The beta-binomial model

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Bayesian Data Analysis and Probabilistic Programming

References

- The Beta-Binomial model: Ch. 3 of `Bayes Rules! An Introduction to Applied Bayesian Modeling`
  - https://www.bayesrulesbook.com/chapter-3.html#chapter-3
  - Alicia A. Johnson, Miles Q. Ott, Mine Dogucu

The bias θ of a coin

- A coin falls heads with probability $\theta \in (0, 1)$
- θ is the *bias* of the coin
  - θ =0: it always lands tails
  - θ =1: it always lands heads
- $\theta \in (0, 1)$ is a continuous parameter

The bias θ of a coin

- First we choose a model of our prior beliefs for each possible value of θ (*prior*).
- Then we collect some data and we express the probability of observing the data given each value of θ (*likelihood*).
- Eventually we use Bayes' rule to obtain the posterior distribution of θ given the data.

The coin problem

- The methodology shown in the following can be used in applications such as estimating:
  - the proportion of supporters of a political party
  - the click-through rate of an online advertisement
  - etc.

Prior density for a continous parameter

- The prior for a continuous parameter is specified by a *probability density function* (pdf), denoted by $f(\theta)$.
- The pdf specifies all possible values of θ and the relative plausibility of each.
- It accounts for all possible values of the parameter and it integrates to 1.
- For θ, the pdf is limited on (0,1)

Properties of $f(\theta)$

- $f(\theta) >= 0$
- $\int f(\theta) d\theta = 1$
- $P(a < \theta < b) = \int_a^b f(\theta) d\theta$
- The underlying area between a and b is the probability of θ being in this range.

Density vs probability

- A continuous pdf is not a probability; we can also have $f(\theta) > 1$ in some points.
- Probabilities are obtained by integrating the pdf over an interval.
- $f(\theta)$ is used to compare the plausibility of different values of θ
  - the greater $f(\theta)$, the more plausible the corresponding value of θ.

Central tendency

- The **mean** or **expected value** of θ is a weighted average: each possible θ value is weighted by its pdf:

$$E[\theta] = \int_x \theta \cdot f(\theta) d\theta$$

- The **mode** is the value of θ at which the pdf is highest.

$$\mathrm{Mode}(\theta) = \text{arg max}_\theta f(\theta)$$

Measures of variability

- The variance measures the expected squared distance of possible θ values from their mean:

$$\text{Var}(\theta) = E((\theta - E(\theta))^2) = \int (\theta - E(\theta))^2 \cdot f(\theta) d\theta.$$

Standard deviation

- The variance has squared units; the standard deviation, which measures the typical unsquared distance of θ values from $E(\theta)$, is easier to interpret.

- The standard deviation measures the expected distance of possible θ values from their mean:

$$\mathsf{SD}(\theta) := \sqrt{\mathsf{Var}(\theta)}$$

The Beta pdf

- Beta(a, b), is a pdf restricted to the $(0, 1)$ interval.
- Its parameters are $a > 0$ and $b > 0$. Parameters used in prior models are referred to as *hyperparameters*.
- The pdf is:

$$f(\theta) = \frac{1}{\underbrace{B(a,b)}_{\text{normalizing constant}}} \theta^{a-1}(1-\theta)^{b-1} \propto \theta^{a-1}(1-\theta)^{b-1}$$

$$a, b > 0$$

- θ is raised to the power of $a - 1$ (not a)
- $1 - \theta$ is raised to the power of $b - 1$ (not b)

Central tendency measures of the Beta

$$\begin{aligned} E(\theta) &= \frac{a}{a+b} \\ \text{Mode}(\theta) &= \frac{a-1}{a+b-2} \quad \text{when } \; a, b > 1. \end{aligned}$$

Variability measures for Beta pdf

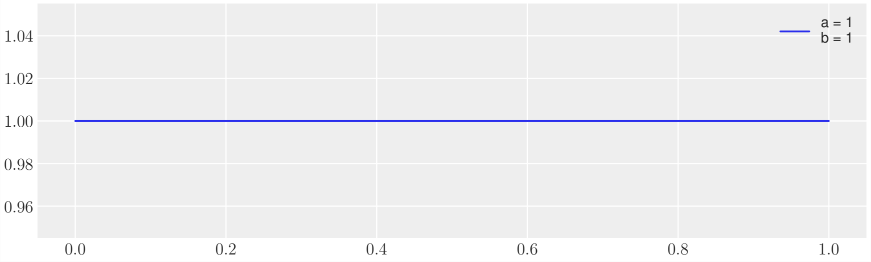
$$\mathrm{VAR}(\theta) = \frac{ab}{(a+b)^2(a+b+1)}$$

$$\mathrm{SD}(\theta) = \sqrt{\frac{ab}{(a+b)^2(a+b+1)}}$$

Uniform distribution: $a = b = 1$

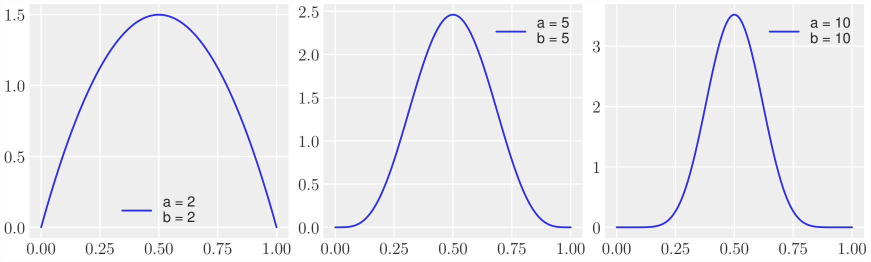
$$\begin{aligned} f(\theta) &\propto \theta^{a-1}(1-\theta)^{b-1} \\ &= \theta^0(1-\theta)^0 \\ &= 1 \end{aligned}$$

- This a *uniform* distribution: all values in $(0, 1)$ are equally probable.
- $E(\theta) = \frac{a}{a+b} = 0.5.$



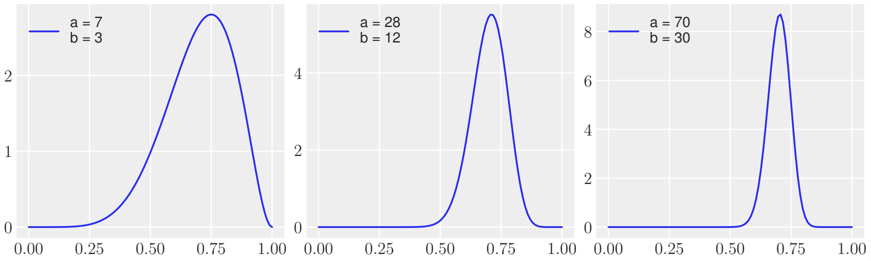
Increasing a and b the prior becomes more concentrated

- We increase both a and b satisfying $a = b$.
- The pdf becomes more concentrated around the expected value $\theta = 0.5$.



If we think the coin is rigged

- If we suspect the coin has 70% chance of landing heads, we set $a = \frac{7}{3}b$.
- We represent more confidence in this statement by setting $a = \frac{7}{3}b$ and increasing b.



How the density changes with a and b

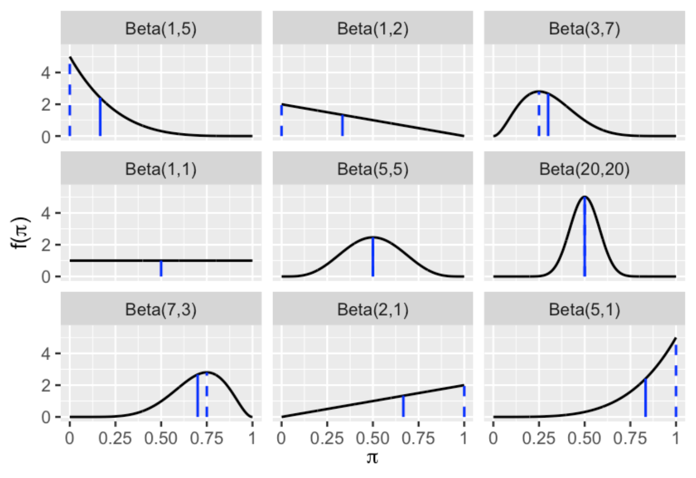


Figure 1: Mean: solid. Mode: dashed.

Quiz yourself

- When $a = b$, the pdf is:
  - Right-skewed, with a mode greater than 0.5.
  - Symmetric with mode 0.5.
  - Left-skewed with mode smaller than 0.5.
- Using the same options as above, discuss the pdf when $a > b$.
- Which pdf has greater variability: Beta(20,20) or Beta(5,5)?

Effect of a and b

- $a > b$: the distribution is right-skewed, the mode is larger than 0.5; vice versa for $b > a$.
- $a = b$: symmetric distribution with mean 0.5.
- Increasing a and b decreases the variance.

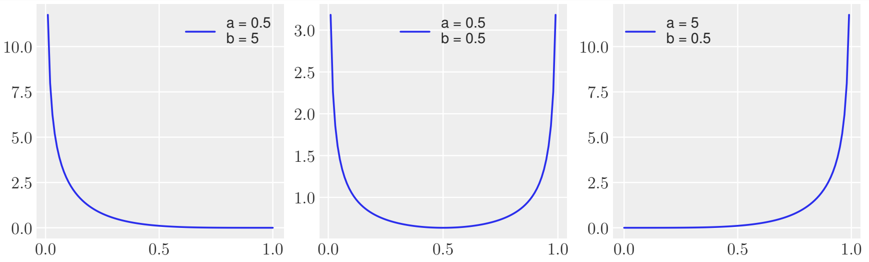
a and/or b <1: convex density

Consider a = b = 0.5

$$f(\theta) \propto \theta^{a-1}(1-\theta)^{b-1}$$

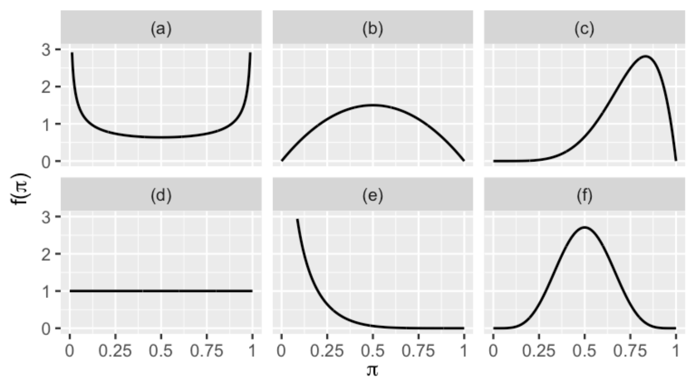
$$f(\theta) \propto \theta^{-0.5}(1-\theta)^{-0.5}$$

$$f(\theta) \propto \frac{1}{\sqrt{\theta}}\frac{1}{\sqrt{1-\theta}}$$



Quiz

- Recognize Beta(0.5,0.5), Beta(1,1), Beta(2,2), Beta(6,6), Beta(6,2), Beta(0.5,6).



Tuning a and b

- The support for a politician is at about 70 percentage points, though he recently polled between 45 and 90 points.
- We set the ratio a/b as follows:

$$\frac{a}{a+b} = .7$$
$$a = \frac{7}{3}b$$

- Feasible pairs of values are for instance (7,3), (14,6), etc.

Tuning a and b

- We check how the 5-th and the 95-th quantile vary with a and b

```
from scipy.stats import beta
q1 = beta.ppf(q=[0.05,0.95],a=7, b=3)
q2 = beta.ppf(q=[0.05,0.95],a=28, b=12)
q1
```

```
## array([0.45035835, 0.90225319])
```

```
q2
```

```
## array([0.57661174, 0.81188104])
```

Tuning a and b

- We try different couples (a, b, $a = \frac{7}{3}b$) to match the variance.

(a,b)	(7, 3)	(28, 12)	(70, 30)
5-th quantile	0.45	0.58	0.62
95-th quantile	0.90	0.81	0.77

- The choice (7, 3) captures the mean and the variability of the polls in this example. Other choices yield too narrow a distribution.

Tune a Beta prior!

- Tune a Beta prior for the cases below:
  - John applies to a job. He thinks I has a 40% chance of getting the job, but he is pretty unsure; he expresses his uncertainty by putting his chance between 20% and 60%.
- There is no single correct prior, but multiple reasonable answers.

The Binomial data model

- After having defined the pdf, the second step of our Bayesian analysis is to collect data.
- We also define the likelihood function, to be used within Bayes' rule.
- In our example, the data collection is done by tossing the coin n times and observing the number y of heads.

Likelihood: assumptions

- Each observation takes a binary value (head or tail; also referred to as *success* and *insuccess*)
- The *success* usually refer to the rarer event among the two.
- The flips are independent: the probability of *heads* at the next flip does not depend on the outcome of the previous flips.
- The success probability θ is constant in all flips.

The binomial distribution

Given θ, a single flip takes:

- *heads* with probability θ
- *tails* with probability $1 - \theta$
- Assuming a constant θ and the independence of the flips, the sequence

$$H \quad T \quad T \quad H \quad H$$

  has probability

$$\theta(1-\theta)(1-\theta)\theta\theta = \theta^3(1-\theta)^2$$

- In general, a sequence containing y heads in n flips has probability

$$\theta^y(1-\theta)^{n-y}$$

- We assume a fixed number n of trials.

Binomial distribution

- We can get $\binom{n}{y} = \frac{n!}{y!(n-y)!}$ sequences containing y successes in n trials.

$$P(y|\theta) = \binom{n}{y} \theta^y (1-\theta)^{n-y}$$

- This is probability of the observing y heads within n flips, given the value of θ.

The binomial distribution - quiz

- Toni Kukoc played for Chicago Bulls in the 90s
- In his career, his 3-point rate is 31%.
- In a match against Orlando Magic, he realized 6 out of 8 shoots.
- Which is the probability of him repeating or improving this performance?

The binomial distribution - quiz

$$P(X = 6) = \binom{8}{6} .31^6 \cdot .69^2 = 0.011$$

$$P(X = 7) = \binom{8}{7} .31^7 \cdot .69 = 0.002$$

$$P(X = 8) = \binom{8}{8} .31^8 \cdot .69^0 = .31^8 = 0.00005$$

$$P(X = 6) + P(X = 7) + P(X = 8) = 0.013 \approx 1/77$$

- A similar performance can be repeated about every 77 matches, that is about once for season.

Checking assumptions

Discuss whether the assumptions of the binomial model are satisfied:

- θ is the same for every shoot
- the outcome of the different shoots is independent from each other
    - Yes, see Tversky, A., and Gilovich, T. “The cold facts about the hot hand in basketball,” Chance 2 (1) 16–21 (1989).
- Kukoc will shoot exactly 8 times in every future game.

Binomial likelihood

- So far, we assumed θ to be fixed and the data to be random. Thus used the binomial to evaluate the plausibility of a certain number of successes given a fixed θ
- From now on, we will instead interpret it as a likelihood. We will keep the data fixed (these are the data we observed) and we will interpret it as a function of θ, to assess which values of θ are more likely to have generated the observed outcome.

The Beta-binomial model

$$\theta \sim \text{Beta}(a, b).$$
$$y|\theta \sim \text{Bin}(n, \theta)$$

- This model applies to any setting where parameter θ lies in [0,1]
  - requires tuning of a Beta prior
  - assumes data y to be the number of "successes" in n independent trials with constant probability of success θ.

Binomial likelihood

- Assume we observe y=6 in n=10 flips.
- The likelihood measures the relative compatibility of the observed data with different $\theta \in [0, 1]$.
- According to the data θ=0.6 is ten times more plausible than θ=0.3:

$$p(y = 6,\ n = 10 \mid \theta = 0.6) = \binom{10}{6} 0.6^6 (0.4)^4 = 0.35$$

$$p(y = 6,\ n = 10, \mid \theta = 0.3) = \binom{10}{6} 0.3^6 (0.7)^4 = 0.037$$

Binomial likelihood

- This a *likelihood* function if interpreted in this way:
  - the probability is a function of θ.
  - the observation y are fixed
- The likelihood function shows how the probability of the observed data varies with θ.
- It does not integrate to 1 over all values of θ!

Binomial likelihood

$$p(y \mid \theta) = \binom{n}{y} \theta^y (1-\theta)^{1-y}$$

- It integrates to 1 if we keep θ fixed and we integrate over possible outcomes y.
- Instead, the likelihood function treats the observed data as fixed and let $P(y \mid \theta)$ vary with θ.

Posterior

- Adopting a beta prior and a binomial *likelihood*, Bayes' rule yields a beta *posterior* density with updated parameters.

- We use f rather than p as it represents a density.

$$
\begin{aligned}
f(\theta) &\propto \theta^{a-1}(1-\theta)^{b-1} && \text{Beta prior} \\
p(y \mid \theta) &\propto \theta^{y}(1-\theta)^{n-y} && \text{Binomial likelihood} \\
f(\theta \mid y) &\propto \theta^{y+a-1}(1-\theta)^{n-y+b-1} && \text{Beta posterior}
\end{aligned}
$$

The beta prior is *conjugate* with the binomial likelihood, as we obtain a posterior Beta pdf.

Conjugacy

- The Beta-binomial model is **conjugate**.
- The prior is conjugated with the likelihood if the posterior has the same functional form of the prior.
- Historically, problems in Bayesian statistics were restricted to the use of conjugate priors, because of mathematical tractability.
- Modern computational techniques allow Bayesian analysis without conjugacy, allowing the resurgence of Bayesian statistics in recent years.

The posterior is a compromise of prior and likelihood

- Given the prior Beta(a,b), the prior mean of θ is:

$$\frac{a}{a+b}$$

- Having observed y tails in n flips, the posterior pdf of θ is Beta($y+a$,$n-y+b$).

- The posterior mean of θ is:

$$E_{\text{post}}[\theta] = \frac{a+y}{a+y+b+n-y} = \frac{a+y}{a+b+n}$$

The posterior is a compromise of prior and likelihood

- Rearranging:

$$\underbrace{\frac{a+y}{a+b+n}}_{\text{posterior}} = \underbrace{\frac{y}{n}}_{\text{observed proportion}} \underbrace{\frac{n}{a+b+n}}_{\text{weight}} + \underbrace{\frac{a}{a+b}}_{\text{prior mean of } \theta} \underbrace{\frac{a+b}{a+b+n}}_{\text{weight of the prior}}$$

- The posterior mean is a weighted average of the prior mean and the observed proportion.

- The weight of the observed proportion increases with n; the weight of the prior mean increases with a and b.

The posterior is a compromise of prior and likelihood

$$\underbrace{\frac{a+y}{a+b+n}}_{\text{posterior}} = \underbrace{\frac{y}{n}}_{\text{observed proportion}} \underbrace{\frac{n}{n+a+b}}_{\text{weight}} + \underbrace{\frac{a}{a+b}}_{\text{prior mean of } \theta} \underbrace{\frac{a+b}{n+a+b}}_{\text{weight of the prior}}$$

- We can interpret the prior as representing an imaginary sample, containing a successes and b insuccesses.
- The larger a and b, the larger the imaginary sample; thus our confidence in the prior increases.

Test yourself!

- Let θ denote the proportion of people that prefer dogs to cats.
- You express your prior beliefs by a Beta(7, 2) model.

- According to your prior, what are reasonable values for θ ?

- In a survey 19 out of 20 people prefer dogs.

- How would that change your understanding about the mean and the certainty of θ?

Sequential updating

- Based on some theoretical studies, a scientist summarizes its belief in the chance θ of a new drug being able to cure a disease as Beta(1,10) distribution.
- In an experimental trial, the drug cures 13/20 persons.
- What's the posterior distribution of θ after the first experiment?
- In a second experiment, the drug cures 20/40 persons.
- What's the posterior distribution of θ after the second experiment?

Sequential updating

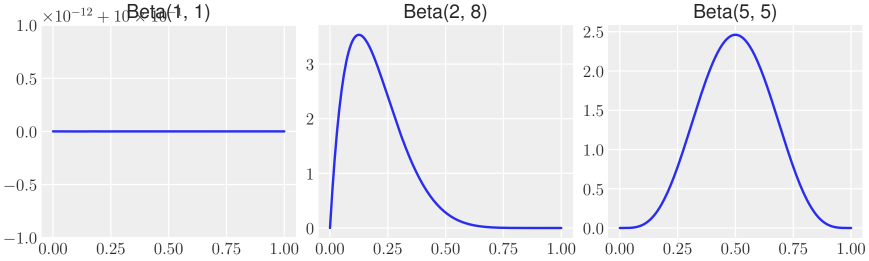
- Prior: Beta(1,10), $E[\theta] = \frac{1}{11} = 0.09$
- After first experiment:
  - $f(\theta|D_1) = \text{Beta}(1 + 13, 10 + 7)$
  - $E[\theta] = \frac{14}{14+17} = 0.45$
  - Thus Beta(14,17) becomes the prior before analyzing the data of the second experiment.

- After second experiment:
  - Beta(14+20,17+20)
  - $E[\theta] = \frac{34}{34+37} = 0.48$

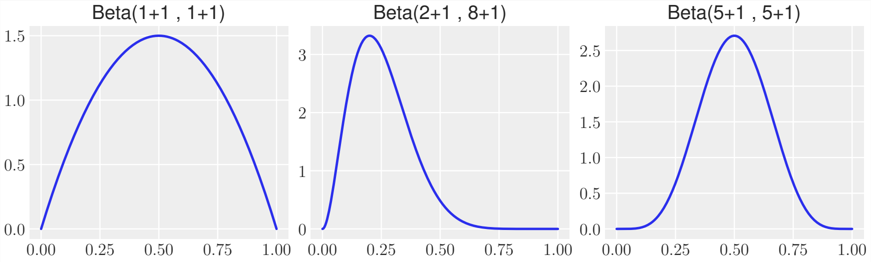
Impact of the prior on the posterior

- It is useful to consider different priors: priors encode domain expertise, and different experts provide you with reasonable but different assessment.

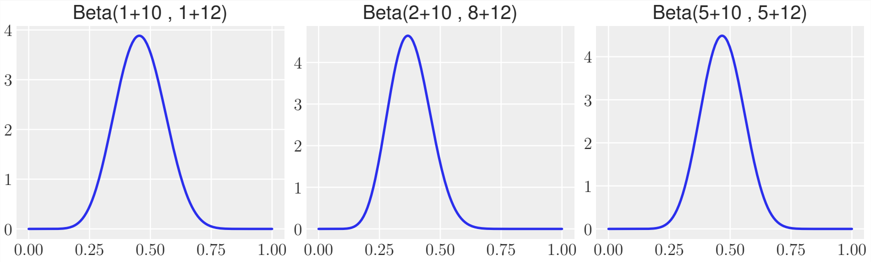
- For instance:



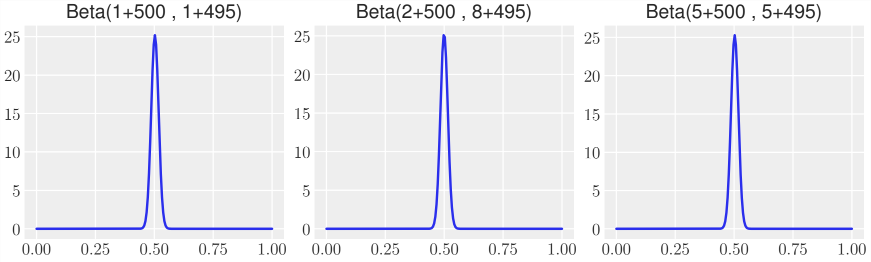
The posterior is prior-sensitive with small data (example: y=1, n=2)



The posterior becomes similar with more data (10 tails, 12 heads)



When data is larger, the posterior is the same whatever the prior



Posterior means $E_{\text{post}}[\theta]$ obtained from different priors:

- $\frac{500+1}{500+1+495+1} = \frac{501}{997} = 0.502$
- $\frac{500+2}{500+2+495+8} = \frac{502}{1005} = 0.499$
- $\frac{500+5}{500+5+495+5} = \frac{505}{1005} = 0.502$
- Also the posterior variances are practically identical.

Test your self!

For each scenario of the next slide, identify whether

- the prior has more influence on the posterior
- the data has more influence on the posterior
- the posterior is an equal compromise between the data and the prior.

Test your self!

- Prior: $\theta \sim$ Beta(1,4), data: y=8, n=10
- Prior: $\theta \sim$ Beta(20,3), data: y=0, n=1
- Prior: $\theta \sim$ Beta(4,2), data: y=1, n=3
- Prior: $\theta \sim$ Beta(20,2), data: y=10, n=200

Sensitivity to the prior

- With a large amount of data, the posterior is practically the same with any prior, but how much data is needed varies with the problem.
- If we only have few data, the posterior can differ depending on the adopted prior; it makes sense to repeat the analysis with different priors (*sensitivity*).
- This is sensible: the prior encodes our previous knowledge and different experts could have different priors.

Discussion

- Priors and likelihood are assumptions which are part of the model.
- Flat priors provide no information (uninformative priors) and should be avoided.
- *Slightly informative* priors are recommended.
- In many cases we known that the parameter can only be positive, or its order of magnitude, etc.
- For instance a Beta(1,1) prior is flat but limits the possible values of θ between 0 and 1.

Priors need a broad support

- In the following slides we discuss some problem which arise if the support of the prior is too small.
- The *support* of a pdf is the set of points where the pdf is >0.

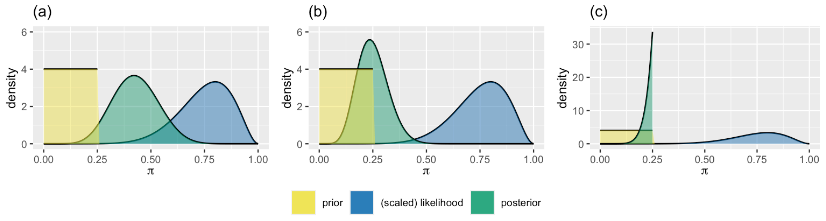
Priors need a broad support

- Bayesian analysis looses its benefits is the prior pdf has a too small support, i.e. it assigns a prior probability of zero also to plausible parameter values.
- For instance a priori we assume π to equally likely be anywhere between 0 and 0.25 and that surely it doesn't exceed 0.25:

$$\pi \sim \text{Unif}(0, 025)$$

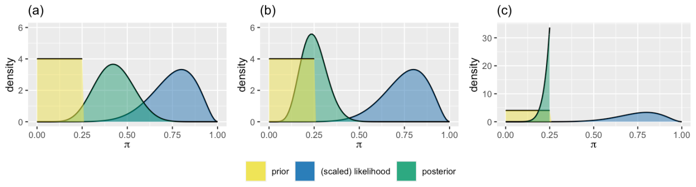
- And assume to observe y=8 successes in n=10 trials.

- The prior pdf, the scaled likelihood and the posterior are shown below. Which is correct plot?



Priors need a broad support

- The correct plot is the third.



- The support of the posterior is inherited from the support of the prior.
- Thus both prior and posterior assigns zero probability to any π >0.25. the posterior model must also assign zero probability to any value in that range.
- No matter how much evidence we will collect, the posterior pdf will be truncated beyond the 0.25 cap.

How to avoid a regrettable prior

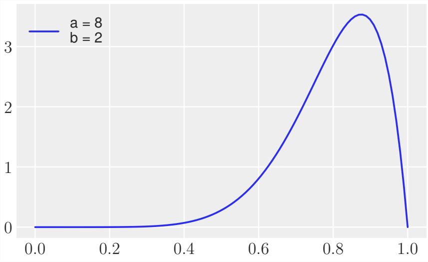
- Let π be the parameter of interest.
- Be sure to assign non-0 pdf to every *possible* value of π.
- For example, if π is a proportion which can range from 0 to 1, the prior model should be defined on this range.

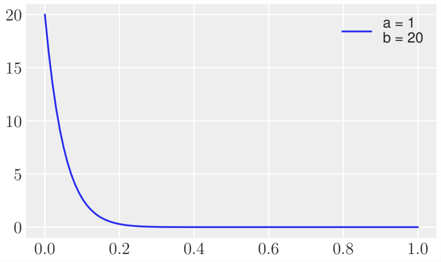
Conclusions

- We have seen how Bayesian inference works when Bayes' rule can be solved analytically (conjugacy).
- Only simple likelihood functions have conjugate priors.
- Complex models have no conjugate priors and requires numerical Markov chain Monte Carlo (MCMC) to get the posterior.

Solution of exercises 3.9 and 3.10 (https://www.bayesrulesbook.com/chapter-3.html#exercises-2)

- plot and summarize the Beta(8,2) and the Beta(1,20) prior





Code of the previous plot

```
plt.figure(figsize=(10, 3))
x = np.linspace(0, 1, 100)

for ind, (a, b) in enumerate([(8, 2), (1, 20)]):
    y = stats.beta.pdf(x, a, b)
    plt.subplot(1, 2, ind+1)
    plt.plot(x, y, label='a = %s\nb = %s' % (a, b))
    plt.legend(fontsize=12)
```

Summarize the Beta (8,2)

```
rv = beta.rvs(a=8, b=2, size=1000)
pd.Series(rv).describe(percentiles=[0.05,0.25,0.50,0.75,0.95])
```

```
## count    1000.000000
## mean        0.802090
## std         0.117228
## min         0.341782
## 5%          0.585287
## 25%         0.732902
## 50%         0.824559
## 75%         0.887608
## 95%         0.957597
## max         0.996715
## dtype: float64
```

Summarize the Beta (1,20)

```
rv = beta.rvs(a=1, b=20, size=1000)
pd.Series(rv).describe(percentiles=[0.05,0.25,0.50,0.75,0.95])
```

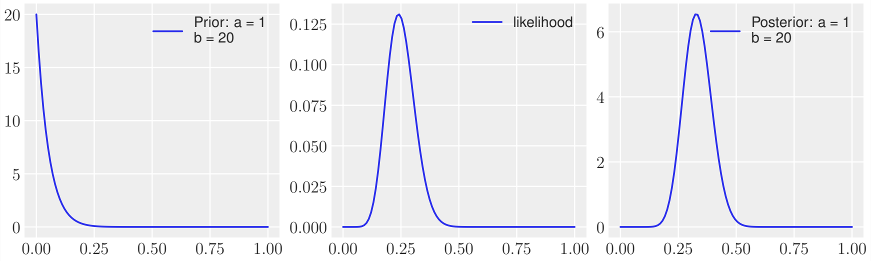
```
## count    1000.000000
## mean        0.045301
## std         0.044351
## min         0.000056
## 5%          0.002456
## 25%         0.014384
## 50%         0.031135
## 75%         0.062790
## 95%         0.136424
## max         0.319482
## dtype: float64
```

Likelihood function: y=12, n=50

```
from scipy.stats import binom
plt.figure(figsize=(10, 3))
theta = np.linspace(0, 1, 100)

#binomial probability mass function for each theta
lik = binom.pmf(12, n=50, p=theta)
```

Prior [Beta(8,2)], likelihood and posterior [Beta(20,40)]



Comparing the two posteriors: Beta(8,2) and Beta(13,58)

- They are similar even though the priors were strongly different.