Math 2135 - Assignment 5

Due October 4, 2024

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Problems 1-5 are review material for the first midterm on September 29. Solve them before Wednesday!

(1) Let

$$A = \begin{bmatrix} 0 & 3 & 1 & 2 \\ 1 & 4 & 0 & 7 \\ 2 & -1 & -3 & 8 \end{bmatrix}, b = \begin{bmatrix} 6 \\ 5 \\ -8 \end{bmatrix}$$

(a) Give the solution for $Ax = b$ in parametrized vector form.
(b) Give vectors that span the null space of A.

Ans:

(a)

- Row reduce augmented matrix:

$$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & \frac{13}{3} & -3 \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Parametric form:

$$x_1 - \frac{4}{3}x_3 + \frac{13}{3}x_4 = -3, x_2 + \frac{1}{3}x_3 + \frac{2}{3}x_4 = 2$$

we have the two free variables $x_4 = t, x_3 = u$

$$x = \begin{bmatrix} -3 \\ 2 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{13}{3} \\ -\frac{2}{3} \\ 0 \\ 1 \end{bmatrix} + u\begin{bmatrix} \frac{4}{3} \\ -\frac{1}{3} \\ 1 \\ 0 \end{bmatrix}$$

(b) $\begin{bmatrix} -\frac{13}{3} \\ -\frac{2}{3} \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} \frac{4}{3} \\ -\frac{1}{3} \\ 1 \\ 0 \end{bmatrix}$ span the nullspace of A because the solution to the equation $Ax = 0$ is the same as the one in part a exept there is no constant term.

(2) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with

$$T(\begin{bmatrix} 1 \\ 2 \end{bmatrix}) = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \text{ and } T(\begin{bmatrix} 3 \\ 4 \end{bmatrix}) = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}.$$

What is the standard matrix of T?

Ans:

$$T(\begin{bmatrix} 3 \\ 4 \end{bmatrix}) - 2T(\begin{bmatrix} 1 \\ 2 \end{bmatrix}) = T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} - \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ -4 \end{bmatrix}$$

$\frac{1}{2}(T(\begin{bmatrix}1\\2\end{bmatrix}) - T(\begin{bmatrix}1\\0\end{bmatrix})) = T(\begin{bmatrix}0\\1\end{bmatrix}) = \frac{1}{2}(\begin{bmatrix}2\\-1\\1\end{bmatrix} - \begin{bmatrix}-4\\3\\-4\end{bmatrix}) = \begin{bmatrix}3\\-2\\\frac{5}{2}\end{bmatrix}$

so the standard matrix is $\begin{bmatrix}-4 & 3\\-1 & -2\\-4 & \frac{5}{2}\end{bmatrix}$

(3) Let $T\colon \mathbb{R}^n \to \mathbb{R}^n, x \mapsto Ax$, be a surjective linear map. Show that T is injective as well.

Ans:

- Since the function is mapping between two sets of the same size ($\mathbb{R}^n$) the standar matrix A will be a squre $n \times n$ matrix.
- Because the function is surjective, every column must be linearly independant in order to make the range and codomain equal.
- This means that there has to be a pivot in every row.
- In a square matrix, this must mean that there is no column without a pivot, which means there are no free variables.
- Since there are no free variables, the function must be injective as well as surjective.

(4) True or false? Explain your answer.
  (a) If $Ax = b$ is inconsistent for some vector b, then A cannot have a pivot in every column.
  (b) If vectors $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent and $\mathbf{v}_3$ is not in the span of $\mathbf{v}_1, \mathbf{v}_2$, then $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is linear independent.
  (c) The range of $T\colon \mathbb{R}^n \to \mathbb{R}^m, x \mapsto Ax$, is the span of the columns of A.

Ans:

(a) FALSE
It can if there are more rows than columns.
(b) TRUE
If we think about it geometricaly, v_3 must be in the plane spaned by the other two vectors to be a linear combination of them. Since it is not, it cannot be a linear combination of them.
(c) TRUE
The linear map outputs linear combinations of the columns of the matricies. The span of these columns represents every possible linear combination, which is the same as the range.

(5) (a) Give examples of square matrices A, B such that neither A nor B is 0 (the matrix with all entries 0) but $AB = 0$.
  (b) If the first two columns of a matrix B are equal, what can you say about the columns of AB?

(c) We can view vectors in $\mathbb{R}^n$ as $n \times 1$ matrices. For $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ compute $\mathbf{u}^T \cdot \mathbf{v}$ and $\mathbf{u} \cdot \mathbf{v}^T$. Interpret the results.

Ans:

(a) $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(b) We know that AB's first two columns will also be the same.

(c) $\begin{bmatrix} 2 & -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$

This is just a 1×1 matrix with the element being the dot product of the two vectors.

$\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 2 \\ 0 & -2 & -1 \\ 0 & 6 & 3 \end{bmatrix}$

This has every combination of multiplying one element of one vector with one element of the other vector, almost like a multiplication table.

(6) Prove for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $ad - bc \neq 0$ that

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Hint: Multiply A with the given matrix and check the result.

Ans:
Multiplying the two matrices, we get

$$\frac{1}{ad - bc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} ad - bc & ba - ab \\ dc - cd & ad - bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Which is the 2×2 identity matrix, showing that the matrices are inverses of each other.

(7) Are the following invertible? Give the inverse if possible.

$$A = \begin{bmatrix} 2 & 1 \\ 4 & -9 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

Ans:

(a) $A^{-1} = \frac{1}{-22} \begin{bmatrix} -9 & -1 \\ -4 & 2 \end{bmatrix}$

(b) B is not invertable because $\frac{1}{ad-bc} = \frac{1}{-12+12} = \frac{1}{0}$

(c) C is not invertable because the first column is all zeroes. This means there is no way for element 1,1 to be 1 no matter what it is multiplied by.

(8) A **diagonal matrix** A has all entries 0 except on the diagonal, that is,

$$A = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}.$$

Under which conditions is A invertible and what is A^{-1}?

Ans:
A as long as none of the elements on the diagonal are zero.

$$A^{-1} = \begin{bmatrix} \frac{1}{a_{11}} & 0 & \dots & 0 \\ 0 & \frac{1}{a_{22}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{a_{nn}} \end{bmatrix}.$$