Problem 57 - Square Root Convergents

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This document originally appeared as a blog post on my website. Find it at gautammanohar.com/euler/57.

1 Problem Statement

It is possible to show that $\sqrt{2}$ has the following infinite continued fraction expansion:

$$\sqrt{2} = 1 + \cfrac{1}{2 + \frac{1}{2 + \frac{1}{2 + \cdots}}}$$

By expanding this for the first four iterations, we get the convergents of $\sqrt{2}$

$$\begin{aligned} 1 + \frac{1}{2} &= \frac{3}{2} \\ 1 + \frac{1}{2 + \frac{1}{2}} &= \frac{7}{5} \\ 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}} &= \frac{17}{12} \\ 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}} &= \frac{41}{29}. \end{aligned}$$

The next three convergents are $\frac{99}{70}$, $\frac{239}{169}$, and $\frac{577}{408}$. However, the eighth convergent, $\frac{1393}{985}$, is the first for which the number of digits in the numerator exceeds the number of digits in the denominator.

Print the convergent numbers $n \leq N$ where this happens.

2 My Algorithm

A common piece of notation for a continued fraction

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}} \tag{1}$$

is $[a_0; a_1, a_2, \ldots]$.

Because all the coefficients $a_i, i > 1$ in our case are 2, the continued fraction expansion for $\sqrt{2}$ is self-similar. In particular, if one convergent is $\frac{p}{q}$, we can write the next as

$$1 + \frac{1}{1 + \frac{p}{q}} = 1 + \frac{q}{p + q} = \frac{p + 2q}{p + q}. \tag{2}$$

Using this recurrence relation, we can easily compute the convergents and count which ones have a numerator with more digits than the denominator. This solution has time complexity $O(N)$.