Math 2135 - Assignment 7

Due October 18, 2024

(1) Explain why the following are not subspaces of $\mathbb{R}^2$. Give explicit counter examples for subspace properties that are not satisfied.

(a) $U = \{\begin{bmatrix} x \\ y \end{bmatrix} \mid x, y \in \mathbb{R}, x \geq 0\}$

(b) $V = \mathbb{Z}^2$ ($\mathbb{Z}$ denotes the set of all integers)

(c) $W = \{\begin{bmatrix} x \\ y \end{bmatrix} \mid x, y \in \mathbb{R}, |x| = |y|\}$

Ans:

(a) This is not a subspace because it is not closed under scaling. For example, $-1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \notin U$ because $-1 < 0$.

(b) This is not a subspace because it is not closed under scaling. For example, $0.2 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix} \notin V$ because $0.2 \notin \mathbb{Z}$

(c) This is not a subspace because it is not closed under addition. For example, $\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \notin W$ because $0 \neq -2$

(2) Which of the following are subspaces of the vector space $\mathbb{R}^{\mathbb{R}} = \{f : \mathbb{R} \to \mathbb{R}\}$ of all functions from $\mathbb{R}$ to $\mathbb{R}$? Check all subspace properties or give one that is not satisfied.

(a) $\{f : \mathbb{R} \to \mathbb{R} \mid f(0) = 1\}$

(b) $\{f : \mathbb{R} \to \mathbb{R} \mid f(3) = 0\}$

(c) $\{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}\}$

Ans:

(a) Not a subspace
This is not a subspace because it is not closed under addition. If we add two functions where $f(0) = 1$, we will get a function where $f(0) = 2$.

(b) Subspace

(i) Addition
$f(3) + f(3) = 0 + 0 = 0$ So it is closed under addition.

(ii) Scaling
$af(3) = a \cdot 0 = 0$ So it is closed under scaling.

(iii) Zero vector
Contains the zero vector $f(x) = 0$.

(c) Subspace

(i) Addition
Two continuous functions added cannot create a discontinuous function (closed under addition).
(ii) Scaling
Scaling a continuous function creates a new continuous function (closed under scaling).
(iii) The zero vector $f(x) = 0$ is continuous.

(3) Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be vectors in a vector space V. Show that $U := \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a subspace of V.

Ans:
U must be a subspace of V, because it can be shown that it has all the required properties:

1. By definition $U \subseteq V$ because $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$
2. Since span is the linear combination of all the vectors, and that includes multiplying them all by zero, U contains the zero element.

3&4. Since span is the linear combination of all the vectors, this includes all possible scalings and additions of those vectors, making their span closed under addition and scaling.

(4) Let $A \in \mathbb{R}^{m \times n}$. Prove that Nul$(A)$ is a subspace of $\mathbb{R}^n$.

Ans:
The nullspace of A must be a subspace of $\mathbb{R}^n$ because it can be represented as the span of some number of n dimensional vectors representing the set of homogenous linear equations defined by $Ax = 0$. As was found in the previous question, the fact that this is a span means that it is a subspace.

(5) Explain whether the following are true or false (give counter examples if possible):
(a) Every vector space is a subspace of itself.
(b) Each plane in $\mathbb{R}^3$ is a subspace.
(c) Let U be a subspace of a vector space V. Any linear combination of vectors of U is also in V.
(d) Let $v_1, \ldots, v_n$ be in a vector space V. Then Span$(v_1, \ldots, v_n)$ is the smallest subspace of V containing $v_1, \ldots, v_n$.

Ans:

(a) **True**
Every vector space is by definition closed under addition and multiplication, has its own zero vector and is a(n impropper) subset of itself.
(b) **False**
Some planes don't have the zero vector in them.Ex: $x = 5$
(c) **True**
The elements in U are closed over linear combination, and U's elements are by definition a subset of V's, so all the linear combinations of elements in U are in V.

(d) **True**
Since the subspace must be closed under addition and scaling, any subspace containing the vectors $v_1, \dots, v_n$ must be at least as large as $\mathrm{Span}(v_1, \dots, v_n)$.

(6) Are the vectors $\mathbf{v}_0 = 1$, $\mathbf{v}_1 = t$, $\mathbf{v}_2 = t^2$ in the vector space $\mathbb{R}^{\mathbb{R}} := \{f\colon \mathbb{R} \to \mathbb{R}\}$ linearly independent?

Ans:
Yes, they are linearly independant because in order to go from $\mathbf{v}_0$ to $\mathbf{v}_1$ to $\mathbf{v}_2$ you would have to multiply by t in each step. The multiplication allowed in vector spaces is only by constant scalars, not variables, and there is no other way to generate these elements from one another.

(7) Which of the following are bases of $\mathbb{R}^3$? Why or why not?

$$A = (\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\3\\4\end{bmatrix}), B = (\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\3\\4\end{bmatrix}, \begin{bmatrix}0\\-1\\4\end{bmatrix}), C = (\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\3\\4\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix})$$

Ans:

(a) Not a basis
The basis for $\mathbb{R}^3$ must have 3 vectors in it because it is 3 dimensional.

(b) Not a basis
$-2 \cdot \begin{bmatrix}1\\2\\0\end{bmatrix} + \begin{bmatrix}2\\3\\4\end{bmatrix} = \begin{bmatrix}0\\-1\\4\end{bmatrix}$ so they are not linearly independant.

(c) Basis
By row reducing the matrix $\begin{bmatrix}1 & 2 & 0\\2 & 3 & 1\\0 & 4 & 1\end{bmatrix}$ We get $\begin{bmatrix}1 & 2 & 0\\0 & -1 & 1\\0 & 0 & 5\end{bmatrix}$. We can see from this that since there are no free variables and a pivot in every row, the vectors are independent and thus are a basis of $\mathbb{R}^3$

(8) Give a basis for $\mathrm{Nul}(A)$ and a basis for $\mathrm{Col}(A)$ for

$$A = \begin{bmatrix}0 & 2 & 0 & 3\\1 & -4 & -1 & 0\\-2 & 6 & 2 & -3\end{bmatrix}$$

Ans:
First, we put A in reduced echelon form:

$$\begin{bmatrix}1 & 0 & -1 & 6\\0 & 1 & 0 & \frac{3}{2}\\0 & 0 & 0 & 0\end{bmatrix}$$

- **Nul(A):** Using the echelon form of A, we can get a set of homogenous equations:
$$x_1 = x_3 - 6x_4, x_2 = -\frac{3}{2}x_4$$
From this we get the solution to the homogenous equation: $x = s \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -6 \\ -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix}$ where $s = x_3$ and $t = x_4$. These two vectors are the basis for the nullspace.
- **Col(A):** We see that there are two pivot columns in the echelon form of the matrix. We take what these columns were in the original matrix as the basis of the column-space. That is, $\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$ is the basis.