Lecture 6 – ANOVA and Linear Models

STAT/BIOF/GSAT 540

Keegan Korthauer

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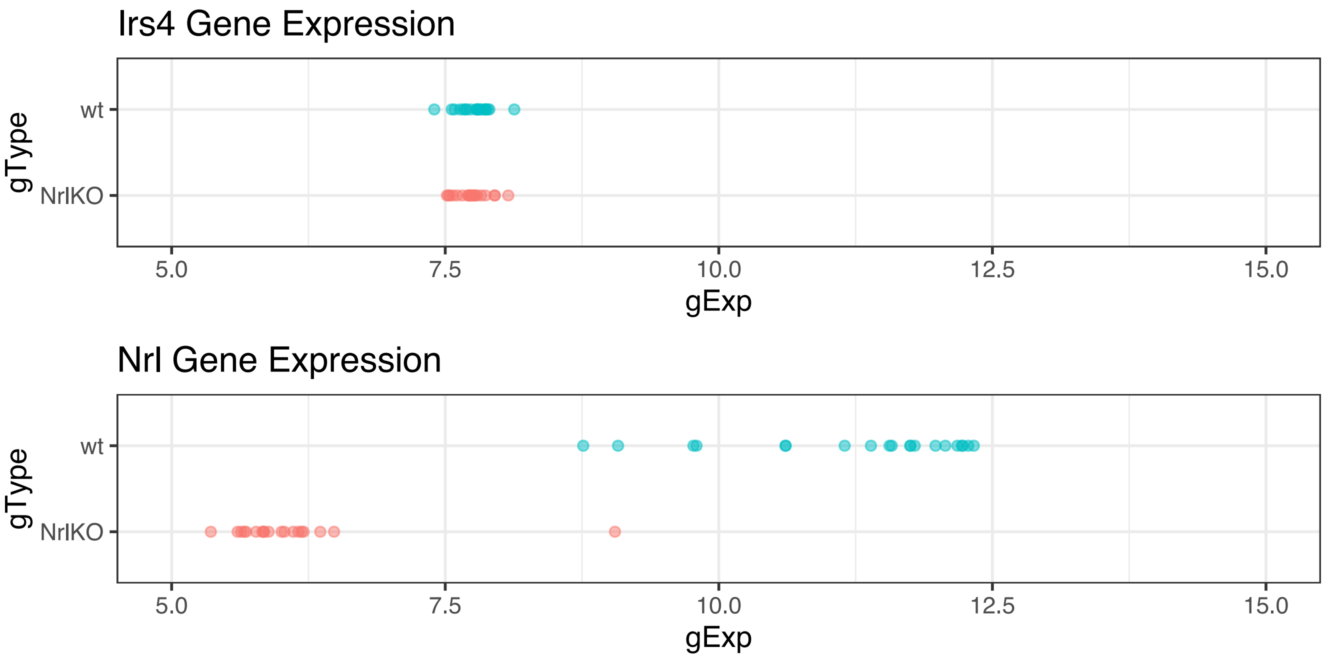
Slides by: Gabriela Cohen Freue with contributions from Jenny Bryan and Keegan Korthauer

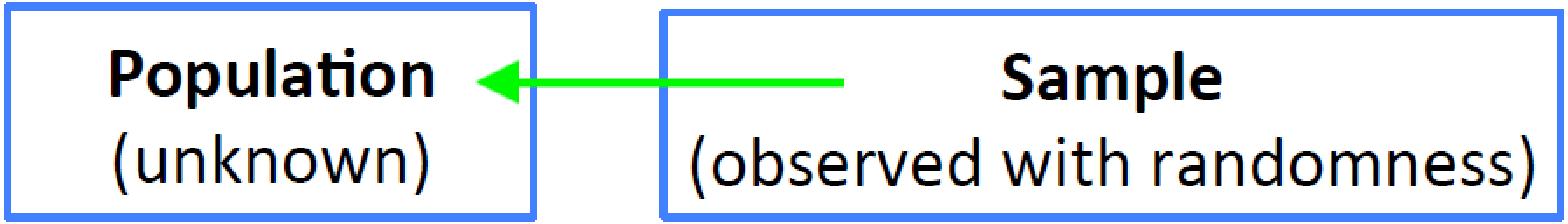
Are these genes truly different in NrlKO compared to WT?

H_0: the expression level of gene A is the same in both conditions.

Is there **enough** evidence in the data to reject H_0?



Statistics: use a random sample to learn about the population



Population
(unknown)

Sample
(observed with randomness)

$$Y \sim F$$

$$Z \sim G$$

$$Y_1,\ Y_2, \ldots, Y_{n_Y}$$

$$Z_1,\ Z_2, \ldots, Z_{n_Z}$$

$$E[Y] = \mu_Y$$

$$E[Z] = \mu_Z$$

$$\hat{\mu}_Y = \bar{Y} = \frac{\sum_{i=1}^{n_Y} Y_i}{n_Y}$$

$$\hat{\mu}_Z = \bar{Z} = \frac{\sum_{i=1}^{n_Z} Z_i}{n_Z}$$

$$H_0 : \mu_Y = \mu_Z$$

$$\bar{Y} \quad \bar{Z}$$

Last class: hypothesis testing

1. Define a test statistic to test

- 2-sample *t*-test
- Welch *t*-test
- Wilcoxon rank-sum test
- Kolmogorov-Smirnov test

2. Compute the observed value for the test statistic

3. Compute the probability of seeing a test statistic as extreme as that observed, under the null sampling distribution (p-value)

4. Make a decision about the significance of the results, based on a pre-specified value (alpha, significance level)

We can run these tests in R

Example: use the `t.test` function to test using a classical 2-sample *t*-test.

```r
miniDat %>% subset(gene == "Irs4") %>% t.test(gExp ~ gType, data = .,
    var.equal = TRUE)
```

```
##
##     Two Sample t-test
##
## data:  gExp by gType
## t = -0.52865, df = 37, p-value = 0.6002
## alternative hypothesis: true difference in means is not equal to 0
## 95 percent confidence interval:
##  -0.12597002  0.07383844
## sample estimates:
## mean in group NrlKO    mean in group wt
##            7.739684            7.765750
```

Today...

- show how to compare means of different groups (2 or more) using a linear regression model

- dummy variables to model the levels of a qualitative explanatory variable

- write a linear model using matrix notation

- understand which matrix is built by R

- distinguish between conditional and marginal effects

- -tests vs -tests

```
> t.test(gExp ~ gType, miniDat,
+        subset = gene == "Irs4", var.equal = TRUE)
```

two sample t test

$$H_0 : \mu_1 = \mu_2$$

```
> summary(aov(gExp ~ gType, miniDat,
+             subset = gene == "Irs4"))
```

(one-way) analysis of variance
“ANOVA”

```
> summary(lm(gExp ~ gType, miniDat,
+            subset = gene == "Irs4"))
```

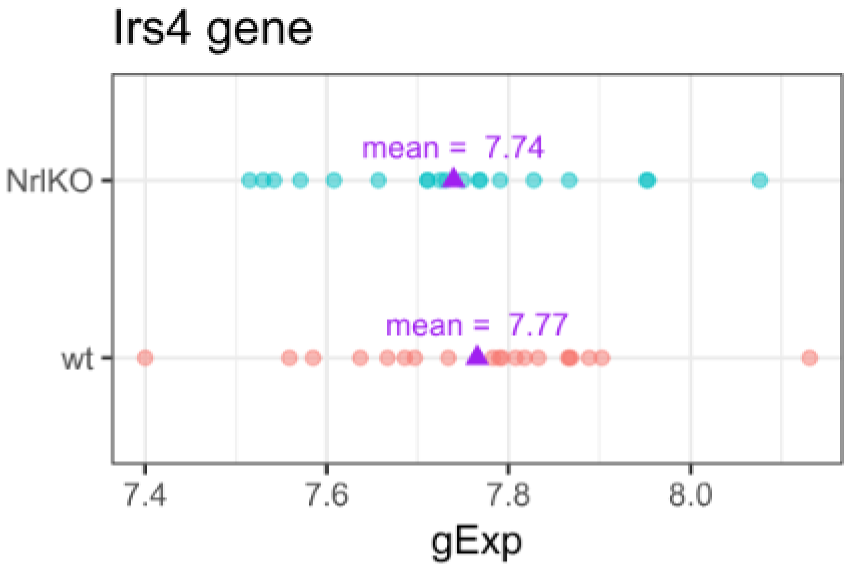
linear model
linear regression

It seems that we can use any of these methods to test

```
> t.test(gExp ~ gType, miniDat,
+          subset = gene == "Irs4", var.equal = TRUE)

    Two Sample t-test

data:  gExp by gType
t = 0.5286, df = 37, p-value = 0.6002
<snip, snip>
sample estimates:
   mean in group wt mean in group NrlKO
           7.765750            7.739684
```


Irs4 gene

```
7.739684 - 7.765750 = -0.026066
```

```
> summary(aov(gExp ~ gType, miniDat,
+              subset = gene == "Irs4"))
            Df Sum Sq Mean Sq F value Pr(>F)
gType        1 0.0066 0.00662   0.279    0.6
Residuals   37 0.8764 0.02369
```

```
-0.5286494 ^ 2 = 0.2794702
```

```
> summary(lm(gExp ~ gType, miniDat,
+             subset = gene == "Irs4"))
<snip, snip>
Coefficients:
```

t-test *vs* linear regression: why the same results?

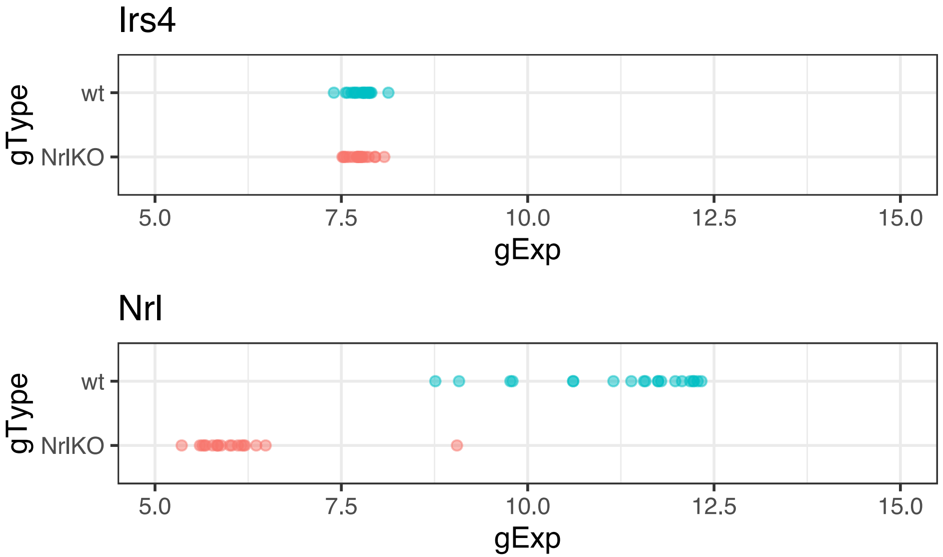
```
irs4Dat <- subset(miniDat,gene=="Irs4")
ttest.irs4<-t.test(gExp ~ gType, irs4Dat, var.equal = TRUE)
list("t value"=ttest.irs4$stat,"p-value"=ttest.irs4$p.value)
```

```
## $t value
##          t
## -0.5286494
##
## $p-value
## [1] 0.6002058
```

```
lm.irs4 <- summary(lm(gExp ~ gType, irs4Dat))
list("t value"=lm.irs4$coeff[2,3],"p-value"=lm.irs4$coeff[2,4])
```

```
## $t value
## [1] 0.5286494
##
## $p-value
## [1] 0.6002058
```

t-test *vs* linear regression: where's the line?



> Note that the -axis in these plots is not numerical, thus a line in this space does not have any mathematical meaning.

Why can we run a -test with a linear regression model?

From *t*-test to linear regression

Let's change the notation to give a common framework to all methods

↓

We can use a subindeces to distinguish observations from each group, i.e.,

where or identifies the groups; and identifies the observations within each group

> For example: is the first observation in group 1 or WT

The goal is to test

using data from the model

where or ; and .

> For simplicity, we assume a common distribution for all groups

Note that the population means are given by , i.e., the model is written with a cell-means - - parametrization

Note that for each group, the population mean is given by

A natural **estimator** of the population mean is the **sample mean**

Classical hypothesis testing methods use the group sample means as estimators

See, for example, the `t.test` function in R:

```
ttest.irs4$estimate
```

```
## mean in group NrlKO     mean in group wt
##            7.739684            7.765750
```

However, the `lm` function reports other estimates, why?

```
(means.irs4 <- as.data.frame(irs4Dat %>% group_by(gType) %>%
    summarize(meanGroups = mean(gExp, digits = 6))))
```

```
##   gType meanGroups
## 1 NrlKO   7.739684
## 2    wt   7.765750
```

```
lm.irs4$coefficients[,1]
```

```
## (Intercept)     gTypewt
##  7.73968421  0.02606579
```

↓

`(Intercept)` is the sample mean of NrlKO group

but `gTypewt` is not the sample mean of the WT group

Parametrizations: which parameters we use to write the model?

By default, the `lm` does not use the cell-means parametrization The goal is to *compare* the means, not to study each in isolation

From **cell-means** - :

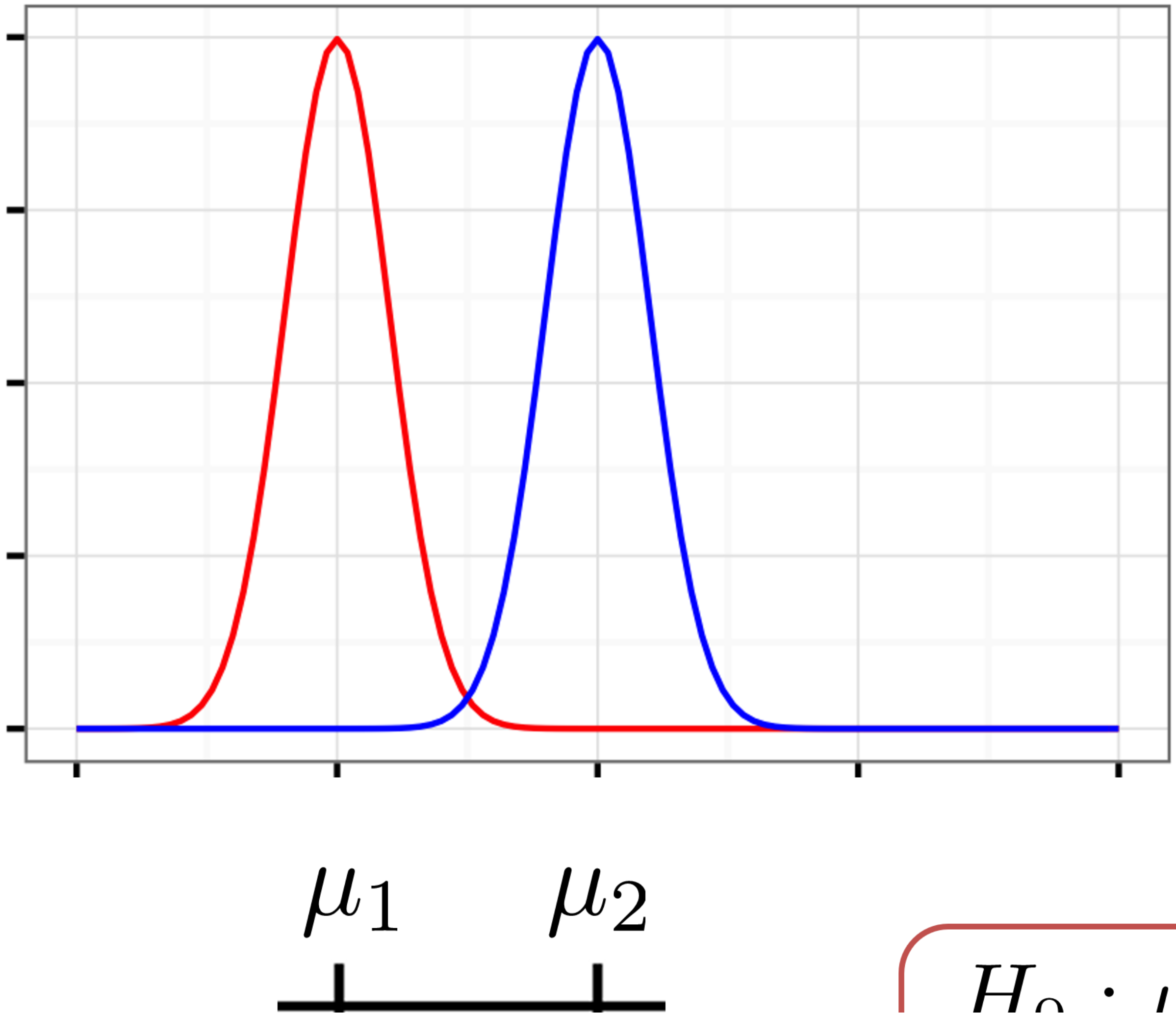
↓

to **reference-treatment effect** - :

Note that for each group, the population mean is given by

and *compares* the means

Relation between parametrizations



$H_0 : \mu_1 = \mu_2$

`lm` reports the sample mean of the reference group (NrlKO):

and the treatment effect, i.e., difference between the sample means of both groups:

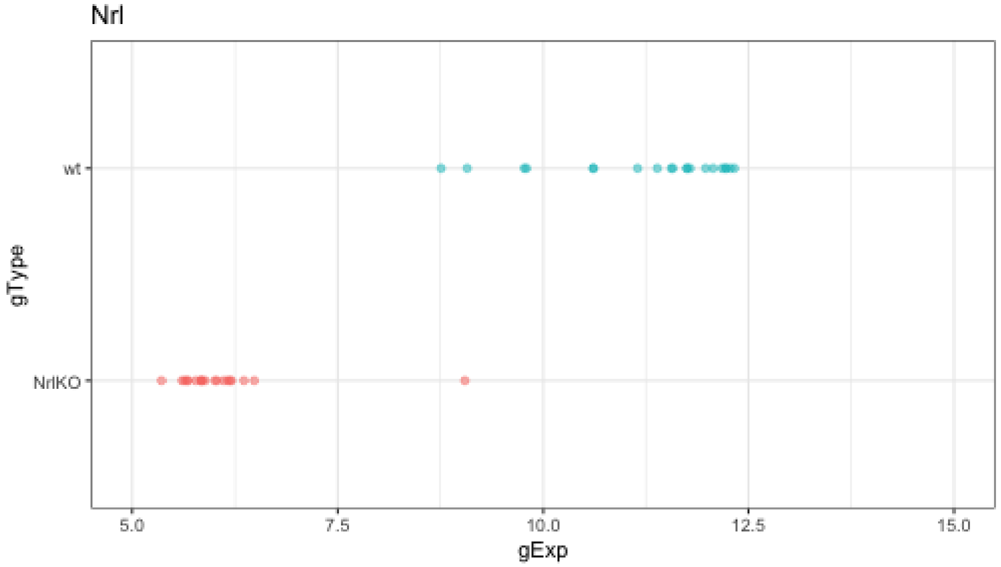
```
lm.irs4$coefficients[, 1]
```

```
## (Intercept)     gTypewt
##  7.73968421  0.02606579
```

```
data.frame(meanWT = means.irs4[1, 2],
    meanDiff = diff(means.irs4$meanGroups))
```

```
##     meanWT   meanDiff
## 1 7.739684 0.02606579
```

We still haven't answered ... where's the line??



Dummy variables

Let's re-write our model using dummy (or indicator) variables:

↓

> Note that , because and and , because (for all)

The second form is written as a linear ( ) regression, with a special (dummy) explanatory variable

Using a dummy variables to model our categorical variables `gtype` we can perform a 2-sample *t*-test with a linear model

```r
list("t value"=ttest.irs4$stat,"p-value"=ttest.irs4$p.value)
```

```
## $t value
##          t
## -0.5286494
##
## $p-value
## [1] 0.6002058
```

```r
list("t value"=lm.irs4$coeff[2,3],"p-value"=lm.irs4$coeff[2,4])
```

```
## $t value
## [1] 0.5286494
##
## $p-value
## [1] 0.6002058
```

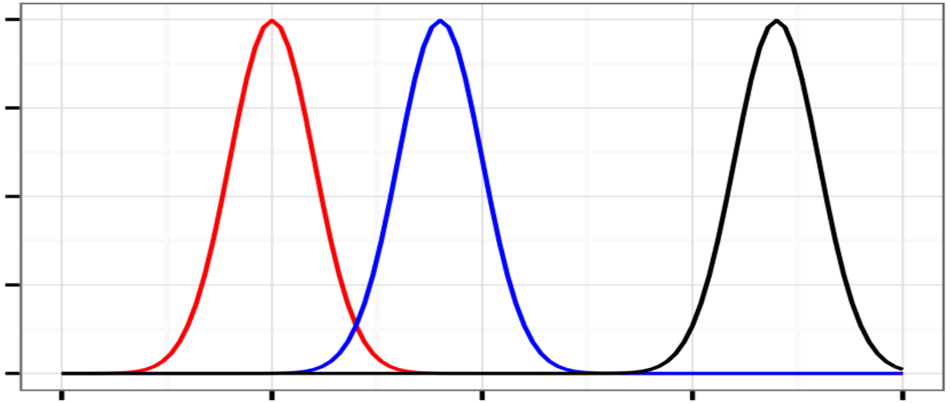
Beyond 2-groups comparisons: difference of means

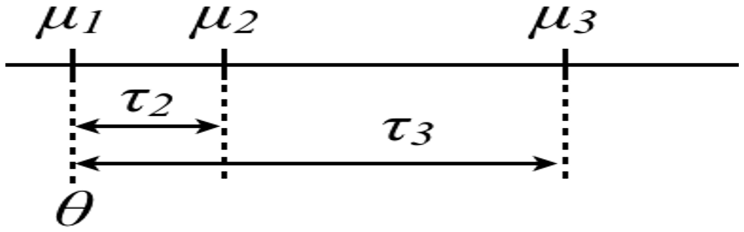
“cell-means”

$$Y_{ij} = \mu_j + \varepsilon_{ij}$$

“reference-treatments”

$$Y_{ij} = \theta + \tau_j + \varepsilon_{ij}, (\tau_1 = 0)$$





More than 2 groups!

Dummy variables can be used to model one *or more* categorical variables with 2 *or more* levels!

2-sample *t*-test using a linear model

1-way ANOVA with many levels using a linear model

This is why R can estimate all of them with `lm()`

in general, *yet* another parametrization is used to present ANOVA

t-test

> Special case of ANOVA, but with ANOVA you can compare **more than two groups** and **more than one factor**.

ANOVA

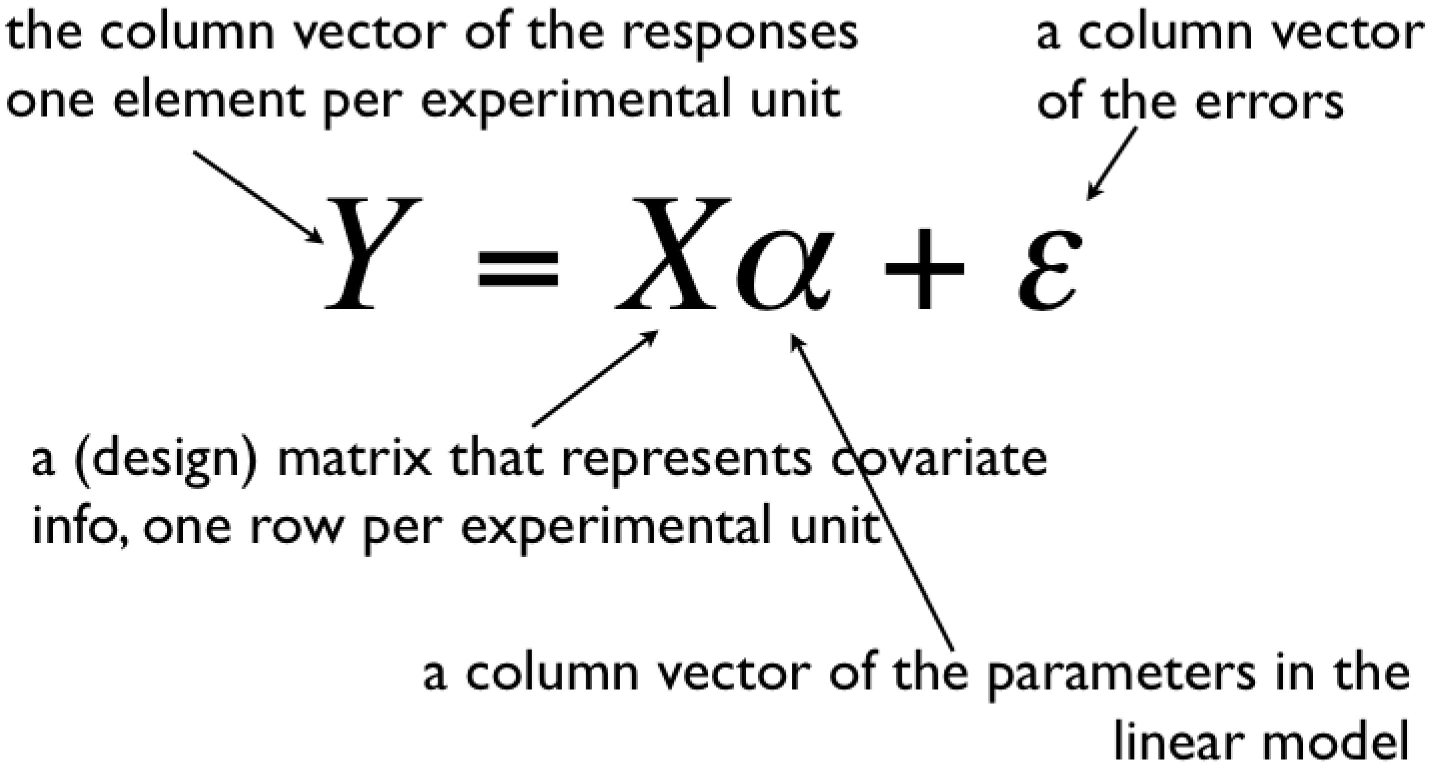
> Special case of linear regression, but with linear regression you can include **quantitative variables** in the model.

Linear regression

> Provides a unifying framework to model the association between a response **many quantitative and qualitative variables**.

In R: all can be computed using the `lm()` function.

Linear models using matrix notation



the column vector of the responses
one element per experimental unit

a column vector
of the errors

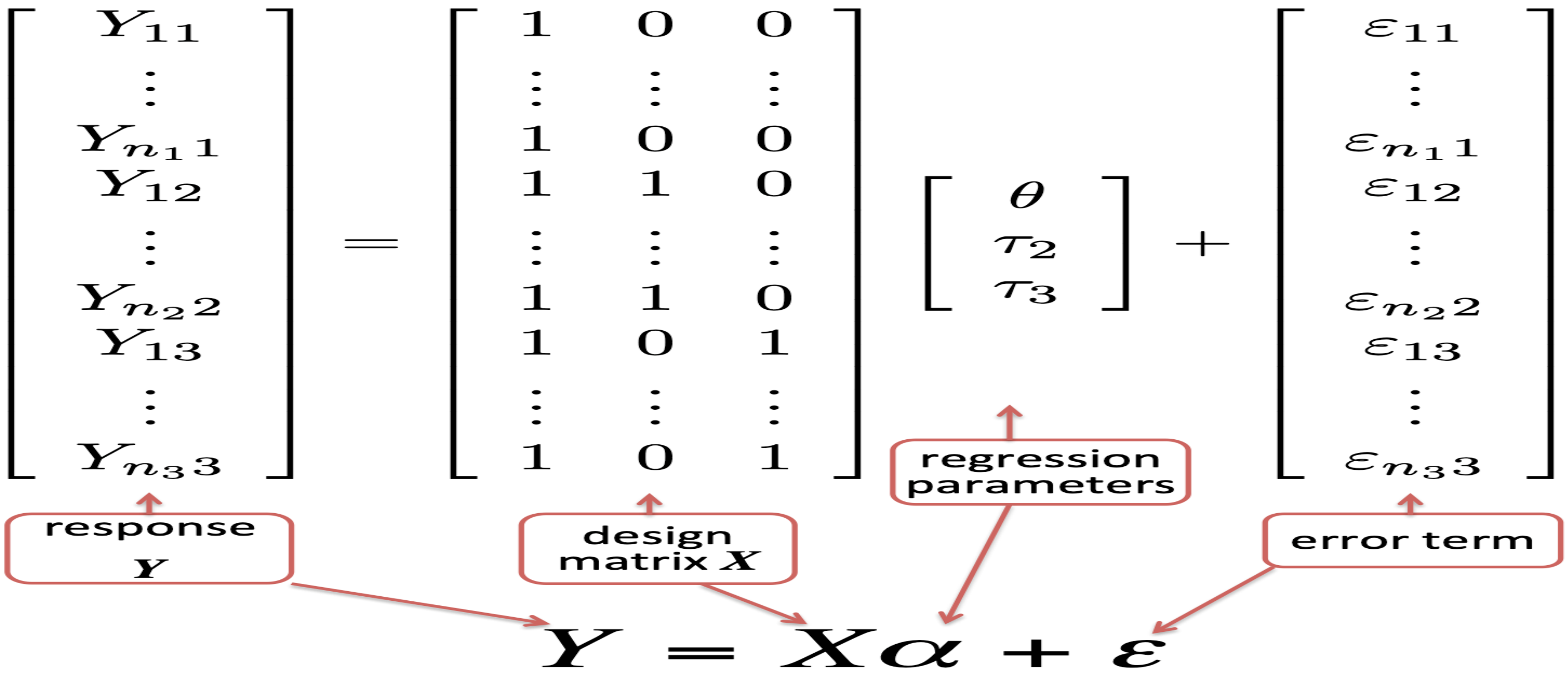
$$Y = X\alpha + \varepsilon$$

a (design) matrix that represents covariate
info, one row per experimental unit

a column vector of the parameters in the
linear model

It will become handy to write our model using matrix notation

Let's form an matrix for a 3-groups comparison:



$$\begin{bmatrix} Y_{11} \\ \vdots \\ Y_{n_1 1} \\ Y_{12} \\ \vdots \\ Y_{n_2 2} \\ Y_{13} \\ \vdots \\ Y_{n_3 3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta \\ \tau_2 \\ \tau_3 \end{bmatrix} + \begin{bmatrix} \varepsilon_{11} \\ \vdots \\ \varepsilon_{n_1 1} \\ \varepsilon_{12} \\ \vdots \\ \varepsilon_{n_2 2} \\ \varepsilon_{13} \\ \vdots \\ \varepsilon_{n_3 3} \end{bmatrix}$$

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\alpha} + \boldsymbol{\varepsilon}$$

Note that and become the 2nd and 3rd columns of : for the reference group; for the 2nd group; and for the 3rd group

$$Y_{ij} = \theta + \tau_2 \times x_{ij2} + \tau_3 \times x_{ij3} + \varepsilon_{ij}$$

$$\left[\begin{array}{c} Y_{11} \\ \hline \vdots \\ Y_{n_1 1} \\ Y_{12} \\ \hline \vdots \\ Y_{n_2 2} \\ Y_{13} \\ \hline \vdots \\ Y_{n_3 3} \end{array}\right] = \left[\begin{array}{ccc} 1 & 0 & 0 \\ \hline \vdots & \vdots & \vdots \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ \hline \vdots & \vdots & \vdots \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ \hline \vdots & \vdots & \vdots \\ 1 & 0 & 1 \end{array}\right] \left[\begin{array}{c} \theta \\ \tau_2 \\ \tau_3 \end{array}\right] + \left[\begin{array}{c} \varepsilon_{11} \\ \hline \vdots \\ \varepsilon_{n_1 1} \\ \varepsilon_{12} \\ \hline \vdots \\ \varepsilon_{n_2 2} \\ \varepsilon_{13} \\ \hline \vdots \\ \varepsilon_{n_3 3} \end{array}\right]$$

> Note that
>
> Note that
>
> Note that

Which is the same as

Which is the same as

$$Y = X\alpha + \varepsilon$$

$$\begin{bmatrix} Y_{11} \\ Y_{21} \\ \vdots \\ Y_{n_3 3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta \\ \tau_2 \\ \tau_3 \end{bmatrix} + \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{21} \\ \vdots \\ \varepsilon_{n_3 3} \end{bmatrix}$$

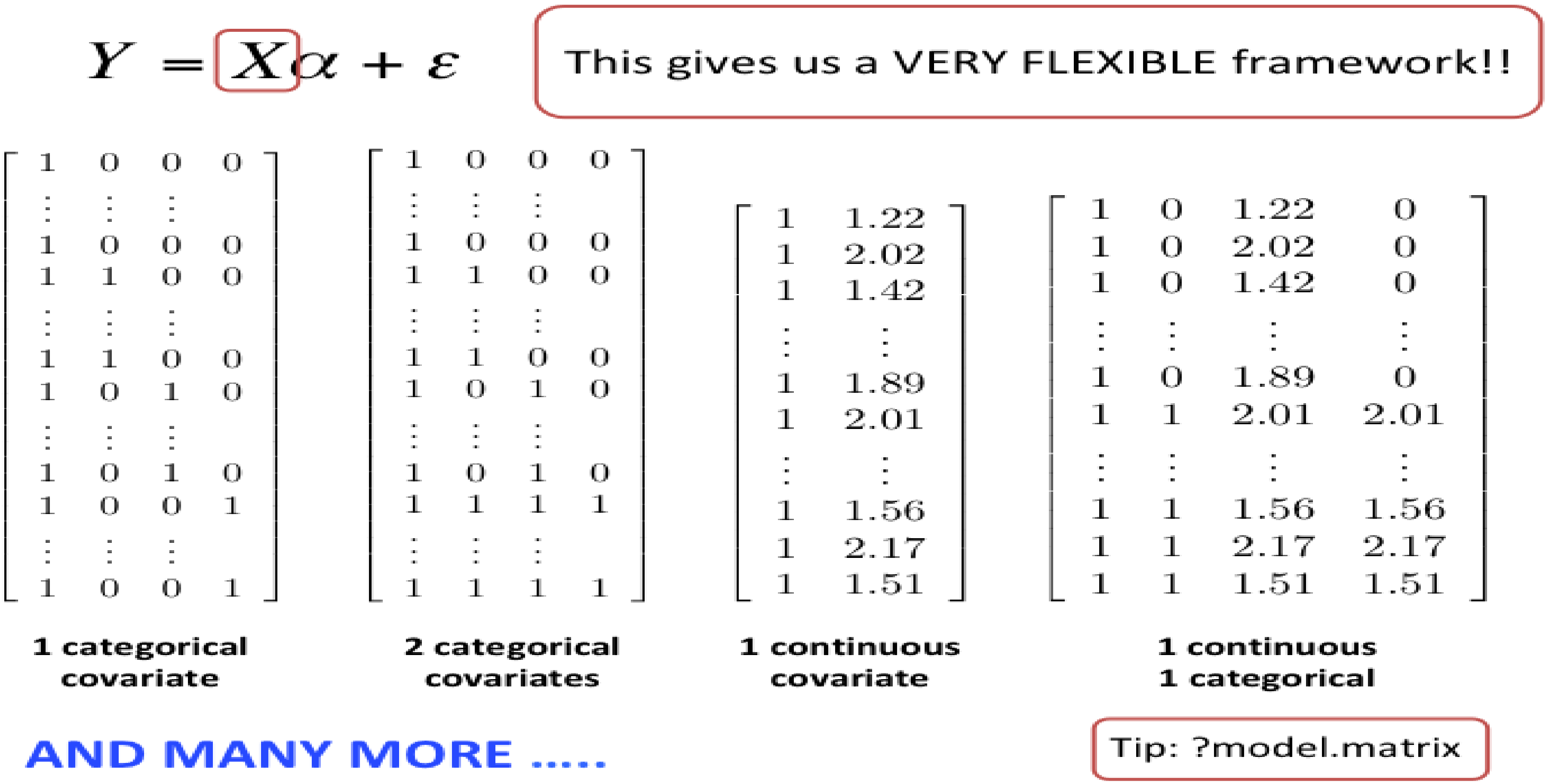
Reference group: μ_1

$\mu_2 - \mu_1$

$\mu_3 - \mu_1$

Note that the model is still written with a reference-treatment parametrization (difference of means)

Linear regression can include **quantitative & qualitative covariates.**



$$Y = \boxed{X}\alpha + \varepsilon$$

This gives us a VERY FLEXIBLE framework!!

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1.22 \\ 1 & 2.02 \\ 1 & 1.42 \\ \vdots & \vdots \\ 1 & 1.89 \\ 1 & 2.01 \\ \vdots & \vdots \\ 1 & 1.56 \\ 1 & 2.17 \\ 1 & 1.51 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 1.22 & 0 \\ 1 & 0 & 2.02 & 0 \\ 1 & 0 & 1.42 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 1.89 & 0 \\ 1 & 1 & 2.01 & 2.01 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1.56 & 1.56 \\ 1 & 1 & 2.17 & 2.17 \\ 1 & 1 & 1.51 & 1.51 \end{bmatrix}$$

1 categorical covariate | **2 categorical covariates** | **1 continuous covariate** | **1 continuous 1 categorical**

AND MANY MORE .....

Tip: ?model.matrix

> Linear in the parameters : can contain , , etc.

How it works in practice using lm() in R

↓

```
lm(y ~ x, data = yourData)
```

y ~ x: formula,
y numeric,
x numeric and/or factor

yourData: data.frame in which x and y are to be found (optional but recommended)

By default, R uses a ref-tx parametrization but you can control that!

- Mathematically, is a numeric matrix
- If your data contains categorical variables (e.g., `gType`), you need to set them as **factors**
- R creates appropriate dummy variables for factors!

```
str(irs4Dat$gType)
```

```
##  Factor w/ 2 levels "NrlKO","wt": 2 2 2 2 1 1 1 2 2 2 ...
```

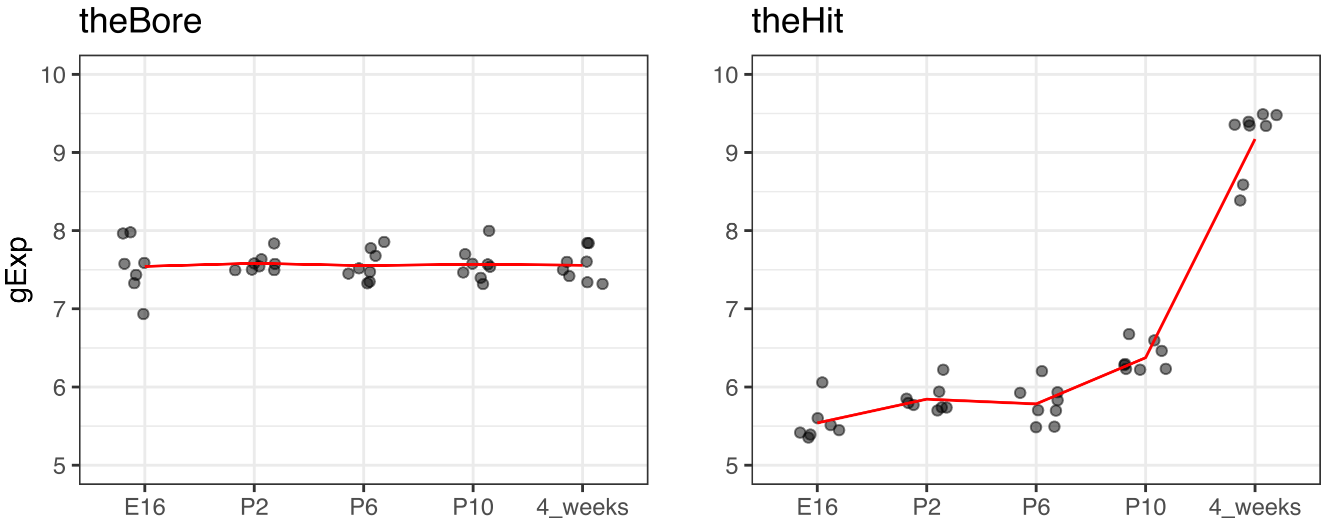
Under the hood, R creates a numeric :

```r
data.frame(X = model.matrix(gExp ~ gType, irs4Dat),
    gType = irs4Dat$gType) %>% head(10)
```

```
##    X..Intercept. X.gTypewt gType
## 1              1         1    wt
## 2              1         1    wt
## 3              1         1    wt
## 4              1         1    wt
## 5              1         0 NrlKO
## 6              1         0 NrlKO
## 7              1         0 NrlKO
## 8              1         1    wt
## 9              1         1    wt
## 10             1         1    wt
```

Beyond 2-group comparisons in our case study:

Is the expression of gene A the same at all developmental stages?

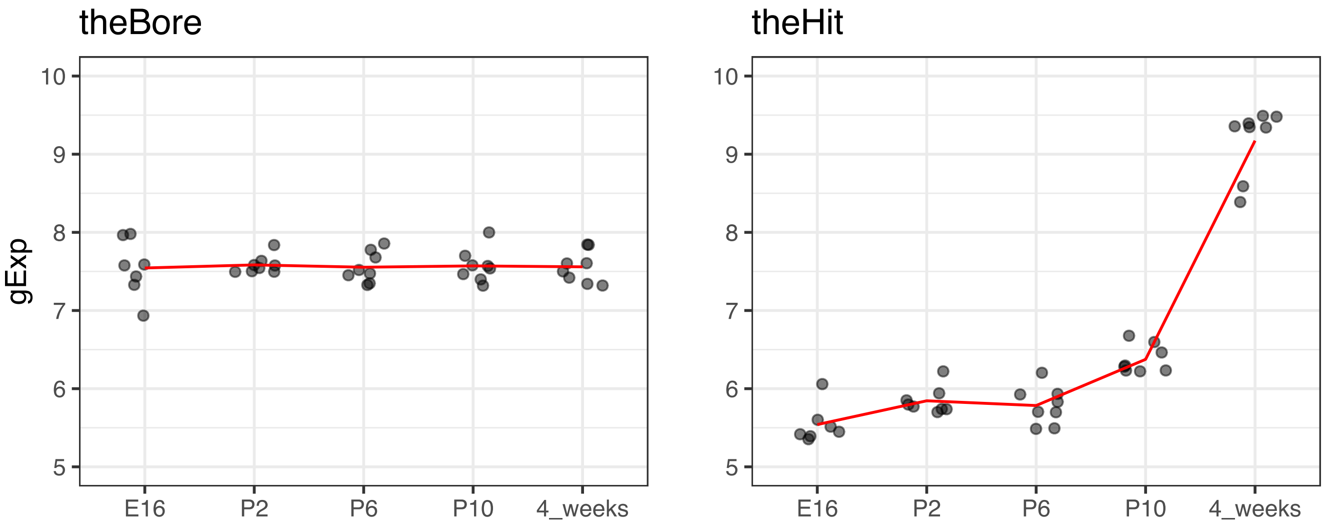


Note: 4W = 4_weeks

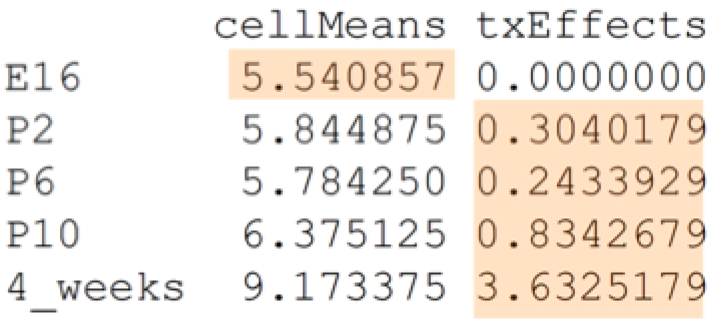
The sample means:

```
with(devDat, tapply(gExp, list(devStage, gene), mean))
```

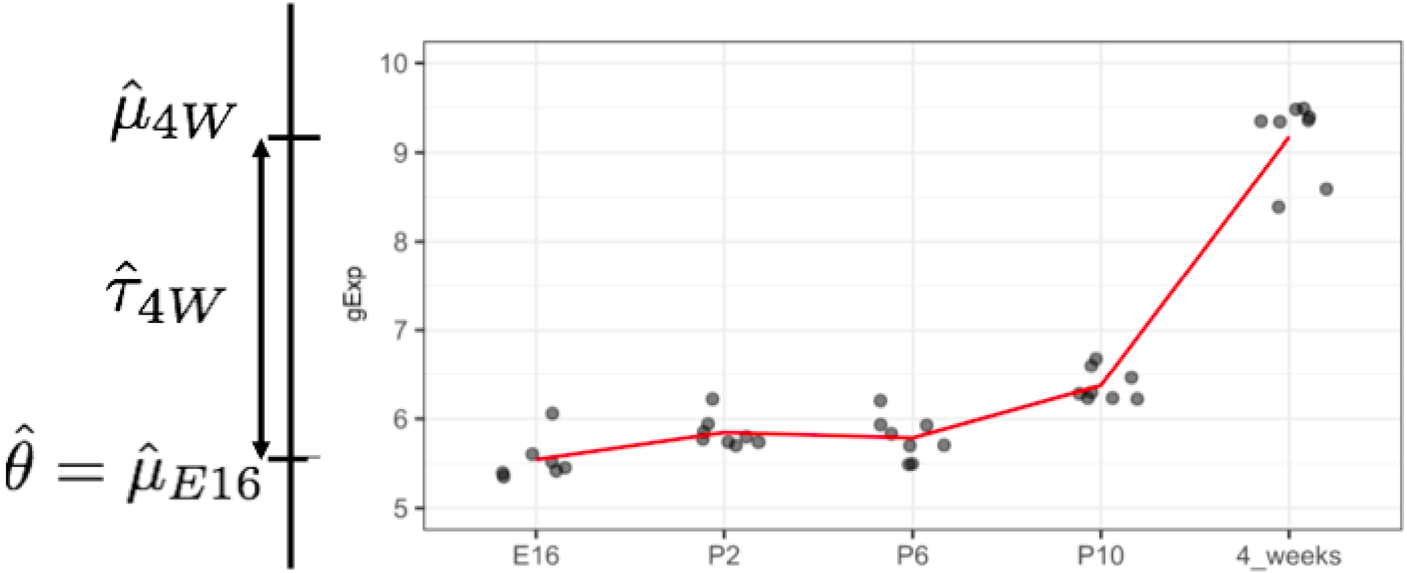
```
##          theBore   theHit
## E16     7.544143 5.540857
## P2      7.583500 5.844875
## P6      7.554000 5.784250
## P10     7.571000 6.375125
## 4_weeks 7.559000 9.173375
```



"theHit" with significant time ("treatment") effect



```
        cellMeans  txEffects
E16      5.540857  0.0000000
P2       5.844875  0.3040179
P6       5.784250  0.2433929
P10      6.375125  0.8342679
4_weeks  9.173375  3.6325179
```



"theHit" with significant time ("treatment") effect

Can you guess the size of the matrix??

> How many dummy variables do we need?

"theHit" with significant time ("treatment") effect

```
##   devStage cellMeans txEffects
## 1      E16  5.540857 0.0000000
## 2       P2  5.844875 0.3040179
## 3       P6  5.784250 0.2433929
## 4      P10  6.375125 0.8342679
## 5  4_weeks  9.173375 3.6325179
```

We need 4 dummy variables to estimate and test 4 time differences:

> : P2 vs E16, : P6 vs E16, : P10 vs E16, : 4W vs E16)

Mathematically:

Notation: , where is an index for the observation, for the level of `devStage`, and for the name of the dummy variable

Under the hood, R creates a numeric :

```
X.matrix <- data.frame(X = model.matrix(gExp ~ devStage, irs4Dat),
    devStage = irs4Dat$devStage)
```

```
##    X..Intercept. X.dStP2 X.dStP6 X.dStP10 X.dS4W  dS
## 1              1       0       0        0      0 E16
## 2              1       0       0        0      0 E16
## 3              1       0       0        0      0 E16
## 4              1       0       0        0      0 E16
## 5              1       0       0        0      0 E16
## 6              1       0       0        0      0 E16
## 7              1       0       0        0      0 E16
## 8              1       1       0        0      0  P2
## 9              1       1       0        0      0  P2
## 10             1       1       0        0      0  P2
## 11             1       1       0        0      0  P2
## 12             1       1       0        0      0  P2
## 13             1       1       0        0      0  P2
## 14             1       1       0        0      0  P2
## 15             1       1       0        0      0  P2
## 16             1       0       1        0      0  P6
```

Note: column names changed and first 16 rows displayed to fit output in the page (code hidden)

```
summary(lm(gExp~devStage,subset(devDat,gene=="theHit")))$coeff
```

```
##                  Estimate  Std. Error   t value     Pr(>|t|)
## (Intercept)     5.5408571  0.1021381 54.248698 1.307554e-34
## devStageP2      0.3040179  0.1398583  2.173756 3.678022e-02
## devStageP6      0.2433929  0.1398583  1.740282 9.085489e-02
## devStageP10     0.8342679  0.1398583  5.965093 9.559065e-07
## devStage4_weeks 3.6325179  0.1398583 25.972843 5.266481e-24
```

```
means.dev %>% mutate(txEffects=cellMeans-cellMeans[1])
```

```
##   devStage cellMeans txEffects
## 1      E16  5.540857 0.0000000
## 2       P2  5.844875 0.3040179
## 3       P6  5.784250 0.2433929
## 4      P10  6.375125 0.8342679
## 5  4_weeks  9.173375 3.6325179
```

Estimate:

or

> we are not usually interested in testing this hypothesis

```
summary(lm(gExp~devStage,subset(devDat,gene=="theHit")))$coeff
```

```
##                    Estimate Std. Error   t value     Pr(>|t|)
## (Intercept)       5.5408571  0.1021381 54.248698 1.307554e-34
## devStageP2        0.3040179  0.1398583  2.173756 3.678022e-02
## devStageP6        0.2433929  0.1398583  1.740282 9.085489e-02
## devStageP10       0.8342679  0.1398583  5.965093 9.559065e-07
## devStage4_weeks 3.6325179  0.1398583 25.972843 5.266481e-24
```

```
means.dev %>% mutate(txEffects=cellMeans-cellMeans[1])
```

```
##   devStage cellMeans txEffects
## 1      E16  5.540857 0.0000000
## 2       P2  5.844875 0.3040179
## 3       P6  5.784250 0.2433929
## 4      P10  6.375125 0.8342679
## 5  4_weeks  9.173375 3.6325179
```

Estimate:

or

> we *are* usually interested in testing this hypothesis: first 2 days after birth

```
summary(lm(gExp~devStage,subset(devDat,gene=="theHit")))$coeff
```

```
##                  Estimate Std. Error   t value     Pr(>|t|)
## (Intercept)     5.5408571  0.1021381 54.248698 1.307554e-34
## devStageP2      0.3040179  0.1398583  2.173756 3.678022e-02
## devStageP6      0.2433929  0.1398583  1.740282 9.085489e-02
## devStageP10     0.8342679  0.1398583  5.965093 9.559065e-07
## devStage4_weeks 3.6325179  0.1398583 25.972843 5.266481e-24
```

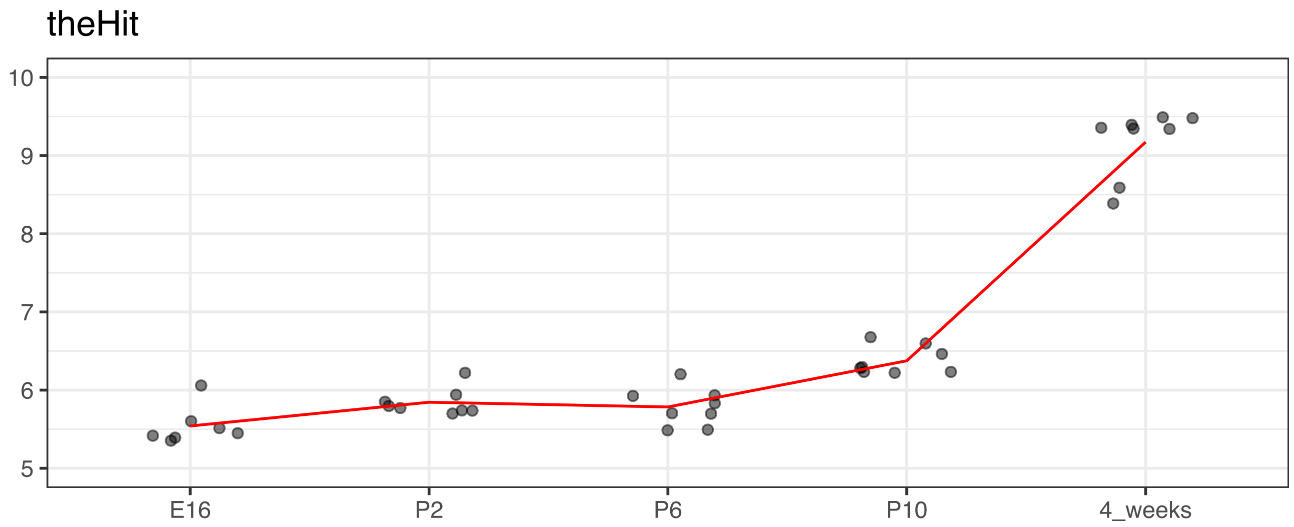
```
means.dev %>% mutate(txEffects=cellMeans-cellMeans[1])
```

```
##   devStage cellMeans txEffects
## 1      E16  5.540857 0.0000000
## 2       P2  5.844875 0.3040179
## 3       P6  5.784250 0.2433929
## 4      P10  6.375125 0.8342679
## 5  4_weeks  9.173375 3.6325179
```

Estimate:

or

> we *are* usually interested in testing this hypothesis: 4 weeks after birth


theHit
10
9
8
7
6
5
E16
P2
P6
P10
4_weeks

We generally test two types of null hypotheses:

vs

for *j* each *j* individually

e.g., Is gene A differencially expressed 2 days after birth?

vs

for all *j* at the same time

e.g., Is gene A significantly affected by time (`devStage`)?

Two types of null hypotheses in R:

$$Y = X\alpha + \varepsilon$$

$$\alpha = (\theta, \tau_{P2}, \tau_{P6}, \tau_{P10}, \tau_{4\_weeks})$$

$H_0 : \tau_j = 0$

vs

$H_0 : \tau_j \neq 0$

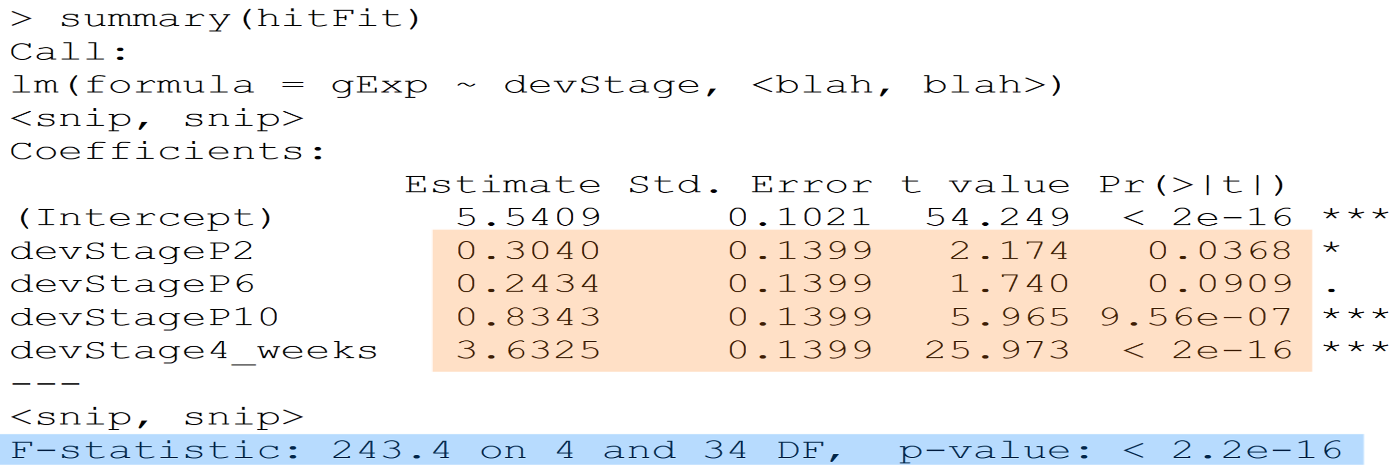
for each j individually

$H_0 : \tau_j = 0$ AND statement

vs

$H_0 : \tau_j \neq 0$ OR statement

for all j at the same time



```
> summary(hitFit)
Call:
lm(formula = gExp ~ devStage, <blah, blah>)
<snip, snip>
Coefficients:
                 Estimate Std. Error t value Pr(>|t|)
(Intercept)        5.5409     0.1021  54.249  < 2e-16 ***
devStageP2         0.3040     0.1399   2.174   0.0368 *
devStageP6         0.2434     0.1399   1.740   0.0909 .
devStageP10        0.8343     0.1399   5.965 9.56e-07 ***
devStage4_weeks    3.6325     0.1399  25.973  < 2e-16 ***
---
<snip, snip>
F-statistic: 243.4 on 4 and 34 DF,  p-value: < 2.2e-16
```

F-test and overall significance of one or more covariates

- the *t*-test in linear regression allows us to test single hypotheses:

- but we often like to test multiple hypotheses *simultaneously*:

  the *F*-test allows us to test such compound tests

To conclude

- we can use different parametrizations to write statistical models

From **cell-means** - :

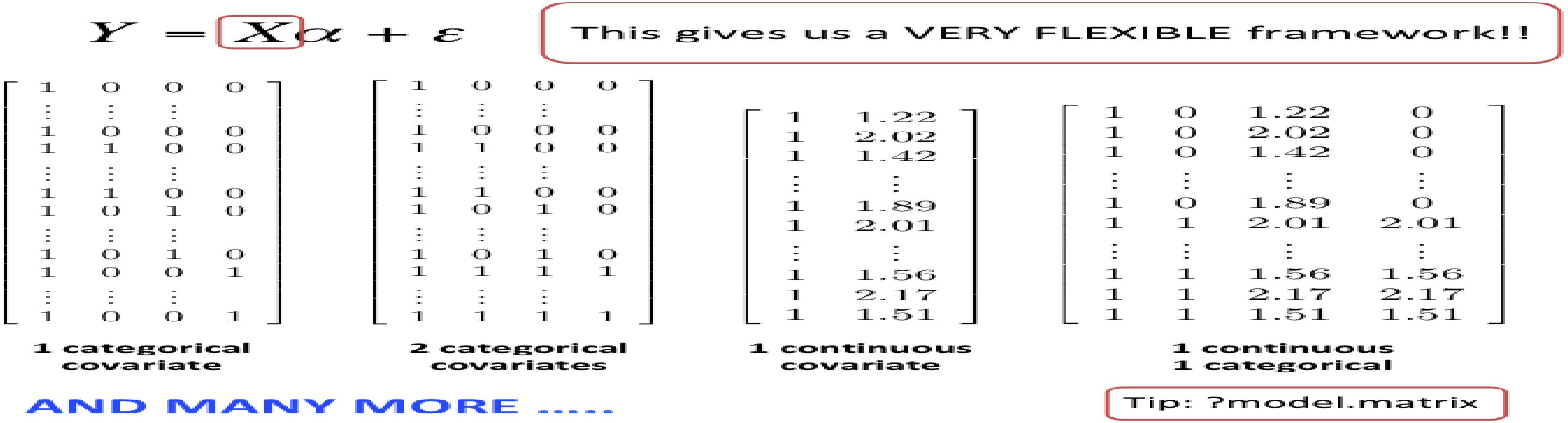
to **reference-treatment effect** - : (used by default by `lm`)

- we can compare group means (2 or more) using a linear model

- **dummy variables** (e.g.,) to model the levels of a qualitative explanatory variables
- qualitative variables need to be set as "factors" in the data --> R creates the dummy variables

- we can write a linear model using matrix notation:

- Linear model can include **quantitative & qualitative covariates**.



$$Y = \boxed{X}\alpha + \varepsilon$$

This gives us a VERY FLEXIBLE framework!!

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1.22 \\ 1 & 2.02 \\ 1 & 1.42 \\ \vdots & \vdots \\ 1 & 1.89 \\ 1 & 2.01 \\ \vdots & \vdots \\ 1 & 1.56 \\ 1 & 2.17 \\ 1 & 1.51 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 1.22 & 0 \\ 1 & 0 & 2.02 & 0 \\ 1 & 0 & 1.42 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 1.89 & 0 \\ 1 & 1 & 2.01 & 2.01 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1.56 & 1.56 \\ 1 & 1 & 2.17 & 2.17 \\ 1 & 1 & 1.51 & 1.51 \end{bmatrix}$$

1 categorical covariate

2 categorical covariates

1 continuous covariate

1 continuous
1 categorical

AND MANY MORE .....

Tip: ?model.matrix

- distinguish between single and joint hypotheses:

    - -tests vs -tests