Problem 27 - Quadratic Primes

Gautam Manohar

13 June 2018

This document originally appeared as a blog post on my website. Find it at gautammanohar.com/euler/27.

1 Problem Statement

Euler published the remarkable quadratic equation

$$n^2 + 41n + 41.$$

It turns out that the formula will produce 40 primes for the consecutive integer values $0 \leq n \leq 39$. However, when $n = 40$, we have $40^2 + 40 + 41 = 41(40 + 1) + 41$, which is divisiblye by 41, and certainly when $n = 41$, the expression $41^2 + 41 + 41$ is clearly divisible by 41.

The incredible formula $n^2 - 79n + 1601$ produces 80 primes for the consecutive values $0 \leq n \leq 79$.

Considering quadratics of the form

$$n^2 + an + b, \quad |a|, |b| \leq N,$$

find the coefficients a, b for the quadratic expression that produces the maximum number of primes with consecutive values of n, starting with $n = 0$.

2 My Algorithm

A naive brute force search over all $-N \leq a, b \leq N$ is too slow. We make two observations that exclude many cases. First, the expression $n^2 + an + b$ is equal to b when $n = 0$. Thus b must be prime. Second, when $n = 1$, we have $1 + a + b$. Above 2, b is odd. So a must also be odd; otherwise, the expression is an even number greater than 2, which is not prime.

We need two things to proceed: a list of primes under N and a way to check if a given n is prime. For the first, we use a Sieve of Eratosthenes, and for the second, we use trial division up to $\sqrt{n}$. We iterate through odd $-N \leq a \leq N$, which takes N steps, and prime $-N \leq bN$, which takes about $\frac{N}{\log N}$. Then, we compute the expression $n^2 + an + b$ until it is not prime; we do this up to N times, with time complexity $\sqrt{N}$ each time. Including the sieve, our solution has time complexity $O\left(N \cdot \frac{N}{\log N} \cdot N\sqrt{N} + \sqrt{N} \log\log N\right) \in O\left(\frac{N^3\sqrt{N}}{\log N}\right)$.