Math 2135 - Assignment 10

Due November 8, 2024

Problems 1-5 are review material for the second midterm on November 6. Solve them before Wednesday!

(1) Let V, W be vector spaces over $\mathbb{R}$ with zero vectors $0_V, 0_W$, respectively. Let $f\colon V \to W$ be linear. Show
 (a) $f(0_V) = 0_W$,
 (b) the kernel $\ker f := \{v \in V \ :\ f(v) = 0_W\}$ of f is a subspace of V.

Ans:

(a) the zero vector in V multiplied by x (that represents the linear map) will always end up being zero (since there is only multiplication) in the output.
(b) We know from the previous part that $\ker f$ is a subset of V.
The kernel is closed under addition because $f(v) = 0$ and $0 + 0 = 0$
The kernel is closed under multiplication because $f(v) = 0$ and $0n = 0$ for all $n \in \mathbb{R}$

(2) Let $T\colon P_2 \to \mathbb{R}, p \mapsto p(3)$, be the map that evaluates a polynomial p at $x = 3$.
 (a) Show that T is linear.
 (b) Determine the kernel of T, that is, $\ker T = \{p \in P_2 \ :\ T(p) = 0\}$, and the image of T, that is, $T(P_2)$.
 (c) Is T injective, surjective, bijective?

Ans:

(a) Property 1: For $a, b \in P_2$ it is true that $a(3) + b(3) = (a + b)(3)$ (since the polynomials don't change each other in the proccess of addition)
Property 2: For $a \in P_2$ and $n \in \mathbb{R}$ it is true that $n \cdot a(3) = (n \cdot a)(3)$ (multiplying the result or all of the terms yeilds the same outcome)
(b) The kernel is where the polynomials equal zero when evaluated at 3 so the kernel is all polynomials of degree 2 with the root $(x - 3)$.
The image is just $\mathbb{R}$ because any real number can be output by a polynomial of degree 2.
(c) It is not injective because many polnomials output the same value for 3, but it is surjective since the range is the same as the codomain.

(3) Let $B = (b_1, b_2)$ with $b_1 = \begin{bmatrix} -5 \\ 11 \\ 5 \end{bmatrix}, b_2 = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ and $C = (\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix})$ be bases of a subspace H of $\mathbb{R}^3$.
 (a) Compute the coordinates $[b_1]_C$ and $[b_2]_C$.
 (b) What is the change of coordinate matrix $P_{C \leftarrow B}$?
 (c) What is the change of coordinate matrix $P_{B \leftarrow C}$?

Ans:

(a) Augmented matrix: $\begin{bmatrix} 1 & 2 & -5 & 3 \\ 1 & -2 & 11 & -1 \\ 3 & 1 & 5 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

So $[b_1]_C = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$, $[b_2]_C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(b) Using the vectors we just found, $P_{C\leftarrow B} = \begin{bmatrix} 3 & 1 \\ -4 & 1 \end{bmatrix}$

(c) Now we just take the inverse of $P_{C\leftarrow B}$ to get $P_{B\leftarrow C} = \frac{1}{7} \cdot \begin{bmatrix} 1 & -1 \\ 4 & 3 \end{bmatrix}$

(4) Let $C = (1+t, t+t^2, 1+t^2)$ be a basis for P_2. Compute the coordinates $[p]_C$ for $p = 2 + t^2$.

Ans:
we can represent this problem as the augmented matrix
$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{2} \end{bmatrix}$ so the answer is $\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$

(5) (a) Show that $A \in \mathbb{R}^{n\times n}$ is invertible iff rank $A = n$.
(b) If A is a 3×4-matrix, what is the largest possible rank of A? What is the smallest possible dimension of Nul A?
(c) If the nullspace of a 4×6-matrix B has dimension 3, what is the dimension of the row space of B?

Ans:

(a) A is invertible iff its columns are linearly independant. A's columns are linearly independant iff rank $A = n$
(b) The largest possible rank of A is 3 (3 rows) and the smallest possible dimension of the nullspace is 1(1 more column than row).
(c) The dimension of the rowspace(/columnspace) and the dimension of the nullspace must add up to the number of columns so Row A has dimension 3.

(6) Compute the determinant of the matrices by cofactor expansion. Pick a row or column that yields the least amount of computation:

$$A = \begin{bmatrix} 0 & 1 & -3 \\ 5 & 4 & -4 \\ 0 & -3 & -4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 3 & 1 & 5 & 1 \\ 2 & 0 & 0 & 0 \\ 7 & 1 & -2 & 5 \end{bmatrix}.$$

Ans:

(a) Using row 2 we just have to do 1 calculation: $\det A = -5\begin{vmatrix} 1 & -3 \\ -3 & -4 \end{vmatrix} = (-5)(-13) = 65$

(b) Using row 3 in B we have to do 1 3×3 matrix, and if we do the first row in that matrix we only have to do one again: $\det B = 2\begin{vmatrix} 0 & -3 & 0 \\ 1 & 5 & 1 \\ 1 & -2 & 5 \end{vmatrix} = (2)(3)\begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} = (2)(3)(4) = 24$

(7) **Rule of Sarrus for the determinant of 3×3-matrices.** Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Prove that

$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$

Hint: Expand $\det A$ across the first row.

Ans:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a33 \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a33 \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a32 \end{vmatrix} =$$

$$a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} =$$

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

(8) Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(a) How does switching the rows effect the determinant? Compare $\det A$ and $\det \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

(b) How does multiplying one row by a scalar effect the determinant? Compare $\det A$ and $\det \begin{bmatrix} ra & rb \\ c & d \end{bmatrix}$.

(c) How does adding a multiple of one row to the other row effect the determinant? Compare $\det A$ and $\det \begin{bmatrix} a & b \\ c + ra & d + rb \end{bmatrix}$.

Ans:

(a) It multiplies the determinant by -1: $\det A = ad - bc$ and $\det \begin{bmatrix} c & d \\ a & b \end{bmatrix} = bc - da = -\det A$

(b) It multiplies the determinant by r: $\det \begin{bmatrix} ra & rb \\ c & d \end{bmatrix} = rad - rbc = r(ad - bc) = r \det A$

(c) $\det \begin{bmatrix} a & b \\ c+ra & d+rb \end{bmatrix} = ad + rab - bc - rab = ad - bc = \det A$