Contents

1 Problem Setup

A ball is thrown with an initial velocity and angle, considering the following forces:

- Gravity: Acts downward with constant acceleration $g \approx 9.81\,\mathrm{m/s}^2$.
- Air resistance (drag): Acts opposite to the direction of motion, affecting both the horizontal and vertical components of the ball's velocity.

2 Breaking Down the Initial Conditions

The initial conditions consist of:

2.1 Initial velocity components:

- Horizontal component:
$$v_{0x} = v_0 \cos(\theta)$$

- Vertical component:
$$v_{0y} = v_0 \sin(\theta)$$

2.2 Initial position:

- Horizontal position:
$$x_0$$
- Vertical position:
$$y_0$$

3 Forces Acting on the Ball

The forces acting on the ball include:

3.1 Gravity (F_g):

- Acts downward with a constant acceleration g, contributing to vertical motion only.

3.2 Drag force ($F_{\textbf{drag}}$):

- Acts opposite to the direction of motion and is proportional to the square of the velocity. This force is decomposed into horizontal and vertical components:
$$F_{\text{drag}} = \frac{1}{2} C_d \rho A v^2$$
Where:
  1. C_d: Drag coefficient,
  2. ρ: Air density,
  3. A: Cross-sectional area of the ball ($A = \pi r^2$, with r as the radius of the ball),
  4. v: Total velocity of the ball,
$$v = \sqrt{v_x^2 + v_y^2}$$

The drag force components are:

- Horizontal drag:
$$F_{\text{drag},x} = \frac{1}{2} C_d \rho A v v_x$$

- Vertical drag:

$$F_{\text{drag},y} = \frac{1}{2} C_d \rho A v v_y$$

4 Equations of Motion with Air Resistance

The motion of the ball is governed by the following equations of motion:

4.1 Horizontal direction (x):

$$m \frac{dv_x}{dt} = -\frac{1}{2} C_d \rho A v v_x$$

$$\frac{dv_x}{dt} = -\frac{C_d \rho A}{2m} v v_x$$

4.2 Vertical direction (y):

$$m \frac{dv_y}{dt} = -mg - \frac{1}{2} C_d \rho A v v_y$$

$$\frac{dv_y}{dt} = -g - \frac{C_d \rho A}{2m} v v_y$$

Where:

- m is the mass of the ball,
- g is the acceleration due to gravity,
- v_x and v_y are the horizontal and vertical velocity components at any time t,
- v is the total velocity,

$$v = \sqrt{v_x^2 + v_y^2}$$

5 Numerical Integration (Euler's Method)

The differential equations are solved numerically using **Euler's method** for time-stepping, where the velocities and positions are updated iteratively over small time steps Δt.

5.1 Initial Conditions

At $t = 0$:

- $$v_x(0) = v_0 \cos(\theta)$$
- $$v_y(0) = v_0 \sin(\theta)$$
- $$x(0) = x_0$$
- $$y(0) = y_0$$

5.2 Update Rules for Velocity and Position

1. **Velocity update**: For each time step Δt, update the velocity components v_x and v_y as follows:

$$v_x(t + \Delta t) = v_x(t) + \frac{dv_x}{dt} \cdot \Delta t$$

$$v_y(t + \Delta t) = v_y(t) + \frac{dv_y}{dt} \cdot \Delta t$$

Where the derivatives of the velocities are given by:

$$\frac{dv_x}{dt} = -\frac{C_d \rho A}{2m} v v_x$$

$$\frac{dv_y}{dt} = -g - \frac{C_d \rho A}{2m} v v_y$$

1. **Position update**: After updating the velocities, update the positions x and y based on the new velocities:

$$x(t + \Delta t) = x(t) + v_x(t) \cdot \Delta t$$

$$y(t + \Delta t) = y(t) + v_y(t) \cdot \Delta t$$

5.3 Repeat for each time step

The process is repeated for subsequent time steps until balls collide. At each step, the velocity and position are updated using the above formulas.

6 Complete Algorithm

1. **Initialize parameters**:
   - Set initial values for v_0, θ, m, r, C_d, ρ, A, and g.
   - Set initial positions: x_0, y_0.
   - Compute initial velocity components:

     $$v_{0x} = v_0 \cos(\theta)$$

     ,

     $$v_{0y} = v_0 \sin(\theta)$$

2. **Define time step** Δt (e.g., 0.01 seconds).
3. **Set initial time** $t = 0$.
4. **At each time step**:
   - Compute total velocity $v = \sqrt{v_x^2 + v_y^2}$.
   - Update velocity components using:

     $$v_x(t + \Delta t) = v_x(t) + \frac{dv_x}{dt} \cdot \Delta t$$

     $$v_y(t + \Delta t) = v_y(t) + \frac{dv_y}{dt} \cdot \Delta t$$

   - Update position using:

     $$x(t + \Delta t) = x(t) + v_x(t) \cdot \Delta t$$

     $$y(t + \Delta t) = y(t) + v_y(t) \cdot \Delta t$$

   - Increment time t by Δt.
5. **Continue until** the balls collide.