Problem 15 - Lattice Paths

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This document originally appeared as a blog post on my website. Find it at gautammanohar.com/euler/15.

1 Problem Statement

Suppose we start at the top-left corner of an $N \times M$ grid and that we can only travel to the right or down. How many routes are there to the bottom-right corner? Output the answer modulo $10^9 + 7$.

2 My Algorithm

In total, we must make $N + M$ moves, of which N are to the right. Choosing these N moves uniquely identifies each possible route, as this is equivalent to choosing the M downwards moves. This is a combinatorial explanation for the fact that

$$\binom{N+M}{N} = \binom{N+M}{M} = \frac{(N+M)!}{N!M!} \tag{1}$$

which is our answer. To make the computation easier, we use the smaller of N and M.

Suppose we know the value of $\binom{a}{b} = \frac{a!}{b!(a-b)!}$. Then we can find the value of

$$\binom{a}{b+1} = \frac{a!}{(b+1)!(a-b-1)!} = \binom{a}{b}\frac{a-b}{b+1} \tag{2}$$

This gives us a recurrence relation. Because we are computing the answer modulo 10^9+7, we must use the modular multiplicative inverse of $b+1$. Because the modulus is prime, this is guaranteed to exist. We can find this using the Extended Euclidean algorithm in $O(\log P)$ time, where $P = 10^9 + 7$.

Starting with $\binom{N+M}{0} = 1$, we iterate over $0 \le k \le \min(N, M) - 1$ use the recurrence relation (2), multiplying by $N + M - k$ and the modular inverse of

$k+1$ under P. At each step, we take the expression modulo P. Our solution has time complexity $O(\min(N, M) \log P)$.