Homework3

Problem 1

Let
$$Y_1, Y_2, ..., Y_n \sim N(\mu, \sigma^2)$$

For now assume $\sigma = 15, \bar{y} = 113, n = 10$

We have

$$p(\mu) = N(\mu_0, \sigma_0^2) = N(100, 15)$$

We know from the slides that
$$E(\mu|y) = \frac{\frac{\mu_0}{15} + \frac{n\bar{y}}{15}}{\frac{n+1}{15}}$$

We can see that the σ terms drop out and we are left with

$$E(\mu|y) = \frac{\mu_0 + n\bar{y}}{n+1} = \frac{100 + 10 * 113}{11} = 111.818$$

Since we know that $\mu|y$ is normally distributed we can construct a 95% credible interval based on

$$\frac{\mu_0 + n\bar{y}}{n+1} - 1.96 * \sigma, \frac{\mu_0 + n\bar{y}}{n+1} + 1.96 * \sigma$$
$$\frac{100 + 10(113)}{11} - 1.96 * \sigma, \frac{\mu_0 + n\bar{y}}{n+1} + 1.96 * \sigma$$

where

$$\sigma = \frac{1}{\frac{1}{15} + \frac{n}{15}} = \frac{15}{n+1} = V = \frac{15}{11} = 1.36$$

so our final CI is

$$(111.818 - 1.96 * 1.36, 111.818 + 1.96 * 1.36) = (109.152, 114.484)$$

Problem 2

We know by definition that bias is

$$E(\hat{\mu}|\mu^*) - \mu^*$$

Let's take a closer look at
$$E(\hat{\mu}|\mu^*)$$

$$= E(\frac{\mu_0 + n\bar{y}}{n+1}|\mu*) = \frac{\mu_0}{n+1} + \frac{n}{n+1}\mu^*$$

by

$$E(\bar{y}) = \mu^*$$

We can see that the Bayesian estimator has higher bias than the frequentist estimator.

We also need to examine

$$Var(\hat{\mu}|\mu^*) = Var(\frac{\mu_0 + n\bar{y}}{n+1}|\mu*) = Var(\frac{n\bar{y}}{n+1}|\mu*)$$

because the first patrt is a constant.

$$= (\frac{n}{n+1})^2\sigma^2$$

We can see that the Bayesian estimator has a smaller variance than the frequentist estimator by $n/n+1 < 1$ We can now consider the MSE of the bayesian estimator

$$= (\frac{n}{n+1})^2\sigma^2 + (\frac{\mu_0}{n+1} + \frac{n}{n+1}\mu*)^2$$

We know by the properties of the sampling distribution that the frequentist MSE is

$$\frac{\sigma^2}{n} + 0^2$$

Lets look at

$$(\frac{n}{n+1})^2\sigma^2 + (\frac{\mu_0}{n+1} + \frac{n}{n+1}\mu*)^2 - \frac{\sigma^2}{n}?0$$

We know that bias^2 term is >0 so we just need to ask

$$(\frac{n}{n+1})^2\sigma^2 - \frac{\sigma^2}{n}?0$$

$$\sigma^2((\frac{n}{n+1})^2 - \frac{1}{n})?0$$

$$\sigma^2((\frac{n^3}{n(n+1)^2}) - \frac{(n+1)^2}{n(n+1)^2})?0$$

$$\sigma^2(\frac{n^3 - (n^2+2n+1)}{n(n+1)^2})?0$$

$$\sigma^2(\frac{n^3 - n^2 - 2n - 1)}{n(n+1)^2})?0$$

which for $n > 2$ is strictly positive.

```
y_bar <- 113
n <- 10
mu0 <- 100
sig2_0 <- 15^2
sig2_mu0 <- 15^2
var.y <- 13^2
nu0 <- 1
```

```r
S <- 1000
nun <- nu0+n

PHI<- matrix(nrow=S,ncol=2)
PHI[1,]<- phi<- c(y_bar,1/var.y)

# Gibbs sampling
set.seed (1)
for(s in 2:S){
# generate a new mu value from its full conditional
  sigma2.mun <- 1/(1/sig2_mu0 + n*phi[2])
  mun <- sigma2.mun*( mu0/ sig2_mu0 + n*y_bar*phi[2])
  phi[1] <- rnorm( 1 , mun, sqrt(sigma2.mun))

# generate a new 1/sigma^2 value from its full conditional
  sig2_n <-(1/nun)*(nu0*sig2_0 + (n-1)*var.y + n*(y_bar - phi[1])^2)
  phi[2] <- rgamma(1, nun/2 , nun*sig2_n/2)
  PHI[s,]<- phi
}


print (signif(mean(PHI[,1]),3))
```

```
## [1] 112
```

```r
print ("95% Credible Interval")
```

```
## [1] "95% Credible Interval"
```

```r
print (signif(quantile(PHI[,1],c(.025,.975)),3))
```

```
##  2.5% 97.5%
##   102   120
```

```r
print (signif(mean(sqrt(1/PHI[,2])),3))
```

```
## [1] 14.6
```

```r
print ("95% Credible Interval")
```

```
## [1] "95% Credible Interval"
```

```r
print (signif(quantile(1/sqrt(PHI [,2]),c(.025,.975))),3)
```

```
##  2.5% 97.5%
##  9.29 24.24
```