Homework 4

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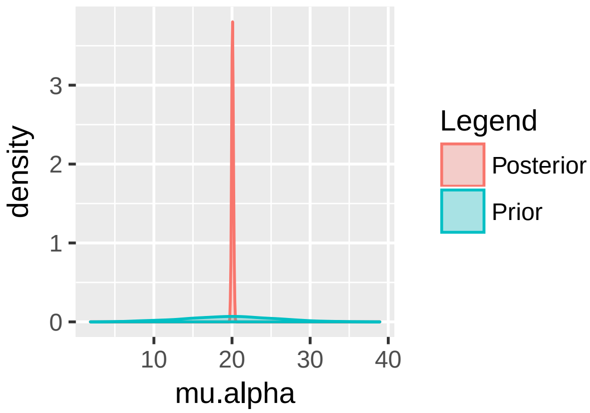
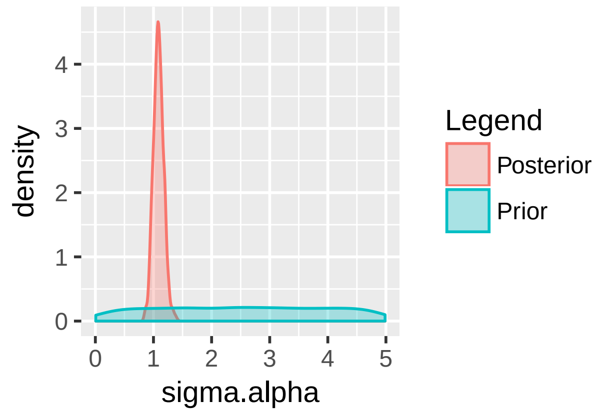
Problem 1

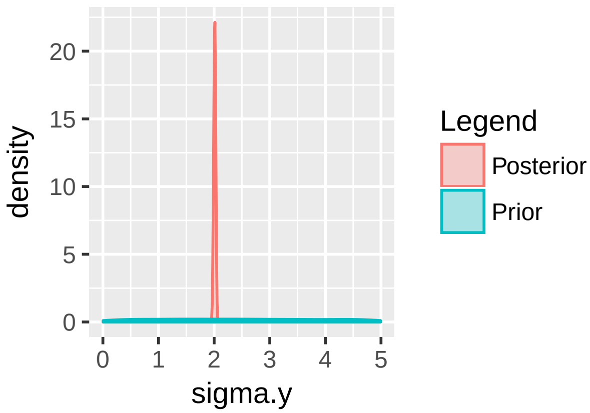
$$m_{\sigma_y} = 5, m_{\sigma_{alpha}} = 5, \mu_0 = 20, \sigma^2_{\mu_0} = 36$$

```
model <-
"model {
for (i in 1:n){
  y.i[i] ~ dnorm(alpha.j[getj.i[i]],tau.y)
}

for (j in 1:J){
 alpha.j[j] ~ dnorm(mu.alpha,tau.alpha)
}

tau.y <- pow(sigma.y, -2)
tau.alpha <-  pow(sigma.alpha, -2)

mu.alpha ~ dnorm(20,1/6^2)
sigma.y ~ dunif(0,5)
sigma.alpha ~ dunif(0,5)

}"
```

```
## Compiling model graph
##    Resolving undeclared variables
##    Allocating nodes
## Graph information:
##    Observed stochastic nodes: 2400
##    Unobserved stochastic nodes: 43
##    Total graph size: 4856
##
## Initializing model
```

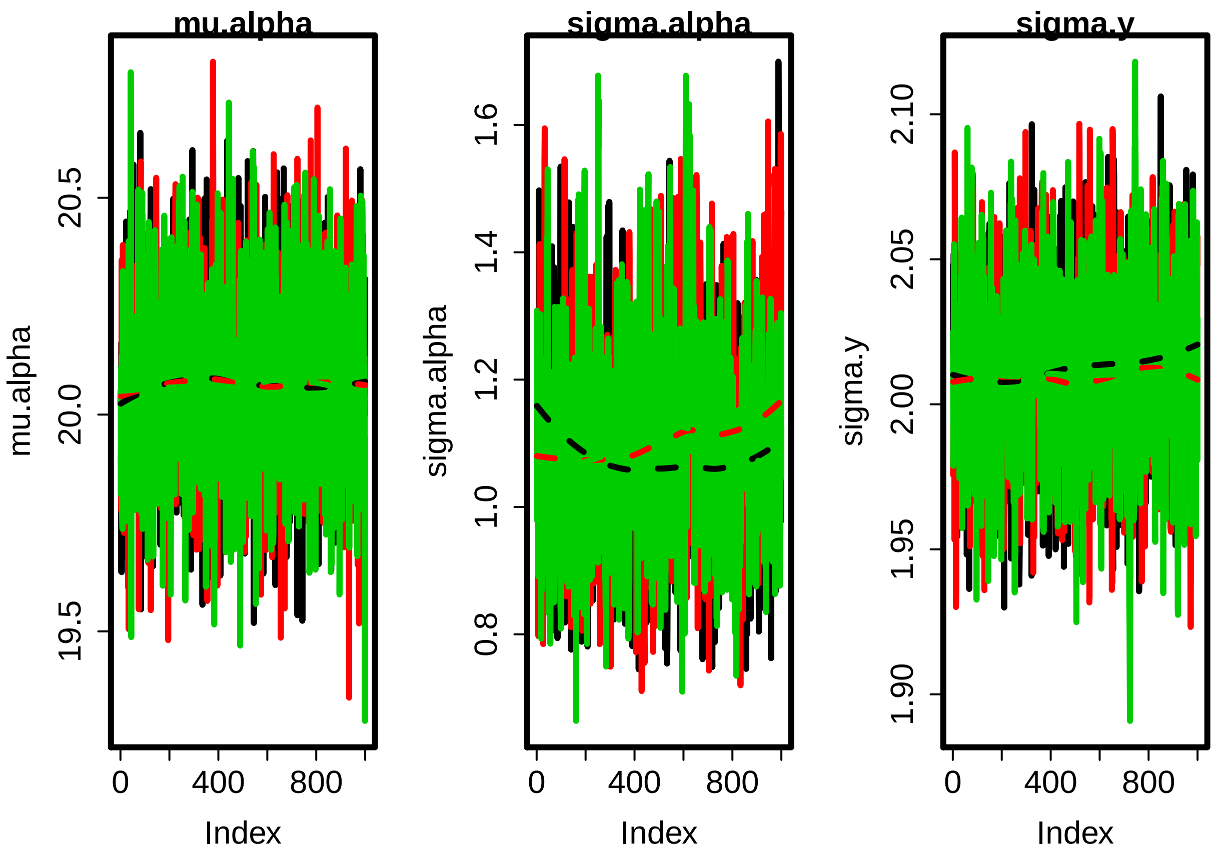






Problem 2

```
##              n.eff  Rhat
## mu.alpha     3000 1.001
## sigma.alpha   560 1.004
## sigma.y      2200 1.001
```



The high N_{eff} values show that the chains have low auto-covariance, which shows that the chains are able to explore the full parameter space. The $\hat{R}$ value of 1 for all params show that all the chains have mixed well, and they did not get stuck in local modes.

Problem 3

a)

```r
print (c(mean(mcmc.array[,,"mu.alpha"]),  quantile(mcmc.array[,,"mu.alpha"],c(.025,.975))     ))
```

```
##              2.5%    97.5%
## 20.06783 19.68833 20.48143
```

```r
print (c(mean(mcmc.array[,,"sigma.alpha"]),  quantile(mcmc.array[,,"sigma.alpha"],c(.025,.975))     ))
```

```
##                2.5%     97.5%
## 1.0880475 0.8260481 1.4396968
```

```r
print (c(mean(mcmc.array[,,"sigma.y"]),  quantile(mcmc.array[,,"sigma.y"],c(.025,.975))     ))
```

```
##              2.5%    97.5%
## 2.010845 1.953183 2.069122
```

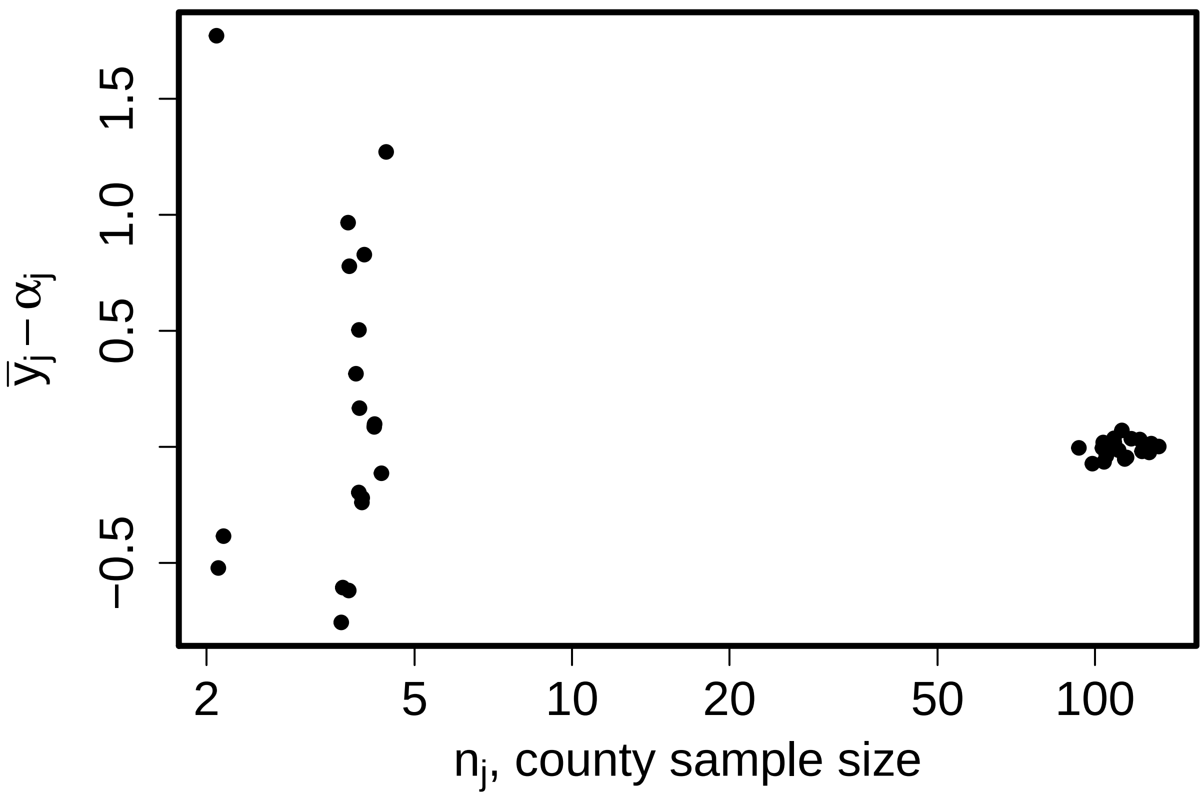
b)

```r
print (round(mod0$BUGSoutput$summary[c("alpha.j[1]"), c("mean", "sd", "2.5%", "97.5%")],1))
```

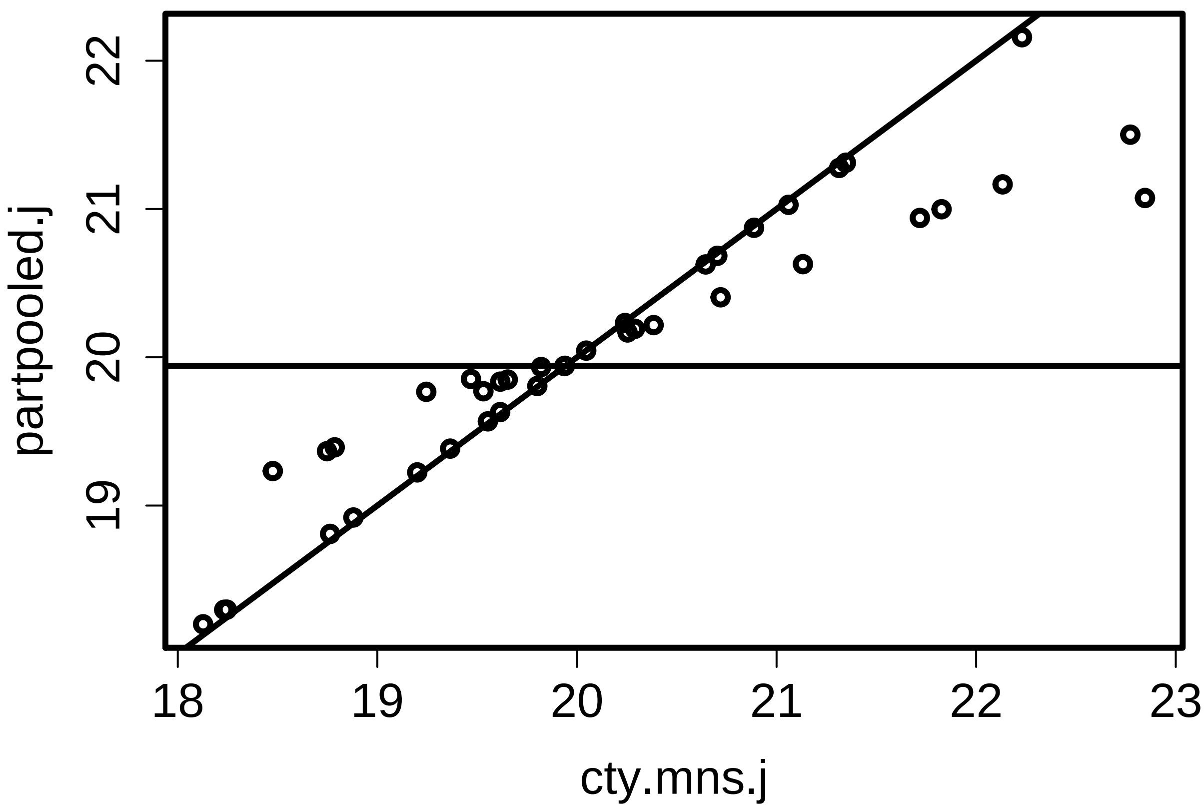
```
##  mean    sd  2.5% 97.5%
##  21.1   0.9  19.4  22.8
```

Problem 4

```
par(lwd = 3, cex.axis = 1.5, cex.lab = 1.5, cex.main = 1.5,mar = c(5,5,1,1))
lims <- range(cty.mns.j-partpooled.j)
plot(c(cty.mns.j-partpooled.j) ~ sample.size.jittered.j, type = "p",
     ylab = expression(bar(y)[j] - alpha[j]),
     xlab = expression(paste(n[j], ", county sample size")), pch = 20,
     log = "x",
     ylim = lims)

abline(h=ybarbar)
```




```
plot(cty.mns.j,partpooled.j,abline(h=ybarbar))
abline(0,1)
```



We can see that there is more variability of $\bar{y_j}$ around α_j when sample size is small, which makes sense because the data does not pull α_j towards $\bar{y_j}$. We can from the second plot that as $\bar{y_j}$ gets larger in magnitude relative to the overall mean, the α_j is pulled down towards the overall mean. Similarly, as $\bar{y}$ gets small in magnitiute relative to the overall mean, α_j gets pulled upwards towards the overall-mean. This is the "shrinkage" effect.

Problem 5

Consider

$$p(\alpha_j|y, \mu_\alpha, \sigma_y, \sigma_\alpha) \propto p(y|\alpha_j, \sigma_y)p(\alpha_j|\mu_\alpha, \sigma_\alpha)$$

$$p(\alpha_j|y, \mu_\alpha, \sigma_y, \sigma_\alpha) \propto (\frac{1}{\sqrt{2\pi\sigma_y}})^n e^{-\frac{1}{2\sigma_y^2}\sum_i (y_i-\alpha_j)^2} \frac{1}{\sqrt{2\pi\sigma_\alpha}} e^{-\frac{1}{2\sigma_\alpha^2}(\alpha_j-\mu_\alpha)^2}$$

For now, let's just focus in on the terms involving e^x

we see the exponent can be written as

$$-\frac{1}{2\sigma_y^2}\sum_i (y_i - \alpha_j)^2 - \frac{1}{2\sigma_\alpha^2}(\alpha_j - \mu_\alpha)^2$$

Expanding out the square we get

$$-\frac{1}{2\sigma_y^2}\sum_i (y^2 - 2y\alpha_j + \alpha_j^2) - \frac{1}{2\sigma_\alpha^2}(\alpha_j^2 - 2\alpha_j\mu_\alpha + \mu_\alpha^2)$$

$$-\frac{1}{2\sigma_y^2}\sum_i y^2 + \frac{n_j\bar{y_j}\alpha_j}{\sigma_y^2} - \frac{n_j\alpha_j^2}{2\sigma_y^2} - \frac{1}{2\sigma_\alpha^2}\alpha_j^2 + \frac{1}{\sigma_\alpha^2}\alpha_j\mu_\alpha - \frac{1}{2\sigma_\alpha^2}\mu_\alpha^2$$

Collecting terms on α_j we can re-write this as

$$-\frac{n_j\alpha_j^2}{2\sigma_y^2} - \frac{1}{2\sigma_\alpha^2}\alpha_j^2 + \frac{1}{\sigma_\alpha^2}\alpha_j\mu_\alpha + \frac{n_j\bar{y_j}\alpha_j}{\sigma_y^2} - \frac{1}{2\sigma_\alpha^2}\mu_\alpha^2 - \frac{1}{2\sigma_y^2}\sum_i y^2$$

$$-\alpha_j^2(\frac{n_j}{2\sigma_y^2} + \frac{1}{2\sigma_\alpha^2}) + \alpha_j(\frac{1}{\sigma_\alpha^2}\mu_\alpha + \frac{n_j\bar{y_j}}{\sigma_y^2}) - \frac{1}{2\sigma_\alpha^2}\mu_\alpha^2 - \frac{1}{2\sigma_y^2}\sum_i y^2$$

We see this is almost a quadratic in α_j, that is we almost have a form of

$$a^2\alpha_j^2 - b\alpha_j + c^2 - c^2 = (a\alpha_j - c)^2$$

If we take

$$b = 2ac \to c = \frac{b}{2a}$$

We know from above that $b = (\frac{1}{\sigma_\alpha^2}\mu_\alpha + \frac{n_j\bar{y_j}}{\sigma_y^2})$

and $a = \sqrt{\frac{n_j}{2\sigma_y^2} + \frac{1}{2\sigma_\alpha^2}}$

So we can choose

$$c = \frac{(\frac{1}{\sigma_\alpha^2}\mu_\alpha + \frac{n_j\bar{y_j}}{\sigma_y^2})}{\sqrt{2(\frac{n_j}{2\sigma_y^2} + \frac{1}{2\sigma_\alpha^2})}}$$

$$= \frac{(\frac{1}{\sigma_\alpha^2}\mu_\alpha + \frac{n_j\bar{y_j}}{\sigma_y^2})}{\sqrt{\frac{n_j}{\sigma_y^2} + \frac{1}{\sigma_\alpha^2}}}$$

plugging this back into the exp we get

$$p(\alpha_j|y,\mu_\alpha,\sigma_y,\sigma_\alpha) \propto \frac{1}{C}e^{-(\sqrt{\frac{n_j}{2\sigma_y^2}+\frac{1}{2\sigma_\alpha^2}}\alpha_j - \frac{(\frac{1}{\sigma_\alpha^2}\mu_\alpha + \frac{n_j\bar{y_j}}{\sigma_y^2})}{\sqrt{\frac{n_j}{\sigma_y^2}+\frac{1}{\sigma_\alpha^2}})^2}}$$

$$p(\alpha_j|y,\mu_\alpha,\sigma_y,\sigma_\alpha) \propto \frac{1}{C}e^{-\frac{1}{\frac{n_j}{2\sigma_y^2}+\frac{1}{2\sigma_\alpha^2}}(\alpha_j - \frac{(\frac{1}{\sigma_\alpha^2}\mu_\alpha + \frac{n_j\bar{y_j}}{\sigma_y^2})}{\frac{n_j}{\sigma_y^2}+\frac{1}{\sigma_\alpha^2}})^2}$$

$$p(\alpha_j|y,\mu_\alpha,\sigma_y,\sigma_\alpha) \propto \frac{1}{C}e^{-\frac{1}{2}\frac{1}{\frac{n_j}{\sigma_y^2}+\frac{1}{\sigma_\alpha^2}}(\alpha_j - \frac{(\frac{1}{\sigma_\alpha^2}\mu_\alpha + \frac{n_j\bar{y_j}}{\sigma_y^2})}{\frac{n_j}{\sigma_y^2}+\frac{1}{\sigma_\alpha^2}})^2}$$

which we can see is the kernel of a normal distribution with mean

$$m = \frac{(\frac{1}{\sigma_\alpha^2}\mu_\alpha + \frac{n_j\bar{y_j}}{\sigma_y^2})}{\frac{n_j}{\sigma_y^2} + \frac{1}{\sigma_\alpha^2}})^2$$

and variance

$$v = \frac{1}{\frac{n_j}{\sigma_y^2} + \frac{1}{\sigma_\alpha^2}}$$