On the Ball in n Euclidean Dimensions

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One of my favourite formulas in mathematics is the one for the area of a circle, πr^2. Its simplicity intrigued my young mind; the circle is, in a sense, the simplest, the *roundest* shape, and this formula seems to encapsulate the circle's aura of "completion." As I grew older, I learned more such formulas for the volumes and surface areas of familiar geometric objects. In particular, I learned of the formula for the volume of a sphere $\frac{4}{3}\pi r^3$. Amused as I was by these clever formulas, my curiosity began to take hold of me. Questions filled my mind: isn't it interesting, that as the dimension of the round shape went from 2 to 3, so did the exponent on the radius? Where did the $\frac{4}{3}$ come from? Why do the formulas stop after circle and sphere? What comes next? Does anything come next? These questions lay dormant in my mind for a while, until my friend Arif Aulakh challenged me to find an expression for the volume of a sphere in an arbitrary number of dimensions. My curiosity rekindled, I set out in search of the higher-dimensional sphere to answer the question: what comes next?

1 Introduction

Before we embark on our journey, we need to lay out some basic definitions. A sphere is the set of all points a given distance (let's call it R) from a given center point. A ball is the space whose boundary is a sphere, or the set of all points whose distance from a given center point is less than or equal to R. For example, a sphere in 2 dimensions is called a circle, and the corresponding ball is called a disk. Upon hearing the word "circle," you might imagine a flat coin-like object. That's not what a mathematician means when they say circle. To a mathematician, a circle is just the boundary of that coin, while a disk is the entirety of the points contained in (and on the boundary of) the coin. Similarly, a sphere is the bounding surface of a ball. So let's call a ball in n Euclidean dimensions with a certain radius the n-ball of radius R. Also, note that a 1-ball is just a line segment. In 2 dimensions, we say closed figures have area, and in 3 dimensions, we say they have volume. The arbitrary-dimensional analogue of this concept is “content.” For example, the 2-ball has content πR^2, the 3-ball $\frac{4}{3}\pi R^2$. We shall find a general formula for the content of an n-ball of radius R. Along the way, we will uncover and explore some interesting related results.

2 A Recurrence Relation

Intutively, we can construct a 3-ball out of a bunch of thin disks (think of slicing a tomato). In the same way, we can construct a disk out of a bunch of thin line segments. And remember, a disk is a 2-ball, and a line segment is a 1-ball. There seems to be a certain modularity to the n-dimensional ball. Let's formalize this idea on the scale of 2 dimensions and then generalize to an arbitrary number of dimensions.

We wish to construct a 2-ball from the interval $[-R, R]$, centered on 0. Let the distance from 0 of a point on this interval be r. Then construct a 1-ball (line segment) of radius $\sqrt{R^2 - r^2}$ centered on and perpendicular to the interval. Doing this for all points on the interval yields a 2-ball, constructed from the sum of infinitely many 1-balls. Similarly, we can construct a 3-ball from infinitely many 2-balls centered on and perpendicular to the interval $[-R, R]$.

Generalizing to n dimensions yields the recurrence relation

$$V_n(R) = \int_{-R}^{R} V_{n-1}\left(\sqrt{R^2 - r^2}\right) \mathrm{d}r. \tag{1}$$

Uniformly scaling any object in n dimensions by R increases the content of that object by a factor of R^n. This result can be shown with linear algebra. We shall prove a weaker version of this statement for the n-ball.

> **Lemma 1. (Proportionality)** Uniformly scaling a n-dimensional ball by R increases the content of the ball by R^n.

As a base case, proportionality clearly holds for $n = 0$, where all balls have content 1. Here scaling by a factor of R scales content by a factor of $R^0 = 1$. Then assume inductively that proportionality holds for an $(n-1)$-ball. Then we can factor R^{n-1} from the integrand in (1):

$$V_n(R) = R^{n-1} \int_{-R}^{R} V_{n-1}\left(\sqrt{1 - \left(\frac{r}{R}\right)^2}\right) \mathrm{d}r. \tag{2}$$

Substituting $x = \frac{r}{R}, \mathrm{d}x = \frac{\mathrm{d}x}{R}$ yields

$$V_n(R) = R^n \int_{-1}^{1} V_{n-1}\left(\sqrt{1 - x^2}\right) \mathrm{d}x = R^n V_n(1), \tag{3}$$

which proves that proportionality holds in n dimensions and, by induction, in all non-negative integer dimensions. $\square$

Then proportionality transforms (3) into

$$\begin{aligned} V_n(R) &= R^n V_{n-1}(1) \int_{-1}^{1} \left(\sqrt{1-x^2}\right)^{n-1} \mathrm{d}x \\ &= V_{n-1}(R) R \int_{-1}^{1} \left(1-x^2\right)^{\frac{n-1}{2}} \mathrm{d}x. \end{aligned} \tag{4}$$

Because the integrand is an even function in x,

$$V_n(R) = 2V_{n-1}(R)R \int_0^1 \left(1-x^2\right)^{\frac{n-1}{2}} \mathrm{d}x. \tag{5}$$

The expression of interest to us is the integral. For notational simplicity, we define

$$I_k = 2\int_0^1 \left(1-x^2\right)^{\frac{k}{2}} \mathrm{d}x. \tag{6}$$

Then

$$V_n(R) = I_{n-1} V_{n-1}(R) R. \tag{7}$$

Let us now solve for I_{n-1}.

3 Solving the Integral

We shall prove the result

$$I_{n-1} = \frac{\sqrt{\pi}\Gamma\left(\frac{n+1}{2}\right)}{\Gamma(\frac{n}{2}+1)} \tag{8}$$

in two ways: analytically and with a probabalistic argument.

3.1 An Analytical Argument

We substitute $x = \cos\theta, \mathrm{d}x = -\sin\theta\,\mathrm{d}\theta$ into (6).

$$\begin{aligned} I_{n-1} &= -2\int_{\frac{\pi}{2}}^{0} \left(1-\cos^2\theta\right)^{\frac{n-1}{2}} \sin\theta\,\mathrm{d}\theta \\ &= 2\int_0^{\frac{\pi}{2}} \left(\sqrt{\sin^2\theta}\right)^{n-1} \sin\theta\,\mathrm{d}\theta \\ &= 2\int_0^{\frac{\pi}{2}} |\sin\theta|^{n-1} \sin\theta\,\mathrm{d}\theta. \end{aligned} \tag{9}$$

Because $\sin\theta$ is non-negative over $[0, \frac{\pi}{2}]$, we have

$$I_{n-1} = 2\int_0^{\frac{\pi}{2}} \sin^n\theta\,\mathrm{d}\theta. \tag{10}$$

We define the function

$$\mathcal{I}(n,m) = 2\int_0^{\frac{\pi}{2}} \sin^n\theta\cos^m\theta\,\mathrm{d}\theta. \tag{11}$$

Then $I_{n-1} = \mathcal{I}(n,0)$. We substitute $t = \tan^2\theta, \mathrm{d}t = 2\tan\theta\sec^2\theta\,\mathrm{d}\theta = 2\sin\theta\sec^3\theta\,\mathrm{d}\theta$ into (11). This gives $\frac{1}{1+t} = \cos^2\theta$ and $1 - \frac{1}{1+t} = \frac{t}{1+t} = \sin^2\theta$.

$$\begin{aligned}
\mathcal{I}(n,m) &= 2\int_0^{\frac{\pi}{2}} (\sin^2\theta)^{\frac{n}{2}}(\cos^2\theta)^{\frac{m}{2}}\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{\frac{n-1}{2}}(\cos^2\theta)^{\frac{m+3}{2}}2\sin\theta\sec^3\theta\,\mathrm{d}\theta \\
&= \int_0^{\infty} \left(\frac{t}{1+t}\right)^{\frac{n-1}{2}}\left(\frac{1}{1+t}\right)^{\frac{m+3}{2}}\mathrm{d}t \\
&= \int_0^{\infty} \frac{t^{\frac{n-1}{2}}}{(1+t)^{\frac{n+m}{2}+1}}\,\mathrm{d}t.
\end{aligned} \tag{12}$$

For simplicity, let $n = 2\alpha - 1, m = 2\beta - 1$. Then

$$\mathcal{I}(n,m) = \int_0^{\infty} \frac{t^{\frac{n-1}{2}}}{(1+t)^{\frac{n+m}{2}+1}}\,\mathrm{d}t = \int_0^{\infty} \frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}}\,\mathrm{d}t. \tag{13}$$

Let us now state two lemmas which we shall prove later.

Lemma 2. Given real numbers α, β, we have

$$\int_0^{\infty} \frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}}\,\mathrm{d}t = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{14}$$

Lemma 3.

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{15}$$

Applying Lemma 2 to (13) gives

$$\mathcal{I}(n,m) = \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{n+m}{2}+1\right)}. \tag{16}$$

Recall that $I_{n-1} = \mathcal{I}(n,0)$. Then applying Lemma 3 gives

$$\begin{aligned}
I_{n-1} &= \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\frac{n}{2}+1)} \\
&= \frac{\sqrt{\pi}\Gamma\left(\frac{n+1}{2}\right)}{\Gamma(\frac{n}{2}+1)},
\end{aligned} \tag{17}$$

which establishes (8). $\square$

3.2 A Statistical Argument

Let "choosing a number" be done with a uniform probability. Let α and β be positive integers. The probability of choosing first choosing a real p in the interval $[0,1]$, then choosing α real numbers in $[0,p]$, and then choosing β real numbers in $[p,1]$ is $\int_0^1 p^\alpha(1-p)^\beta \,\mathrm{d}p$. Then the probability of choosing $\alpha+\beta+1$ real numbers in $[0,1]$ such that the first number is p, some α of the remaining numbers are in $[0,p]$, and some β of them are in $[p,1]$ is $\binom{\alpha+\beta}{\alpha}\int_0^1 p^\alpha(1-p)^\beta$. But this is the same as the probability that, after p and $\alpha+\beta$ real numbers in $[0,1]$ are placed in increasing order, p happens to be in the $(\alpha+1)$-th position, or $\frac{1}{\alpha+\beta+1}$. This yields

$$\begin{aligned}\binom{\alpha+\beta}{\alpha}\int_0^1 p^\alpha(1-p)^\beta \,\mathrm{d}p &= \frac{1}{\alpha+\beta+1}\\ \frac{(\alpha+\beta)!}{\alpha!\beta!}\int_0^1 p^\alpha(1-p)^\beta \,\mathrm{d}p &= \frac{1}{\alpha+\beta+1}\\ \int_0^1 p^\alpha(1-p)^\beta \,\mathrm{d}p &= \frac{\alpha!\beta!}{(\alpha+\beta+1)!}.\end{aligned} \tag{18}$$

Generalizing with the gamma function $\Gamma(n+1)=n!$, we have

$$\int_0^1 p^\alpha(1-p)^\beta \,\mathrm{d}p = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}. \tag{19}$$

Substituting $p=x^2, \mathrm{d}p = 2x\,\mathrm{d}x$ yields

$$2\int_0^1 x^{2\alpha+1}(1-x^2)^\beta \,\mathrm{d}x = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \tag{20}$$

Setting $\alpha=-\frac{1}{2}, \beta=\frac{n-1}{2}$ yields the desired (8):

$$\begin{aligned}2\int_0^1 (1-x^2)^{\frac{n-1}{2}} \,\mathrm{d}x &= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}\\ I_{n-1} &= \frac{\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}.\end{aligned} \tag{21}$$

□

4 A General Formula

Now let's put it all together.

Theorem 1. The content of an n-ball of radius R, denoted $V_n(R)$, is given by

$$V_n(R) = \frac{R^n \pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}. \tag{22}$$

Expanding (7) using (8), we have

$$\begin{aligned}
V_n(R) &= I_{n-1}V_{n-1}(R)R \\
&= R\frac{\sqrt{\pi}\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}V_{n-1}(R) \\
&= \left(R\frac{\sqrt{\pi}\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}\right)\left(R\frac{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}+1\right)}\right)\cdots\left(R\frac{\sqrt{\pi}\Gamma\left(\frac{n+2-m}{2}\right)}{\Gamma\left(\frac{n-m}{2}+1\right)}\right)V_{n-m}(R) \\
&= R^m\pi^{\frac{m}{2}}\left(\frac{\Gamma\left(\frac{n-1}{2}+1\right)}{\Gamma\left(\frac{n}{2}+1\right)}\right)\left(\frac{\Gamma\left(\frac{n-2}{2}+1\right)}{\Gamma\left(\frac{n-1}{2}+1\right)}\right)\cdots\left(\frac{\Gamma\left(\frac{n-m}{2}+1\right)}{\Gamma\left(\frac{n-m+1}{2}+1\right)}\right)V_{n-m}(R).
\end{aligned} \tag{23}$$

The m-th numerator is the same as the $(m+1)$-st denominator, so

$$V_n(R) = V_{n-m}(R)R^m\pi^{\frac{m}{2}}\frac{\Gamma\left(\frac{n-m}{2}+1\right)}{\Gamma\left(\frac{n}{2}+1\right)}. \tag{24}$$

Because a 0-dimensional ball of any radius consists of only one point, $V_0(R) = 1$. Setting $m = n$, we obtain our general expression for the content of an n-dimensional ball and establish Theorem 1.

5 Proofs and Other Results

5.1 A Trigonometric Integral Identity

From (16), it follows that $\mathcal{I}(n,m) = \mathcal{I}(m,n)$. So $\frac{1}{2}\mathcal{I}(n,0) = \frac{1}{2}\mathcal{I}(0,n)$. Then (11) implies the interesting identity

$$\int_0^{\frac{\pi}{2}} \sin^n\theta \,\mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \cos^n\theta \,\mathrm{d}\theta. \tag{25}$$

5.2 Identities Involving the Gamma Function

5.2.1 Gamma Fraction Identity: Proof of Lemma 2

Lemma 2, which states that

$$\int_0^{\infty} \frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}}\,\mathrm{d}t = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{26}$$

was key in our analytical argument. We shall prove it here.

The gamma function is defined as

$$\Gamma(z) = \int_0^{\infty} e^{-t}t^{z-1}\,\mathrm{d}t. \tag{27}$$

In (27) we make the substitution $t = pq$, $\mathrm{d}t = q\,\mathrm{d}p$.

$$\begin{aligned}\Gamma(\alpha) &= \int_0^\infty e^{-pq}(pq)^{\alpha-1} q\,\mathrm{d}p \\ &= \int_0^\infty q^\alpha e^{-pq} p^{\alpha-1}\,\mathrm{d}p.\end{aligned} \tag{28}$$

We multiply both sides of (28) by $\Gamma(\beta) = \int_0^\infty e^{-q} q^{\beta-1}\,\mathrm{d}q$.

$$\begin{aligned}\Gamma(\alpha)\Gamma(\beta) &= \left(\int_0^\infty q^\alpha e^{-pq} p^{\alpha-1}\,\mathrm{d}p\right)\left(\int_0^\infty e^{-q} q^{\beta-1}\,\mathrm{d}q\right) \\ &= \int_0^\infty \int_0^\infty e^{-q-pq} q^{\alpha+\beta-1} p^{\alpha-1}\,\mathrm{d}p\,\mathrm{d}q \\ &= \int_0^\infty p^{\alpha-1}\,\mathrm{d}p \int_0^\infty e^{-q(1+p)} q^{\alpha+\beta-1}\,\mathrm{d}q.\end{aligned} \tag{29}$$

In the inner integral we substitute $q = \frac{s}{1+p}, \mathrm{d}q = \frac{\mathrm{d}s}{1+p}$. Then

$$\begin{aligned}\int_0^\infty e^{-q(1+p)} q^{\alpha+\beta-1}\,\mathrm{d}q &= \int_0^\infty e^{-s}\left(\frac{s}{1+p}\right)^{\alpha+\beta-1} \frac{\mathrm{d}s}{1+p} \\ &= \frac{1}{(1+p)^{\alpha+\beta}} \int_0^\infty e^{-s} s^{\alpha+\beta-1}\,\mathrm{d}s \\ &= \frac{\Gamma(\alpha+\beta)}{(1+p)^{\alpha+\beta}}.\end{aligned} \tag{30}$$

Substituting (30) into (29), we have

$$\begin{aligned}\Gamma(\alpha)\Gamma(\beta) &= \int_0^\infty p^{\alpha-1} \frac{\Gamma(\alpha+\beta)}{(1+p)^{\alpha+\beta}}\,\mathrm{d}p \\ \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} &= \int_0^\infty \frac{p^{\alpha-1}}{(1+p)^{\alpha+\beta}}\,\mathrm{d}p,\end{aligned} \tag{31}$$

which establishes the desired result. □

5.2.2 Gamma of One-Half: Proof of Lemma 3

We shall prove Lemma 3, that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. In (14), set $\alpha = \beta = \frac{1}{2}$. Then

$$\begin{aligned}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} &= \int_0^\infty \frac{t^{-\frac{1}{2}}}{1+t}\,\mathrm{d}t \\ \Gamma^2\left(\frac{1}{2}\right) &= \int_0^\infty \frac{1}{\sqrt{t}(1+t)}\,\mathrm{d}t.\end{aligned} \tag{32}$$

We substitute $t = x^2, \mathrm{d}t = 2x\,\mathrm{d}x = 2\sqrt{t}\,\mathrm{d}x$:

$$\begin{aligned}\Gamma^2\left(\frac{1}{2}\right) &= 2\int_0^\infty \frac{1}{1+x^2}\,\mathrm{d}x \\ &= 2\left(\arctan x\big|_0^\infty\right) \\ &= 2\cdot\frac{\pi}{2} \\ &= \pi,\end{aligned} \tag{33}$$

so

$$\Gamma\left(\frac{1}{2}\right) = \pm\sqrt{\pi}. \tag{34}$$

But in the integral representation of $\Gamma(z)$, the integrand $e^t t^{z-1}$ is positive if $z > 0$. This yields the desired result,

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{35}$$

A consequence of this result is the value of the famous Gaussian integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x, \tag{36}$$

whose integrand has no elementary antiderivative. Substituting $t = x^2, \mathrm{d}t = 2t^{\frac{1}{2}}\,\mathrm{d}x$ into the integral representation of $\Gamma\left(\frac{1}{2}\right)$, we have

$$\begin{aligned}\sqrt{\pi} &= \Gamma\left(\frac{1}{2}\right) \\ &= \int_0^\infty e^{-t}t^{-\frac{1}{2}}\,\mathrm{d}t \\ &= 2\int_0^\infty e^{-x^2}\,\mathrm{d}x.\end{aligned} \tag{37}$$

Because the integrand e^{-x^2} is an even function, it follows that

$$\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}. \tag{38}$$

5.2.3 Gamma Function Doubling Formula

We have the following result.

Theorem 2. (Gamma Function Doubling Formula)

$$\Gamma(2z) = \frac{2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)}{\sqrt{\pi}} \tag{39}$$

To prove this result, we shall use integration by parts and (8). Let

$$A_k = \int_0^1 \left(1-x^2\right)^k \mathrm{d}x. \tag{40}$$

Then, integrating by parts, we have

$$\begin{aligned}
A_k &= x(1-x^2)^k\big|_0^1 - \int_0^1 -2xk(1-x^2)^{k-1}x \,\mathrm{d}x \\
&= 2k\int_0^1 x^2(1-x^2)^{k-1}\,\mathrm{d}x \\
&= 2k\int_0^1 (1-(1-x^2))(1-x^2)^{k-1}\,\mathrm{d}x \\
&= 2k\int_0^1 (1-x^2)^{k-1}-(1-x^2)^k\,\mathrm{d}x \\
&= 2kA_{k-1} - 2kA_k \\
&= \frac{2k}{2k+1}A_{k-1}.
\end{aligned} \tag{41}$$

Expanding the recurrence relation yields

$$A_k = \frac{(2k)(2k-2)\cdots 4\cdot 2}{(2k+1)(2k-1)\cdots 3\cdot 1}. \tag{42}$$

Let us first deal with the numerator. We know that

$$k! = k(k-1)\cdots 2\cdot 1. \tag{43}$$

Multiplying each of the k terms on the right-hand side by 2 yields the numerator. So the numerator is equal to $2^k k!$. Then we deal with the denominator. We know that

$$(2k+1)! = (2k+1)(2k)\cdots 2\cdot 1. \tag{44}$$

Dividing both sides of this equation by $2^k k!$ yields the denominator. So the denominator is equal to $\frac{(2k+1)!}{2^k k!}$. Then

$$\begin{aligned}
A_k &= \frac{2^k k!}{\frac{(2k+1)}{2^k k!}} \\
&= \frac{(2^k k!)^2}{(2k+1)!}.
\end{aligned} \tag{45}$$

Generalising using the gamma function $\Gamma(n+1) = n!$ yields

$$A_k = \frac{2^{2k}\Gamma^2(k+1)}{\Gamma(2k+2)}. \tag{46}$$

From (6) and (40), $I_{n-1} = 2A_{\frac{n-1}{2}}$, so by (46),

$$I_{n-1} = \frac{2^n \Gamma^2\left(\frac{n+1}{2}\right)}{\Gamma(n+1)}. \tag{47}$$

Equating (47) with (8) and making the substitution $z = \frac{n+1}{2}$ establishes Theorem 2.

$$\begin{aligned} \frac{2^{2z-1}\Gamma^2(z)}{\Gamma(2z)} &= \frac{\sqrt{\pi}\Gamma(z)}{\Gamma(z+\frac{1}{2})} \\ \Gamma(2z) &= \frac{2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)}{\sqrt{\pi}}. \end{aligned} \tag{48}$$

□

This result is very powerful. It implies that, combined with the recurrence relation $\Gamma(n+1) = n\Gamma(n)$, we can compute $\Gamma(\frac{n}{2})$ for any positive integer n. Then we can also compute all expressions of the form $\Gamma(\frac{n}{4})$, because the right-hand side of (39) will be in terms of expressions of the form $\Gamma(\frac{n}{2})$. Thus for any positive real number r whose binary expansion is finite—that is, for any positive rational r—we can thus compute $\Gamma(r)$. And because every irrational number can be written as the limit of the partial sums of a sequence of rational numbers, we can compute $\Gamma(r)$ for any positive real r.

5.2.4 Approximating the Logarithmic Derivative of the Gamma Function

Consider the logarithmic derivative of the gamma function, which we shall call $\mathcal{G}(x)$:

$$\mathcal{G}(x) = \frac{\mathrm{d}}{\mathrm{d}x} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}. \tag{49}$$

For the sake of this section, let $\mathcal{G} = \frac{\Gamma'(x)}{\Gamma(x)}$. Then we shall prove

Theorem 3. The following upper and lower bounds hold for the logarithmic derivative of the gamma function:

$$\ln x - \frac{1}{x} < \mathcal{G}(x) < \ln x. \tag{50}$$

To do this, we first show the following lemma, which may seem intuitive.

Lemma 4. Let $f(x)$ be a differentiable function defined on $[a, b]$. Then if $f'(x) > 0$ for all $x \in [a, b]$, then $f(x)$ is strictly increasing.

By the Fundamental Theorem of Calculus,

$$\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a). \tag{51}$$

If $f'(x) > 0$ for all x, then because the integral of a positive integrand is also positive, $f(b) > f(a)$ for all x. That is, $f(x)$ is strictly increasing. $\square$

We also use the following lemma for continuous functions, which is a special case of the Cauchy-Schwarz inequality.

Lemma 5. Let f and g be continuous functions defined over $[a, b]$. Then we have

$$\left(\int_a^b fg \,\mathrm{d}t\right)^2 \leq \left(\int_a^b f^2 \,\mathrm{d}t\right)\left(\int_a^b g^2 \,\mathrm{d}t\right). \tag{52}$$

In particular, equality holds if and only if f is proportional to g; that is, there exists some z such that for all $x \in [a, b]$, we have $zf(x) = g(x)$.

Let's take some contant z. Then $0 \leq (fz - g)^2$. And because the integral of a non-negative function is also non-negative, for all real z, we have

$$0 \leq \int_a^b (fz - g)^2 \,\mathrm{d}t = \left(\int_a^b f^2 \,\mathrm{d}t\right) z^2 - \left(\int_a^b 2fg \,\mathrm{d}t\right) z + \left(\int_a^b g^2 \,\mathrm{d}t\right). \tag{53}$$

Because this quadratic is non-negative, its discriminant must be non-positive, such that

$$4\left(\int_a^b fg \,\mathrm{d}t\right)^2 \leq 4\left(\int_a^b f^2 \,\mathrm{d}t\right)\left(\int_a^b g^2 \,\mathrm{d}t\right). \tag{54}$$

Equality only holds in (54) if $\int_a^b (fz - g)^2 \,\mathrm{d}t = 0$. Consider a function h' continuous over $[a, b]$ such that $\int_a^b h'(x) \,\mathrm{d}x = 0$. By the Fundamental Theorem of Calculus, this integral equals $h(b) - h(a)$, so $h(a) = h(b)$; that is, h is a constant function, so the integrand h' is the zero function. Thus equality only holds in (54) if $fz = g$. $\square$

Lemma 6. For non-negative real x, $\mathcal{G}(x)$ is a strictly increasing function.

We will prove Lemma 6 by showing that $\mathcal{G}'(x)$ is positive everywhere. We have

$$\begin{aligned} \Gamma'(x) &= \frac{\mathrm{d}}{\mathrm{d}x} \int_0^\infty e^{-t} t^{x-1} \,\mathrm{d}t \\ &= \int_0^\infty \frac{\partial}{\partial x} e^{-t} t^{x-1} \,\mathrm{d}t \\ &= \int_0^\infty e^{-t} t^{x-1} \ln t \,\mathrm{d}t, \end{aligned} \tag{55}$$

and, similarly,

$$\Gamma''(x) = \int_0^\infty e^{-t} t^{x-1} \ln^2 t \,\mathrm{d}t. \tag{56}$$

Furthermore,

$$\mathcal{G}'(x) = \frac{\Gamma(x)\Gamma''(x) - (\Gamma'(x))^2}{\Gamma^2(x)}. \tag{57}$$

Substituting $\alpha = \sqrt{e^{-t}t^{x-1}}$ and $\beta = \ln t\sqrt{e^{-t}t^{x-1}}$ into (57) yields

$$\mathcal{G}'(x) = \frac{\left(\int_0^\infty \alpha^2\,\mathrm{d}t\right)\left(\int_0^\infty \beta^2\,\mathrm{d}t\right) - \left(\int_0^\infty \alpha\beta\,\mathrm{d}t\right)^2}{\Gamma^2(x)}. \tag{58}$$

We note that functions α and β are continuous and not proportional (because $\ln t$ is not constant with respect to t) to one another over $[0, \infty)$. Then Lemma 5 shows, with strict inequality, that

$$\left(\int_0^\infty \alpha\beta\,\mathrm{d}t\right)^2 < \left(\int_0^\infty \alpha^2\,\mathrm{d}t\right)\left(\int_0^\infty \beta^2\,\mathrm{d}t\right). \tag{59}$$

The denominator of (58) is always positive, so (59) shows that (57) is positive for all non-negative real x. Finally, Lemma 4 establishes that $\mathcal{G}(x)$ is a strictly increasing function. □

Lemma 7. We have $\mathcal{G}(x+1) = \mathcal{G}(x) + \frac{1}{x}$.

We have $\Gamma(x+1) = x\Gamma(x)$, so $\ln(\Gamma(x+1)) = \ln\Gamma(x) + \ln x$. Differentiating establishes the result. □

Lemma 8. We have the following inequality: $\mathcal{G}(x) < \ln x < \mathcal{G}(x+1)$.

By the Mean Value Theorem, there exists some z in $(x, x+1)$ such that

$$\begin{aligned}\mathcal{G}(z) &= \frac{\ln\Gamma(x+1) - \ln\Gamma(x)}{x+1-x} \\ &= \ln(x\Gamma(x)) - \ln\Gamma(x) \\ &= \ln x.\end{aligned} \tag{60}$$

In particular, note that $x < z < x+1$. By Lemma 6, $\mathcal{G}(x)$ is strictly increasing. So $\mathcal{G}(x) < \mathcal{G}(z) = \ln x < \mathcal{G}(x+1)$, and we establish Lemma 8. □

Finally, we can establish Theorem 3 by applying Lemma 7 and Lemma 8:

$$\begin{aligned}\mathcal{G}(x) < \ln x < \mathcal{G}(x+1) &= \mathcal{G}(x) + \frac{1}{x} \\ \ln x < \mathcal{G}(x+1) &< \ln x + \frac{1}{x} \\ \ln x - \frac{1}{x} < \mathcal{G}(x) &< \ln x.\end{aligned} \tag{61}$$

□

5.3 Surface Area of Arbitrary Dimensional Spheres

We can build a the 2-ball, or disk, from the union of concentric 2-sphere (circle) shells all evenly spaced by a distance $\mathrm{d}r$. This is like the layers of an onion. This generalizes to n dimensions, where an n-ball is the union of concentric n-sphere shells. As $\mathrm{d}r$ tends to 0, we find that the content of a n-ball is the sum of the surface areas of the infinitely many n-spheres with radii ranging from 0 to R. That is,

$$V_n(R) = \int_0^R S_n(r)\,\mathrm{d}r, \tag{62}$$

which gives

$$S_n(R) = \frac{\mathrm{d}V_n(R)}{\mathrm{d}R}. \tag{63}$$

Substituting (22) yields

$$\begin{aligned} S_n(R) &= \frac{nR^{n-1}\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} \\ &= \frac{nR^{n-1}\pi^{\frac{n}{2}}}{\frac{n}{2}\Gamma\left(\frac{n}{2}\right)} \\ &= \frac{2R^{n-1}\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}. \end{aligned} \tag{64}$$

5.4 Implications of the Content of Balls of Arbitrary Dimension

5.4.1 Small Dimensions

n	V	$\approx V$
0	1	1
1	$2R$	$2R$
2	πR^2	$3.14R^2$
3	$\frac{4}{3}\pi R^3$	$4.19R^3$
4	$\frac{\pi^2}{2}R^4$	$4.94R^4$
5	$\frac{8\pi^2}{15}R^5$	$5.26R^5$
6	$\frac{\pi^3}{6}R^6$	$5.17R^6$
7	$\frac{16\pi^3}{105}R^7$	$4.73R^7$
8	$\frac{\pi^4}{24}R^8$	$4.06R^8$

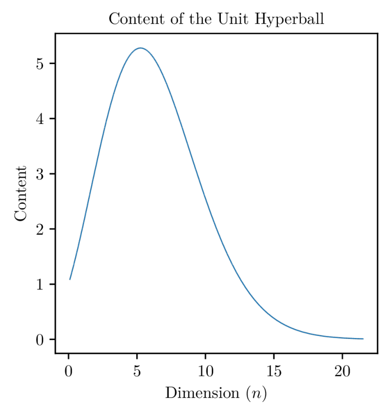

Figure 1. The dimensionless content of the unit ball in n dimensions. Note the global maximum, after which the content quickly decreases.

We shall ignore the "unit" of content and instead compare contents as dimenionless quantities. The integer dimension that maximizes the content of the unit ball is evidently $n = 5$. The figure reveals that this maximum is actually at $n \approx 5.26$. In fact, the global maximum of the function $n \mapsto \frac{R^n \pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}, n \geq 0$ depends on R and is given by

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}n}\left(\frac{R^n \pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}\right) &= 0 \\ \frac{\Gamma\left(\frac{n}{2}+1\right)(R\sqrt{\pi})^n \ln(R\sqrt{\pi}) - \frac{1}{2}(R\sqrt{\pi})^n \Gamma'\left(\frac{n}{2}+1\right)}{\Gamma^2\left(\frac{n}{2}+1\right)} &= 0 \\ \Gamma\left(\frac{n}{2}+1\right)\ln(R\sqrt{\pi}) - \frac{1}{2}\Gamma'\left(\frac{n}{2}+1\right) &= 0 \\ \ln(\pi R^2) &= \frac{\Gamma'\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}+1\right)}. \end{aligned} \tag{65}$$

This equation cannot be solved for n analytically, so we approximate. From Theorem 3, we have

$$\ln n < \frac{\Gamma'(n+1)}{\Gamma(n+1)} < \ln n + \frac{1}{n}, \tag{66}$$

which means that the maximal content of a ball of radius R is attained in

$$\begin{aligned} \ln\left(\frac{n}{2}\right) &\sim \ln(\pi R^2) \\ n &\sim 2\pi R^2. \end{aligned} \tag{67}$$

5.4.2 Large Dimensions

For a given radius R, the content of an n-ball goes to 0 as n goes to infinity. That is,

$$\lim_{n\to\infty} \frac{R^n \pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} = 0. \tag{68}$$

In fact, this result holds for any expression of the form $\frac{a^n}{\Gamma(kn+1)} = \frac{a^n}{(kn)!}$ for positive real a and k. Define the function $f(n) = \frac{a^n}{(kn)!}$. Consider the limiting ratio $\frac{f(n+1)}{f(n)}$:

$$\begin{aligned} \lim_{n\to\infty} \frac{\frac{a^{n+1}}{(k(n+1))!}}{\frac{a^n}{(kn)!}} &= \lim_{n\to\infty} \frac{a^{n+1}(kn)!}{a^n(kn+k)!} \\ &= \lim_{n\to\infty} \frac{a}{(kn+1)(kn+2)\cdots(kn+k)} \\ &= 0. \end{aligned} \tag{69}$$

Then by the ratio test, the series $\sum_{n=0}^{\infty} \frac{a^n}{(kn)!}$ converges because its limiting ratio is 0. Because the terms of a convergent series must tend to 0, $\frac{a^n}{(kn)!}$ tends to 0. Thus we have proven that for a given R, $V_n(R)$ tends to 0 as n increases).

6 Conclusion

Our most important result was a general expression for the content of an n-ball of radius R:

$$V_n(R) = \frac{R^n \pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}. \tag{70}$$

We also derived two useful identities related to the gamma function:

$$\Gamma(2z) = \frac{2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)}{\sqrt{\pi}} \tag{71}$$

and

$$\ln x - \frac{1}{x} < \frac{\Gamma'(x)}{\Gamma(x)} < \ln x. \tag{72}$$

Our next steps would be to examine the properties of other geometric figures in n Euclidean dimensions, such as hypercubes and simplices. We could also analyze balls in non-Euclidean spaces, such as hyperbolic space.