Bayes' rule

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Bayesian Data analysis and Probabilistic Programming

Credits

- Chap. 2 of *Bayes Rules! An Introduction to Applied Bayesian Modeling*
    - https://www.bayesrulesbook.com/chapter-2.html

Events

- An event is a set of outcomes of an experiment to which a probability is assigned.
- Examples of events and probabilities related to a fair dice:
  - E_1: the dice lands 4 (P=1/6)
  - E_2: the dice lands odd: 1 or 3 or 5. (P=1/2)
  - E_3: the dice lands 2 or 4. (P=1/3)

Conditional vs unconditional probability

- Let A and B be two events.
- $P(A)$: unconditional probability of A. It measures the probability of observing A, without any knowledge of B.
- $P(A|B)$: conditional probability of A given B: probability of observing A once B occurred (probability of A *given* B).

Conditional probability

- A is not observed.
- Comparing $P(A|B)$ vs $P(A)$ reveals how the observation of B informs us about A.
- $P(A|B)$ can be larger, smaller or equal to $P(A)$.

Conditional probability

- Probability of joining an orchestra, given that one practices clarinet every day:

$$P(\text{orchestra} \mid \text{practice}) > P(\text{orchestra})$$

Conditional probability

- Probability of getting the flu given that one washes thoroughly his hands:

$$P(\text{flu} \mid \text{wash hands}) < P(\text{flu})$$

$P(A|B) \neq P(B|A)$

- Roughly 100% of puppies are adorable:

$$P(\text{adorable} \mid \text{puppy}) = 1$$

- But an adorable object is not necessarily a puppy:

$$P(\text{puppy} \mid \text{adorable}) < 1$$

$P(A|B) > P(A)$?

- A = you will enjoy the newest novel from a certain author.
- B = you just finished reading a book from the same author and you enjoyed it.

$P(A|B) > P(A)$?

- B = it’s 0 degrees Celsius tonight.
- A = tomorrow it will be very warm.

$P(A|B) > P(A)$?

Consider a woman who is mother of two children.

- A = the second child will be a girl
- B = the first child is a boy

Independent events

- A and B are independent if the occurrence of B doesn't tell us anything about the occurrence of A:

$$P(A|B) = P(A)$$

- Unconditional (i.e., *marginal*) and conditional probability are identical.

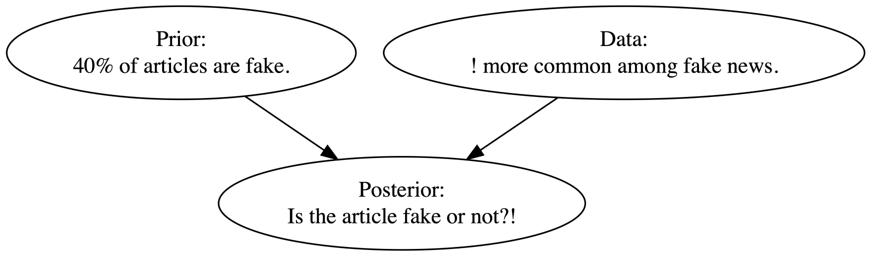
- For instance:
  - event A: rider Q wins the next motoGP race
  - event B: my coin lands tails
  - Since the coin does not affect the performance of rider Q, $P(A|B) = P(A)$.

Motivating example

- According to a study, 40% of the articles taken from a certain website are fake news and 60% are real.
- The usage of an exclamation point in the title is however uncommon in real articles.
  - 26.67% of fake news titles use an exclamation point
  - only 2.22% of real news titles use it.

Bayesian knowledge-building

- Given that an article contains the exclamation point, which is its probability of being fake news?



Notation

- We denote random variable with uppercase letters and their specific values by lowercase letters.
- Let us consider discrete variables, which have a finite set of possible outcomes.
- $P(Y)$: probability distribution for variable Y, i.e, a table containing as many elements as the possible values of Y.
  - If Y is discrete, the distribution is also called a probability mass function (pmf).

Notation

- Specific values are denote by lowercase; for instance, y.
- $P(y)$ denotes the probability of $P(Y = y)$, where $\sum_y p(y) = 1$.
- We use moreover the shortcut notation
  $P(x|y) = P(X = x \mid Y = y)$

Prior probability

- From now we will deal with random variables, not event.
- Consider the random variable A (article) with possible states {*real*, *fake*}.
- *Prior* probability, i.e., before having seen the article:
  - $P(A = \text{fake})$ = 0.4
  - $P(A = \text{real})$ = 0.6

A further random variable

- Variable B refers to whether the article title contains or not an explanation point.
- Its possible states are {!, ~!}

Conditional probability

- The exclamation point in the title is more compatible with fake news than with real news.
- If an article is fake, there is 26.67% chance it uses exclamation points in the title.
- The conditional probability $P(!|fake)$ is 0.2667.
- The conditional probability $P(!|real)$ is instead 0.0222.

Likelihood

- The likelihood measures the compatibility of the observation (article contains exclamation point) with the different values of the unobserved variable (article is fake or real).
- $P(A|b)$: table containing the probability of the each value of A, given the observation $B = b$.

Prior probabilities and likelihoods of fake news.

	fake	real
prior probability	$P(\text{fake})$=0.40	$P(\text{real})$=0.6
likelihood	$P(! \mid \text{fake})$=0.26	$P(! \mid \text{real})$=0.02

- Prior probabilities add up to 1 but the likelihoods do not.
- The likelihood function is not a probability function; instead it is a way to measure the compatibility of the observation (title contains !) with the different states of the unobserved variable A (fake or real).

Posterior probability that the article is fake

- We want the posterior probability of the article being fake given that it uses exclamation points.

$$P(\text{fake } |!) = \frac{P(\text{fake})P(! \mid \text{fake})}{P(!)} = \frac{0.4 \cdot 0.2667}{0.12} = 0.889$$

.

- Notice:

$$P(!) = \underbrace{P(! \mid \text{real})P(\text{real})}_{P(!,\text{real})} + \underbrace{P(! \mid \text{fake})P(\text{fake})}_{P(!,\text{fake})}$$

Bayes' theorem - recap

- Unobserved variable A, whose set of possible states is $\mathcal{A}$.
- Observation: $B = b$
- Given $B = b$, the posterior probability of $A = a$ is:

$$P(a|b) = \frac{P(a)P(b \mid a)}{P(b)} = \frac{P(a)P(b \mid a)}{\sum_{a' \in \mathcal{A}} P(a', b)}$$

$$P(b) > 0$$

Prior

- $P(A)$ represents how likely are the different values of A, according to our beliefs, before we see any data.

Likelihood

- When we have a specific observation b, we get the likelihood $P(a|b)$ for each $a \in \mathcal{A}$.
- The likelihood is not a probability distribution; it does not sum to one.

Marginal likelihood

- The denominator of Bayes' rule is a normalizing constant, referred to as the marginal likelihood.
- It marginalizes the likelihood over the states of the unobserved variable A:

$$P(b) = \sum_a P(a)P(b \mid a) = \sum_a P(a, b)$$

Joint probability

	fake	real
!		
~!		

- The joint probability $P(a, b)$ is the probability of observing both $A = a$ and $B = b$.
- We can compute it as

$$P(a, b) = P(a|b)P(b)$$

Joint probability

$$
\begin{aligned}
P(\text{fake}, !) &= P(\text{fake}) P(! \mid \text{fake}) \\
&= 0.4 \cdot 0.2667 = 0.1067
\end{aligned}
$$

	fake	real
!	0.1067	
~!		

Joint probability

$$
\begin{aligned}
P(\text{fake}, \sim !) &= P(\text{fake})P(\sim ! \mid \text{fake}) \\
&= 0.4 \cdot (1 - 0.2667) = 0.2933
\end{aligned}
$$

	fake	real
!	0.1067	
~ !	0.2933	

Joint probability

$$P(\text{real}, !) = P(\text{real})P(! \mid \text{fake})$$
$$= 0.6 \cdot 0.0222 = 0.0133$$

	fake	real
!	0.1067	0.0133
~ !	0.2933	

Joint probability

$$
\begin{aligned}
P(\text{real}, \text{no !}) &= P(\text{real})P(\text{no !} \mid \text{fake}) \\
&= 0.6 \cdot (1 - 0.0222) \\
&= 0.6 \cdot 0.9778 = 0.5867
\end{aligned}
$$

	fake	real
!	0.1067	0.0133
~ !	0.2933	0.5867

- The joint probability sums up to 1.

Marginal distribution and marginal probability

- The marginal distribution of A is obtained by summing the joint distribution over all states of B.

	fake	real
!	0.1067	0.0133
~ !	0.2933	0.5867
marginal	**0.4**	**0.6**

- The marginal distribution of A is {fake=0.4; real=0.6}.

- Moreover,

$$P(A = \text{fake}) = P(\text{fake}, !) + P(\text{fake}, \sim !)$$

.

Marginal distribution and marginal probability

- The marginal distribution of B is obtained by summing the joint distribution over all states of A.

	fake	real
!	0.1067	0.0133
~ !	0.2933	0.5867

- The marginal distribution of B is {! = 0.12; ~ ! = 0.88}.
- A randomly chosen article has 0.12 probability of containing the exclamation point in the title.

Sum rule

- Given the joint distribution $P(X, Y)$
- The marginal probability $P(X = x)$ is given by the **sum rule**
$$P(x) = \sum_y P(x, y)$$
- The summation is over all possible values of Y.

- Assume you want to compute $P(b)$ from $P(A)$ and $P(B|A)$, rather than from $P(A, B)$.
- The marginal $P(B = b)$ is $P(b) = \sum_a P(b, a) = \sum_a P(b \mid a)P(a)$.

$$\begin{aligned} P(\text{!}) &= P(\text{real}, \text{!}) + P(\text{fake}, \text{!}) \\ &= P(\text{real})P(\text{!} \mid \text{real}) + P(\text{fake})P(\text{!} \mid \text{real}) \\ &= 0.4 \cdot 0.0222 + 0.6 \cdot 0.2667 \\ &= 0.1067 + 0.0133 = 0.12 \end{aligned}$$

Testing for Covid 19 (Murphy, Sec 3.2.1)

- You decide to take a diagnostic test to check if you have contracted Covid. You want to make inference about your health H whose possible states are:
  - infected
  - healthy

The diagnostic test

The test T can be either:

- positive
- negative

We want to determine the probability distribution $P(H|positive)$.

Test performance: conditional probability

Assume the conditional probability of the test outcome, given an infected person, to be:

	test negative	test positive
$P(T \mid \text{infected})$	0.125	.875

- Thus the probability of a positive test for an infected persons is 87.5%.

Test performance: conditional probability

Assume the conditional probability of the test outcome, given an healthy person, to be:

	test negative	test positive
$P(T \mid \text{healthy})$	.975	.025

- Thus the probability of a negative test for a healthy patient, 97.5%

Prior

- The *prevalence* is the percentage of persons affected by the disease.
- The covid prevalence in New York City 2020 was 10%.

	H=healthy	H=infected
probability	0.9	.1

- This is our prior, before observing the outcome of the test.

P(infected | positive)

$$P(\text{infected} \mid \text{positive}) = \frac{\overbrace{P(\text{infected})}^{\text{prior: prevalence}}\ \overbrace{P(\text{positive}|\text{infected})}^{\text{likelihood}}}{\underbrace{P(\text{positive})}_{\text{prob of observing a positive test: marginal likelihood}}}$$

Denominator, a.k.a. marginal likelihood

- Total probability of having a positive test:
  - probability of testing positive while infected + …
  - probability of testing positive while healthy

$$
\begin{aligned}
P(\text{positive}) &= P(\text{positive, infected}) + P(\text{positive, healthy}) \\
&= P(\text{positive} \mid \text{infected})P(\text{infected}) + P(\text{positive} \mid \text{healthy})P(\text{he} \\
&= 0.875 \times 0.1 \times +0.025 \times 0.9 \\
&= 0.11
\end{aligned}
$$

Posterior

$$\begin{aligned} P(\text{infected}|\text{positive}) &= \frac{P(\text{infected})P(\text{positive} \mid \text{infected})}{P(\text{positive})} \\ &= \frac{0.1 \times 0.875}{0.11} \\ &= 0.795 \end{aligned}$$

- The posterior probability of being healthy is:

$$P(\text{healthy} \mid \text{positive}) = 1 - .795 = .205$$

- The positive test increases your probability of being infected of about 8 times.

Exercise

- Work out the probability of being infected if you test negative
  - 0.014

Yet another exercise

- We have two coins:
  - the first coins lands heads or tails with equal probability
  - the second coin is rigged and always lands heads.
- We take one coin at random and we get heads. What is the probability that this coin is the rigged one?

Solution

Since the coins are randomly chosen, the prior is:

$$P(\text{fair}) = P(\text{rigged}) = 0.5$$

The likelihood is:

$$P(\text{head} \mid \text{fair}) = 0.5$$

$$P(\text{head} \mid \text{rigged}) = 1$$

Solution - 2

The posterior probability of the coin being rigged is:

$$
\begin{aligned}
P(\text{rigged} \mid \text{head}) &= \frac{P(\text{rigged})P(\text{head} \mid \text{rigged})}{P(\text{head})} \\
&= \frac{0.5 \times 1}{P(\text{head})}
\end{aligned}
$$

Computing the denominator

$$P(\text{head}) = P(\text{head}, \text{rigged}) + P(\text{head}, \text{fair})$$

$$\begin{aligned} P(\text{head}, \text{rigged}) &= P(\text{head}|\text{rigged})P(\text{rigged}) \\ &= 1 \times 0.5 \end{aligned}$$

$$\begin{aligned} P(\text{head}, \text{fair}) &= P(\text{head}|\text{fair})P(\text{fair}) \\ &= 0.5 \times 0.5 \end{aligned}$$

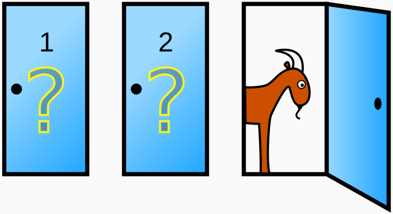
$$P(\text{head}) = 0.5 \times 1 + 0.5 \times 0.5 = 0.75$$

Conclusion

The posterior probability of the coin which has landed tail to be the rigged one is:

$$\begin{aligned} P(\text{rigged} \mid \text{head}) = & \frac{P(\text{rigged})P(\text{head} \mid \text{rigged})}{P(\text{head})} \\ & = \frac{0.5 \times 1}{0.75} = \frac{2}{3} \end{aligned}$$

Your turn: Monty Python problem



- You’re given the choice of three doors:
  - Behind one door is a car
  - Behind the others, goats.
- You pick door 1 and the host, who knows what’s behind the doors, opens door 3, which has a goat.
- He then says “Do you want to pick door No. 2?” Should you switch your choice?

Solutions

Solutions of the Monty Python problem

- The prior probability of each door having the car is uniform:

$$P(D_1) = P(D_2) = P(D_3) = 1/3$$

- Let us assume you choose door 1 and the presenter open door 3.

- To update our prior beliefs, we need the conditional probabilities of opening door 2, given that the car is behind any given door:

$$P(O_3 \mid D_1), P(O_3 \mid D_2), P(O_3 \mid D_3)$$

.

Conditional probability

- If the car is behind door 1, the presenter opens with equal probability D_2 or D_3:
$$P(O_3 \mid D_1) = 0.5$$
- If the car is behind door 2, the presenter will surely open door 3:
$$P(O_3 \mid D_2) = 1$$
- If the car is behind door 3, the presenter will never open door 3:
$$P(O_3 \mid D_3) = 0$$

Posterior probability of D_1

$$P(D_1 \mid O_3) = \frac{P(D_1)P(O_3 \mid D_1)}{P(O_3)}$$

- The denominator is decomposed as:
  $P(O_3) = P(O_3 \mid D_1)P(D_1)+P(O_3 \mid D_2)P(D_2)+P(O_3 \mid D_3)P(D_3)$
- The prior probabilities of all doors are the same and they cancel out:

$$\begin{aligned} P(D_1 \mid O_3) &= \frac{P(O_3 \mid D_1)}{P(O_3 \mid D_1) + P(O_3 \mid D_2) + P(O_3 \mid D_3)} \\ &= \frac{1/2}{3/2} = 1/3 \end{aligned}$$

Posterior probability of D_2

- The posterior probability of D_3 is obviously 0 (it has been opened).
- The posterior probability of D_2 is thus 2/3, but let us compute it.

$$
\begin{aligned}
P(D_2 \mid O_3) &= \frac{P(O_3 \mid D_3)}{P(O_3 \mid D_1) + P(O_3 \mid D_2) + P(O_3 \mid D_3)} \\
&= \frac{1}{3/2} = 2/3
\end{aligned}
$$

Conclusions

- The optimal strategy is swapping doors as it has a 2/3 probability of winning.
- In contrast, the probability of winning with the original door is 1/3.

Further considerations (from Kevin Murphy, Sec. 2.3.2)

- Consider a game is played with a million doors. The contestant chooses one door, then the game show host opens 999,998 doors in such a way as not to reveal the prize, leaving the contestant's selected door and one other door closed.
- The contestant may now stick or switch. Where do you think the prize is?