Problem 65 - Convergents of e

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16 July 2018

This document originally appeared as a blog post on my website. Find it at `gautammanohar.com/euler/65`.

1 Problem Statement

The constant e can be written as the infinite continued fraction $[2; 1, 2, 1,\ 1, 4, 1,\ 1, 6, 1, \ldots, 1, 2k, \ldots]$. Find the sum of the digits in the N-th convergent of the continued fraction expansion for e.

2 A Recurrence Relation

We will show the following theorem. Let the the n-th convergent of a continued fraction $[a_0; a_1, a_2, \ldots]$ be $\frac{P_n}{Q_n}$. Then the $(n+1)$-th convergent is

$$\frac{a_{n+1}P_n + P_{n-1}}{a_{n+1}Q_n + Q_{n-1}}. \tag{1}$$

We use induction. Because the 0-th convergent is a_0, we define $P_0 = a_0, Q_0 = 1$. Then the first convergent is

$$a_0 + \frac{1}{a_1} = \frac{a_0 a_1 + 1}{a_1}, \tag{2}$$

so $P_1 = a_0 a_1 + 1, Q_1 = a_1$. As a base case, we show that the formula holds for

$n = 2$:

$$\begin{aligned}
a_0 + \frac{1}{a_1 + \frac{1}{a_2}} &= a_0 + \frac{a_2}{a_1 a_2 + 1} \\
&= \frac{a_0 a_1 a_2 + a_0 + a_2}{a_1 a_2 + 1} \\
&= \frac{a_2(a_0 a_1 + 1) + a_0}{a_2 a_1 + 1} \\
&= \frac{a_2 P_1 + P_0}{a_2 Q_1 + Q_0}.
\end{aligned} \tag{3}$$

Now suppose inductively that

$$P_n = a_n P_{n-1} + P_{n-2}, Q_n = a_n Q_{n-1} + Q_{n-2}. \tag{4}$$

The n-th convergent is

$$\frac{P_n}{Q_n} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}} \tag{5}$$

Turning (5) into the $(n+1)$-th convergent involves replacing a_n with $a_n + \frac{1}{a_{n+1}}$. Using the recursive formula, this gives

$$\begin{aligned}
\frac{P_{n+1}}{Q_{n+1}} &= \frac{\frac{P_{n-1}}{a_{n+1}} + a_n P_{n-1} + P_{n-2}}{\frac{Q_{n-1}}{a_{n+1}} + a_n Q_{n-1} + Q_{n-2}} \\
&= \frac{\frac{P_{n-1}}{a_{n+1}} + P_n}{\frac{Q_{n-1}}{a_{n+1}} + Q_n} \\
&= \frac{a_{n+1} P_n + P_{n-1}}{a_{n+1} Q_n + Q_{n-1}},
\end{aligned} \tag{6}$$

which proves the theorem.

3 My Algorithm

Let the n-th coefficient in the continued fraction expansion for e be a_n. Then $a_0 = 2$. From then on, $a_1 = a_3 = 1$, and $a_2 = 2$. In particular, $a_n = 1$ if n is congruent to 0 or 1 mod 3 (unless $n = 0$). And so

$$a_n = \begin{cases} 2 & n = 0, \\ 2\left(\lfloor \frac{n}{3} \rfloor + 1\right) & n \equiv 2 \pmod 3, \\ 1 & \text{otherwise.} \end{cases} \tag{7}$$

Using this and our theorem, we iteratively calculate the numerator of the N-th convergent by

$$P_n = a_n P_{n-1} + P_{n-2}. \tag{8}$$

We cache the previous two convergents at each step. Finally, we take the digit sum. Our solution has time complexity $O(N)$.