of obtaining an elegant closed-form representation for N^{S}(A,L,MC). This issue is avoided by partitioning the set of profiles in N^{S}(A,L,MC) into 13 subspaces, which do not contain Min or Max arguments. Representations are then obtained for each subspace, which are then accumulated to obtain the desired result, with:

$$
N^{\mathrm{S}}(\mathrm{A}, L, \mathrm{MC}) = \frac{2L}{3} + \frac{107L^2}{45} + \frac{91L^3}{24} + \frac{239L^4}{72} + \frac{37L^5}{24} + \frac{109L^6}{360}, \quad \text{for } L = 3, 4, 5, 6, \ldots
$$

Due to the symmetry of MC with respect to candidates $P^{\mathrm{S}}(3, L, \mathrm{MC}), = \frac{3N^{\mathrm{S}}(A, L, MC)}{(L+1)^6}$, which reduces to

$$P^{\mathrm{S}}(3, L, MC) = \frac{L(109L^4 + 446L^3 + 749L^2 + 616L + 240)}{120(L+1)^5},$$
$$L = 3, 4, 5, 6...$$

Following the development of IAC probabilities above, it is possible to develop representations for $P^{\#1}(3, L, MC)$ and $P^{\mathrm{W}}(3, L, MC)$ for the case of MC. However, we wish to avoid the cumbersome partitioning of subspaces of profiles, as in Gehrlein and Lepelley (1997), to obtain these representations.

A procedure described in Gehrlein (2001a) uses partial computer enumeration to initialize a process that then solves seven equations for seven unknowns to obtain representations for MC probabilities. This procedure was used to obtain:

$$P^{\#1}(3, L, MC) =$$
$$\frac{109L^5 + 644L^4 + 1541L^3 + 1894L^2 + 1212L + 360}{360(L+1)^5},$$
$$L = 3, 4, 5, 6...$$

$$P^{\mathrm{W}}(3, L, MC) =$$
$$\frac{109L^5 + 578L^4 + 1157L^3 + 1168L^2 + 588L + 120}{120(L+1)^5},$$
$$L = 3, 4, 5, 6, ...$$

Since the number of voters is not fixed with MC, there are no odd-even effects, as with IAC. The following results are obtained by taking derivatives of the representations:

THEOREM 1 (MC). $P^{s}(3, L, MC) < P^{S}(3, L+1, MC)$, for all L$\geqslant$ 3.

THEOREM 2 (MC). $P^{\#1}(3, L, MC) > P^{\#1}(3, L+1, MC)$, for all L$\geqslant$ 3.

TABLE III

Probabilities with Maximum Culture Condition (MC)

L	$P^{\mathrm{S}}(3,n,\mathrm{MC})$	$P^{\#1}(3,n,\mathrm{MC})$	$P^{\mathrm{W}}(3,n,\mathrm{MC})$
3	0.7251	0.3833	0.9517
4	0.7588	0.3650	0.9461
5	0.7819	0.3535	0.9417
6	0.7988	0.3456	0.9382
7	0.8117	0.3398	0.9354
8	0.8218	0.3354	0.9330
9	0.8301	0.3319	0.9310
10	0.8368	0.3291	0.9293
11	0.8426	0.3268	0.9278
20	0.8700	0.3162	0.9203
40	0.8885	0.3096	0.9147
50	0.8923	0.3082	0.9135
∞	0.9083	0.3028	0.9083

THEOREM 3 (MC). $P^W(3, L, MC) > P^W(3, L+1, MC)$ for all L$\geqslant$ 3.

Table 3 lists computed values of $P^{\mathrm{S}}(3, n, MC)$, $P^{\#1}(3, n, MC)$ and $P^{\mathrm{W}}(3, n, MC)$ for various values of L. The computed probability values approach their limiting values slowly as L increases, to suggest that the convergence to the limiting probability values is quite slow as $E(n)$ increases. $P^{\mathrm{S}}(3,n,\mathrm{MC})$ is relatively small for small values of $E(n)$.

4.4. *Expected balance relative to each individual voter's preferences*

Another view of balanced preferences considers an expected balance of preferences between pairs of candidates within the preference rankings of individual voters. In this situation, we assume that each voter has a preference ranking that is arrived at from some probability distribution over the possible preference rankings. We let **p** denote a six-dimensional vector, where p_i denotes the probability that a randomly selected voter will have the corresponding preference ranking, as defined above. That is, a randomly selected voter will have the transitive preference ranking $A \succ B \succ C$ with probability p_1. We also assume that each voter's preferences are independent of the other voters' preferences.

For any particular $\mathbf{p}$ with independent voters, the probability, $P^S(3,n,\mathbf{p})$, that there is a strict PMRW, for odd n, is given in Gehrlein and Fishburn (1976b) as

$$P^S(3, n, \mathbf{p}) = \sum_{m_1=0}^{\frac{n-1}{2}} \sum_{m_2=0}^{\frac{n-1}{2}-m_1} \sum_{m_3=0}^{\frac{n-1}{2}-m_1} \frac{n!}{m_1!m_2!m_3!m_4!} \times \left\{ \begin{array}{l} (p_5+p_6)^{m_1} p_3^{m_2} p_4^{m_3} (p_1+p_2)^{m_4} + \\ (p_2+p_4)^{m_1} p_1^{m_2} p_6^{m_3} (p_3+p_5)^{m_4} + \\ (p_1+p_3)^{m_1} p_5^{m_2} p_2^{m_3} (p_4+p_6)^{m_4} \end{array} \right\}. \tag{2}$$

Here, $m_4 = n - m_1 - m_2 - m_3$.

Following earlier discussion related to IAC, we have an expected balance on pairs of candidates within an individual voter's preference rankings on candidates if each possible ranking has the same probability of being observed as its reversed, or dual, ranking. That is, when $p_1 = p_6$, $p_2 = p_5$ and $p_3 = p_4$, which is defined as the Dual Culture Condition (DC) in Gehrlein (1978).

Sen (1970) considers an example that is very much in the spirit of an extreme case of the dual culture condition. That is, a two-class society in which the classes have radically different interests is discussed. For this 'class war' condition we would expect to have voter profiles containing only two different rankings on alternatives, some ranking and its dual. It is noted that there will always be a transitive majority rule relation for an odd number of voters in this particular situation with two possible rankings. DC would assume that the two classes contain the same number of members in Sen's example. Arrow and Raynaud (1986) summarize a number of other conditions on preference rankings that will ensure transitivity of PMR.

Gehrlein (1978) develops a closed form representation of the limiting probability as $n \to \infty$, $P^S(3,\infty,\text{DC})$, that a PMRW exists for three alternatives for a combination of p_i's meeting DC as

$$P^S(3, \infty, DC) = \frac{3}{4} + \frac{1}{2\pi}\sum_{i=1}^{3} \mathrm{Sin}^{-1}(1 - 4\mathrm{p}_i)$$

Gehrlein (1978) also proves that $P^{\mathrm{S}}(3{,}\infty{,}\mathrm{DC})$ is minimized for DC for the special case in which $p_i = \frac{1}{6}$ for all $i = 1, 2, 3, 4, 5, 6$. This special case has been widely referred to as the Impartial Culture Condition (IC) and it has received a great deal of attention. Guilbaud (1952) considered the case of IC and developed a representation for $P^{\mathrm{S}}(3{,}\infty{,}\mathrm{IC})$ like the representation for $P^{\mathrm{S}}(3{,}\infty{,}\mathrm{DC})$ above, which is evaluated as 0.9123 for IC.

Tsetlin et al. (2001) show that IC minimizes the probability that a PMR winner exists for a broader set of **p** vectors than DC includes. They consider the case as $n \to \infty$ with the set of **p** vectors such that the probability that a PMR winner exists is not always 0 or 1 in the corresponding profiles. In their study, some individual voter indifference between candidates is permitted.

Kelly (1974) and Buckley and Westen (1979) considered the general behavior of various aspects of the probability that a PMRW exists under IC. Some results were proved and a number of conjectures were given regarding this behavior. Fishburn et al. (1979a,b) then proved some of the conjectures for the case of three candidates, and a summary of known results is given by:

THEOREM 1 (IC). $P^{\mathrm{S}}(3, n, IC) > P^S(3, n+2, IC)$, for all odd $n \geqslant 1$.

THEOREM 2 (IC). $P^S(3, n, IC) < P^S(3, n+2, IC)$ for all even $n \geqslant 2$.

THEOREM 3 (IC). $P^S(3, n, IC) = 3P^{\#1}(3, n, IC) = P^W(3, n, IC)$ for all odd $n \geqslant 1$.

THEOREM 4 (IC). $P^{\#1}(3, n, IC) > P^{\#1}(3, n+2, IC)$, for all even $n \geqslant 2$.

THEOREM 5 (IC). $P^{\mathrm{W}}(3, n, IC) > P^{\mathrm{W}}(3, n+2, IC)$, for all even n greater than some integer N.

TABLE IV

Probabilities with Impartial Culture Condition (IC)

Voters (n)	$P^{S}(3,n,\mathrm{IC})$	$P^{\#1}(3,n,\mathrm{IC})$	$P^{W}(3,n,\mathrm{IC})$
3	0.9444	0.3148	0.9444
4	0.4444	0.5231	1.0000
5	0.9306	0.3102	0.9306
6	0.5087	0.4821	0.9961
7	0.9250	0.3083	0.9250
8	0.5519	0.4574	0.9920
9	0.9220	0.3073	0.9220
10	0.5834	0.4406	0.9882
11	0.9202	0.3067	0.9202
20	0.6686	0.3991	0.9750
21	0.9163	0.3054	0.9163
40	0.7346	0.3702	0.9616
41	0.9143	0.3048	0.9143
∞	0.9123	0.3041	0.9123

Table 4 shows computed values of $P^{S}(3,n,\mathrm{IC})$, $P^{\#1}(3,n,\mathrm{IC})$ and $P^{W}(3,n,\mathrm{IC})$ for various values of n. Values of $P^{S}(3,n,\mathrm{IC})$ for odd n were computed by using (2) with IC. Values of $P^{S}(3,n,\mathrm{IC})$ for even n were computed with a simple modification of (2). Values of $P^{\#1}(3,n,\mathrm{IC})$ and $P^{W}(3,n,\mathrm{IC})$ were computed in the same fashion as in the development of $P^{\#1}(3,n,\mathrm{IAC})$ and $P^{W}(3,n,\mathrm{IAC})$, but with multinomial probabilities for IC replacing the simple counting techniques of IAC.

The computed values in Table 4 are close to Monte-Carlo simulation estimates of these probabilities in Buckley and Westen (1979). As in the IAC results, we see different behavior depending upon whether n is odd or even. The probabilities converge to their limiting values quite quickly for odd n. The convergence is much slower for even n, with rather small values of $P^{S}(3,n,\mathrm{IC})$ for small even values of n.

A different perspective on expected balance between pairs of candidates within a voter's preferences on pairs of candidates is the

Uniform Culture Condition (UC), which is in the nature of IAC and DC. In particular, all **p** vectors with $\sum_{i=1}^{6} p_i = 1$ are assumed to be equally likely. As in the case of IAC, there is then a matching of pairs of vectors given by $p_1 \leftrightarrow p_6$, $p_2 \leftrightarrow p_5$, $p_3 \leftrightarrow p_4$. If both vectors in the matching pair are equally likely to be observed, then the expected probability that $A \succ B$ in a voter's preference ranking is the same as the expected probability that $B \succ A$. The expected probability that there is a strict PMRW under UC is then given by $E[P^{\mathrm{S}}(3, n, \mathrm{UC})]$. Following work in Gehrlein (1982), the probability density function, $f(\mathbf{p})$, for UC has $f(\mathbf{p}) = 120$.

To obtain a representation for $E[P^{\mathrm{s}}(3, n, \mathrm{UC})]$, we first substitute for p_1 in the representation for $P^{\mathrm{S}}(3, n, \mathbf{p})$ with $p_1 = 1 - p_2 - p_3 - p_4 - p_5 - p_6$, and then find

$$E[P^{\mathrm{S}}(3, n, \mathrm{UC})] = \int_{p_6=0}^{1} \int_{p_5=0}^{1-p_6} \int_{p_4=0}^{1-p_6-p_5} \int_{p_3=0}^{1-p_6-p_5-p_4} \times$$

$$\int_{p_2=0}^{1-p_6-p_5-p_4-p_3} P^{\mathrm{S}}(3, n, p) 120 dp_6 dp_5 dp_4 dp_3 dp_2.$$

Gehrlein (1981) sequentially reduces this integral to show that

$$E[P^{\mathrm{S}}(3, n, UC)] = P^{\mathrm{S}}(3, n, IAC).$$

In a later study, Gehrlein (1984) shows that this result is easily generalized, so that we have

$$E[P^{\#1}(3, n, UC)] = P^{\#1}(3, n, IAC)$$
$$E[P^{W}(3, n, UC)] = P^{W}(3, n, IAC).$$

Thus the same relationships that hold for IAC and IC are valid on an expected value basis for UC. In addition, IC represents the DC special case with the greatest likelihood of exhibiting a pairwise majority rule cycle.

5. OTHER IAC – IC LINKS

Berg (1985) uses Pólya–Eggenberger (P–E) models (Johnson and Kotz 1977) to evaluate the probability that a PMRW exists. To describe these urn models in the context of voting probabilities for three candidate elections, we start with an urn containing chips of six different colors. For each different preference ranking, there are A_i chips of a particular color corresponding to the i^{th} preference ranking. A chip is drawn at random and the corresponding preference ranking is assigned to the first voter. The chip is then replaced, along with σ additional chips of the same color. A second chip is then drawn, the corresponding ranking is assigned to the second voter, and the chip is replaced with additional chips of the same color. The process is repeated n times, to obtain a preference ranking for each of the n voters. Since the color of the chip drawn for the first voter will have an increased likelihood, when $\sigma > 0$, of representing the color of the chip drawn for the second voter, and so on, probability models of this type have some dependence among the voters' preferences. However, there is no dependence for the particular case when $\sigma = 0$.

With P–E models, the probability, $P(n, \sigma)$, of observing a given voter profile is given by

$$P(n, \sigma) = \frac{n!}{A^{[n,\sigma]}} \prod_{i=1}^{6} \frac{A_i^{[n_i,\sigma]}}{n_i!}$$

Here, $A = \sum_{i=1}^{6} A_i$ and $A^{[k,\sigma]}$ is the generalized ascending factorial with

$$A^{[k,\sigma]} = A(A+\sigma)(A+2\sigma)\ldots(A+(k-1)\sigma).$$

By definition, $A^{[k,\sigma]} = A$ for $k = 0$ and $k = 1$.

We give particular attention to the P–E probability $P^1(n, \sigma)$ which has $A_i = 1$ for all $i = 1, 2, 3, 4, 5, 6$. When we consider the special cases of $\sigma = 0$ and $\sigma = 1$, we obtain

$$P^1(n, 0) = \frac{n!}{n_1!n_2!n_3!n_4!n_5!n_6!} \frac{1}{6^n}$$

$$P^1(n, 1) = \frac{120}{(n+1)(n+2)(n+3)(n+4)(n+5)}.$$

Thus, we find that P–E probability model with $\sigma = 0$ is equivalent to an independent voter model with a multinomial probability for profiles, with equally likely preference rankings. That is, when $\sigma = 0$ we have the equivalent of IC. When $\sigma = 1$, each possible preference profile is equally likely to be observed, given n. That is, when $\sigma = 1$ we have the equivalent of IAC, and the direct implication that IAC represents a situation of some dependence among voters' preferences. Berg (1985) and Stensholt (1999b) give various other interpretations of the IAC assumption, particularly with regard to the small degree of dependence between voters' preferences that it implies.

Berg and Bjurulf (1983) do a study of the probability that there is a PMRW with IAC and make a number of observations. An analogy is drawn with the subject of statistical mechanics in physics, which considers the behavior of collections of particles. In particular, physicists do computations in statistical mechanics, and the approach that is used depends upon whether or not it is possible to distinguish one particle from another. When particles are indistinguishable, the use of Bose–Einstein statistics is applicable. When particles are distinguishable, the use of Maxwell–Bolzmann statistics is applicable. In the study of probabilities, the assumption of IC is equivalent to the use of Maxwell–Bolzmann statistics and the assumption of IAC is equivalent to using Bose–Einstein statistics. As a result, the term Impartial 'Anonymous' Culture, as coined by Gehrlein and Fishburn (1976a), is very appropriate, since it is equivalent to the concept of dealing with 'indistinguishable' particles in statistical mechanics.

Berg and Bjurulf (1983) show results to suggest that any differences between IC and IAC should become small for $m = 4$, and insignificant for $m \geqslant 5$. The connection between IAC and IC with the notions of statistical mechanics is also discussed in Feix and Rouet (1999). Meyer and Brown (1998) also discuss the notion of majority rule cycles in the context of statistical mechanics.

Feix and Rouet (1999) present Monte-Carlo simulation estimates for $P^S(m, \infty, \text{IAC})$ and $P^S(m, \infty, \text{IC})$ with $m = 3, 4, 5, \ldots, 8$, as shown in Table 5. These simulation results compare very closely to exact computations that are already known for the IC case for $m > 3$ (Gehrlein and Fishburn 1979), and for the IAC case with four candidates (Gehrlein 2001b). This gives credence to the claim

TABLE V

Simulation estimates of $P^S(m, \infty, \text{IAC})$ and $P^S(m, \infty, \text{IC})$.

3	0.9376	0.9123
4	0.8384	0.8244
5	0.7523	0.7484
6	0.6857	0.6848
7	0.6309	0.6306
8	0.583	0.586

by Berg and Bjurulf (1983) that IC and IAC probabilities converge to very similar values for m greater than 4 or 5.

6. CONCLUSION

We have observed very similar behavior for the probability that a PMRW exists under the different methods of considering balanced preferences in three candidate elections: IAC, MC, IC, DC and UC. The situation developed by Condorcet, which has a balance in social outcomes, suggests that there should be widespread occurrences of majority rule cycles, if that assumption is valid. However, this assumption tends to ignore a certain amount of coherence among voter preferences. We have typically found the greatest likelihood for majority rule cycles to exist with a small number of voters. For very large electorates we expect to have a PMRW with probability approaching 15/16 = 0.9375 with IAC and UC, and approaching 109/120 = 0.9083 for MC. The results of Guilbaud (1952) show that a PMRW exists with probability approaching 0.9123 for large electorates with IC.

Some studies suggest that these different assumptions give extraordinarily small estimates of the probability that a PMRW exists (Stensholt 1999b, for example). This should not seem surprising, since none of the studies referenced above have ever suggested that IAC, IC, DC or UC reflect reality in any particular situation. As was suggested in the introduction, they have considered instead the likelihood that a PMRW exists under various interpretations of balanced preferences. If indeed balanced preferences are most likely to produce a majority rule cycle, then each of these cases represent situations in which the probability that a PMRW exists would tend to be at a minimum. In all of these different situations that were contrived to make majority rule cycles as likely as is possible, we still expect to have a PMRW in over 90 percent of cases for large electorates with three candidates. That probability would certainly be significantly greater for typical situations that are not contrived to make majority rule cycles most likely to occur.

Given all of this, if we wonder whether or not majority rule cycles should be expected to be routinely observed in three-candidate elections, the answer is clearly 'No.' On the other hand, if we wonder whether or not majority rule cycles ever exist, the answer is clearly 'Yes.' These theoretical models for the probability that a PMRW exists must be taken for what they were meant to be. They were never meant to be a predictor of the probability that a PMRW exists for any particular specific situation in reality. They were intended to give us some idea of the lower bound on the likelihood that a PMRW exists.

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