Comparing e^π and π^e

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This document originally appeared as a blog post on my website. Find it at `gautammanohar.com/e-pi`.

The constants e and π are everywhere in mathematics. Determining the greater of the two expressions e^π and π^e (without using a calculator, of course...) is a fun puzzle that you can approach in many ways. I'd like to discuss my solutions.

1 Differentiation

We shall perform the same operations on the two expressions:

$$\begin{aligned} e^{\pi} &\odot \pi^{e} \\ e^{\frac{\pi}{e}} &\odot \pi^{\frac{e}{e}} \\ e^{\frac{1}{e}} &\odot \pi^{\frac{1}{\pi}}. \end{aligned} \tag{1}$$

To show that $e^\pi > \pi^e$, it suffices to show that $e^{\frac{1}{e}} > \pi^{\frac{1}{\pi}}$. Let $y = x^{\frac{1}{x}}$. Then we can implicitly differentiate to find the critical points.

$$\begin{aligned} \ln y &= \ln x^{\frac{1}{x}} \\ \ln y &= \frac{\ln x}{x} \\ \frac{\mathrm{d}}{\mathrm{d}x} \ln y &= \frac{\mathrm{d}}{\mathrm{d}x} \frac{\ln x}{x} \\ \frac{1}{y} y' &= \frac{x\left(\frac{1}{x}\right) - \ln x \cdot 1}{x^2} \\ y' &= x^{\frac{1}{x}} \frac{1 - \ln x}{x^2}. \end{aligned} \tag{2}$$

The expressions $x^{\frac{1}{x}}$ and x^2 are always positive, so there is only critical point: when $1 - \ln x = 0$, or when $x = e$. We must find whether this point is a global minimum or a maximum. When $x = 1 < e$, we have $1 - \ln x = 1$, so the function is increasing. The value $x = e^2 > e$ gives $1 - \ln x = -1$, which means the function is decreasing. Thus $x^{\frac{1}{x}}$ has a global maximum at $x = e$. And so $e^{\frac{1}{e}} > \pi^{\frac{1}{\pi}}$, which shows that $e^{\pi} > \pi^{e}$.

2 Inequality

If we use the inequality $1 + x < e^x$ (I will present three proofs of this below), then a very simple solution presents itself. The equality holds for all x, but we only require it to hold for positive x. Make the substitution $x = \frac{\pi}{e} - 1$, in an effort to cancel out the 1 on the right side of the inequality and introduce π. Because $\pi > e$, $\frac{\pi}{e} - 1 > 0$, and so

$$\begin{aligned} 1 + \frac{\pi}{e} - 1 &< e^{\frac{\pi}{e} - 1} \\ \pi \cdot \frac{1}{e} &< e^{\frac{\pi}{e}} \cdot \frac{1}{e} \\ \pi &< e^{\frac{\pi}{e}} \\ \pi^e &< e^{\pi}. \end{aligned} \tag{3}$$

Wonderful, isn't it?

2.1 Taylor Series

This is the most standard proof I have; I think it's the least exciting. We only prove the equality for positive x. We know

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{4}$$

Thus for $x > 0$, all the terms on the right side will be positive, and so $e^x > 1 + x$.

2.2 Concavity

This method requires a little more "geometric intuition" than the last. At $x = 0$, we have $e^x = 1$. At this point $(0, 1)$, the tangent line to e^x has slope 1 and has the equation $y = 1 + x$. Because $(e^x)'' = e^x > 0$, e^x is always concave up, so it is always above its tangent line. Therefore, $1 + x < e^x$.

2.3 AM-GM

This is my favourite proof. It's a little less intuitive than the others, but I think it's beautiful. We use the arithmetic-geometric mean inequality.

$$\begin{aligned}
\sqrt[n]{1+x} &= \sqrt[n]{\underbrace{1\cdot 1\cdots 1}_{n-1\text{ times}}\cdot(1+x)} \\
&\leq \frac{\overbrace{1+\cdots+1}^{n-1\text{ times}}}{n} \\
&= \frac{\overbrace{1+\cdots+1}^{n-1\text{ times}}+(1+x)}{n} \\
&= \frac{\overbrace{1+\cdots+1}^{n\text{ times}}+x}{n} \\
&= 1+\frac{x}{n}.
\end{aligned} \tag{5}$$

Strict equality in the AM-GM inequality only holds when all the terms are equal. In this case, $x > 0$, so $1 + x \neq 1$, so we have strict inequality. This gives $\sqrt[n]{1+x} < 1 + \frac{x}{n}$. Raising both sides to the n-th power gives

$$1 + x < \left(1 + \frac{x}{n}\right)^n. \tag{6}$$

Taking the limit as n approaches ∞ on both sides yields

$$\begin{aligned}
\lim_{n\to\infty}(1+x) &< \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \\
1+x &< e^x.
\end{aligned} \tag{7}$$

Using the limit definition of e, we conclude our proof.

I get the feeling that there are many other ways to attack this problem. If you can solve it with a method that I have not shown, please let me know!