Problem 2 - Even Fibonacci Numbers

Gautam Manohar

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This document originally appeared as a blog post on my website. Find it at
gautammanohar.com/euler/2.

1 Problem Statement

Each new term in the Fibonacci sequence is generated by adding the previous
two terms. By starting with 1 and 2, the first 10 terms will be:

$$1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

By considering the terms in the Fibonacci sequence whose values do not exceed
N, find the sum of the even-valued terms.

2 My Algorithm

For the Project Euler problem, $N = 4 \times 10^6$. Let's list the Fibonacci numbers
with $F(0) = 0$:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots \tag{1}$$

Using the definition of the Fibonacci sequence:

$$\begin{aligned}
F(n+3) &= F(n+2) + F(n+1) \\
&= F(n+1) + F(n) + F(n+1) \\
&= 2F(n+1) + F(n).
\end{aligned} \tag{2}$$

This shows that if $F(n)$ is even, then so is $F(n+3)$. Because $F(0) = 0$ is even,
then the even Fibonacci numbers are $F(3k)$ for positive integers k.

Further applying the Fibonacci recurrence relation gives

$$\begin{aligned}
F(3k) &= 2F(3k-2) + F(3k-3) \\
&= 3F(3k-3) + 2F(3k-4) \\
&= 3F(3k-3) + 2F(3k-5) + 2F(3k-6).
\end{aligned} \tag{3}$$

From (2), we have $F(3k - 3) = 2F(3k - 5) + F(3k - 6)$, so

$$F(3k) = 4F(3k - 3) + F(3k - 6) = 4F(3(k - 1)) + F(3(k - 2)). \qquad (4)$$

To solve the problem, we begin with $0, 2$. Using (4), we add to a sum the next even Fibonacci number, keeping track of the last two even Fibonacci numbers. We continue this until the next even Fibonacci number exceeds N. This solution performs one-third as many steps as there are Fibonacci numbers up to N, so it is $O(\log n)$.

2.1 Other Solutions

The optimization of only computing even Fibonacci numbers does not change the time complexity of the solution, but it does lend an improvement by a constant factor of 3. Without this optimization, the solution is still $O(\log n)$. Because there are only 28 even Fibonacci numbers under 4×10^{16}, preprocessing can be used to find the prefix sums of the even Fibonacci numbers. This solution is $O(1)$.

2.2 Mathematical Solution

A more mathematically oriented $O(1)$ solution (my personal favourite) is also possible. Unfortunately, the floating point calculations involved become too inaccurate with large values of N, and using a high precision floating point library (such as Python's `decimal`) is too slow. Regardless, a famous formula for the value of the n-th Fibonacci number is Binet's formula:

$$F(n) = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}. \qquad (5)$$

Suppose we know that we need to sum the Fibonacci numbers $F(3k)$ for integers k from 0 to some maximum M to get our desired sum S. Then we have two geometric series:

$$S = \frac{1}{\sqrt{5}} \left(\sum_{k=0}^{M} \varphi^{3k} - \sum_{k=0}^{M} (-\varphi)^{-(3k)} \right). \qquad (6)$$

We can find the first term of each series by setting $k = 0$; they are both 1. In the first series, the common ratio is φ^3, in the second $(-\varphi)^{-3}$. We use the formula for the sum of a geometric series with first term a and common ratio r with n terms:

$$a + ar + ar^3 + \cdots + ar^{n-1} = a \left(\frac{r^n - 1}{r - 1} \right). \qquad (7)$$

Now we can rewrite (6), noting that each sum contains the first $M + 1$ terms of a geometric series:

$$S = \frac{1}{\sqrt{5}} \left(\frac{\varphi^{3(M+1)} - 1}{\varphi^3 - 1} + \frac{(-\varphi)^{-3(M+1)} - 1}{\varphi^{-3} + 1} \right). \qquad (8)$$

It's nice to have from a mathematical point of view, but the second term in (8) quickly vanishes. Excluding it gives an absolute error of less than 1, so we can ignore it and floor the expression:

$$S = \left\lfloor \frac{\varphi^{3(M+1)} - 1}{\sqrt{5}(\varphi^3 - 1)} \right\rfloor. \tag{9}$$

Now we need to find what this maximum M actually is. We know that $F(3M)$ is the largest even Fibonacci number less than or equal to N. Using the truncated version of Binet's formula, we know that

$$F(n) \approx \frac{\varphi^n}{\sqrt{5}}, \tag{10}$$

so $n \approx \log_\varphi(\sqrt{5}F(n))$. Therefore, $M \approx \left\lfloor \frac{1}{3} \log_\varphi(N\sqrt{5}) \right\rfloor$. And so our desired answer is

$$S = \left\lfloor \frac{\varphi^{3\left(\left\lfloor \frac{1}{3} \log_\varphi(N\sqrt{5})\right\rfloor + 1\right)} - 1}{\sqrt{5}(\varphi^3 - 1)} \right\rfloor. \tag{11}$$