Finding cuts of bounded degree: complexity, FPT and exact algorithms, and kernelization

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Matching Cut

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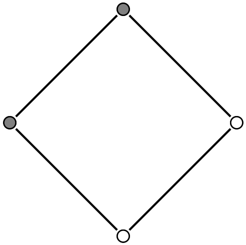

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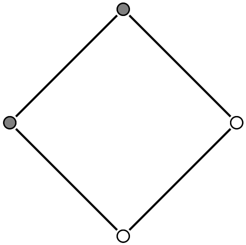

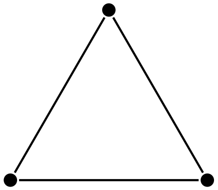

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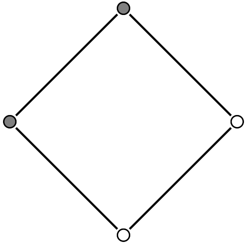

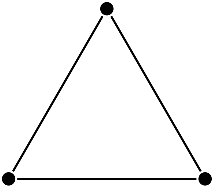

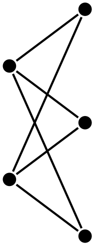

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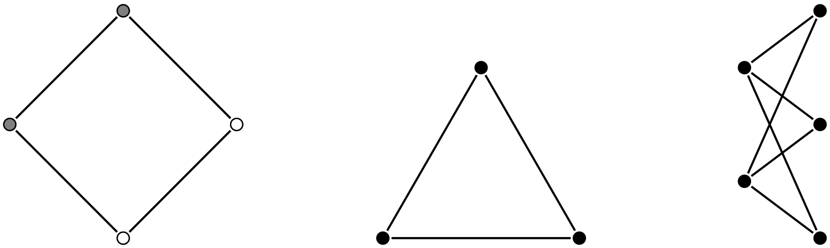

Ron L Graham. "On primitive graphs and optimal vertex assignments". In: *Annals of the New York academy of sciences* 175.1 (1970), pp. 170–186

Which graphs admit a matching cut?

Matching Cut and graph classes

Vasek Chvátal. "Recognizing decomposable graphs". In: *Journal of Graph Theory* 8.1 (1984), pp. 51–53

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...G is planar...

Van Bang Le and Bert Randerath. “On stable cutsets in line graphs”. In: *Theoretical Computer Science* 301.1-3 (2003), pp. 463–475

...or G is bipartite.

Algorithmic results for Matching Cut

Dániel Marx, Barry O'Sullivan, and Igor Razgon. "Treewidth Reduction for Constrained Separation and Bipartization Problems". In: *Proc. of the 27th International Symposium on Theoretical Aspects of Computer Science, (STACS)*. vol. 5. LIPIcs. 2010, pp. 561–572

FPT parameterized by the number of edges crossing the cut.

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Dieter Kratsch and Van Bang Le. "Algorithms solving the matching cut problem". In: *Theoretical Computer Science* 609 (2016), pp. 328–335

FPT parameterized by vertex cover; $\mathcal{O}^*\left(2^{n/2}\right)$ exact exponential algorithm.

Algorithmic results for MATCHING CUT

N. R. Aravind, Subrahmanyam Kalyanasundaram, and Anjeneya Swami Kare. "On Structural Parameterizations of the Matching Cut Problem". In: *Proc. of the 11th International Conference on Combinatorial Optimization and Applications (COCOA)*. vol. 10628. LNCS. 2017, pp. 475–482

FPT parameterized by treewidth, neighborhood diversity, or twin cover.

Algorithmic results for MATCHING CUT

Christian Komusiewicz, Dieter Kratsch, and Van Bang Le. “Matching Cut: Kernelization, Single-Exponential Time FPT, and Exact Exponential Algorithms”. In: *Proc. of the 13th International Symposium on Parameterized and Exact Computation (IPEC)*. vol. 115. LIPIcs. 2018, 19:1–19:13

- FPT parameterized by distance to cluster, distance to co-cluster;
- Quadratic kernel for distance to cluster, linear kernel for distance to clique;
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During their presentation, asked for results on the generalization of Matching Cut we discuss here.

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Option 3

Look for a ***p*-partition** such that each pair of parts forms a matching cut.

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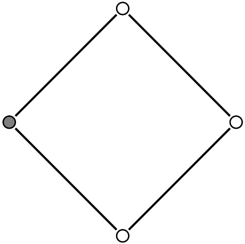

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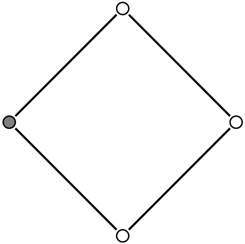

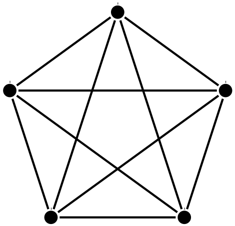

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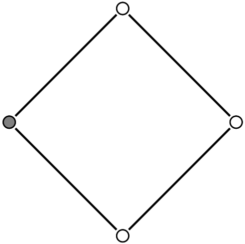

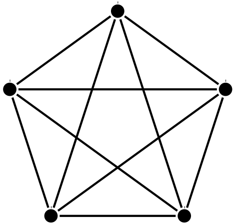

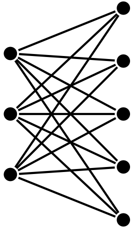

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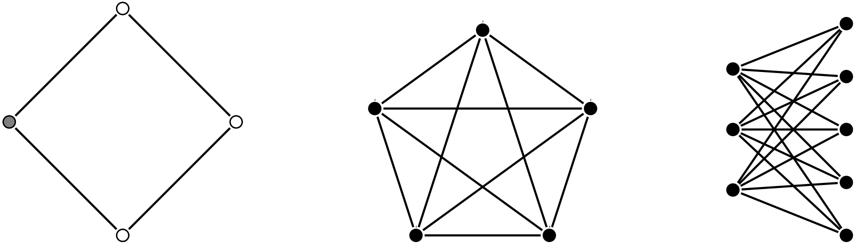

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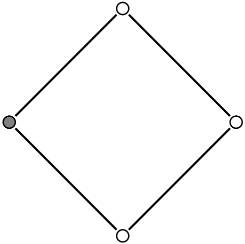
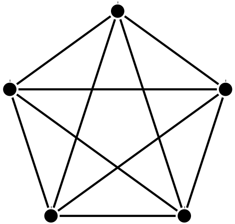
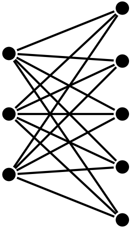


$K_{2d+1} = K_5$





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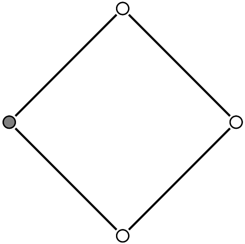

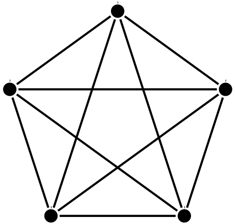

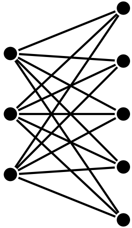

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3-Uniform Hypergraph Bicoloring

Instance: A hypergraph $\mathcal{H}$ with three vertices in each hyperedge.
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Heavily inspired on the reduction given by: Vasek Chvátal. "Recognizing decomposable graphs". In: *Journal of Graph Theory* 8.1 (1984), pp. 51–53.

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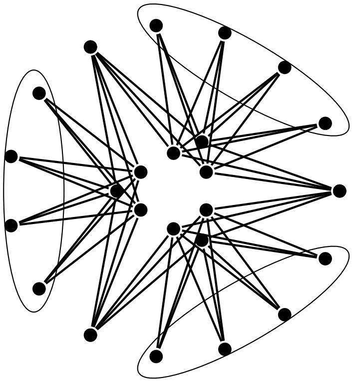

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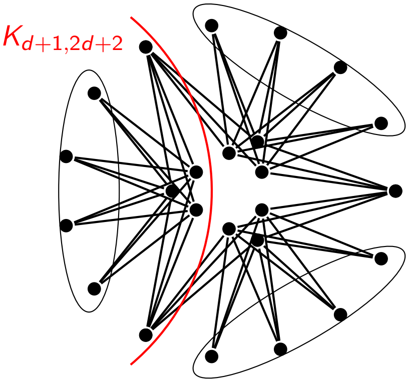

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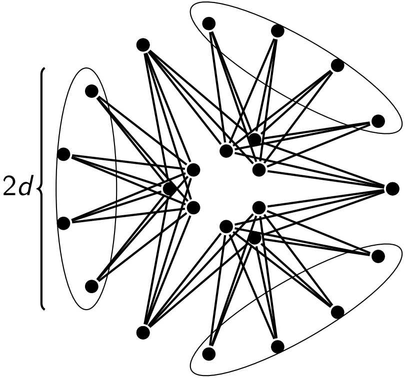

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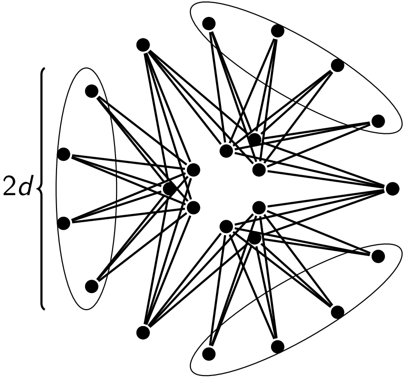

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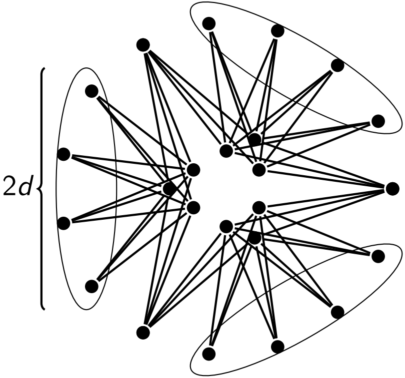

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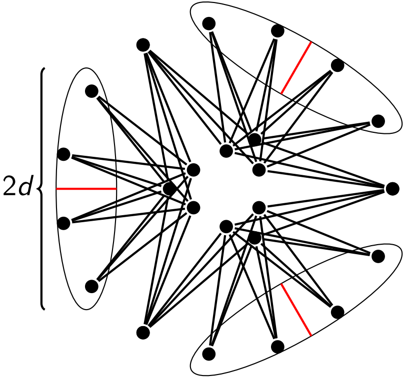

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- Divide each of them in two equal sized sets. Add edges within and between these sets to encode the hyperedges and achieve regularity.

Regularity

There is a very similar problem known as INTERNAL PARTITIONS, where we want a bipartition of $V(G)$ such that every vertex has at least half of its neighbors on its own part.

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Conjecture

For every r, there is a constant n_r such that every r-regular graph with at least n_r vertices has an internal partition (open since 2002). Known to hold for $r = 3, 4, 6$.

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Improving upon our reduction would imply that the conjecture is false (or $\mathrm{P} = \mathrm{NP}$), while polynomial algorithms would likely yield a proof.

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If $\Delta(G) \leq d + 2$, d-CUT can be solved in polynomial time.

Sketch of the proof: find a shortest cycle C. If there is some vertex with too many neighbors in C, we either find a shorter cycle, or we have that $d = 2$ and extend C to Q until $(Q, G - Q)$ is a d-cut or G is monochromatic.

Parameterized algorithms

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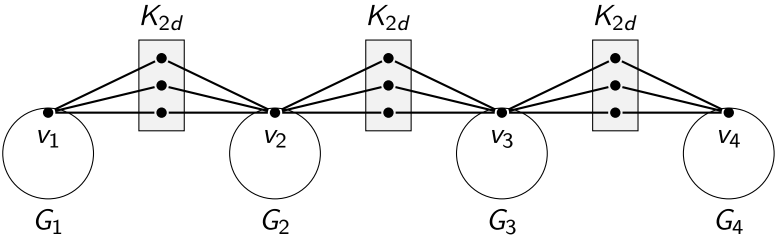

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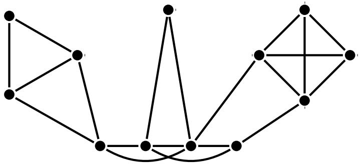

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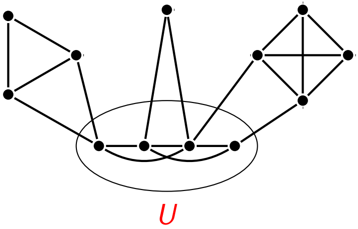

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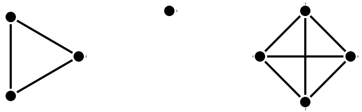

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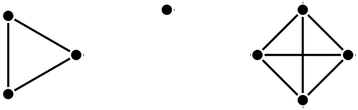

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A co-cluster graph is the complement graph of a cluster graph.

Goal

Partition U in monochromatic sets $\{U_1, \ldots, U_\ell\}$ and merge them until we get a kernel.

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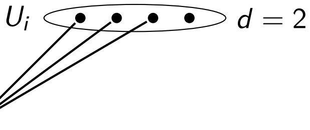

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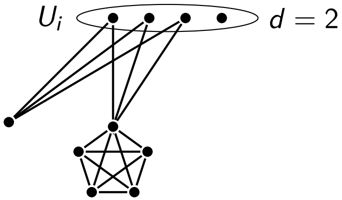

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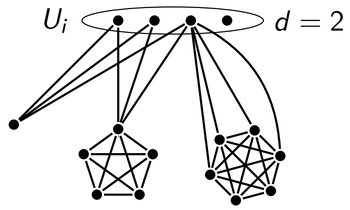

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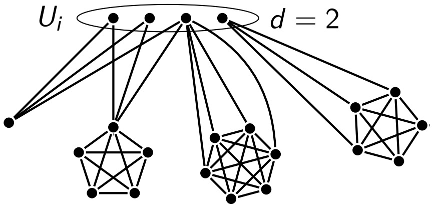
3. v is in a cluster C and some vertex in U_i has $2d$ neighbors in C; or
4. v is in a cluster C of size at least $2d+1$ and there is some vertex in U_i with $d+1$ neighbors in C.



The kernel

Theorem

*When parameterized by distance to cluster, d-*CUT *admits a polynomial kernel with* $\mathcal{O}\left(d^2 \mathrm{dc}(G)^{2d+1}\right)$ *vertices that can be computed in* $\mathcal{O}\left(d^4 \mathrm{dc}(G)^{2d+1}(n+m)\right)$ *time.*

Number of crossing edges

We can solve d-CUT in FPT time using the treewidth reduction technique.

Dániel Marx, Barry O'Sullivan, and Igor Razgon. "Treewidth Reduction for Constrained Separation and Bipartization Problems". In: *Proc. of the 27th International Symposium on Theoretical Aspects of Computer Science, (STACS)*. vol. 5. LIPIcs. 2010, pp. 561–572

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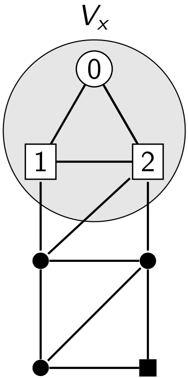
Theorem

d-Cut is FPT paramterized by the number of crossing edges.

Treewidth

Dynamic programming on a nice tree decomposition.



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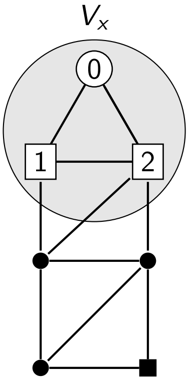

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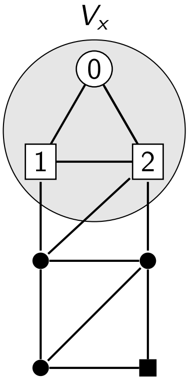

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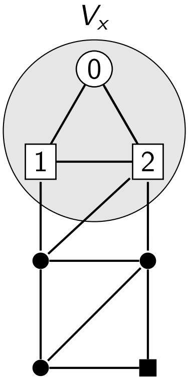

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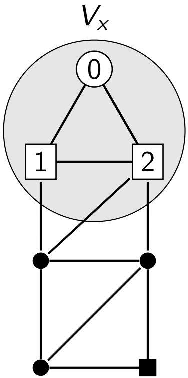

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- $t = 1$ if A and B are non-empty.
- $f_x(S, \alpha, t) = 1$ iff the subtree rooted at bag x has a partition that satisfies all of the above.

Treewidth

Theorem

For every $d \geq 1$, d-$\textsc{Cut}$ *can be solved in* $\mathcal{O}^*\left(2^{\mathsf{tw}(G)}(d+1)^{2\mathsf{tw}(G)}\right)$.

Treewidth

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Corollary

Matching Cut *can be solved in* $\mathcal{O}^*\left(8^{\mathrm{tw}(G)}\right)$. *(Improvement upon* $12^{\mathrm{tw}(G)}$*).*

N. R. Aravind, Subrahmanyam Kalyanasundaram, and Anjeneya Swami Kare. "On Structural Parameterizations of the Matching Cut Problem". In: *Proc. of the 11th International Conference on Combinatorial Optimization and Applications (COCOA)*. vol. 10628. LNCS. 2017, pp. 475–482

Other results

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Explore the other generalizations of MATCHING CUT, in particular Option 3, which seems considerably harder than the other two.

Option 3

Look for a ***p*-partition** such that each pair of parts forms a matching cut.

Thank you!