Differential Equation Solution Methods

1. Find a general solution to the following differential equations.

(a) $4x^3 + 2xy^2 + (2x^2y + 4y^3)\frac{dy}{dx} = 0$

$M \qquad N$

$N_x = 4xy$

$M_y = 4xy$

$\Rightarrow$ Exact

$\phi_y = 2x^2y + 4y^3$

$\phi_x = 4x^3 + 2xy^2$

$\Rightarrow \quad \phi = \boxed{x^2y^2 + x^4 + y^4 = c}$

(b) $y^3\frac{dy}{dx} = (y^4 + 1)\cos x$

$\frac{1}{4}(y^4+1)' = (y^4+1)\cos x$

$\int \frac{d(y^4+1)}{y^4+1} = \int 4\cos x\, dx$

$\boxed{\ln(y^4+1) = 4\sin x + c}$

or

$\int \frac{y^3}{y^4+1}dy = \int \cos x\, dx$

$u = y^4+1$

$\ldots$

(c) $\left(\frac{y}{x} + 6x\right) + \frac{dy}{dx}(\ln x - 2) = 0$

$M \qquad N$

$N_x = M_y \implies$ Exact

$\phi = \boxed{y \ln x + 3x^2 - 2y = c}$

(d) $y' + 5y = e^{-x} \sin x$

$uy' + 5uy = ue^{-x} \sin x$

end goal: $(uy)' = uy' + u'y \to u' = 5u$

$u = e^{5x}$

$(e^{5x} y)' = e^{5x} e^{-x} \sin x$

$\boxed{e^{5x} y = \frac{1}{15} e^{4x} (4 \sin x - \cos x)}$

$y = \frac{1}{15e^x} (4 \sin x - \cos x)$

$\int e^{4x} \sin x \, dx = \frac{1}{4} e^{4x} \sin x - \int \frac{1}{4} e^{4x} \cos x \, dx$

$= \frac{1}{4} e^{4x} \sin x - \frac{1}{16} e^{4x} \cos x + \int \frac{1}{16} e^{4x} \sin x \, dx$

$\int e^{4x} \sin x \, dx = \frac{1}{15} e^{4x} (4 \sin x - \cos x)$

2. Estimate y(3) using Euler's method with a stepsize of $\Delta t = 1$ for the differential equation

$$\frac{dy}{dt} = 3t - y, \quad y(0) = -2.$$

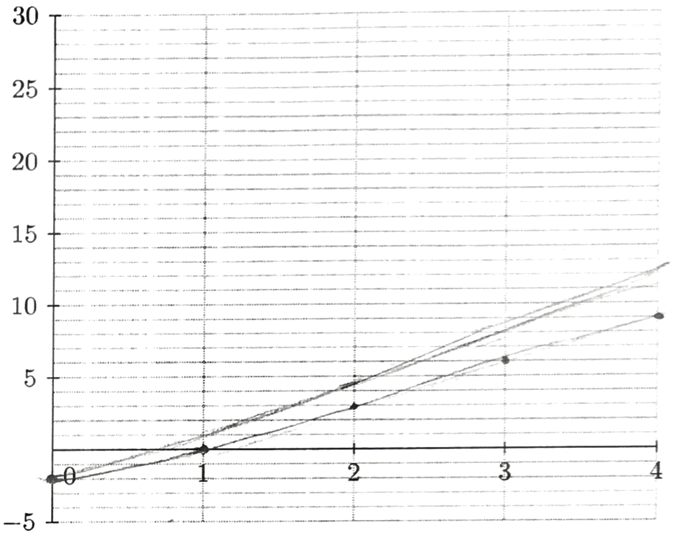
(a) Plot your solution on the graph below.

$$\left.\frac{dy}{dt}\right|_{(0,-2)} = 2$$

$$\left.\frac{dy}{dt}\right|_{(1,0)} = 3$$

$$\left.\frac{dy}{dt}\right|_{(2,3)} = 3$$

$$\left.\frac{dy}{dt}\right|_{(3,6)} = 3$$



(b) Find the solution $y(t)$ to the initial value problem and plot it on the graph as well. Does Euler's method give and overestimate or underestimate?

$$y + y' = 3t$$

$$(e^t y)' = 3te^t$$

$$e^t y = e^t(3t - 3) + c$$

$$e^0(-2) = e^0(0-3) + c$$

$$c = -2 + 3 = 1$$

$$y = 3(t-1) + e^{-t}$$

Underestimate

$$y'' = 3 - \underset{\to 0}{y'} = 3 - (3t - y) = y - 3t + 3$$

3. Find the solutions to the following initial value problems:

(a) $y'' + y' = 1 + 2e^{-x}$, $y(0) = y'(0) = 1$

$y_h'' + y_h' + 0y_h = 0$ $\qquad y_h = c_1 + c_2 e^{-x}$ $\qquad x(1-x)$

$y_p = c_1 x + c_2 x e^{-x}$ $\qquad e^{-x}(x-1-1)$

$y_p'' + y_p' = c_2 e^{-x}(\cancel{x}-1-1) + c_1 + c_2 e^{-x}(1-\cancel{x}) = 1 + 2e^{-2x}$

$c_1 = 1$ $\qquad -c_2 = 2$

$c_2 = -2$

$y = c_1 + c_2 e^{-x} + x - 2xe^{-x}$

$y(0) = c_1 + c_2 = 1$ $\qquad c_2 = -2$

$y'(0) = -c_2 + \cancel{1} - 2 = \cancel{1}$ $\qquad c_1 = 3$

$$\boxed{y = 3 + x - 2e^{-x}(1+x)}$$

(b) $y'' + 2y' + 5y = 0$, $y(0) = 1, y'(0) = 2$

$\lambda^2 + 2\lambda + 5 = 0$

$\lambda = -1 \pm \sqrt{1-5}$

$= -1 \pm 2i$

$y = e^{-x}(c_1 \cos 2x + c_2 \sin 2x)$

$y(0) = c_1 = 1$

$y'(0) = e^{-x}(-c_1 \cos 2x + 2c_2 \cos 2x)\Big|_{x=0}$

$= -c_1 + 2c_2 = 2$

$c_2 = \frac{3}{2}$

$$\boxed{y(x) = e^{-x}\left(\cos 2x + \tfrac{3}{2}\sin 2x\right)}$$

Undetermined Coefficients Review

4. Find the form of the particular solution for the following differential equations.

(a) $y'' + 4y' + 4y = \sin x + x^2 e^{-2x}$

$\lambda^2 + 4\lambda + 4 = 0$ $\quad y_h = c_1 e^{-2x} + c_2 x e^{-2x}$

$\lambda = -2$

$y_p = A\sin x + B\cos x + Cx^2 e^{-2x}$

(b) $y''' + 4y'' + 5y' = x^3 + e^{2x}\sin x$

$\lambda^3 + 4\lambda^2 + 5\lambda = 0$

$\lambda = 0, \frac{-4 \pm \sqrt{16-20}}{2}$

$-2 \pm i$

$y_p = Ax^4 + Bx^3 + Cx^2 + Dx + e^{2x}(E\sin x + F\cos x)$

(c) $y'''' + 2y'' + y = \sin x + x^2 \cos x$

$\lambda^4 + 2\lambda^2 + 1 = 0$

$(\lambda^2 + 1)^2 = 0$

$\lambda = \pm i$

$y_p = (A\sin x + B\cos x)(x^4 + Cx^3 + Dx^2)$

Variation of Parameters

5. Solve the following differential equations.

(a) $y'' - 4y = \sinh(2x)$

$\lambda^2 - 4 = 0$

$y_h = c_1 \sinh(2x) + c_2 \cosh(2x)$

$\lambda = \pm 2$

$y_p = \cancel{k_1(x)\sinh(2x) + k_2(x)\cosh(2x)} = k_1(x) e^{2x} + k_2(x) e^{-2x}$

$P(k_1'' y_1 + 2k_1' y_1' + k_1 y_1'') + Q(k_1' y_1 + k_1 y_1') + R(k_1 y_1) = g(x)$

$+ P(\quad) + Q(\quad) + R(\quad)$

$P(k_1' y_1 + k_2' y_2)' + k_1'(P y_1' + Q y_1) + k_2'(P y_2' + Q y_2) = g(x)$

$k_1' y_1 + k_2' y_2 = 0$

$k_1' y_1' + k_2' y_2' = \frac{g}{P}$

$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} k_1' \\ k_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g/P \end{bmatrix}$

$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} \vec{k} \end{bmatrix}' = \begin{bmatrix} 0 \\ \sinh 2x \end{bmatrix}$

$k_1' = + \frac{y_2 \sinh(2x)}{W(y_1, y_2)} = +2$

$k_1 = \frac{1}{8}\cosh^2 2x$

$k_2' = \frac{y_1 \sinh(2x)}{-2} = -\sinh^2 2x = -\frac{\cosh 4x - 1}{2}$

$k_2 = -\frac{1}{8}\sinh 4x - 1$

$= -\frac{1}{8}\sinh 2x \cosh 2x + \frac{x}{4}$

$\boxed{y_p = \frac{x}{4}\cosh(2x)}$

(b) $y'' + 4y = \sin^2 x = \frac{1 - \cos 2x}{2}$

$\lambda^2 + 4 = 0$

$\lambda = \pm 2i$

$y_h = c_1 \sin 2x + c_2 \cos 2x$

$k_1' = \frac{\sin^2 x \cos 2x}{2} \Rightarrow k_1 = -\frac{1}{32} - \frac{1}{8}x + \frac{1}{8}\sin 2x$

$k_2' = \frac{\sin^2 x \cdot \sin 2x}{-2} \Rightarrow k_2 = -\frac{1}{4}\sin^4 x$

$y_p = \sin 2x \left(-\frac{1}{32} - \frac{1}{8}x + \frac{1}{8}\sin 2x\right)$

$+ \cos 2x \left(-\frac{1}{4}\sin^4 x\right)$

$\hookrightarrow$ simplify?