Problem 25 - 1000-digit Fibonacci Number

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This document originally appeared as a blog post on my website. Find it at gautammanohar.com/euler/25.

1 Problem Statement

What is the first term in the Fibonacci sequence to contain N digits?

2 My Algorithm

We will use Binet's formula for the n-th fibonacci number

$$F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}. \tag{1}$$

The $(-\varphi)^{-n}$ quickly becomes negligible. So we can make the approximation

$$F_n \approx \frac{\varphi^n}{\sqrt{5}}. \tag{2}$$

A number n has $\lfloor \log_{10} n \rfloor + 1$ digits. So we must solve the following equation such that n is minimized:

$$\lfloor \log_{10} F_n \rfloor + 1 = N. \tag{3}$$

Recall that $\lfloor x \rfloor$ is defined as the greatest integer less or equal to than x. That is,

$$x \leq \lfloor x \rfloor < x + 1. \tag{4}$$

Applying (2) and (4) to (3) gives

$$
\begin{aligned}
N &\leq 1 + \log_{10} F_n < N + 1 \\
N - 1 &\leq \log_{10} \frac{\varphi^n}{\sqrt{5}} \\
N - 1 + \frac{\log_{10} 5}{2} &\leq n \log_{10} \varphi \\
\frac{N - 1 + \frac{\log_{10} 5}{2}}{\log_{10} \varphi} &\leq n.
\end{aligned} \tag{5}
$$

We see that n is minimized when it is the smallest integer greater than the left side of (5). This is precisely the definition of the ceiling function. And so we have our desired answer:

$$
n = \left\lceil \frac{N - 1 + \frac{\log_{10} 5}{2}}{\log_{10} \varphi} \right\rceil. \tag{6}
$$

Because our solution is just a computation, it has time complexity $O(1)$. In our code, we just need to handle the corner case where $N = 1$; here our approximation (2) introduces enough error to return an incorrect answer.