Sequential Stein Variational Gradient Descent for Time Series Model Estimation

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Introduction

Particle filtering suffers from two main practical disadvantages. The first is particle depletion, where the number of effective particles with non-neglibile weight becomes too small. This has the effect of concentrating the mass around a small number of particles, leading to poor estimates of the target distribution. The second is the running time of the algorithm. A cursory analysis reveals that each particle is updated once per time step in the time series, and once per re-sampling step, to mitigate the issue above. If we imagine the order of particles is close to the length of the time series, we see that run-time is $O(n^3)$.

We propose another approach that we hope will do better than particle filtering in practice. In this approach, Stein Variational Gradient Descent (SVGD) is used to sequentially estimate the distribution of state variables in each time step, conditional on observed data up through that time. This method should overcome problems with particle depletion and excessive run-times for long time-series.

Overview of SVGD

Stein Variational Gradient Descent can be used to estimate a continuous distribution by a set of samples. By iteratively transporting samples from an initial distribution in the direction of the likelihood, we are able to generate a sample from the target distribution. The usefullness of this approximation is apparent in Bayesian statistics, where the usually intractable normalizing constant disappears in the gradient. The particles are subject to the following gradient ascent procedure.

$$x_t^{(i)} \leftarrow x_{t-1}^{(i)} + \frac{1}{n}\sum_{j=1}^{n}[k(x_j, x_{t-1}^{(i)}) * \nabla log\ p(x_j) + \nabla k(x_j, , x_{t-1}^{(i)})]$$

Sequential Stein Variational Gradient Descent

Suppose we are given a time series $Y_1, Y_2, ..., Y_t$ for $Y \in \mathbb{R}$. We model the sequence as a state-space model parameterized by an observation density $p(y_t|x_t)$ and a transition density $p(x_t|x_{t-1})$ Figure 1.

We are interested in the filtering distribution $p(x_1, ..., x_n|y_1, ..., y_n)$ which by Bayes formula is

$$p(x_1, ..., x_n|y_1, ..., y_n) = \frac{p(y_1, ..., y_n|x_1, ..., x_n)p(x_1, ..., x_n)}{Z}$$

.

Because computing the normalizing constant Z is intractable for many choices of $p(y_t|x_t)$ and $p(x_t|x_{t-1})$, we must resort to monte carlo algorithms. The classic approach that incorporates the sequential nature of the data is given by the particle filtering algorithm. Particle filtering approximates the filtering density using sequential importance sampling. We instead focus on the following recursion.

$$p(x_t|y_{1:t}) = \int p(x_{0:t}|y_{1:t})d_{x_0:t-1}$$

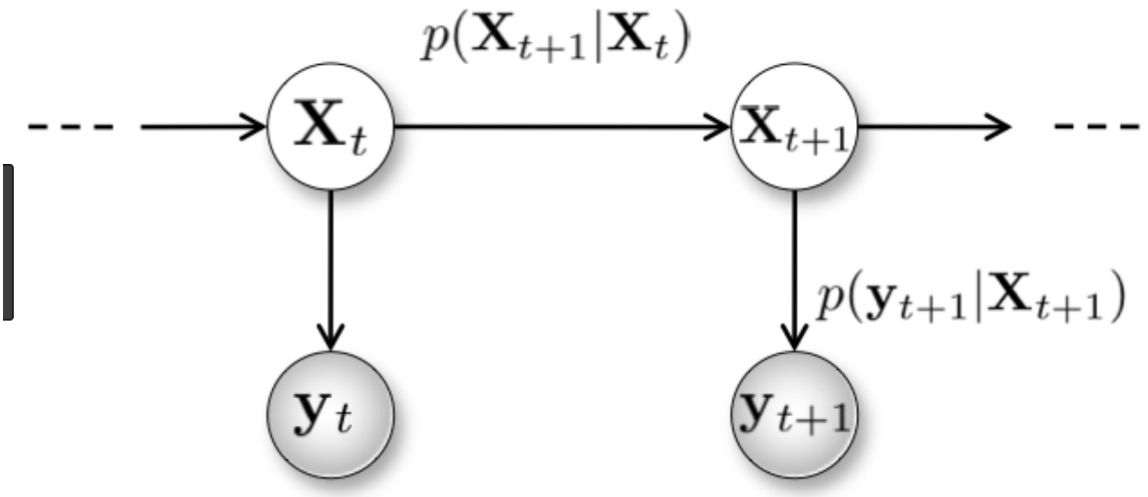


Figure 1: Caption for the picture.

$$= \frac{p(y_t|x_t)}{\int p(y_t|x_t)p(x_t|y_{1:t-1})dx_t} p(x_t|y_{1:t-1})$$

$$\propto p(y_t|x_t)p(x_t|y_{1:t-1})$$
$$\propto p(y_t|x_t)p(x_t|y_{1:t-1})$$
$$\propto p(y_t|x_t) \int_{x_{t-1}} p(x_t, x_{t-1}|y_{1:t-1}) d_{x_{t-1}}$$

$$\propto p(y_t|x_t) \int_{x_{t-1}} p(x_t|x_{t-1})p(x_{t-1}|y_{1:t-1}) d_{x_{t-1}}$$

which we can approximate using svgd as

$$\approx p(y_t|x_t) \frac{1}{n} \sum_{i=1}^{n} p(x_t|x_{t-1}^{(i)})$$

We can now estimate $p(x_{t+1}|y_{1:t+1})$ using the same algebra as above. (proof in apendix A)

Model Structure

States:

- $X_1 \sim g_1(x_1; \xi)$
- $X_t|X_{t-1} \sim g(x_t|x_{t-1}; \xi)$ for all $t = 2, \ldots, T$

Observations:

- $Y_t|X_t \sim h(y_t|x_t; \zeta)$

Here, $g_1(\cdot)$ and $g(\cdot)$ are appropriately defined probability density functions depending on parameters ξ and $h(\cdot)$ is an appropriately defined probability density function or probability mass function depending on parameters ζ.

Define $\theta = (\xi, \zeta)$ to be the full set of model parameters.

Filtering

There are two types of filtering:

1. sample of particles $x_{1:T}^{(k)} \sim f(x_{1:T}|y_{1:T})$
2. sample of particles $x_t^{(k)} \sim f(x_t|y_{1:t})$ for each $t = 1, \ldots, T$

Let's look at the second one. Assume we have a sample $x_{t-1}^{(k)} \sim f(x_{t-1}|y_{1:t-1})$

$$\begin{aligned}
p(x_t|y_{1:t}) &= \frac{f(x_t, y_t|y_{1:t-1})}{f(y_t|y_{1:t-1})} \\
&\propto f(x_t, y_t|y_{1:t-1}) \\
&= f(y_t|x_t) f(x_t|y_{1:t-1}) \\
&= f(y_t|x_t) \int f(x_t, x_{t-1}|y_{1:t-1}) dx_{t-1} \\
&= f(y_t|x_t) \int f(x_t|x_{t-1}) f(x_{t-1}|y_{1:t-1}) dx_{t-1} \\
&\approx f(y_t|x_t) \sum_{x_{t-1}^{(k)}} f(x_t|x_{t-1}^{(k)})
\end{aligned}$$

So $\log\{p(x_t|y_{1:t})\}$ is approximately proportional to $\log\{f(y_t|x_t)\} + \log\{\sum_{x_{t-1}^{(k)}} f(x_t|x_{t-1}^{(k)})\}$

Locally Level Gaussian Noise Model

In order to demonstrate that the approximation is reasonable we evaluate the predictive accuracy under an analytically tractable model, the locally level Gaussian model. This model takes the form

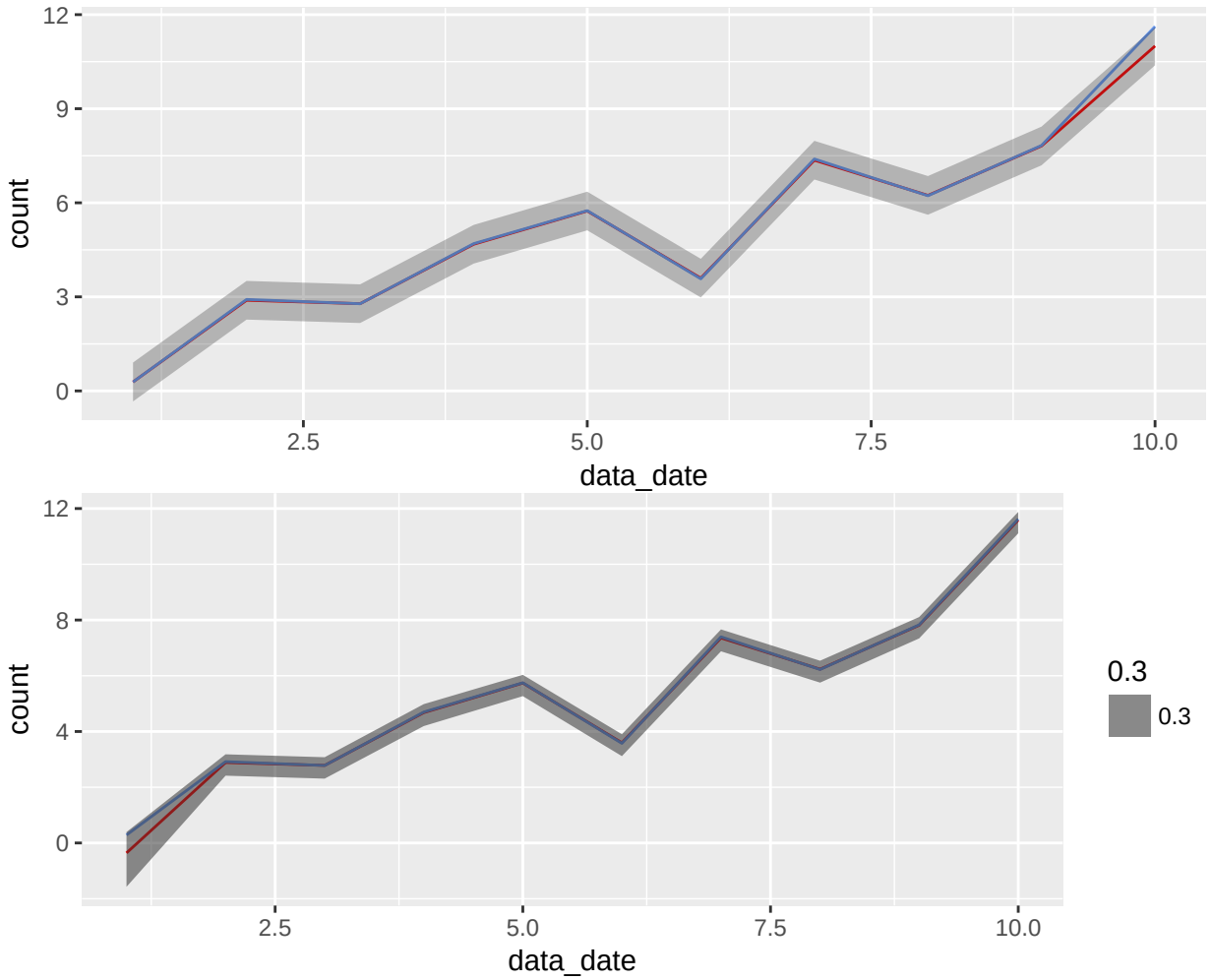
$$X_t \sim N(X_{t-1}, \sigma_1^2)$$

$$Y_t \sim N(X_t, \sigma_2^2)$$

```
##
## Attaching package: 'dlm'
```

```
## The following object is masked from 'package:ggplot2':
##
##     %+%
```

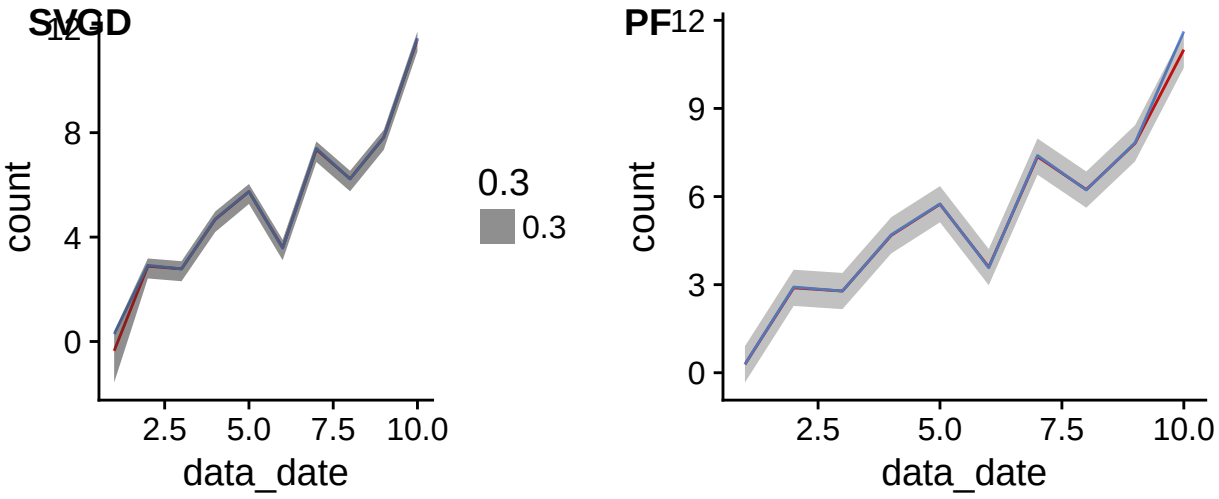
```
## [1] "python /Users/gcgibson/Stein-Variational-Gradient-Descent/python/locally_level_gaussian.py  0.28
```




```
## Warning: package 'cowplot' was built under R version 3.4.3
```

```
##
## Attaching package: 'cowplot'
```

```
## The following object is masked from 'package:ggplot2':
##
##     ggsave
```



Poisson Observation Model With Seasonal State-Space Dynamics

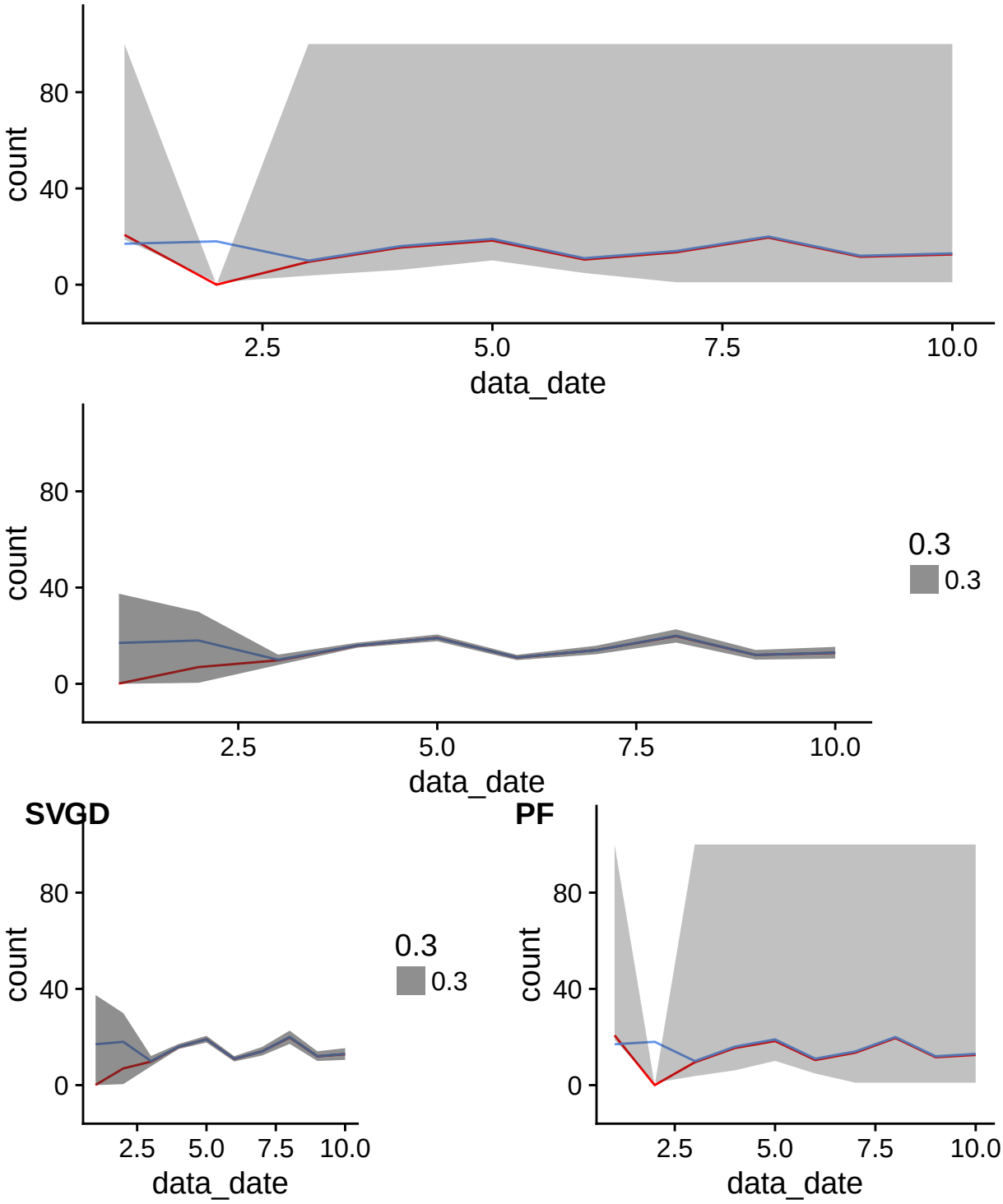
In order to evaluate the performance on more involved dynamics we consider the following state-space model.

$$\begin{pmatrix} X_{t,1} \\ X_{t,2} \end{pmatrix} = \begin{pmatrix} cos(2\pi/s) & sin(2\pi/s) \\ -sin(2\pi/s) & cos(2\pi/s) \end{pmatrix} \begin{pmatrix} X_{t-1,1} \\ X_{t-1,2} \end{pmatrix}$$

$$Y_t \sim Pois(e^{X_{t,1}})$$

```
## Loading required package: rbiips
## Loading required package: coda
## Loading required package: MASS
##
## Attaching package: 'MASS'
## The following object is masked from 'package:dplyr':
##
##     select
## ##
## ## Markov Chain Monte Carlo Package (MCMCpack)
## ## Copyright (C) 2003-2018 Andrew D. Martin, Kevin M. Quinn, and Jong Hee Park
## ##
## ## Support provided by the U.S. National Science Foundation
## ## (Grants SES-0350646 and SES-0350613)
## ##
## * Parsing model in: /Users/gcgibson/Stein-Variational-Gradient-Descent/bug_files/seasonal_pois.bug
## Warning in biips_model(seasonal_model_file, data = seasonal_data,
## sample_data = FALSE): Unused variables in data: G, mean_sigma_init,
## cov_sigma_init, mean_x_init
## * Compiling model graph
##   Declaring variables
##   Resolving undeclared variables
##   Allocating nodes
##   Graph size: 121
```

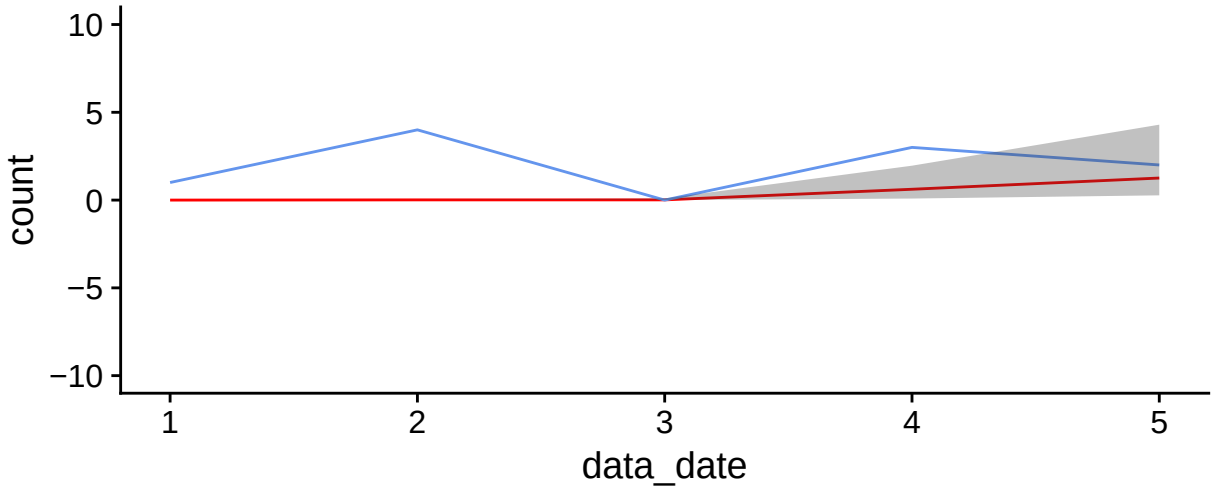
```
## * Assigning node samplers
## * Running SMC forward sampler with 100000 particles
##   |-------------------------------------------------| 100%
##   |*************************************************| 11 iterations in 20.80 s
## [1] "python /Users/gcgibson/Stein-Variational-Gradient-Descent/python/seasonal.py  17, 18, 10, 16, 19
```

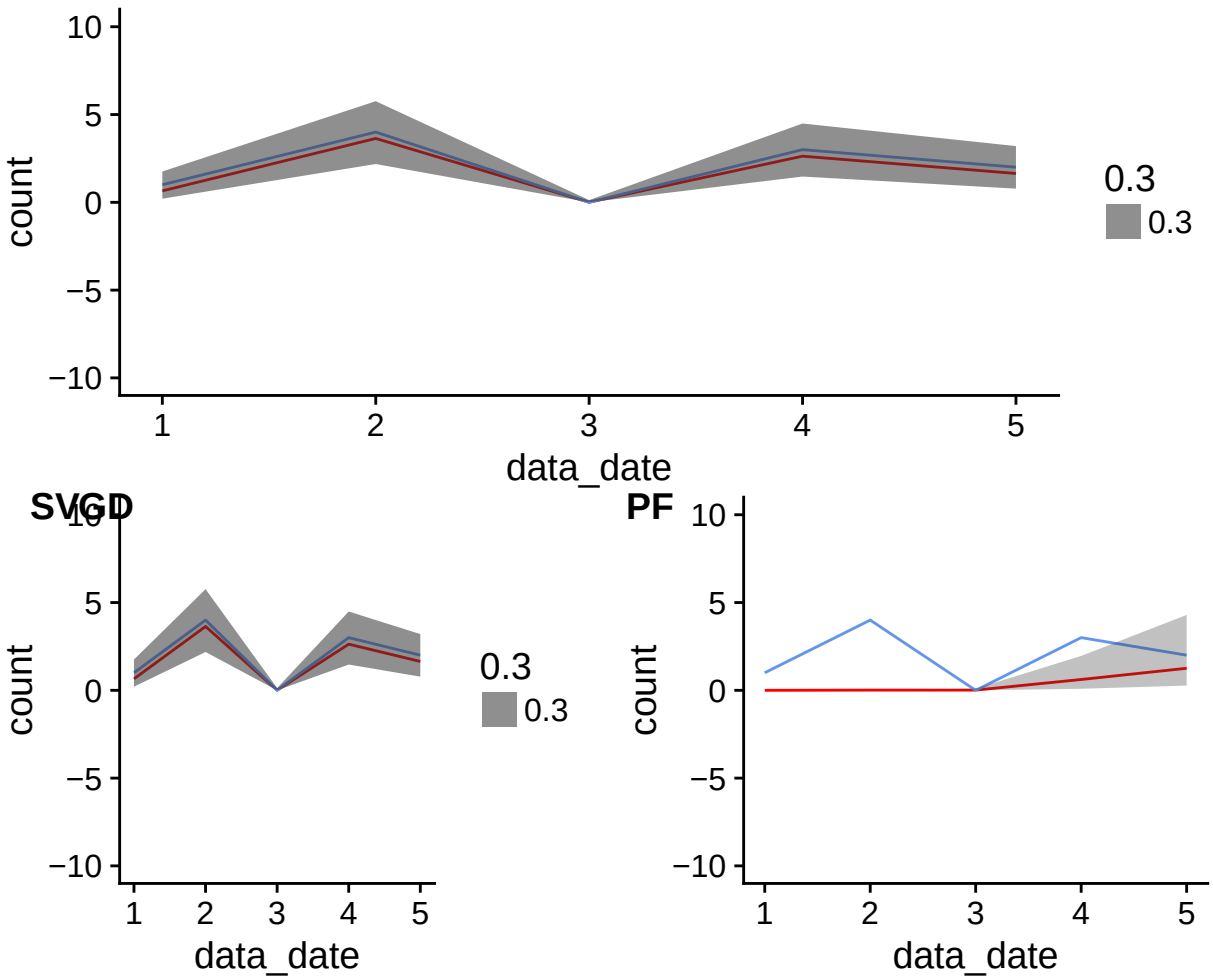


Divergent Particle Filter

We next investigate the ability of SSVGD to perform in the presence of poor initialization. This is a well known issue with current particle filter implemntations: starting far from a plausabile value of x_0 forces all particles to receive weight 0 under the likelihood, leading to a degenerate filtering distribution. However, under SSVGD, we can simply increase the number of iterations, allowing for arbitrarily poor starting points.

```
## * Parsing model in: /Users/gcgibson/Stein-Variational-Gradient-Descent/bug_files/locally_level_1.bug
## * Compiling model graph
##   Declaring variables
##   Resolving undeclared variables
##   Allocating nodes
##   Graph size: 18

## * Assigning node samplers
## * Running SMC forward sampler with 10000 particles
##   |-------------------------------------------------| 100%
##   |*************************************************| 5 iterations in 0.24 s

## * Diagnosis of variable: x[1:5]
##   Filtering: POOR
##     The minimum effective sample size is too low: 8.788576
##     Estimates may be poor for some variables.
##     You should increase the number of particles
## .  Smoothing: POOR
##     The minimum effective sample size is too low: 3.664264
##     Estimates may be poor for some variables.
##     You should increase the number of particles
## .

## [1] 0.0001262422 0.0115295532 0.0112004317 0.6160331587 1.2560522188

## [1] "python /Users/gcgibson/Stein-Variational-Gradient-Descent/python/locally_level.py  1, 4, 0, 3, 1
```





Results

Paragraph, referencing one figure and one table, summarizing results

Application

Example model with real data. fairly real model, but not thaaaaaat complex.

Discussion

Bibliography