Math 2135 - Assignment 12

Due Nov 22, 2024
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(1) Are the matrices A, B, C, D in (5), (6), (7) of assignment 11 diagonalizable? How?

Ans:
Both B and C are diagonalizable but A and D are not because they do not have as many eigenspaces as they have rows/columns.
The diagonalized matrices of B and C would just be the eigenvalues of those matrices put in the diagonal of ann by n matrix with all the other entries set to 0.

(2) Let A be an $n \times n$-matrix. Are the following true or false? Explain why:
  (a) If A has n eigenvectors, then A is diagonalizable.
  (b) If a 4×4-matrix A has two eigenvalues with eigenspaces of dimension 3 and 1, respectively, then A is diagonalizable.
  (c) A is diagonalizable iff A has n eigenvalues (counting multiplicities).
  (d) If $\mathbb{R}^n$ has a basis of eigenvectors of A, then A is diagonalizable.
  (e) Every triangular matrix is diagonalizable.

Ans:

(a) ?

(b) TRUE

(c) FALSE

(d) TRUE

(e) FALSE

(3) Let A be the standard matrix for the reflection t of $\mathbb{R}^2$ on some line g throught the origin. What are the eigenvalues, eigenvectors and eigenspaces of A? Can A be diagonalized?
Hint: Consider what a reflection does to specific vectors.

(4) As the previous problem for a rotation r of $\mathbb{R}^2$ by an angle φ around the origin.
Hint: Consider $\varphi = 0, \pi$ separately.

(5) Consider a population of owls feeding on a population of squirrels. In month k, let o_k denote the number of owls and s_k the number of squirrels. Assume that the populations change every month as follows:

$$o_{k+1} = 0.3o_k + 0.4s_k$$
$$s_{k+1} = -0.4o_k + 1.3s_k$$

That is, if there would be no squirrels to hunt, only 30% of the owls would survive to the next month; if there were no owls that hunted squirrels, then the squirrel population would grow by factor 1.3 every month.

Let $x_k = \begin{bmatrix} o_k \\ s_k \end{bmatrix}$. Express the population change from x_k to x_{k+1} using a matrix A. Diagonalize A.

Ans:
The matrix is $A = \begin{bmatrix} 0.3 & 0.4 \\ -0.4 & 1.3 \end{bmatrix}$ (from the cofficients of the equations) This means that the eigenvalues are represented by the equation $(0.3 - \lambda)(1.3 - \lambda) + 0.16 = \lambda^2 - 1.6\lambda + 0.64 - 0.09 = (\lambda - 0.8)^2 - 0.09 = 0$ so $\lambda = 0.8 \pm 0.3$ This means that the diagonalized matrix is $D = \begin{bmatrix} 1.1 & 0 \\ 0 & 0.5 \end{bmatrix}$ To get the P matrix we take the $D - I\lambda$ matrices and find their nullspaces: $0.5 : \begin{bmatrix} -0.2 & 0.4 \\ -0.4 & 0.8 \end{bmatrix} \to \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $1.1 : \begin{bmatrix} -0.8 & 0.4 \\ -0.4 & 0.2 \end{bmatrix} \to \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ meaning the P matrix is $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

(6) Continue the previous problem: Let the starting population be $x_1 = \begin{bmatrix} o_1 \\ s_1 \end{bmatrix} = \begin{bmatrix} 20 \\ 100 \end{bmatrix}$.

(a) Give an explicit formula for the populations in month $k + 1$.
(b) Are the populations growing or decreasing over time? By which factor?
(c) What is ratio of owls to squirrels after 12 months? After 24 months? Can you explain why?

Ans:

(a) First we find the starting parameters in terms of the eigenvectors which is simply $\begin{bmatrix} o_{k+1} \\ s_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1.1 & 0 \\ 0 & 0.5 \end{bmatrix}^n \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 20 \\ 100 \end{bmatrix}$

(b)

(c)