

Problem 14 - Longest Collatz Sequence

Gautam Manohar

10 June 2018

This document originally appeared as a blog post on my website. Find it at gautammanohar.com/euler/14.

1 Problem Statement

The following iterative sequence is defined for the set of positive integers:

$$\begin{cases} n \rightarrow \frac{n}{2} & n \text{ even,} \\ n \rightarrow 3n + 1 & n \text{ odd} \end{cases}$$

Using the rule above and starting with 13 generates the following sequence:

$$13 \rightarrow 40 \rightarrow 20 \rightarrow 10 \rightarrow 5 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$$

This sequence contains 10 terms. It is conjectured that every starting number produces a sequence that ends in 1.

Which starting number at most N produces the longest chain? If there are many possible such numbers, print the maximum one.

2 My Algorithm

This problem is perfect for caching (or memoization) and recursion. We define a recursive function that returns the length of a Collatz chain with a given starting number. The function also store the chain length for each starting number (and, because the function is recursive, each number in the chain) in a list so that we do not have to compute it again. Because of memory constraints, we do not store chain lengths for numbers greater than the maximum given N . Instead, we compute these values on the fly.

We can then iterate over all possible starting numbers and keep a list of starting numbers which produce a right-maximal chain length (that is, they produce a

chain length greater than or equal to all lengths produced by smaller starting numbers). Finally, we perform a binary search for the greatest element in this list less than each query. This solution has time complexity $O(M + T \log M)$, where M is the maximum possible starting value and T is the number of queries.

2.1 Other Solutions

Solutions which do not use memoization are too slow.