Math 2135 - Assignment 5

Due October 4, 2024

Problems 1-5 are review material for the first midterm on September 29. Solve them before Wednesday!

(1) Let

$$A = \begin{bmatrix} 0 & 3 & 1 & 2 \\ 1 & 4 & 0 & 7 \\ 2 & -1 & -3 & 8 \end{bmatrix}, b = \begin{bmatrix} 6 \\ 5 \\ -8 \end{bmatrix}$$

(a) Give the solution for $Ax = b$ in parametrized vector form.
(b) Give vectors that span the null space of A.

Ans:

(a)
- Row reduce augmented matrix:

$$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & \frac{13}{3} & -3 \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Parametric form:

$$x_1 - \frac{4}{3}x_3 + \frac{13}{3}x_4 = -3, x_2 + \frac{1}{3}x_3 + \frac{2}{3}x_4 = 2$$

we have the two free variables $x_4 = t, x_3 = u$

$$x = \begin{bmatrix} -3 \\ 2 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{13}{3} \\ -\frac{2}{3} \\ 0 \\ 1 \end{bmatrix} + u \begin{bmatrix} \frac{4}{3} \\ -\frac{1}{3} \\ 1 \\ 0 \end{bmatrix}$$

(b) $\begin{bmatrix} -\frac{13}{3} \\ -\frac{2}{3} \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} \frac{4}{3} \\ -\frac{1}{3} \\ 1 \\ 0 \end{bmatrix}$ span the nullspace of A because the solution to the equation $Ax = 0$ is the same as the one in part a exept there is no constant term.

(2) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with

$$T(\begin{bmatrix} 1 \\ 2 \end{bmatrix}) = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \text{ and } T(\begin{bmatrix} 3 \\ 4 \end{bmatrix}) = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}.$$

What is the standard matrix of T?

Ans:

$$T(\begin{bmatrix} 3 \\ 4 \end{bmatrix} - 2T(\begin{bmatrix} 1 \\ 2 \end{bmatrix}) = T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} - \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \\ -4 \end{bmatrix}$$

$\frac{1}{2}(T(\begin{bmatrix}1\\2\end{bmatrix}) - T(\begin{bmatrix}1\\0\end{bmatrix})) = T(\begin{bmatrix}0\\1\end{bmatrix}) = \frac{1}{2}(\begin{bmatrix}2\\-1\\1\end{bmatrix} - \begin{bmatrix}-4\\-1\\-4\end{bmatrix}) = \begin{bmatrix}3\\0\\\frac{5}{2}\end{bmatrix}$

so the standard matrix is $\begin{bmatrix}-4 & 3\\-1 & 0\\-4 & \frac{5}{2}\end{bmatrix}$

(3) Let $T\colon \mathbb{R}^n \to \mathbb{R}^n, x \mapsto Ax$, be a surjective linear map. Show that T is injective as well.

Ans:

- Since the function is mapping between two sets of the same size ($\mathbb{R}^n$) the standar matrix A will be a squre $n \times n$ matrix.
- Because the function is surjective, every column must be linearly independant in order to make the range and codomain equal.
- This means that there has to be a pivot in every row.
- In a square matrix, this must mean that there is no column without a pivot, which means there are no free variables.
- Since there are no free variables, the function must be injective as well as surjective.

(4) True or false? Explain your answer.

(a) If $Ax = b$ is inconsistent for some vector b, then A cannot have a pivot in every column.

(b) If vectors $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent and $\mathbf{v}_3$ is not in the span of $\mathbf{v}_1, \mathbf{v}_2$, then $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is linear independent.

(c) The range of $T\colon \mathbb{R}^n \to \mathbb{R}^m, x \mapsto Ax$, is the span of the columns of A.

Ans:

(a) FALSE

(b) TRUE

(c) TRUE

(5) (a) Give examples of square matrices A, B such that neither A nor B is 0 (the matrix with all entries 0) but $AB = 0$.

(b) If the first two columns of a matrix B are equal, what can you say about the columns of AB?

(c) We can view vectors in $\mathbb{R}^n$ as $n \times 1$ matrices. For $\mathbf{u} = \begin{bmatrix}2\\-1\\3\end{bmatrix}, \mathbf{v} = \begin{bmatrix}0\\2\\1\end{bmatrix}$ compute $\mathbf{u}^T \cdot \mathbf{v}$ and $\mathbf{u} \cdot \mathbf{v}^T$. Interpret the results.

Ans:

(a) $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(b) We know that AB's first two columns will also be the same.

(c) $\begin{bmatrix} 2 & -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$

$\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 2 \\ 0 & -2 & -1 \\ 0 & 6 & 3 \end{bmatrix}$

(6) Prove for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $ad - bc \neq 0$ that

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Hint: Multiply A with the given matrix and check the result.

(7) Are the following invertible? Give the inverse if possible.

$$A = \begin{bmatrix} 2 & 1 \\ 4 & -9 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

(8) A **diagonal matrix** A has all entries 0 except on the diagonal, that is,

$$A = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}.$$

Under which conditions is A invertible and what is A^{-1}?