This document originally appeared as a blog post on my website. Find it at gautammanohar.com/.

1 Problem Statement

If we list all the natural numbers below 10 that are multiples of 3 or 5, we get 3, 5, 6 and 9. The sum of these multiples is 23.

Find the sum of all the multiples of 3 or 5 below n.

2 My Algorithm

For the Project Euler problem, $n = 1000$.

In general, the sum of the natural numbers up to n is the n-th *triangular number* (see here). Let's call this $T(n)$. A well known formula for this is

$$T(n) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \tag{1}$$

The sum of all multiples of 3 below n looks like this:

$$1 \cdot 3 + 2 \cdot 3 + 3 \cdot 3 + \ldots + \left\lfloor \frac{n-1}{3} \right\rfloor \cdot 3. \tag{2}$$

We can factor 3 out and write (2) as $3T \left\lfloor \frac{n-1}{3} \right\rfloor$, which we know how to find with (1). We can do the same thing with 5.

But now we've overcounted! Each multiple of 15 under n has been counted twice: once as a multiple of 3, then again as a multiple of 5. We can fix this by subtracting the sum of all multiples of 15 under n. And so our desired answer is

$$3T \left\lfloor \frac{n-1}{3} \right\rfloor + 5T \left\lfloor \frac{n-1}{5} \right\rfloor - 15T \left\lfloor \frac{n-1}{15} \right\rfloor. \tag{3}$$

The complexity of this solution is $O(1)$, because our answer is just a computation.

2.1 Other Solutions

A brute-force solution that adds each number i from 1 to $n-1$ to a count if i is divisible by 3 *or* 5 would have time complexity $O(n)$. With the large input sizes of the Hackerrank problem ($n \leq 10^9$) this solution is too slow, but it easily passes the original Project Euler problem.