# The normal-normal model

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Bayesian Data Analysis and Probabilistic Programming

## Credits

- Chap. 5 of *Bayes Rules! An Introduction to Applied Bayesian Modeling*
  - https://www.bayesrulesbook.com/chapter-5.html

## The Normal model

- Let $Y$ be a continuous random variable which can take values in $(-\infty,\infty)$
- The variability of Y might be well represented by a Normal model
$$Y \sim N(\mu, \sigma^2)$$

## The Normal model

- The Normal pdf is

$$f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(y-\mu)^2}{2\sigma^2}\right]$$

- With:

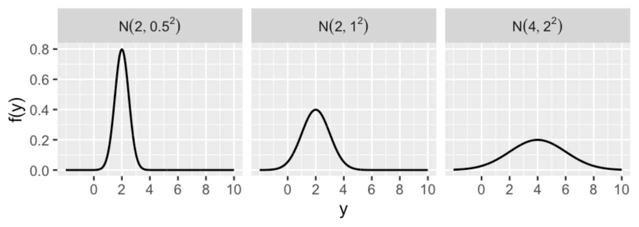
$$\begin{aligned} E(Y) = Mode(Y) &= \mu \\ Var(Y) &= \sigma^2 \\ SD(Y) &= \sigma \end{aligned}$$

## Standard deviation $\sigma$

- $\sigma$ provides a sense of scale for $Y$.
- Roughly 95% of $Y$ values are within 2 standard deviations of $\mu$:
$$\mu \pm 2\sigma$$
- Roughly 99% of $Y$ values are within 3 standard deviations of $\mu$:
$$\mu \pm 3\sigma$$

# The normal model

- The Normal model is bell-shaped and symmetric around $\mu$.
- As σ gets larger, the pdf becomes more spread out.
- Though a Normal variable is defined in $(-\infty, \infty)$, the plausibility of values that are more than 3 standard deviations σ from the mean $\mu$ is negligible.



## Example

- The volume of the hippocampus (a part of the brain) is researched in studies about the effect of concussions.
- In the general population, both halves of the hippocampus have a volume between 3.0 and 3.5 cm$^3$.
- Thus, the hippocampal volume is thought to vary, within the population, between 6 and 7 cm$^3$.
- The average volume $\mu$ is thought to be between 6.4 and 6.6 cm$^3$.

## Normal prior

- Assuming symmetry, we formalize our prior information about $\mu$ as:

$$\mu \sim N(\mu', \sigma_\mu)$$

which in this example is :

$$\mu \sim N(6.5, 0.05)$$

- $\mu'$ is our prior guess on the value of $\mu$.
- $\sigma_\mu$ represents our uncertainty on the guess $\mu'$.
- According to this prior, $\mu$ lies with 95% probability in (6.4, 6.6).
- We allow the volume to range over $(-\infty, \infty)$, but values beyond $\mu \pm 3\sigma$ are given negligible probability.

# Normal likelihood

- We now define a model for the distribution of the observations.
- We make a *second* assumption of normality.
- The hyppocampal volumes observed in $n$ subjects ($y_1, y_2, ..., y_n$) are normally distributed $N(\mu, \sigma)$.

## Normal likelihood

- $\mu$ is the mean volume in the population.
- $\sigma$ expresses the spread of the measures within the population.
- We expect $y$ to vary in (6-7); we interpret this interval as $\mu \pm 2\sigma$, hence it has length of $4\sigma$.
- We thus set $\sigma$=0.25.

- We morever assume the observations $y_1, ..., y_n$ to be *independent* samples from $N(\mu, \sigma)$.
- This is realistic: the measure $y_i$ tells us nothing about the measure $y_{i+1}$ (assuming they refer to different subjects)

Assuming independence, the joint pdf of the $n$ measures ($y_1, y_2, ..., y_n$) is the product of the unique Normal pdfs $f(y_i \mid \mu)$:

$$f(\vec{y}|\mu) = \prod_{i=1}^{n} f(y_i|\mu) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(y_i-\mu)^2}{2\sigma^2}\right].$$

- $\vec{y}$ is the vector containing the measures $y_1, ....y_n$.

$$\mu \sim N(\mu', \sigma_\mu)$$
$$\vec{y} \sim N(\mu, \sigma)$$

- We treat $\mu'$, $\sigma_\mu$ and $\sigma$ as fixed numbers.
- The likelihood assumes independence of the observations $y_1, y_2, \ldots, y_n$
- The only parameter of the model is $\mu$.
- Later we will treat also $\sigma$ as a parameter.

## Your turn: normal likelihood functions

- For a Normal random sample $y_i \sim N(\mu, \sigma)$ with $\sigma$=10 we observe:

$$y_1, y_2, y_3 = (-4.3, 0.7, -19.4)$$

- Specify and plot the corresponding likelihood function of $\mu$.

## Conjugacy of the normal-normal model

- Denote the sample mean as $\bar{y} = \frac{1}{n} \sum_i y_i$.
- The posterior density of $\mu$ is normal with updated parameters:

$$\mu | \vec{y} \ \sim \ N\left( \underbrace{\mu' \frac{\sigma^2}{n\sigma_\mu^2 + \sigma^2} + \bar{y} \frac{n\sigma_\mu^2}{n\sigma_\mu^2 + \sigma^2}}_{\text{posterior mean}}, \underbrace{\frac{\sigma_\mu^2 \sigma^2}{n\sigma_\mu^2 + \sigma^2}}_{\text{posterior variance}} \right).$$

# Posterior mean

$$\mu|\vec{y} \sim N\left(\mu' \underbrace{\frac{\sigma^2}{n\sigma_\mu^2 + \sigma^2}}_{w} + \bar{y} \underbrace{\frac{n\sigma_\mu^2}{n\sigma_\mu^2 + \sigma^2}}_{1-w}, \ \frac{\sigma_\mu^2 \sigma^2}{n\sigma_\mu^2 + \sigma^2}\right).$$

- The posterior mean is a weighted average of the prior mean $\mu'$ and the sample mean $\bar{y}$.
- As $n$ increases, the posterior mean converges to $\bar{y}$.
- As $n$ increases, the posterior variance decreases.
- The normal-normal is a *conjugate* model, since the posterior density is normal like the prior.

## Your turn

- Which is the posterior mean, if we did 5 measures with $\bar{y} = 6.7$?
- Which is the posterior mean, if we did 35 measures with $\bar{y} = 6.7$?

## Your turn

- Let $\mu$ be the average 3 p.m. temperature in Lugano.
- Your friend's prior understanding is that $\mu$ is around 15 degrees Celsius, though might be anywhere between 5 and 25 degrees.
- To learn about $\mu$, he will analyze 1000 days of temperature data.
- Letting $y_i$ denote the 3 p.m. temperature on day $i$, they'll assume that daily temperatures vary Normally around $\mu$ with a standard deviation of 5 degrees.
- Formalize a normal-normal model.

# Your turn

- Solve exercises 5.9 and 5.10 from:

https://www.bayesrulesbook.com/chapter-5.html#exercises-4

# Treating $\sigma$ as a parameter

- A more sophisticated approach is to treat $\sigma$ as a parameter.
- We assigning a prior to it; it should cover a wide range of plausible values for $\sigma$, leaving out however values that make no sense.
- In this case there is no closed-form expression of the posterior.

## Half-normal distribution

- $\sigma$ is strictly positive; a suitable prior is the *half-normal* distribution.
- The half-normal is a Gaussian restricted to positive values.
- Sample $s$ from a half-normal are obtained by:
  - sampling from a normal distribution
  - applying the absolute value to the sampled values
  - $s \sim |N(0, \xi)|$, where $\xi$ is the standard deviation of the underlying normal. It is referred to as the *scale* of the half-normal.

## The half-normal distribution

- It is asymmetric and right-skewed.
- It has long tails which are much larger than the median.

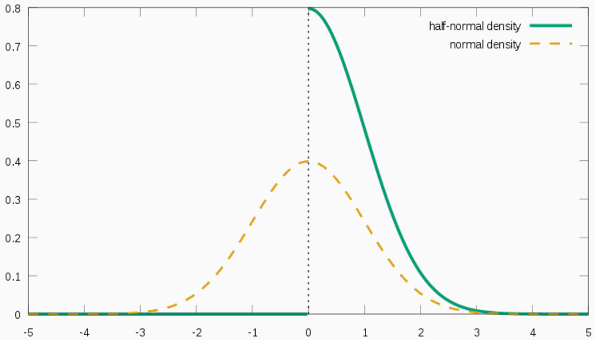
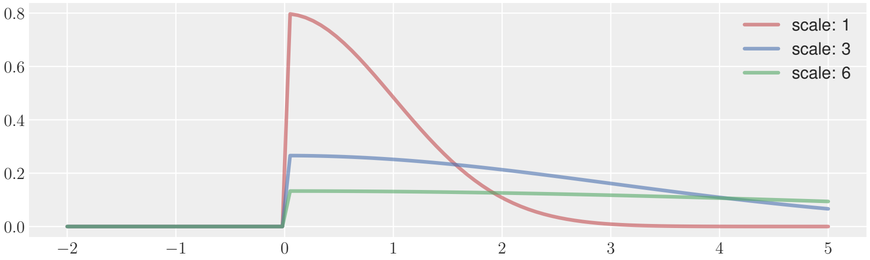



**Figure 1:** from wikipedia

## Effect of the scale parameter

- The half-normal pdf is characterized by a scale parameter (the standard deviation of the underlying normal).



## Tuning the half-normal distribution

- You can tune the scale of the HN by considering a plausible value of $\sigma$, and choose the scale so that it is close to the median of the HN.
- E.g., assume a plausible value of $\sigma$ is 7.5.
- With 95% probability the measures are lie in an interval of $\pm 15$ around the mean.
- But we are uncertain about this statement, as the interval could be well of $\pm 30$ .

# Tuning the half-normal distribution

- We try different scales, until the median is about 7.5.
- Notice the long tails of the distribution, which allows to model to correct if our prior median guess (7.5) is underestimated.

```
pd.DataFrame(halfnorm.rvs(size=1000, scale=11)).describe()
```

```
##                  0
## count  1000.000000
## mean      8.523807
## std       6.585748
## min       0.001333
## 25%       3.307487
## 50%       6.967453
## 75%      12.516742
## max      30.973912
```

## Probabilistic model with $\sigma$ as parameter

| | |
|---|---|
| $\mu \sim N(\mu', \sigma_\mu)$ | prior beliefs about $\mu$ |
| $\sigma \sim$ Half-Normal$(\xi)$ | prior beliefs about $\sigma$ |
| $y \sim N(\mu, \sigma)$ | the observation are normally distributed$\sigma$ |

- We cannot treat this model analytically, as the prior are no longer conjugates.
- We will implement it later via probabilistic programming.

## Conceptual exercise

- Try to define a probabilistic model of the distribution of height of adult males in Switzerland

# Prior for $\mu$

- The mean height of the population could be 175.
- Keeping our prior broad, we state the mean height of the population to lie with 99% probability between 160 and 190 cm (the 99% interval roughly corresponds to $\mu \pm 3\sigma$).
  - $\mu \sim N(175, 5)$

# Prior for $\sigma$

- We shall now assign a prior to $\sigma$. Within the population, we assume the height to lie with 99% probability between 100 and 250 (broad but realistic range).
- Hence the corresponding value of the standard deviation is (250-100)/6 = 25.

# Tuning the half-normal

- A half-normal distribution with scale 35 has roughly median 25:

```
pd.DataFrame(halfnorm.rvs(size=1000, scale=35)).describe()
```

```
##                  0
## count  1000.000000
## mean     28.276113
## std      20.973013
## min       0.001678
## 25%      11.481879
## 50%      24.567303
## 75%      40.408515
## max     116.372805
```

# Likelihood (distribution of the data)

- Under the assumption of normality and independence, the likelihood is:

$$y \sim \mathcal{N}(\mu, \sigma)$$

- No further specification is required.

# The resulting model

$$\mu \sim N(175, 5)$$
$$\sigma \sim \text{half-normal}(35)$$
$$\vec{y} \sim N(\mu, \sigma)$$

# Solution of the likelihood exercise

- Compute the likelihood as a function of $\mu$ for a normal sample with $\sigma$=10, given the observations:

$$y_1, y_2, y_3 = (-4.3, 0.7, -19.4)$$

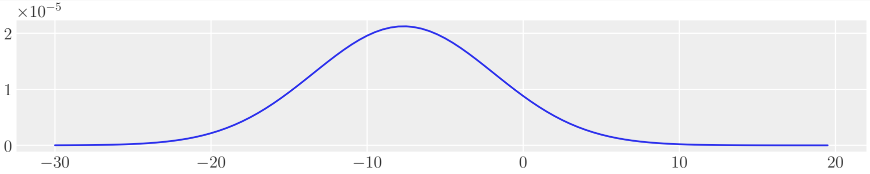
# Solution of the likelihood exercise

```
#based on the observations, plausible values of mu range between -30 and 20.
mu = np.arange(-30, 20, 0.5)
sigma = 10

#a likelihood value for each value of mu
lik = norm.pdf(-4.3, loc=mu, scale=sigma)

#under independence, the likelihood of each observation multiplies
lik = lik * norm.pdf(0.7, loc=mu, scale=sigma)
lik = lik * norm.pdf(-19.4, loc=mu, scale=sigma)

plt.figure(figsize=(10, 2))
plt.plot(mu, lik)
```




- The function has its maximum in correspondence of $\bar{y}$.
- Small numerical values (10 e-5)

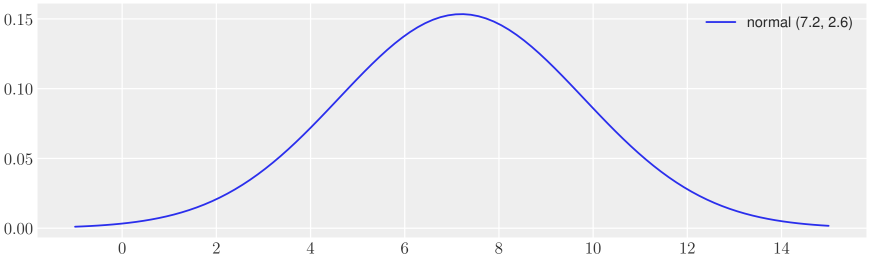
## Solution

- The values in the previous slide are numerically small. With more data, and more likelihood multiplication, it will become numerically untractable.
- For this reason it is numerically better to work with the log of the likelihood (log-likelihood) and exponentiate back the results.
- You will see how do this in the labs.

# Solution of exercises 5.9 and 5.10

- https://www.bayesrulesbook.com/chapter-5.html#exercises-4

```
plt.figure(figsize=(10, 3))
x = np.linspace(-1, 15, 100)
mu = 7.2
sigma = 2.6
y = stats.norm.pdf(x, loc = mu, scale = sigma)
plt.plot(x, y, label='normal (%s, %s)' % (mu, sigma))
plt.legend(fontsize=12)
```



# Questions b,c,d,e

- $P(X) >= 7.6, P(X) >= 4, P(X) < 0, P(X) > 8$

```
mu = 7.2
sigma = 2.6
p1 = 1 - stats.norm.cdf(7.6, loc = mu, scale = sigma)
p2 = 1 - stats.norm.cdf(4, loc = mu, scale = sigma)
p3 = stats.norm.cdf(0, loc = mu, scale = sigma)
p4 = 1 - stats.norm.cdf(8, loc = mu, scale = sigma)
p1
```

```
## 0.43886552075085816
```

```
p2
```

```
## 0.8907954066903792
```

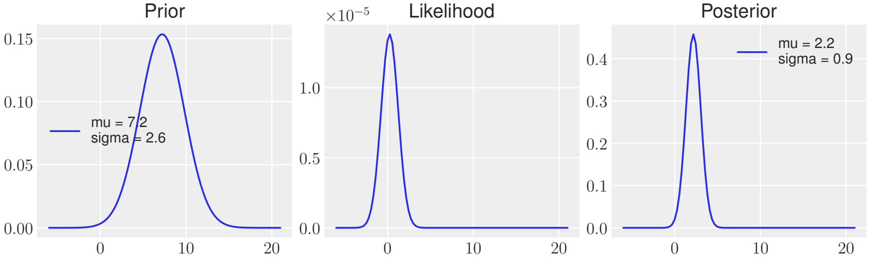
```
p3
```

```
## 0.00280944107441954
```

```
p4
```

```
## 0.3791582367631452
```

# Prior, likelihood and posterior



# Code of the previous figure

```
plt.figure(figsize=(10, 3))
mu = np.linspace(-6, 21, 100)

prior_mean = 7.2
prior_sigma = 2.6
sigma_lik = 2

#prior
prior = stats.norm.pdf(mu, prior_mean, prior_sigma)
plt.subplot(1, 3, 1)
plt.plot(mu, prior, label='mu = %s\n sigma = %s'
  % (prior_mean, prior_sigma))

plt.title('Prior')
plt.legend(fontsize=12)
```

# Code of the previous figure (cont'd)

```
#likelihood
#code below could be  vectorized
y   = np.array([-0.7, 1.2, 4.5, -4])
lik = norm.pdf (y[0], loc = mu, scale = sigma_lik)
lik = lik * norm.pdf (y[1], loc = mu, scale = sigma_lik)
lik = lik * norm.pdf (y[2], loc = mu, scale = sigma_lik)
lik = lik * norm.pdf (y[3],  loc = mu, scale = sigma_lik)


plt.subplot(1, 3, 2)
plt.plot(mu, lik)
plt.title('Likelihood')
```

# Code of the previous figure (cont'd)

```
#posterior
y_bar     = np.mean(y)
n         = len(y)
w_prior   = sigma_lik**2 / (n*prior_sigma + sigma_lik**2)

post_mean = prior_mean * w_prior + y_bar * (1 - w_prior)

post_var  = (prior_sigma**2 * sigma_lik**2)/
            (n * prior_sigma**2 + sigma_lik**2)
post_s    = np.sqrt(post_var)
posterior = stats.norm.pdf(mu, post_mean, post_var)

 plt.subplot(1, 3, 3)
plt.plot(mu, posterior, label='mu = %s\n sigma = %s' % (post_mean, post_s))
plt.legend(fontsize=12)
plt.title('Posterior')
```