

# Problem 21 - Amicable Numbers

Gautam Manohar

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## 1 Problem Statement

Let  $d(n)$  be defined as the sum of proper divisors of  $n$  (numbers less than  $n$  that divide evenly into  $n$ ). If  $d(a) = b$  and  $d(b) = a$ , where  $a \neq b$ , then  $a$  and  $b$  are an amicable pair and each of  $a$  and  $b$  are amicable numbers.

For example, the proper divisors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55 and 110; therefore  $d(220) = 284$ . The proper divisors of 284 are 1, 2, 4, 71 and 142; so  $d(284) = 220$ .

Evaluate the sum of the proper divisors under  $N$ .

## 2 My Algorithm

The first thing we need is a way to calculate  $d(n)$ . A brute force method that tests all numbers from 1 to  $n$  and checks whether they divide  $n$  is too slow. But if a number  $d$  divides  $n$ , then  $\frac{n}{d}$  also divides  $n$ . In particular, one of  $n$  and  $\frac{n}{d}$  is less or equal than  $\sqrt{n}$ , and the other is greater than or equal to  $\sqrt{n}$ . To see this, consider what would happen if both were smaller than  $\sqrt{n}$ . Their product would be less than  $n$ ; a similar contradiction arises if both were greater than  $\sqrt{n}$ .

And so we only need to check each number less than  $\sqrt{n}$ . If it divides  $n$ , we add it to the sum that we will return. We therefore find  $d(n)$  in  $O(\sqrt{n})$  time.

Then we can use preprocessing to answer each query in  $O(1)$  time. For each  $1 \leq x \leq N_{\max}$ , if  $x = d(d(x))$  and  $x \neq d(x)$ , then we add  $x$  to a list of amicable numbers. To answer each query, we simply add the amicable numbers less than the given  $N$ . Our solution has time complexity  $O(N\sqrt{N} + T)$ , where  $N$  is the maximum possible input value and  $T$  is the number of queries.