Problem 28 - Number Spiral Diagonals

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1 Problem Statement

Starting with the number 1 and moving to the right in a clockwise direction, a 5-by-5 spiral is formed:

It can be verified that the sum of the numbers on the diagonals is 101. What is the sum of the numbers on the diagonals in an $N \times N$ spiral (N odd)? Report your answer modulo $10^9 + 7$.

2 My Algorithm

We can think of the spiral as being made up of a central 1 surrounded by n rings of side length $2n + 1$, where $n = \frac{N-1}{2}$. Let us derive a formula for the sum of the numbers in the corners of each ring—these make up the diagonals of the spiral. The number in the top-right corner of the n-th ring is $(2n + 1)^2$, the area of the ring. Going counter-clockwise, the number decreases by one less than the side length, or $2n + 1 - 1 = 2n$, each time. And so the n-th ring has corner sum

$$4(2n + 1)^2 - 6(2n) = 16n^2 + 4n + 4. \tag{1}$$

To find the sum of the corners of all rings, we can use the formula for the sum of the first n natural numbers

$$\sum_{i=0}^{n} i = \frac{n(n + 1)}{2} \tag{2}$$

and their squares:

$$\sum_{i=0}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6}. \tag{3}$$

To get the sum of the diagonals, we just add 1 to the sum of the corners, to account for the central 1.

And so our desired answer S is

$$
\begin{aligned}
S &= 1 + \sum_{i=0}^{\frac{N-1}{2}} (16n^2 + 4n + 4) \\
&= 1 + 16 \cdot \frac{\frac{N-1}{2}\left(\frac{N-1}{2}+1\right)\left(2 \cdot \frac{N-1}{2}+1\right)}{6} + 4 \cdot \frac{\frac{N-1}{2}\left(\frac{N-1}{2}+1\right)}{2} + 4\frac{N-1}{2} \\
&= 1 + 4 \cdot \frac{(N-1)(N-1+2)(N-1+1)}{6} + \frac{(N-1)(N-1+2)}{2} + 2(N-1) \\
&= 1 + \frac{2N(N-1)(N+1)}{3} + \frac{(N-1)(N+1)}{2} + 2(N-1) \\
&= \frac{4N(N-1)(N+1) + 3(N-1)(N+1) + 12(N-1) + 6}{6} \\
&= \frac{4N^3 + 3N^2 + 8N - 9}{6}.
\end{aligned}
\tag{4}
$$

We report our answer modulo $10^9 + 7$. Because there is a division by 6 in our answer, we must use the multiplicative modular inverse of 6 modulo $10^9 + 7$, which we find in $O(\log P)$ time. And so our solution has time complexity $O(1)$.