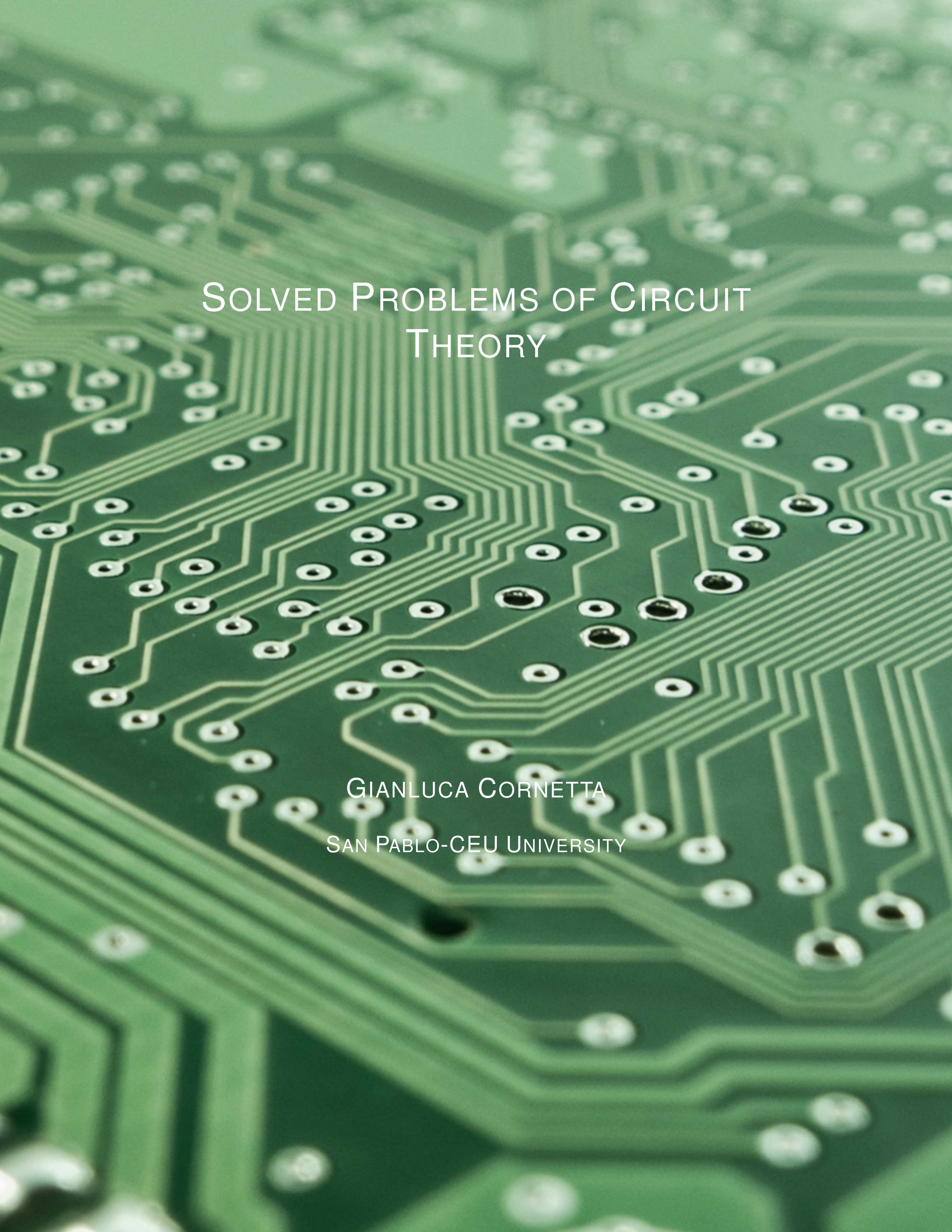

Solved Problems of Circuit Theory

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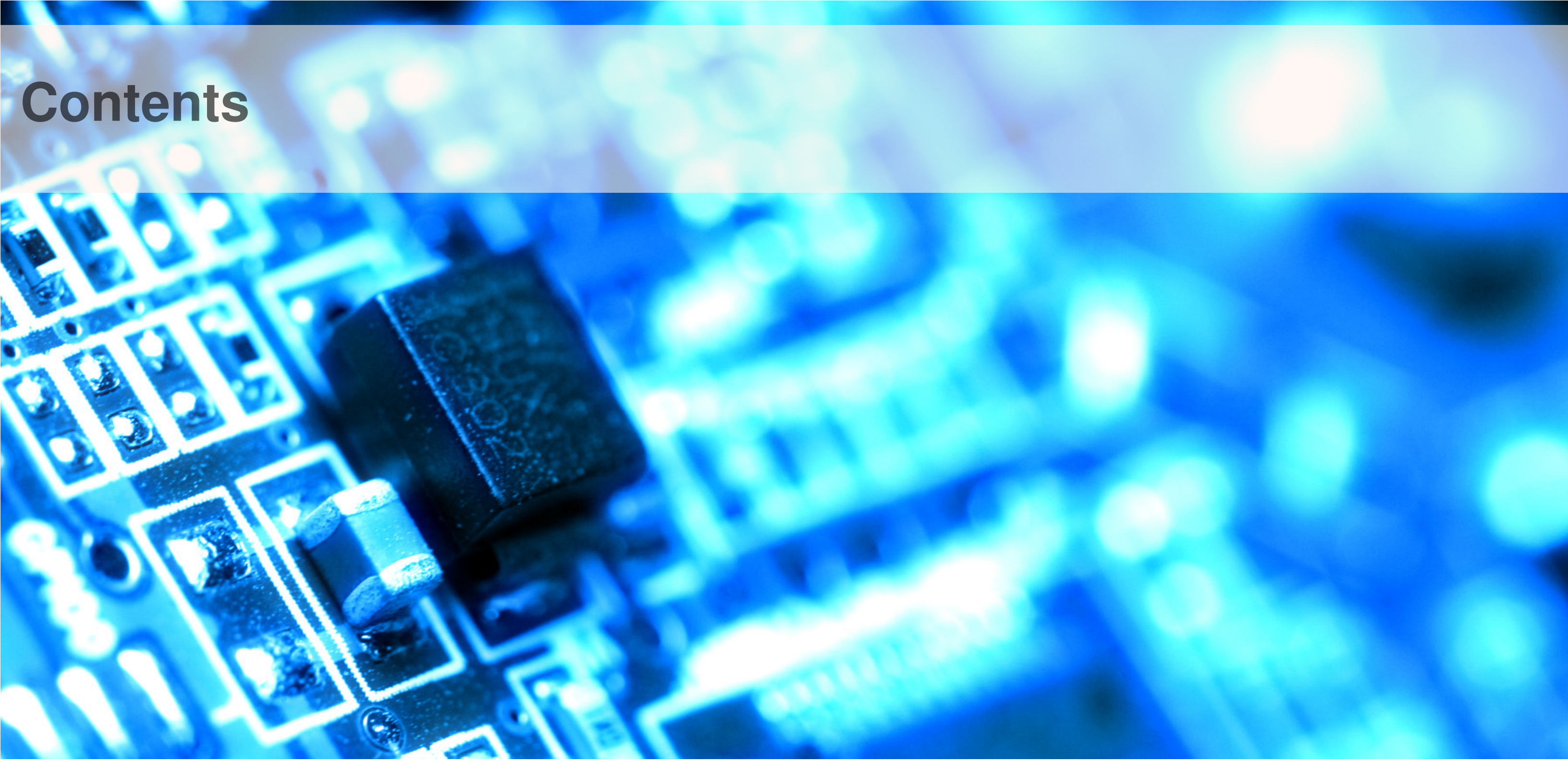

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1. Basic Concepts

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An electrical circuit is a network of interconnected components such as resistors, capacitors, inductors, and voltage sources. Few basic fundamental laws characterise the behaviour of these components. These laws are the baseline of analysis methods and mathematical relationships that are known as ***circuit theory****.*

1.1 Units of Measurement

When you analyse or design a circuit, you are typically required to compute voltages, currents or powers. However, your answer must also include a unit. The system of units used for this purpose is the *International System* (SI). The SI is a unified system of measurement that includes not only the MKS (meters, kilograms, seconds) units for length, mass, and time; but also units to characterise electrical and magnetic quantities. Table 1.1 reports some SI base units.

Magnitude	Symbol	Unit	Abbreviation
Length	l	meter	m
Mass	m	kilogram	Kg
Time	t	second	s
Electric current	I, i	ampere	A
Temperature	T	kelvin	K

Table 1.1: Some SI base units

There are also derived units, namely, units that represent a given physical magnitude but that can be derived from the SI base units. For example, capacity is measured in farads (F). A farad can be expressed as CV^{-1}. However, a volt is dimensionally equal to JC^{-1}, thus:

$$1\,F = \frac{C^2}{J} \tag{1.1}$$

The coulomb (C) can be expressed as A s, whereas a joule is N m. By definition, the newton (N) is the force necessary to give to a mass of 1 Kg, an acceleration of $1\,\mathrm{ms}^{-2}$. Hence, substituting these definitions into equation 1.1 yelds:

$$1\,\mathrm{F} = \frac{\mathrm{C}^2}{\mathrm{J}} = \frac{\mathrm{A}^2\,\mathrm{s}^2}{\mathrm{N\,m}} = \frac{\mathrm{A}^2\,\mathrm{s}^2}{\mathrm{Kg\,m}^2\,\mathrm{s}^{-2}} = \frac{\mathrm{A}^2\,\mathrm{s}^4}{\mathrm{Kg\,m}^2} \tag{1.2}$$

Table 1.2 reports some SI derived units.

Magnitude	**Symbol**	**Unit**	**Abbreviation**
Force	F	newton	N
Energy	W	joule	J
Power	P, p	watt	W
Voltage	V, v, E, e	volt	V
Charge	Q, q	coulomb	C
Resistance	R	ohm	Ω
Capacitance	C	farad	F
Inductance	L	henry	H
Frequency	f	hertz	Hz
Magnetic flux	Φ	weber	Wb
Magnetic flux density	B	tesla	T

Table 1.2: Some SI base units

Exercises

Exercise 1.1 Capacitance is measured in farads (F). However this is a quite large unit. Express the following values in powers of 10 and write them in their abbreviated forms:
(a) 0.000015 F (b) 0.0001 F (c) 0.000000005 F

solution:

(a) 0.000015 F $= 15 \times 10^{-6}$ F $= 15$ μF
(b) 0.0001 F $= 0.1 \times 10^{-3}$ F $= 0.1$ mF
(c) 0.000000005 F $= 5 \times 10^{-9}$ F $= 5$ nF

Exercise 1.2 Electric inductance is measured in henry (H). Express the following values in powers of 10 and write them in their abbreviated forms:
(a) 0.0015 H (b) 0.025 H (c) 0.000075 H

solution:

(a) 0.0015 H $= 1.5 \times 10^{-3}$ H $= 15$ mH
(b) 0.025 H $= 0.25 \times 10^{-3}$ H $= 0.25$ mF
(c) 0.000075 H $= 75 \times 10^{-6}$ H $= 75$ μH

Exercise 1.3 Frequency is measured in hertz (Hz). Express the following values in powers of 10 and write them in their abbreviated forms:
(a) 5 000 Hz (b) 8 750 000 Hz (c) 750 000 000 Hz

solution:

(a) 5000 Hz $= 5 \times 10^3$ Hz $= 5$ KHz
(b) 8750000 Hz $= 8.75 \times 10^6$ Hz $= 8.75$ MHz
(c) 750000000 Hz $= 0.75 \times 10^9$ Hz $= 0.75$ GHz

Exercise 1.4 Current is measured in amperes (A). Knowing that an ampere can be expressed as the flow of charge per second (C/s), how many electrons pass trhough a conductor carrying a 5-A current in 20 s (recall that the charge of an electron is approximately 1.6×10^{-19} C).

solution:

The total charge flowing in the conducor is:

$$Charge = Current \times time = 5 \times 20 = 100 \text{ C}$$

The charge of a single electron is $e = 1.6 \times 10^{-19}$, thus the overall charge of 100 C corresponds to:

$$\frac{Charge}{e} = \frac{240}{1.6 \times 10^{-19}} = 62.5 \times 10^{19} = 625 \times 10^{18} \text{ electrons}$$

Exercise 1.5 The current in an electric circuit rises exponentially according to the following law: $i(t) = 5(1 - e^{-3t})$ A. Calculate the charge flowing through the circuit in 200 ms.

solution:

$$i(t) = \frac{dq}{dt} \longrightarrow q = \int i(t)\,dt =$$

$$\int_0^{0.2} 5(1 - e^{-3t})\,dt =$$

$$5\left[t + \frac{e^{-3t}}{3}\right]_0^{0.2} =$$

$$5\left(0.2 + \frac{1}{3}e^{-0.6} - 0 - \frac{1}{3}\right) = 0.248 \text{ C}$$

Exercise 1.6 The unit of force is the newton (N) and work is measured in newton-meter (N $\cdot$ m), which is also the unit of energy. Alternatively, energy can be also expressed in Joules (J) where $1\,\text{J} = 1\,\text{N} \cdot \text{m}$. Determine the work for moving an electric charge

$Q = 50\ \mu$C in the direction of a uniform electric field $E = 20$ KVm^{-1} through a conductor whose length is $L = 25$ cm.

solution:
The electric force F is:

$$F = Q \times E = (50 \times 10^{-6}) \times (20 \times 10^{3}) = 1 \text{ N}$$

The work W done is by the electric field is:

$$W = F \times L = 1 \times (25 \times 10^{-2}) = 0.25 \text{ J}$$

Exercise 1.7 Electric potential difference between to points is measured in volts (V), and it is defined as the work necessary to move a unit positive charge from one point to another. What is the potential difference between the two points if it requires 100 μJ to move a 10 μC charge between the two points?

solution:

$$1\,\text{V} = 1\,\frac{\text{J}}{\text{C}} \longrightarrow \text{V} = \frac{100 \times 10^{-6}}{10 \times 10^{-6}} = 10 \text{ V}$$

Exercise 1.8 Calculate the potential difference across a resistor dissipating 20 W while absorbing a 4-A current. Compute also the resistance value of the device.

solution:
Observe that voltage may be also expressed as follows:

$$1\,\text{V} = 1\,\frac{\text{J}}{\text{C}} = \frac{\text{J/s}}{\text{C/s}}\frac{\text{W}}{\text{A}}$$

Thus:

$$\text{V} = \frac{\text{W}}{\text{I}} = \frac{20}{4} = 5 \text{ V}$$

Finally:

$$\text{V} = \text{R} \times \text{I} \longrightarrow \text{R} = \frac{\text{W}}{\text{I}^2} = \frac{20}{(4)^2} = 1.25\ \Omega$$

Exercise 1.9 The voltage and current in a circuit element are respectively $v(t) = 10\sqrt{2}\sin t\,\text{V}$ and $i(t) = 2\sqrt{2}\sin t\,\text{A}$. Calculate the instantaneous and the average power delivered to the circuit.

solution:
The instantaneous power $p(t)$ is:

$$p(t) = v(t) \times i(t) = 10\sqrt{2}\sin t \times 2\sqrt{2}\sin t \ = 40\sin^2 t\,\text{W}$$

Thus:

$$p(t) = 40 \times \frac{1}{2}(1 - \cos 2t) = 20 - 20\cos 2t\,\text{W}$$

Average power p_{avg} over a period T can be computed as:

$$p_{avg} = \frac{1}{T}\int_T p(t)\,dt = 20\,\mathrm{W}$$

Since the cosine function averages to zero over a period T.

Exercise 1.10 A resistor at a voltage $v(t) = 100\sin\omega t\,\mathrm{V}$ draws a current $i(t) = 4\sin\omega t\,\mathrm{A}$. Calculate the energy consumed by the resistor over one period of the current wave. Hence determine the average power dissipated by the resistor.

solution:
The period T is such that when $t = T$ it results:

$$\omega T = 2\pi \longrightarrow T = \frac{2\pi}{\omega}$$

Thus energy E results:

$$E = \int_0^{2\pi/\omega} v(t)i(t)\,dt = \int_0^{2\pi/\omega}(100\sin\omega t)(4\sin\omega t)\,dt = \frac{400\pi}{\omega}\,\mathrm{J}$$

Average power p_{avg} over a period T can be computes as:

$$p_{avg} = \frac{E}{T} = \frac{E}{2\pi/\omega} = \frac{400\pi}{\omega(2\pi/\omega)} = 200\,\mathrm{W}$$

Exercise 1.11 The voltage $v(t)$ and the current $i(t)$ in an AC circuit are respectively $v(t) = 10\sin 50t\,\mathrm{V}$ and $i(t) = 2\sin(50t - 60°)\,\mathrm{A}$. Calculate the instantaneous and the average power delivered to the circuit.

solution:
The instantaneous power is:

$$\begin{aligned}
p(t) = v(t)\times i(t) = &\\
&10\sin 50t \times 2\sin(50t - 60°) =\\
&20\sin 50t \times \sin(50t - 60°) =\\
&20\frac{1}{2}[cos(50t - 50t + 60°) - cos(50t + 50t - 60°)] =\\
&10[cos(60°) - cos(100t - 60°)]
\end{aligned}$$

Average power p_{avg} depends only on DC cpmponent of $p(t)$ since the AC component averages to zero over a period. Thus:

$$p_{avg} = 10\,cos(60°) = 5\,\mathrm{W}$$

Problems

Problem 1.1 Convert 2.5 minutes to milliseconds.

Problem 1.2 Convert 4 kilometers to centimeters.

Problem 1.3 Convert 25 centimeters to millimeters.

Problem 1.4 Which is the current flowing in a conductor through which 25×10^{20} electrons pass during 5 s? (Assume the charge of an electron equal approximatively to 1.6×10^{-19} C).

Problem 1.5 A charge of 400 C passes through a conductor in 15 s. What is the corresponding current in Ampere?.

Problem 1.6 A 100-W electric bulb draws a 780 mA current from the supply. How long will it take to pass a 30-C through the bulb?

Problem 1.7 An energy of 10 J is necessary to move a 5-C charge from infinity fo a point A. Determine the potential at point A assuming infinity at zero potential.

Problem 1.8 The potential difference between two conductors is 50 V. How much work is done to move a 10-C charge from one conductor to another?

Problem 1.9 Determine the charge that require 1-KJ energy to be moved from infinity to a point having a 10-V potential.

Problem 1.10 The energy capacity or rating of a battery is generally expressed in Ampere-hour (Ah). A battery is required to supply 1.5 A continuosly for five days. What must be the rating of the battery?

Problem 1.11 The decay of charge in an electric circuit is given by $q = 20e^{-100t}$ μC. Determine the resulting current.

Problem 1.12 The voltage v and the current i in a circuit are given respectively by $v = 20 \sin t\,\mathrm{V}$ and $4 \cos t\,\mathrm{A}$. Determine the instantaneous and average powers and explain your result.

1.2 Resistance and Ohm's Law

Current involves the displacement of charge in a material. In a conductor, the charge carriers are the free electrons which are displaced by a difference of potential created by an external voltage source. As these electrons move through the material, they collide with atoms of the crystalline lattice and other electrons. In a process similar to friction, the moving electrons release some of their energy in the form of heat. Hence these collisions represent an opposition to charge movement that is called resistance. The greater the opposition (i.e., the greater the resistance), the smaller will be the current for a given applied voltage.

Circuit components (called resistors) are specifically designed to possess resistance and are used in almost all electronic and electrical circuits. Resistance is represented by the symbol R and is measured in units of Ohms (after Georg Simon Ohm). The symbol for ohms is the capital Greek letter omega (Ω).

Exercises

Exercise 1.12 A copper cable, whose length is $l = 2$ m, has a circular cross section of 2 mm diamenter. Calculate the resistance of the cable at 20°C knowing that that the resistivity of copper at 20°C is $\rho = 1.72 \times 10^{-8}\Omega \cdot \mathrm{m}$.

solution:
Let $A = \pi\, r^2 = \pi \times (1 \times 10^{-3})^2 = \pi\, 10^{-6}\,\mathrm{m}^2$ the cable cross section area.
The resistance of the cable results:

$$R = \frac{\rho\, l}{A} = \frac{(1.72 \times 10^{-8}) \times 2}{\pi\, 10^{-6}} = 5.47\,\mathrm{m}\Omega$$

The resistance of a material is directly proportional to its resistivity ρ and to its length, and inversely proportional to the area of its cross section.

Exercise 1.13 A 10-m long metallic conductor with cross section area of 1 mm^2 has a resistance of 2 Ω. Determine the conductivity of the metal.

solution:

$$\sigma = \frac{l}{R\, A} = \frac{10}{2 \times (10^{-3})^2} = 5\,\mathrm{MS/m}$$

Note that 1 siemens (S) = 1 Ω^{-1}.

Observe that the **conducivity** *of a material is the inverse of its resistivity, namely $\sigma = \rho^{-1}$.*

Exercise 1.14 The temperature coefficient α expresses the variation of the resistance with temperature. More concretely, the resistance R_T at a temperature of $T\,^{circ}$C is related with the resistance R_0 at $0\,°$C by $R_T = R_0(1 + \alpha_0\, T)$, where α_0 is the temperature coefficient at $0\,°$C. Calculate the resistance of a copper wire at $T_1 = -10$°C assuming that the resistance is zero at -273°C and that $R_0 = 10\,\Omega$.

solution:

$$\frac{R_0}{273 + T_0} = \frac{R_1}{273 + T_1}$$

Hence, resistance R_1 at temperature T_1 is:

$$R_1 = \frac{R_0(273 + T_1)}{273 + T_0} = \frac{10(273 - 10)}{(273 + 0)} = 9.63\,\Omega$$

Exercise 1.15 A metallic conductor has a resistance of $10\,\Omega$ at $0°$C. At $30°$C the resistance becomes $11\,\Omega$. Determine the temperature coefficient of the metal at $30°$C.

solution:

$$R_0 = R_1[1 + \alpha_1(0 - 30)] \longrightarrow 10 = 11[1 + \alpha_1(-30)]$$

Hence at $30°$C, the temperature coefficient results:

$$\alpha_1 = \frac{10 - 11}{11 \times (-30)} \approx 0.003\,°\mathrm{C}^{-1}$$

Exercise 1.16 For the metallic metallic conductor of Exercise 2.4, determine the temperature coefficient at $0°$C.

solution:

$$R_T = R_0[1 + \alpha_0 T] \longrightarrow 11 = 10[1 + \alpha_0(30)]$$

Hence at $0°$C, the temperature coefficient results:

$$\alpha_0 = \frac{11 - 10}{10 \times (30)} = 0.003\,°\mathrm{C}^{-1}$$

Exercise 1.17 Obtain a general relationship between the temperature coefficients α_0 and α_T.

solution:

$$R_T = R_0[1 + \alpha_0 T] \tag{1.3}$$

and

$$R_0 = R_T[1 - \alpha_T T] \tag{1.4}$$

Solving equation 1.4 for α_T leads to:

$$\alpha_T = \frac{R_T - R_0}{T\,R_T} \tag{1.5}$$

Substituting R_T from equation 1.3 yelds:

$$\alpha_T = \frac{R_0(1 + \alpha_0 T) - R_0}{T\,R_0(1 + \alpha_0 T)} = \frac{\alpha_0}{1 + \alpha_0 T} \tag{1.6}$$

Problems

Problem 1.13 Calculate the length of a copper wire having a diamenter of 4 mm and a resistance of 4 Ω. Conductivity of copper is 5.8×10^7 S/m.

1.3 Series and Parallel Resistive Circuits

In the previous sections we examined the interrelation of current, voltage, resistance, and power in simple resistive circuits. In this section we will expand these basic concepts to examine the behavior of circuits having several resistors in series or in parallel. We will use Ohm's law to derive the **voltage divider rule** and to verify Kirchhoff's voltage law. A good understanding of these important principles provides the fundamental concepts upon which more advanced circuit analysis techniques are built. Kirchhoff's voltage law and Kirchhoff's current law, which will be covered in the next section, are fundamental for understanding all electrical and electronic circuits.

Exercises

Exercise 1.18 A 10-V voltage source provides supply to two resistors $R_1 = 10\,\Omega$ and $R_2 = 20\,\Omega$ Connected in parallel. Find the total parallel resistance R_p of the parallel connection and the current through each resistor and the total current supplied by the source

solution:
The total parallel resistance is:

$$\frac{1}{R_p} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{10} + \frac{1}{20} = \frac{3}{20}$$

Thus:

$$R_p \approx 6.67\,\Omega$$

The current through R_1 is:

$$I_{R_1} = \frac{V}{R_1} = \frac{10}{10} = 1\,\mathrm{A}$$

Analogously, the current through R_2 is:

$$I_{R_2} = \frac{V}{R_2} = \frac{10}{20} = 0.5\,\mathrm{A}$$

The total current results:

$$I = I_{R_1} + I_{R_2} = 1 + 0.5 = 1.5\,\mathrm{A}$$

Exercise 1.19 A voltage source V provides supply to the series connection of resistors R_1 and R_2. Which are the voltages across the resistors?

solution:

The total series resistance is $R_s = R_1 + R_2$, thus the current flowing through the circuit is:

$$I = \frac{V}{R_s} = \frac{V}{R_1 + R_2}$$

The voltage across R_1 is:

$$V_1 = R_1 \times I = \frac{R_1}{R_1 + R_2} V$$

Similarly:

$$V_2 = R_2 \times I = \frac{R_2}{R_1 + R_2} V$$

The result of Exercise 1.19 is also known as **voltage divider rule** *and can be generalized as follows: "the voltage dropped across any series resistor is proportional to the magnitude of the resistor. The total voltage dropped across all resistors must equal the applied voltage".*

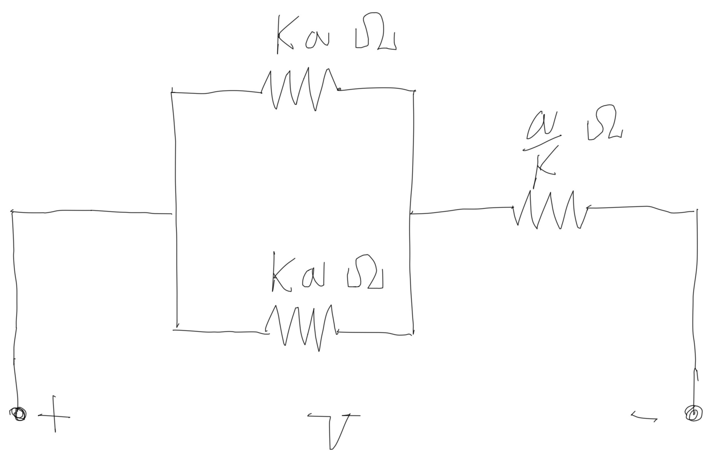


Figure 1.1: Resistive circuit of Exercise 1.20

Exercise 1.20 For the circuit depicted in Figure 1.1, determine the value of k so that the total resistance is minimum.

solution:
The total resistance of the circuit is:

$$R = (ka \parallel ka) + \frac{a}{k} = \frac{ka}{2} + \frac{a}{k}$$

Thus:

$$R = \frac{k^2 a + 2a}{2k} \tag{1.7}$$

Observe that R is a function of k; hence, the resistance is minimum when $\frac{\partial R}{\partial k} = 0$, which implies that:

$$2ka(2k) - 2(k^2 a + 2a) = 0$$

Which yelds to:

$$2k^2 - k^2 - 2 = 0 \longrightarrow k = \sqrt{2} = 1.414$$

Exercise 1.21 What is the maximum power that can be absorbed by the resistors of the circuit of Figure 1.1 when it is connected to supply voltage V? Calculate the input current at maximum power condition.

solution:
The power P dissipated on the total resistance R of the circuit is $P = \frac{V^2}{R}$. Hence power is inversely proportional to resistance R.
In exercise 1.20 we found that R is minimum when $k = \sqrt{2}$. Substituting in Equation 1.7 yelds:

$$R = \frac{2a}{\sqrt{2}}$$

Hence:

$$P = \frac{\sqrt{2}V^2}{2a} = \frac{V^2}{\sqrt{2}a}\ \mathrm{W}$$

Finally, recalling that $P = V\,I$, it follows that:

$$I = \frac{P}{V} = \frac{V}{\sqrt{2}a}\ \mathrm{A}$$

Observe that the power dissipated by a resistive circuit is always proportional to the square of the applied voltage.

Exercise 1.22 Formulate the **current division rule** among three resistors R_1, R_2, and R_3 connected in parallel and with a total input current I.

solution:
The total parallel resistance R_p is sucha that:

$$\frac{1}{R_p} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

Hence, the voltage drop V across the resistors is:

$$V = R_p \times I$$

Consequently, the currents I_1, I_2,vand I_3 flowing into the individual resistors are:

$$I_1 = \frac{V}{R_1} = \frac{R_p}{R_1}I$$
$$I_2 = \frac{V}{R_2} = \frac{R_p}{R_2}I$$
$$I_3 = \frac{V}{R_3} = \frac{R_p}{R_3}I$$

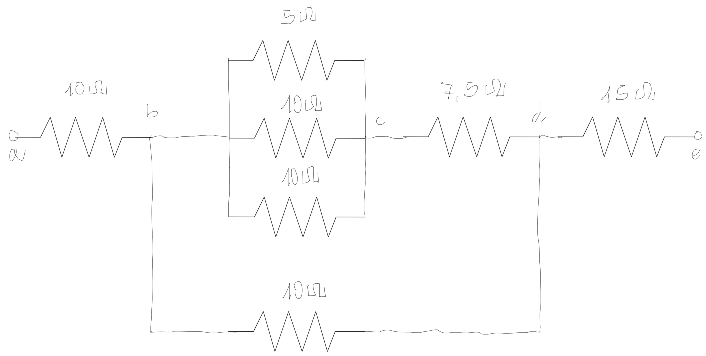


Figure 1.2: Resistive circuit of Exercise 1.23

Exercise 1.23 Reduce the circuit of Figure 1.2 between terminals a and e To a signle resistor.

solution:
The resistance R_{bc} between points b and c is:

$$\frac{1}{R_{bc}} = \frac{1}{10} + \frac{1}{10} + \frac{1}{5} = \frac{4}{10}\,\Omega$$

The series resistance R'_{bd} between points b and d is:

$$R'_{bd} = R_{bc} + R_{cd} = \frac{10}{4} + \frac{30}{4} = \frac{40}{4}\,\Omega$$

Hence, the overall resistance R_{bd} is:

$$\frac{1}{R_{bd}} = \frac{1}{R'_{bd}} + \frac{1}{10} = \frac{4}{40} + \frac{1}{10} = \frac{2}{10}\,\Omega$$

Finally R_{ae} results:

$$R_{ae} = R_{ab} + R_{bd} + R_{de} = 10 + \frac{10}{2} + 15 = 30\,\Omega$$

Exercise 1.24 Two resistors, R_1 and R_2, are connected in paralell and consume equal power at $20\,°\mathrm{C}$. The resistors are made of different materials and the temperature coefficients of R_1 and R_2 are respectively $\alpha_1 = 0.002\,°\mathrm{C}^{-1}$ and $\alpha_2 = 0.004\,°\mathrm{C}^{-1}$. What is the ratio of the powers dissipated at $80\,°\mathrm{C}$by resistances R_2 and R_1 respectively?

solution:
At $20\,°\mathrm{C}$ the two resistors dissipate the same power which implies that $R_1 = R_2$; hence:

$$R_{01}(1 + 20\alpha_1) = R_{02}(1 + 20\alpha_2) \longleftarrow \frac{R_{01}}{R_{02}} = \frac{1 + 20\alpha_2}{1 + 20\alpha_1}$$

Consequently, the power ratio at 80 °C is:

$$\frac{V^2/R_2}{V^2/R_1} = \frac{R_1}{R_2} = \frac{R_{01}(1+60\alpha_1)}{R_{02}(1+60\alpha_2)} = \frac{(1+10\alpha_2)(1+60\alpha_1)}{(1+10\alpha_1)(1+60\alpha_2)}$$

Substituting the numerical values yelds 0.920.

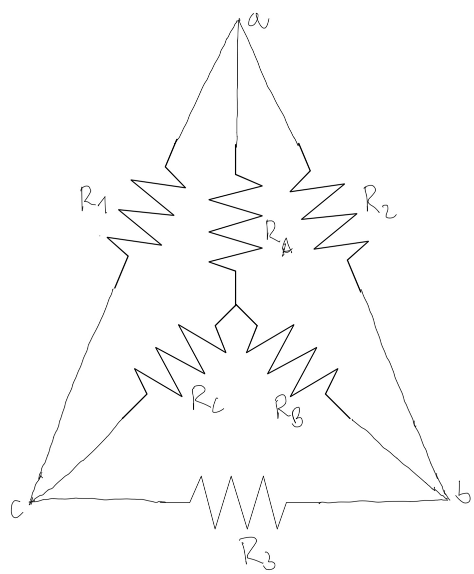


Figure 1.3: Resistive circuit of Exercise 1.25

Exercise 1.25 Convert the star-connected resistors of Figure 1.3 into an equivalent triangle-connected bank.

solution:
For equivalence the resistance seen from any two terminals for both the triangle and the star configurations must be the same. Thus:

$$\begin{aligned}
R_{ac} = R_A + R_C = R_1 \parallel (R_2 + R_3) &= \frac{R_1(R_2+R_3)}{R_1+R_2+R_3} \\
R_{ab} = R_A + R_B = R_2 \parallel (R_1 + R_3) &= \frac{R_2(R_1+R_3)}{R_1+R_2+R_3} \\
R_{bc} = R_B + R_C = R_3 \parallel (R_1 + R_2) &= \frac{R_3(R_1+R_2)}{R_1+R_2+R_3}
\end{aligned}$$

Solving first for R_A, R_B and R_C yelds:

$$\begin{aligned}
R_A &= \frac{R_1 R_2}{R_1+R_2+R_3} \\
R_B &= \frac{R_2 R_3}{R_1+R_2+R_3} \\
R_C &= \frac{R_1 R_3}{R_1+R_2+R_3}
\end{aligned} \tag{1.8}$$

Note that Equations 1.8 are used to transform a triangle connection into a star connection. Now, solving [a] equations 1.8 for $(R_1 + R_2 + R_3)$ leads to the equations for star to triangle transformation:

$$R_1 = \frac{1}{R_B}(R_A R_B + R_A R_C + R_B R_C)$$
$$R_2 = \frac{1}{R_C}(R_A R_B + R_A R_C + R_B R_C) \quad (1.9)$$
$$R_3 = \frac{1}{R_A}(R_A R_B + R_A R_C + R_B R_C)$$

[a]**Help:** obtain, for example, R_A in Equation 1.8 in two steps as $(R_A+R_C)-(R_B+R_C) = (R_A-R_B)$, and $(R_A + R_B) + (R_A - R_B)$. Then solve all Equations 1.8 for $(R_1 + R_2 + R_3)$ and equate the results to find, for example, R_1 and R_2 as a function of R_A, R_B and R_3. Substitute in the expression for R_A to find R_3. Repeat the same procedure to compute R_1 and R_2.

The triangle-star transformation is known in the literature with several different names. Some authors refer to it as triangle-delta (or delta-triangle) transformation, wye-tee transformation, pi-tee transformation or Kennelly Theorem.

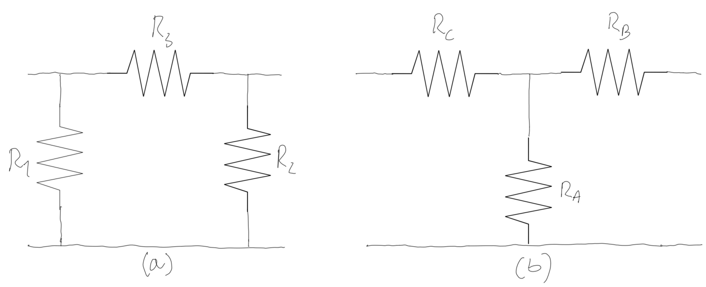


Figure 1.4: Resistive circuit of Exercise 1.26

Exercise 1.26 Convert the pi-connected resistors with $R_1 = 10\,\Omega$, $R_2 = 15\,\Omega$, and $R_3 = 5\,\Omega$ of Figure 1.4 (a) into the equivalent tee-connected bank of Figure 1.4 (b).

solution:
Observe that pi- and tee-connections are the same of triangle- and star-connections. Thus:

$$R_A = \frac{R_1 R_2}{R_1 + R_2 + R_3} = \frac{10 \times 15}{10 + 15 + 5} = \frac{150}{30} = 5\,\Omega$$
$$R_B = \frac{R_2 R_3}{R_1 + R_2 + R_3} = \frac{15 \times 5}{10 + 15 + 5} = \frac{75}{30} = 2.5\,\Omega$$
$$R_C = \frac{R_1 R_3}{R_1 + R_2 + R_3} == \frac{10 \times 5}{10 + 15 + 5} = \frac{50}{30} = 1.67\,\Omega$$

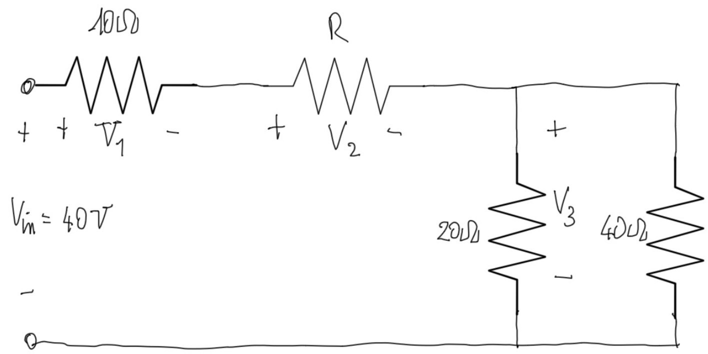


Figure 1.5: Resistive circuit of Exercise 1.27

Exercise 1.27 For the circuit of Figure 1.5 determine the value of R So that the power dissipated in the $20 - \Omega$ resitor is $20\,\mathrm{W}$.

solution:
The following relation must hold:

$$P_{20\,\Omega} = \frac{V_3^2}{20} = 20\,\mathrm{W} \longrightarrow V_3 = \sqrt{(P_{20\,\Omega} \times 20)} = 20\,\mathrm{V}$$

The current flowing through the $20 - \Omega$ resistor is:

$$I_{20\,\Omega} = \frac{V_3}{20} = \frac{20}{20} = 1\,\mathrm{A}$$

Similarly the current through the $40 - \Omega$ resistor is:

$$I_{40\,\Omega} = \frac{V_3}{40} = \frac{20}{40} = 0.5\,\mathrm{A}$$

Hence the total current I flowing through the circuit is:

$$I = I_{20\,\Omega} + I_{40\,\Omega} = 1 + 0.5 = 1.6\,\mathrm{A}$$

The voltage drop V on the remaining resistance is:

$$V = V_1 + V_2 = V_{in} - V_3 = 40 - 20 = 20\,V$$

More specifically:

$$V = (10 + R) \times I = 10I + RI \longrightarrow R = \frac{V - 10I}{I} = \frac{20 - 15}{1.5} \approx 3.3\,\Omega$$

Exercise 1.28 The resistance R of a coil is measured experimentally using the voltmeter-ammeter method. Figure 1.6 depicts two possible arrangements. The internal resistance of the voltmenter and of the ammeter are $20\,\mathrm{K\Omega}$ And $0.1\,\Omega$ respectively. What is the value of R For the set-up of Figure 1.6 (a) assuming that the voltmeter reads $4\,\mathrm{V}$ and the ammeter reads $16\,\mathrm{A}$?

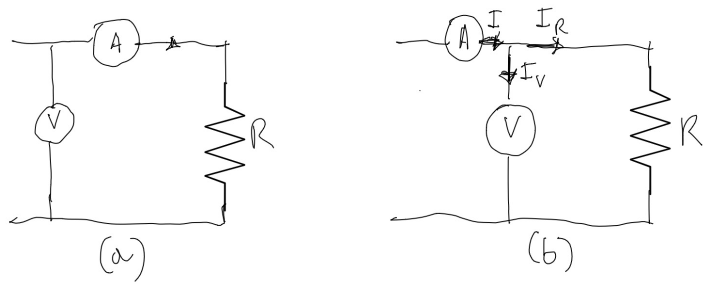


Figure 1.6: Resistive circuit of Exercise 1.28

solution:
Applying the Ohm's law:

$$4 = (1 + R) \times 16 \longrightarrow R = \frac{4}{16} - 0.1 = 0.15\,\Omega$$

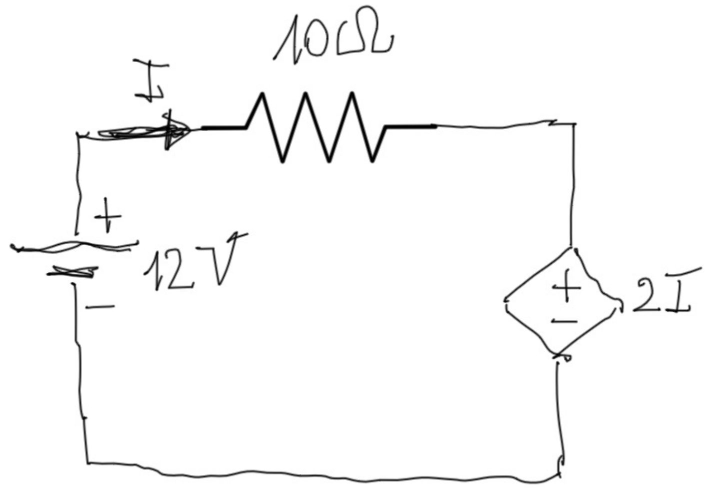


Figure 1.7: Resistive circuit of Exercise 1.29

Exercise 1.29 For the circuit of Figure 1.7 determine the power supplied by the 12-V source and by the $2I$-dependent voltage source.

solution:
Applying the Ohm's law:

$$(12 - 2I) = 10I \longrightarrow I = \frac{10 + 2}{12} = 1\,\text{A}$$

Thus the power delivered by the 12-V source is $12 \times 1 = 12\,\text{W}$. Conversely, the power edlivered by the current-controlled dependent voltage source is $(2 \times 1) \times (-1) = -2\,\text{W}$. Note that for the dependent source negative sign is used for the current because is going into the source. This means that the dependent source is absobring (rather than delivering) power.

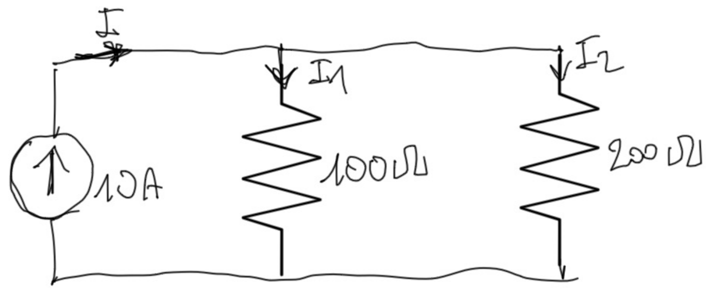


Figure 1.8: Resistive circuit of Exercise 1.30

Exercise 1.30 The parallel combination of Figure 1.8 is fed by a 10-A current source. Calculate the power absorbed by each resistor.

solution:
Let $R_1 = 100\,\Omega$ and $R_2 = 100\,\Omega$. The current division rule yelds:

$$I_1 = \frac{R_2}{R_1 + R_2} I = \frac{200}{100 + 200} 10 = 6.67\,\mathrm{A}$$
$$I_2 = \frac{R_1}{R_1 + R_2} I = \frac{100}{100 + 200} 10 = 3.33\,\mathrm{A}$$

Thus the respective power lossess can be computes as:

$$P_1 = R_1 \times I_1^2 = 100 \times (6.67)^2 = 4.448\,\mathrm{KW}$$
$$P_2 = R_2 \times I_2^2 = 200 \times (3.33)^2 = 2.217\,\mathrm{KW}$$

Problems

Problem 1.14 If a voltage supply V is connected at the input of the circuit of Figure 1.1, find the condition for which the maximum power is supplied to the resistors.

Problem 1.15 A 100-V voltage supply is connected to four series resistors whose values are 10, 20, 25, and 50 Ω respectively.

Problem 1.16 Determine the voltages and the currents through three parallel resistors connected to a 10-V voltage supply and whose values are 10, 25, and 50 Ω respectively.

Problem 1.17 A battery has an internal resistance R_i and a teminal voltage V_t. Show that the power supplied to a resistive load R_L cannot exceed $V^2/2R_i$.

Problem 1.18 A battery has an internal resistance of $1\,\Omega$ and an open circuit voltage of $10\,\mathrm{V}$. The battery supplies a $4-\Omega$ load. Detrmine (i) the power lost within the battery, and (ii) the terminal voltage on the load.

Problem 1.19 Suppose the resistors R_1, R_2 and R_3 form a triangle connection as depicted in Figure 1.3. Obtain the equivalent wye-connected configuration.

Problem 1.20 If the ammeter reading in Figure 1.6 (b) is 16 A and the resistance values are the same as in Exercise 1.28. Determine the voltmeter reading.

Problem 1.21 Based on the results of Exercise 1.28 and Problem 1.20, if the resistance is measured as the ratio of the voltmeter and ammeter readings, state which one fo the two configurations depiceted in Figure 1.6 is preferred for the measurement of a low resistance and which one is suitable for the measurement of a high resistance.

Problem 1.22 Determine the voltage across the resistors of the circuit of Figure 1.8. Verify that the power supplied by the source is the same as the total power dissipated on the resistors.

1.4 Kirchoff's Law

Next to Ohm's law, Kirchoff's voltage (KVL) and current (KCL) laws are ones of the most important laws of electricity. KVL states that *the algebraic summation of voltage drops around a closed loop is equal to zero.* Namely:

$$\sum_i V_i = 0 \tag{1.10}$$

Kirchoff's voltage law is extremely useful to understand the operarion of a series circuits. In a similar manner, Kirchoff's current law is the underlying principle to explain the operation of a parallel circuit. KCL states that *the algebraic summation of the currents entering a node is equal to zero.* Namely:

$$\sum_i I_i = 0 \tag{1.11}$$

Alternatively, KCL may be formulated stating that *the summation of the currents entering a node is equal to the summation of the currents leaving that node.*

Exercises

Exercise 1.31 Apply KVL to the circuit of Figure 1.9.

solution:
The direction of current I is arbitrary and it is chosen as indicated in Figure 1.9. The voltage across each resistor is assigned a polarity. Application of Ohm's law yelds $V_i = -R_i I$ if I enters the positive terminal of resistor R_i, and $V_i = R_i I$ If I enters the negative terminal. Hence:

$$V - V_1 - V_2 - V_3 = 0 \longrightarrow V = V_1 + V_2 + V_3 = R_1 I + R_2 I + R_3 I = (R_1 + R_2 + R_3)\, I$$

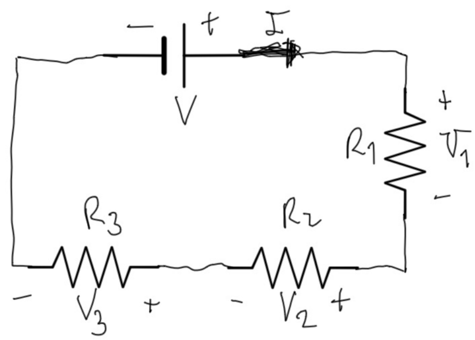


Figure 1.9: Resistive circuit of Exercise 1.31

Exercise 1.32 Write the Kirchoff's voltage equations for the two loops of the circuit of Figure 1.10. Assume the polarities indicated in the Figure.

solution:
The Kirchoff's equations are:

$$V_a - V_1 - V_2 - V_b = 0$$
$$V_b + V_2 - V_3 - V_4 - V_c = 0$$

Which yelds:

$$(V_a - V_b) = V_1 + V_2 = R_1 I_1 + R_2(I_1 - I_2)$$
$$(V_b - V_c) = -V_2 + V_3 + V_4 = R_2(I_2 - I_1) + R_3 I_2 + R_4 I_2$$

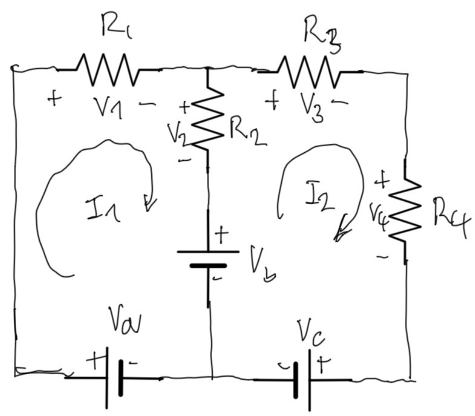


Figure 1.10: Resistive circuit of Exercise 1.32

Exercise 1.33 Apply the KCL to the node of Figure 1.11 to find the magnitude and the direction of current I. Assume the polarities indicated in the Figure.

solution:

The Kirchoff current equation for the indicated directions is:

$$-4-2-6+3-I+1-8=0 \longrightarrow I=-4-2-6+3+1-8=-16\,\mathrm{A}$$

Thus current I Enters the node.

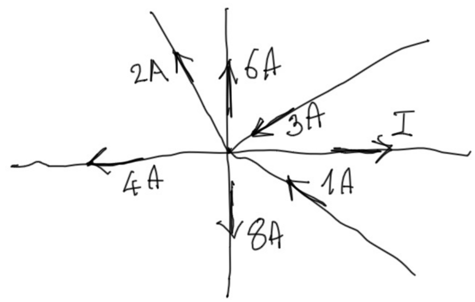


Figure 1.11: Node of Exercise 1.33

Exercise 1.34 Determine I_1, I_2, and I_3 for the circuit of Figure 1.12 by mesh analysis.

solution:
Applying the KVL to the three loops indicated yelds:

$$\begin{aligned} 10-4I_1-(I_1-I_2)&=0\\ (I_1-I_2)-4I_2-2(I_2-I_3)&=0\\ 2(I_2-I_3)+8I_3-12&=0 \end{aligned}$$

Namely:

$$\begin{aligned} 5I_1-I_2&=10\\ I_1-7I_2+2I_3)&=0\\ I_2+3I_3&=6 \end{aligned}$$

Which yelds $I_1=2.16\,\mathrm{A}$, $I_1=0.804\,\mathrm{A}$, and $I_3=1.732\,\mathrm{A}$.

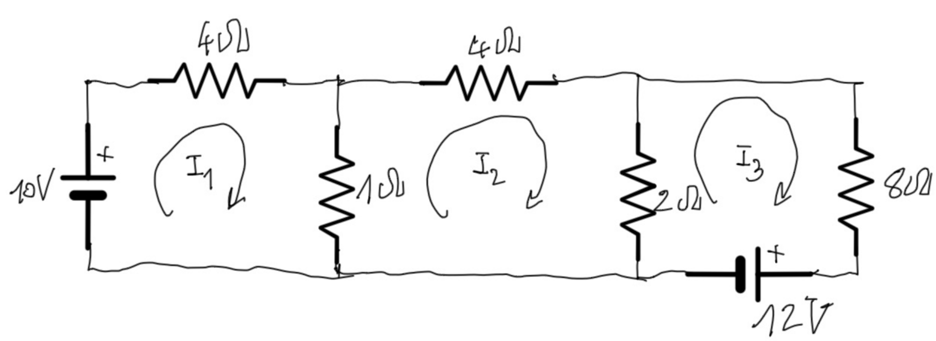


Figure 1.12: Circuit for Exercise 1.34

Exercise 1.35 Determine current i_x and i_y for the circuit of Figure 1.13.

solution:
Applying the KVL to the three loops yelds:

$$
\begin{aligned}
20 - 10i_1 - 10i_x &= 0 \\
10i_x + 10 - 5(i_1 - i_x) + 5i_y &= 0 \\
5i_y + 10 - 10(i_1 - i_x + i_y) &= 0
\end{aligned}
$$

Which finally leads to $i_x = 0.25\,\mathrm{A}$ and $i_y = -1\,\mathrm{A}$. Thus, current i_y flows the opposite direction with respect the one indicated in Figure 1.13.

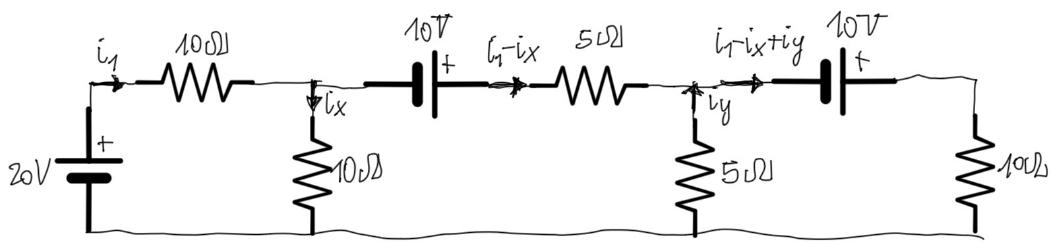
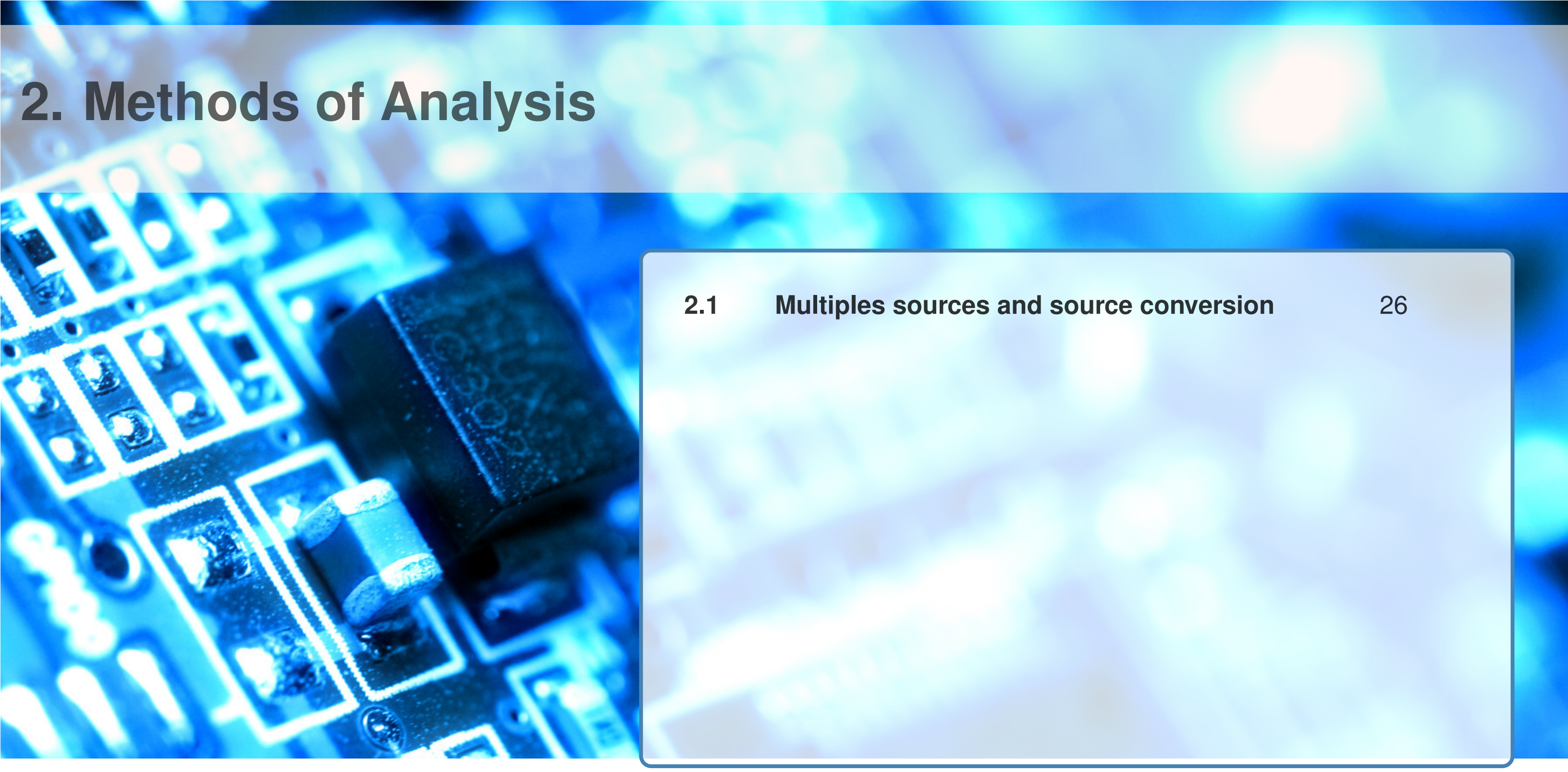


Figure 1.13: Circuit for Exercise 1.35

Problems

Problem 1.23 Apply KVL to obtain an expression for the equivalent resistance formed by n resistances $R_1, R_2, \ldots, R_n$ connected in series.

Problem 1.24 Apply KCL to obtain an expression for the equivalent resistance formed by n resistances $R_1, R_2, \ldots, R_n$ connected in parallel.



2. Methods of Analysis

2.1 Multiples sources and source conversion 26

The networks studied so far have, in general, a single voltage or current source and can be easily analysed using Kirchoff's volatge and current laws. In this chapter, you will learn how to analyse circuits with more than one source and that cannot be easily analysed using the techniques developed in the previous chapter.

The methods used to analyse complex networks include branch current analysis, mesh (or loop) analysis and nodal analysis.

The methods outlined above can be applied to ***linear bilateral networks****. The components of a* ***linear*** *network exhibit a linear voltage-current charachteristic. The term* ***bilateral*** *indicates that each component will have a characteristic that is independent of the direction of the currnt flow and of the voltage drop across its termnals.*

2.1 Multiples sources and source conversion

Most of the circuits presented so far used **constant voltage sources** to provide power to the circuit. Recall that a voltage source supplies a constant voltage regardless of how the components are connected and of the current drawn by the circuit. Conversely, **constant current sources** supply a constant current regardless of how the components are connected to the source and of the voltage drop across the current source.

In the analysis of certain circuits it is easier to work with current sources rather than with voltage sources; thus, it is interesting to know how to perform source conversion and transform a voltage source into a current one and viceversa. This technique is depicted in Figure 2.1 and applies only to **real sources.** So far, we have considered only ideal sources. An ideal voltage source has a zero series resistance and an ideal current source has a infinite shunt resistance. Conversely real voltage and current sources have finite series and shunt resistance respectively.

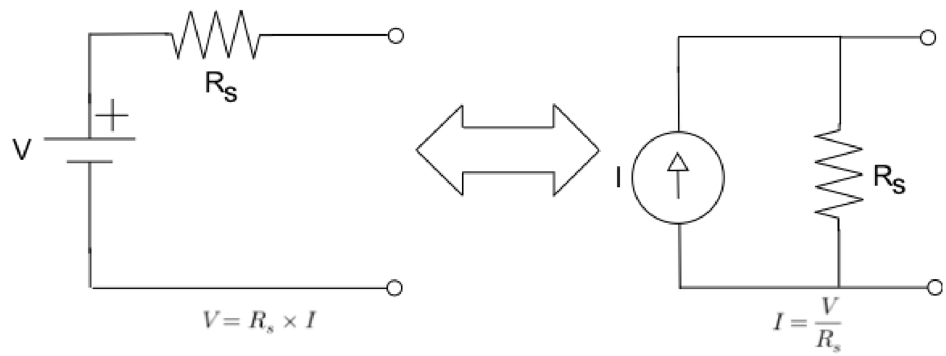


Figure 2.1: Source conversion.

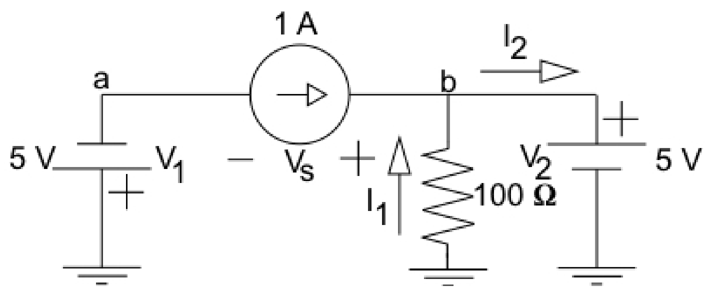


Figure 2.2: Circuit of Exercise 2.1.

Exercises

Exercise 2.1 Calculate the currents I_1, I_2, and the voltage V_s for the circuit of Figure 2.2.
solution:
Source V_2 sets the voltage drop across the resistor, hence, applying the Ohm's law yelds:

$$I_1 = \frac{5}{100} = 50\,\text{mA}$$

Applying the Kirchoff's current law at node b yelds:

$$I_2 = I_1 + 1 = 1.05\,\text{mA}$$

Finally, applying the Kirchoff's voltage law yelds:

$$\sum_i V_i = -5 + V_s - 5 = 0 \longrightarrow V_s = 5 + 5 = 10\,\text{V}$$

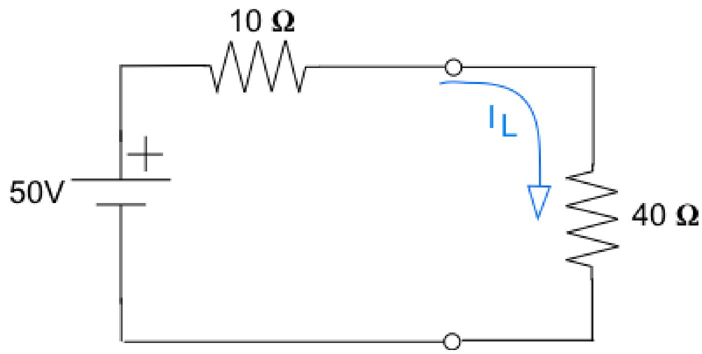


Figure 2.3: Circuit of Exercise 2.2.

Exercises

Exercise 2.2 Convert the voltage source of Figure 2.3 into a current source and verify that the load current I_L is the same for both sources.
solution:
The equivalent current source will have a current magnitude given by:

$$I = \frac{50}{10} = 5\,\mathrm{A}$$

For the voltage source of Figure 2.2, the current I_L through the $40\,\Omega$ load resistance is:

$$I_L = \frac{50}{10+50} = 1\,\mathrm{A}$$

The load current for the equivalent current source can be computed using the *current divider rule*, yelding:

$$I_L = \frac{10}{10+40} I = \frac{10}{50} 5 = 1\,\mathrm{A}$$

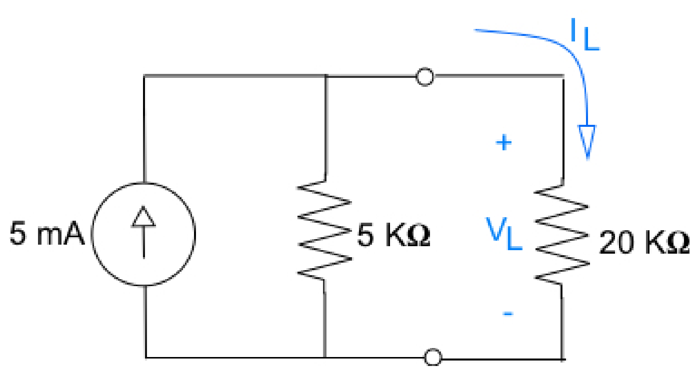


Figure 2.4: Circuit of Exercise 2.3.

Exercises

Exercise 2.3 Convert the current source of Figure 2.4 into a voltage source and verify that the load voltage V_L is the same for both sources.
solution:
The equivalent voltage source will have a voltage magnitude given by:

$$V = 5\,\text{mA} \times 5\,\text{K}\Omega = (5 \times 10^{-3}\text{A}) \times (5 \times 10^{3}\Omega) = 25\,\text{V}$$

For the current source of Figure 2.3, the voltage V_L through the $10\,\text{K}\Omega$ load resistance can be computed as follows:

$$I_L = \frac{5\,\text{K}\Omega}{(5+20)\,\text{K}\Omega} 5\text{mA} = 1\,\text{mA}$$

Which yelds:

$$V_L = 20\,\text{K}\Omega \times \ 1\,\text{mA} = 20\,\text{V}$$

The load voltage for the equivalent voltage source can be computed using the *voltage divider rule*, yelding:

$$V_L = \frac{20\,\text{K}\Omega}{(5+20)\,\text{K}\Omega} V = \frac{20}{25} 25 = 20\,\text{V}$$

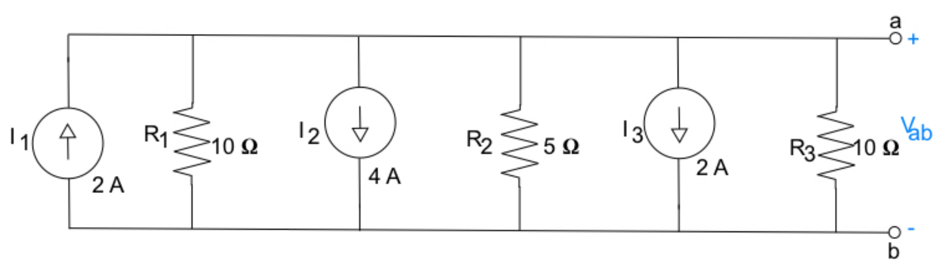


Figure 2.5: Circuit of Exercise 2.4.

Exercises

Exercise 2.4 Simplify the circuit of Figure 2.5 and determine the output voltage V_{ab}.
solution:
Since all the current sources I_1, I_2, and I_3 are in parallel they are equivalent to a new current source I whose value is (applying the KCL) the algebraic sum of all the source currents:

$$I = I_1 - I_2 - I_3 = 2 - 4 - 2 = -4\,\text{A}$$

The resulting current I is negative, thus the current flows downwards from node a to node b. The equivalent resistance of the circuit is:

$$R_{eq} = R_1 \parallel R_2 \parallel R_3 = 10\,\Omega \parallel 5\,\Omega \parallel 10\,\Omega = 2.5\,\Omega$$

Finally:

$$V_{ab} = R_eq \times I = (2.5\,\Omega) \times (-4\,\text{A}) = -10\,\text{V}$$

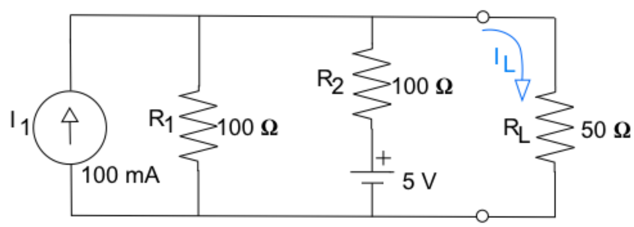


Figure 2.6: Circuit of Exercise 2.5.

Exercises

Exercise 2.5 Simplify the circuit of Figure 2.6 into a single current source and determine load current through resistor R_L.
solution:
First, transform the voltage source into its Norton equivalent,i.e., the parallel connection between resistance R_2 and the equivalent current source I computed as:

$$I = \frac{5}{R_2} = 50\,\text{mA}$$

Current sources are in parallel, hence the equivalent source I_s is:

$$I_s = I_1 + I = 100\,\text{mA} + 50\,\text{mA} = 150\,\text{mA}$$

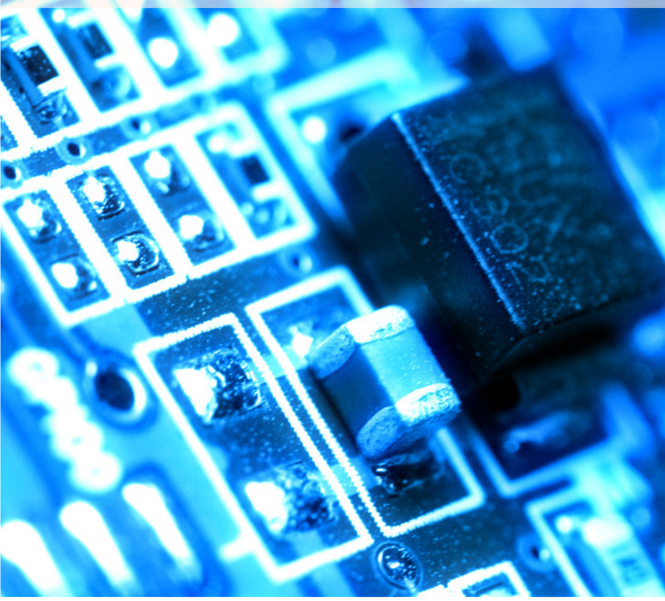
The equivalent resistance is:

$$R_{eq} = R_1 \parallel R_2 = 100\,\Omega \parallel 100\,\Omega = 50\,\Omega$$

Finally, the load current I_L can be computed by applying the *current divider rule*:

$$I_L = \frac{R_{eq}}{R_L + R_{eq}}\, I_s = \frac{50\Omega}{50\Omega + 50\Omega}\, 150\text{mA} = 75\text{mA}$$

A. Fundamentals of Mathematics



A.1	Trigonometry	32
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A.5	Complex numbers	40
A.6	Complex numbers algebra	41

This annex covers much of the basic mathematics useful for analysing electronic circuits. Nowadays it is not strictly necessary to have an in-depth mathematical background for circuit analysis, since electronic designers may rely on mathematical packages and electronic simulation software. However, a good knowledge of mathematical techniques may help to gain a better understanding of circuits working principles, that are somehow abstracted by the simulation software. Mathematical circuit analysis is challenging and it is very easy making mistakes. Thus, great care must be put in writing the equations and checking the coherence and the consistency of the units. Moreover, in some cases, approximations must be made to deal with complexity.

A.1 Trigonometry

Trigonometry knowledge is essential in circuit analysis, since in many cases we assume sinusoidal inputs. Sinusoidal functions can be defined in term of the sides of the right-angled triangle depicted in Figure A.1, where a is the opposite side, b is the adjacent side and c the hypotenuse:

$$\begin{aligned} \sin\theta &= \frac{a}{c} \\ \cos\theta &= \frac{b}{c} \\ \tan\theta &= \frac{a}{b} \end{aligned} \tag{A.1}$$

A common way to represent a sinusoidal signal is through a rotating unit vector OA around the origin O At a given rate ω (given in radians per second) and take the projection of OA on the y axis as a function of time as depicted in Figure A.2. The corresponding projection over the x axis will produce a cosine wave. This allows us to see graphically the values of the functions at particular angles, namely $\omega t = \frac{pi}{2}, \frac{2pi}{2}, 2\pi$, as well as the signs in the four quadrants Q. Table A.1 summarises values and signs of the trigonometric functions in the four quadrants.

Finally, reported in the sequel, are some useful trigonometrical relationships and expressions.

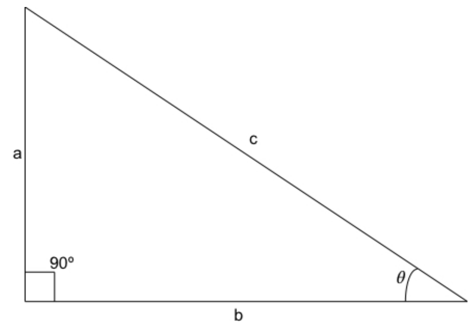


Figure A.1: Right-angled triangle

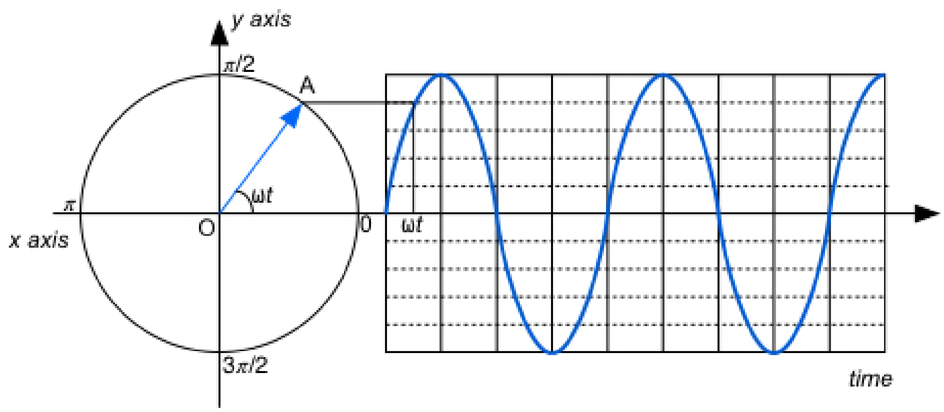


Figure A.2: Projection of the rotating vector for sin function.

Angle	0	$\pi/2$	π	$3\pi/2$	2π	1Q	2Q	3Q	4Q
sin	0	1	0	1	0	+	+	−	−
cos	1	0	1	0	1	+	−	−	+
tan	0	∞	0	∞	0	+	−	+	−

Table A.1: Values and signs of the trigonometric functions in the four quadrants.

$$
\begin{aligned}
&\text{(a)}\ \sin(-\theta) = -\sin\theta \\
&\text{(b)}\ \cos(-\theta) = \cos\theta \\
&\text{(c)}\ \tan(-\theta) = -\tan\theta \\
&\text{(d)}\ \cos(\theta+\phi) = \cos\theta\cdot\cos\phi - \sin\theta\cdot\sin\phi \\
&\text{(e)}\ \cos(\theta-\phi) = \cos\theta\cdot\cos\phi + \sin\theta\cdot\sin\phi \\
&\text{(f)}\ \sin(\theta+\phi) = \sin\theta\cdot\cos\phi + \cos\theta\cdot\sin\phi \\
&\text{(g)}\ \sin(\theta-\phi) = \sin\theta\cdot\cos\phi - \cos\theta\cdot\sin\phi \\
&\text{(h)}\ \sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\
&\text{(i)}\ \sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \\
&\text{(j)}\ \cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\
&\text{(k)}\ \cos\alpha - \cos\beta = 2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \\
&\text{(l)}\ \sin\alpha\cdot\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right] \\
&\text{(m)}\ \cos\alpha\cdot\sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right] \\
&\text{(n)}\ \cos\alpha\cdot\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right] \\
&\text{(o)}\ \sin\alpha\cdot\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right] \\
&\text{(p)}\ \sin^2\theta + \cos^2\theta = 1 \\
&\text{(q)}\ \cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta \\
&\text{(r)}\ 1 + \cos\theta = 2\cos^2\frac{\theta}{2} \\
&\text{(s)}\ 1 - \cos\theta = 2\sin^2\frac{\theta}{2} \\
&\text{(t)}\ \sin 2\theta = 2\sin\theta\cos\theta \\
&\text{(u)}\ \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = \lim_{\theta\to 0}\frac{\tan\theta}{\theta} = 1
\end{aligned}
\tag{A.2}
$$

A.2 Series expansion

In circuit theory and electronics, is sometimes more convenient to perform a linear approximation of a nonlinear response. In some cases it also allows us to obtain a relationship between apparently inconnected parameters.

Linear approximation can be performed by truncated to the linear terms a series expansion.

The more useful expansions are:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$
$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \tag{A.3}$$

Where $n! = n(n-1)(n-2)\cdots 1$ is known as the *factorial* of n, and θ is in radians. Two other very common series in circuit theory and electronics are the *binomial series* and the *geometric series*.

The binomial series is given by:

$$(1+x)^2 = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \ldots \tag{A.4}$$

valid for both positive and negative n and for $|x| < 1$. This series is most frequently used for $x \ll 1$ so that:

$$(1+x)^n \cong 1 + nx \tag{A.5}$$

The geometric series in x has a sum S_n of the first n terms equal to:

$$S_n a + ax^2 + ax^3 + \cdots + ax^n = \frac{a(1-x^n)}{1-x} \tag{A.6}$$

With a being a constant. If $|x| < 1$ the sum S for an infinite number of terms is:

$$S = \frac{a}{1-x} \tag{A.7}$$

In some cases, when we need to find the value of some function when the variable goes to a limit (e.g., zero or infinity), we might obtain an indeterminate value (e.g., $0/0$, ∞/∞, or $0 \times \infty$). In such cases, if the limit exists, we may obtaining it by observing how the function approaches to the limit rather than substituting the limiting value intthe variable.

Consider, for example:

$$\lim_{x\to 0} \frac{1-e^x}{x} \tag{A.8}$$

Limit A.8 gives $0/0$ for $x = 0$. However, if we expand the exponential and remember that x must be very small. The previous limit becomes:

$$\lim_{x\to 0} \frac{1-e^x}{x} = \lim_{x\to 0} \frac{1 - \left(1 + x + \frac{x^2}{2!} + \cdots\right)}{x} = \lim_{x\to 0}\left(-1 - \frac{x}{2!}\right) = -1 \tag{A.9}$$

Expansion in terms of a series is generally very useful when we want to study the behaviour of a respons when $x \to 0$, since in the limit the series is reduced to a constant term. However, there is another approach, known as *de l'Hôpital's rule*, which leverages of Taylor series expansion in terms of derivatives. Namely:

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} \tag{A.10}$$

The Taylor's series allows expanding a function $f(x)$ around a point $x = x_0$:

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \cdots + \frac{(x-x_0)^n}{n!} f^n(x_0) \tag{A.11}$$

If the expansion is around the origin $x = 0$, the series is referred as *Maclaurin's series*.

A.3 Logarithms

Logarithms are widely used in electronics to cope with numbers spanning over very wide intervals. In a Bode plot of gain, for example, a linear scale will show only a small part of the entire range with an accurate resolution. Thus, log-log scales and decibel representation are used to improve the visualisation of circuit frequency response.

The logarithm of number y in a base b is defined as:

$$y = b^x \longrightarrow \log_b y = x \tag{A.12}$$

Logarithm has a number of interesting properties that are valid fir any base b:

$$\begin{aligned} \log_b(\alpha\beta) &= \log_b \alpha + \log_b \beta \\ \log_b\left(\frac{\alpha}{\beta}\right) &= \log_b \alpha - \log_b \beta \end{aligned} \tag{A.13}$$

Thus the logarithm function has the property to transform a multiplication into a sum and a division into a subtraction of logarithms. Moreover:

$$\begin{aligned} \log_b(\alpha^n) &= n \times \log_b \alpha \\ \log_b\left(\alpha^{\frac{1}{n}}\right) &= \frac{1}{n}\log_b \alpha \end{aligned} \tag{A.14}$$

To convert a logarithm between bases a and b:

$$\log_b a = \frac{\log_a a}{\log_a b} \tag{A.15}$$

Finally:

$$\begin{aligned} \log_b 1 &= 0 \\ \log_b 0 &= -\infty \end{aligned} \tag{A.16}$$

The two common bases are $b = 10$ (decimal logarithm is often denoted simply as log) and $b = e$, being e the *Euler's number*. Natural or neperian logarithms are denote with ln.

There are some additional properties for natural logarithms:

$$\begin{aligned} \ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \\ \ln(n+1) &= \ln\, n + 2\left[\frac{1}{(2n+1)} + \frac{1}{3\frac{1}{(2n+1)^3}} + \frac{1}{5}\frac{1}{(2n+1)^5} + \dots\right] \text{ for} n > 0 \end{aligned} \tag{A.17}$$

Finally:

$$\begin{aligned} \frac{d}{dx}(\ln x) &= \frac{1}{x} \\ \int \frac{dx}{x} &= \ln\, x \end{aligned} \tag{A.18}$$

The very wide range of many quantities met in electronics together with the convenience of transforming gain multiplication in cascaded amplifiers into simple additions when expressed in a logarithmic scale has fostered the use of logarithms measures.

In many technical fields, the ratio of two homogeneous values x_1 and x_2 (e.g., the radiation or signal power) is expressed in **bels** (B) in honour of Alexander Graham Bell. The bel is defined as:

$$L_{bel} = \log_{10} \frac{x_1}{x_2} \tag{A.19}$$

However with the passing of time, instead of bel, its submultiple the decibel (dB) has gained wide acceptance and is commonly used.

$$1\,\text{dB} = 0.1\,\text{B}$$

The ratio of two physical magnitudes can be also expressed in ***Nepers*** *(Np), defined as:*

$$L_{Np} = \frac{1}{2} \ln x_1 x_2$$

In electronics, the decibel is defined as a power ratio or gain G_p between input and output powers of a circuit, P_{in} and P_{out} respectively:

$$G_p = 10 \log_{10} \left(\frac{P_{out}}{P_{in}} \right) \text{dB} \tag{A.20}$$

If $P_{out} > P_{in}$, G_p is expressed in positive dBs; conversely, if $P_{in} > P_{out}$ G_p is expressed in negative dBs.

Recalling that $P = \frac{V^2}{R}$, G_p can be also written as:

$$G_p = 10 \log_{10} \frac{V_{out}^2}{V_{in}^2} = 20 \frac{V_{out}}{V_{in}} \text{dB} \tag{A.21}$$

Observe that the derivation of Equation A.21 assumes equal input and output impedance levels (i.e., $R_{in} = R_{out}$); however, in the practice, when using Equation A.21 the usual difference between input and output impedance levels is ignored.

Many derivative "units" of dB has been defined such as the dBm which refers to a power gain where the reference level is 1 mW instead of the input power P_{in}, so that 0 dBm = 1 mW.

A.4 Exponentials

The exponential function is frequently found in circuit theory and electronics. It represents phenomena where the rate of change of a variable x is proportional to the value of that variable. Namely, these phenomena are described by a differenctial equation like Equation A.22.

$$\frac{dx}{dt} = kx(t) \tag{A.22}$$

Where k Is a constant. An example could be the charging of a capacitor. Consider the simple RC circuit depicted in Figure **??**. Although this is not essential, for the sake of simplicity we assume that the capacitor is initially discharged and that at time $t = 0$ the switch is closed. At this instant, the voltage drop $v_C(t)$ across the capacitor is 0 V and the voltage drop across the resistor R is hence V_{in}. ence, the current i through the mesh is $i(t) = i_0 = \frac{V_{in}}{R}$. At any

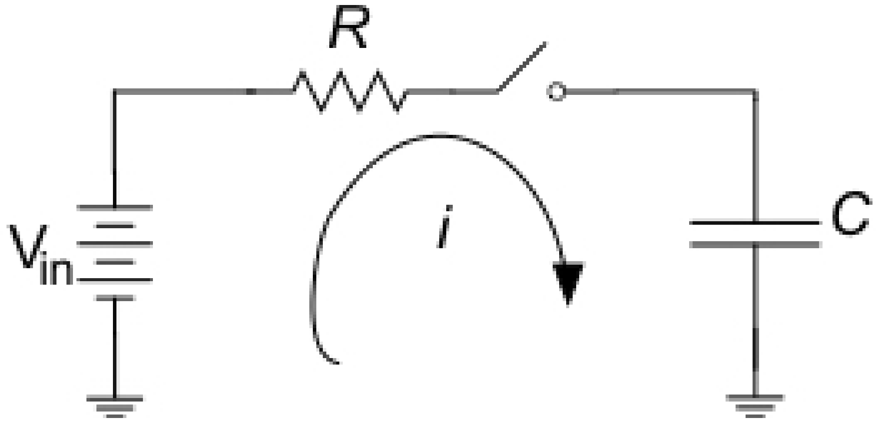


Figure A.3: Current flow in a first-order RC circuit.

time, when the current is $i(t)$ and the charge stored in the capacitor is $Q(t)$, $v_C(t)$ will be given by:

$$v_C(t) = \frac{Q}{C} \longrightarrow dV_c(t) = \frac{dQ(t)}{C} = \frac{i(t)dt}{C}$$

Where:

$$i(t) = \frac{V_{in} - v_C(t)}{R}$$

Thus

$$\frac{dv_C(t)}{dt} = \frac{i(t)}{C} = \frac{V_{in} - v_C(t)}{RC} \tag{A.23}$$

Equation A.23 matches the assuntion above about the relation on the rate of change except that in this case $\frac{dv_C(t)}{dt}$ is proportional to $V_{in} - v_C(t)$. Thus, when $v_C(t)$ increases towards its final values V_{in}, the rate of change of $v_C(t)$ will decrease.

There is a formal method to solve Equation A.23 for $v_C(t)$; however, in this case we will proceed by guessing the solution and showing that it agrees with Equation A.23. Let us check that:

$$v_C(t) = V_{in}\left(1 - e^{-\frac{t}{RC}}\right) \tag{A.24}$$

is a solution of Equation A.23. To prove that, let us differentiate Equation A.24. This yelds:

$$\begin{aligned} \frac{dv_C(t)}{dt} &= 0 + \frac{V_{in}}{RC}e^{-\frac{t}{RC}} \\ &= \frac{V_{in}}{RC}\left(\frac{V_{in} - v_C(t)}{V_{in}}\right) = \frac{V_{in} - v_C(t)}{RC} \end{aligned} \tag{A.25}$$

Where we have substituted $e^{-\frac{t}{RC}}$ from Equation A.24. For the circuit of Figure A.3 we also have that:

$$\frac{V_{in} - v_C(t)}{R} = \frac{V_{in}}{R}e^{-\frac{t}{RC}} = i_0 e^{-\frac{t}{RC}} \tag{A.26}$$

Where $i_0 = \frac{V_{in}}{R}$.

The initial slope of $v_C(t)$ is given by the value of $\frac{dv_C(t)}{dt}$ computed for $t = 0$. From Equation A.25 we have:

$$\left(\frac{dv_C(t)}{dt}\right)_{t=0} = \frac{V_{in}}{RC} \tag{A.27}$$

Time τ	0	1	2	3	4	5	6	7
$v_C(t)/V_{in}$	0	0.632	0.865	0.950	0.982	0.993	0.998	0.999
$v_R(t)/V_{in}$	1	0.368	0.135	0.050	0.018	0.007	0.002	0.001

Table A.2: Convergence of the RC first order circuit response to the final value.

The quantity $\tau = RC$ is called the *time constant*. It easy to check that RC is dimensionally a time, since $C = \frac{Q}{V} = \frac{A \times s}{V}$ and $R = \frac{V}{A}$, hence:

$$RC = \frac{A \times s}{V} \times \frac{V}{A} = s$$

The initial slope tangent will reach V_{in} at time τ. The voltage V_R across resistor R Is the voltage difference between V_{in} and $v_C(t)$; thus:

$$V_R = V_{in} - v_C(t) = V_{in} e^{-\frac{t}{RC}} \tag{A.28}$$

Responses $\frac{v_C(t)}{V_{in}}$ and $\frac{v_R(t)}{V_{in}}$ of the first-order RC circuit are depicted in Figure A.4 for $\tau = 1$. The voltage drop on the capacitor $v_C(t)$ (i.e., the circuit response), asyntothically converges to the final value V_{in} but theroretically never reaches it. The time to reach a given percentage p of V_{in} can be computed as:

$$t = -\tau \ln p \tag{A.29}$$

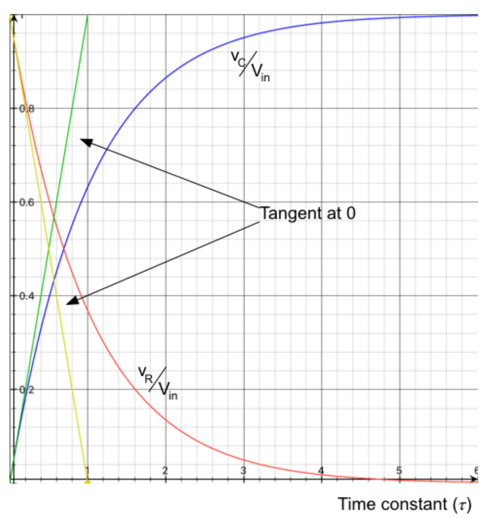


Figure A.4: Exponential responses of the first-order RC circuit with $\tau = RC = 1$.

Table A.2 depicts how $v_C(t)$ and $v_R(t)$ converge to the final value as a function of the time constant τ. For example, after $t = \tau$, the level of the response is $\frac{v_C(t)}{V_{in}} = e^{-1} = 0.632$. Namely the voltage drop on the capacitor $v_C(t)$ id approximately the 63% of the final value V_{in}.

Similarly, if we consider a first-order RL circuit obtained by replacing the capacitor with an inductor in Figure A.3, a similar analysis leads to:

$$i_L(t) = i_0 \left(1 - e^{-t\frac{R}{L}}\right) \tag{A.30}$$

With $i_0 = \frac{V_{in}}{R}$, and in this case $\tau = \frac{L}{R}$.

A.5 Complex numbers

The $\sqrt{-1}$ is one of the most important numbers in mathematics,and like other important numbers such as e or π, it has its own symbol; namely, i. In electronics this notation coincides with the current symbol i, for this reason it has been replaced with j.

Geometrically, a complex number represents a rotation on the *Argand plane*. The Argand plane (or complex plane) is the plane formed by complex numbers with a cartesian coordinate system such that the x-axis is the locus of the real numbers, whereas the y-axis is the locus of immaginary numbers.

Let us consider a number $n \in \mathbb{R}$. Applying twice the operator j to n we obtain:

$$j \times j \times n = \sqrt{-1} \times \sqrt{-1} \times n = -n$$

Which is equivalent to a rotation of 180° on the complex plane. This means that the immaginary number jn is equivalent to 90° rotation.

A plot of a complex number z in the Argand plane is known as *Argand diagram* (see Figure **??**). The number z can be expressed either in cartesian or in polar form. In cartesian

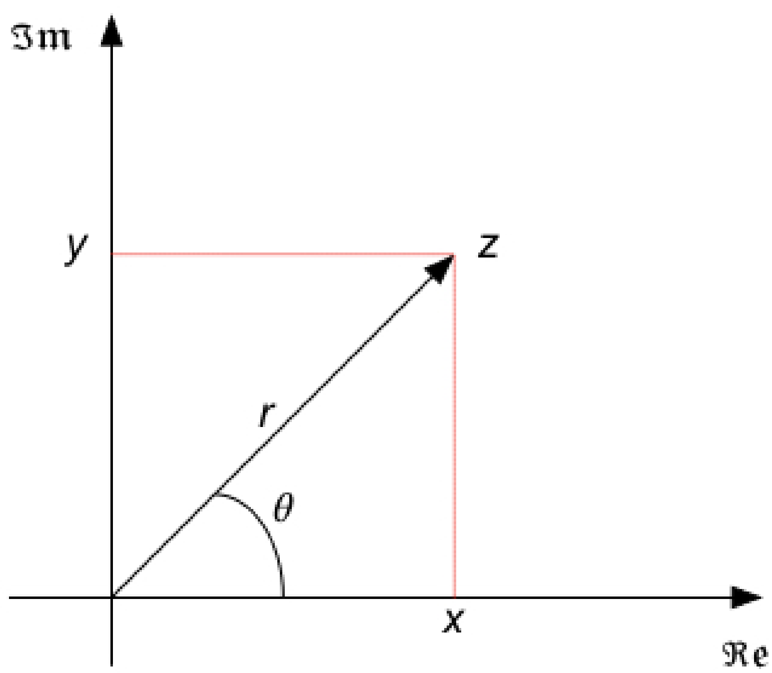


Figure A.5: Argand diagram of a complex number in cartesian and polar form.

form , z can be expressed as:

$$z = x + jy$$

Where $\Re(z) = x = r\cos\theta$ and $\Im(z) = y = r\sin\theta$. Thus, the complex number z can be also expressed in polar form as:

$$z = r\ (\cos\theta + j\sin\theta) = r\, e^{j\theta}$$

Where $r = |z| = \sqrt{x^2 + y^2}$ is called the *modulus* or *absolute value* of z, and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ is called the *argument* or *phase* of z.

Also the trigonometric funtions can be expanded in series. Namely:

$$\begin{aligned}\cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots \\ \sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots\end{aligned} \tag{A.31}$$

Thus, the complex exponential can be expressed as:

$$\begin{aligned}e^{j\theta} &= \cos\theta - j\sin\theta \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots\right) + j\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots\right) \\ &= 1 - j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \dots\end{aligned} \tag{A.32}$$

Recall that Equation A.32 is known as *Euler's formula* and it is a very handy way to represent complex numbers in circuit analysis. Conversely, it is also possible to express trigonometric functions in terms of complex exponentials as follows:

$$\begin{aligned}\cos\theta &= \frac{e^{j\theta} + e^{-j\theta}}{2} \\ \sin\theta &= \frac{e^{j\theta} - e^{-j\theta}}{2j}\end{aligned} \tag{A.33}$$

A.6 Complex numbers algebra

Let us consider now algebraic operations on complex numbers; namely, how can we add, subtract, multiply and divide them.

Sum operation is straightforward and is the same as adding two vectors. Let us consider two complex numbers in cartesian form, $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$, then:

$$z = z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

Subtraction works in a similar way and can be expressed as $z = z_1 + (-z_2)$. Hence, as depicted in Figure A.6, the first step consists in finding $-z_2$ (this is equivalent to rotating z_2 by 180°) and then add it to z_1:

$$z = z_1 + (-z_2) = (x_1 - x_2) + j(y_1 - y_2)$$

Some algebraic operations are much easier to handle when complex numbers are represented in exponential form rather than in cartesian or trigonometric form. Namely, using Equation A.32:

$$z = r\,(\cos\theta + j\sin\theta) = re^{j\theta} \tag{A.34}$$

Which allows us to write the nth power of a complex number as:

$$\begin{aligned}z^n &= \left(re^{j\theta}\right)^n \\ &= r^n\,(\cos\theta + j\sin\theta)^n \\ &= r^n\,(\cos n\theta + j\sin n\theta)\end{aligned} \tag{A.35}$$

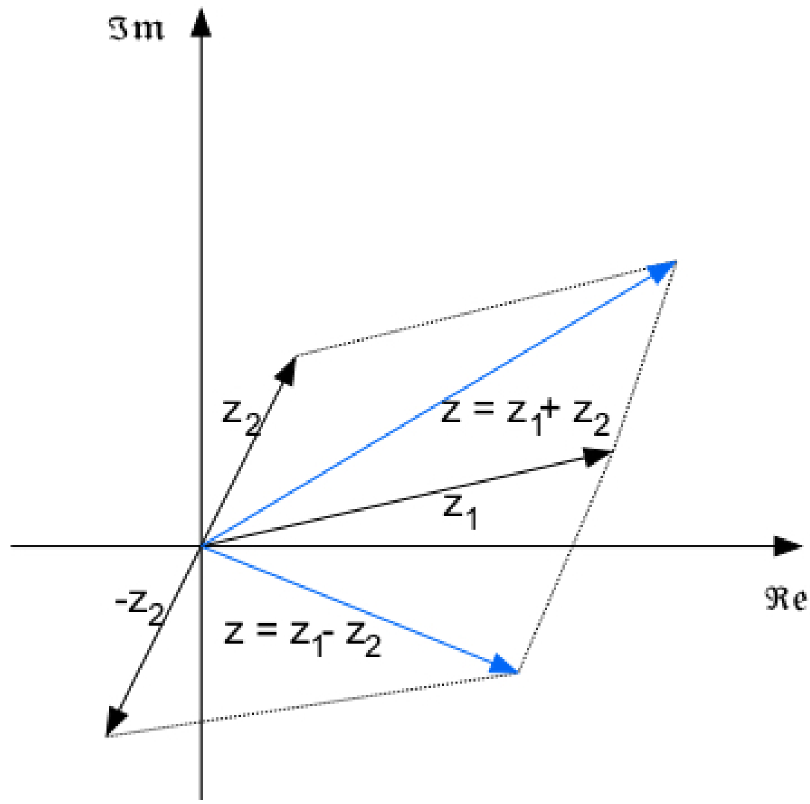


Figure A.6: Addition and subtraction of complex numbers.

Which is known as *De Moivre's theorem*. Observe that n can be a fraction. The exponential form allows us to perform multiplications and divisions much more easily than in cartesian form. Namely, taking $z_1 = x_1 + jy_1 = r_1 e^{j\theta_1}$ and $z_2 = x_2 + jy_2 = r_2 e^{j\theta_2}$, yelds:

$$z = z_1 \times z_2 = (x_1 + jy_1)(x_2 + jy_2) = r_1 r_2 e^{j(\theta_1 + \theta_2)}$$

Thus the modulus of the product z is equal to the product of the moduli of z_1 and $z2$, whereas the argument is the sum of the arguments of z_1 and $z2$. Namely:

$$\begin{aligned} |z| &= |z_1 \times z_2| = |z_1| \times |z_2| \\ \angle z &= \angle(z_1 \times z_2) = \angle z_1 + \angle z_2 \end{aligned} \tag{A.36}$$

Similarly, for division we have:

$$z = \frac{z_1}{z_2} = \frac{x_1 + jy_1}{x_2 + jy_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}$$

Thus the modulus of the quotient z is equal to the quotient of the moduli of z_1 and $z2$, whereas the argument is the difference of the arguments of z_1 and $z2$. Namely:

$$\begin{aligned} |z| &= \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \\ \angle z &= \angle\left(\frac{z_1}{z_2}\right) = \angle z_1 - \angle z_2 \end{aligned} \tag{A.37}$$

Finally the *complex conjugate* z^* of the complex number z can be formed by taking the negative of the immaginary part. Thus:

$$z = x + jy = r(\cos\theta + j\sin\theta) = re^{j\theta} \longrightarrow z^* = x - jy = r(\cos\theta - j\sin\theta) = re^{-j\theta}$$

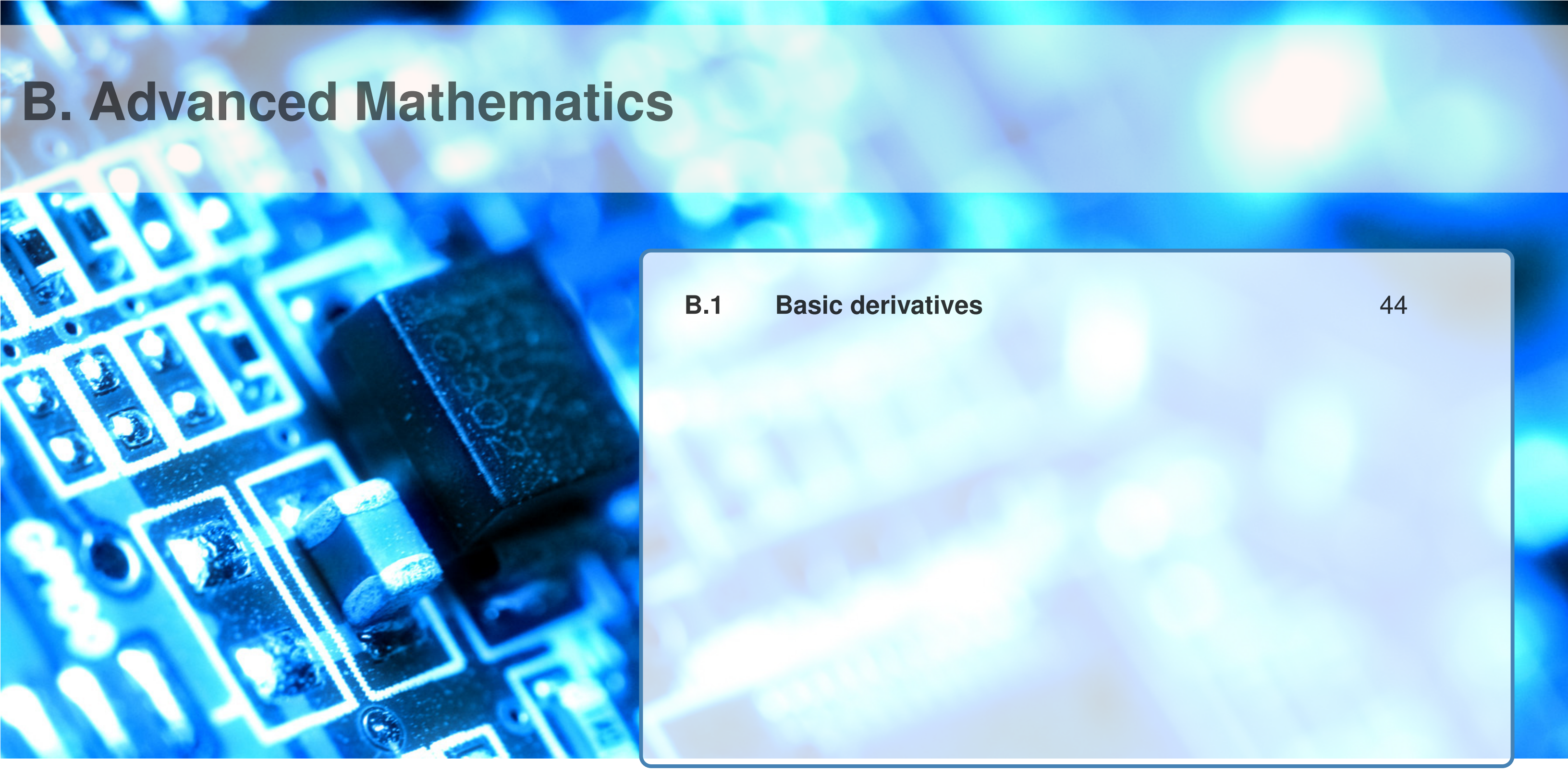
The complex conjugate provide a straightforward way to find the modulus of a complex number as:

$$|z| = r = \sqrt{z \times z^*}$$

The complex conjugate also provides a convenient way of rationalizing a complex quotient as follows:

$$\begin{aligned}
z &= \frac{z_1}{z_2} \\
&= \frac{z_1 z_2^*}{z_2 z_2^*} \\
&= \frac{(x_1 + jy_1)(x_2 - jy_2)}{x_2^2 + y_2^2} \\
&= \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}\right) + j\left(\frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2}\right)
\end{aligned} \tag{A.38}$$

Complex numbers are widely used in AC circuit analysis as well as for describing the behaviour of dielectrics and magnetic materials.



B. Advanced Mathematics

B.1 Basic derivatives 44

This annex covers more advanced mathematics used in electronics and circuit theory. More specifically, we will deal with differentiation, integration and transformation. In electronics, differentiation is a mathematical process that gives information about the rate of change of physical magnitude (voltage, current, charge, etc.).

B.1 Basic derivatives

In the sequel, some of the differentials commonly encountered in electronics are listed.

$$
\begin{aligned}
&f(x) = kx^n \quad \frac{df(x)}{dx} = knx^{n-1}, \text{for } k \text{ and } n \text{ constants} \\
&f(x) = ke^{nx} \quad \frac{df(x)}{dx} = kne^{nx} \\
&f(x) = ke^{g(x)} \quad \frac{df(x)}{dx} = ke^{g(x)}\frac{dg(x)}{dx} \\
&f(x) = \ln(x) \quad \frac{df(x)}{dx} = \frac{1}{x} \\
&f(x) = \sin(x) \quad \frac{df(x)}{dx} = \cos(x) \\
&f(x) = \cos(x) \quad \frac{df(x)}{dx} = -\sin(x) \\
&f(x) = a^x \quad \frac{df(x)}{dx} = a^x \ln(a), \text{with } a \neq 1 \\
&f(x) = \tan^{-1}(x) \quad \frac{df(x)}{dx} = \frac{1}{1+x^2}
\end{aligned}
\tag{B.1}
$$

The derivative is a linear operator, thus the following properties apply:

$$\frac{d(k\,f(x))}{dx} = k\,\frac{df(x)}{dx}, \text{with } k \text{ constant} \tag{B.2}$$

and

$$\frac{d(f(x)+g(x))}{dx} = \frac{df(x)}{dx} + \frac{dg(x)}{dx} \tag{B.3}$$

In addition, the differentiation of the product of two functions $f(x)$ and $g(x)$ is:

$$\frac{d(f(x) \times g(x))}{dx} = f(x)\frac{dg(x)}{dx} + g(x)\frac{d(f(x))}{dx} \tag{B.4}$$

The differentiation of the quotient of two functions is:

$$\frac{d(f(x)/g(x))}{dx} = \frac{g(x)\frac{df(x)}{dx} - f(x)\frac{dg(x)}{dx}}{g^2(x)} \tag{B.5}$$

Finally, the differentiation of a function of a function follows the rule in the sequel:

$$\frac{df(g(x))}{dx} = \frac{df(g)}{dg}\frac{g(x)}{dx} \tag{B.6}$$

If n successive derivatives must be performed, then the differential is written as:

$$\frac{df^n(x)}{dx^n} \equiv \frac{d^{n-1}}{x^{n-1}}\left(\frac{df(x)}{dx}\right) \tag{B.7}$$

Geometrically, the differential represents the slope of a funtion in a given point. This, in turn, allows us to find the turning points of a function, namely its maximum and minimum, since at those points the slope will be zero. A function $f(x)$ is increasing (namely, $f(x)$ increases as x increases) if the slope of the tangent in x is positive. Conversely, a function $f(x)$ is decreasing (namely, $f(x)$ decreases as x increases) if the slope of the tangent in x is negative.

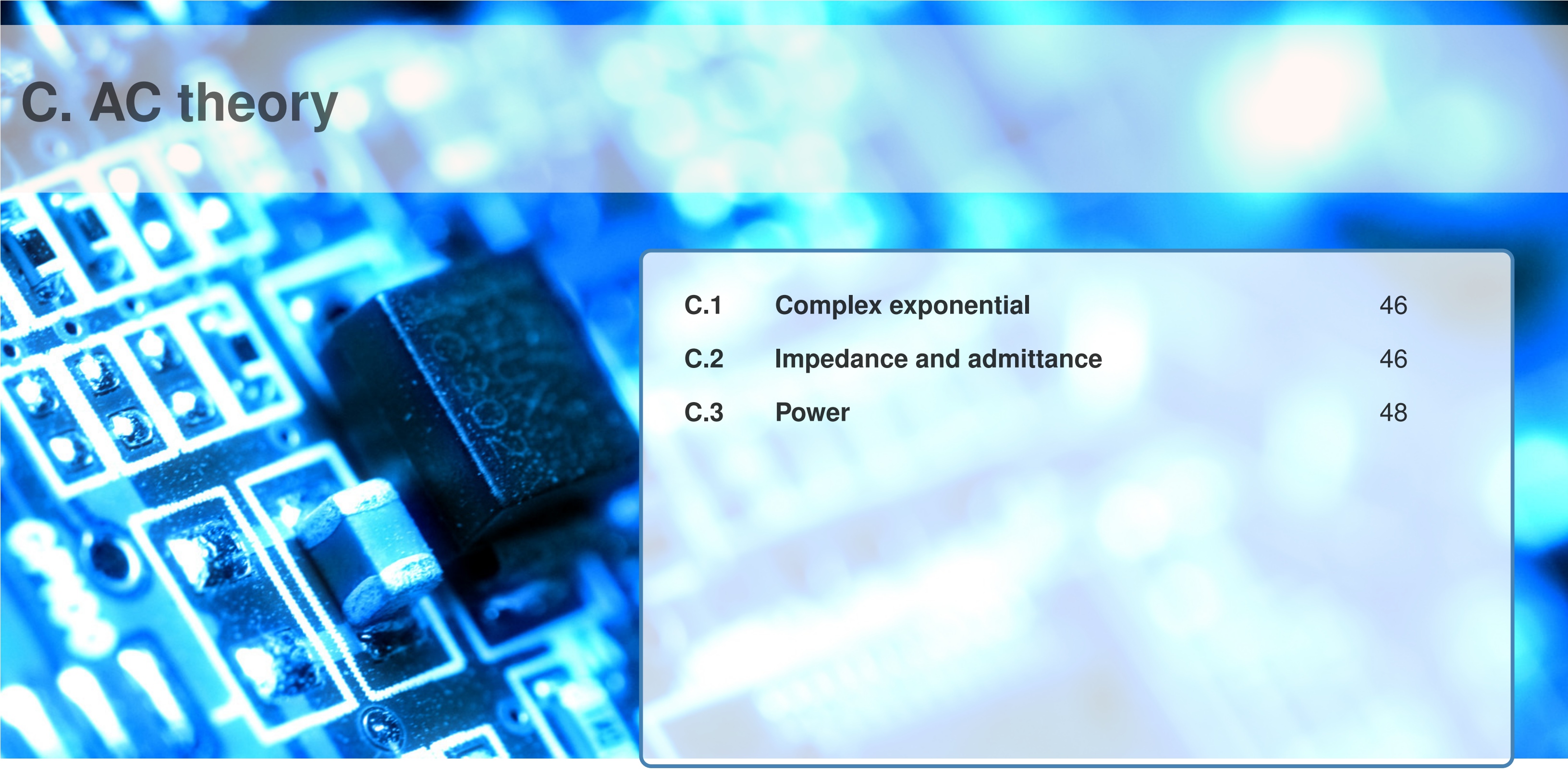
A shortand for derivative is the following:

$$\frac{df(x)}{dx} \equiv f'(x), \frac{df^2(x)}{dx^2} \equiv f''(x), \ldots$$

When $x(t)$ is a funtion of time, dot notation is used:

$$\frac{dx(t)}{dt} \equiv \dot{x}, \frac{dx^2(t)}{dt^2} \equiv \ddot{x}, \ldots$$

In the case of multivariate function, when we are interested in the variation with respect just one variable, we use partial differentials to indicate this. Namely, for $f(x,t)$, the differential is either $\frac{\partial f}{\partial x}$ keeping t fixed, or $\frac{\partial f}{\partial t}$ keeping x fixed.



C. AC theory

C.1 Complex exponential 46

C.2 Impedance and admittance 46

C.3 Power 48

The analysis od AC circuits is fundamental in circuit theory. When a sinusoidal input is applied to a linear circuit, the shape of the signal is preserved at the circuit output. Namely, adding, subtracting, multiplying, dividing, differentiating and integrating will still produce a sinusoidal signal.

More complex non-sinusoidal signals can be represented as Fourier series so that this important property is still preserved at the cost of a considerable mathematical effort. Trigonometrical functions are not easy to use; however, working with trigonometrical functions can become more handy by expressing them in term of complex quantities and exponential notation.

C.1 Complex exponential

Sinusoidally varying waveforms can be expressed both in trigonometrical or in complex form. For example, voltage $v(t)$ can be represented as:

$$v(t) = V_0\,(\cos(\omega t) + j\sin(\omega t)) = V_0\,e^{j\omega t} \tag{C.1}$$

Where V_0 is the peak value of the voltage waveform.

The corresponding current $i(t)$ is, generally, not in phase with the voltage. If ϕ is the phase difference among the voltage and current waveforms, then current $i(t)$ may be written as:

$$i(t) = I_0\,(\cos(\omega t - \phi) + j\sin(\omega t - \phi)) = I_0\,e^{(j\omega t - \phi)} \tag{C.2}$$

Where I_0 is the peak value of the current waveform.

C.2 Impedance and admittance

The complex notation of the current and voltage waveforms has allowed the introduction of the concept of *impedance* Z. The impedance is a complex quantity that can be considered as a generalization of the concept of resistance R for AC circuits. The impedance is defined as:

$$Z = \frac{v(t)}{i(t)} = \frac{V_0\,e^{j\omega t}}{I_0\,e^{(j\omega t - \phi)}} = \frac{V_0}{I_0}e^{j\phi} = \frac{V_0}{I_0}\,(\cos\phi + j\sin\phi) \tag{C.3}$$

Like every complex number, the impedance Z can be also expressed in terms of **modulus** $|Z|$ and its **phase** (or *argument*) $\angle Z$. Namely:

$$|Z| = \sqrt{\frac{V_0^2}{I_0^2}\left(\cos^2\phi + \sin^2\phi\right)} = \frac{V_0}{I_0} \tag{C.4}$$

And

$$\angle Z = \tan^{-1}\left(\frac{\sin\phi}{\cos\phi}\right) = \tan^{-1}(\tan\phi) = \phi \tag{C.5}$$

The impedance Z can be also expressed in cartesian form as:

$$Z = R + jX \tag{C.6}$$

Where the real part $R = \Re Z$ is called the *resistance*, and the immaginary part $X = \Im Z$ is called the *reactance*. More precisely:

$$R = \frac{V_0}{I_0}\cos\phi$$
$$X = \frac{V_0}{I_0}\sin\phi$$

The magnitude of impedance Z is:

$$|Z| = \sqrt{(R + jX)(R - jX)} = \sqrt{(R^2 + X^2)} \tag{C.7}$$

The phase difference among the voltage and the current waveforms is given by:

$$\angle Z = \tan^{-1}\left(\frac{X}{R}\right) \tag{C.8}$$

In some cases, for example when computing a parallel impedance, it is more convenient to work with the reciprocals of the impedances since this reduces the parallel to a simple sum. The reciprocal Y of the impedance Z is called the *admittance*; namely:

$$Y = \frac{1}{Z} = G + jB \tag{C.9}$$

Where G is called the *conductance* and B the *susceptance*, measured in siemens S. Some old text refer to the unit of admittance as mho (℧), to pinpoint that the unit of admittance is the reciprocal of the ohm.

From Equations C.6 and C.9 it results:

$$G + jB = \frac{1}{R + jX} = \frac{R - jX}{(R + jX)(R - jX)} = \frac{R - jX}{R^2 + X^2} \tag{C.10}$$

Thus:

$$G = \frac{R}{R^2 + X^2}$$
$$B = \frac{-X}{R^2 + X^2}$$

Similarly:

$$R = \frac{G}{G^2 + B^2}$$
$$X = \frac{-B}{G^2 + B^2}$$

Measurement in AC circuits are given in terms of *root-mean-square* (r.m.s.) values. Root mean square is defined as the square root of the mean square (the arithmetic mean of the squares) of the measured values. For this reason, the r.m.s. is also known as the *quadratic mean*. In many cases averagong signals is more significant than considering their instantaneous values since in AC ratios and powers may vary over a wide range during a cycle. The root mean square of a periodic voltage signal $v(t)$ of period T is defined as:

$$\begin{aligned} V_{rms} &= \left\langle v^2(t) \right\rangle^{\frac{1}{2}} \\ &= \sqrt{\frac{1}{T}\int_0^T V_0^2 \cos^2(\omega t)\, dt} \\ &= \sqrt{\frac{V_0^2}{T}\int_0^T \frac{1}{2}\left[1 + \cos(2\omega t)\right]\, dt} \\ &= \sqrt{\frac{V_0^2}{2T}\left[t + \sin(2\omega t)\right]_0^T} \\ &= \sqrt{\frac{V_0^2}{2}} = \frac{V_0}{\sqrt{2}} = 0.707 V_0 \end{aligned}$$

Hence the root-mean-square value of the AC voltage $v(t)$ is equal to its peak value V_0 divided by $\sqrt{2}$. An identical realtion holds for the current.

It must be pinpointed that this result holds only for pure sinusoidal signals. For arbitrary signals or sinusoidal signals with significant higher-order harmonic content, the r.m.s. Value may change significantly. For example, the *crest factor*, namely the ratio between the peak and the r.m.s. value of a signal, is (as seen above) $\sqrt{2}$ for a sinusoidal signal, 1 for a symmetrical square wave and 1.73 For a triangle wave.

C.3 Power

The instantaneous power dissipated in AC by a circuit, varies during the period T, thus it is more significant considering average powers.

Assuming a phase difference ϕ between the voltage and the current waveforms, the instantaneous power is given by:

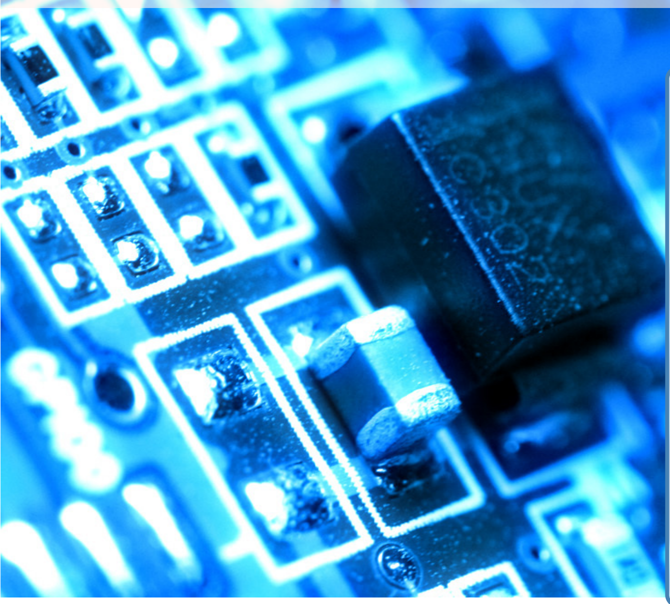
$$p(t) = V_0 \cos(\omega t) \times I_0 \sin(\omega t - \phi) \tag{C.11}$$
$$= \frac{1}{2} V_0 I_0 \left[\cos(2\omega t - \phi) + \cos\phi\right] \tag{C.12}$$

The first term is periodic and will average zero over the cycle T, hence the average power P is:

$$P = \frac{1}{T}\int_0^T p(t)\, dt = \frac{1}{2} V_0 I_0 \cos\phi = \frac{V_0}{\sqrt{2}}\frac{I_0}{\sqrt{2}}\cos\phi = V_{rms} I_{rms} \cos\phi \tag{C.13}$$

Where $\cos\phi$ is known as the *power factor*.

D. Phasors



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Phasors are a very useful mathematical tool for representing phase shifts by means of phasor diagrams. The diagram represents an instantaneous plot of the magnitudes and relative pahses of voltages and currents in a circuit. By default, increasing time is represented by counterclockwise rotation (i.e., increasing angle). Conversely, decreasing time is represented by a clockwise rotation (i.e., decreasing angle).

D.1 The basics

A phasor is a complex number that represents any sinusoidal function with amplitude A, angular frequency ω, and an initial phase ϕ. Though it may seem difficult at first, it makes the mathematics involved in tha analysis of systems with sinusoidal inputs much simpler. Let us consider a sinusoidal voltage $v(t)$ given by:

$$v(t) = A\cos(\omega t + \phi) \tag{D.1}$$

Voltage $v(t)$ can be represented in phasor form as:

$$\boldsymbol{V} = A(\cos\phi + j\sin\phi) = Ae^{j\phi} \tag{D.2}$$

Thus, a sinusoidal signal can be represented by a vector $\boldsymbol{V}$ in the complex plane. Note that angular frequency ω is not explicitly included in the phasor notation, since it is implicit in the concept of phasor. If we multiply the phasor by $e^{j\omega t}$, we are simply rotating the phasor by an angle ωt. Thus:

$$\boldsymbol{V}e^{j\omega t} = Ae^{j\phi}e^{j\omega t} = Ae^{j(\omega t+\phi)}$$

The time-domain function $v(t)$ can be obtained by taking the real part of this rotating vector; namely:

$$\begin{aligned} v(t) &= \Re\left(\boldsymbol{V}e^{j\omega t}\right) \\ &= \Re\left(Ae^{j\phi}e^{j\omega t}\right) \\ &= A\Re\left[\cos(\omega t+\phi) + j\sin(\omega t+\phi)\right] \\ &= A\cos(\omega t+\phi) \end{aligned} \tag{D.3}$$

One of the major advantages of using phasors is their capability to turn a differentiation or an integration problem into an algebraic problem. These properties are summarised in Table D.1.

Time domain	Phasor domain
$f(t)$	$\boldsymbol{F}$
$\frac{d}{dt} f(t)$	$j\omega \boldsymbol{F}$
$\int f(t)\, dt$	$\frac{1}{j\omega}\boldsymbol{F}$

Table D.1: Some phasors properties.

D.2 The phasor diagram

The difficulty behind phasor diagrams is choosing which phasor to start with, since a wrong choice may lead into difficulty. As an example, consider the simple RC low-pass filter of Figure D.1 (a). The current i is the quantity that is common to both resistor R and capacitor C, so it is a good choice to start with, since we know the relationships between i and the voltage drops v_R and v_C on the resistor and the capacitor respectively. Observe that $i =$

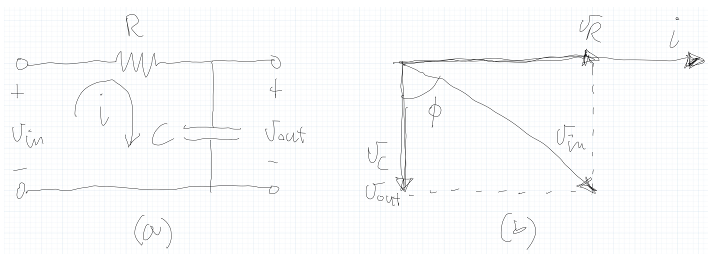

Figure D.1: (a) Simple RC circuit, and (b) Phasor diagram

$i_R = i_C = C\frac{dv_C}{dt} = j\omega C v_C$. Thus, in the phasor domain we obtain $\boldsymbol{V_R} = R \times \boldsymbol{I} = j\omega RC\boldsymbol{V_C}$. Similarly, $v_C = v_{out} = \frac{1}{C}\int i_C\, dt$; thus, in the phasor domain we obtain $\boldsymbol{V_C} = \boldsymbol{V_{out}} = \frac{1}{j\omega C}\boldsymbol{I}$. This means that the phasor $\boldsymbol{V_C}$ (and hence also $\boldsymbol{V_{out}}$) have a phase lag of $\frac{\pi}{2}$ radians with respect the phasors of current $\boldsymbol{I}$ and of voltage across R (namely, $\boldsymbol{V_R}$).

Finally from inspection of the circuit of Figure D.1 (a) it results that $\boldsymbol{V_{in}} = \boldsymbol{V_R} + \boldsymbol{V_C} = \boldsymbol{V_R} + \boldsymbol{V_{out}}$; hence, the phasor of the input voltage $v_{in}(t)$ can be determined by composing the two vectors $\boldsymbol{V_R}$ and $\boldsymbol{V_C}$ using the parallelogram rule as depicted in Figure D.1 (b).

From the discussion above, the relationship between the input and output voltage of hte simple RC filter of Figure D.1 (a) can be written as:

$$\begin{aligned} \boldsymbol{V_{in}} &= \boldsymbol{V_R} + \boldsymbol{V_{out}} \\ &= R\boldsymbol{I} + \boldsymbol{V_{out}} \\ &= R\left(j\omega C\boldsymbol{V_{out}}\right) + \boldsymbol{V_{out}} \\ &= \left(1 + j\omega RC\right)\boldsymbol{V_{out}} \end{aligned} \tag{D.4}$$

Equation D.4 yelds to the following expression for the gain G of the circuit of Figure D.1 (a):

$$G = \frac{\boldsymbol{V_{out}}}{\boldsymbol{V_{in}}} = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j\frac{\omega}{\omega_0}} \tag{D.5}$$

Where $\omega_0 = \frac{1}{RC}$.

The magnitude of the gain is:

$$|G| = \frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_0}\right)^2}} \tag{D.6}$$

and the phase shift is:

$$\angle G = \phi = -\tan^{-1}\left(\frac{\omega/\omega_0}{1}\right) = -\tan^{-1}\left(\frac{\omega}{\omega_0}\right) \tag{D.7}$$

At the corner frequency $\omega = \omega_0$, the gain modulus is $|G| = (1+1)^{-\frac{1}{2}} = \frac{1}{\sqrt{2}} = 0.707$ (or $|G| = 20\,\log(0.707) \approx -3\,\mathrm{dB}$), and the phase is $\phi = -\tan^{-1}(1) = 45°$. Considering the plot

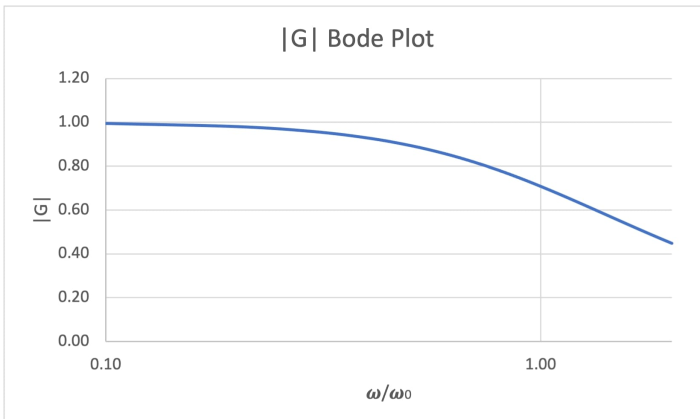

Figure D.2: Plot of the modulus of gain G of the RC low-pass filter

of $|G|$ depicted in Figure D.2 and taking two frecuencies a factor or 2 (i.e., an octave) or a factor of 10 (i.e., a decade) apart, it is possible to determine the slope of the plot:

$$\begin{aligned}|G_{2\omega}| - |G_\omega| &= -20\left[\log\left(\frac{2\omega}{\omega_0}\right) - \log\left(\frac{\omega}{\omega_0}\right)\right] \\ &= -20\log\left(\frac{2\omega}{\omega_0}\cdot\frac{\omega_0}{\omega}\right) \\ &\approx -6 \text{ dB/octave or } -20 \text{ dB/decade}\end{aligned} \tag{D.8}$$