Ex. 1

$P_{ab} \to 2^N \times 2^N$ matrix $\quad P_{ab} = Pr(\text{meas. bitstring } a \mid \text{bitstring } b)$

Error-free prob. for each bitstring a : v_a

MEASURED prob. " " " " : v'_a

1) If v'_a : $\sum_a v'_a = 1 \implies \sum v_a = 1$

$$v'_a = \sum_b P_{ab} v_b$$

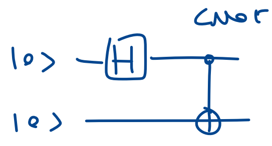
$$\sum_a v'_a = \sum_a \sum_b P_{ab} v_b = 1 = \sum_b v_b \Big(\underbrace{\sum_a P_{ab}}_{=1} \Big) = 1$$

$\sum_a P_{ab}$ has to be 1 because given starting bitstring b, we will end up in one of the 2^N possible ones.

$$\implies \sum_b v_b = 1$$

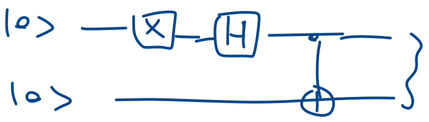
Exercise 2: Use CNOT, H, and X gates to come up with the quantum circuits to prepare the four EPR pairs in Eq. (4.2). Suppose you are now given an unknown EPR state:
(a) How do you measure the state to infer the label?
(b) As discussed above, depending on the measurement outputs from Alice, a unitary is performed on Bob's qubit to decode the message. List the decoding unitary in each case.

$|\phi^+\rangle$:



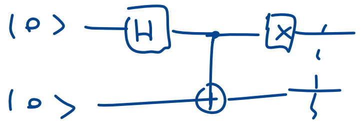
$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$|\phi^-\rangle$:



$|\phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$

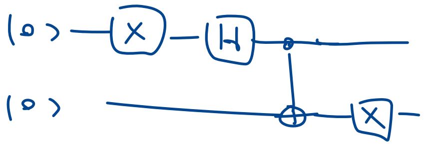
$|\psi^+\rangle$:



$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

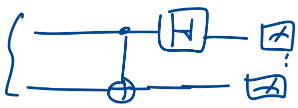
$$|\psi_{\phi^+}\rangle = (X \otimes \mathbb{1})(|00\rangle + |11\rangle) = |10\rangle + |01\rangle$$

$|\psi^-\rangle$:



$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$

(a) You apply EPR $|\psi\rangle$



$|\phi^+\rangle \longrightarrow |00\rangle$

$|\phi^-\rangle \longrightarrow |10\rangle$

$|\psi^+\rangle : \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}(|11\rangle + |01\rangle) \xrightarrow{H} |01\rangle$

$|\psi^-\rangle \longrightarrow |11\rangle$

b) If Alice measures

$|\phi^+\rangle \leftrightarrow |00\rangle \Longrightarrow$ the state Bob has is $|\psi_B\rangle = \alpha|0\rangle + \beta|1\rangle \rightarrow U = \mathbb{1}$

$|\phi^-\rangle \leftrightarrow |10\rangle \Longrightarrow |\psi_B\rangle = \alpha|0\rangle - \beta|1\rangle \longrightarrow U = Z$

$|\psi^+\rangle \leftrightarrow |01\rangle \Longrightarrow |\psi_B\rangle = \alpha|1\rangle + \beta|0\rangle \longrightarrow U = X$

$|\psi^-\rangle \leftrightarrow |11\rangle \Longrightarrow |\psi_B\rangle = \alpha|1\rangle - \beta|0\rangle \longrightarrow U = ZX = iY$

Ex. 4

$$p(0) = |\langle 0|\psi(t)\rangle|^2 = |\alpha(t)|^2 = \cos^2\left(\frac{\Omega t}{2}\right) + \frac{\Delta^2}{\Omega^2}\sin^2\left(\frac{\Omega t}{2}\right)$$

$$= \frac{\Delta^2}{\Omega^2} + \left(1 - \frac{\Delta^2}{\Omega^2}\right)\cos^2\left(\frac{\Omega t}{2}\right)$$

$$\Omega^2 - \Delta^2 = \omega_1^2 + \Delta^2 - \Delta^2 = \omega_1^2 \longrightarrow = \frac{\Delta^2}{\Omega^2} + \frac{\omega_1^2}{\Omega^2}\cos^2\left(\frac{\Omega t}{2}\right)$$

$$p(1) = |\langle 1|\psi(t)\rangle|^2 = |\beta(t)|^2 = \frac{\omega_1^2}{\Omega^2}\sin^2\left(\frac{\Omega t}{2}\right)$$

Ex. 6

$$|\psi\rangle = \underbrace{\cos\frac{\theta}{2}}_{\alpha}|0\rangle + \underbrace{e^{i\varphi}\sin\frac{\theta}{2}}_{\beta}|1\rangle$$

$$\rho = \begin{pmatrix} \cos^2\theta/2 & e^{-i\varphi}\cos\theta/2\,\sin\theta/2 \\ e^{i\varphi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{pmatrix}$$

$e^{-i\varphi}\cos\theta/2\,\sin\theta/2$

$\cos\varphi\,\cos\frac{\theta}{2}\,\sin\frac{\theta}{2}$

$\frac{\cos(\omega t)}{2}\ \sin($

$$\alpha = \sqrt{\rho_{00}} = |\cos\theta/2|$$

$$\alpha\beta^* = \rho_{01} \rightarrow \beta = \frac{\rho_{01}^*}{\alpha^*} = \frac{\rho_{01}^*}{\sqrt{\rho_{00}}}$$

$$\beta = \frac{\rho_{01}^*}{\alpha} \rightarrow \alpha = \frac{\beta}{\rho_{01}^*}$$

$$|\psi\rangle = e^{i\varphi}\left(e^{-i\varphi}\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle\right)$$

$$\alpha(t)\,|0\rangle + \beta(t)\,|1\rangle$$

$$e^{i\omega t/2}\,\tilde{\alpha}(t)\,|0\rangle + e^{-i\omega t/2}\,\tilde{\beta}(t)\,|1\rangle$$

$$= e^{-i\omega t/2}\left(e^{i\omega t}\,\tilde{\alpha}(t)\,|0\rangle + \tilde{\beta}(t)\,|1\rangle\right)$$

$$= e^{-i\omega t/2}\left(e^{i\omega t}\,\tilde{\alpha}(t)\,|0\rangle + e^{-i\pi/2}\underbrace{\tilde{\tilde{\beta}}(t)}_{\text{reale}}|1\rangle\right) =$$

$$e^{-i\frac{\omega t}{2} - i\pi/2}\left(e^{i(\omega t + i\frac{\pi}{2})}\,\tilde{\alpha}(t)\,|0\rangle + \tilde{\tilde{\beta}}(t)\,|1\rangle\right)$$

$$\rho = \begin{pmatrix} |\alpha(t)|^2 & \alpha(t)\,\beta^*(t) \\ \alpha^*(t)\,\beta(t) & |\beta(t)|^2 \end{pmatrix}$$

$$\alpha(t)\,\beta^*(t) + \alpha^*(t)\,\beta(t) =$$

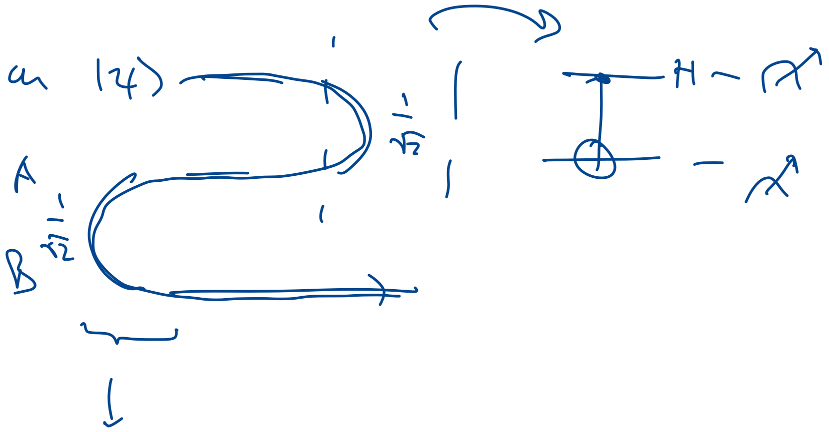
$$\rho_{01} = \alpha(t)\cdot\beta^*(t) = e^{i\omega t}\,\tilde{\alpha}(t)\,\tilde{\beta}^*(t)$$

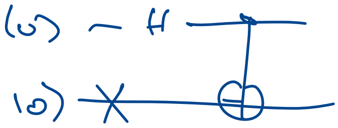
$$\Longrightarrow \quad e^{i\omega t}\,\tilde{\alpha}(t) = \frac{\rho_{01}}{\tilde{\beta}^*}$$

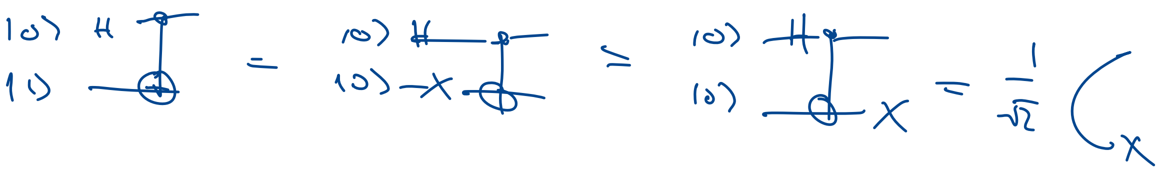
$$\alpha(t)\,\beta^*(t) = \underbrace{e^{i(\omega t + i\frac{\pi}{2})}\,\tilde{\alpha}(t)}_{\substack{\| \\ \rho_{01}/\tilde{\tilde{\beta}}(t)}}\,\tilde{\tilde{\beta}}(t)$$

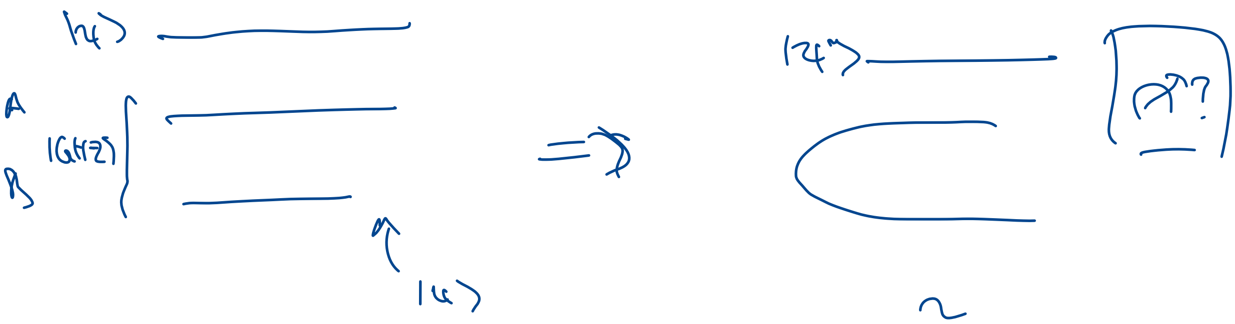
UNA VOLTA CHE $\tilde{\tilde{\beta}}$ è reale, $\alpha(t)$ HA UNA FASE BEN DEFINITA!

$$\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \longrightarrow \frac{1}{\sqrt{2}} \begin{array}{c} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \end{array} \begin{array}{l} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} \longrightarrow \begin{array}{c} \\ |0\rangle \\ |1\rangle \end{array} \begin{array}{c} |0\rangle\,|1\rangle \\ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \end{array} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \mathbb{1}$$

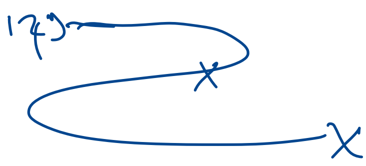








$\approx$



DAY 2

Exercise 1: Show that $\lambda(t)$ and $g(t)$ are related via $\lambda(t) = 2\,\mathrm{Re}[g(t)]$. The exponential dependence of $\mathcal{G}(t)$ and $\mathcal{L}(t)$ on system size N in Eq. (3.6) and Eq. (3.7) is only in general valid if the quench changes the energy density, i.e. inserts an extensive amount of energy into the system. Can you come up with an intuitive reason for this (no calculations necessary)?

$$G(t) = \langle \psi_0 | \psi_0(t) \rangle = \langle \psi_0 | e^{-iHt} | \psi_0 \rangle$$

$$L(t) = \langle \psi_0 | e^{-iHt} | \psi_0 \rangle \langle \psi_0 | e^{iHt} | \psi_0 \rangle = |G(t)|^2$$

$$G(t) = e^{-Ng(t)} \longrightarrow L(t) = |e^{-Ng(t)}|^2 = |e^{-N(\mathrm{Re}(g(t)) + i\,\mathrm{Im}(g(t)))}|^2$$

$$= |e^{-N(\mathrm{Re}(g(t)))} e^{i(-N\,\mathrm{Im}(g(t)))}|^2$$

$$= e^{-2N\,\mathrm{Re}(g(t))} = e^{-N\lambda(t)}$$

$$\Rightarrow \lambda(t) = 2\,\mathrm{Re}(g(t))$$

I don't insert extensive amount of energy

$\Downarrow$

I am changing the Hamiltonian only locally

E.g. ISING MODEL : adding only local excitations

Ground state ↑ ↑ ↑ ↑ ↑ ↑ - - - ↑

New ground state ↑ ↑ ↑ ↓ ↓ ↑ . . . ↑

Exercise 2: Based on the above definitions, show that in the thermodynamic limit $N \to \infty$, the Loschmidt rate $\lambda(t)$ reduces to the minimum function applied to the set of individual rates $\lambda_i(t)$, i.e. $\lambda(t) = \min_i [\lambda_i(t)]$.

$$\mathcal{L}(t) = \sum_{i=0}^{N_{gs}-1} |\langle \varphi_i | \Psi_0(t) \rangle|^2 = \sum_{i=0}^{N_{gs}-1} e^{-N\lambda_i(t)}$$

$$\lambda(t) = -\lim_{N\to\infty} \frac{1}{N} \log \mathcal{L}(t) = \min_i \left[\lambda_i(t) \right] = \lambda_{min}$$

$$= -\lim_{N\to\infty} \frac{1}{N} \cdot \log\left(\sum_{i=0}^{N_{gs}-1} e^{-N\lambda_i(t)} \right)$$

$$= -\lim_{N\to\infty} \frac{1}{N} \log\left(e^{-N\lambda_{min}} \left(1 + \sum_{i\neq i_{min}} e^{-N(\lambda_i(t)-\lambda_{min})} \right) \right)$$

$$= -\lim_{N\to\infty} \frac{1}{\cancel{N}} (-\cancel{N}\lambda_{min}) + \underbrace{\frac{1}{N}}_{\to 0} \underbrace{\log\left(1 + \underbrace{\sum_{i\neq i_{min}} e^{-N\overbrace{(\lambda_i(t)-\lambda_{min})}^{>0\ \forall i}}}_{\to 0} \right)}_{\to 0}$$

$$= \lambda_{min} = \min_i \left[\lambda_i(t) \right]$$

Exercise 3: Show that the parity operator

$$\hat{P} = \prod_i X_i \tag{4.2}$$

commutes with the Hamiltonian, i.e. $[\hat{H}, \hat{P}] = 0$. What is the spectrum of $\hat{P}$, i.e. its possible eigenvalues? Determine the exact ground states of $\hat{H}$ in the two limiting cases $g = 0$ and $g \to \infty$. Notice that for $g = 0$, the symmetry $\hat{P}$ is spontaneously broken, i.e. an infinitesimally small longitudinal field $\pm\epsilon \sum_i Z_i$ induces a finite expectation value of the ground state magnetization $m_z = \frac{1}{N}\sum_i Z_i$. What is the expectation value of m_z in the respective symmetry broken ground states at $g = 0$, and what is $\langle m_z \rangle$ for the ground state at $g \to \infty$?

$$[X_i, X_j] = 0$$

$$[Z_i, X_j] = 2i Y_i \delta_{ij}$$

$$\to \{Z_i, X_i\} = 0$$

$$Z_i X_i - X_i Z_i$$

$$[\hat{H}, \hat{P}] = 0$$

$$\hat{H} = -\frac{1}{2}\sum_{\langle i,j \rangle} Z_i Z_j - \frac{g}{2}\sum_i X_i$$

1) $$\left[\sum_i X_i, \prod_j X_j\right] = \sum_i \left[X_i, \prod_j X_j\right] = \sum_i \left(X_i \prod_j X_j - \prod_j X_j X_i\right)$$

X_i commutes with every X_j $\longrightarrow$ $$= \sum_i \left(\prod_j X_j X_i - \prod_j X_j X_i\right) = 0$$

2) $$\left[\sum_{\langle i,j \rangle} Z_i Z_j, \prod_k X_k\right] = \sum_{\langle i,j \rangle} \left[Z_i Z_j, \prod_k X_k\right] = \sum_{\langle i,j \rangle} \left(Z_i Z_j \prod_k X_k - \prod_k X_k Z_i Z_j\right)$$

Let's just look at one exemplary case with $j > i$ without loss of generality

$\Rightarrow j = i+1$

$$\left[Z_i Z_{i+1}, \prod_k X_k\right] = \prod_{k=1}^{i-1} X_k \, Z_i X_i Z_{i+1} X_{i+1} \prod_{k'=i+2} X_{k'} - \prod_{k=1}^{i-1} X_k \, X_i Z_i X_{i+1} Z_{i+1} \prod_{k'=i+2} X_{k'}$$

$$= \prod_{k=1}^{i-1} X_k \underbrace{\left((-X_i Z_i)\cdot(-Z_{i+1} X_{i+1}) - X_i Z_i X_{i+1} Z_{i+1}\right)}_{=0} \prod_{k'=i+2} X_{k'}$$

$$= 0$$

$g = 0$: We have 2 possible ground states
$$|\psi_0^1\rangle = |\uparrow\uparrow\uparrow\uparrow \dots \uparrow\rangle = |0 \dots 0\rangle$$
$$|\psi_0^2\rangle = |\downarrow\downarrow \dots \downarrow\downarrow\rangle = |1 \dots 1\rangle$$

$g \to \infty$ $\Longrightarrow$ $-\frac{g}{2}\sum_i X_i$ prevailing term: $$|\psi_0^\infty\rangle = |+++++ +\rangle$$

$$M_z = \frac{1}{N} \sum_i Z_i$$

$$\langle M_z \rangle_{\psi_0^1} = \langle \psi_0^1 | \left(\frac{1}{N} \sum_i Z_i \right) | \psi_0^1 \rangle = \frac{1}{N} \cdot \sum_i \langle \psi_0^1 | \overbrace{Z_i | \psi_0^1 \rangle}^{+1\, |\psi_0^1\rangle} =$$

$$= \frac{N}{N} = 1$$

$$\langle M_z \rangle_{\psi_0^2} = -1$$

$$\langle M_z \rangle_{\psi_0^\infty} = \langle \psi_0^\infty | \left(\frac{1}{N} \sum_i Z_i \right) | \psi_0^\infty \rangle$$

$$= \frac{1}{N} \sum_i \langle \psi_0^\infty | Z_i | \psi_0^\infty \rangle$$

$$= \frac{1}{N} \frac{1}{2} \left(\langle 0| + \langle 1| \right) \left(|0\rangle\langle 0| - |1\rangle\langle 1| \right) \left(|0\rangle + |1\rangle \right)$$

$$= \frac{1}{2N} \cdot (1 - 1) = 0$$