Problem 14 - Longest Collatz Sequence

Gautam Manohar

10 June 2018

This document originally appeared as a blog post on my website. Find it at

1   Problem Statement

The following iterative sequence is defined for the set of positive integers:

$$\begin{cases} n \to \frac{n}{2} & n \text{ even}, \\ n \to 3n+1 & n \text{ odd} \end{cases}$$

Using the rule above and starting with 13 generates the following sequence:

$$13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1 \tag{1}$$

This sequence contains 10 terms. It is conjectured that every starting number produces a sequence that ends in 1.

Which starting number at most N produces the longest chain? If there are many possible such numbers, print the maximum one.

2   My Algorithm

This problem is perfect for caching (or memoization) and recursion. We define a recursive function that returns the length of a Collatz chain with a given starting number. The function also store the chain length for each starting number (and, because the function is recursive, each number in the chain) in a list so that we do not have to compute it again. Because of memory constraints, we do not store chain lengths for numbers greater than the maximum given N. Instead, we compute these values on the fly.

We can then iterate over all possible starting numbers and keep a list of starting numbers which produce a right-maximal chain length (that is, they produce a

chain length greater than or equal to all lenghts produced by smaller starting numbers). Finally, we perform a binary search for the greatest element in this list less than each query. This solution has time complexity $O(M + T \log M)$, where M is the maximum possible starting value and T is the number of queries.

2.1 Other Solutions

Solutions which do not use memoization are too slow.