Problem 5 - Smallest Multiple

Gautam Manohar

10 June 2018

This document originally appeared as a blog post on my website. Find it at gautammanohar.com/euler/5.

1 Problem Statement

2520 is the smallest number that can be divided by each of the numbers from 1 to 10 without any remainder.

What is the smallest positive number that is evenly divisible by all of the numbers from 1 to N?

2 My Algorithm

For a number a to divide another number b, each of the prime powers that divide a must also divide b. That is, the exponent on each of the prime powers that divide a is less than or equal to the corresponding exponent on a prime power that divides b. For each number from 1 to N to divide a number M, each prime power less than N must also divide M. To minimize M, we have each prime $p \leq N$ divide M as many times as it does the largest prime power of p at most N. That is

$$M = \prod_{p \leq N} p^{\lfloor \log_p N \rfloor}. \tag{1}$$

To compute our answer M, we need a list of the primes less than N, for which we can use the Sieve of Eratosthenes. And so our solution has time complexity $O(n \log \log n)$.

2.1 Other Solutions

This problem can also be phrased as finding the lowest common multiple of $1, \ldots, N$, for which the classic formula is $\operatorname{lcm}(1, \ldots, N) = \frac{N!}{\gcd(1,\ldots,N)}$. Because the numerator grows very quickly, we can compute the LCM iteratively, making use of the fact that $\mathrm{lcm}(a, b, c) = \mathrm{lcm}(\mathrm{lcm}(a, b), c)$ and storing the latest LCM with each step. This algorithm computes the greatest common denominator of two numbers at most N, which can be done in $O(\log n)$ time, N times. And so this solution has time complexity $O(n \log n)$. However, it has much better space complexity, at $O(1)$, than the sieve of Eratosthenes, which needs $O(n)$ space to store an array of size n. So for large n, this alternative solution is preferred.