Math 2135 - Assignment 3

Due September 20, 2024

Completed September 19, 2024, Maxwell Rodgers

(1) Which of the following sets of vectors are linearly independent?

(a) $\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -11 \\ 0 \end{bmatrix}$

Ans:

(a)
- Create Matrix:

$$\begin{bmatrix} 0 & 2 & 1 \\ -1 & 1 & 0 \\ 4 & 3 & -2 \end{bmatrix}$$

- Row Reduce:

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -\frac{11}{2} \end{bmatrix}$$

These Vectors are linearly independant because the row reduction of them in matrix form leads to a pivot in every column and every row (this means they only have the trivial solution to $Ax = \mathbf{0}$)

(b)
- Create Matrix:

$$\begin{bmatrix} 1 & 2 & -1 \\ -3 & 1 & 11 \\ 2 & 3 & 0 \end{bmatrix}$$

- Row Reduce:

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

These vectors are linearly dependant because the row reduction of the matrix with them as columns leads to a system with non trivial solutions to $Ax = \mathbf{0}$ (because one of the rows is all zeroes.

(2) Explain whether the following are true or false:

(a) Vectors $\mathbf{v}_1, \mathbf{v}_2, v_3$ are linearly dependent if $\mathbf{v}_2$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_3$.

(b) A subset $\{\mathbf{v}\}$ containing just a single vector is linearly dependent iff $\mathbf{v} = \mathbf{0}$.

(c) Two vectors are linearly dependent iff they lie on a line through the origin.

(d) There exist four vectors in $\mathbb{R}^3$ that are linearly independent.

Ans:

(a) **True**: If $\mathbf{v}_2$ is a linear combination of two other vectors, then there will be at least one non-trivial solution to $Ax = \mathbf{0}$. We can see this because $n\mathbf{v}_1 + m\mathbf{v}_3 = \mathbf{v}_2, m, n \in \mathbb{R}$ (since $\mathbf{v}_2$ is a linear combination of the other two) which implies $n\mathbf{v}_1 + m\mathbf{v}_3 - \mathbf{v}_2 = 0$.

(b) **True**: There is no non-trivial solution for one non-zero vector, and $Ax = \mathbf{0}$ is certainly true for x being any non-trivial multiple of the zero vector.

(c) **True**: If the vector lie on the same line, they are going to be multiples of each other such that $n\mathbf{v}_1 = \mathbf{v}_2$, which means that each is a linear combination of the other.

(d) **False**: If we row reduce the matrix formed by these 4 vectors, there will always be a column without a pivot, which means that there will always be at least one free variable in $Ax = \mathbf{0}$. This means there is a non-trivial solution and the vectors must be linearly dependant.

(3) Show: If any of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is the zero vector (say $\mathbf{v}_i = \mathbf{0}$ for $i \leq n$), then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent.

Ans:
Say we have some vector $\mathbf{v}_i = \mathbf{0}$ and some other vector $\mathbf{v}_j \neq \mathbf{0}$. Since $0\mathbf{v}_j = \mathbf{0} = \mathbf{v}_i$, $\mathbf{v}_i$ is a linear combination of the other vectors and thus the set of vectors is linearly dependant.

(4) Show: If $n > m$, then any n vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^m$ are linearly dependent.

Ans:

- We start by taking the matrix that has $\mathbf{a}_1, \ldots, \mathbf{a}_n$ as columns:
$$\begin{bmatrix} \mathbf{a}_{1,1} & \cdots & \mathbf{a}_{n,1} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{1,m} & \cdots & \mathbf{a}_{n,m} \end{bmatrix}$$
- We then row reduce it into reduced row echelon form:
$$\begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & \mathbf{a}'_{n,1} \\ 0 & 1 & \cdots & 0 & \cdots & \mathbf{a}'_{n,2} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & \mathbf{a}'_{n,m} \end{bmatrix}$$
- We can see that there will always be at least one free variable (unless a vector is a zero vector, but that is covered by the previous question). This free variable means that there will be infinite solutions to $Ax = \mathbf{0}$, meaning the vectors are linearly dependant.

(5) Show that the following maps are not linear by giving concrete vectors for which the defining properties of linear maps are not satisfied.

(a) $f : \mathbb{R}^2 \to \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x+1 \\ y+3 \end{bmatrix}$

(b) $g : \mathbb{R}^2 \to \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} xy \\ y \end{bmatrix}$

(c) $h : \mathbb{R}^2 \to \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} |x| + |y| \\ 2x \end{bmatrix}$

Ans:
Transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ $n, m \in \mathbb{R}$ is linear if:

1: $T(v_1) + T(v_2) = T(v_1 + v_2)$ $v_1, v_2 \in \mathbb{R}^n$
2: $cT(v_3) = T(cv_3)$ $v_3 \in \mathbb{R}^n$, $c \in \mathbb{R}$

(a) $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, c = 2$

$T(v_1) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, T(v_2) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$T(v_1) + T(v_2) = \begin{bmatrix} 5 \\ 9 \end{bmatrix} \neq T(v_1 + v_2) = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$ (violates 1)

$cT(v_1) = \begin{bmatrix} 4 \\ 8 \end{bmatrix} \neq T(cv_1) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ (violates 2)

(b) $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, c = 2$

$T(v_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, T(v_2) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

$T(v_1) + T(v_2) = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \neq T(v_1 + v_2) = \begin{bmatrix} 9 \\ 3 \end{bmatrix}$ (violates 1)

$cT(v_1) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \neq T(cv_1) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ (violates 2)

(c) $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, c = -1$

$T(v_1) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, T(v_2) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

$T(v_1) + T(v_2) = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \neq T(v_1 + v_2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (violates 1)

$cT(v_1) = \begin{bmatrix} -2 \\ -2 \end{bmatrix} \neq T(cv_1) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ (violates 2)

(6) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map such that

$$T(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}) = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \ T(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

Use the linearity of T to compute $T(\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix})$ and $T(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix})$. What is the issue with the latter?

Ans:

- Vector 1:
$T(\begin{bmatrix}2\\3\\0\end{bmatrix}) = 2T(\begin{bmatrix}1\\0\\0\end{bmatrix}) + 3T(\begin{bmatrix}0\\1\\0\end{bmatrix}) = \begin{bmatrix}-2\\4\\0\end{bmatrix} + \begin{bmatrix}-9\\0\\3\end{bmatrix} = \begin{bmatrix}-11\\4\\3\end{bmatrix}$
- Vector 2:
The problem with the second vector is that it is not a linear combination of the two unit vectors that we have the value of T for, so we cannot find it.

(7) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear map such that

$$T(\begin{bmatrix}1\\2\end{bmatrix}) = \begin{bmatrix}2\\0\\-3\end{bmatrix},\ T(\begin{bmatrix}3\\2\end{bmatrix}) = \begin{bmatrix}-2\\2\\1\end{bmatrix}.$$

(a) Use the linearity of T to find $T(\begin{bmatrix}1\\0\end{bmatrix})$ and $T(\begin{bmatrix}0\\1\end{bmatrix})$.

(b) Determine $T(\begin{bmatrix}x\\y\end{bmatrix})$ for arbitrary $x, y \in \mathbb{R}$.

Ans:

(a)
- $\frac{1}{2}(\begin{bmatrix}3\\2\end{bmatrix} - \begin{bmatrix}1\\2\end{bmatrix}) = \begin{bmatrix}1\\0\end{bmatrix}$
$\frac{1}{2}(T(\begin{bmatrix}3\\2\end{bmatrix}) - T(\begin{bmatrix}1\\2\end{bmatrix})) = T(\begin{bmatrix}1\\0\end{bmatrix})$
$\frac{1}{2}(\begin{bmatrix}-2\\2\\1\end{bmatrix} + \begin{bmatrix}-2\\0\\3\end{bmatrix}) = T(\begin{bmatrix}1\\0\end{bmatrix}) = \begin{bmatrix}-2\\1\\2\end{bmatrix}$
- $-\frac{1}{4}(\begin{bmatrix}3\\2\end{bmatrix} - 3\begin{bmatrix}1\\2\end{bmatrix}) = \begin{bmatrix}0\\1\end{bmatrix}$
$-\frac{1}{4}(T(\begin{bmatrix}3\\2\end{bmatrix}) - 3T(\begin{bmatrix}1\\2\end{bmatrix})) = T(\begin{bmatrix}0\\1\end{bmatrix})$
$-\frac{1}{4}(\begin{bmatrix}-2\\2\\1\end{bmatrix} + \begin{bmatrix}-6\\0\\9\end{bmatrix}) = T(\begin{bmatrix}0\\1\end{bmatrix}) = \begin{bmatrix}2\\-\frac{1}{2}\\-\frac{5}{2}\end{bmatrix}$

(b) $T(\begin{bmatrix}x\\y\end{bmatrix}) = \begin{bmatrix}2y-2x\\-\frac{y}{2}+x\\-\frac{5y}{2}+2x\end{bmatrix}$

(8) Give the standard matrices for the following linear transformations:

(a) $T: \mathbb{R}^2 \to \mathbb{R}^3$, $\begin{bmatrix}x\\y\end{bmatrix} \mapsto \begin{bmatrix}2x+y\\x\\-x+y\end{bmatrix}$;

(b) the function S on $\mathbb{R}^2$ that scales all vectors to half their length.

Ans:

(a) $\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ -1 & -1 \end{bmatrix}$ (from the coefficients of x and y, x in left column y in right.)

(b) $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ x is multiplied by 1/2 and put in the top, y is also multiplied by 1/2 in the bottom this gives the whole vector multiplied by the scalar 1/2