Problem 21 - Amicable Numbers

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1 Problem Statement

Let $d(n)$ be defined as the sum of proper divisors of n (numbers less than n that divide evenly into n). If $d(a) = b$ and $d(b) = a$, where $a \neq b$, then a and b are an amicable pair and each of a and b are amicable numbers.

For example, the proper divisors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55 and 110; therefore $d(220) = 284$. The proper divisors of 284 are 1, 2, 4, 71 and 142; so $d(284) = 220$.

Evaluate the sum of the proper divisors under N.

2 My Algorithm

The first thing we need is a way to calculate $d(n)$. A brute force method that tests all numbers from 1 to n and checks whether they divide n is too slow. But if a number d divides n, then $\frac{n}{d}$ also divides n. In particular, one of n and $\frac{n}{d}$ is less or equal than $\sqrt{n}$, and the other is greater than or equal to $\sqrt{n}$. To see this, consider what would happen if both were smaller than $\sqrt{n}$. Their product would be less than n; a similar contradiction arises if both were greater than $\sqrt{n}$.

And so we only need to check each number less than $\sqrt{n}$. If it divides n, we add it to the sum that we will return. We therefore find $d(n)$ in $O(\sqrt{n})$ time.

Then we can use preprocessing to answer each query in $O(1)$ time. For each $1 \leq x \leq N_{\max}$, if $x = d(d(x))$ and $x \neq d(x)$, then we add x to a list of amicable numbers. To answer each query, we simply add the amicable numbers less than the given N. Our solution has time complexity $O(N\sqrt{N} + T)$, where N is the maximum possible input value and T is the number of queries.