Math 2135 - Assignment 11

Due November 15, 2021

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(1) Compute the determinants by row reduction to echelon form:

$$A = \begin{bmatrix} 3 & 3 & -3 \\ 3 & 4 & -4 \\ 2 & -3 & -5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 7 & 6 & -3 \\ -3 & -10 & -7 & 2 \end{bmatrix}$$

Ans:

(a) $|A| = \begin{vmatrix} 3 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -8 \end{vmatrix} = (3)(1)(-8) = -24$

(b) $|B| = \begin{vmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & -10 \end{vmatrix} = (-10)(1)(1)(1) = -10$

(2) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} u & v \\ w & x \end{bmatrix}$. Show

$$\det(AB) = \det(A)\det(B).$$

Ans:
$\det(A)\det(B) = (ad - bc)(ux - vw) = adux - advw - bcux + bcvw = (acuv) + adux + bcvw + (bdwx) - (acuv) - advw - bcux - (bdwx) = (au + bw)(cv + dx) - (av + bx)(cu + dw) = \begin{vmatrix} au + bw & av + bx \\ cu + dw & cv + dx \end{vmatrix} = \det(AB)$

(3) Let $A \in \mathbb{R}^{n\times n}$. Are the following true or false? Explain why:

(a) If two rows or columns of A are identical, then $\det A = 0$.
(b) For $c \in \mathbb{R}$, $\det(cA) = c \det A$.
(c) If A is invertible, then $\det A^{-1} = \frac{1}{\det A}$.
(d) A is invertible iff 0 is not an eigenvalue of A.

Ans:

(a) TRUE
This means that the row reduced form of the matrix will have a zero row which means det A = 0.

(b) FALSE
Since the determinant multiplies elements of the matrix, $\det(cA)$ will be more like $c^n \det A$

(c) TRUE
Since we just showed that $\det(AB) = \det(A)\det(B)$, $\det(AA^{-1}) = \det(I) = 1 = \det(A)\det(A^{-1})$ meaning that the determinant of A and the inverse of A must be multiplicative inverses of each other.

(d) TRUE
If A has an eigenvalue of 0 this means that the nullspace of A is non-trivial (bc. there exists $x \neq 0$ s.t. $Ax = 0$) and therfore the columns of A are not linearly independant and A doesn't have an inverse.

(4) Eigenvalues, -vectors and -spaces can be be defined for linear maps just as for matrices.

Let $h\colon V \to W$ be a linear map for vector spaces V, W over F. Show that the eigenspace for $\lambda \in F$,

$$E_{h,\lambda} := \{x \in V \ : \ h(x) = \lambda x\},$$

is a subspace of V.

Ans:
We know that $E_{h,\lambda}$ must be a subspace of V because
1) all the elements of $E_{h,\lambda}$ are in V as well
2) since $E_{h,\lambda}$ is an eigenspace it must be closed under addition and scalar multiplication.
3) it includes the zero element of v ($\lambda(x)$ for $\lambda = 0$ or equivilant)

(5) Give all eigenvalues and bases for eigenspaces of the following matrices. Do you need the characteristic polynomials?

$$A = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 3 \end{bmatrix}$$

Ans:
The Eigenvalue of A is -3 and the Eigenvalues for B are 0,2, and 3.
This means that the eigenspace of A is spanned by the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (from row reduced augmented matrix $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$)
and the eigenspaces of B are spanned by the vectors $\left[\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 1 \\ \frac{1}{2} \\ 1 \end{smallmatrix}\right]$, and $\left[\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right]$
(from the nullspaces of $0\colon \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}, 2: \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}, 3: \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$)
We don't need to use the characteristic polynomial because these matrices are both triangular.

(6) Give the characteristic polynomial, all eigenvalues and bases for eigenspaces for $C = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$.

Ans:
$(1-\lambda)^2 - 6 = 0)$ is the characteristic polynomial so the eigenvalues are $1 \pm \sqrt{6}$
We can find the eigenvectors by reducing the matrix $\begin{bmatrix} \mp\sqrt{6} & 2 \\ 3 & \mp\sqrt{6} \end{bmatrix}$
$\begin{bmatrix} \mp\sqrt{6} & 2 \\ 0 & 0 \end{bmatrix}$ which gives the eigenvectors $\begin{bmatrix} \frac{2}{\mp\sqrt{6}} \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{6}}{3} \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{6}}{3} \\ 1 \end{bmatrix}$

(7) Compute eigenvalues and eigenvectors for $D = \begin{bmatrix} -1 & 4 & 1 \\ 6 & 9 & 2 \\ 0 & 0 & -3 \end{bmatrix}$.

Ans:
Polynomial: $(-3-\lambda)((-1-\lambda)(9-\lambda) - 24) = -(\lambda+3)(\lambda+3)(\lambda-11) = 0$
This means the eigenvalues are -3 and 11. Taking the matrices $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 3 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ which corespond to the eigenvectors $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{3} \\ 1 \\ 0 \end{bmatrix}$

(8) Are the matrices A, B, C, D in (5), (6), (7) diagonalizable? How?

Ans:
Both B and C are diagonalizable but A and D are not because they do not have as many eigenspaces as they have rows/columns.
The diagonalized matrices of B and C would just be the eigenvalues of those matrices put in the diagonal of an n by n matrix with all the other entries set to 0.