Almost There I

The Pattern That Sincs Eventually

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2 March 2018

This document originally appeared as a blog post on my website. Find it at gautammanohar.com/almost-patterns-sinc.

Mathematics students are often told not to carelessly extrapolate small sample sizes into general claims. Indeed, this is a general truth of life: isolated examples do not constitute general proof. Say you wanted to convince me of a fact about numbers. You could show me that your fact was true for all numbers up to ten, a hundred, a million, or to whatever ridiculously huge number you could think of—but it wouldn't constitute a mathematical proof.

This post will hopefully be the first in a series called "Almost There." I will share interesting patterns that, although they seem to *surely* continue, break down eventually.

You may have noticed that I spelled "sink" incorrectly in the title. Sharp eye. That's because we're investigating some interesting properties the sinc function:

$$\operatorname{sinc} x = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \tag{1}$$

It looks like this:

This function famously has no elementary antiderivative, yet its definite integral over the real line evaluates to

$$\int_{-\infty}^{\infty} \operatorname{sinc} x \, \mathrm{d}x = \pi. \tag{2}$$

Stay tuned for a proof of this fact.

A great tool for playing around with computations in math (and, thanks to the wonders of technology, a lot more) is Wolfram Alpha. We can compute some pretty complicated integrals with it, like

$$\int_{-\infty}^{\infty} \operatorname{sinc} x \operatorname{sinc} \frac{x}{3} \, \mathrm{d}x = \pi, \tag{3}$$

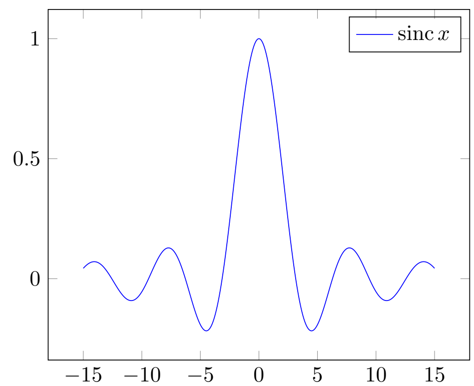

Figure 1. The graph of $\operatorname{sinc} x$. Note that at $x = 0$, this function is defined to be equal to 1.

and

$$\int_{-\infty}^{\infty} \operatorname{sinc} x \operatorname{sinc} \frac{x}{3} \operatorname{sinc} \frac{x}{5} \, \mathrm{d}x = \pi. \tag{4}$$

Even as we multiply the integrand by another sinc function, the value of the definite integral remains constant at π. The pattern continues for quite a while. In fact,

$$\int_{-\infty}^{\infty} \operatorname{sinc} x \operatorname{sinc} \frac{x}{3} \operatorname{sinc} \frac{x}{5} \cdots \operatorname{sinc} \frac{x}{13} \, \mathrm{d}x = \pi. \tag{5}$$

But with one more term, the pattern fails spectacularly:

$$\int_{-\infty}^{\infty} \operatorname{sinc} x \operatorname{sinc} \frac{x}{3} \operatorname{sinc} \frac{x}{5} \cdots \operatorname{sinc} \frac{x}{15} \, \mathrm{d}x = \frac{467807924713440738696537864469}{467807924720320453655260875000} \pi. \tag{6}$$

This value is 4.62×10^{-11} less than π.

What happened? After some research, I found that such integrals were documented in this paper. It turns out that, in general, for real numbers $a_1, \ldots, a_n$,

$$\int_{-\infty}^{\infty} \prod_{k=1}^{n} \operatorname{sinc} a_k x \, \mathrm{d}x \tag{7}$$

evaluates to π if $\sum_k a_k \leq 2$. In particular, $1 + \frac{1}{3} + \cdots + \frac{1}{13} = \frac{88069}{45045} = 2 - \frac{2021}{45045}$, but adding $\frac{1}{15}$ pushes it over the edge. So we can construct sequences of numbers a_k such that the pattern holds for arbitrarily many terms before failing, when the partial sum of the numbers a_k crosses 2. Similarly, we can make it so the pattern always holds. For example, the series $1 + \frac{1}{2} + \frac{1}{4} + \cdots$ famously converges to 2, so if we set a_k to be the reciprocal powers of two, the corresponding integral will be equal to π... forever.

But not all patterns hold forever—certainly not this one!