# Problem 5 - Smallest Multiple

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## 1 Problem Statement

2520 is the smallest number that can be divided by each of the numbers from 1 to 10 without any remainder.

What is the smallest positive number that is evenly divisible by all of the numbers from 1 to $N$?

## 2 My Algorithm

For a number $a$ to divide another number $b$, each of the prime powers that divide $a$ must also divide $b$. That is, the exponent on each of the prime powers that divide $a$ is less than or equal to the corresponding exponent on a prime power that divides $b$. For each number from 1 to $N$ to divide a number $M$, each prime power less than $N$ must also divide $M$. To minimize $M$, we have each prime $p \leq N$ divide $M$ as many times as it does the largest prime power of $p$ at most $N$. That is

$$M = \prod_{p \leq N} p^{\lfloor \log_p N \rfloor}. \tag{1}$$

To compute our answer $M$, we need a list of the primes less than $N$, for which we can use the Sieve of Eratosthenes. And so our solution has time complexity $O(n \log \log n)$.

### 2.1 Other Solutions

This problem can also be phrased as finding the lowest common multiple of $1, \ldots, N$, for which the classic formula is $\text{lcm}(1, \ldots, N) = \frac{N!}{\gcd(1, \ldots, N)}$. Because the numerator grows very quickly, we can compute the LCM iteratively, making use of the fact that $\operatorname{lcm}(a, b, c) = \operatorname{lcm}(\operatorname{lcm}(a, b), c)$ and storing the latest LCM with each step. This algorithm computes the greatest common denominator of two numbers at most $N$, which can be done in $O(\log n)$ time, $N$ times. And so this solution has time complexity $O(n \log n)$. However, it has much better space complexity, at $O(1)$, than the sieve of Eratosthenes, which needs $O(n)$ space to store an array of size $n$. So for large $n$, this alternative solution is preferred.