Problem 6 - Sum Square Difference

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This document originally appeared as a blog post on my website. Find it at `gautammanohar.com/euler/6`.

1 Problem Statement

The sum of the squares of the first ten natural numbers is

$$1^2 + 2^2 + \cdots + 10^2 = 385. \tag{1}$$

The square of the sum of the first ten natural numbers is

$$(1 + 2 + \cdots + 10)^2 = 55^2 = 3025 \tag{2}$$

Hence the difference between the sum of the squares of the first ten natural numbers and the square of the sum is $3025 - 385 = 2640$. Find the absolute difference between the sum of the squares of the first N natural numbers and the square of the sum.

2 My Algorithm

We make use of the following two formulas, which can be proven using induction.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \tag{3}$$

and

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}. \tag{4}$$

Applying these two formulas, the desired value is

$$\begin{aligned}\left|\frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6}\right| &= \left|\frac{n^4+2n^3+n^2}{4} - \frac{2n^3+3n^2+n}{6}\right| \\ &= \left|\frac{3n^4+6n^3+3n^2}{12} - \frac{4n^3+6n^2+2n}{12}\right| \\ &= \left|\frac{3n^4+2n^3-3n^2-2n}{12}\right|\end{aligned} \tag{5}$$

This value is non-negative for all positive integers, so the absolute value bars are not necessary. Our algorithm is just a computation, so it has time complexity $O(1)$.

2.1 Other Solutions

We did not need to simplify the expression as in (5); we could have simply subtracted (4) from the square of (3). An $O(n)$ solution is also possible: simply add the first N numbers, square them, and subtract the sum of the squares of the first N numbers.