Math 2135 - Assignment 13

Due December 9, 2024
Maxwell Rodgers

(1) (a) Give 3 vectors of length 1 in $\mathbb{R}^3$ that are orthogonal to $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.
(b) Which of the following sets are orthogonal? Orthonormal?

$$A = \{\begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}, \begin{bmatrix} 0.8 \\ -0.6 \end{bmatrix}\}, \qquad B = \{\frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{18}}\begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}\}$$

Ans:

(a) We need to solve the equation $x - y + 2z = 0$ by setting each variable equal to zero in turn, we can easily find orthogonal vectors: $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ and now we just have to normalize them: $\frac{\sqrt{5}}{5}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \frac{\sqrt{5}}{5}\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \frac{\sqrt{5}}{5}\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

(b) Set A is orthonormal because $\begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix} \cdot \begin{bmatrix} 0.8 \\ -0.6 \end{bmatrix} = 0.48 - 0.48 = 0$(orthogonal) and both vectors have length $\sqrt{0.36 + 0.64} = 1$(normal).
Set B is also orthonormal because $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} = 4 - 2 - 2 = 0$ (orthogonal). and both have length of 1. $\frac{\sqrt{1+4+4}}{3} = \frac{3}{3} = 1$, $\frac{\sqrt{16+1+1}}{\sqrt{18}} = \frac{\sqrt{18}}{\sqrt{18}} = 1$

(2) (a) Let W be the subspace of $\mathbb{R}^3$ with orthonormal basis $B = (\frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix})$. Compute the coordinates $[x]_B$ for $x = \begin{bmatrix} 7 \\ 4 \\ 4 \end{bmatrix}$ in W using dot products.
(b) Give a basis for $W^{\perp}$.
(c) Find the closest point to $y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in W. What is the distance from y to W?

Ans:

(a) $[x]_B = \begin{bmatrix} x \cdot \mathbf{u}_1 \\ x \cdot \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}((7)(2) + (4)(-1) + (2)(4)) \\ \frac{1}{\sqrt{5}}((7)(1) + (4)(2) + (4)(0)) \end{bmatrix} = \begin{bmatrix} 6 \\ 3\sqrt{5} \end{bmatrix}$

(b) Since W is 2 dimensional and in $\mathbb{R}^3$ we know that $W^{\perp}$ is 1 dimensional. we can use the cross product of the basis vectors of W: $\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 5 \end{bmatrix}$ =basis of $W^{\perp}$.

(c) We find the point at the tip of the projection of y onto W: $\frac{1}{9}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}) + \frac{1}{5}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}) = \frac{2}{3}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 7 \\ 4 \\ 4 \end{bmatrix} = (\frac{7}{3}, \frac{4}{3}, \frac{4}{3})$
The distance from y to this point is $|\frac{1}{3}(\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} - \begin{bmatrix} 7 \\ 4 \\ 4 \end{bmatrix})| = \frac{1}{3}\sqrt{-4^2 + 2^2 + 5^2} = \sqrt{5}$

(3) True or false. Explain your answers.
(a) Every orthogonal set is also orthonormal.
(b) Not every orthonormal set in $\mathbb{R}^n$ is linearly independent.
(c) For each x and each subspace W, the vector $x - \mathrm{proj}_W(x)$ is orthogonal to W.

Ans:

(a) FALSE
It is possible for vectors to be perpendicular to each other and not have magnitude of 1.

(b) FALSE
In order to be an orthonormal set all of the vectors must be perpendicular to each other which means that they cannot be linear combinations of any of the other vectors.

(c) TRUE
This is just the same as representing x as the sum of 2 vectors, one of them being in W, forcing the other one to be perpendicular to the first and therefore perpendicular to the whole of W.

(4) Let W be a subset of $\mathbb{R}^n$. Show that its orthogonal complement

$$W^\perp := \{x \in \mathbb{R}^n \mid x \text{ is orthogonal to all } w \in W\}$$

is a subspace of $\mathbb{R}^n$.

Ans:
Take $\mathbf{u} \in W \;\; \mathbf{v}, \mathbf{w} \in W^\perp \;\; c \in \mathbb{R}$

- Since $\mathbf{v}$ and $\mathbf{u}$ are in perpendicular spaces, they must be perpendicular to each other, and thus $\mathbf{u} \cdot \mathbf{v} = 0$. From the properties of dot products we can see that $\mathbf{u} \cdot c\mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = 0(c) = 0$ The same can be said for $\mathbf{u}$ and $\mathbf{w}$. This means that $c\mathbf{v} \in W^\perp$
- We can also see then that $\mathbf{u} \cdot (\mathbf{w} + \mathbf{v}) = \mathbf{u} \cdot \mathbf{w} + \mathbf{u} \cdot \mathbf{v} = 0 + 0 = 0$ so $(\mathbf{w} + \mathbf{v}) \in W^\perp$

Thus $W^\perp$ is a subspace of $\mathbb{R}^n$ because all of its elements are from $\mathbb{R}^n$, and it is closed under addition and scalar multiplication.

(5) Let W be a subspace of $\mathbb{R}^n$. Show that
(a) $W \cap W^\perp = 0$
(b) $\dim W + \dim W^\perp = n$
Hint: Let $w_1, \ldots, w_k$ be a basis of W. Use that $x \in W^\perp$ iff x is orthogonal to $w_1, \ldots, w_k$.

Ans:

(a) Since all of the elements in $W^\perp$ have to be perpendicular to the elements in W, the only element that they could share would be an element that is perpedicular to itself. This would mean that the dot product between this element and itself would have to be zero, meaning that it must be the zero vector 0.

(b) Take an orthogonal basis of $\mathbb{R}^n$, $(u_1, \ldots, u_n)$. This basis can be split into two parts by chosing any subset and its compliment. Each of these subsets is an orthogonal basis of a subspace. Additionaly, the subspaces defined by these basis would be perpendicular because every basis vector in each one would be orthogonal to every basis vector in the other (because we started with an orthogonal basis). Thus we get two subspaces W and $W^{\perp}$. Since the bases of theses subspaces have a total of n elements between them, $\dim W + \dim W^{\perp} = n$.

(6) Find the least squares solutions of $Ax = b$.

(a) $A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$, $b = \left[\begin{smallmatrix} 5 \\ 1 \\ 0 \end{smallmatrix}\right]$ (b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, $b = \left[\begin{smallmatrix} -1 \\ 2 \\ -3 \\ 4 \end{smallmatrix}\right]$

Ans:
From theorem 13 we know that the least squares solutions can be found by solving $A^TAx = A^Tb$

(a) $A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}$

$A^Tb = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \left[\begin{smallmatrix} 5 \\ 1 \\ 0 \end{smallmatrix}\right] = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$

So now we can row reduce the augmented matrix $\begin{bmatrix} 3 & 3 & 6 \\ 3 & 11 & 14 \end{bmatrix} \to \begin{bmatrix} 3 & 3 & 6 \\ 0 & 8 & 8 \end{bmatrix} \to$ $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ so $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(b) $A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$

$A^Tb = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \left[\begin{smallmatrix} -1 \\ 2 \\ -3 \\ 4 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 2 \\ 1 \\ 1 \end{smallmatrix}\right]$

So now we row reduce the augmented matrix $\begin{bmatrix} 4 & 2 & 2 & 2 \\ 2 & 2 & 0 & 1 \\ 2 & 0 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

So $x = \left[\begin{smallmatrix} \frac{1}{2} \\ 0 \\ 0 \end{smallmatrix}\right] + x_3 \left[\begin{smallmatrix} -1 \\ 1 \\ 1 \end{smallmatrix}\right]$

(7) True or false for $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Explain your answers.
(a) A least squares solution of $Ax = b$ is an $\hat{x}$ such that $A\hat{x}$ is as close as possible to b.
(b) A least squares solution of $Ax = b$ is an $\hat{x}$ such that $A\hat{x} = \hat{b}$ for $\hat{b}$ the orthogonal projection of b onto $\operatorname{Col} A$.
(c) The point in $\operatorname{Col} A$ closest to b is a least squares solution of $Ax = b$.
(d) If $Ax = b$ is consistent, then every solution x is a least squares solution.

Ans:

(a) TRUE
Least squares means that the euclidian distance between the solution's output and the desired output is minimized.

(b) TRUE
Since a) is true this is also true because $\hat{b}$ is the closest possible solution to b.

(c) TRUE
See two explanations above.

(d) TRUE
The difference can't be less than 0 so I guess this is true.