Problem 24 - Lexicographic Permutations

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1 Problem Statement

A permutation is an ordered arrangement of objects. For example, dabc is a permutation of the word abcd. If the permutations are listed alphabetically, we call it lexicographic order. The lexicographic permutations of abc are:

$$\text{abc}, \text{acb}, \text{bac}, \text{bca}, \text{cab}, \text{cba}.$$

What is the N-th lexicographic permutation of the word abcdefghijklm?

2 My Algorithm

Suppose our word has n letters. Then the first $(n-1)!$ lexicographic permutations begin with a, the next $(n-1)!$ with b, and so on. Consider the first $(n-1)!$ lexicographic permutations. They consist of a followed by a lexicographic permutation of the word without a. Of these, the first $(n-2)!$ begin with b.

We can write N as a unique sum

$$N = \sum_{i=0}^{n-1} c_i \cdot i!, \tag{1}$$

where $0 \le c_i \le i+1$. This is a kind of "base-factorial" expansion of N. Once we do this, we use the procedure above. Starting from $n-1$ and going down to 0, the N-th lexicographic permutation has the c_{n-1}-th letter in the first position, the c_{n-2}-th letter of those remaining in the second position, and so on.

And so our algorithm is as follows. Write N as a sum of factorials. Maintain a list of the letters in the word, in alphabetical order. For $n-1 \ge i \ge 0$, delete the

c_i-th element from the list and add it to the string representing the lexicographic permutation. This solution has time complexity $O(L)$, where L is the length of the given word.