Problem 71 - Ordered Fractions

Gautam Manohar

21 July 2018

This document originally appeared as a blog post on my website. Find it at gautammanohar.com/euler/71.

1 Problem Statement

The fraction $\frac{a}{b}$ is called a reduced proper fraction if a and b are positive integers with $a < b$ and $\gcd(a, b) = 1$. If we list the reduced proper fractions for $b \leq 8$ in increasing order, we get

$$\frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}$$

It can be seen that $\frac{2}{5}$ is the fraction immediately to the left of $\frac{3}{7}$.

By listing the set of reduced proper fractions with denominator at most N in increasing order, find the numerator and denominator of the fraction immediately to the left of $\frac{a}{b}$.

2 My Algorithm

Let the sequence of reduced proper fractions with denominator at most n listed in increasing order be the n-th Farey sequence, F_n. We will show that if $\frac{c}{d}$ directly precedes $\frac{a}{b}$ in F_n, then $ad - bc = 1$.

Because $\gcd(a, b) = 1$, the equation

$$ay - bx = 1 \tag{1}$$

has infinitely many integer solutions, by Bezout's theorem. In particular, if (x_0, y_0) is a solution, so is $(x_0 + ak, y_0 + bk)$. Let us choose k such that $n - b < y_0 + br \leq n$. And so there exists a solution (x, y) such that

$$0 \leq n - b < y \leq n. \tag{2}$$

Suppose $\gcd(x, y) = d$. Because $d|x, d|y$, we have $d|(ay - bx)$, and so $d|1$. This implies that x and y are coprime. Because of this and the fact that $y \leq n$, we have $\frac{x}{y} \in F_n$. Furthermore,

$$\frac{a}{b} > \frac{a}{b} - \frac{1}{by} = \frac{ay-1}{by} = \frac{bx}{by} = \frac{x}{y}. \tag{3}$$

And so $\frac{x}{y}$ precedes $\frac{a}{b}$.

Suppose $(x, y) \neq (c, d)$. Then $\frac{x}{y}$ precedes $\frac{c}{d}$, and

$$\frac{c}{d} - \frac{x}{y} = \frac{cy - dx}{dy} \geq \frac{1}{dy}. \tag{4}$$

On the other hand,

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd} \geq \frac{1}{bd}. \tag{5}$$

Applying (1) gives

$$\frac{1}{xy} = \frac{ay - bx}{by} = \frac{a}{b} - \frac{x}{y}. \tag{6}$$

Applying the sum of (4) and (5) to (6) gives

$$\begin{aligned} \frac{a}{b} - \frac{x}{y} &\geq \frac{1}{dy} + \frac{1}{bd} \\ &= \frac{b+y}{bdy}. \end{aligned} \tag{7}$$

By (2) and the above, we thus have $\frac{1}{by} > \frac{n}{bdy}$. Because $d < n$, we obtain the contradiction $\frac{1}{by} > \frac{1}{by}$. Thus $(x, y) = (c, d)$. This proves the result that $ad - bc = 1$.

Given a, b, we must solve for $ad - bc = 1$ such that d, the denominator, is maximized. We can do this using the extended Euclidean algorithm. Suppose k is the multiplicative inverse of a modulo b. Then $ak \equiv 1 \pmod{b}$. This means $ak - bj = 1$, for some positive integer j. We wish to maximize k. Given that (k, j) is a solution, so is $(k + nb, j + na)$. Thus we set

$$\begin{aligned} k + nb &\leq N \\ n &\leq \frac{N-k}{b} \\ n &= \left\lfloor \frac{N-k}{b} \right\rfloor. \end{aligned} \tag{8}$$

Solving for j, we get $j = \frac{a\left\lfloor \frac{N-k}{b} \right\rfloor + ak - 1}{b}$. solution is

$$(c, d) = \left(\left\lfloor \frac{ab\left\lfloor \frac{N-k}{b} \right\rfloor + ak - 1}{b} \right\rfloor, b\left\lfloor \frac{N-k}{b} \right\rfloor + k \right). \tag{9}$$

The only expensive part of this procedure is calculating the modular inverse, and so our solution has $O(\log b)$ time complexity.