4. Markov Chains (10/13/05, cf. Ross)

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4.1. Introduction

Definition: A *stochastic process* (SP) $\{X(t) : t \in T\}$ is a collection of RV's. Each $X(t)$ is a RV; t is usually regarded as "time."

Example: $X(t) =$ the number of customers in line at the post office at time t.

Example: $X(t) =$ the price of IBM stock at time t.

T is the *index set* of the process. If T is countable, then $\{X(t) : t \in T\}$ is a *discrete-time* SP. If T is some continuum, then $\{X(t) : t \in T\}$ is a *continuous-time* SP.

Example: $\{X_n : n = 0, 1, 2, \ldots\}$ (index set of non-negative integers)

Example: $\{X(t) : t \geq 0\}$ (index set is $\Re_+$)

The *state space* of the SP is the set of all possible values that the RV's $X(t)$ can take.

Example: If $X_n = j$, then the process is in *state* j at time n.

Any realization of $\{X(t)\}$ is a *sample path*.

Definition: A *Markov chain* (MC) is a SP such that whenever the process is in state i, there is a fixed *transition probability* P_{ij} that its next state will be j.

Denote the "current" state (at time n) by $X_n = i$.

Let the event $A = \{X_0 = i_0, X_1 = i_1, \ldots X_{n-1} = i_{n-1}\}$ be the previous history of the MC (before time n).

$\{X_n\}$ has the *Markov property* if it forgets about its past, i.e.,

$$\Pr(X_{n+1} = j | A \cap X_n = i) = \Pr(X_{n+1} = j | X_n = i).$$

$\{X_n\}$ is *time homogeneous* if

$$\Pr(X_{n+1} = j | X_n = i) = \Pr(X_1 = j | X_0 = i) = P_{ij},$$

i.e., if the transition probabilities are independent of n.

Recap: A *Markov chain* is a SP such that

$$\text{Pr}(X_{n+1} = j | A \cap X_n = i) \ = \ P_{ij},$$

i.e., the next state depends *only* on the current state (and is indep of the time).

Since P_{ij} is a probability, $0 \leq P_{ij} \leq 1$ for all i, j.

Since the process has to go from i to *some* state, we must have $\sum_{j=0}^{\infty} P_{ij} = 1$, for all i. Note that it may be possible to go from i to i (i.e., "stay" at i).

Definition: The *one-step transition matrix* is

$$\mathbf{P} = \begin{pmatrix} P_{00} & P_{01} & P_{02} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

Example: A frog lives in a pond with three lily pads (1,2,3). He sits on one of the pads and periodically rolls a die. If he rolls a 1, he jumps to the lower numbered of the two unoccupied pads. Otherwise, he jumps to the higher numbered pad. Let X_0 be the initial pad and let X_n be his location just after the nth jump. This is a MC since his position only depends on the current position, and the P_{ij}'s are independent of n.

$$\mathbf{P} = \begin{pmatrix} 0 & 1/6 & 5/6 \\ 1/6 & 0 & 5/6 \\ 1/6 & 5/6 & 0 \end{pmatrix}. \quad \diamondsuit$$

Example: Let X_i denote the weather (rain or sun) on day i. We'll think of X_{i-1} as yesterday, X_i as today, and X_{i+1} as tomorrow. Suppose that

$$\begin{aligned}
\Pr(X_{i+1} = R \mid X_{i-1} = R, X_i = R) &= 0.7 \\
\Pr(X_{i+1} = R \mid X_{i-1} = S, X_i = R) &= 0.5 \\
\Pr(X_{i+1} = R \mid X_{i-1} = R, X_i = S) &= 0.4 \\
\Pr(X_{i+1} = R \mid X_{i-1} = S, X_i = S) &= 0.2
\end{aligned}$$

$X_0, X_1, \ldots$ isn't quite a MC, since the probability that it'll rain tomorrow depends on X_i *and* X_{i-1}.

We'll transform the process into a MC by defining the following states in terms of today *and* yesterday.

$$
\begin{aligned}
0: &\quad X_{i-1} = R,\ X_i = R \\
1: &\quad X_{i-1} = S,\ X_i = R \\
2: &\quad X_{i-1} = R,\ X_i = S \\
3: &\quad X_{i-1} = S,\ X_i = S
\end{aligned}
$$

Thus, we have, e.g.,

$$\Pr(X_{i+1} = R \mid X_{i-1} = R, X_i = R) \; = \; P_{00} \; = \; 0.7$$
$$\Pr(X_{i+1} = S \mid X_{i-1} = R, X_i = R) \; = \; P_{02} \; = \; 0.3$$

Using similar reasoning, we get

$$\mathbf{P} = \begin{pmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{pmatrix}. \quad \diamondsuit$$

Example: A MC whose state space is given by the integers is called a *random walk* if $P_{i,i+1} = p$ and $P_{i,i-1} = 1 - p$.

$$\mathbf{P} = \begin{pmatrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1-p & 0 & p & 0 & 0 & \cdots \\ \cdots & 0 & 1-p & 0 & p & 0 & \cdots \\ \cdots & 0 & 0 & 1-p & 0 & p & \cdots \\ \cdots & 0 & 0 & 0 & 1-p & 0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \quad \diamondsuit$$

Example (Gambler's Ruin): Every time a gambler plays a game, he wins \$1 w.p. p, and he loses \$1 w.p. $1-p$. He stops playing as soon as his fortune is either \$0 or \$N. The gambler's fortune is a MC with the following P_{ij}'s:

$$
\begin{aligned}
P_{i,i+1} &= p, \quad i = 1,2,\ldots,N-1 \\
P_{i,i-1} &= 1-p, \quad i = 1,2,\ldots,N-1 \\
P_{0,0} &= P_{N,N} = 1
\end{aligned}
$$

0 and N are *absorbing* states — once the process enters one of these states, it can't leave. $\diamondsuit$

Example (Ehrenfest Model): A random walk on a finite set of states with "reflecting" boundaries. Set of states is $\{1, 2, \ldots, a\}$.

$$P_{ij} = \begin{cases} \frac{a-i}{a} & \text{if } j = i+1 \\ \frac{i}{a} & \text{if } j = i-1 \\ 0 & \text{otherwise} \end{cases}$$

Idea: Suppose A has i marbles, B has $a-i$. Select a marble at random, and put it in the other container. $\diamondsuit$

4.2 Chapman-Kolmogorov Equations

Definition: The n-*step transition probability* that a process currently in state i will be in state j after n additional transitions is

$$P_{ij}^{(n)} \equiv \text{Pr}(X_n = j | X_0 = i), \quad n, i, j \geq 0.$$

Note that $P_{ij}^{(1)} = P_{ij}$, and

$$P_{ij}^{(0)} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} .$$

Theorem (C-K Equations):

$$P_{ij}^{(n+m)} = \sum_{k=0}^{\infty} P_{ik}^{(n)} P_{kj}^{(m)}.$$

Think of going from i to j in $n+m$ steps with an intermediate stop in state k after n steps; then sum over all possible k values.

Proof: By definition,

$$
\begin{aligned}
& P_{ij}^{(n+m)} \\
&= \Pr(X_{n+m}=j|X_0=i) \\
&= \sum_{k=0}^{\infty} \Pr(X_{n+m}=j \cap X_n=k|X_0=i) \quad \text{(total prob)} \\
&= \sum_{k=0}^{\infty} \Pr(X_{n+m}=j|X_0=i \cap X_n=k)\Pr(X_n=k|X_0=i) \\
&\quad \text{(since } \Pr(A \cap C|B) = \Pr(A|B \cap C)\Pr(C|B)\text{)} \\
&= \sum_{k=0}^{\infty} \Pr(X_{n+m}=j|X_n=k)\Pr(X_n=k|X_0=i) \\
&\quad \text{(Markov property).} \quad \diamondsuit
\end{aligned}
$$

Definition: The n-*step transition matrix* is

$$\mathbf{P}^{(n)} = \begin{pmatrix} P_{00}^{(n)} & P_{01}^{(n)} & P_{02}^{(n)} & \cdots \\ P_{10}^{(n)} & P_{11}^{(n)} & P_{12}^{(n)} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

The C-K equations imply $\mathbf{P}^{(n+m)} = \mathbf{P}^{(n)}\mathbf{P}^{(m)}$.

In particular, $\mathbf{P}^{(2)} = \mathbf{P}^{(1)}\mathbf{P}^{(1)} = \mathbf{P}\mathbf{P} = \mathbf{P}^2$.

By induction, $\mathbf{P}^{(n)} = \mathbf{P}^n$.

Example: Let $X_i = 0$ if it rains on day i; otherwise, $X_i = 1$. Suppose $P_{00} = 0.7$ and $P_{10} = 0.4$. Then

$$\mathbf{P} = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}.$$

Suppose it rains on Monday. Then the prob that it rains on Friday is $P_{00}^{(4)}$. Note that

$$\mathbf{P}^{(4)} = \mathbf{P}^4 = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}^4 = \begin{pmatrix} 0.5749 & 0.4251 \\ 0.5668 & 0.4332 \end{pmatrix},$$

so that $P_{00}^{(4)} = 0.5749$. $\diamondsuit$

Unconditional Probabilities

Suppose we know the "initial" probabilities,

$$\alpha_i \equiv \Pr(X_0 = i), \quad i = 0, 1, \ldots.$$

(Note that $\sum_i \alpha_i = 1$.) Then by total probability,

$$\begin{aligned}
\Pr(X_n = j) &= \sum_{i=0}^{\infty} \Pr(X_n = j \cap X_0 = i) \\
&= \sum_{i=0}^{\infty} \Pr(X_n = j | X_0 = i) \Pr(X_0 = i) \\
&= \sum_{i=0}^{\infty} P_{ij}^{(n)} \alpha_i.
\end{aligned}$$

Example: In the above example, suppose $\alpha_0 = 0.4$ and $\alpha_1 = 0.6$. Find the prob that it will not rain on the 4th day after we start keeping records (assuming nothing about the first day).

$$
\begin{aligned}
\Pr(X_4 = 1) &= \sum_{i=0}^{\infty} P_{i1}^{(4)} \alpha_i \\
&= P_{01}^{(4)} \alpha_0 + P_{11}^{(4)} \alpha_1 \\
&= (0.4251)(0.4) + (0.4332)(0.6) \\
&= 0.4300. \quad \diamondsuit
\end{aligned}
$$

4.3 Types of States

Definition: If $P_{ij}^{(n)} > 0$ for some $n \geq 0$, state j is *accessible* from i.

Notation: $i \to j$.

Definition: If $i \to j$ and $j \to i$, then i and j *communicate*.

Notation: $i \leftrightarrow j$.

Theorem: Communication is an *equivalence relation*:

(i) $i \leftrightarrow i$ for all i (reflexive).

(ii) $i \leftrightarrow j$ implies $j \leftrightarrow i$ (symmetric).

(iii) $i \leftrightarrow j$ and $j \leftrightarrow k$ imply $i \leftrightarrow k$ (transitive).

Proof: (i) and (ii) are trivial, so we'll only do (iii). To do so, suppose $i \leftrightarrow j$ and $j \leftrightarrow k$. Then there are n, m such that $P_{ij}^{(n)} > 0$ and $P_{jk}^{(m)} > 0$. So by C-K,

$$P_{ik}^{(n+m)} = \sum_{r=0}^{\infty} P_{ir}^{(n)} P_{rk}^{(m)} \geq P_{ij}^{(n)} P_{jk}^{(m)} > 0.$$

Thus, $i \to k$. Similarly, $k \to i$. $\diamondsuit$

Definition: An *equivalence class* consists of all states that communicate with each other.

Remark: Easy to see that two equiv classes are disjoint.

Example: The following $\mathbf{P}$ has equiv classes $\{0,1\}$ and $\{2,3\}$.

$$\mathbf{P} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 3/4 & 1/4 \\ 0 & 0 & 1/4 & 3/4 \end{pmatrix}. \quad \diamondsuit$$

Example: **P** again has equiv classes $\{0,1\}$ and $\{2,3\}$ — note that 1 isn't accessible from 2.

$$\mathbf{P} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/4 & 1/4 & 0 \\ 0 & 0 & 3/4 & 1/4 \\ 0 & 0 & 1/4 & 3/4 \end{pmatrix}. \quad \diamondsuit$$

Definition: A MC is *irreducible* if there is only one equiv class (i.e., if all states communicate).

Example: The previous two examples are **not** irreducible. $\diamondsuit$

Example: The following $\mathbf{P}$ is irreducible since all states communicate ("loop" technique: $0 \to 1 \to 0$).

$$\mathbf{P} = \begin{pmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{pmatrix}. \quad \diamondsuit$$

Example: $\mathbf{P}$ is irreducible since $0 \to 2 \to 1 \to 0$.

$$\mathbf{P} = \begin{pmatrix} 1/4 & 0 & 3/4 \\ 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix}. \quad \diamondsuit$$

Definition: The probability that the MC eventually returns to state i is

$$f_i \equiv \Pr(X_n = i \text{ for some } n \geq 1 | X_0 = i).$$

Example: The following MC has equiv classes $\{0,1\}$, $\{2\}$, and $\{3\}$, the latter of which is absorbing.

$$\mathbf{P} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have $f_0 = f_1 = 1$, $f_2 = 1/4$, $f_3 = 1$. $\diamondsuit$

Remark: The f_i's are usually hard to compute.

Definition: If $f_i = 1$, state i is *recurrent*. If $f_i < 1$, state i is *transient*.

Theorem: Suppose $X_0 = i$. Let N denote the number of times that the MC is in state i (before leaving i forever). Note that $N \geq 1$ since $X_0 = i$. Then i is recurrent iff $\mathsf{E}[N] = \infty$ (and i is transient iff $\mathsf{E}[N] < \infty$).

Proof: If i is recurrent, it's easy to see that the MC returns to i an infinite number of times; so $\mathsf{E}[N] = \infty$. Otherwise, suppose i is transient. Then

$$
\begin{aligned}
\Pr(N = 1) &= 1 - f_i \quad \text{(never returns)} \\
\Pr(N = 2) &= f_i(1 - f_i) \quad \text{(returns exactly once)} \\
&\vdots \\
\Pr(N = k) &= f_i^{k-1}(1 - f_i) \quad \text{(returns } k-1 \text{ times)}
\end{aligned}
$$

So $N \sim \text{Geom}(1 - f_i)$. Finally, since $f_i < 1$, we have $\mathsf{E}[N] = \frac{1}{1-f_i} < \infty$. $\diamondsuit$

Theorem: i is recurrent iff $\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty$. (So i is transient iff $\sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty$.)

Proof: Define the event

$$A_n \equiv \begin{cases} 1 & \text{if } X_n = i \\ 0 & \text{if } X_n \neq i \end{cases}.$$

Note that $N \equiv \sum_{n=1}^{\infty} A_n$ is the number of returns to i.

Then by the trick that allows us to treat the expected value of an indicator function as a probability, we have. . .

$$
\begin{aligned}
\sum_{n=1}^{\infty} P_{ii}^{(n)} &= \sum_{n=1}^{\infty} \mathsf{Pr}(X_n = i | X_0 = i) \\
&= \sum_{n=1}^{\infty} \mathsf{E}[A_n | X_0 = i] \quad \text{(trick)} \\
&= \mathsf{E}\Big[\sum_{n=1}^{\infty} A_n \Big| X_0 = i \Big] \\
&= \mathsf{E}[N | X_0 = i] \quad (N = \text{number of returns}) \\
&= \infty \\
&\Leftrightarrow i \text{ is recur (by previous theorem).} \quad \diamondsuit
\end{aligned}
$$

Corollary 1: If i is recur and $i \leftrightarrow j$, then j is recur.

Proof: See Ross. $\diamondsuit$

Corollary 2: In a MC with a *finite* number of states, not all of the states can be transient.

Proof: Suppose not. Then the MC will run out of states not to go to an infinite number of times. This is a contradiction. $\diamondsuit$

Corollary 3: If one state in an equiv class is transient, then all states are trans.

Proof: Suppose not, i.e., suppose there's a recur state. Since all states in the equiv class communicate, Corollary 1 implies all states are recur. This is a contradiction. $\diamondsuit$

Corollary 4: All states in a finite irreducible MC are recurrent.

Proof: Suppose not, i.e., suppose there's a trans state. Then Corollary 3 implies *all* states are trans. But this contradicts Corollary 1. $\diamondsuit$

Definition: By Corollary 1, all states in an equiv class are recur if one state in that class is recur. Such a class is a *recurrent equiv class*.

By Corollary 3, all states in an equiv class are trans if one state in that class is trans. Such a class is a *transient equiv class*.

Example: Consider the prob transition matrix

$$\mathbf{P} = \begin{pmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{pmatrix}.$$

Clearly, all states communicate. So this is a finite, irreducible MC. So Corollary 4 implies all states are recurrent. $\diamondsuit$

Example: Consider

$$\mathbf{P} = \begin{pmatrix} 1/4 & 0 & 0 & 3/4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Loop: $0 \to 3 \to 2 \to 1 \to 0$. Thus, all states communicate; so they're all recurrent. $\diamondsuit$

Example: Consider

$$\mathbf{P} = \left(\begin{array}{c|c|c|c|c} 1/4 & 0 & 3/4 & 0 & 0 \\ \hline 0 & 1/2 & 0 & 1/2 & 0 \\ \hline 1/2 & 0 & 1/2 & 0 & 0 \\ \hline 0 & 1/2 & 0 & 1/2 & 0 \\ \hline 1/5 & 1/5 & 0 & 0 & 3/5 \end{array}\right).$$

The equiv classes are $\{0,2\}$ (recur), $\{1,3\}$ (recur), and $\{4\}$ (trans). $\diamondsuit$

Example: Random Walk: A drunk walks on the integers $0, \pm 1, \pm 2, \ldots$ with transition probabilities

$$\begin{aligned} P_{i,i+1} &= p \\ P_{i,i-1} &= q = 1 - p \end{aligned}$$

(i.e., he steps to the right w.p. p and to the left w.p. $1 - p$).

The prob transition matrix is

$$\mathbf{P} = \begin{pmatrix} & & & \vdots & & & \\ & q & 0 & p & 0 & 0 & \\ & 0 & q & 0 & p & 0 & \\ \ldots & 0 & 0 & q & 0 & p & \ldots \\ & 0 & 0 & 0 & q & 0 & \\ & & & \vdots & & & \end{pmatrix}.$$

Are the states recurrent or transient?

Clearly, all states communicate. So Corollary 1 implies that **if** one of the states are recur, then they all are. Otherwise, all states will be transient.

Consider a typical state 0. If 0 is recurrent [transient], then all states will be recurrent [transient]. We'll find out which is the case by calculating $\sum_{n=1}^{\infty} P_{00}^{(n)}$.

Suppose the drunk starts at 0. Since it's impossible for him to return to 0 in an odd number of steps, we see that $P_{00}^{(2n+1)} = 0$ for all $n \geq 0$.

So the only chance he has of returning to 0 is if he's taken an even number of steps, say $2n$. Of these steps, n must be taken to the left, and n to the right. So, thinking binomial, we have

$$P_{00}^{(2n)} = \binom{2n}{n} p^n q^n = \frac{(2n)!}{n!n!} p^n q^n, \quad n \geq 1.$$

Aside: For large n, Stirling's approximation says that

$$n! \approx \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n}.$$

After the smoke clears,

$$P_{00}^{(2n)} \approx \frac{[4p(1-p)]^n}{\sqrt{\pi n}},$$

so that

$$\begin{aligned}\sum_{n=1}^{\infty} P_{00}^{(n)} &= \sum_{n=1}^{\infty} P_{00}^{(2n)} \\ &= \sum_{n=1}^{\infty} \frac{[4p(1-p)]^n}{\sqrt{\pi n}} \begin{cases} = \infty & \text{if } p = 1/2 \\ < \infty & \text{if } p \neq 1/2 \end{cases}.\end{aligned}$$

So the MC is recur if $p = 1/2$ and trans otherwise.

$\diamondsuit$

Definition: If $p = 1/2$, the random walk is *symmetric*.

Remark: A 2-dimensional r.w. with probability 1/4 of going each way yields a recurrent MC.

A 3-dimensional r.w. with probability 1/6 of going each way (N, S, E, W, up, down) yields a transient MC.

4.4 Limiting Probabilities

Example: Note that the following matrices appear to be converging. . . .

$$\mathbf{P} = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}, \quad \mathbf{P}^{(2)} = \begin{pmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{pmatrix},$$

$$\mathbf{P}^{(4)} = \begin{pmatrix} 0.575 & 0.425 \\ 0.567 & 0.433 \end{pmatrix}, \quad \mathbf{P}^{(8)} = \begin{pmatrix} 0.572 & 0.428 \\ 0.570 & 0.430 \end{pmatrix}, \ldots$$

Definition: Suppose that $P_{ii}^{(n)} = 0$ whenever n is not divisible by d, and suppose that d is the largest integer with this property. Then state i has *period* d. Think of d as the greatest common divisor of all n values for which $P_{ii}^{(n)} > 0$.

Example: All states have period 3.

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \quad \diamondsuit$$

Definition: A state with period 1 is *aperiodic*.

Example:

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

Here, states 0 and 1 have period 2, while states 2 and 3 are aperiodic. $\diamondsuit$

Definition: Suppose state i is recurrent and $X_0 = i$. If the expected time until the process returns to i is finite, then i is *positive recurrent*.

Remark: It turns out that...

(1) In a *finite* MC, all recur states are positive recur.

(2) In an ∞-state MC, there may be some recur states that are not positive recur. Such states are *null recur*.

Definition: Pos recur, aperiodic states are *ergodic*.

Theorem: For an irreducible, ergodic MC,

(1) $\pi_j \equiv \lim_{n\to\infty} P_{ij}^{(n)}$ *exists* and is *independent* of i.

(The π_j's are called *limiting probabilities*.)

(2) π_j is the *unique*, nonnegative solution of

$$\begin{cases} \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, & j \geq 0 \\ 1 = \sum_{j=0}^{\infty} \pi_j \end{cases}.$$

In vector notation, this can be written as $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$.

"Heuristic "proof": see Ross. $\diamondsuit$

Remarks: (1) π_j is also the long-run proportion of time that the MC will be in state j. The π_j's are often called *stationary* probs — since **if** $\Pr(X_0 = j) = \pi_j$, then $\Pr(X_n = j) = \pi_j$ for all n.

(2) In the irred, pos recur, *periodic* case, π_j can only be interpreted as the long-run proportion of time in j.

(3) Let $m_{jj} \equiv$ expected number of transitions needed to go from j to j. Since, on average, the MC spends 1 time unit in state j for every m_{jj} time units, we have $m_{jj} = 1/\pi_j$.

Example: Find the limiting probabilities of

$$\mathbf{P} = \begin{pmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{pmatrix}.$$

Solve $\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}$ ($\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$), i.e.,

$$\begin{aligned} \pi_0 &= \pi_0 P_{00} + \pi_1 P_{10} + \pi_2 P_{20} = 0.5\pi_0 + 0.3\pi_1 + 0.2\pi_2, \\ \pi_1 &= \pi_0 P_{01} + \pi_1 P_{11} + \pi_2 P_{21} = 0.4\pi_0 + 0.4\pi_1 + 0.3\pi_2, \\ \pi_2 &= \pi_0 P_{02} + \pi_1 P_{12} + \pi_2 P_{22} = 0.1\pi_0 + 0.3\pi_1 + 0.5\pi_2, \end{aligned}$$

and $\pi_0 + \pi_1 + \pi_2 = 1$. Get $\boldsymbol{\pi} = \left\{\frac{21}{62}, \frac{23}{62}, \frac{18}{62}\right\}$. $\diamondsuit$

Definition: A transition matrix **P** is *doubly stochastic* if each column (and row) sums to 1.

Theorem: If, in addition to the conditions of the previous theorem, **P** is a doubly stochastic $n \times n$ matrix, then $\pi_j = 1/n$ for all j.

Proof: Just plug in $\pi_j = 1/n$ for all j into $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$ to verify that it works. Since this solution must be unique, we're done. $\diamondsuit$

Example: Find the limiting probabilities of

$$\mathbf{P} = \begin{pmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.2 & 0.3 & 0.5 \end{pmatrix}.$$

This is a doubly stochastic matrix, so we immediately have that $\pi_0 = \pi_1 = \pi_2 = 1/3.$ $\diamondsuit$

4.5 Gambler's Ruin Problem

Each time a gambler plays, he wins \$1 w.p. p and loses \$1 w.p. $1-p=q$. Each play is independent. Suppose he starts with \$$i$. Find the probability that his fortune will hit \$$N$ (i.e., he breaks the bank) before it hits \$0 (i.e., he is ruined).

Let X_n denote his fortune at time n. Clearly, $\{X_n\}$ is a MC.

Note $P_{i,i+1} = p$ and $P_{i,i-1} = q$ for $i = 1, 2, \ldots N - 1$.

Further, $P_{00} = 1 = P_{NN}$.

We have 3 equiv classes: $\{0\}$ (recur), $\{1, 2, \ldots, N-1\}$ (trans), and $\{N\}$ (recur).

By a standard one-step conditioning argument,

$$
\begin{aligned}
P_i &\equiv \text{Pr(Eventually hit \$}N|X_0 = i) \\
&= \text{Pr(Event. hit } N|X_1 = i+1 \textbf{ and } X_0 = i) \\
&\qquad \times \text{Pr}(X_1 = i+1|X_0 = i) \\
&\quad + \text{Pr(Event. hit } N|X_1 = i-1 \textbf{ and } X_0 = i) \\
&\qquad \times \text{Pr}(X_1 = i-1|X_0 = i) \\
&= \text{Pr(Event. hit } N|X_1 = i+1)p \\
&\qquad + \text{Pr(Event. hit } N|X_1 = i-1)q \\
&= pP_{i+1} + qP_{i-1}, \quad i = 1, 2, \ldots, N-1.
\end{aligned}
$$

Since $p + q = 1$, we have

$$pP_i + qP_i \;=\; pP_{i+1} + qP_{i-1}$$

iff

$$p(P_{i+1} - P_i) \;=\; q(P_i - P_{i-1})$$

iff

$$P_{i+1} - P_i \;=\; \frac{q}{p}(P_i - P_{i-1}), \qquad i = 1, 2, \ldots, N-1.$$

Since $P_0 = 0$, we have

$$\begin{aligned}
P_2 - P_1 &= \frac{q}{p} P_1 \\
P_3 - P_2 &= \frac{q}{p}(P_2 - P_1) = \left(\frac{q}{p}\right)^2 P_1 \\
&\vdots \\
P_i - P_{i-1} &= \frac{q}{p}(P_{i-1} - P_{i-2}) = \left(\frac{q}{p}\right)^{i-1} P_1.
\end{aligned}$$

Summing up the LHS terms and the RHS terms,

$$\sum_{j=2}^{i} (P_j - P_{j-1}) = P_i - P_1 = \sum_{j=1}^{i-1} \left(\frac{q}{p}\right)^j P_1.$$

This implies that

$$P_i = P_1 \sum_{j=0}^{i-1} \left(\frac{q}{p}\right)^j = \begin{cases} \frac{1-(q/p)^i}{1-(q/p)} P_1 & \text{if } q \neq p \ (p \neq 1/2) \\ \\ iP_1 & \text{if } q = p \ (p = 1/2) \end{cases}.$$

In particular, note that

$$1 = P_N = \begin{cases} \frac{1-(q/p)^N}{1-(q/p)} P_1 & \text{if } p \neq 1/2 \\ \\ NP_1 & \text{if } p = 1/2 \end{cases}.$$

Thus,

$$P_1 = \begin{cases} \frac{1-(q/p)}{1-(q/p)^N} & \text{if } p \neq 1/2 \\ \\ 1/N & \text{if } p = 1/2 \end{cases},$$

so that

$$P_i = \begin{cases} \frac{1-(q/p)^i}{1-(q/p)^N} & \text{if } p \neq 1/2 \\ \\ i/N & \text{if } p = 1/2 \end{cases}. \quad \diamondsuit$$

By the way, as $N \to \infty$,

$$P_i \to \begin{cases} 1-(q/p)^i & \text{if } p > 1/2 \\ \\ 0 & \text{if } p \leq 1/2 \end{cases}. \quad \diamondsuit$$

Example: A guy can somehow win any blackjack hand w.p. 0.6. If he wins, he fortune increases by \$100; a loss costs him \$100. Suppose he starts out with \$500, and that he'll quit playing as soon as his fortune hits \$0 or \$1500. What's the probability that he'll eventually hit \$1500?

$$P_5 = \frac{1-(0.4/0.6)^5}{1-(0.4/0.6)^{15}} = 0.870. \quad \diamondsuit$$

4.6 First Passage Time from State 0 to State N

$$P_{ij}^{(n)} \equiv P(X_n = j | X_0 = i)$$

Definition: The probability that the *first passage time* from i to j is n is

$$f_{ij}^{(n)} \equiv P(X_n = j | X_0 = i, X_k \neq j, k = 0, 1, \ldots, n-1).$$

This is the probability that the MC goes from i to j in exactly n steps (without passing thru j along the way).

Remarks:

(1) By definition, $f_{ij}^{(1)} = P_{ij}^{(1)} = P_{ij}$

(2) $f_{ij}^{(n)} = P_{ij}^{(n)} - \sum_{k=1}^{n-1} f_{ij}^{(k)} P_{jj}^{(n-k)}$

$P_{ij}^{(n)}$ = Prob. of going from i to j in n steps
$f_{ij}^{(k)}$ = Prob. of i to j for first time in k steps
$P_{jj}^{(n-k)}$ = Prob. of j to j in remaining $n-k$ steps

Special Case: Start in state 0 and state N is an absorbing ("trapping") state.

$$
\begin{aligned}
f_{0N}^{(1)} &= P_{0N}^{(1)} = P_{0N} \\
f_{0N}^{(2)} &= P_{0N}^{(2)} - f_{0N}^{(1)} P_{NN}^{(1)} = P_{0N}^{(2)} - P_{0N}^{(1)} \\
f_{0N}^{(3)} &= P_{0N}^{(3)} - f_{0N}^{(1)} - f_{0N}^{(2)} \\
&= P_{0N}^{(3)} - P_{0N}^{(1)} - (P_{0N}^{(2)} - P_{0N}^{(1)}) = P_{0N}^{(3)} - P_{0N}^{(2)} \\
&\vdots \\
f_{0N}^{(n)} &= P_{0N}^{(n)} - P_{0N}^{(n-1)}
\end{aligned}
$$

$f_{0N}^{(n)}$'s can be calculated iteratively starting at $f_{0N}^{(1)}$.

Define $T \equiv$ first passage time from 0 to N

$$\begin{aligned}
\mathrm{E}(T^k) &= \sum_{n=1}^{\infty} n^k \Pr(T = n) = \sum_{n=1}^{\infty} n^k f_{0N}^{(n)} \\
&= \sum_{n=1}^{\infty} n^k (P_{0N}^{(n)} - P_{0N}^{(n-1)})
\end{aligned}$$

Usually use a computer to calculate this.

(WARNING! Don't break this up into 2 separate ∞ summations!) Stop calculating when $f_{0N}^{(n)} \approx 0$.

2nd Special Case: 2 absorbing states N, N'

Same procedure as before but divide each $f_{0N}^{(n)}$, $f_{0N'}^{(n)}$ by the probs. of being trapped. So probs. of first passage times to N, N' in n steps are

$$\frac{f_{0N}^{(n)}}{\sum_{k=1}^{\infty} f_{0N}^{(k)}} \quad \text{and} \quad \frac{f_{0N'}^{(n)}}{\sum_{k=1}^{\infty} f_{0N'}^{(k)}}.$$

4.7 Branching Processes $\leftarrow$ Special class of MC's

Suppose X_0 is the number of individuals in a certain population. Suppose the probability that any individual will have exactly j offspring during its lifetime is P_j, $j \geq 0$. (Assume that the number of offspring from one individual is independent of the number from any other individual.)

$X_0 \equiv$ size of the 0^{th} generation

$X_1 \equiv$ size of the 1^{st} gener'n $=$ # kids produced by individuals from 0^{th} gener'n.

$\vdots$

$X_n \equiv$ size of the n^{th} gener'n $=$ # kids produced by indiv.'s from $(n-1)^{st}$ gener'n.

Then $\{X_n : n \geq 0\}$ is a MC with the non-negative integers as its state space. $P_{ij} \equiv P(X_{n+1} = j | X_n = i)$.

Remarks:

(1) 0 is recurrent since $P_{00} = 1$.

(2) If $P_0 > 0$, then all other states are transient.

(Proof: If $P_0 > 0$, then $P_{i0} = P_0^i > 0$. If i is recurrent, we'd eventually go to state 0. Contradiction.)

These two remarks imply that the population either dies out or its size $\to \infty$.

Denote $\mu \equiv \sum_{j=0}^{\infty} jP_j$, the mean number of offspring of a particular individual.

Denote $\sigma^2 \equiv \sum_{j=0}^{\infty}(j-\mu)^2 P_j$, the variance.

Suppose $X_0 = 1$. In order to calculate E$[X_n]$ and Var(X_n), note that

$$X_n = \sum_{i=1}^{X_{n-1}} Z_i$$

where Z_i is the # of kids from indiv. i of gener'n $(n-1)$.

Since X_{n-1} is indep of the Z_i's,

$$\begin{aligned} \mathsf{E}[X_n] &= \mathsf{E}\left[\sum_{i=1}^{X_{n-1}} Z_i\right] \\ &= \mathsf{E}[X_{n-1}]\mathsf{E}[Z_i] \\ &= \mu \mathsf{E}[X_{n-1}]. \end{aligned}$$

Since $X_0 = 1$,

$$\begin{aligned} \mathsf{E}[X_1] &= \mu \\ \mathsf{E}[X_2] &= \mu \mathsf{E}[X_1] = \mu^2 \\ &\vdots \\ \mathsf{E}[X_n] &= \mu^n. \end{aligned}$$

Similarly,

$$\text{Var}(X_n) = \begin{cases} \sigma^2\mu^{n-1}\left(\frac{\mu^n-1}{\mu-1}\right), & \text{if } \mu \neq 1 \\ n\sigma^2, & \text{if } \mu = 1 \end{cases}$$

Denote $\pi_0 \equiv \lim_{n\to\infty} \Pr(X_n = 0|X_0 = 1)$ = prob that the population will eventually die out (given $X_0 = 1$).

Fact: If $\mu < 1$, then $\pi_0 = 1$.

Proof:

$$\begin{aligned}\Pr(X_n \geq 1) &= \sum_{j=1}^{\infty} \Pr(X_n = j) \\ &\leq \sum_{j=1}^{\infty} j\Pr(X_n = j) \\ &= \mathrm{E}[X_n] = \mu^n \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Fact: If $\mu = 1$, then $\pi_0 = 1$.

What about the case when $\mu > 1$?

Here, it turns out that $\pi_0 < 1$, i.e., the prob. population dies out is < 1.

$$\begin{aligned} \pi_0 &= \text{Pr(pop'n dies out)} \\ &= \sum_{j=0}^{\infty} \underbrace{\text{Pr(pop'n dies out}|X_1 = j)}_{\pi_0^j} \underbrace{\Pr(X_1 = j)}_{P_j}, \end{aligned}$$

where π_0^j implies that the families started by the j members of the first generation all die out (indep'ly).

Summary:

$$\pi_0 = \sum_{j=0}^{\infty} \pi_0^j P_j \quad (*)$$

For $\mu > 1$, π_0 is the smallest positive number satisfying (*).

Example: Suppose $P_0 = \frac{1}{4}, P_1 = \frac{1}{4}, P_2 = \frac{1}{2}$.

$$\mu = \sum_{j=0}^{\infty} jP_j = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{2} = \frac{5}{4} > 1$$

Furthermore, (*) implies

$$\pi_0 = \pi_0^0 \cdot \frac{1}{4} + \pi_0^1 \cdot \frac{1}{4} + \pi_0^2 \cdot \frac{1}{2} = \frac{1}{4} + \frac{1}{4}\pi_0 + \frac{1}{2}\pi_0^2$$

$$\Leftrightarrow 2\pi_0^2 - 3\pi_0 + 1 = 0$$

Smallest positive sol'n is $\pi_0 = \frac{1}{2}$.