Math 2135 - Assignment 3

Due September 20, 2024
Completed September 19, 2024, Maxwell Rodgers

(1) Which of the following sets of vectors are linearly independent?

(a) $\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -11 \\ 0 \end{bmatrix}$

Ans:

(a)
- Create Matrix:

$$\begin{bmatrix} 0 & 2 & 1 \\ -1 & 1 & 0 \\ 4 & 3 & -2 \end{bmatrix}$$

- Row Reduce:

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -\frac{11}{2} \end{bmatrix}$$

These Vectors are linearly independant because the row reduction of them in matrix form leads to a pivot in every column and every row (this means they only have the trivial solution to $Ax = \mathbf{0}$)

(b)
- Create Matrix:

$$\begin{bmatrix} 1 & 2 & -1 \\ -3 & 1 & 11 \\ 2 & 3 & 0 \end{bmatrix}$$

- Row Reduce:

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

These vectors are linearly dependant because the row reduction of the matrix with them as columns leads to a system with non trivial solutions to $Ax = \mathbf{0}$ (because one of the rows is all zeroes.

(2) Explain whether the following are true or false:

(a) Vectors $\mathbf{v}_1, \mathbf{v}_2, v_3$ are linearly dependent if $\mathbf{v}_2$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_3$.
(b) A subset $\{\mathbf{v}\}$ containing just a single vector is linearly dependent iff $\mathbf{v} = \mathbf{0}$.
(c) Two vectors are linearly dependent iff they lie on a line through the origin.
(d) There exist four vectors in $\mathbb{R}^3$ that are linearly independent.

Ans:

(a) **True**: If $\mathbf{v}_2$ is a linear combination of two other vectors, then there will be at least one non-trivial solution to $Ax = \mathbf{0}$. We can see this because $n\mathbf{v}_1 + m\mathbf{v}_3 = \mathbf{v}_2, m, n \in \mathbb{R}$ (since $\mathbf{v}_2$ is a linear combination of the other two) which implies $n\mathbf{v}_1 + m\mathbf{v}_3 - \mathbf{v}_2 = 0$.

(b) **True**: There is no non-trivial solution for one non-zero vector, and $Ax = \mathbf{0}$ is certainly true for x being any non-trivial multiple of the zero vector.

(c) **True**: If the vector lie on the same line, they are going to be multiples of each other such that $n\mathbf{v}_1 = \mathbf{v}_2$, which means that each is a linear combination of the other.

(d) **False**: If we row reduce the matrix formed by these 4 vectors, there will always be a column without a pivot, which means that there will always be at least one free variable in $Ax = \mathbf{0}$. This means there is a non-trivial solution and the vectors must be linearly dependant.

(3) Show: If any of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is the zero vector (say $\mathbf{v}_i = \mathbf{0}$ for $i \leq n$), then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent.

Ans:
Say we have sove vector $\mathbf{v}_i = \mathbf{0}$ and some other vector $\mathbf{v}_j \neq \mathbf{0}$. Since $0\mathbf{v}_j = \mathbf{0} = \mathbf{v}_i$, $\mathbf{v}_i$ is a linear combination of the other vectors and thus the set of vectors is linearly dependant.

(4) Show: If $n > m$, then any n vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^m$ are linearly dependent.

(5) Show that the following maps are not linear by giving concrete vectors for which the defining properties of linear maps are not satisfied.

(a) $f : \mathbb{R}^2 \to \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x+1 \\ y+3 \end{bmatrix}$

(b) $g : \mathbb{R}^2 \to \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} xy \\ y \end{bmatrix}$

(c) $h : \mathbb{R}^2 \to \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} |x| + |y| \\ 2x \end{bmatrix}$

Ans:
Transformation $T : \mathbb{R}^n \to \mathbb{R}^m \ n, m \in \mathbb{R}$ is linear if:

1: $T(v_1) + T(v_2) = T(v_1 + v_2) \ v_1, v_2 \in \mathbb{R}^n$
2: $cT(v_3) = T(cv_3) \ v_3 \in \mathbb{R}^n, \ c \in \mathbb{R}$

(a) $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, c = 2$

$T(v_1) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, T(v_2) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$T(v_1) + T(v_2) = \begin{bmatrix} 5 \\ 9 \end{bmatrix} \neq T(v_1 + v_2) = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$ (violates 1)

$cT(v_1) = \begin{bmatrix} 4 \\ 8 \end{bmatrix} \neq T(cv_1) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ (violates 2)

(b) $v_1 = \begin{bmatrix}1\\1\end{bmatrix}, v_2 = \begin{bmatrix}2\\2\end{bmatrix}, c = 2$

$T(v_1) = \begin{bmatrix}1\\1\end{bmatrix}, T(v_2) = \begin{bmatrix}4\\2\end{bmatrix}$

$T(v_1) + T(v_2) = \begin{bmatrix}5\\3\end{bmatrix} \neq T(v_1 + v_2) = \begin{bmatrix}9\\3\end{bmatrix}$ (violates 1)

$cT(v_1) = \begin{bmatrix}2\\2\end{bmatrix} \neq T(cv_1) = \begin{bmatrix}4\\2\end{bmatrix}$ (violates 2)

(c) $v_1 = \begin{bmatrix}1\\1\end{bmatrix}, v_2 = \begin{bmatrix}-1\\-1\end{bmatrix}, c = -1$

$T(v_1) = \begin{bmatrix}2\\2\end{bmatrix}, T(v_2) = \begin{bmatrix}2\\-2\end{bmatrix}$

$T(v_1) + T(v_2) = \begin{bmatrix}4\\0\end{bmatrix} \neq T(v_1 + v_2) = \begin{bmatrix}0\\0\end{bmatrix}$ (violates 1)

$cT(v_1) = \begin{bmatrix}-2\\-2\end{bmatrix} \neq T(cv_1) = \begin{bmatrix}2\\-2\end{bmatrix}$ (violates 2)

(6) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map such that

$$T(\begin{bmatrix}1\\0\\0\end{bmatrix}) = \begin{bmatrix}-1\\2\\0\end{bmatrix}, \quad T(\begin{bmatrix}0\\1\\0\end{bmatrix}) = \begin{bmatrix}-3\\0\\1\end{bmatrix}.$$

Use the linearity of T to compute $T(\begin{bmatrix}2\\3\\0\end{bmatrix})$ and $T(\begin{bmatrix}1\\2\\3\end{bmatrix})$. What is the issue with the latter?

(7) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear map such that

$$T(\begin{bmatrix}1\\2\end{bmatrix}) = \begin{bmatrix}2\\0\\-3\end{bmatrix}, \quad T(\begin{bmatrix}3\\2\end{bmatrix}) = \begin{bmatrix}-2\\2\\1\end{bmatrix}.$$

(a) Use the linearity of T to find $T(\begin{bmatrix}1\\0\end{bmatrix})$ and $T(\begin{bmatrix}0\\1\end{bmatrix})$.

(b) Determine $T(\begin{bmatrix}x\\y\end{bmatrix})$ for arbitrary $x, y \in \mathbb{R}$.

(8) Give the standard matrices for the following linear transformations:

(a) $T : \mathbb{R}^2 \to \mathbb{R}^3, \begin{bmatrix}x\\y\end{bmatrix} \mapsto \begin{bmatrix}2x + y\\x\\-x + y\end{bmatrix}$;

(b) the function S on $\mathbb{R}^2$ that scales all vectors to half their length.