

Problem 6 - Sum Square Difference

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This document originally appeared as a blog post on my website. Find it at gautammanohar.com/euler/6.

1 Problem Statement

The sum of the squares of the first ten natural numbers is

$$1^2 + 2^2 + \cdots + 10^2 = 385.$$

The square of the sum of the first ten natural numbers is

$$(1 + 2 + \cdots + 10)^2 = 55^2 = 3025$$

Hence the difference between the sum of the squares of the first ten natural numbers and the square of the sum is $3025 - 385 = 2640$. Find the absolute difference between the sum of the squares of the first N natural numbers and the square of the sum.

2 My Algorithm

We make use of the following two formulas, which can be proven using induction.

$$\sum_{i=1}^n i = \frac{n(n+1)}{2} \tag{1}$$

and

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}. \tag{2}$$

Applying these two formulas, the desired value is

$$\begin{aligned}
\left| \frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6} \right| &= \left| \frac{n^4 + 2n^3 + n^2}{4} - \frac{2n^3 + 3n^2 + n}{6} \right| \\
&= \left| \frac{3n^4 + 6n^3 + 3n^2}{12} - \frac{4n^3 + 6n^2 + 2n}{12} \right| \quad (3) \\
&= \left| \frac{3n^4 + 2n^3 - 3n^2 - 2n}{12} \right|
\end{aligned}$$

This value is non-negative for all positive integers, so the absolute value bars are not necessary. Our algorithm is just a computation, so it has time complexity $O(1)$.

2.1 Other Solutions

We did not need to simplify the expression as in (3); we could have simply subtracted (2) from the square of (1). An $O(n)$ solution is also possible: simply add the first N numbers, square them, and subtract the sum of the squares of the first N numbers.