Math 2135 - Assignment 4

Due September 27, 2024
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(1) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear map such that

$$T(\begin{bmatrix}1\\2\end{bmatrix}) = \begin{bmatrix}2\\0\\-3\end{bmatrix},\ T(\begin{bmatrix}3\\2\end{bmatrix}) = \begin{bmatrix}-2\\2\\1\end{bmatrix}.$$

(a) Use linearity to find $T(e_1)$ and $T(e_2)$ for the unit vectors e_1, e_2 in $\mathbb{R}^2$.

(b) Give the standard matrix for T and determine $T(\begin{bmatrix}x\\y\end{bmatrix})$ for arbitrary $x, y \in \mathbb{R}$.

Ans:

(a)

- $\frac{1}{2}(\begin{bmatrix}3\\2\end{bmatrix} - \begin{bmatrix}1\\2\end{bmatrix}) = \begin{bmatrix}1\\0\end{bmatrix}$

  $\frac{1}{2}(T(\begin{bmatrix}3\\2\end{bmatrix}) - T(\begin{bmatrix}1\\2\end{bmatrix})) = T(\begin{bmatrix}1\\0\end{bmatrix})$

  $\frac{1}{2}(\begin{bmatrix}-2\\2\\1\end{bmatrix} + \begin{bmatrix}-2\\0\\3\end{bmatrix}) = T(\begin{bmatrix}1\\0\end{bmatrix}) = T(e_1) = \begin{bmatrix}-2\\1\\2\end{bmatrix}$

- $-\frac{1}{4}(\begin{bmatrix}3\\2\end{bmatrix} - 3\begin{bmatrix}1\\2\end{bmatrix}) = \begin{bmatrix}0\\1\end{bmatrix}$

  $-\frac{1}{4}(T(\begin{bmatrix}3\\2\end{bmatrix}) - 3T(\begin{bmatrix}1\\2\end{bmatrix})) = T(\begin{bmatrix}0\\1\end{bmatrix})$

  $-\frac{1}{4}(\begin{bmatrix}-2\\2\\1\end{bmatrix} + \begin{bmatrix}-6\\0\\9\end{bmatrix}) = T(\begin{bmatrix}0\\1\end{bmatrix}) = T(e_2) = \begin{bmatrix}2\\-\frac{1}{2}\\-\frac{5}{2}\end{bmatrix}$

(b) Standard Matrix: $\begin{bmatrix}-2 & 2\\1 & -\frac{1}{2}\\2 & -\frac{5}{2}\end{bmatrix}$

$$T(\begin{bmatrix}x\\y\end{bmatrix}) = \begin{bmatrix}2y-2x\\-\frac{y}{2}+x\\-\frac{5y}{2}+2x\end{bmatrix}$$

(2) Is the following injective, surjective, bijective? What is its range?

$$T : \mathbb{R}^3 \to \mathbb{R}^2,\ x \mapsto \begin{bmatrix}0 & 2 & -1\\0 & 0 & 3\end{bmatrix} \cdot x$$

Ans:
The matrix is already in echelon form. If we put it in reduced echelon form we get this: $\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

The columns of the matrix are the zero vector and unit vectors of $\mathbb{R}^2$, which of course span $\mathbb{R}^2$. This means that the function is surjective. Because one of the columns is the zero vector, the vectors cannot be linearly independant.

Therefore the function T is only Surjective. Since the function is surjective, the range is the codomain which is $\mathbb{R}^2$.

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(3) Is the following injective, surjective, bijective?

$$T : \mathbb{R}^3 \to \mathbb{R}^3,\ x \mapsto \begin{bmatrix} 1 & -1 & 2 \\ -2 & 0 & 1 \\ 3 & -1 & 1 \end{bmatrix} \cdot x$$

Ans:

- Row reduce:
  $\begin{bmatrix} 1 & -1 & 2 \\ 0 & -2 & 5 \\ 0 & 0 & 0 \end{bmatrix}$
- Since an entire row is zeroes, the columns of this matrix cannot span $\mathbb{R}^3$ and thus it isn't surjective. Since there are 3 columns and each only has 2 or fewer of the same rows with non-zero elements, the columns cannot linearly independant and thus T cannot be injective.
- **The fuction is not injective or surjective.**

(4) True or False? Explain why and correct the false statements to make them true.

(a) A linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is completely determined by the images of the unit vectors in $\mathbb{R}^n$.

(b) Not every linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ can be written as $T(x) = Ax$ for some matrix A.

(c) The composition of any two linear transformations is linear as well.

Ans:

(a) **True**
Because the properties of linear transformations allow us to determine the output for any input that is a linear combination of inputs with known output, and the unit vectors of $\mathbb{R}^n$ span $\mathbb{R}^n$, the unit vectors are enough to determine the output of any input.

(b) **False**
Every linear transformation can be writen as $T(x) = Ax$ for some matrix A
If we have some vector x we can write it as each element multiplied by the unit vector for the coresponding row. If we then apply a linear transformation to this sum of scaled vectors, and use the properties of linear transformations to seperate the vectors and pull out the scalars, we get the elements of x multiplied by the transformations of the unit vectors. This is equivilant to Ax if A's columns are just the transformed unit vectors.

(c) **True**
If we take any two linear transformations, given their domain and ranges are compatible we can compose them. Since every linear transformation has an associated matrix A we can show that the composition of the two linear maps has such a matrix. Take maps T and S, arbitrary vector x,

and composition $T \bigcirc S(x)$. The composition is equivilant to $A_T(A_S x)$. Because we already stipulated that the functions have compatible dimensions, matrix multiplication in this case is associative and the new matrix for the composed linear transformation is $A_{T\bigcirc S(x)} = A_T A_S$.

(5) True or False? Explain why and correct the false statements to make them true.
  (a) $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto $\mathbb{R}^m$ if every vector $x \in \mathbb{R}^n$ is mapped onto some vector in $\mathbb{R}^m$.
  (b) $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if every vector $x \in \mathbb{R}^n$ is mapped onto a unique vector in $\mathbb{R}^m$.
  (c) A linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ cannot be one-to-one.
  (d) There is a surjective linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^4$.

Ans:

(a) **False**
This is just a description of a function, and says nothing of the range. $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto $\mathbb{R}^m$ if every vector $x \in \mathbb{R}^n$ is mapped onto some vector in $\mathbb{R}^m$ **and** every vector in $\mathbb{R}^m$ is mapped to by a vector in $\mathbb{R}^n$.

(b) **True**
This is what one-to-one means. Every possible input has a unique output.

(c) **True**
The associated matrix to the map T would have to be a 2×3 matrix. This ensures that there would be at least one free variable meaning that the function could not be one-to-one.

(d) **False**
T is surjective iff its columns span $\mathbb{R}^4$. There are only 3 columns since the matrix must map from $\mathbb{R}^3$ and 3 vectors cannot span $\mathbb{R}^4$.

(6) If defined, compute the following for the matrices

$$A = \begin{bmatrix} 2 & 1 & -4 \\ 3 & -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ -3 & 4 \\ 2 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix}$$

Else explain why the computation is not defined.

(a) AB (b) BA (c) AC (d) $A + C$ (e) $AB + 2C$

Ans:

(a) $\begin{bmatrix} 2-3-8 & -2+4+0 \\ 3+3+2 & -3-4+0 \end{bmatrix} = \begin{bmatrix} -9 & 2 \\ 8 & -7 \end{bmatrix}$

(b) $\begin{bmatrix} 2-3 & 1+1 & -4-1 \\ -6+12 & -3-4 & 12+4 \\ 4+0 & 2+0 & -8+0 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -5 \\ 6 & -7 & 16 \\ 4 & 2 & -8 \end{bmatrix}$

(c) Not possible. The height of C is not the same as the width of A.

(d) Not possible. A and C are not the same size.

(e) $\begin{bmatrix} -9-4 & 2+2 \\ 8+4 & -7-6 \end{bmatrix} = \begin{bmatrix} -13 & 4 \\ 12 & -13 \end{bmatrix}$

(7) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ first rotate points around the origin by $60°$ counter clockwise and then reflect points at the line with equation $y = x$. Give the standard matrix for T.

(a) Recall the standard matrix A for the rotation R by $60°$ from class.
(b) Determine the standard matrix B for the reflection S at the line with equation $y = x$ (a sketch will help).
(c) Since T is the composition of S and R, compute the standard matrix C of T as the product of B and A. Careful about the order!

Ans:

(a) $A = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

(b) $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(c) $BA = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$

(8) Continuation of (7): What is the standard matrix for $U\colon \mathbb{R}^2 \to \mathbb{R}^2$ which first reflects points at the line with equation $y = x$ and then rotates points around the origin by $60°$ counter clockwise? Compare T and U.

Ans:
$AB = \begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ The matrices are the same exept for the signs of the top left and bottom right elements being switched.