Relatively Prime, Relatively Often

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17 April 2018

This document originally appeared as a blog post on my website. Find it at gautammanohar.com/coprime-prob.

Say you choose two integers. What's the chance that they're coprime? My intuition says that it can't be more than one-half, because half of all numbers share a factor of 2. But I'm thoroughly wrong here. The probability is actually $\frac{6}{\pi^2}$, which is about 61%.

We wish to find the probability that two randomly selected integers m and n are coprime. If this is true, then $\gcd(m,n) = 1$. Let's say that $p = \mathrm{P}(\gcd(m,n) = 1)$.

Consider $p_k = \mathrm{P}(\gcd(a,b) = k)$ for some positive integer k. For this to be true, k must divide both a and b, and $\frac{a}{k}$ and $\frac{b}{k}$ must be coprime. And so $p_k = \mathrm{P}(k|a \,\text{and}\, k|b) \cdot \mathrm{P}\left(\gcd\left(\frac{a}{k}, \frac{b}{k}\right)\right)$. One in every k integers is divisible by k. So the first probability, namely that two random integers are both divisible by k, is $\frac{1}{k^2}$. The second probability is just p—because a and b were random, so are $\frac{a}{k}$ and $\frac{b}{k}$. And so $p_k = \frac{p}{k^2}$

Two numbers must have some greatest common divisor. And so the sum of p_k over all k must be 1. That is,

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{p}{k^2} &= 1 \\ p &= \frac{1}{\sum\limits_{k=1}^{\infty} \frac{1}{k^2}}. \end{aligned} \tag{1}$$

The denominator is famously equal to $\frac{\pi^2}{6}$, as shown by Euler. And so we conclude that $p = \frac{6}{\pi^2}$.