Problem 69 - Totient Maximum

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1 Problem Statement

Euler's totient function, $\varphi(n)$, counts the number of positive integers less than n that are coprime to n.

n	Coprime	$\varphi(n)$	$\frac{n}{\varphi(n)}$
2	1	1	2
3	1,2	2	1.5
4	1,3	2	2
5	1,2,3,4	4	1.25
6	1,5	2	3
7	1,2,3,4,5,6	6	1.16
8	1,3,5,7	4	2
9	1,2,4,5,7,8	6	1.5
10	1,3,7,9	4	2.5

Figure 1. Note that 6 produces a maximum for the desired ratio in the range of this table.

Given N, find the smallest value n such that $\frac{n}{\varphi(n)}$ achieves a maximum.

2 My Algorithm

We use Euler's product formula for the totient function,

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right), \tag{1}$$

taken over the distinct primes dividing n. This formula gives

$$\frac{n}{\varphi(n)} = \frac{1}{\prod\limits_{p|n}\left(1 - \frac{1}{p}\right)}. \tag{2}$$

Therefore, to maximize $\frac{n}{\varphi(n)}$, we must minimize the product. This is done when n is a product of many distinct primes. We want as many of these primes as possible to be small, so that their reciprocals are big. And so n should be the product of the first k primes, for some k. That is, n is the k-th primorial number. The first primorial number greater than 10^18 is the product of the primes up to 41.

Our solution consists of multiplying primes until the given upper bound is reached. Because the n-th primorial has size about e^n, our solution is $O(\log n)$.