Laplace Transforms

1. To solve the problem $y'' + 3y' + 2y = e^t, y(0) = 1, y'(0) = -1$.

   (a) First find the Laplace transform of the entire equation.

   $$\mathcal{L}\{y'' + 3y' + 2y\} = \mathcal{L}\{e^t\}$$

   $$\boxed{\begin{aligned} &s^2 Y - sy(0) - y'(0) \\ &+ 3sY - 3y(0) \quad = \frac{1}{s-1} \\ &+ 2Y \end{aligned}}$$

   (b) Then use the initial conditions and solve for $\mathcal{L}(y)$.

   $$(s^2 + 3s + 2)\,Y = \frac{1}{s-1} + (+s + 2)$$

   $$\boxed{Y = \frac{1}{(s-1)(s+2)(s+1)} + \frac{1}{s+1}}$$

   (c) Then find the partial fractions of the right side of your equation and use the Laplace table to find y.

   $$Y = \frac{1}{(s+1)(s+2)(s-1)} + \frac{1}{s+1}$$

   $$= \frac{1/3}{s+2} + \frac{-1/2}{(s+1)} + \frac{1/6}{s-1} + \frac{1}{s+1}$$

   $\lambda^2 + 3\lambda + 2 = 0$

   $(\lambda + 2)(\lambda + 1) = 0$

   $\lambda = -1, -2$

   $\Downarrow$

   makes sense

   $$\boxed{y(t) = \frac{1}{3}e^{-2t} + \frac{1}{6}e^{t} + \frac{1}{2}e^{-t}}$$

2. Solve the following differential equations using Laplace transforms.

(a) $y'' + 3y' + 2y = 3\sin 2t, y(0) = 1, y'(0) = 0.$

$$s^2Y - s + 3(sY - 1) + 2Y = \frac{6}{s^2+4}$$

$$(s^2+3s+2)Y = \frac{6}{s^2+4} + s+3$$

$$Y = \frac{6}{(s^2+4)(s+1)(s+2)} + \frac{s+3}{(s+2)(s+1)}$$

$$= \frac{6/5}{s+1} + \frac{-3/4}{s+2} + \frac{-9s-6}{20(s^2+4)} + \frac{2}{s+1} + \frac{-1}{s+2}$$

$$\boxed{y = \frac{16}{5}e^{-t} - \frac{7}{4}e^{-2t} - \frac{9}{20}\cos 2t - \frac{3}{20}\sin 2t}$$

(b) $y'' + 3y' + 2y = 3u(t-2), y(0) = 0, y'(0) = 2.$

$$s^2Y - 2 + 3sY + 2Y = \frac{3e^{-2s}}{s}$$

$$(s^2+3s+2)Y = \frac{3e^{-2s}}{s} + 2$$

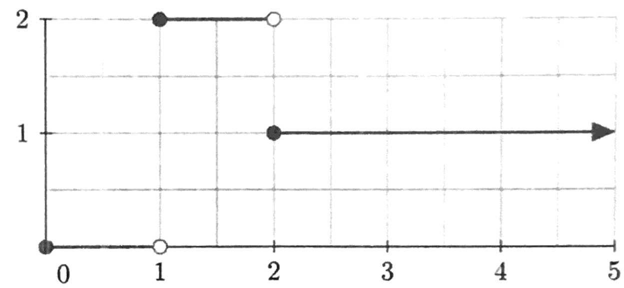
$$Y = e^{-2s}\frac{3}{s(s+1)(s+2)} + \frac{2}{(s+1)(s+2)}$$

$$= e^{-2s}\left(\frac{3/2}{s} + \frac{-3}{s+1} + \frac{3/2}{s+2}\right) + \frac{1}{s+1} + \frac{-1}{s+2}$$

$$= \boxed{u(t-2)\left[\frac{3}{2} - 3e^{-t} + \frac{3}{2}e^{-2t}\right] + e^{-t} - e^{-2t}}$$

3. Solve the following initial value problem

$$y'' + 2y' + 5y = \begin{cases} 0 & 0 \le t < 1 \\ 2 & 1 \le t < 2 \\ 1 & 2 \le t \end{cases}, \quad y(0) = 1, y'(0) = -1.$$



(a) First show that the right hand side, which is shown on the graph above, can be written as $2u(t-1) - u(t-2)$.

Inspection

(b) Then use Laplace transforms to solve the initial value problem.

$$\begin{aligned} &s^2 Y - s + 1 \\ &+ 2(sY - 1) \quad = \frac{2e^{-t}}{s} - \frac{e^{-2t}}{s} \\ &+ 5Y \end{aligned}$$

$$(s^2+2s+5)Y = \frac{2e^{-t}}{s} - \frac{e^{-2t}}{s} + s + 1$$

$$Y = e^{-t}\frac{2}{s(s^2+2s+5)} + e^{-2t}\frac{1}{s(s^2+2s+5)} + \frac{s+1}{s^2+2s+5}$$

$$= \left(\frac{1/5}{s} - \frac{1}{10}\frac{2s+4}{\underbrace{s^2+2s+5}_{(s+1)^2+4}}\right)\left(2e^{-t} + e^{-2t}\right) + \frac{s+1}{s^2+2s+5}$$

$$y = \left[\frac{1}{5} - \frac{1}{10}\left(2e^{-t}\cos 2t + 1e^{-t}\sin 2t\right)\right]\left(2u(t-1) + u(t-2)\right) + e^{-t}\cos 2t$$