

Math 2135 - Assignment 1

Due September 6, 2024

Completed September 5th by Maxwell Rodgers

Solve all systems of linear equations by row reduction (Gaussian elimination) and give the solutions in parametric vector form.

1. Do the following 4 planes intersect in a point? Which?

$$\begin{aligned}x + 5y + 3z &= 16 \\2x + 10y + 8z &= 34 \\4x + 20y + 15z &= 67 \\x + 6y + 5z &= 21\end{aligned}$$

Ans:

- Turn equations into augmented matrix:

$$\begin{array}{l} I \\ II \\ III \\ IV \end{array} \left[\begin{array}{cccc} 1 & 5 & 3 & 16 \\ 2 & 10 & 8 & 34 \\ 4 & 20 & 15 & 67 \\ 1 & 6 & 5 & 21 \end{array} \right]$$

- Row reduce and re-arrange:
(subtract 2I, 4I and I from II, III and IV respectively, multiply II by $\frac{1}{2}$, subtract 3II from I, then move IV between I and II)

$$\begin{array}{l} I \\ II \\ III \\ IV \end{array} \left[\begin{array}{cccc} 1 & 5 & 3 & 16 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

- Solve for x , y and z :
 $z = 1$ from III.
 $y = 5 - 2 = 3$ from II and z .
 $x = 16 - 3 - 15 = -2$ from I y and z .
So the point at which all the given planes intersect at is $(-2, 3, 1)$.

2. Add an equation of a line to the equation

$$2x + 3y = 4$$

such that the resulting system has (a) no solution, (b) exactly one solution, (c) infinitely many solutions.

Ans:

(a) No Solution: $-2x - 3y = 0$

$$\begin{array}{rcl} 2x + 3y & = & 4 \\ -2x - 3y & = & 0 \\ \hline 0 & = & 4 \end{array}$$

The contradiction shows that there are no solutions

(b) One Solution: $-2x - 2y = -3$

$$\begin{array}{rcl} 2x + 3y & = & 4 \\ -2x - 2y & = & -3 \\ \hline y & = & 1 \end{array}$$

This implies that $x = \frac{1}{2}$, leading to the single solution $(\frac{1}{2}, 1)$

(c) Infinite Solutions: $4x + 6y = 8$

$$\begin{array}{rcl} 2x + 3y & = & 4 \\ -\frac{1}{2}(4x + 6y = 8) & & \\ \hline 0 & = & 0 \end{array}$$

The result of $n = n$ where in this case $n = 0$ implies the lines are the same and thus have infinite solutions.

3. Solve the system of linear equations with augmented matrix

$$\begin{array}{l} I \\ II \\ III \end{array} \left[\begin{array}{cccc} 0 & 0 & 2 & 4 \\ 2 & -4 & 1 & 0 \\ -3 & 6 & 2 & 7 \end{array} \right]$$

Ans:

• Row Reduce:

do $\frac{3}{2}II + III$, then do $\frac{2}{7}III$ and $\frac{1}{2}I$. Next, do $III-I$ and $I-II$ followed by $\frac{1}{2}I$. Finally, swap I and II to get the following matrix.

$$\begin{array}{l} I \\ II \\ III \end{array} \left[\begin{array}{cccc} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

• Solve Equations:

$x_3 = 2$ and $x_1 = 2x_2 - 1$. Taking $x_2 = t$ as the parameter, we get the

following equation:

$$x = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

4. Solve the system of linear equations with augmented matrix

$$\begin{array}{l} I \\ II \\ III \end{array} \left[\begin{array}{ccccc} 2 & 2 & 1 & 8 & 2 \\ 2 & 0 & 0 & 8 & 2 \\ 2 & 6 & 3 & 8 & 1 \end{array} \right]$$

Ans:

- Row Reduce:

Start by subtracting II from I and III. Then subtract 3I from III. at this point I stopped the row reduction leaving me with the following matrix.

$$\left[\begin{array}{ccccc} 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & 0 & 8 & 2 \\ 0 & 0 & 0 & 0 & -1 \end{array} \right]$$

- Solve Equations:

I stopped the row reduction because based on the final row of the matrix, we have the equation $0 = -1$ indicating that this system has no solutions.

5. Let $A \in \mathbb{R}^{3 \times 2}$. Explain why $Ax = b$ cannot have a solution x for all $b \in \mathbb{R}^3$.

Ans: Since A represents three equations that have only two variables, the most lenient case is all the equations being the same (since none would be limited by the others). This would lead to the solution being an equation with two variables. This cannot represent the entirety of \mathbb{R}^3 as having two variables is essentially having two degrees of freedom. This means that the most that A can represent is \mathbb{R}^2 , and if it were graphed in 3D it would at most be a plane.

6. [1, cf. Section 1.5, Ex 17] Let

$$A = \begin{bmatrix} 2 & 2 & 4 \\ -4 & -4 & -8 \\ 0 & -3 & -3 \end{bmatrix}, \quad b = \begin{bmatrix} 8 \\ -16 \\ 12 \end{bmatrix}, \quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solve $Ax = b$ and $Ax = \mathbf{0}$. Express both solution sets in parametric vector form. Give a geometric description of the solution sets.

Ans:

- $Ax = b$:

$$\begin{array}{l} I \\ II \\ III \end{array} \left[\begin{array}{cccc} 2 & 2 & 4 & 8 \\ -4 & -4 & -8 & -16 \\ 0 & -3 & -3 & 12 \end{array} \right]$$

- Row Reduce:

Start by doing $\frac{1}{2}I$, $\frac{1}{4}II$ and $\frac{-1}{3}III$. Then add I to II and subtract III from I. Finally, swap II and III to get the following matrix.

$$\left[\begin{array}{cccc} 1 & 0 & 1 & 8 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

- Solve Equations:

The two equations are $x + y = 8$ and $z + y = -4$. Using $y = t$ we get the following equation:

$$x = \begin{pmatrix} 8 \\ 0 \\ -4 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

- $Ax = 0$:

$$\begin{array}{l} I \\ II \\ III \end{array} \left[\begin{array}{cccc} 2 & 2 & 4 & 0 \\ -4 & -4 & -8 & 0 \\ 0 & -3 & -3 & 0 \end{array} \right]$$

- Row Reduce:

Start by doing $\frac{1}{2}I$, $\frac{1}{4}II$ and $\frac{-1}{3}III$. Then add I to II and subtract III from I. Finally, swap II and III to get the following matrix.

$$\left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

- Solve Equations:

The two equations are $x_1 + x_2 = 0$ and $x_3 + x_2 = 0$. Using $x_2 = t$ we get the following equation:

$$x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

7. [1, cf. Section 1.5, Ex 11] Let

$$A = \begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solve the equations $Ax = b$ and $Ax = \mathbf{0}$. Express both solution sets in parametric vector form.

Ans:

- $Ax = b$:

$$\begin{array}{l} I \\ II \\ III \end{array} \begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -4 & 1 \end{bmatrix}$$

- Row Reduce:

Start by adding 2II to I then subtract 3III from I.

$$\begin{bmatrix} 1 & -4 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -4 & 1 \end{bmatrix}$$

- Solve Equations:

The three equations are $x_1 = 4x_2 - 5x_6$, $x_3 = x_6 + 1$, and $x_5 = 4x_6 + 1$.

Using $x_2 = t$, $x_6 = u$, and $x_4 = v$ we get the following equation:

$$x = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + u \begin{pmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 4 \\ 1 \end{pmatrix} + v \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

- $Ax = \mathbf{0}$:

$$\begin{array}{l} I \\ II \\ III \end{array} \begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -4 & 0 \end{bmatrix}$$

- Row Reduce:

Start by adding 2II to I then subtract 3III from I.

$$\begin{bmatrix} 1 & -4 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -4 & 0 \end{bmatrix}$$

- Solve Equations:

The three equations are $x_1 = 4x_2 - 5x_6$, $x_3 = x_6$, and $x_5 = 4x_6$. Using $x_2 = t$, $x_6 = u$, and $x_4 = v$ we get the following equation:

$$x = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + u \begin{pmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 4 \\ 1 \end{pmatrix} + v \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

8. Are the following true or false? Explain your answers.

- (a) Any system of linear equations with strictly less equations than variables has infinitely many solutions.
- (b) Different sequences of elementary row reductions can transform one matrix to distinct matrices in row echelon form.
- (c) A consistent system has exactly one solution.

Ans:

- (a) True

Each equation only allows you to solve for one variable, potentially in terms of other variables. If there are n equations and $n + 1$ variables, there will always be one variable that cannot be solved for.

- (b) True

It is possible to end up with distinct but equivalent matrices. i.e. some rows might be multiplied by a factor in one matrix compared to another matrix, still representing the same system of equations. If we add the restriction of the pivot being 1, then it is no longer possible, as this does not allow any row to be multiplied by any factor other than the one that leads the pivot to equal 1.

- (c) False

All that a consistent system is is a system that is not unsolvable. This means that a consistent system could have infinitely many or finitely many solutions.

References

- [1] David C. Lay, Steven R. Lay, and Judi J. McDonald. Linear Algebra and Its Applications. Addison-Wesley, 5th edition, 2015.