Harmonic Numbers and Binomial Coefficients

Gautam Manohar

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The n-th harmonic number is defined as the sum of reciprocals of the first n natural numbers:

$$\sum_{k=1}^{n} \frac{1}{k}. \tag{1}$$

We wish to prove the following identity involving harmonic numbers and binomial coefficients:

$$\sum_{k=1}^{n} \frac{1}{k} = -\sum_{k=1}^{n} \frac{1}{k} \binom{n}{k} (-1)^k. \tag{2}$$

I have two solutions: one with calculus, and one by induction.

1 Calculus

First, we notice that

$$\int_0^1 x^{k-1} \, \mathrm{d}x = \left. \frac{x^k}{k} \right|_0^1 = \frac{1}{k} - \frac{0}{k} = \frac{1}{k}. \tag{3}$$

Then, substituting (3) into (2) gives

$$\sum_{k=1}^{n} \frac{1}{k} = \sum_{k=1}^{n} \int_0^1 x^{k-1} \, \mathrm{d}x. \tag{4}$$

Because this is a finite sum of converging integrals, we can switch the sum and the integral to get

$$\int_0^1 \sum_{k=1}^{n} x^{k-1} \, \mathrm{d}x. \tag{5}$$

This sum is a geometric series! We can use the formula for the n-th partial sum of a geometric series.

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{1-x^n}{1-x}. \tag{6}$$

Then we can solve the integral, using the substitution $x = 1 - u, \mathrm{d}x = -\,\mathrm{d}u$.

$$\int_0^1 \frac{1-x^n}{1-x}\,\mathrm{d}x = -\int_1^0 \frac{1-(1-u)^n}{u}\,\mathrm{d}u \tag{7}$$

Now the motivation for substituting $x = 1 - u$ becomes clear. We can expand the $(1-u)^n$ term using the binomial theorem:

$$\begin{aligned}
-\int_1^0 \frac{1-(1-u)^n}{u}\,\mathrm{d}u &= \int_0^1 \frac{1}{u}\left(1 - \sum_{k=0}^{n} \binom{n}{k}(-1)^k u^k\right)\mathrm{d}x \\
&= -\int_0^1 \frac{1}{u}\left(\sum_{k=0}^{n} \binom{n}{k}(-1)^k u^k - 1\right)\mathrm{d}x \\
&= -\int_0^1 \frac{1}{u}\sum_{k=1}^{n} \binom{n}{k}(-1)^k u^k\,\mathrm{d}x \\
&= -\int_0^1 \sum_{k=1}^{n} \binom{n}{k}(-1)^k u^{k-1}\,\mathrm{d}x \\
&= -\sum_{k=1}^{n} \binom{n}{k}(-1)^k \int_0^1 u^{k-1}\,\mathrm{d}x \\
&= -\sum_{k=1}^{n} \frac{1}{k}\binom{n}{k}(-1)^k.
\end{aligned} \tag{8}$$

This concludes our proof of (2).

2 Induction

We first show that

$$\frac{1}{k+1}\binom{n}{k} = \frac{1}{n+1}\binom{n+1}{k+1}. \tag{9}$$

We can expand the binomial coefficients with their factorial representation. Then the left-hand side equals

$$\frac{1}{k+1}\binom{n}{k} = \frac{1}{k+1}\frac{n!}{k!(n-k)!} = \frac{n!}{(k+1)!(n-k)!}, \tag{10}$$

while the right-hand side equals

$$\frac{1}{n+1}\binom{n+1}{k+1} = \frac{1}{n+1}\frac{(n+1)!}{(k+1)!(n-k)!} = \frac{n!}{(k+1)!(n-k)!}. \tag{11}$$

So (9) holds. Next we show that

$$\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}(-1)^k = \frac{1}{n+1}. \tag{12}$$

We use (9):

$$\begin{aligned}
\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}(-1)^k &= \sum_{k=0}^{n} \frac{1}{n+1}\binom{n+1}{k+1}(-1)^k \\
&= -\frac{1}{n+1}\sum_{k=1}^{n+1}\binom{n+1}{k}(-1)(-1)^k \\
&= -\frac{1}{n+1}\left(\sum_{k=0}^{n+1}\binom{n+1}{k}(-1)^k - \binom{n+1}{0}(-1)^0\right) \\
&= -\frac{1}{n+1}(0-1) \\
&= \frac{1}{n+1}.
\end{aligned} \tag{13}$$

We can now use induction to prove our main claim. Let's first show the base case for $n = 1$. The left side of (2) is clearly 1, while the right side is $-(-1) = 1$. So the base case holds. Assume inductively that

$$\sum_{k=1}^{n} \frac{1}{k} = -\sum_{k=1}^{n} \frac{1}{k}\binom{n}{k}(-1)^k. \tag{14}$$

Then we wish to show that

$$\sum_{k=1}^{n+1} \frac{1}{k} = -\sum_{k=1}^{n+1} \frac{1}{k}\binom{n+1}{k}(-1)^k. \tag{15}$$

We have

$$\begin{aligned}
-\sum_{k=1}^{n+1} \frac{1}{k}\binom{n+1}{k}(-1)^k &= -\sum_{k=1}^{n} \frac{1}{k}\binom{n+1}{k}(-1)^k - \frac{1}{n+1}\binom{n+1}{n+1}(-1)^{n+1} \\
&= -\sum_{k=1}^{n} \frac{1}{k}\binom{n+1}{k}(-1)^k + \frac{1}{n+1}(-1)^n.
\end{aligned} \tag{16}$$

Using a standard identity of binomial coefficients

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1} \tag{17}$$

(a recurrence relation that is the defining feature of Pascal's triangle) yields

$$\begin{aligned}
&\frac{1}{n+1}(-1)^n - \sum_{k=1}^{n}\binom{n}{k}\frac{1}{k}(-1)^k - \sum_{k=1}^{n}\binom{n}{k-1}\frac{1}{k}(-1)^k \\
&\quad= \frac{1}{n+1}(-1)^n - \sum_{k=1}^{n}\binom{n}{k}\frac{1}{k}(-1)^k + \sum_{k=0}^{n-1}\binom{n}{k}\frac{1}{k+1}(-1)^k \\
&\quad= \frac{1}{n+1}(-1)^n - \sum_{k=1}^{n}\binom{n}{k}\frac{1}{k}(-1)^k + \sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}(-1)^k - \binom{n}{n}\frac{1}{n+1}(-1)^n.
\end{aligned} \tag{18}$$

Applying (12) yields

$$-\sum_{k=1}^{n}\binom{n}{k}\frac{1}{k}(-1)^k + \frac{1}{n+1}. \tag{19}$$

By the inductive hypothesis, we have

$$\begin{aligned}
-\sum_{k=1}^{n}\binom{n}{k}\frac{1}{k}(-1)^k + \frac{1}{n+1} &= \sum_{k=1}^{n}\frac{1}{k} + \frac{1}{n+1} \\
&= \sum_{k=1}^{n+1}\frac{1}{k},
\end{aligned} \tag{20}$$

which closes the induction.