Math 2135 - Assignment 9

Completed October 31, 2024
Maxwell Rodgers

(1) Let $b_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $b_2 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$, $b_3 = \begin{bmatrix} 1 \\ 2.5 \\ -5 \end{bmatrix}$.

(a) Find vectors $u_1, \ldots, u_k$ such that $(b_1, b_2, u_1, \ldots, u_k)$ is a basis for $\mathbb{R}^3$.

(b) Find vectors $v_1, \ldots, v_\ell$ such that $(b_3, v_1, \ldots, v_\ell)$ is a basis for $\mathbb{R}^3$.

Prove that your choices for (a) and (b) form a basis.

Ans:

(a) The only vector that is needed to make a basis with b_1 and b_2 is $b_1 \times b_2 = \begin{bmatrix} 7 \\ -4 \\ 1 \end{bmatrix} = u_1$ We know this is a basis because there are three vectors and they are linearly independant. We can see this in the row reduction of the matrix $\begin{bmatrix} 1 & 1 & 7 \\ 2 & 1 & -4 \\ -1 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 7 \\ 0 & -1 & -18 \\ 0 & 0 & -64 \end{bmatrix}$ because there is a pivot in every column.

(b) We need two linearly independant vectors who's spanned plane does not conatin b_3. The vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ fit this description. This is a basis because there are three vectors and they are linearly independant. We can see this by looking at the matrix of the basis $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2.5 \\ 0 & 0 & -5 \end{bmatrix}$ which is already in echelon form and has a basis in every column.

(2) A 25×35 matrix A has 20 pivots. Find $\dim \operatorname{Nul} A$, $\dim \operatorname{Col} A$, $\dim \operatorname{Row} A$, and $\operatorname{rank} A$.

Ans:

$\dim \operatorname{Nul} A = 35 - 20 = 15$

$\dim \operatorname{Col} A = \dim \operatorname{Row} A = \operatorname{rank} A = 20$

(3) True or false? Explain.

(a) A basis of B is a set of linear independent vectors in V that is as large as possible.

(b) If $\dim V = n$, then any n vectors that span V are linearly independent.

(c) Every 2-dimensional subspace of $\mathbb{R}^2$ is a plane.

Ans:

(a) True

A basis must be linearly independant and have n vectors for an n dimensional space. A set of linearly independant vectors in n dimensional space V can have at most n vectors.

(b) True

If n vectors are not linearly independant then the space they span is less that n dimensional so they cannot span V.

(c) True
Every 2 dimensional subspace of $\mathbb{R}^2$ is just $\mathbb{R}^2$, but $\mathbb{R}^2$ is a plane so technicaly this is true.

(4) Let P_3 the vector space of polynomials of degree ≤ 3 over $\mathbb{R}$ with basis $B = (1, x, x^2, x^3)$.
(a) Find the matrix $d_{B\leftarrow B}$ for the derivation map $d\colon P_3 \to P_3, p \to p'$.
(b) Use $d_{B\leftarrow B}$ to compute $[p']_B$ and p' for the polynomial p with $[p]_B = (-3, 2, 0, 1)$.

Ans:

(a) Since we know how each of the unit vectors transform (what their derivatives are) we can easily make a matrix: $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
(b) $d_{B\leftarrow B}[p]_B = [p']_B = (2, 0, 3, 0)$ which means that $p' = 2 + 3x^2$

(5) Let $B = (\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix})$ and $C = (\begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix})$ be bases of $\mathbb{R}^2$, let E be the standard basis of $\mathbb{R}^2$.
(a) Find the change of coordinates matrix $P_{E\leftarrow B}$ for $f: [u]_B \mapsto [u]_E$.
(b) Find the change of coordinates matrix $P_{C\leftarrow E}$ for $g: [u]_E \mapsto [u]_C$.
(c) Find the change of coordinates matrix $P_{C\leftarrow B}$ for $h: [u]_B \mapsto [u]_C$.
Hint: h is the composition of g and f, $h([u]_B) = g(f([u]_B))$.

Ans:

(a) This is just the basis vectors of B in a matrix because $[b_1]_B = e_1$. $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
(b) This is just the inverse of the matrix of the basis vectorss of C: $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$
(c) Since this is the composition of g and f we can just multiply the previously found matrices:
$\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -3 & -7 \end{bmatrix}$

(6) Determine the standard matrix for the reflection t of $\mathbb{R}^2$ at the line $3x + y = 0$ as follows:
(a) Find a basis B of $\mathbb{R}^2$ whose vectors are easy to reflect.
(b) Give the matrix $t_{B\leftarrow B}$ for the reflection with respect to the coordinate system determined by B.
(c) Use the change of coordinate matrix to compute the standard matrix $t_{E\leftarrow E}$ with respect to the standard basis $E = (e_1, e_2)$.

Ans:

(a) A vector on the line ($b_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$) and a vector orthogonal to the line ($b_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$) form a basis that is easy to reflect.

(b) We find that since $T(b_1) = b_1$ and $T(b_2) = -b_2$, these end up as e_1 and $-e_2$ when transformed to B. This leaves us with the matrix $t = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(c) To do this, we need a matrix to go from E to B, then B to B, then B to E (right to left). the B to E and E to B matrices will be inverses of each other and the B to E matrix will just be the basis vectors. $t_{E\leftarrow E} = P_{E\leftarrow B}T_{B\leftarrow B}P_{B\leftarrow E} = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{10} & -\frac{3}{10} \\ \frac{3}{10} & \frac{1}{10} \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} & -\frac{3}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{bmatrix}$

(7) (a) Determine the standard matrix A for the rotation r of $\mathbb{R}^3$ around the z-axis through the angle $\pi/3$ counterclockwise.
Hint: Use the matrix for the rotation around the origin in $\mathbb{R}^2$ for the xy-plane. What happens to e_3 under this rotation?

(b) Consider the rotation s of $\mathbb{R}^3$ around the line spanned by $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ through the angle $\pi/3$ counterclockwise. Find a basis of $\mathbb{R}^3$ for which the matrix $s_{B\leftarrow B}$ is equal to A from (a).

(c) Give the standard matrix $s_{E\leftarrow E}$ for the standard basis E (You do not need to actually multiply and invert the involved matrices; the product formula is enough).

Ans:

(a) The matrix for the rotation arround the origin of $\frac{\pi}{3}$ radians is $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$
from this we can extrapolate to the rotation arround the z axis
x and y don't depend on z and z just remains the same (because e_3 does not change): $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) We know that $[e_3]_B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ so this means we just need to transform the other unit vectors in the same way. Since all the vectors started the same length they also have to be the same length after the transform. (???)

(c) ...

(8) The *kernel* of a linear map $h\colon V \to W$ is the subspace of V,

$$\{v \in V \mid h(v) = 0\}.$$

(a) Determine the kernel and the image of $d\colon P_3 \to P_3, p \to p'$.

(b) Is d injective, surjective, bijective?

Ans:

(a) The kernel in this case is just the polynomials that only have a constant term (because the derivative of a constant is zero), so the basis is just the vector $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

The image is just P_2 because these are all the polynomials that you can get by deriving polynomials in P_3

(b) No, the range (P_2) is smaller than the codomain and the domain (both P_3) so it cannot be injective or surjective.