Grothendieck Duality on Diagrams of Schemes

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Abstract

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The Du Bois complex and Du Bois singularities, which extend results of Hodge theory to singular complex varieties, can be defined in terms of a cubical hyperresolution. In this dissertation I further develop the language of diagrams of schemes and then prove analogues of Grothendieck duality and other cohomological theorems for cubical diagrams. I then demonstrate the use of these by revisiting some known results about the Du Bois complex from a different perspective, and proving new results concerning the cohomology of a contraction.

TABLE OF CONTENTS

Page

Chapter 1

INTRODUCTION

1.1 Motivation

The de Rham complex $\Omega^\bullet_{\tilde{X}}$ of a smooth complex manifold consists of the vector spaces of degree p differential forms for each p, and exterior differentiation. Results from Hodge theory give that the hypercohomology of this complex is isomorphic to the singular cohomology of the manifold, and furthermore the spectral sequence computing the hypercohomology is degenerate, so that in fact the singular cohomology is given by a direct sum of $H^q(\tilde{X}, \Omega^p_{\tilde{X}})$ terms. The Du Bois complex, introduced in [Du 81], generalizes the de Rham complex and Hodge theory to singular complex varieties: the idea is to resolve the singularities of the variety and then push down the de Rham complex of the smooth variety through the resolution map, however a traditional resolution of singularities is inadequate for this process as topological information is lost. Instead, hyperresolutions are a refinement which resolve the singularities of a variety while preserving the topology.

As an example of this, consider the resolution of the cone over $\mathbb{C}$, $f : \tilde{X} \to X$ obtained by blowing up the cone point, as seen in the figure on the right. These topologically differ, as the cylinder has a non-trivial $H^1(\tilde{X}, \mathbb{C})$, while $H^1(X, \mathbb{C}) = 0$, pushing the de Rham complex to X naively would lose the property that $H^i(\tilde{X}, \mathbb{C}) \cong \mathbb{H}^i(\tilde{X}, \Omega^\bullet_{\tilde{X}})$.

Instead, for a hyperresolution the goal is to "remember" the fact that there is a $\mathbb{P}^1$ in the cylinder which collapses to a point. This is achieved by replacing the single scheme $\tilde{X}$ with a collection of schemes and maps between them which collectively make up the resolution.
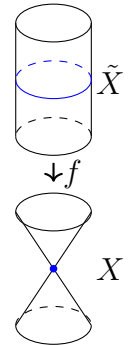
Since the original definition, several implementations of the hyperresolution have been

developed, each resulting in the same Du Bois complex. The most tractable definition is the cubical hyperresolution which uses diagrams of schemes, and such diagrams are the main topic of this dissertation.

1.2 Outline

Chapter 2 serves as a mostly self-contained introduction to diagrams and develops the tools we will use to study them. The main reference for diagrams of schemes is Guillién, et. al [GNAPGP88], and a restricted category of diagrams–only shape-preserving maps (defined in 2.1.2) are allowed–is studied in Lipman and Hashimoto [LH09]; this chapter builds on the framework of these, compiling results and then adding what is necessary for the following chapters.

The main results come in chapter 3, and consist of generalizations of important cohomological results from schemes into the diagram setting. This includes the flat base change (theorem 3.2.2), the projection formula (theorem 3.2.3), and Grothendieck duality (corollary 3.2.4). In particular, the following theorem holds:

Theorem 1.2.1 (Theorem 3.2.3 and Corollary 3.2.4)**.** *Let* $f_\bullet : X_\bullet \to Y_\bullet$ *be a cubical, proper morphism of noetherian diagrams of schemes. Then for each* $F_\bullet \in D_{lc}^-(X_\bullet)$ *and* $G_\bullet \in D_{lqc}^+(X_\bullet)$*, there are isomorphisms*

$$G_\bullet \otimes^{\mathbf{L}} \boldsymbol{R} f_{\bullet *} F_\bullet \xrightarrow{\sim} \boldsymbol{R} f_{\bullet *}(\boldsymbol{L} f_\bullet^* G_\bullet \otimes^{\mathbf{L}} F_\bullet)$$

$$\boldsymbol{R} f_{\bullet *} \boldsymbol{R} \mathcal{H} om(F_\bullet, f_\bullet^! G_\bullet) \xrightarrow{\sim} \boldsymbol{R} \mathcal{H} om(\boldsymbol{R} f_{\bullet *} F_\bullet, G_\bullet).$$

Chapter 4 then introduces the Du Bois complex and singularities, lists some known results, and then applies Grothendieck duality to studying the complex. In particular, there is an analogue of a dualizing complex on diagrams (herein called the canonical complex since it is not dualizing in the traditional sense), which when pushed to the resolved variety becomes $\underline{\omega}_X^\bullet$ the Grothendieck dual of the complex $\underline{\Omega}_X^0$ which was studied somewhat in [KS11].

Finally, in chapter 5 there is some demonstration of these results outside the realm of diagrams of schemes, including the following theorem:

Theorem 1.2.2 (Corollary 5.1.2). *Let X be a proper, reduced, Cohen-Macaulay $\mathbb{C}$-scheme of dimension n with Du Bois singularities, Y smooth, with $f : Y \to X$ a proper morphism which is as isomorphism outside of a closed $p \in X$. Then, for $0 < i < n - 1$, the exceptional locus Y_p satisfies*

$$H^i(Y_p, \underline{\Omega}^0_{Y_p}) = \begin{cases} \mathbb{C} & \text{if } i = 0 \\ 0 & \text{if } 0 < i < n - 1. \end{cases}$$

In particular, this implies that Y_p is connected (a fact already known by Zariski's Main Theorem) and if Y_p has Du Bois singularities $H^i(Y_p, \mathcal{O}_{Y_p}) = 0$ for $0 < i < n - 1$.

This vanishing on Y_p appears to fit in well with a well-known result that a similar vanishing holds for Y.

Chapter 2

DIAGRAMS OF SCHEMES AND THEIR SHEAVES

2.1 Diagram Categories and Diagrams of Schemes

In general, a diagram in a given category A will be a collection of objects in morphisms in that category; the idea is that we want to deal with that whole collection simultaneously. For example, in a cubical hyperresolution the diagram will contain both the resolution of singularities *and* information about how the resolution map transforms the space.

These collections will be defined by a functor from another category I to A, and in general any small I and functor can work, but for our purposes we can restrict I to a much simpler case:

Definition 2.1.1. A *diagram shape category* is a finite, thin (any two objects have at most one morphism between them) category without loops.

It is clear why requiring finiteness makes things easier, the primary reason for requiring thinness is so that we can think of slice categories and over categories as fully faithful subcategories, which will make some constructions simpler. Finally, loops in a thin category are somewhat silly (each map in the loop must be an isomorphism), so categories with loops contain no further information than the category with the loops collapsed, and doing so will be useful for lemma 2.6.1. These conditions are particularly reasonable given the only categories we will be studying in much detail are given in definition 2.1.3.

Definition 2.1.2. Let I be a diagram shape category and A be any category. A *diagram* in A of shape I (or an I-diagram) is a functor

$$X_\bullet : I^{op} \to A.$$

For each α and φ an object and morphism from I respectively, we'll write X_α and X_φ.

A *morphism* between an I-diagram $X_\bullet$ and a J-diagram $Y_\bullet$ is given by a *shape functor* $\Gamma : I^{op} \to J^{op}$ and a natural transformation $\eta : X_\bullet \to Y_\bullet \circ \Gamma$.

In simpler terms, this is a collection of morphisms between the components of $X_\bullet$ and $Y_\bullet$ such that the total diagram with all the arrows from the morphism and both diagrams drawn is commutative. Usually we'll denote a morphism as $f_\bullet : X_\bullet \to Y_\bullet$ with Γ implicit and for each $\alpha \in I$, $f_\alpha : X_\alpha \to Y_{\Gamma(\alpha)}$.

Call a morphism *shape-preserving* if Γ is the identity, and *shape-embedding* if it is fully faithful.

The category of diagrams in A will be denoted $\underline{\mathrm{Diag}}\, A$, and the category of I-diagrams with shape-preserving morphisms in A will be denoted $\underline{\mathrm{Diag}}_I\, A$.

Finally, there is a diagonal functor for each I, $- \times I : A \to \underline{\mathrm{Diag}}_I\, A$ sending each X to the diagram $X \times I$ with shape I, all components X, and all morphisms the identity.

The following definition introduces all the shapes we'll want to consider in detail.

Definition 2.1.3. Let $\underline{1}$ be the category $\{0\}$, and $\underline{2}$ be the category $\{0 \to 1\}$. For $n \geq -1$, let $\square_n^+$ be the product of $n+1$ copies of $\underline{2}$. The elements here are tuples $\alpha = (\alpha_0, \ldots, \alpha_n)$, with $\alpha_i \in \{0, 1\}$, and it is useful to define $|\alpha| = \sum \alpha_i$. Let $\square_n$ be the full subcategory without the initial object $(0, \ldots, 0)$. See figure 2.1.

A morphism of diagrams whose shape functor is the projection $\square_n^{op} \times I^{op} \to I^{op}$ will be called a cubical morphism.

With these generalities taken care of, we specialize to diagrams of schemes.

Definition 2.1.4. A diagram of schemes is an object in the category $\underline{\mathrm{Diag}}\, \mathbf{Sch}$. We'll say that a diagram of schemes $X_\bullet$ over I (respectively, a morphism $f_\bullet : X_\bullet \to Y_\bullet$) has a given property $\mathcal{P}$ (e.g., smooth, noetherian) if for all α, X_α (resp., f_α) has $\mathcal{P}$.

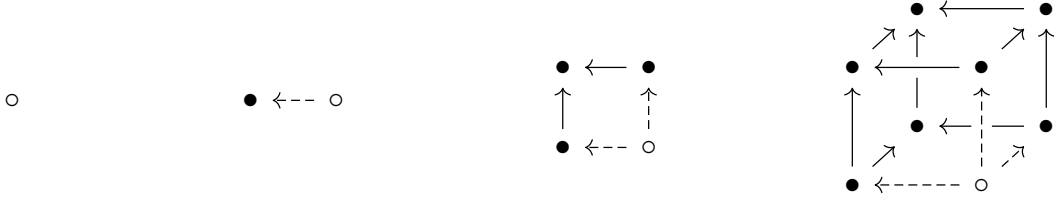The following constructions will be needed later:

Figure 2.1: For $n \in \{-1, 0, 1, 2\}$, $\square_n^+$ is the diagram including the dashed arrows and an initial object, $\square_n$ is the diagram without

Proposition 2.1.5. *Let $f_\bullet : X_\bullet \to Y_\bullet$ and $g_\bullet : Y'_\bullet \to Y_\bullet$ be morphisms of diagrams of schemes. Then there is an inverse limit in the category of diagrams of schemes, the fibered product denoted $X_\bullet \times_{Y_\bullet} Y'_\bullet$ and fitting into the diagram*

$$
\begin{array}{ccc}
X_\bullet \times_{Y_\bullet} Y'_\bullet & \xrightarrow{\;f'_\bullet\;} & Y'_\bullet \\
\downarrow{\scriptstyle g'_\bullet} & & \downarrow{\scriptstyle g_\bullet} \\
X_\bullet & \xrightarrow{\;f_\bullet\;} & Y_\bullet.
\end{array}
$$

If each of the scheme maps in $f_\bullet$ has some property preserved under base change of schemes, then the scheme maps in $f'_\bullet$ have the same property.

Proof. First, we fix some notation: let I be the shape of $X_\bullet$, J the shape of $Y'_\bullet$, and K the shape of $Y_\bullet$. Let $\Gamma : I \to K$ be associated to $f_\bullet$ and $\Lambda : J \to K$ be associated to $g_\bullet$. Then there is a fibered product $I \times_K J$ in the category of diagram shape categories: the full subcategory with objects $\{(i, j) \in I \times J : \Gamma(i) = \Lambda(j)\}$ in $I \times J$.

Define an $I \times_K J$-scheme $X'_\bullet$ as follows: for $(i, j) \in I \times_K J$, let $X'_{(i,j)} = X_i \times_{Y_{\Gamma(i)}} Y_j$ and let the morphisms be the product of those in $X_\bullet$ and $Y_\bullet$. Then the collection of projections to each coordinate give the maps $f'_\bullet$ and $g'_\bullet$ in the diagram, and it is clear now $f'_\bullet$ preserves any property which is preserved under base change.

Finally, we show this satisfies the universal property of the fibered product. If $Z_\bullet$ were an L-scheme with compatible morphisms to $X_\bullet$ and $Y_\bullet$, by definition it would have $\Theta_1 : L \to I$

and $\Theta_2 : L \to J$. For each $l \in L$, define a map $Z_l \to X'_{(\Theta_1(l),\Theta_2(l))}$ by the product of the morphisms to $X_{\Theta_1(l)}$ and $Y_{\Theta_2(l)}$. $\qquad\square$

Definition 2.1.6. Let $X_\bullet$ be an I-scheme. Suppose $\alpha \in I$ and $U \subseteq X_\alpha$. Define the *over-scheme of* U, $X_\bullet/U$ as a diagram of schemes over α/I, the under category consisting of (β, ϕ) with $\beta \in I$ and $\phi : \alpha \to \beta$. The components are

$$(X_\bullet/U)_{(\beta,\phi)} = X_\phi^{-1}(U) \subseteq X_\beta,$$

and the maps are the restrictions of the maps of $X_\bullet$. There is a shape-embedding (by thinness) open immersion $i_U : X_\bullet/U \to X_\bullet$ mapping $X_\phi^{-1}(U) \hookrightarrow X_\beta$ for all (β, ϕ).

Definition 2.1.7. Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism of diagrams of schemes, with $U \subseteq Y_\alpha$. Define $f_\bullet^{-1}(U) = X_\bullet \times_{Y_\bullet} Y_\bullet/U$, and let $i_{f_\bullet^{-1}(U)}$ be the projection onto $X_\bullet$, which is a shape-embedding open immersion by thinness and proposition 2.1.5.

2.2   Sheaves on Diagrams

With a definition of diagrams of schemes, we can now consider sheaves over diagrams, which can be thought of as compatible collections of sheaves and morphisms over the schemes in the diagram.

Definition 2.2.1 ([Kol13, 10.84]). Let $X_\bullet$ be an I-scheme. A sheaf $\mathscr{F}_\bullet$ on $X_\bullet$ is a collection of sheaves $\mathscr{F}_\alpha$ on X_α with, for each $\phi_{\alpha\beta} : X_\alpha \to X_\beta$ in the diagram, morphisms $\mathscr{F}_\beta \to f_* \mathscr{F}_\alpha$, or equivalently $f^* \mathscr{F}_\beta \to \mathscr{F}_\alpha$. This is an abelian category, so we can also take the derived category of such sheaves, $D(X_\bullet)$.

Example 2.2.2. For each morphism $\phi_{\alpha\beta} : X_\alpha \to X_\beta$, there is an induced map $\mathcal{O}_{X_\beta} \to \phi_{\alpha\beta*}\mathcal{O}_{X_\alpha}$, thus there is a sheaf $\mathcal{O}_{X_\bullet}$ on $X_\bullet$ given by $(\mathcal{O}_{X_\bullet})_\alpha = \mathcal{O}_{X_\alpha}$.

Given a map of diagrams, there are several associated functors of sheaves, analogous to those on literal schemes. The easiest to construct is the pullback

Definition 2.2.3. Let $X_\bullet$ be an I-scheme and $Y_\bullet$ be a J-scheme, $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism of diagrams of schemes over the functor $\Phi : I^{op} \to J^{op}$. Then if $\mathscr{G}_\bullet$ is a sheaf over $Y_\bullet$, define the pullback $f_\bullet^* \mathscr{G}_\bullet$ a sheaf on $X_\bullet$ as

$$(f_\bullet^* \mathscr{G}_\bullet)_\alpha = f_\alpha^* \mathscr{G}_{\Phi(\alpha)}.$$

This functor is right-exact and has a left-derived functor on the derived categories given on each component by

$$(\mathbf{L}f_\bullet^* G_\bullet)_\alpha = \mathbf{L}f_\alpha^* G_{\Phi(\alpha)}.$$

On the other hand, the pushforward is more mysterious, the problem being for each scheme in the target diagram, there are possibly many schemes mapping to it. However, if this is the case then they form a diagram of sheaves over that target scheme, and we can define the pushforward as the limit of this diagram.

Definition 2.2.4. Let $X_\bullet$ be an I-scheme and $Y_\bullet$ a J-scheme, $f_\bullet : X_\bullet \to Y_\bullet$ a morphism of diagrams of schemes over the functor $\Phi : I^{op} \to J^{op}$. Then if $\mathscr{F}_\bullet$ is a sheaf over $X_\bullet$, define the pushforward $f_{\bullet*} \mathscr{F}_\bullet$ a sheaf on $Y_\bullet$ as

$$(f_{\bullet*} \mathscr{F}_\bullet)_\alpha = \varprojlim_{(\beta,\phi)} Y_{\phi*} f_{\beta*} \mathscr{F}_\beta,$$

where (β, ϕ) ranges over $\beta \in I^{op}$ and $\phi : \Phi(\beta) \to \alpha$ a morphism in J^{op}. This functor is left-exact and has a right-derived functor on the derived categories given on each component by

$$(\mathbf{R}f_{\bullet*} F_\bullet)_\alpha = \mathbf{R} \varprojlim_{(\beta,\phi)} \mathbf{R}Y_{\phi*} \mathbf{R}f_{\beta*} F_\beta.$$

Proposition 2.2.5 ([GNAPGP88, I 5.5]). *With these definitions and using this notation, $f_{\bullet*}$ is the right adjoint of $f_\bullet^*$ and $\mathbf{R}f_{\bullet*}$ is the right adjoint of $\mathbf{L}f_\bullet^*$.*

The main case of interest is a hyperresolution of a literal scheme, so it is worth considering in the case where $J = \underline{1}$, and even more particular where $I = \square_n$. After this first restriction, the functor $\mathbf{R}f_{\bullet*}$ can be written as

$$\mathbf{R}f_{\bullet*} F_\bullet = \mathbf{R} \varprojlim_\beta \mathbf{R}f_{\beta*} F_\beta$$

Restricting to $I = \square_n$ is very powerful, as we will see in section 2.3.

The following gives a notion of global sections for sheaves on diagrams:

Definition 2.2.6. If $X_\bullet$ is a diagram of schemes, $\mathscr{F}_\bullet$ a sheaf on $X_\bullet$, then define the functor

$$\Gamma(X_\bullet, \mathscr{F}_\bullet) = \varprojlim \Gamma(X_\alpha, \mathscr{F}_\alpha).$$

Similarly, if $F_\bullet \in D(X_\bullet)$, define

$$\mathbf{R}\Gamma(F_\bullet) = \mathbf{R}\varprojlim \mathbf{R}\Gamma(F_\alpha).$$

Note that if $f_\bullet : X_\bullet \to \operatorname{Spec} k$ is a morphism to a field, $\Gamma = f_{\bullet *}$.

Given $\mathscr{F}_\bullet$ and $\mathscr{G}_\bullet$, sheaves on $X_\bullet$, we can define $\operatorname{Hom}_{X_\bullet}(\mathscr{F}_\bullet, \mathscr{G}_\bullet)$ as the morphisms in the category of sheaves, and this has many of the properties as before, however defining a sheaf $\mathcal{H}om_{X_\bullet}(\mathscr{F}_\bullet, \mathscr{G}_\bullet)$ is more subtle as the most obvious choice does not have the natural maps required.

Definition 2.2.7. Let $X_\bullet$ be a diagram of schemes, $\mathscr{F}_\bullet$ and $\mathscr{G}_\bullet$ two sheaves over $X_\bullet$. Define $\mathcal{H}om_{X_\bullet}(\mathscr{F}_\bullet, \mathscr{G}_\bullet)$ as follows: for each $\alpha \in I$, and $U \subseteq X_\alpha$ open, let

$$(\mathcal{H}om_{X_\bullet}(\mathscr{F}_\bullet, \mathscr{G}_\bullet))_\alpha(U) = \operatorname{Hom}_{X_\bullet/U}(i_U^* \mathscr{F}_\bullet, i_U^* \mathscr{G}_\bullet).$$

Note that $X_\bullet/U$ and i_U are defined in 2.1.6. This gives a sheaf on each X_α. For $X_\phi : X_\alpha \to X_\beta$, $U \subseteq X_\beta$, there is a map $X_\bullet/\phi^{-1}(U) \to X_\bullet/U$ such that the i maps commute, and putting this together gives maps $(\mathcal{H}om_{X_\bullet}(\mathscr{F}_\bullet, \mathscr{G}_\bullet))_\beta(U) \to (\mathcal{H}om_{X_\bullet}(\mathscr{F}_\bullet, \mathscr{G}_\bullet))_\alpha(X_\phi^{-1}(U))$ as needed.

Proposition 2.2.8. *Let $X_\bullet$ be a diagram of schemes, $\mathscr{F}_\bullet$ and $\mathscr{G}_\bullet$ two sheaves over $X_\bullet$. Then*

$$\Gamma(X_\bullet, \mathcal{H}om_{X_\bullet}(\mathscr{F}_\bullet, \mathscr{G}_\bullet)) = \operatorname{Hom}_{X_\bullet}(\mathscr{F}_\bullet, \mathscr{G}_\bullet),$$

and similarly if $F_\bullet, G_\bullet \in D^+(X_\bullet)$

$$\mathbf{R}\Gamma(\mathbf{R}\mathcal{H}om_{X_\bullet}(F_\bullet, G_\bullet)) = \mathbf{R}\operatorname{Hom}_{X_\bullet}(F_\bullet, G_\bullet).$$

Proof. Note that $\mathrm{Hom}_{X_\bullet/X_\alpha}(i^*_{X_\alpha}\mathscr{F}_\bullet, i^*_{X_\alpha}\mathscr{G}_\bullet)$ consists of compatible morphisms of sheaves $\mathscr{F}_\beta \to \mathscr{G}_\beta$ for each β under α, but then it is clear the inverse limit of these groups will be the collection of compatible morphisms $\mathscr{F}_\beta \to \mathscr{G}_\beta$ for all β, which is precisely $\mathrm{Hom}_{X_\bullet}(\mathscr{F}_\bullet, \mathscr{G}_\bullet)$.

The derived category version is simply by the composition of derived functors. $\qquad\square$

The following proposition can be thought of as a generalization of the above, where Γ is thought of as a special case of the pushforward functor. The natural map obtained by this characterization is used in the sheaf version of duality.

Proposition 2.2.9. *Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism of diagrams of schemes, and $\mathscr{F}_\bullet, \mathscr{G}_\bullet$ two $X_\bullet$ sheaves. Then there is a natural isomorphism, for each $U \subseteq Y_\alpha$*

$$f_{\bullet*}\mathcal{H}om_{X_\bullet}(\mathscr{F}_\bullet, \mathscr{G}_\bullet)(U) \xrightarrow{\sim} \mathrm{Hom}_{f_\bullet^{-1}(U)}(i^*_{f_\bullet^{-1}(U)}\mathscr{F}_\bullet, i^*_{f_\bullet^{-1}(U)}\mathscr{G}_\bullet).$$

Furthermore, there is a natural map

$$f_{\bullet*}\mathcal{H}om_{X_\bullet}(\mathscr{F}_\bullet, \mathscr{G}_\bullet) \to \mathcal{H}om_{Y_\bullet}(f_{\bullet*}\mathscr{F}_\bullet, f_{\bullet*}\mathscr{G}_\bullet),$$

and its derived version

$$\boldsymbol{R}f_{\bullet*}\boldsymbol{R}\mathcal{H}om_{X_\bullet}(F_\bullet, G_\bullet) \to \boldsymbol{R}\mathcal{H}om_{Y_\bullet}(\boldsymbol{R}f_{\bullet*}F_\bullet, \boldsymbol{R}f_{\bullet*}G_\bullet).$$

Proof. Working through the definitions,

$$f_{\bullet*}\mathcal{H}om_{X_\bullet}(\mathscr{F}_\bullet, \mathscr{G}_\bullet)(U) = \varprojlim_{(\beta,\phi)} \mathrm{Hom}_{X_\bullet/f_\beta^{-1}(Y_\phi^{-1}(U))}(i^*_{f_\beta^{-1}(Y_\phi^{-1}(U))}\mathscr{F}_\bullet, i^*_{f_\beta^{-1}(Y_\phi^{-1}(U))}\mathscr{G}_\bullet).$$

The elements on the left are tuples of compatible morphisms of sheaves $(\psi_{(\beta,\phi)})$, where (β,ϕ) ranges over $I \times_J \alpha/J$. $f_\bullet^{-1}(U)$ is a $I \times_J \alpha/J$ scheme, and working through the definitions, $(p^*_{f_\bullet^{-1}(U)}\mathscr{F}_\bullet)_{(\beta,\phi)} = \mathscr{F}_\beta|_{f_\beta^{-1}(Y_\phi^{-1}(U))}$, then $\psi_{(\beta,\phi)}$ gives a morphism from this to $(p^*_{f_\bullet^{-1}(U)}\mathscr{G}_\bullet)_{(\beta,\phi)}$ for each (β,ϕ), and these maps are compatible.

The inverse is the restriction of the morphism $i^*_{f_\bullet^{-1}(U)}\mathscr{F}_\bullet \to i^*_{f_\bullet^{-1}(U)}\mathscr{G}_\bullet$ to each of the components of the limit.

To define the natural map, we can define compatible maps of the U sections, and by the first part and the definition, these are maps

$$\mathrm{Hom}_{f_\bullet^{-1}(U)}(i^*_{f_\bullet^{-1}(U)}\mathscr{F}_\bullet, i^*_{f_\bullet^{-1}(U)}\mathscr{G}_\bullet) \to \mathrm{Hom}_{Y_\bullet/U}(i^*_U f_{\bullet *}\mathscr{F}_\bullet, i^*_U f_{\bullet *}\mathscr{G}_\bullet).$$

On each of the components, i^*_U and $i^*_{f_\bullet^{-1}(U)}$ are just the restrictions, which $f_{\bullet *}$ is compatible with, so we can rewrite the right side as

$$Hom_{Y_\bullet/U}(f_{\bullet *} i^*_{f_\bullet^{-1}(U)}\mathscr{F}_\bullet, f_{\bullet *} i^*_{f_\bullet^{-1}(U)}\mathscr{G}_\bullet).$$

Then it is clear the natural map is just the application of the functor $f_{\bullet *}$.

The derived version is then tediously obtained by the sheaf version. $\qquad\square$

Proposition 2.2.10. *Let $X_\bullet$ be a diagram of schemes, $i_\bullet : U_\bullet \hookrightarrow X_\bullet$ a flat shape-embedding. Then $i^*_\bullet$ is an exact functor.*

Proof. Let the shape functor of $i_\bullet$ be the inclusion of $J \hookrightarrow I$. Treat $i_\bullet$ as a composition of a flat, shape-preserving map $U_\bullet \to X_\bullet|_J$ and the inclusion $X_\bullet|_J \to X_\bullet$. By [LH09, II 6], the restriction functor to any subdiagram is exact (indeed, it has both left and right adjoints), and the pullback by a shape-preserving map is just the pullback on each component, and on each component it is exact. $\qquad\square$

It's tempting to think of complexes in the derived category of sheaves on a diagram as a sort of diagram of complexes in the derived categories of sheaves on the component schemes: an object $F_\alpha \in D(X_\alpha)$ and compatible morphisms $\mathbf{R}X_{\phi_{\beta\alpha}*}F_\beta \to F_\alpha$. This is a subtle issue: the complex category $C(\mathrm{Sh}(X_\bullet))$ is indeed this sort of construction, but the chain homotopies over the entire diagram are not the same as the chain homotopies over each component combined, they must be compatible as well. Here is a simple example of this phenomenon:

Example 2.2.11. Consider the diagram of schemes $X_\bullet = (\mathrm{Spec}\, k \to \mathrm{Spec}\, k)$ over the category $\underline{2}$, and the following elements, of the complex category, $F_\bullet$:

$$
\begin{array}{ccccccccccc}
\cdots & \to & 0 & \to & 0 & \to & k & \to & k & \to & 0 & \to & \cdots \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \to & 0 & \to & 0 & \to & k & \to & 0 & \to & 0 & \to & \cdots,
\end{array}
$$

and $G_\bullet$:

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow k \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$
$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$
$$\cdots \longrightarrow 0 \longrightarrow k \longrightarrow k \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots .$$

The chain maps between $F_\bullet$ and $G_\bullet$ are exactly what we think, chain maps between the two top rows and chain maps between the two bottom that are compatible, and thus $\mathrm{Hom}_{C(X_\bullet)}(F_\bullet, G_\bullet) = k$. Next, to compute the Hom in the derived category, since everything is injective here, we need only mod out by chain homotopy. When we ignore the compatibilities, and just consider each of $F_\bullet$ and $G_\bullet$ as two elements of $D(\mathrm{Spec}\,k)$ and a morphism between them, then we can replace the top row of $F_\bullet$ and the bottom row of $G_\bullet$ with 0 as these rows are quasi-isomorphic to 0, thus the Hom considered this way would itself be zero. However, the only nontrivial maps of the homotopy must fit into diagrams like

$$\begin{array}{ccc} k & \xrightarrow{\;\;h\;\;} & k \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & k, \end{array}$$

and therefore must be zero, so in fact $\mathrm{Hom}_{D(X_\bullet)}(F_\bullet, G_\bullet) = k$.

Another thing to note about this is that $F_\bullet$ is not quasi-isomorphic to the complex

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$
$$\downarrow \qquad \downarrow \qquad \downarrow$$
$$\cdots \longrightarrow 0 \longrightarrow k \longrightarrow 0 \longrightarrow \cdots ,$$

as any morphism to or from this complex runs into the same issue: some subdiagram commuting meaning the map must be zero.

2.3 Cubical Descent

As previously mentioned, we are particularly interested in the case of a $\square_n$-diagram. This has particularly nice properties in that the $\mathbf{R} \varprojlim$ functor which is relatively mysterious in general becomes very concrete and even computable, it is given by the so-called simple functor, which appears in [GNAPGP88, I 6] and is developed further in [GNA02] (an English reference is given by [PRP09]).

For notational convenience, we'll consider $\Box_n = \{(\alpha_0, \ldots \alpha_n) : \alpha_i \in \{0, 1\}\})$ as a subset of $\mathbb{Z}^n$. Let e_i be the ith basis vector of $\mathbb{Z}^n$, and let the morphism from α to $\alpha + e_i$ (which exists if and only if $\alpha_i = 0$ be denoted by $\phi_{(\alpha, i)}$.

Definition 2.3.1 ([GNAPGP88, I 6]). Let $F_\bullet : \Box_n \to C(A)$ be a diagram (over $\Box_n^{op}$) in the category of complexes in an abelian category A, where the differential of F_α is d_α. Define the *simple complex* of $s(F_\bullet)$ to be the object in $C(A)$ with

$$s(F_\bullet)^p = \bigoplus_\alpha F_\alpha^{p+1-|\alpha|}$$

and such that the differential on each $F_\alpha^{p+1-|\alpha|}$ is

$$d = (\bigoplus_{\{i : \alpha_i = 0\}} (-1)^{\sum_{j<i} \alpha_j} F_{\phi_{(\alpha, i)}}) \oplus (-1)^{|\alpha|} d_\alpha.$$

In more tangible terms, complex is obtained by combining all the complexes, staggering them to the right by the value of $|\alpha|$. To illustrate, suppose Φ was the $\Box_1$-diagram

$$\begin{array}{ccc} C^\bullet & \longleftarrow & A^\bullet \\ \uparrow & & \\ B^\bullet. & & \end{array}$$

Then to obtain the simple complex, we arrange the complexes, written out, as

$$\cdots \xrightarrow{-} A^0 \xrightarrow{-} A^1 \xrightarrow{-} A^2 \xrightarrow{-} \cdots$$

and then take the direct sum of each column, with the signs above indicating the sign in the differential.

This can also be interpreted as taking a succession of mapping cones: treat $F_\bullet$ as a $(\Box_n^+)^{op}$-diagram with a 0 at the initial object, then since by definition $\Box_n^+ = \prod \underline{2}$, s on $\underline{2}$ is the mapping cone, and by either direct computation or as in [GNA02] s factors over products.

Proposition 2.3.2 ([GNAPGP88, I 6.3], [GNA02, 1.3.1])**.** *Let A be an abelian category, then the simple complex is a functor $s : \underline{\mathrm{Diag}}_{\square_n} C(A) \to C(A)$ which takes quasi-isomorphisms to quasi-isomorphisms.*

Then this functor descends to the derived categories $D(\underline{\mathrm{Diag}}_{\square_n} A)$ and $\underline{\mathrm{Diag}}_{\square_n} D(A)$ and is the same when considering an object of the first as an object of the latter (so that the issues arising in example 2.2.11 vanish when this functor is applied).

Proposition 2.3.3. *Let $X_\bullet$ be a $\square_n$-scheme with a map $f_\bullet : X_\bullet \to X$ to a scheme, and let $F_\bullet$ be an element of the bounded below derived category of sheaves on $\square_n$ Then $\boldsymbol{R}f_{\bullet*}F_\bullet$ is naturally quasi-isomorphic to the simple complex formed by the $\square_n$ diagram of $\boldsymbol{R}f_{\alpha*}F_\alpha$ for each $\alpha \in \square_n$.*

Proof. This is [GNAPGP88, I 6.6] in the special case of $I = \underline{1}$, and $K_\bullet$ the functor sending 0 to $\square_n$. $\qquad\square$

This works just as well for functors $\square_n \to C(A)$ (i.e. objects of $\underline{\mathrm{Diag}}_{\square_n^{op}} C(A)$). To avoid confusion, we notate this differently:

Definition 2.3.4. Let $F_\bullet \in \underline{\mathrm{Diag}}_{\square_n} C(A)$, define the complex $s^*(F_\bullet)$ via

$$s^*(F_\bullet)^p = \oplus_\alpha F_\alpha^{p-1+|\alpha|}$$

with differential on each $F_\alpha^{p-1+|\alpha|}$

$$d = \bigoplus_{(i : \alpha_i = 0)} (-1)^{\sum_{j<i} \alpha_j} F_{\phi(\alpha,i)} \oplus (-1)^{|\alpha|} d_\alpha.$$

Proposition 2.3.3 holds for s^* as well.

The functors s and s^* have a sort of duality theorem:

Theorem 2.3.5. *Let A be an abelian category, $F_\bullet \in D(\underline{\mathrm{Diag}}_{\square_n^{op}} A)$ and $G \in D(A)$, then $\boldsymbol{R}\mathrm{Hom}(F_\bullet, G \times \square_n^{op})$ is a $\square_n$-diagram, $\boldsymbol{R}\mathrm{Hom}(G \times \square_n, F_\bullet)$ is a $\square_n^{op}$-diagram and*

$$\boldsymbol{R}\mathrm{Hom}(s(F_\bullet), G) = s^*(\boldsymbol{R}\mathrm{Hom}(F_\bullet, G \times \square_n^{op}))$$

$$\boldsymbol{R}\mathrm{Hom}(G, s(F_\bullet)) = s(\boldsymbol{R}\mathrm{Hom}(G \times \square_n^{op}, F_\bullet))$$

If $F \in D(A)$, $G_\bullet \in D(\underline{\mathrm{Diag}}_{\square_n} A)$, then the analogous equalities hold:

$$\boldsymbol{R}\operatorname{Hom}(F, s^*(G_\bullet)) = s^*(\boldsymbol{R}\operatorname{Hom}(F \times \square_n, G_\bullet))$$

$$\boldsymbol{R}\operatorname{Hom}(s^*(G_\bullet), F) = s(\boldsymbol{R}\operatorname{Hom}(G_\bullet, F \times \square_n))$$

Furthermore, this continues to hold if A is a category of sheaves and we use sheaf $\mathcal{H}om$.

Proof. These are all similar proofs, both in technique and tediousness, so we check the first only.

Choose representatives of $F_\bullet$ and G in $K(\underline{\mathrm{Diag}}_{\square_n^{op}} A)$ and $K(A)$ respectively, with G K-injective. Then $\boldsymbol{R}\operatorname{Hom}(F, s^*(G_\bullet)$ is represented by the complex with degree p given by

$$
\begin{aligned}
\boldsymbol{R}\operatorname{Hom}(F, s^*(G_\bullet))^p &= \bigoplus_{i-j=p} \operatorname{Hom}(s(F_\bullet)^j, G^i) \\
&= \bigoplus_{i-j=p} \operatorname{Hom}(\bigoplus_\alpha F_\alpha^{j+1-|\alpha|}, G^i) \\
&= \bigoplus_\alpha \bigoplus_{i-j=p} \operatorname{Hom}(F_\alpha^{j+1-|\alpha|}, G^i) \\
&= \bigoplus_\alpha \bigoplus_{i-k=p-1+|\alpha|} \operatorname{Hom}(F_\alpha^k, G^i) \\
&= \bigoplus_\alpha \boldsymbol{R}\operatorname{Hom}(F_\alpha, G)^{p-1+|\alpha|} \\
&= s^*(\boldsymbol{R}\operatorname{Hom}(F_\bullet, G \times \square_n^{op}))^p.
\end{aligned}
$$

Furthermore, the differentials of the two complexes also agree after enough identifications.

$\square$

We can skip the simple complex in actual computations by using the following spectral sequence.

Proposition 2.3.6 ([GNAPGP88, I 6.7])**.** *Let $X_\bullet$ be a $\square_n$-scheme, and $F_\bullet \in D(X_\bullet)$. Then there is a spectral sequence computing the cohomology of $F_\bullet$:*

$$E_1^{pq} = \bigoplus_{|\alpha|=p+1} \mathbb{H}^q(X_\alpha, F_\alpha) \Rightarrow \mathbb{H}^{p+q}(X_\bullet, F_\bullet).$$

For a morphism $\pi_\bullet : X_\bullet \to X$ to a scheme X, this generalizes to a spectral sequence

$$E_1^{pq} = \bigoplus_{|\alpha|=p+1} \boldsymbol{R}^q \pi_{\alpha*} F_\alpha \Rightarrow \boldsymbol{R}^{p+q} \pi_{\bullet*} F_\bullet.$$

2.4   Cubical Hyperresolutions

The basic unit of a cubical hyperresolution is a 2-resolution: this can be thought of a resolution of singularities which keeps track of the fiber over the singular locus.

Definition 2.4.1 ([Kol13, 10.76])**.** If X is an I-scheme, then a *2-resolution* of S is a $\square_1^+ \times I$-scheme $X_\bullet$ as in the diagram

$$\begin{array}{ccc} X_{11} & \lhook\joinrel\longrightarrow & X_{01} \\ \downarrow & & \downarrow{\scriptstyle f} \\ X_{10} & \lhook\joinrel\longrightarrow & X_{00} \end{array}$$

such that the horizontal maps are closed immersions, f is a resolution of singularities which is an isomorphism outside of $X_{10} \subseteq X_{00}$, and X_{11} is the fibered product.

Any resolution of X can be completed to a 2-resolution by adding the singular set and its pullback. For instance, we can improve the resolution of the cone by forming the square as pictured on the right.

Now if we consider the $\square_1$-shaped diagram without the cone and dashed maps, all the varieties are smooth and yet the necessary information about the singularity is preserved, in fact the cone is now the pushout of this diagram.

In this example, and for any variety X where $\operatorname{Sing} X$ and it's pullback is smooth, the first 2-resolution will be enough, but in general the process will need to continue to resolve the singularities of $\operatorname{Sing} X$. Since we would like the final product to be a diagram, we first define a way to combine successive 2-resolutions into a larger diagram
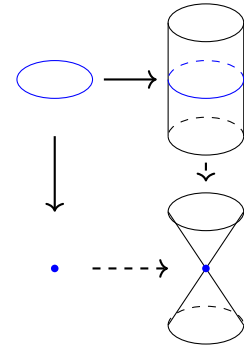
Definition 2.4.2 ([Kol13, 10.78])**.** Suppose $n \leq 1$, $X_\bullet^r$ a $\square_r^+ \times I$-scheme for $1 \leq r \leq n$ such that $X_{00\bullet}^{r+1}$ is equal to $X_{1\bullet}^r$. That is, each term in the sequence is a 2-resolution of the left

part of the previous diagram. Then define the *reduction*, a $\square_n^+ \times I$-scheme $\text{red}(X_\bullet^1, \ldots, X_\bullet^n)$ inductively as follows:

$$\text{red}(X_\bullet^1, X_\bullet^2) = \begin{cases} X_{0\beta}^1 & \alpha = (0,0) \\ X_{\alpha\beta}^2 & \alpha \in \square_1 \end{cases}$$

$$\text{red}(X_\bullet^1, \ldots, X_\bullet^n) = \text{red}(\text{red}(X_\bullet^1, \ldots, X_\bullet^{n-1}), X_\bullet^n).$$

Now we come to the definition of a hyperresolution:

Definition 2.4.3 ([Kol13, 10.79])**.** Let X be an I-scheme. Then the $\square_n^+ \times I$-scheme $\text{red}(X_\bullet^1, \ldots, X_\bullet^n)$ is a *cubic hyperresolution of* X if $X_\bullet^1$ is a 2-resolution of X, each successive step is a 2-resolution of the left part of the last, and each scheme in the final diagram is smooth. Often, this will be written as a map $f_\bullet : X_\bullet \to X$, where $X_\bullet$ is a $\square_n \times I$-scheme, and the total diagram of $X_\bullet \to X$ is the cubic hyperresolution.

Theorem 2.4.4 ([GNAPGP88, I 2.15])**.** *Suppose I is a finite, ordered category, and X is an I-scheme over a field of characteristic 0. Then there exists a cubic hyperresolution $X_\bullet$ of X such that for each $\alpha \in \square_n$*

$$\dim X_\alpha \le \dim X - |\alpha| + 1.$$

Proposition 2.4.5. *Suppose $\pi_\bullet : Y_\bullet \to X_\bullet$ is a cubical hyperresolution of the I-scheme $X_\bullet$, and $\overline{X}_\bullet \to X_\bullet$ is a shape-embedding morphism of diagrams, an isomorphism on each component of $\overline{X}_\bullet$. Then $Y_\bullet \times_{X_\bullet} \overline{X}_\bullet \to \overline{X}_\bullet$ is a cubical hyperresolution of $\overline{X}_\bullet$.*

Proof. Take the sequence of 2-resolutions which gives $\pi_\bullet$ as cubical hyperresolutions, then the fibered product of each of these remains a 2-resolution as closed immersions, birational maps, and fibered products are all preserved. Finally, the reduction of this sequence gives $Y_\bullet \times_{X_\bullet} \overline{X}_\bullet$. $\square$

2.5 Grothendieck Topology

Everything done so far can be recovered by viewing diagrams of schemes as a certain type of site and using the definition of sheaves on a site. For instance, the sheaf-hom defined above is exactly the sheaf-hom over a site.

Definition-Proposition 2.5.1. For any diagram of schemes $X_\bullet$, define the Zariski site $\text{Zar}(X_\bullet)$ to be the category of shape-embedding open immersions $U_\bullet \to X_\bullet$, with morphisms compatible shape-embedding open immersions $U_\bullet \to V_\bullet$.

For each $U_\bullet$, a set of maps $\{U_{\bullet i} \to U_\bullet\}$ in the category is a covering if each component of $U_\bullet$, U_α is covered by the set of components $\{U_{\alpha i}\}$. These form a Grothendieck topology, and a morphism of the diagrams gives an associated morphism of the Zariski sites, and the category of sheaves over $X_\bullet$ is equivalent to the category of sheaves on $\text{Zar}(X_\bullet)$, and each morphism of diagrams has an associated morphism of sites.

Proof. An isomorphism is certainly a covering, as are refinements. Finally, a base change of a shape-preserving open immersion retains these properties and component-wise it is the base change of a covering in the usual topology, so this defines a topology.

Define a functor $\Phi : \text{Sh}(X_\bullet) \to \text{Sh}(\text{Zar}(X_\bullet))$ by $\Phi(\mathscr{F}_\bullet)(U_\bullet) = \varprojlim \mathscr{F}_\alpha(U_\alpha)$, a limit ranging over the shape of $U_\bullet$. This functor is an equivalence, with $\Phi^{-1}(\mathscr{G})_\alpha = (U \mapsto \mathscr{G}(U \hookrightarrow X_\bullet))$.

For each $f_\bullet : X_\bullet \to Y_\bullet$ we can associate morphism of sites which takes $U_\bullet \hookrightarrow Y_\bullet$ to $U_\bullet \times_{Y_\bullet} X_\bullet \hookrightarrow X_\bullet$. $\qquad\square$

Where the site viewpoint begins to differ is that many of the concepts from sites tend to imply a strong condition called equivariance, which in general we'd like to avoid where possible.

Definition 2.5.2. A sheaf $\mathscr{F}_\bullet$ on a diagram is called equivariant if for each X_ϕ in the diagram, the map $X_\phi^* \mathscr{F}_\alpha \to \mathscr{F}_\beta$ is an isomorphism.

Definition 2.5.3 ([Sta15, Tag 0408])**.** A sheaf $\mathcal{L}_\bullet$ on a diagram is called invertible if for each $U \in \text{Zar}(X_\bullet)$, there is a covering $\{U_i \to U\}$ and isomorphisms $\mathcal{O}_{X_\bullet/U_i} \xrightarrow{\sim} \mathcal{L}_\bullet$. An invertible

sheaf is equivariant (by a similar argument to [LH09, II 7.3]), and if $\mathcal{L}_\bullet^{-1} = \mathcal{H}om(\mathcal{L}_\bullet, \mathcal{O}_{X_\bullet})$, then the natural map $\mathcal{L}_\bullet \otimes \mathcal{L}_\bullet^{-1} \to \mathcal{O}_{X_\bullet}$ is an isomorphism.

Equivalently, an invertible sheaf is an equivariant sheaf which is invertible on every component.

Note that this differs from the convention of "$\mathscr{F}_\bullet$ has property P if each $\mathscr{F}_\alpha$ does," in this case we need equivariance in order to take maps $X_\phi^* \mathcal{L}_\beta \to \mathcal{L}_\alpha$ and give a map from $X_\phi^* \mathcal{L}_\beta^{-1}$ to $\mathcal{L}_\alpha$.

2.6 Local Quasi-coherentness

The quasi-coherentness of sheaves on diagrams is very important for later theorems. Here we prove some results guaranteeing quasi-coherent properties, starting with the following technical lemma.

Lemma 2.6.1. *Let I be a diagram shape category and with an initial object. Then if X is a quasi-compact I-scheme, there is a finite affine covering of X in the Zariski site.*

Proof. First we note that the objects in I can be partially-ordered such that each object only maps to the objects of larger order, that is, $\mathrm{Ob}\, I$ has a bijection with $\{0, \ldots, N\}$ such that if there is a morphism $i \to j$, then $i \leq j$ (note that the definition of a diagram shape category is essential for this construction). The initial object must be 0.

Thus we can construct X inductively as follows: let $X^0 = X_0$, and let X^1 be the full subdiagram of X with only X_0 and X_1, etc., so that X^n is obtained from X^{n-1} by taking the diagram containing X^{n-1}, X_n, and all the maps $f_{n\alpha} : X_n \to X_\alpha$ out of X_n in the diagram X, by construction these all map to objects in X^{n-1}.

Construct V_i^n as follows: for X^0 choose an open affine cover of X_0, and for X_n, for each V_i^{n-1} choose an open affine cover U_{ij} of $\cap_\alpha f_{n\alpha}^{-1}(V_{i\alpha})$ (here we take an empty intersection to be the entire space). Let V_{ij}^n be the diagram $U_{ij} \xrightarrow{\{f_{n\alpha}\}_\alpha} V_i$.

We wish to prove that for each n, this construction forms a cover as desired for X^n. This would follow by the construction and the following:

Claim: For each n, $\cap_\alpha f_{n\alpha}^{-1}(V_{i\alpha})$ covers X_n.

We prove this by induction. For X_0, the intersection the entire space, so this is true. In general, suppose $p \in X_n$. Let $\alpha_1, \ldots, \alpha_m$ be the objects in I which n maps to, in their order. We must show that there is an element of V_i^{n-1} which has $f_{\alpha_j}(p)$ in each $V_{i\alpha_j}^{n-1}$.

There is a $V_{i_1}^{\alpha_1}$ containing $f_{\alpha_1}(p)$, by construction. Inducting on the α_i's, assume now that there is a $V_{i_{k-1}}^{\alpha_{k-1}}$ containing each of the $\{f_{\alpha_1}(p), \ldots, f_{\alpha_{k-1}}(p)\}$. Then by the hypotheses, if g_β's are the maps out of X_{α_k}, $\cap_\beta g_\beta^{-1}(V_{i_{k-1}\beta})$ is a cover of X_{α_k}, thus by construction there is a $V_{i_{k-1}j}^{\alpha_k}$ including $\{f_{\alpha_1}(p), \ldots, f_{\alpha_k}(p)\}$. $\qquad\square$

Definition 2.6.2. A sheaf on $X_\bullet$ will be called (quasi-)coherent if it is as a sheaf on the Zariski site, and locally (quasi-)coherent if each component of the sheaf is (quasi-)coherent. Being (quasi-)coherent in this sense implies locally (quasi-)coherent, but not the other way around: as in [LH09, II 7.3], (quasi-)coherent is equivalent locally (quasi-)coherent and equivariance.

Definition-Proposition 2.6.3. Let $X_\bullet$ be an I-scheme, and $\alpha \in I$. Define the inclusion $j_\alpha : X_\alpha \hookrightarrow X_\bullet$ (note this is different than the previously defined $i_\alpha : X_\bullet/X_\alpha \hookrightarrow X_\bullet$).

Then $(-)_\alpha = j_\alpha^*$, which has a right adjoint $j_{\alpha*}$ and a left adjoint $j_{\alpha!}$ as follows: For any $\mathcal{O}_{X_\alpha}$-module $\mathscr{F}_\alpha$, define $j_{\alpha!}\mathscr{F}_{\alpha\bullet}$ a $\mathcal{O}_{X_\bullet}$-module as

$$j_{\alpha!}\mathscr{F}_{\alpha\beta} = \begin{cases} X_{\phi_{\alpha\beta}}^* \mathscr{F}_\alpha & \text{if } \beta \in \alpha/I \\ 0 & \text{if } \beta \notin \alpha/I, \end{cases}$$

with diagram maps the obvious isomorphisms. Furthermore $\mathcal{H}om_{X_\bullet}(\mathscr{F}_{\alpha\bullet}^\uparrow, -) = i_{X_\alpha*}\mathcal{H}om(i_{X_\alpha}^*\mathscr{F}_{\alpha\bullet}^\uparrow, i_{X_\alpha}^* -)$.

Finally, for $F_\alpha \in D(X_\alpha)$, the derived version becomes $\mathbf{R}\,\mathrm{Hom}_{X_\bullet}(\mathbf{L}j_{\alpha!}F_{\alpha\bullet}, G_\bullet) = \mathbf{R}\,\mathrm{Hom}_{X_\alpha}(F_\alpha, G_\alpha)$ and $\mathbf{R}\mathcal{H}om_{X_\bullet}(\mathbf{L}j_{\alpha!}F_{\alpha\bullet}, G_\bullet) = \mathbf{R}i_{X_\alpha*}\mathbf{R}\mathcal{H}om(i_{X_\alpha}^*\mathbf{L}j_{\alpha!}F_{\alpha\bullet}, i_{X_\alpha}^*G_\bullet)$.

Proof. For adjointness, we must show that $\mathrm{Hom}_{X_\alpha}(\mathscr{F}_\alpha, \mathscr{G}_\alpha) = \mathrm{Hom}_{X_\bullet}(j_{\alpha!}\mathscr{F}_{\alpha\bullet}, \mathscr{G}_\bullet)$.

For each $\Phi \in \mathrm{Hom}_{X_\bullet}(j_{\alpha!}\mathscr{F}_{\alpha\bullet}, \mathscr{G}_\bullet)$, and $\beta \in \alpha/I$, the compatibility ensures that Φ_β is determined by Φ_α, and for $\eta \notin \alpha/I$, this must be zero, which does not restrict the morphisms elsewhere since there are no maps from any such β to any such η.

For the second part, since the definition of $j_{\alpha!}$ commutes with restriction, the first part shows this is true restricted to X_α. If $\beta \in \alpha/I$, then restricting everything to $X_\bullet/X_\beta$, this becomes the same situation for $X^*_{\phi_{\alpha\beta}}\mathscr{F}_\alpha$. If $\beta \in I/\alpha$, then maps of sheaves on $X_\bullet/U$ for $U \subseteq X_\beta$ are determined by the maps on $X_\bullet/X^{-1}_{\phi_{\beta\alpha}}(U)$, and finally if β is in neither subcategory then there are no maps in $X_\bullet/X_\beta$ as $j_{\alpha!}\mathscr{F}_{\alpha\bullet}$ is zero everywhere.

Finally, $\mathbf{L}j_{\alpha!}$ is adjoint with the restriction functor (which is exact), and continues to commute with restriction to an open subset, so the derived versions hold. $\qquad\square$

Definition-Proposition 2.6.4 ([LH09, II 6.7]). Let $X_\bullet$ be an I scheme, with $\alpha \to \beta$ a morphism in I. Define the inclusion $j_{\alpha\beta} : (X_\beta \to X_\alpha) \to X_\bullet$ of diagrams of schemes. Then the restriction map $j^*_{\alpha\beta}$ has a right adjoint on sheaves $\mathscr{F}_\bullet$ on $(X_\beta \to X_\alpha)$ computed as follows:

$$(j_{\alpha\beta!}\mathscr{F}_\bullet)_\gamma = \begin{cases} X^*_{\phi_{\gamma\beta}}\mathscr{F}_\beta & \text{if } \gamma \in \beta/I \\ X^*_{\phi_{\gamma\alpha}}\mathscr{F}_\alpha & \text{if } \gamma \in (\alpha/I) \setminus (\beta/I) \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.6.5. *If $X_\bullet$ has affine arrows, then for any $\mathscr{F}_\bullet$ locally coherent, $\mathscr{G}_\bullet$ locally quasi-coherent, $\mathcal{H}om_{X_\bullet}(\mathscr{F}_\bullet, \mathscr{G}_\bullet)$ is locally quasi-coherent. If in addition, $\mathscr{G}_\bullet$ is locally coherent, $X_\bullet$ is noetherian, and the arrows are proper, then $\mathcal{H}om_{X_\bullet}(\mathscr{F}_\bullet, \mathscr{G}_\bullet)$ is locally coherent.*

Proof. Enough to show that for each X_α, that the sheaf $(\mathcal{H}om_{X_\bullet}(\mathscr{F}_\bullet, \mathscr{G}_\bullet))_\alpha$ is (quasi-)coherent, which can be done locally. Furthermore for each $U \subseteq X_\alpha$, the definition of $\mathcal{H}om$ only depends on the diagram $X_\bullet/U$ so we can restrict to proving this for a terminal scheme in a diagram, and furthermore restrict that this scheme is affine. Because $X_\bullet$ has affine arrows, the entire diagram must be affine.

Note that

$$\mathrm{Hom}_{X_\bullet}(j_{\alpha!}\mathcal{O}_{X_\alpha\bullet}, \mathscr{G}_\bullet) = \mathscr{G}_\alpha(X_\alpha).$$

Furthermore,

$$\mathcal{H}om_{X_\bullet}(j_{\alpha!}\mathcal{O}_{X_\alpha\bullet}, \mathcal{G}_\bullet)_\beta = \begin{cases} \mathcal{G}_\beta & \text{if } \beta \in \alpha/I \\ X_{\phi*}\mathcal{G}_\alpha & \text{if } \phi : \beta \to \alpha , \\ 0 & \text{otherwise} \end{cases}$$

so this is locally quasi-coherent, and if the additional conditions are met, it is locally coherent.

Each $\mathcal{F}_\alpha$ is coherent on an affine scheme, so is the cokernel of a map of finite rank free modules, these give sequences, surjective on the α component of the diagram:

$$j_{\alpha!}\mathcal{O}_{X_\alpha\bullet}^{\oplus m_\alpha} \to j_{\alpha!}\mathcal{O}_{X_\alpha\bullet}^{\oplus n_\alpha} \to \mathcal{F}_\bullet.$$

Combining these gives a $\mathcal{F}$ as a cokernel:

$$\bigoplus_\alpha j_{\alpha!}\mathcal{O}_{X_\alpha\bullet}^{\oplus m_\alpha} \to \bigoplus_\alpha j_{\alpha!}\mathcal{O}_{X_\alpha\bullet}^{\oplus n_\alpha} \to \mathcal{F}_\bullet \to 0.$$

Applying $\mathcal{H}om_{X_\bullet}(-, \mathcal{G}_\bullet)$,

$$0 \to \mathcal{H}om_{X_\bullet}(\mathcal{F}_\bullet, \mathcal{G}_\bullet) \to \mathcal{H}om_{X_\bullet}(\bigoplus j_{\alpha!}\mathcal{O}_{X_\alpha\bullet}, \mathcal{G}_\bullet) \to \mathcal{H}om_{X_\bullet}(\bigoplus j_{\alpha!}\mathcal{O}_{X_\alpha\bullet}, \mathcal{G}_\bullet),$$

and since the direct sums are finite this can be rewritten as

$$0 \to \mathcal{H}om_{X_\bullet}(\mathcal{F}_\bullet, \mathcal{G}_\bullet) \to \bigoplus \mathcal{H}om_{X_\bullet}(j_{\alpha!}\mathcal{O}_{X_\alpha\bullet}, \mathcal{G}_\bullet) \to \bigoplus \mathcal{H}om_{X_\bullet}(j_{\alpha!}\mathcal{O}_{X_\alpha\bullet}, \mathcal{G}_\bullet).$$

Now since finite direct sums and kernels of (quasi-)coherent sheaves are (quasi-)coherent, we are done. $\square$

Theorem 2.6.6. *Let $X_\bullet$ be a concentrated (that is, quasi-compact and quasi-separated) I-scheme. Then if $\mathcal{F}_\bullet$ is locally coherent and $\mathcal{G}_\bullet$ is locally quasi-coherent, $\mathcal{H}om_{X_\bullet}(\mathcal{F}_\bullet, \mathcal{G}_\bullet)$ is locally quasi-coherent. If in addition $\mathcal{G}_\bullet$ is locally coherent, and $X_\bullet$ is noetherian with proper arrows, then $\mathcal{H}om_{X_\bullet}(\mathcal{F}_\bullet, \mathcal{G}_\bullet)$ is locally coherent.*

Proof. As before, reduce to the case with $X_\bullet$ a diagram with an affine, terminal component X_0. Then apply lemma 2.6.1 to $X_\bullet$ to have an affine diagram cover $\{V_{i\bullet}\}$. Each intersection

$V_{i\bullet} \cap V_{j\bullet}$ is quasi-compact by quasi-separability, so we can reapply lemma 2.6.1 to have affine diagram covers $\{V_{ijk\bullet}\}$.

Have maps $\mathcal{H}om_{X_\bullet}(\mathscr{F}_\bullet, \mathscr{G}_\bullet)_0 \to \mathcal{H}om_{V_{i\bullet}}(\mathscr{F}_\bullet|_{V_{i\bullet}}, \mathscr{G}_\bullet|_{V_{i\bullet}})_0$ given by restriction, and in fact if we are given $\phi_i \in \mathcal{H}om_{V_{i\bullet}}(\mathscr{F}_\bullet|_{V_{i\bullet}}, \mathscr{G}_\bullet|_{V_{i\bullet}})_0$ for each i such that $\phi_i|_{V_{ijk\bullet}} = \phi_j|_{V_{ijk\bullet}}$ for each i, j, k, then the morphisms glue uniquely to give $\psi \in \mathcal{H}om_{X_\bullet}(\mathscr{F}_\bullet, \mathscr{G}_\bullet)_0$.

This establishes the equalizer diagram

$$\mathcal{H}om_{X_\bullet}(\mathscr{F}_\bullet, \mathscr{G}_\bullet)_0 \to \bigoplus_i \mathcal{H}om_{V_{i\bullet}}(\mathscr{F}_\bullet|_{V_{i\bullet}}, \mathscr{G}_\bullet|_{V_{i\bullet}})_0 \rightrightarrows \bigoplus_{i,j,k} \mathcal{H}om_{V_{ijk\bullet}}(\mathscr{F}_\bullet|_{V_{ijk\bullet}}, \mathscr{G}_\bullet|_{V_{ijk\bullet}})_0$$

by proposition 2.6.5, the right two terms are (quasi-)coherent, so as is the term on the left. $\square$

Theorem 2.6.7. *Suppose $\mathscr{G}_\bullet$ is an $\mathcal{O}_{X_\bullet}$-module. Then the functors $\mathcal{H}om_{X_\bullet}(\mathscr{G}_\bullet, -)$ and $- \otimes \mathscr{G}_\bullet$ are adjoint.*

Proof. Let $\mathscr{F}_\bullet$ and $\mathscr{H}_\bullet$ be arbitrary $\mathcal{O}_{X_\bullet}$-modules, we will describe an isomorphism

$$\mathrm{Hom}(\mathscr{F}_\bullet, \mathcal{H}om_{X_\bullet}(\mathscr{G}_\bullet, \mathscr{H}_\bullet)) \xrightarrow{\sim} \mathrm{Hom}(\mathscr{F}_\bullet \otimes \mathscr{G}_\bullet, \mathscr{H}_\bullet)$$

Note that if $\mathscr{F}_\bullet \otimes^p \mathscr{G}_\bullet$ is the presheaf tensor product, then there is a natural isomorphism $\mathrm{Hom}(\mathscr{F}_\bullet \otimes \mathscr{G}_\bullet, \mathscr{H}_\bullet) \cong \mathrm{Hom}(\mathscr{F}_\bullet \otimes^p \mathscr{G}_\bullet, \mathscr{H}_\bullet)$, so it is enough to show for the presheaf tensor product, and in this case $X_{\phi*}(\mathscr{F}_\alpha \otimes^p \mathscr{F}_\beta) \cong X_{\phi*}\mathscr{F}_\alpha \otimes^p X_{\phi*}\mathscr{G}_\alpha$, compatible with the arrows of $\mathscr{F}_\bullet \otimes^p \mathscr{G}_\bullet$.

Given $\psi \in \mathrm{Hom}(\mathscr{F}_\bullet, \mathcal{H}om_{X_\bullet}(\mathscr{G}_\bullet, \mathscr{H}_\bullet))$, with restriction to each component α denoted ψ_α, define a morphism $\mathscr{F}_\alpha \otimes^p \mathscr{G}_\alpha \to \mathscr{H}_\alpha$ by $s \otimes t \mapsto \psi_\alpha(s)(t)$. It remains to check that these give compatible morphisms, namely to show the following diagram commutes:

$$
\begin{array}{ccc}
\mathscr{F}_\alpha \otimes^p \mathscr{G}_\alpha & \xrightarrow{s \otimes t \mapsto \psi_\alpha(s)(t)} & \mathscr{H}_\alpha \\
\downarrow{\scriptstyle \mathscr{F}_\phi \otimes \mathscr{G}_\phi} & & \downarrow{\scriptstyle \mathscr{H}_\phi} \\
X_{\phi*}\mathscr{F}_\beta \otimes^p X_{\phi*}\mathscr{G}_\beta & \xrightarrow{s \otimes t \mapsto \psi_\beta(s)(t)} & X_{\phi*}\mathscr{H}_\beta.
\end{array}
$$

Then we must show that for $s \otimes t$ in $\mathscr{F}_\alpha \otimes^p \mathscr{G}_\alpha$, $\mathscr{H}_\phi(\psi_\alpha(s)(t)) = \psi_\beta(\mathscr{F}_\phi s)(\mathscr{G}_\phi t)$, or more

abstractly that the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{H}om_{X_\bullet/X_\alpha}(\mathscr{G}_\bullet,\mathscr{H}_\bullet)_\alpha & \longrightarrow & \mathcal{H}om_{X_\alpha}(\mathscr{G}_\alpha,\mathscr{H}_\alpha) \\
\downarrow & & \\
& & \mathcal{H}om_{X_\alpha}(\mathscr{G}_\alpha, X_{\phi*}\mathscr{H}_\beta) \\
& & \uparrow \\
X_{\phi*}\mathcal{H}om_{X_\bullet/X_\beta}(\mathscr{G}_\bullet,\mathscr{H}_\bullet)_\beta \longrightarrow X_{\phi*}\mathcal{H}om_{X_\beta}(\mathscr{G}_\beta,\mathscr{H}_\beta) \longrightarrow \mathcal{H}om_{X_\alpha}(X_{\phi*}\mathscr{G}_\beta, X_{\phi*}\mathscr{H}_\beta).
\end{array}
$$

Taking a morphism $\varphi \in \mathcal{H}om_{X_\bullet/X_\alpha}(\mathscr{G}_\bullet,\mathscr{H}_\bullet)_\alpha$, the result of the map to the right is, for each $U \subseteq X_\alpha$, a module homomorphism $\mathscr{G}_\alpha(U) \to \mathscr{H}_\alpha(U)$. The result of the composition to the bottom right is, for each $U \subseteq X_\alpha$, a module homomorphism $\mathscr{G}_\beta(X_\phi^{-1}(U)) \to \mathscr{H}_\beta(X_\phi^{-1}(U))$. Then they map to the same morphism in $\mathcal{H}om_{X_\alpha}(\mathscr{G}_\alpha, X_{\phi*}\mathscr{H}_\beta)$ by the compatibilities of the original map.

Now given a $\psi \in \mathrm{Hom}(\mathscr{F}_\bullet \otimes^p \mathscr{G}_\bullet, \mathscr{H}_\bullet)$, define for each $U \subseteq X_\alpha$, $s \in \mathscr{F}_\alpha(U)$, and $\phi : \alpha \to \beta$, a map $\mathscr{G}_\alpha(X_\phi^{-1}(U)) \to \mathscr{H}_\alpha(X_\phi^{-1}(U))$ by $t \mapsto \psi_\alpha(\mathscr{F}_\phi(s) \otimes t)$, and further a map on $V \subseteq X_\phi^{-1}(U)$ by restricting $\mathscr{F}_\phi(s)$. This defines an element of $\mathcal{H}om_{X_\bullet/X_\alpha}(\mathscr{G}_\bullet,\mathscr{H}_\bullet)$, as these morphisms are compatible by the compatibility of ψ and $\mathscr{F}_\phi$.

By careful consideration the two operations described are inverses, so the functors are adjoint. $\square$

Theorem 2.6.8. *The derived functor $\mathbf{R}\mathcal{H}om_{X_\bullet}(-,-)$ takes $D^-(X_\bullet)^{op} \times D^+(X_\bullet)$ to $D^+(X_\bullet)$. Furthermore, it takes $D_c^-(X_\bullet)^{op} \times D_{qc}^+(X_\bullet)$ to $D_{qc}^+(X_\bullet)$.*

Proof. For each $F_\bullet \in D^-(X_\bullet)$, $G_\bullet \in D^+(X_\bullet)$, $\mathbf{R}\mathcal{H}om_{X_\bullet}(F_\bullet, G_\bullet)$ is defined by replacing $G_\bullet$

with an injective resolution $I_\bullet$, and then taking the total complex of the double complex

$$
\begin{array}{ccccccc}
& & \vdots & & \vdots & & \\
& & \downarrow & & \downarrow & & \\
\cdots \longrightarrow & \mathcal{H}om_{X_\bullet}(F_\bullet^i, I_\bullet^j) & \longrightarrow & \mathcal{H}om_{X_\bullet}(F_\bullet^{i-i}, I_\bullet^j) & \longrightarrow & \cdots \\
& \downarrow & & \downarrow & & \\
\cdots \longrightarrow & \mathcal{H}om_{X_\bullet}(F_\bullet^i, I_\bullet^{j+1}) & \longrightarrow & \mathcal{H}om_{X_\bullet}(F_\bullet^{i-i}, I_\bullet^{j+1}) & \longrightarrow & \cdots \\
& \downarrow & & \downarrow & & \\
& & \vdots & & \vdots & &
\end{array}
$$

As the diagonals move to the left, eventually each term must be zero by the bounds on the original complexes. Furthermore, if each term in $F_\bullet$ is coherent and each term in $G_\bullet$ is quasi-coherent, then $I_\bullet$ can be taken to be a locally quasi-coherent injective resolution and thus each term in the double complex is locally quasi-coherent. $\qquad\square$

Theorem 2.6.9. *The derived functor* $-\otimes^{\mathbf{L}} -$ *takes* $D(X_\bullet) \times D(X_\bullet)$ *to* $D(X_\bullet)$. *Furthermore, it takes* $D_{lqc}(X_\bullet) \times D_{lqc}(X_\bullet)$ *to* $D_{lqc}(X_\bullet)$ *and* $D_{lc}(X_\bullet) \times D_{lc}(X_\bullet)$ *to* $D_{lc}(X_\bullet)$.

Proof. This follows immediately from $(F_\bullet \otimes^{\mathbf{L}} G_\bullet)_\alpha = F_\alpha \otimes^{\mathbf{L}} G_\alpha$ and the corresponding fact on schemes. $\qquad\square$

Theorem 2.6.10. *Suppose* $G_\bullet \in D(X_\bullet)$. *Then the functors* $\mathbf{R}\mathcal{H}om_{X_\bullet}(G_\bullet, -)$ *and* $- \otimes^{\mathbf{L}} G_\bullet$ *are adjoint.*

Proof. This is an application of [Sta15, Tag 09T5]. $\qquad\square$

Theorem 2.6.11. *Let* $f_\bullet : X_\bullet \to Y_\bullet$ *be a concentrated, cubical morphism. Then* $\mathbf{R}f_{\bullet*} : D(X_\bullet) \to D(X_\bullet)$ *maps* $D_{lqc}(X_\bullet)$ *to* $D_{lqc}(X_\bullet)$, $D^-(X_\bullet)$ *to* $D^-(X_\bullet)$, *and if* $f_\bullet$ *is a proper morphism of noetherian schemes,* $D_c(X_\bullet)$ *to* $D_c(X_\bullet)$.

Proof. The derived limit on each component preserves (quasi-)coherence and is bounded above, so by the simple complex construction, the derived limit of the total pushforward on each component Y_α also has these properties. $\qquad\square$

Chapter 3

GROTHENDIECK DUALITY ON DIAGRAMS

In this chapter we prove our main results, which are a diagram versions of well-known cohomological results on schemes.

3.1 Abstract $f_\bullet^!$ for an Morphism of Diagrams

For morphisms of proper schemes over a field k, $f : X \to Y$, Grothendieck duality can be stated as the existence of a right adjoint to $\mathbf{R}f_*$, written $f^!$. By the various properties of these maps, we then have an isomorphism for $F \in D_c^+(X), G \in D_c^+(Y)$

$$\mathbf{R}f_*\mathbf{R}\mathcal{H}om_X(F, f^!G) \cong \mathbf{R}\mathcal{H}om_Y(\mathbf{R}f_*F, G)$$

The theory of this adjoint is developed in detail in [Har66].

We wish to develop a theory of Grothendieck duality for morphisms of diagrams of schemes $f_\bullet : X_\bullet \to Y_\bullet$. Some progress on this is made in [LH09], however this is restricted to the case where the shape of $X_\bullet$ and $Y_\bullet$ is the same. The first step in this is proving the existence of an adjoint. Instead of a direct proof as in [Har66], we can use Neeman's proof of Grothendieck duality via Brown representability in [Nee96], though this does not help with construction. Below is the critical existence theorem

Theorem 3.1.1 ([Nee96, 4.1]). *Let $\mathcal{C}$ be a compactly generated triangulated category, $\mathcal{D}$ any triangulated category, and let $F : \mathcal{C} \to \mathcal{D}$ be a triangulated functor which respects coproducts. Then F has a right adjoint.*

Corollary 3.1.2. *Let $f_\bullet : X_\bullet \to Y_\bullet$ be a separated morphism of diagrams of quasi-compact, separated schemes. Then $\mathbf{R}f_{\bullet*}$ has a right adjoint $f_\bullet^!$. Furthermore when the adjoint is taken with $\mathbf{R}f_{\bullet*}$ a functor on $D_{lqc}(X_\bullet)$, $f_\bullet^!$ maps $D_{lqc}^+(X_\bullet)$ to $D_{lqc}^+(X_\bullet)$.*

Proof. The category $D^+_{lqc}(X_\bullet)$ is compactly generated by diagrams of perfect complexes (see [LH09, II 17.1]). As a derived functor of $f_{\bullet*}$, $\mathbf{R}f_{\bullet*}$ is triangulated. For any component morphism ϕ, $\mathbf{R}\phi_*$ respects coproducts by [Nee96, 1.4]. Finally, so does $\mathbf{R}\varprojlim$, so $\mathbf{R}f_{\bullet*}$ has a right adjoint by 3.1.1. The last statement can be shown by the boundedness of the pushforward in Theorem 2.6.11 and using [LH09, Lemma 4.1.8]. $\qquad\qquad\square$

Definition 3.1.3. Let $p_\bullet : X_\bullet \to \operatorname{Spec} k$ be the structure map of a concentrated diagram of schemes. Define the *canonical complex* to be $\omega^\bullet_{X_\bullet} = p^!_\bullet k$

This is analogous to the dualizing complex on schemes, however it is rarely dualizing (in the sense of $\mathbf{R}\mathcal{H}om(\omega^\bullet_{X_\bullet}, \omega^\bullet_{X_\bullet}) \simeq_{qis} \mathcal{O}_{X_\bullet}$), so we will not call it that.

Our next goal is to compute this object. We can first try our hand at the simplest possible diagram:

Theorem 3.1.4. *Consider the diagram of schemes $X_\bullet$ given by $X \xrightarrow{f} Y$, where X and Y are concentrated over a field k. The canonical complex of $X_\bullet$ is given by 0 over X and $\omega^\bullet_Y$ over Y, with the zero map $\omega^\bullet_Y \to 0$.*

Proof. The canonical complex represents the functor

$$F_\bullet \mapsto \mathbf{R}\Gamma(F_\bullet)^*$$
$$\mapsto (\varprojlim(\mathbf{R}\Gamma(F_Y) \to \mathbf{R}\Gamma(F_X)))^*$$
$$\mapsto \mathbf{R}\Gamma(F_Y)^*.$$

By usual duality on schemes, the last module is in fact $\operatorname{Hom}(\mathscr{F}_Y, \omega^\bullet_Y)$, but this is $\operatorname{Hom}(\mathscr{F}_\bullet, \omega^\bullet_{X_\bullet})$, where $\omega^\bullet_{X_\bullet}$ is as described. $\qquad\qquad\square$

Now we take on cubical diagrams, for which we will need the following construction.

Definition 3.1.5. Let $X_\bullet$ be an I-scheme and $\alpha \in I$. Define an object in the derived category of diagrams of sheaves on X_α over I as $\hat{\omega}^\alpha_\bullet \in D(X_\bullet)$

$$\hat{\omega}^\alpha_\beta = \begin{cases} \mathbf{R}X_{\phi_{\beta\alpha}*}\omega^\bullet_{X_\beta} & \text{if } \beta \in \alpha/I \\ 0 & \text{otherwise,} \end{cases}$$

where the nonzero diagram maps are the pushforwards of the natural maps $\mathbf{R}X_{\phi_{\gamma\beta}*}\omega^{\bullet}_{X_\gamma} \to \omega_{X_\beta}$.

The direction of the maps here can be confusing: note that diagrams are contravariant functors. For any arrow $\phi_{\gamma\beta} : \beta \to \gamma$ in I, there is a morphism $X_{\phi_{\gamma\beta}} : X_\gamma \to X_\beta$ and a morphism $\hat{\omega}^\alpha_\gamma \to \hat{\omega}^\alpha_\beta$. This is in contrast with a sheaf $\mathscr{F}_\bullet$ on the diagram of schemes with morphisms $\mathscr{F}_\beta \to X_{\phi_{\gamma\beta}*}\mathscr{F}_\gamma$.

Theorem 3.1.6. *For any $X_\bullet$ a proper, noetherian $\square_n$-scheme over a field k, and $\alpha \in \square_n$, there is an isomorphism*

$$(\omega^{\bullet}_{X_\bullet})_\alpha \cong s^*(\hat{\omega}^\alpha_\bullet)$$

Proof. For any $F \in D(X_\alpha)$, we have

$$\begin{aligned}
\operatorname{Hom}(F, (\omega^{\bullet}_{X_\bullet})_\alpha) &= \operatorname{Hom}(\mathbf{L}j_{\alpha!}F_\bullet, \omega^{\bullet}_{X_\bullet}) \\
&= \operatorname{Hom}(\mathbf{R}\Gamma(\mathbf{L}j_{\alpha!}F_\bullet), k) \\
&= h^0(\mathbf{R}\operatorname{Hom}(s(\mathbf{R}\Gamma(\mathbf{L}j_{\alpha!}F_\beta)), k)) \\
&= h^0(s^*\mathbf{R}\operatorname{Hom}(\mathbf{R}\Gamma(\mathbf{L}j_{\alpha!}F_\beta), k))
\end{aligned}$$

using theorem 2.3.5.

For $\beta \in \alpha/I$, $\mathbf{L}j_{\alpha!}F_\beta = \mathbf{L}X^*_{\phi_{\beta\alpha}}F$, so using Grothendieck duality

$$\begin{aligned}
\mathbf{R}\operatorname{Hom}(\mathbf{R}\Gamma(j_{\alpha!}F_\beta), k) &= \mathbf{R}\operatorname{Hom}(\mathbf{L}X^*_{\phi_{\beta\alpha}}F, \omega^{\bullet}_{X_\alpha}) \\
&= \mathbf{R}\operatorname{Hom}(F, \mathbf{R}X_{\phi_{\beta\alpha}*}\omega^{\bullet}_{X_\alpha}) \\
&= \mathbf{R}\operatorname{Hom}(F, \hat{\omega}^\alpha_\beta)
\end{aligned}$$

similarly, if $\beta \notin \alpha/I$, both $\mathbf{R}\operatorname{Hom}(\mathbf{R}\Gamma(j_{\alpha!}F_\beta), k)$ and $\operatorname{Hom}(F, \hat{\omega}^\alpha_\beta)$ are zero.

Thus, continuing from above,

$$\begin{aligned}
h^0(s^*\mathbf{R}\operatorname{Hom}(\mathbf{R}\Gamma(j_{\alpha!}F_\beta), k)) &= h^0(s^*\mathbf{R}\operatorname{Hom}(F \times \square_n, \hat{\omega}^\alpha_\bullet)) \\
&= \operatorname{Hom}(F, s^*(\hat{\omega}^\alpha_\bullet))
\end{aligned}$$

again using 2.3.5.

Finally, we are done by the Yoneda lemma. $\qquad\square$

3.2 Projection Formula and Duality

The critical step in our method of approaching Grothendieck duality is the conjugacy of the duality and the projection formula map, so that we can approach the complicated hom-sheaves of duality via the much simpler (at least on diagrams) tensor product.

Theorem 3.2.1. *Let $f_\bullet : X_\bullet \to Y_\bullet$ be a proper morphism of noetherian diagrams of schemes. Suppose that for $F_\bullet \in D_{lc}^-(X_\bullet)$, the projection formula natural transformation of functors $D_{lqc}(Y_\bullet) \to D_{lqc}(Y_\bullet)$*

$$- \otimes^{\mathbf{L}} \mathbf{R}f_{\bullet *}F_\bullet \to \mathbf{R}f_{\bullet *}(\mathbf{L}f_\bullet^* - \otimes^{\mathbf{L}} F_\bullet)$$

is an isomorphism. Then the Grothendieck duality natural transformation of functors $D_{lqc}^+(Y_\bullet) \to D_{lqc}^+(Y_\bullet)$

$$\mathbf{R}f_{\bullet *}\mathbf{R}\mathcal{H}om_{X_\bullet}(F_\bullet, f_\bullet^! -) \to \mathbf{R}\mathcal{H}om_{Y_\bullet}(\mathbf{R}f_{\bullet *}F_\bullet, -)$$

is an isomorphism.

Proof. By theorem 2.6.11, $\mathbf{R}f_{\bullet *}$ maps $D_c^-(X_\bullet)$ to $D_c^-(X_\bullet)$. Thus each of these is well-defined as functors to the indicated categories.

Next we note that if $E_\bullet \in D_{lqc}(X_\bullet)$, then for any $G_\bullet \in D_{lqc}^+(X_\bullet)$, the two transformations are conjugate and fit into the diagram

$$
\begin{array}{ccc}
\mathrm{Hom}(\mathbf{R}f_{\bullet *}(\mathbf{L}f_\bullet^* E_\bullet \otimes^{\mathbf{L}} F_\bullet), G_\bullet) & \longrightarrow & \mathrm{Hom}(E_\bullet \otimes^{\mathbf{L}} \mathbf{R}f_{\bullet *}F_\bullet, G_\bullet) \\
\downarrow{\wr} & & \downarrow{\wr} \\
\mathrm{Hom}(E_\bullet, \mathbf{R}f_{\bullet *}\mathbf{R}\mathcal{H}om_{X_\bullet}(F_\bullet, f_\bullet^! G_\bullet) & \longrightarrow & \mathrm{Hom}(E_\bullet, \mathbf{R}\mathcal{H}om_{Y_\bullet}(\mathbf{R}f_{\bullet *}F_\bullet, G_\bullet)).
\end{array}
$$

By hypothesis, the top arrow is an isomorphism, so as is the bottom arrow. Then by Yoneda's lemma, the Grothendieck duality transformation is an isomorphism. $\square$

With this established, we can focus on deriving the projection formula. The following is a cubical version of the flat base change theorem: the key to this is the commutation of the simple functor with a pullback, so from now on we will restrict to cubical morphisms. It is likely that this commutation holds on a wider class of shapes, however it certainly does not always hold as we will show in example 3.2.5.

Theorem 3.2.2 (Flat Base Change for Cubical Morphisms). *Let* $u_\bullet : U_\bullet \to X_\bullet$ *and* $f_\bullet :$
$Y_\bullet \to X_\bullet$ *be morphisms of diagrams of schemes, and* $V_\bullet = U_\bullet \times_{X_\bullet} Y_\bullet$, *with projections* $g_\bullet$ *to*
$U_\bullet$ *and* $v_\bullet$ *to* $Y_\bullet$. *Then there is a natural transformation of functors* $D_{lqc}(Y_\bullet) \to D_{lqc}(U_\bullet)$

$$\boldsymbol{L}u_\bullet^* \boldsymbol{R}f_{\bullet*} \to \boldsymbol{R}g_{\bullet*} \boldsymbol{L}v_\bullet^*,$$

and if $u_\bullet$ *is flat and* $f_\bullet$ *is cubical, this is an isomorphism.*

Proof. The natural map is the composition

$$\mathbf{L}u_\bullet^* \mathbf{R}f_{\bullet*} \to \mathbf{R}g_{\bullet*} \mathbf{L}g_\bullet^* \mathbf{L}u_\bullet^* \mathbf{R}f_{\bullet*}$$

$$\to \mathbf{R}g_{\bullet*} \mathbf{L}v_\bullet^* \mathbf{L}f_\bullet^* \mathbf{R}f_{\bullet*}$$

$$\to \mathbf{R}g_{\bullet*} \mathbf{L}v_\bullet^*.$$

Now suppose $F_\bullet \in D_{lqc}(Y_\bullet)$ and $u_\bullet$ is a functor over $\Phi : I \to J$. Then for each $\alpha \in I$, by definition,

$$(\mathbf{R}g_{\bullet*} \mathbf{L}v_\bullet^* F_\bullet)_\alpha = \mathbf{R} \varprojlim_{\beta \in \square_n} \mathbf{R}g_{(\alpha,\beta)*} \mathbf{L}v_{(\alpha,\beta)}^* F_\beta,$$

then by usual flat base change for schemes, this becomes

$$\mathbf{R} \varprojlim_{\beta \in \square_n} \mathbf{L}u_\alpha^* \mathbf{R}f_{\beta*} F_\beta.$$

By definition,

$$(\mathbf{L}u_\bullet^* \mathbf{R}f_{\bullet*} F_\bullet)_\alpha = \mathbf{L}u_\alpha^* \mathbf{R} \varprojlim_{\beta \in \square_n} \mathbf{R}f_{\beta*} F_\beta,$$

so we have reduced to showing that for a flat morphism of schemes $u : U \to X$, a $\square_n$-diagram
of complexes $F_\bullet$ on X, $\mathbf{L}u^* \mathbf{R} \varprojlim F_\bullet = \mathbf{R} \varprojlim \mathbf{L}u^*$.

By flatness, $\mathbf{L}u^* = u^*$, and by using the simple complex construction $\mathbf{R} \varprojlim F_\bullet$ becomes a
complex with each degree the direct sum of F_α^ps and differential maps the direct sum of maps
$F_\alpha \to F_\beta$. Since u^* commutes with all of these, $u^* \mathbf{R} \varprojlim F_\bullet$ is in fact the simple complex of
$u^* F_\bullet$. $\qquad\square$

Theorem 3.2.3 (Projection Formula for Cubical Morphisms). *Let $f_\bullet : X_\bullet \to Y_\bullet$ be a cubical morphism. Then for each $F_\bullet \in D_{lqc}(X_\bullet)$, the projection formula natural transformation of functors on $D_{lqc}(X_\bullet)$*

$$- \otimes^{\mathbf{L}} \boldsymbol{R}f_{\bullet *}F_\bullet \to \boldsymbol{R}f_{\bullet *}(\boldsymbol{L}f_\bullet^* - \otimes^{\mathbf{L}}F_\bullet)$$

is an isomorphism.

Proof. It is enough to show this over each component of $Y_\bullet$, and $\mathbf{R}f_{\bullet *}$ is computed independently on each component of Y via the stratification of $f_\bullet$, so we can assume Y is a scheme, and $f_\bullet : X_\bullet \to Y$

Then this morphism is local in Y using theorem 3.2.2, so we can reduce to the case where Y is affine. Here, $\mathcal{O}_Y$ is ample and generates the category $D_{qc}(Y)$, and since each functor is triangulated from $D_{qc}(Y)$ to $D_{qc}(Y)$, and preserves coproducts, it is in fact enough to show just for $\mathcal{O}_Y$, where it is trivial. $\square$

Corollary 3.2.4 (Grothendieck Duality for Cubical Morphisms). *If $f_\bullet : X_\bullet \to Y_\bullet$ is a cubical, proper morphism of noetherian diagrams of schemes, then for any $F_\bullet \in D_{lc}^-(X_\bullet)$, $G_\bullet \in D_{lqc}^+(X_\bullet)$, the Grothendieck duality map is an isomorphism:*

$$\boldsymbol{R}f_{\bullet *}\boldsymbol{R}\mathcal{H}om(F_\bullet, f_\bullet^! G_\bullet) \xrightarrow{\sim} \boldsymbol{R}\mathcal{H}om(\boldsymbol{R}f_{\bullet *}F_\bullet, G_\bullet).$$

Furthermore,

$$\boldsymbol{R}f_{\bullet *}\boldsymbol{R}\mathcal{H}om(F_\bullet, \omega_{X_\bullet}^\bullet) \xrightarrow{\sim} \boldsymbol{R}\mathcal{H}om(\boldsymbol{R}f_{\bullet *}F_\bullet, \omega_{Y_\bullet}^\bullet).$$

The following example exhibits that the shape of the diagrams is of critical importance for this result: these all fail for arbitrary diagrams.

Example 3.2.5. Let X be a scheme, and $Y_\bullet$ be the diagram $X \to X$ with the identity map (for convenience, labeled $X_1 \to X_2$). Then we have a morphism of diagrams $f_\bullet$ as follows:

$$X \xrightarrow{\mathrm{id}} X_1$$
$$\downarrow_{\mathrm{id}}$$
$$X_2.$$

We will construct a counterexample to the projection formula and Grothendieck duality for this morphism of diagrams. Let $\mathscr{F}_\bullet$ be the sheaf over $Y_\bullet$ which is 0 on X_1 and $\mathcal{O}_{X_2}$ on X_2. Now consider the natural morphism

$$\mathbf{R}f_{\bullet *}(\mathbf{L}f_\bullet^* \mathscr{F}_\bullet) \to \mathscr{F}_\bullet \otimes \mathbf{R}f_{\bullet *}\mathcal{O}_X$$

Using the definition of the pushforward, $(\mathbf{R}f_{\bullet *}\mathcal{O}_X)_1 = \mathbf{R}\varinjlim \mathcal{O}_{X_1} = \mathcal{O}_{X_1}$ and $(\mathbf{R}f_{\bullet *}\mathcal{O}_X)_2 = \mathbf{R}\varinjlim \mathcal{O}_{X_2} = \mathcal{O}_{X_2}$, so the right side is $\mathscr{F}_\bullet$. On the other hand, $\mathbf{L}f_\bullet^* \mathscr{F}_\bullet = \mathbf{L}\operatorname{id}^* 0 = 0$ so the left side is 0 and this is not a natural isomorphism.

Then we can do the same for the Grothendieck duality map

$$\mathbf{R}f_{\bullet *}\mathbf{R}\mathcal{H}om(\mathcal{O}_X, f_\bullet^! \mathscr{F}_\bullet) \to \mathbf{R}\mathcal{H}om(\mathbf{R}f_{\bullet *}\mathcal{O}_X, \mathscr{F}_\bullet) :$$

by the previous computation this is morphism $\mathbf{R}f_{\bullet *}f_\bullet^! \mathscr{F}_\bullet \to \mathscr{F}_\bullet$, and the right hand side gives the same sheaf when restricted to either X_1 or X_2, while $\mathscr{F}_\bullet$ is not the same on each.

Chapter 4

THE DU BOIS COMPLEX

Cubical hyperresolutions were defined to provide a simpler construction of the Du Bois complex, which is a singular version of the de Rham complex of smooth complex manifolds. Using this complex, Hodge-theoretic results can be extended somewhat to singular varieties.

4.1 Construction and Properties

If X is a smooth variety of dimension n over $\mathbb{C}$, then using analytic methods we can construct the de Rham complex

$$0 \to \mathcal{O}_X \to \Omega^1_X \to \cdots \to \Omega^n_X \to 0$$

with maps given by exterior differentiation.

This can be viewed as an object $\Omega^\bullet_X \in D(X)$ and has an associated filtration induced by truncation. Then $\mathrm{Gr}^p \Omega^\bullet_X[p] = \Omega^p_X$.

If X is any variety over $\mathbb{C}$, then by theorem 2.4.4, there is a hyperresolution $f_\bullet : X_\bullet \to X$ composed of smooth varieties over $\mathbb{C}$. The de Rham complex is functorial, so this gives an object $\Omega^\bullet_{X_\bullet}$ of the derived category of sheaves over $X_\bullet$, where the corresponding sheaf on X_α is the de Rham complex $\Omega^\bullet_{X_\alpha}$ for each α.

Definition 4.1.1. Let X be a variety over $\mathbb{C}$ with a hyperresolution $f_\bullet : X_\bullet \to X$. Then the *Du Bois complex* of X is an object in the derived category of sheaves over X defined by

$$\underline{\Omega}^\bullet_X = \mathbf{R}f_{\bullet*}\Omega^\bullet_{X_\bullet}$$

This has a filtration induced by the truncation filtrations on the de Rham complexes, so further define

$$\underline{\Omega}^p_X = \mathrm{Gr}^p \underline{\Omega}^\bullet_X[p] \simeq_{qis} \mathbf{R}f_{\bullet*}\Omega^p_{X_\bullet}$$

It should be noted that this theory can be developed for pairs, but we are choosing to restrict our attention to the simpler case.

The following is an omnibus theorem stating all the basic properties of the Du Bois complex which come from pushing forward the properties of the de Rham complexes. Importantly, this theorem establishes the Du Bois complex as intrinsic to X rather than the above definition which is dependent on the hyperresolution.

Theorem 4.1.2 ([Kol13, 6.5]). *Let X be a variety over $\mathbb{C}$. Then $\underline{\Omega}_X^\bullet \in D_{filt}(X)$ satisfies the following*

1. *The complex is quasi-isomorphic to the constant sheaf*

$$\underline{\Omega}_X^\bullet \simeq_{qis} \mathbb{C},$$

   and in particular there is an isomorphism for each i

$$\mathbb{H}^i(X, \underline{\Omega}_X^\bullet) \cong \mathrm{H}^i(X, \mathbb{C}).$$

2. *$\underline{\Omega}_X^\bullet$ is independent of hyperresolution: if $f_\bullet : X_\bullet \to X$ is any hyperresolution of X, then $\boldsymbol{R}f_{\bullet *}\Omega_X^\bullet$ is isomorphic to $\underline{\Omega}_X^\bullet$ as objects in $D_{filt}(X)$.*

3. *If X is smooth, then $\underline{\Omega}_X^\bullet$ is isomorphic to $\Omega_X^\bullet$ as objects in $D_{filt}(X)$.*

4. *If $\varphi : Y \to X$ is a morphism of varieties, then there exists a natural map of filtered complexes*

$$\varphi^* : \underline{\Omega}_X^\bullet \to \boldsymbol{R}\varphi_* \underline{\Omega}_Y^\bullet.$$

5. *If $U \subseteq X$ is an open subscheme of X, then*

$$\underline{\Omega}_X^\bullet|_U \simeq_{qis,filt} \underline{\Omega}_U^\bullet.$$

6. *Let $H \subseteq X$ be a general member of a base point free linear system. Then*

$$\underline{\Omega}_H^\bullet \simeq_{qis,filt} \underline{\Omega}_X^\bullet \otimes^{\boldsymbol{L}} \mathcal{O}_H.$$

7. *There is a natural map $\mathcal{O}_X \to \underline{\Omega}^0_X$ compatible with the maps above.*

8. *If X is proper, there is a spectral sequence degenerating at E_1 and computing the singular cohomology with compact support of X*

$$E_1^{pq} = \mathbb{H}^q(X, \underline{\Omega}^p_X) \Rightarrow \mathrm{H}_c^{p+q}(X, \mathbb{C}).$$

9. *Suppose $\pi : \tilde{X} \to X$ is projective and birational, an isomorphism outside of the reduced closed subscheme $\Sigma \subseteq X$. Let $E = \pi^{-1}(\Sigma)_{red}$. Then there is a distinguished triangle*

$$\underline{\Omega}^\bullet_X \to \underline{\Omega}^\bullet_\Sigma \oplus \boldsymbol{R}\pi_*\underline{\Omega}^\bullet_{\tilde{X}} \to \boldsymbol{R}\pi_*\underline{\Omega}^\bullet_E \xrightarrow{+1},$$

where the first arrow is the sum of the maps from functoriality, and the second arrow is the difference. In addition, for each p there is a distinguished triangle

$$\underline{\Omega}^p_X \to \underline{\Omega}^p_\Sigma \oplus \boldsymbol{R}\pi_*\underline{\Omega}^p_{\tilde{X}} \to \boldsymbol{R}\pi_*\underline{\Omega}^p_E \xrightarrow{+1},$$

Note that because of (1), the complex $\underline{\Omega}^\bullet_X \in D(X)$ is neither new nor interesting. The important part is the *filtration*. Still, in general, these properties of $\underline{\Omega}^\bullet_X$ may not be helpful if we have no way to compute the actual sheaves involved. This is where Du Bois singularities come in, as in many cases we care about, we can connect these sheaves with those we already know.

4.2 Du Bois Singularities

Definition 4.2.1. We say a variety X over $\mathbb{C}$ has *Du Bois singularities* (or is DB) if the natural map in (7) is a quasi-isomorphism. Equivalently by working through the definitions, X has Du Bois singularities if for a hyperresolution $f_\bullet : X_\bullet \to X$, the natural map $\mathcal{O}_X \to \mathbf{R}f_{\bullet*}\mathcal{O}_{X_\bullet}$ is a quasi-isomorphism.

While the first interpretation is useful as many of the nice properties are a result of the connection with the Du Bois complex, this second interpretation provides a strong analogy with rational singularities by simply exchanging a hyperresolution with a resolution of

singularities. In fact we have the following theorem, showing Du Bois singularities are a generalization of well-known classes of singularities.

Theorem 4.2.2 ([Kol13, 6.25 and 6.32])**.** *If X is a complex variety with rational singularities or semi-log canonical singularities, then X has Du Bois singularities.*

An alternative characterization of Du Bois singularities is given below.

Theorem 4.2.3 ([Sch07])**.** *Let X be a reduced separated scheme of finite type over a field of characteristic zero. Suppose $X \subseteq Y$ where Y is smooth and suppose that $\pi : \tilde{Y} \to Y$ is a log resolution of X in Y, an isomorphism outside of X. If E is the reduced preimage of X in $\tilde{Y}$, then $\boldsymbol{R}\pi_*\mathcal{O}_E \simeq_{qis} \underline{\Omega}_X^0$ and in particular X has Du Bois singularities if and only if the natural map $\mathcal{O}_X \to \boldsymbol{R}\pi_*\mathcal{O}_E$ is a quasi-isomorphism.*

As an example of the use of this condition, here are two results well-known for smooth complex varieties which apply in the Du Bois setting:

Proposition 4.2.4. *Let X be a complex variety with Du Bois singularities. Then,*

1. *The natural morphism $H^i(X, \mathbb{C}) \to H^i(X, \mathcal{O}_X)$ is a surjection for each i.*

2. *If X is Cohen-Macaulay, than for every big and nef line bundle $\mathcal{L}$ on X, $H^i(X, \mathcal{L}^{-1}) = 0$ for $i < \dim X$, or equivalently, $H^i(X, \mathcal{L} \otimes \omega_X) = 0$ for $i > 0$.*

Proof. The first is a result of the spectral sequence in 8. The second follows from the proof of [KSS10, 6.6]. $\qquad\qquad\square$

4.3   Duality and Hyperresolutions

Now we turn to applying the results of chapter 3 to hyperresolutions, in particular we can find a dual version of the definition of Du Bois singularities using the pushforward of the canonical complex.

Proposition 4.3.1. *Let* $\pi_\bullet : X_\bullet \to X$ *be a cubical hyperresolution. Then* $\boldsymbol{R}\pi_{\bullet*}\omega_{X_\bullet}^\bullet = \boldsymbol{R}\mathcal{H}om(\underline{\Omega}_X^0, \omega_X^\bullet)$ *and this is independent of choice of hyperresolution, denoted* $\underline{\omega}_X^\bullet$, *(agreeing with the use of this notation in [KS11]).*

Proof. The equality of the two complexes is a direct result of Grothendieck duality. Since the latter complex is independent of choice of hyperresolution owing to theorem 4.1.2, the former is as well. □

Corollary 4.3.2 (c.f. [KS11]). *Let* X *be a reduced scheme, finite-type over* $\mathbb{C}$, *and let* $f_\bullet : X_\bullet \to X$ *be a cubic hyperresolution. Then* X *has Du Bois singularities if and only if the natural map*

$$\underline{\omega}_X^\bullet \xrightarrow{\sim} \omega_X^\bullet$$

is a quasi-isomorphism.

Proof. Suppose X has Du Bois singularities, so that the natural map

$$\mathcal{O}_X \xrightarrow{\sim} \mathbf{R}f_{\bullet*}\mathcal{O}_{X_\bullet}$$

is a quasi-isomorphism. Then applying the functor $\mathbf{R}\mathcal{H}om(-, \omega_X^\bullet)$, and using corollary 3.2.4,

$$\omega_X^\bullet \xleftarrow{\sim} \mathbf{R}\mathcal{H}om(\mathbf{R}f_{\bullet*}\mathcal{O}_{X_\bullet}, \omega_X^\bullet)$$
$$\xleftarrow{\sim} \mathbf{R}f_{\bullet*}\mathbf{R}\mathcal{H}om(\mathcal{O}_{X_\bullet}, \omega_{X_\bullet}^\bullet)$$
$$\xleftarrow{\sim} \mathbf{R}f_{\bullet*}\omega_{X_\bullet}^\bullet$$

Now assume this map is an isomorphism. Applying the same functor and using the fact that $\omega_X^\bullet$ is dualizing,

$$\mathbf{R}\mathcal{H}om(\omega_X^\bullet, \omega_X^\bullet) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om(\mathbf{R}f_{\bullet*}\omega_{X_\bullet}^\bullet, \omega_X^\bullet)$$
$$\xrightarrow{\sim} \mathbf{R}\mathcal{H}om(\mathbf{R}f_{\bullet*}\mathbf{R}\mathcal{H}om(\mathcal{O}_{X_\bullet}, \omega_{X_\bullet}^\bullet), \omega_X^\bullet)$$
$$\xrightarrow{\sim} \mathbf{R}\mathcal{H}om(\mathbf{R}\mathcal{H}om(\mathbf{R}f_{\bullet*}\mathcal{O}_{X_\bullet}, \omega_X^\bullet), \omega_X^\bullet)$$
$$\mathcal{O}_X \xrightarrow{\sim} \mathbf{R}f_{\bullet*}\mathcal{O}_{X_\bullet}$$

□

Corollary 4.3.3. *Suppose $\pi_\bullet : X_\bullet \to X$ is a cubical hyperresolution. Then X has Du Bois singularities if and only if the unit of adjunction*

$$\mathrm{id} \to \boldsymbol{R}\pi_{\bullet *}\boldsymbol{L}\pi_\bullet^*$$

is a natural isomorphism. Dually, X has Du Bois singularities if and only if the counit of adjunction

$$\boldsymbol{R}\pi_{\bullet *}\pi_\bullet^! \to \mathrm{id}$$

is a natural isomorphism.

Proof. By the projection formula, $\mathbf{R}\pi_{\bullet *}\mathbf{L}\pi_\bullet^* F \simeq_{qis} F \otimes^{\mathbf{L}} \mathbf{R}\pi_{\bullet *}\mathcal{O}_{X_\bullet}$ for each F. The morphism $F \to F \otimes^{\mathbf{L}} \mathbf{R}\pi_{\bullet *}\mathcal{O}_{X_\bullet}$ is an isomorphism for all F if and only if X is Du Bois.

The second part is then conjugate to the first. $\qquad\square$

The following are some useful vanishing properties of this complex.

Theorem 4.3.4 ([KS11, 3.3, 3.5]). *Let X be a complex variety of dimension n. Then the natural map $h^j(\underline{\omega}_X^\bullet) \to h^j(\omega_X^\bullet)$ is an injection, and if X is Cohen-Macaulay then*

$$\underline{\omega}_X^\bullet \simeq_{qis} h^{-n}(\underline{\omega}_x^\bullet)[n].$$

Lemma 4.3.5 (proof of [KS11, 3.3]). *Let X be a variety over $\mathbb{C}$, $\mathcal{L}$ an ample line bundle. Then for all $j \in \mathbb{Z}$, the natural map $\mathbb{H}^j(X, \underline{\omega}_X^\bullet \otimes \mathcal{L}) \hookrightarrow \mathbb{H}^j(X, \omega_X^\bullet \otimes \mathcal{L})$ is an injection.*

Lemma 4.3.6. *Let X be a Cohen-Macaulay variety of dimension n, and $\Sigma_d \subseteq X$ the smallest closed set such that $X \setminus \Sigma_d$ is Du Bois. Then $h^i(\underline{\Omega}_X^0) = 0$ for $0 < i < n - \dim \Sigma_d - 1$.*

Proof. This is a Du Bois version of [Kov99, Lemma 3.3], the proof is identical in method.

Using 4.3.4, we have an exact sequence

$$0 \to h^{-n}(\underline{\omega}_X^\bullet) \to \omega_X \to \mathscr{Q} \to 0$$

for some sheaf $\mathscr{Q}$. Now apply $\mathbf{R}\mathcal{H}om(-, \omega_X)$ and use Grothendieck duality to arrive at the distinguished triangle

$$\mathbf{R}\mathcal{H}om(\mathscr{Q}, \omega_X) \to \mathcal{O}_X \to \underline{\Omega}_X^0 \xrightarrow{+1},$$

which realizes $\mathbf{R}\mathcal{H}om(\mathcal{Q}, \omega_X) \simeq_{qis} \underline{\Omega}_X^{\times}[-1]$ (c.f. [Kol13, 6.12]). Taking the cohomology of the complexes involved, since $h^i(\mathcal{O}_X) = 0$ for $i \neq 0$, $h^i(\underline{\Omega}_X^0) \cong \mathscr{E}xt^{i+1}(\mathcal{Q}, \omega_X)$ for $i > 0$.

For any closed $x \in X$, $\mathcal{O}_{X,x}$ is a local Cohen-Macaulay ring of dimension n, and $h^i(\underline{\Omega}_X^0)_x \cong$ $\mathrm{Ext}^{i+1}(\mathcal{Q}_x, \omega_{\mathcal{O}_{X,x}})$ which is zero for $1 < i + 1 < n - \dim \mathcal{Q}_x$ by [BH93, 3.5.11]. Since the support of $\mathcal{Q}$ is contained in Σ_d, we are done. $\square$

Chapter 5

AN APPLICATION OF DUALITY

5.1 Cohomology of Contractions

In the case that the singularities on X are isolated, we can understand the cohomology on X of somewhat mysterious complexes by using the cohomology on the fiber of the blow-up. The next theorem encapsulates this to be used in later computations.

Theorem 5.1.1. *Let X be a proper, reduced $\mathbb{C}$-scheme of dimension n, Y smooth, with $f : Y \to X$ a proper morphism which is an isomorphism outside of a closed $p \in X$. Then for $0 \le i \le n$ and any big and nef line bundle $\mathcal{L}$ on X, there are isomorphisms*

$$
H^i(X, \mathcal{L}^{-1} \otimes \underline{\Omega}^0_X) = \begin{cases} 0 & \text{if } i = 0 \\ H^0(Y_p, \mathbb{C})/\mathbb{C} & \text{if } i = 1 \\ H^{i-1}(Y_p, \underline{\Omega}^0_{Y_p}) & \text{if } 2 \le i \le n-1 \\ H^n(Y, f^*\mathcal{L}^{-1}) \oplus H^{n-1}(Y_p, \underline{\Omega}^0_{Y_p}) & \text{if } i = n \end{cases}
$$

where Y_p is the fiber of f over p.

Proof. First, we set up the notation: by construction, X has the 2-resolution

$$
\begin{array}{ccc}
Y_p & \longrightarrow & Y \\
\downarrow & & \downarrow \\
\{p\} & \longrightarrow & X,
\end{array}
$$

which we'll denote $f_\bullet : Y_\bullet \to X$ and by definition, if $Z_\bullet$ is a hyperresolution of Y_p, $\{p\}_\bullet$ denotes the diagram of schemes of the same shape as $Z_\bullet$ but with each component p, then

the diagram of schemes

$$
\begin{array}{ccc}
Z_\bullet & \longrightarrow & Y \\
\downarrow & & \downarrow \\
\{p\}_\bullet & \longrightarrow & X
\end{array}
$$

which we'll denote $\pi_\bullet : X_\bullet \to X$ (note that we are thinking of this as a diagram of schemes not a diagram of diagrams, writing $Z_\bullet$ is just notational convenience).

The hyperresolution $\pi_\bullet$ factors through $f_\bullet$ via a $g_\bullet$ mapping $Z_\bullet \to Y_p$, $\{p\}_\bullet \to \{p\}$ and $Y \to Y$.

Using the projection formula, $H^i(X, \mathcal{L}^{-1} \otimes \underline{\Omega}_X^0)$ becomes $H^i(X_\bullet, \pi_\bullet^* \mathcal{L}^{-1})$, but this is $H^i(Y_\bullet, \mathbf{R}g_{\bullet *}\pi_\bullet^* \mathcal{L}^{-1})$, so we can first find $\mathbf{R}g_{\bullet *}\pi_\bullet^* \mathcal{L}^{-1}$ and then compute its cohomology.

Since all maps except f in the diagram factor through the point p, $\pi_\bullet^* \mathcal{L}^{-1}$ is $\mathcal{O}$ on all components of $X_\bullet$ except Y. Now since we can write $g_\bullet$ stratified over the projection of $\square_m \leftarrow \square_m \to \underline{1}$ to $\underline{1} \leftarrow \underline{1} \to \underline{1}$, [GNAPGP88, I 5.14] allows $\mathbf{R}g_{\bullet *}$ to computed separately over the components of $Y_\bullet$.

Thus, $\mathbf{R}g_{\bullet *}\pi_\bullet^* \mathcal{L}^{-1}|_Y = f^* \mathcal{L}^{-1}$ since Y itself is the only scheme in $X_\bullet$ over Y in $Y_\bullet$. $\pi_\bullet^* \mathcal{L}^{-1}|_{Z_\bullet} = \mathcal{O}_{Z_\bullet}$ as above and by definition, the pushforward is $\underline{\Omega}_{Y_p}^0$. Since $\{p\}_\bullet \to \{p\}$ is a hyperresolution of a smooth scheme, we know that over $\square_m$ $s(\mathbb{C}) = \mathbb{C}$, and thus the pushforward is $\mathbb{C}$ on $\{p\}$. Let $\mathscr{F}_\bullet$ denote this complex on $Y_\bullet$. To compute this, we use the spectral sequence $E_1^{pq} = \bigoplus_{|\alpha|=p+1} H^q(Y_\alpha, \mathscr{F}_\alpha)$, and the facts that $H^i(\{p\}, \mathbb{C}) = 0$ for $i > 0$ by Grothendieck vanishing, $H^i(Y_p, \underline{\Omega}_{Y_p}^0)$ for $i > n - 1$ by 4.1.2 (8), and $H^i(Y, f^* \mathcal{L}^{-1}) = 0$ for $i < n$ by Kawamata-Viehweg. All together, the E_1 page is

$$
\begin{array}{ccc}
H^n(Y, f^* \mathcal{L}^{-1}) & \longrightarrow & 0 \\[2ex]
0 & \longrightarrow & H^{n-1}(Y_p, \underline{\Omega}_{Y_p}^0) \\[2ex]
\vdots & & \vdots \\[2ex]
0 & \longrightarrow & H^1(Y_p, \underline{\Omega}_{Y_p}^0) \\[2ex]
\mathbb{C} & \longrightarrow & H^0(Y_p, \underline{\Omega}_{Y_p}).
\end{array}
$$

Then the results for $i > 1$ follow from this degenerate sequence, and $i = 0$ is obtained by using 4.1.2 (8) to see $H^0(Y_p, \underline{\Omega}_{Y_p}) = H^0(Y_p, \mathbb{C})$. $\qquad\square$

With some assumptions on X this theorem provides cohomological constraints on the possible exceptional locus of a resolution f.

Corollary 5.1.2. *Let X be a proper, reduced, Cohen-Macaulay $\mathbb{C}$-scheme of dimension n with Du Bois singularities, Y smooth, with $f : Y \to X$ a proper morphism which is as isomorphism outside of a closed $p \in X$. Then, for $0 \leq i \leq n - 2$, the exceptional locus Y_p satisfies*

$$H^i(Y_p, \underline{\Omega}^0_{Y_p}) = \begin{cases} \mathbb{C} & \text{if } i = 0 \\ 0 & \text{if } 1 \leq i \leq n - 2. \end{cases}$$

In particular, Y_p is connected and if Y_p has Du Bois singularities $H^i(Y_p, \mathcal{O}_{Y_p}) = 0$ for $0 < i < n - 1$.

Proof. Under these hypotheses, for any big and nef $\mathcal{L}$ on X, $H^i(X, \mathcal{L}^{-1} \otimes \underline{\Omega}^0_X) = H^i(X, \mathcal{L}^{-1}) = 0$ for $i < n$ since Kodaira vanishing holds for X. The results follow by utilizing theorem 5.1.1. $\qquad\square$

Remark 5.1.3. The hypotheses imply that X is normal, so the connectedness portion of this just gives an alternate proof of Zariski's main theorem in this case. Furthermore, there appear to be parallels between this result and the well-known result that in this setup, $\mathbf{R}f_*\mathcal{O}_Y = 0$ for $0 < i < n - 1$ (c.f. [Kov99, Lemma 3.3]).

The Cohen-Macaulay hypothesis is necessary here since two planes intersecting at a point will provide a counterexample to the connectedness statement, and furthermore one can calculate that $h^i(X, \mathcal{L}^{-1} \otimes \underline{\Omega}^0_X) = 1$ for every $\mathcal{L}$.

Another result of our calculation is that we have an analogue of Kodaira vanishing on cones without assuming Du Bois.

Corollary 5.1.4. *Let Z be a smooth projective complex variety of dimension $n - 1$, X the projective cone over Z, and Y the blow-up of X at the cone point p. Assume either that Z*

is connected and $H^i(Z, \mathcal{O}_Z) = 0$ for $0 < i < n-2$ or X is Cohen-Macaulay. Then for any $\mathcal{L}$ big an nef, $H^i(X, \mathcal{L}^{-1} \otimes \underline{\Omega}_X^0) = 0$ for $i < n$, or equivalently $H^i(X, \mathcal{L} \otimes \underline{\omega}_X^{\bullet}) = 0$ for $i > -n$.

Proof. Note that X being Cohen-Macaulay implies that Z is connected and $H^i(Z, \mathcal{O}_Z) = 0$ by Hartshorne's connectedness theorem and [Pat13, Lemma 4.3].

With this condition, we simply compute based on theorem 5.1.1. $\square$

This provides numerous examples of varieties which are not Du Bois but have this vanishing property, for instance any cone over a smooth, connected plane curve, only those of degree at most 3 are Du Bois. This also suggests further examples of the failure of 5.1.2, for instance take a cone over a surface ruled over a curve of genus $g \geq 1$, this has non-zero h^1 and thus this vanishing does not hold.

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