Math 2135 - Assignment 2

Due September 14, 2024
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1. Is $\mathbf{b}$ a linear combination of the vectors $\mathbf{a}_1, \mathbf{a}_2$?

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} -2 \\ -3 \\ 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

Ans:

- Create matrix to represent system:

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & -2 \\ 1 & 3 & 3 \end{bmatrix}$$

- Row reduce:

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & 22 \end{bmatrix}$$

- $\mathbf{b}$ is not a linear combination of the vectors because the the system represented by the augmented matrix is inconsistent.

2. Is $\mathbf{b} \in \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ for

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}?$$

Ans:

- Create matrix to represent system:

$$\begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 1 & -2 & 5 & 9 \end{bmatrix}$$

- Row reduce:

$$\begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 0 & 0 & 11 & -2 \end{bmatrix}$$

- $\mathbf{b}$ is a linear combination of the vectors because the the system represented by the augmented matrix is consistent.

3. For which values of a is $\mathbf{b}$ in the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$?

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} a \\ -3 \\ -5 \end{bmatrix}$$

Ans:

- Create matrix to represent system:

$$\begin{bmatrix} 1 & -2 & a \\ 0 & 1 & -3 \\ -2 & 7 & -5 \end{bmatrix}$$

- Row reduce:

$$\begin{bmatrix} 1 & -2 & a \\ 0 & 1 & -3 \\ 0 & 0 & 4+2a \end{bmatrix}$$

- The system must be consistent for $\mathbf{b}$ to be in the plane. The last row represents the equation $0 = 4 + 2a$, so if we solve this, we find the value of a that allows the system to be consistent. When the equation is solved we get $a = -2$ as the solution.

4. Find vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^3$ that span the plane in $\mathbb{R}^3$ with equation $x - 2y + 3z = 0$. How many do you need?

Hint: Write down a parametrized solution for the equation.

Ans:

- We can see from the equation that the plane goes through $(0, 0, 0)$. We can choose two arbitrary non-colinear points and the vectors to those points will span the plane.

- Setting $x = 0$ and $y = 3$, we get $z = 2$ and the point $(0, 3, 2)$. Setting $y = 0$ and $x = 3$ we get $z = -1$ and the point $(3, 0, -1)$.
- The vectors from the origin to these point are $\begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$.
- Since these vectors both lie on the plane and are not colinear, they span the plane.

5. Are the following true or false? Explain your answers.

   (a) For every $A \in \mathbb{R}^{2\times 3}$ with 2 pivots, $Ax = 0$ has a nontrivial solution.

   (b) For every $A \in \mathbb{R}^{2\times 3}$ with 2 pivots and every $\mathbf{b} \in \mathbb{R}^2$, $Ax = \mathbf{b}$ is consistent.

   (c) The vector $3\mathbf{v}_1$ is a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2$.

   (d) For $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^3$, $\text{Span}(\mathbf{v}_1, \mathbf{v}_2)$ is always a plane through the origin.

Ans:

(a) False, A in reduced echelon form will always be the same, and so the augmented matrix that represents this system will also always be the same. This matrix is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Which has the only solution $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which is the trivial solution.

(b) True, this will always produce an augmented matrix in row echelon form that looks like this:

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix}, a, b \in \mathbb{R}$$

The last row is all zeroes because $\mathbf{b} \in \mathbb{R}^2$ not $\mathbb{R}^3$,$\dot{\text{T}}$his will always have the solution $\begin{bmatrix} a \\ b \end{bmatrix}$.

(c) True, $3\mathbf{v}_1 + 0\mathbf{v}_2 = 3\mathbf{v}_1$

(d) False, if the vectors are colinear then their span will only be a line.

6. [1, cf. Section 1.5, Ex 17] Let

$$A = \begin{bmatrix} 2 & 2 & 4 \\ -4 & -4 & -8 \\ 0 & -3 & -3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 8 \\ -16 \\ 12 \end{bmatrix}, \quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solve the equations $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$. Express both solution sets in parametric vector form. Give a geometric description of the solution sets.

Ans:

- Augmented Matrix:

$$\begin{bmatrix} 2 & 2 & 4 & 8 \\ -4 & -4 & -8 & -16 \\ 0 & -3 & -3 & 12 \end{bmatrix}$$

- Row reduce:

$$\begin{bmatrix} 1 & -1 & 0 & 12 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Solve for $Ax = b$:
Equation 2: $x_2 = -x_3 - 4$ Equation 1: $x_1 = 12 + x_2 = 8 - x_3$ if we set $t = x_3$ we get the following parametric equation: $x = \begin{bmatrix} 8 \\ -4 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$
The system of equations represents a line that is the intersection of 2 planes. The line goes through the point $(8, -4, 0)$.

(b) Solve for $Ax = 0$;
The row reduced array is the same exept the rightmost column is all zeroes.
Equation 2: $x_2 = -x_3$ Equation 1: $x_1 = x_2 = -x_3$
if we set $t = x_3$ we get the following parametric equation: $x = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$
The system of equations represents a line that is the intersection of 2 planes. The line goes through the origin.

7. (a) Which of the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are in the nullspace of A, Null A?

$$\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -2 \\ 0 \\ 4 \\ -2 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 2 & 4 \\ 2 & -4 & 1 & 0 \\ -3 & 6 & 2 & 7 \end{bmatrix}$$

(b) Solve $A\mathbf{x} = \mathbf{0}$ and give the solution in parametric vector form.

(c) Find vectors $v_1, \ldots, v_k \in \mathbb{R}^4$ such that $\text{Null}\, A = \text{Span}\{v_1, \ldots, v_k\}$.

Ans:

(a) i.

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

is in the nullspace because it is clear that if we multiply A by all zeroes we will get the zero vector.

ii.

$$\begin{bmatrix} -2 \\ 0 \\ 4 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 2 & 4 \\ 2 & -4 & 1 & 0 \\ -3 & 6 & 2 & 7 \end{bmatrix} = \begin{bmatrix} 0+0+8-8 \\ -4+0+4+0 \\ 6+0+8-14 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

so the vector is in the nullspace of A.

iii.

$$\begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 2 & 4 \\ 2 & -4 & 1 & 0 \\ -3 & 6 & 2 & 7 \end{bmatrix} = \begin{bmatrix} 0+0-4+4 \\ 2-4-2+0 \\ -3+6-4+7 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \\ 6 \end{bmatrix}$$

so the vector is not in the nullspace of A.

(b)

- Augmented matrix:

$$\begin{bmatrix} 0 & 0 & 2 & 4 & 0 \\ 2 & -4 & 1 & 0 & 0 \\ -3 & 6 & 2 & 7 & 0 \end{bmatrix}$$

- Row reduce:

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Equations:
  Equation 2: $x_3 = -2x_4$. Equation 1: $x_1 = 2x_2 + x_4$. x_2 and x_4 are both free so we set $t = x_2$ and $u = x_4$.
- Parametric equation:

$$x = t \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + u \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

(c) The span of a system is just the set of vectors for which all possible linear combination of them is equivilant to the system's solutions. Therefore we can just use the vector from the previous question, so

$$\text{Null}\, A = \text{Span}\{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}\}$$

8. Show the following:

Theorem. Suppose $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{p}$. Then the set of all solutions of $A\mathbf{x} = \mathbf{b}$ is

$$\mathbf{p} + \text{Null}\, A = \{\mathbf{p} + \mathbf{v} \mid \mathbf{v} \in \text{Null}\, A\}.$$

Hint: For the proof suppose $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{p}$ and use 2 steps:

(a) Show that if $\mathbf{v}$ is in $\text{Null}\, A$, then $\mathbf{p} + \mathbf{v}$ is also a solution for $A\mathbf{x} = \mathbf{b}$.

(b) Show that if $\mathbf{q}$ is a solution for $A\mathbf{x} = \mathbf{b}$, then $\mathbf{q} - \mathbf{p}$ is in $\text{Null}\, A$.

Ans:

- First, suppose that $A\mathbf{x} = \mathbf{b}$ has some solution $\mathbf{x} = \mathbf{p}$.
- If we then take any $\mathbf{v} \in \text{Null}\, A$, by definition $A\mathbf{v} = 0$. Thus, if we take $A \cdot (\mathbf{p} + \mathbf{v})$, we get $A\mathbf{v} + A\mathbf{p} = 0 + \mathbf{b} = \mathbf{b}$.
- This means that every vector that can be represented in the form $\mathbf{p}+\mathbf{v}$ is a solution for $A\mathbf{x} = \mathbf{b}$.
- If we then take $\mathbf{q} - \mathbf{p}$ s.t. $\mathbf{q}$ is a solution of $A\mathbf{x} = \mathbf{b}$, we see that $A \cdot (\mathbf{q} - \mathbf{p}) = A\mathbf{q} + A(-\mathbf{p}) = \mathbf{b} - \mathbf{b} = 0$.
- This means that every vector that can be represented in the form $\mathbf{q}-\mathbf{p}$ is in $\text{Null}\, A$.
- Together, these two facts show us that $\mathbf{p}+\mathbf{v} = \mathbf{q}$, which means that the sum of an arbitrary solution of A and an arbitrary member of $\text{Null}\, A$ generate another solution of A.
- This means that if we take $\text{Null}\, A$, then $\mathbf{p} + \text{Null}\, A = \mathbf{Q}$ where $\mathbf{Q}$ is the set of all solutions of $A\mathbf{x} = \mathbf{b}$.

References

[1] David C. Lay, Steven R. Lay, and Judi J. McDonald. Linear Algebra and Its Applications. Addison-Wesley, 5th edition, 2015.