Problem 9 - Special Pythagorean Triplets

Gautam Manohar

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This document originally appeared as a blog post on my website. Find it at `gautammanohar.com/euler/9`.

1 Problem Statement

A Pythagorean triplet is a set of three natural numbers, $a < b < c$, for which,

$$a^2 + b^2 = c^2 \tag{1}$$

For example, $3^2 + 4^2 = 9 + 16 = 25 = 52$.

Given N, find the maximal product abc such that a, b, c form a Pythagorean triplet for which $a + b + c = N$. If no such triplet exists, output -1.

2 My Algorithm

We know

$$a + b + c = N \tag{2}$$

and

$$a^2 + b^2 = c^2. \tag{3}$$

From (2) we get $c = N - a - b$, which we substitute into (3) to get

$$\begin{aligned}
a^2 + b^2 &= (N - (a + b))^2 \\
a^2 + b^2 &= N^2 - 2N(a + b) + (a + b)^2 \\
a^2 + b^2 &= a^2 + b^2 + 2ab + N^2 - 2Na - 2Nb \\
0 &= b(2a - 2N) + N^2 - 2Na \\
b(2a - 2N) &= 2Na - N^2 \\
b &= \frac{N^2 - 2Na}{2N - 2a}.
\end{aligned} \tag{4}$$

Now we have expressions for b, c in terms of a. Note furthermore that because $a < b < c$ and $a+b+c = N$, none of the sides can exceed $\frac{N}{3}$. Now we can iterate over $1 \leq a \leq \frac{N}{3}$ and store the maximal product abc for which $a^2 + b^2 = c^2$. This solution is $O(n)$.

2.1 Other Solutions

We can also use preprocessing for an $O(N^2 + T)$ solution, where T is the number of queries. For all $a < b < 3000$, we check whether $c = \sqrt{a^2 + b^2}$ is an integer. If so, we check whether abc is greater than our existing answer for the perimeter $a + b + c$, which is by default -1. To answer the queries, we look up the corresponding value in $O(1)$ time.