Hypothesis testing for two samples

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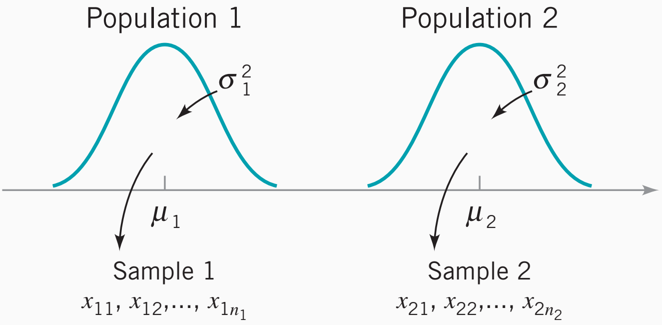
Bayesian Data Analysis and Probabilistic Programming

Credits

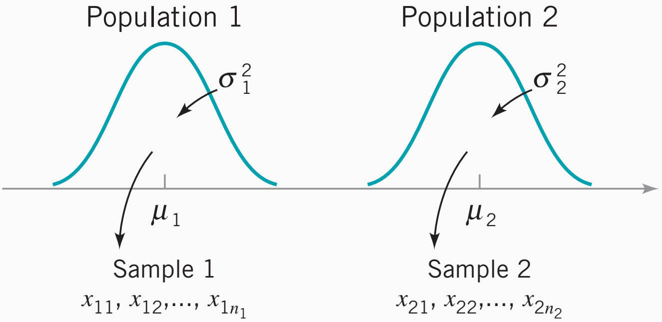
- The examples are mostly from D. P. Montgomery, *Introduction to Statistical Process Control*, 6th Edition, Wiley.

How to compare two populations



- The first population has mean μ_1 and variance σ_1^2.
- The second population has mean μ_2 and variance σ_2^2.

How to compare two populations



- The sample sizes are n_1 e n_2.
- We assume the samples of the populations to be *independent* from each other.

The assumption of equal variances

- We assume $\sigma_1^2 = \sigma_2^2$.
- This allows estimating σ^2 as a weighted average of s_1^2 e s_2^2. This is generally more accurate than estimating the two variances separately.

Comparing the mean of two populations

- The two-tailed test is:

$$
\begin{aligned}
H_0 &: \mu_1 = \mu_2 \\
H_1 &: \mu_1 \neq \mu_2
\end{aligned}
$$

Comparing the mean of two populations

We have:

- $\bar{x}_1$ e $\bar{x}_2$: empirical means of the two samples
- s_1^2 e s_2^2: empirical variances of the two samples.

- It is the distribution of $\bar{x}_1 - \bar{x}_2$ if we extract many times two samples of size n_1 e n_2 from the two populations.
- Assuming
  - $\sigma_1^2 = \sigma_2^2 = \sigma^2$.
  - $n_1 > 10$ and $n_2 > 10$, to have the normality of $\bar{x}_1$ e $\bar{x}_2$):

$$\bar{x}_1 - \bar{x}_2 \sim N\left(\mu_1 - \mu_2, \sigma^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\right)$$

Test statistic, assuming σ to be known

- Given:

$$\bar{x}_1 - \bar{x}_2 \sim N\left(\mu_1 - \mu_2, \sigma^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\right)$$

under H_0 we have:

$$\frac{\bar{x}_1 - \bar{x}_2 - \overbrace{(\mu_1 - \mu_2)}^{\text{assumed 0 by } H_0}}{\sigma\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{\bar{x}_1 - \bar{x}_2}{\sigma\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \sim N(0, 1)$$

- Yet, σ is unknown and we cannot use this statistic.

- The statistic of the t-test is:

$$t = \frac{\bar{x}_1 - \bar{x}_2}{s_P \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

- It follows a t distribution with $(n_1 + n_2 - 2)$ degrees of freedom.
- s_p replaces σ. Thus, the statistic only requires information from the samples.

Pooled variance

- We estimate σ^2 as a weighted average of s_1^2 and s_2^2:

$$s_P^2 = \frac{(n_1 - 1)}{n_1 + n_2 - 2} \cdot s_1^2 + \frac{(n_2 - 1)}{n_1 + n_2 - 2} \cdot s_2^2$$
$$s_P = \sqrt{s_P^2}$$

- s_P^2: *pooled* variance:
  - the weight are ~ proportional to the sample sizes (actually, they are proportional to the degrees of freedom).
  - if $n_1 = n_2$, s_p^2 is the simple mean of s_1^2 and s_2^2.

The test statistic

$$t = \frac{\bar{x}_1 - \bar{x}_2}{s_P\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

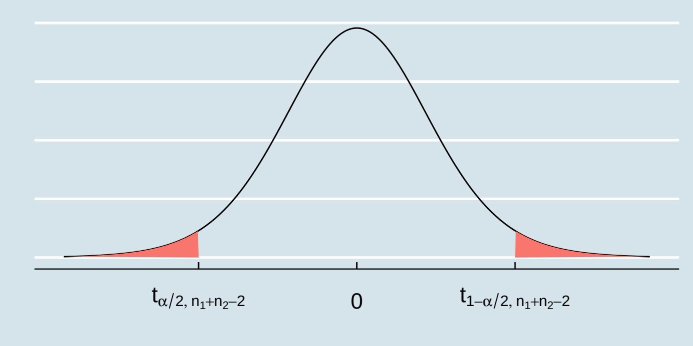
- $\bar{x}_1 - \bar{x}_2$ is the sample estimate of $\mu_1 - \mu_2$.
- $s_P\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$ is the *standard error* of $\bar{x}_1 - \bar{x}_2$, i.e., a measure of how the estimate $\bar{x}_1 - \bar{x}_2$ is spread around the actual value of $\mu_1 - \mu_2$
- Practical interpretation of the standard error: $\mu_1 - \mu_2$ lies with probability 95% in the interval $\bar{x}_1 - \bar{x}_2 \pm 2$standard errors.

Rejection region

- The rejection region is a set of values of the test statistic for which the null hypothesis is rejected.

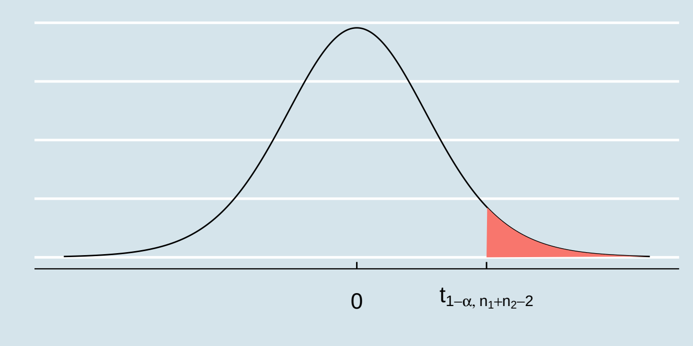
Two-tailed test

- $H_0 : \mu_1 = \mu_2$
- $H_1 : \mu_1 \neq \mu_2$
  - Rejection region: $t_0 < t_{n_1+n_2-2,\alpha/2}$ e $t_0 > t_{n_1+n_2-2,1-\alpha/2}$
  - Each tails contains probability $\alpha/2$



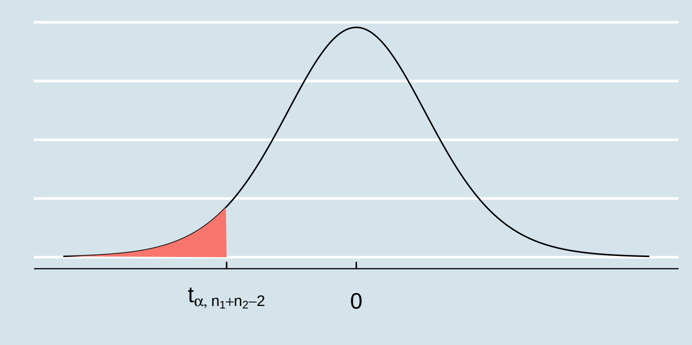
Right-tailed test

- $H_0 : \mu_1 \leq \mu_2$
- $H_1 : \mu_1 > \mu_2$
  - The most *positive* values of the statistic support H_1.
  - Rejection region: $t_0 > t_{n_1+n_2-2,\alpha}$ (contains probability α).



Left-tailed test

- $H_0 : \mu_1 \geq \mu_2$
- $H_1 : \mu_1 < \mu_2$
  - The most *negative* values of the statistic support H_1.
  - Rejection region: $t_0 < -t_{n_1+n_2-2,\alpha}$ (contains probability α)



Example: comparing mean yields of catalysts

- Two catalysts are being compared: catalyst 1 is currently in use.
- Catalyst 2 is cheaper: it should be adopted, providing it does not change the process yield.
- An experiment is run in the pilot plant and results are in the next slide. Is there any difference between the mean yields?
- The two-tailed test is:

$$H_0 : \mu_1 = \mu_2$$
$$H_1 : \mu_1 \neq \mu_2$$

Comparing mean yields of catalysts

$$
\begin{aligned}
n_1 &= 8 & n_2 &= 8 \\
\bar{x}_1 &= 92.25 & s_1 &= 2.39 \\
\bar{x}_2 &= 92.73 & s_2 &= 2.98
\end{aligned}
$$

- We adopt α=0.05.

Comparing mean yields of catalysts

- Since $n_1 = n_2$, s_p^2 is the average of s_1^2 and s_2^2:

$$s_p^2 = \frac{7}{14}s_1^2 + \frac{7}{14}s_2^2 = \frac{2.39^2 + 2.98^2}{2} = 7.3$$
$$s_p = \sqrt{s_p^2} = \sqrt{7.3} = 2.7$$

Statistic and critical values

$$t_0 = \frac{\bar{x}_1 - \bar{x}_2}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

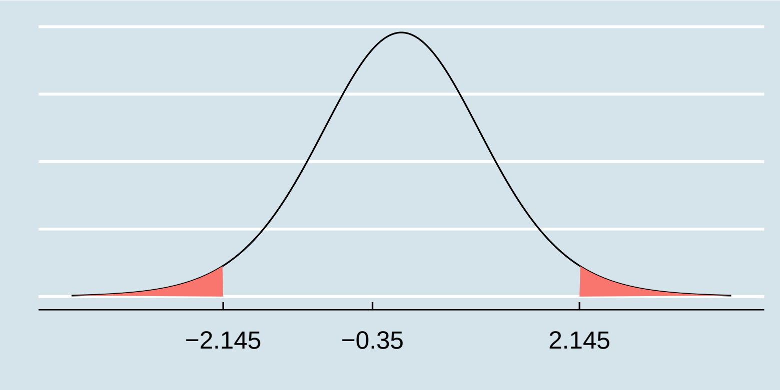
$$= \frac{92.25 - 92.73}{2.7\sqrt{\frac{1}{8} + \frac{1}{8}}}$$

$$= -0.35$$

- The critical values are $\pm t_{.975,14} = \pm 2.145$.

Decision

- The statistic is in *non-rejection* region: we do not have strong evidence that the mean yield of the two catalysts is different.



Decision

Recall that the frequentist test can either:

- reject the null hypothesis (strong decisions)
- not reject (*fail to reject*) the null hypothesis (weak decisions)

  Failing to reject H_0 implies that we have not found sufficient evidence to reject H0, that is, to make a strong statement. Failing to reject H0 does not necessarily mean that there is a high probability that H0 is true. It may simply mean that more data are required to reach a strong conclusion. (Montgomery, 9.1.2)

Confidence interval (CI) of $\mu_1 - \mu_2$

- The CI contains the plausible values of $\mu_1 - \mu_2$:

$$\bar{x}_1 - \bar{x}_2 \pm t_{1-\alpha/2, n_1+n_2-2} \cdot s_P \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

- $t_{1-\alpha/2, n_1+n_2-2}$: quantile $(1-\alpha/2)$ of the t distributon with $(n_1 + n_2 - 2)$ degrees of freedom; it is the critical value of the test.
- $s_P \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$ is the standard error of $(\bar{x}_1 - \bar{x}_2)$

CI vs hypothesis test

Consider the null hypothesis $\mu_1 = \mu_2$.

- If the two-tailed test fails to reject H_0, the CI contains 0.
- If the two-tailed test *rejects* H_0 the CI does *not* contain 0.

Confidence interval (CI)

- The degrees of freedom are 8-1+8-1 = 14

$$\bar{x}_1 - \bar{x}_2 \pm t_{.975,14} \cdot \overbrace{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}^{\text{sd error}}$$

$$(92.25 - 92.73) \pm 2.145 \cdot 2.7\sqrt{\frac{1}{8} + \frac{1}{8}} = (-3.38, 2.42)$$

- 0 is a plausible value for $\mu_1 - \mu_2$, as it is within the CI.
- Indeed the test does not refuse H_0.

Example of one-tailed test

- A study reports the weight of calcium in standard cement and cement doped with lead, after a stress test.
- Reduced levels of calcium imply low hydration in the cement, possibly allowing water to attack various the cement structure.
- The dopes cement is more expensive, and we want evidence (α=0.05) of its higher performance compared to standard cement.

One-tailed test

- The alternative hypothesis H_1 is what we try to demonstrate (*doped* cement has higher performance).

$$H_0 : \mu_{\text{standard}} \geq \mu_{\text{doped}}$$
$$H_1 : \mu_{\text{standard}} < \mu_{\text{doped}}$$

Data

$$
\begin{aligned}
n_{\text{standard}} &= 10 \\
\bar{x}_{\text{standard}} &= 87.0 \\
s_{\text{standard}} &= 5.0
\end{aligned}
$$

$$
\begin{aligned}
n_{\text{doped}} &= 15 \\
\bar{x}_{\text{doped}} &= 90.0 \\
s_{\text{doped}} &= 4.0
\end{aligned}
$$

Test statistic

$$
\begin{aligned}
s_P^2 &= \frac{9 \cdot (5)^2 + 14 \cdot (4)^2}{10 + 15 - 2} = 19.52 \\
s_P &= \sqrt{19.52} = 4.4
\end{aligned}
$$

The statistic is:

$$
\begin{aligned}
t_0 &= \frac{\bar{x}_{\text{standard}} - \bar{x}_{\text{doped}}}{s_p \sqrt{\frac{1}{n_{\text{standard}}} + \frac{1}{n_{\text{doped}}}}} \\
&= \frac{87 - 90}{4.4\sqrt{\frac{1}{10} + \frac{1}{15}}} = -1.67
\end{aligned}
$$

Rejection region

- If $\bar{x}_{\text{doped}} > \bar{x}_{\text{standard}}$, the statistic is negative and thus in favor of H_1.
- Actually we reject H_0 if $t_0 < t_{0.05,23} = -1.71$.
- The statistic (-1.67) is in rejection region: there is no strong evidence that *doped* cement performs better than standard cement.

Comparing two proportions

Hypothesis test for two proportions

- The term success and failure refer to the outcome being 1 or 0.
- We denote by π_1 e π_2 the proportion of successes in two populations.
- We want to check whether π_1 e π_2 are significantly different.
- π_1 e π_2 cannot be observed.
- We observe instead the *sample* proportions, $p_1 = \frac{X_1}{n_1}$ and $p_2 = \frac{X_2}{n_2}$.

Comparing two proportions

- We have two samples of size n_1 e n_2, containing X_1 and X_2 successes.
- The two-tailed test is:

$$\begin{aligned} H_0 &: \pi_1 = \pi_2 \\ H_1 &: \pi_1 \neq \pi_2 \end{aligned}$$

The test statistic

$$Z = \frac{(p_1 - p_2)}{\sqrt{\bar{p}(1-\bar{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

with :

$$\bar{p} = \frac{X_1 + X_2}{n_1 + n_2}$$

The test statistic

- If both samples contains at least 5 successes and failures, $p_1 - p_2$ is approximately normal.

- Under H_0, $\pi_1 = \pi_2 = \pi$.

$$p_1 - p_2 \sim N(0, \sqrt{\pi(1-\pi)(\frac{1}{n_1} + \frac{1}{n_2})})$$

$$Z \sim N(0, 1)$$

Sampling distribution of the test statistic

- If we extract many times two samples of size n_1 and n_2 from two populations with $\pi_1 = \pi_2$ and compute the statistic, it will be different every time, following approximately a $N(0, 1)$ distribution.

Rejection regions

H_1	Rejection region	p-value
$\pi_1 \neq \pi_2$	$z < z_{\alpha/2}$ e $z > z_{1-\alpha/2}$	$2(1 - \Phi(\lvert z \rvert))$
$\pi_1 > \pi_2$	$z > z_{1-\alpha}$	$1 - \Phi(z)$
$\pi_1 < \pi_2$	$z < z_{\alpha}$	$\Phi(z)$

CI of $\pi_1 - \pi_2$

- If we extract many times two samples of size n_1 and n_2 from two populations with $\pi_1 = \pi_2$ and compute the CI, it will contain the actual value of $\pi_1 - \pi_2$ in $(1 - \alpha)$ of the experiments.

- The CI is:

$$p_1 - p_2 \pm z_{1-\alpha/2}\sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}$$

- The CI and the two-tailed test approximate differently the standard error and they might sometimes draw inconsistent conclusions.

Example: assess the effect of a process innovation

- From a traditional production process we have 262 boards, of which 154 without any defect.
- From a innovative production process we have 227 boards, of which 163 without any defect.
- Is the new process significantly more accurate than the previous one?

Comparing the two proportions

- We take as success the board being without any defect.
- Both samples contain more than 5 successes and 5 failures; the normal approximation is sound.
- The test is:

$$
\begin{aligned}
H_0 &: \pi_{\text{new}} \leq \pi_{\text{old}} \\
H_1 &: \pi_{\text{new}} > \pi_{\text{old}}
\end{aligned}
$$

- and we use $\alpha = 0.01$.

Comparing the two proportions

- The rejection region contains positive values of $p_{\text{new}} - p_{\text{old}}$ and thus also of the statistic.
- Rejection region: $Z_0 > \Phi^{-1}(.99)$ = 2.33

Comparing the two proportions

$$
\begin{aligned}
p_{\text{new}} &= 163/227 = 0.72 \\
p_{\text{old}} &= 154/262 = 0.59 \\
\bar{p} &= (163+154)/(227+262) = 0.65 \\
Z &= \frac{p_{\text{new}} - p_{\text{old}}}{\sqrt{\bar{p}\cdot(1-\bar{p})\cdot 1/n}} = 3.01 > 2.33
\end{aligned}
$$

- We reject H_0.
- Frequentist tests do not allow however modelling any prior knowledge.

Beauty and sex ratio

- Keep in mind these example, we will re-analyze the data in a later lecture using a Bayesian approach.

Taken from: > Chap. 9.4 of “Regression and other stories”, A. Gelman, J. Hill, A. Vehtari

- Book published from Cambridge University Press (2020).
- The book is also freely available online.

Beauty and sex ratio

- A researcher analyzed data from a survey of 3000 Americans and observed a correlation between attractiveness of parents and the sex of their children.
- Among the 3000 couple of parents, 300 are classified as highly attractive.
- The proportion of girls among the children of "highly attractive" parents is 56% ($X = 168, n = 300$).
- The proportion of girls among the children of "standard" parents is 48% ($X = 1296, n = 2700$).

Is the difference significant?

The test is:

$$\begin{aligned} H_0 &: \pi_{\text{attr}} \leq \pi_{\text{std}} \\ H_1 &: \pi_{\text{attr}} > \pi_{\text{std}} \end{aligned}$$

Is the difference significant?

$$
\begin{aligned}
p_{\text{attr}} &= 0.56 \\
p_{\text{std}} &= 0.48 \\
\bar{p} &= \frac{168 + 1296}{300 + 2700} = 0.488 \\
\text{sd err} &= \sqrt{\bar{p} \cdot (1 - \bar{p}) \cdot (\frac{1}{n_1} + \frac{1}{n_2})} = 0.03 \\
Z &= \frac{p_{\text{new}} - p_{\text{old}}}{\text{sd err}} = 2.63 > 2.33,
\end{aligned}
$$

where 2.33 is the 99-th quantile (α=0.01).

- The difference in the proportion of girls between the two groups is statistically *significant*.