A Robust Framework for Risk Parity Portfolios

Abstract

We propose a robust formulation of the traditional risk parity problem by introducing an uncertainty structure specifically tailored to capture the intricacies of risk parity. Typical minimum variance portfolios attempt to introduce robustness by computing the worst-case estimate of the risk measure, which is not intuitive for risk parity. Instead, our motivation is to shield the risk parity portfolio against noise in the estimated asset risk contributions. Thus, we present a novel robust risk parity model that introduces robustness around both the overall portfolio risk and the assets' marginal risk contributions. The proposed robust model is highly tractable and is able to retain the same level of complexity as the original problem. We provide a general procedure by which to create an uncertainty structure around the asset covariance matrix. We quantify this uncertainty as a perturbation on the nominal covariance estimate, which allows us to intuitively embed robustness during optimization. We then propose a specific procedure to construct a robust risk parity portfolio through a factor model of asset returns. Computational experiments show that the robust formulation yields a higher risk-adjusted rate of return than the nominal model while maintaining a sufficiently risk-diverse portfolio.

Keywords: Robust Optimization, Risk Parity, Asset Allocation, Uncertainty, Factor Model.

1 Introduction

Optimal portfolio construction has become an academic discipline since the introduction of Modern Portfolio Theory in Markowitz (1952). Mean-variance optimization (MVO) has the power to yield portfolios that minimize risk while maintaining a target expected return. However, in practice, these 'optimal' portfolios suffer from considerable drawbacks. First, optimal solutions often yield over-concentrated and counter-intuitive portfolios (Chopra and Ziemba, 1993). The second issue pertains to the sensitivity of the optimal solution to estimation errors arising from data uncertainty. Naive optimization methods assume that data, and any corresponding estimated parameters, are deterministic. However, it is well-known that data only provides partial evidence of some latent parameter in many practical applications. In the case of traditional portfolio optimization, the parameters of interest are the asset expected returns and covariance matrix, which quantify our financial reward and risk, respectively. A naive approach, where data uncertainty is not taken into account, may lead to unreliable estimates of the expected returns and covariance matrix. Moreover, naive optimization portfolio methods are highly sensitive to errors in their input parameters. Thus, using unreliable parameters may lead to what is sometimes referred to as 'error maximization' (Chopra and Ziemba, 1993; Michaud, 1989). The sensitivity to estimation errors in portfolio optimization has been widely explored in literature (Best and Grauer, 1991; Broadie, 1993; Merton, 1980). In particular, Chopra and Ziemba (1993) found that estimation errors in the expected returns can have an impact ten times larger than errors in the covariance matrix in MVO. Uncertainty arising from parameter estimation during portfolio construction is a well-known issue, and several strategies exist to mitigate or bypass it. For example, we can avoid parameter estimation altogether if we employ the classic '$1/n$' asset allocation strategy (DeMiguel et al., 2009), where wealth is distributed equally among all assets.

Another strategy used to address the issue of uncertainty is through the application of robust optimization methods. These methods work by creating a 'robust' counterpart to the naive nominal problem, where the parameter uncertainty is explicitly quantified and incorporated into the optimization model in a deterministic fashion. In the context of portfolio optimization, robustness is introduced by defining uncertainty sets around noisy input parameters, namely the expected returns and covariance matrix. These uncertainty sets bound the ‘true’ (but latent) parameters within a given distance from their estimates, either using linear (i.e., ‘box’) or ellipsoidal uncertainty sets (Ceria and Stubbs, 2006; Lobo and Boyd, 2000; Tütüncü and Koenig, 2004). Guastaroba et al. (2011) provide an extensive analysis that compares both linear and ellipsoidal robust formulations in the context of portfolio optimization. Alternatively, Goldfarb and Iyengar (2003) propose a robust formulation based on the estimation error arising from factor models, i.e., they quantify the error from regression coefficients, and use this information to define the uncertainty set around the expected returns and covariance matrix. A more general approach involves taking a distributionally robust approach, where the assumption is that the user is not even certain about the probability distribution that governs the asset returns (Delage and Ye, 2010). A pragmatic approach to robust MVO, which relies on a resampling method with repeated optimization, is presented in Michaud and Michaud (2007).

Unlike the previous robust models, a modern attempt to mitigate estimation error and create diversified portfolios is the introduction of risk parity portfolios, also known as equal risk contribution (ERC) portfolios. These portfolios do not set targets on the expected portfolio return or risk, but instead yield portfolios where resources are allocated based solely on the risk measure such that the risk contribution of each asset is the same (Maillard et al., 2010). Thus, by design, the construction of a risk parity portfolio does not require the estimation of expected returns, thereby reducing the most significant source of estimation noise. Moreover, since every asset must contribute the same level of risk, the resulting portfolios tend to be well diversified.

The only input parameter required for risk parity portfolio optimization is an estimate of the risk measure, such as the estimated covariance matrix. Thus, it follows that the risk measure is the sole source of uncertainty in this optimization model. One possible avenue to mitigate the impact of uncertainty and the model risk associated with parameter misspecification is to use different parameter estimation methods. For example, we could estimate the asset covariance matrix using a factor model, covariance shrinkage, or the dynamic conditional correlation proposed in Engle (2002).

The impact of using shrinkage as a covariance estimation method on risk-based portfolios is studied in Neffelli (2018), who finds that shrinkage can improve the estimation of the covariance. However, it also shows that risk parity portfolios are not very sensitive to covariance misspecification. This finding is echoed in Ardia et al. (2017), demonstrating that risk parity portfolios are relatively robust to both asset covariance and asset correlation misspecification under multiple parameter estimation methods when compared against other risk-based investment strategies. The robustness intrinsic to risk parity portfolios is also discussed in Nakagawa et al. (2018), where risk parity portfolios built using different methods to estimate the covariance matrix are shown to have similar performance. In other words, their findings reiterate that risk parity is quite robust to misspecification in the estimated covariance matrix. However, this is not meant to suggest that risk parity portfolios cannot benefit from more accurate estimates of the covariance matrix. For example, Costa and Kwon (2019) show that risk parity portfolios can have better risk-adjusted out-of-sample performance when market regime information is encoded into the estimated covariance matrix through a regime-switching factor model.

Regardless of the statistical procedure used to estimate the asset covariance matrix, any estimate derived from data will always have some degree of estimation error. It follows that the uncertainty surrounding our estimate can usually be statistically quantified, and this information can be exploited during the subsequent optimization step. Our proposed robust risk parity framework accepts the estimation error as a secondary input parameter. This robust framework is specifically tailored towards risk parity. From a risk parity perspective, our objective is to equalize the asset risk contributions. Thus, we are not only concerned with the overall portfolio risk, but we are also concerned with our measure of individual risk contributions. As such, our contribution is to design an optimization model that uses the estimation error of the risk measure to introduce robustness around the assets' marginal risk contributions.

This manuscript uses the portfolio variance as the risk measure, which is a typical risk measure used when discussing individual asset risk contributions. Alternative risk measures where risk contributions are concerned are discussed in Ji and Lejeune (2018) and Mausser and Romanko (2018).

We note that the robust risk parity model presented in this paper is able to accommodate any generic estimation procedure of the covariance matrix, provided this allows us to specify both the nominal estimate of the covariance and its corresponding uncertainty set. After we estimate the covariance matrix and quantify its estimation error, we then develop a robust optimization framework based on the methodology described in Bertsimas and Sim (2006). The nominal[1] risk parity model is formulated as a second-order cone program (SOCP). Thus, as outlined in Bertsimas and Sim (2006), we are able to formulate a robust optimization framework that retains the same level of complexity as the original model (i.e., the robust counterpart is also a SOCP). Computational results show this robust model is able to improve both portfolio returns and risk-adjusted returns relative to the nominal model. Hereafter, we provide a description of each section in this paper.

Section 2 briefly introduces the nominal risk parity non-linear program (NLP) from Maillard et al. (2010). We then propose a novel yet equivalent version of this NLP that reduces its complexity by reformulating it as a quadratically-constrained linear program (QCLP). We justify our preference for this QCLP by conducting a brief computational performance test.

In Section 3 we introduce a simple box uncertainty set on the covariance matrix. The simplicity of such a model is key because it allows us to quantify the estimation error as a perturbation of our nominal estimate, which aligns well with many statistical estimation procedures. Such an uncertainty structure can be derived from any estimate of the covariance matrix that has upper and lower bounds. An example of such an uncertainty set is derived from a factor model of asset returns. Proceeding as in Goldfarb and Iyengar (2003), we use the estimation error of the regression coefficients to formulate the upper and lower bounds of the asset covariance matrix.

Section 4 presents the robust optimization problem specifically designed to capture the intricacies of risk parity. Unlike traditional MVO portfolios, risk parity is not explicitly concerned with minimizing portfolio risk. Instead, risk parity aims to attain perfect risk diversification by equalizing the asset risk contributions. Thus, mitigating the effect of misspecification on the marginal

[1]The authors note the use of the word *nominal* throughout this paper in the context of the Operations Research discipline, where it serves to differentiate the basis model from its *robust* counterpart.

risk contributions is paramount when constructing risk parity portfolios. We begin by introducing a SOCP version of the nominal risk parity model as in Mausser and Romanko (2014). We proceed to introduce robustness by targeting the two constraints pertinent to the risk measure: the overall portfolio risk and the marginal risk contributions. The rationale for selecting a SOCP formulation is that it allows for a tractable introduction of robustness while still maintaining the same level of complexity as the original problem (i.e., the robust problem is also a SOCP).

The robust instance of most MVO models is attained when the worst-case estimate of the risk measure is assumed (Delage and Ye, 2010; Lobo and Boyd, 2000; Tütüncü and Koenig, 2004). In the case of portfolios where short positions are not allowed, a robust 'worst-case' estimate of the variance is attained simply by assuming the upper bound estimate of the covariance matrix. This shields a portfolio whose objective is to minimize risk from estimation error in the risk measure. This same approach can be directly applied to risk parity, as proposed in Kapsos et al. (2018), where a robust model is formulated by taking the worst-case estimate of the covariance matrix subject to a risk diversification constraint. In addition, Costa and Kwon (2019) propose a robust risk parity model that implements the uncertainty structure proposed in Goldfarb and Iyengar (2003). This method relies on deriving a worst-case estimate of the covariance matrix arising from the errors in the regression coefficients.

Introducing robustness by taking the 'worst-case' instance of the portfolio variance is an intuitive approach when we only seek to minimize our financial risk. However, this is not tailored towards an optimization model where the objective is to fully diversify our risk, regardless of the overall portfolio risk. Therefore, our proposed robust formulation deviates away from traditional robust methods based solely on worst-case estimates of the covariance matrix. The aforementioned techniques may be unable to fully capture the intricacies behind a risk parity portfolio, where overall portfolio risk is not a concern, but where we seek to correctly estimate the risk contribution per asset in order to allocate wealth accordingly. Thus, we propose a model that relaxes the 'worst-case' variance assumption by focusing not only on the overall risk of the portfolio, but also on the more relevant marginal risk contributions.

A set of computational experiments is shown in Section 5, which test the performance of a robust risk parity portfolio when compared against both the nominal and worst-case variance portfolios. Experimentally, the cost of robustness manifests through an increase in the period-over-period turnover rate and a slight deviation from perfect risk diversification (i.e., we must deviate away from the nominal portfolio in order to attain robustness). Nevertheless, the experimental results show that our robust SOCP is able to consistently outperform both the nominal and worst-case variance portfolios. This enhanced performance is particularly evident during periods of market distress, where the robust portfolio is able to attain a higher risk-adjusted rate of return.

The improved performance can be explained by the design of the robust problem. First, this formulation places a restriction on assets with a high level of noise in their marginal risk contributions, thereby reducing the exposure to noisier assets. Second, it implicitly reduces our exposure to assets with larger marginal risk contributions. These two characteristics of our robust risk parity problem ensures the resulting portfolios are shielded against errors in the allocation of risk among its constituents. In general, risk parity has become increasingly popular because it yields well-diversified portfolios; indeed, by definition, it yields the most risk-diverse portfolio. Thus, the proposed robust risk parity portfolio is able to reduce exposure to riskier assets while maintaining a sufficient degree of diversification from a risk contribution perspective.

2 Risk Parity Portfolios

The principle of risk parity portfolios is to determine resource allocation by distributing wealth in such a way that each asset has an equal contribution to the overall portfolio risk. Fundamentally, the concept mirrors the classic '$1/n$' portfolio, where wealth is distributed equally among all assets. For this reason, risk parity portfolios are sometimes referred to as ERC portfolios.

The distribution of wealth by equal risk contribution is entirely dependent on the measure of risk selected. The typical choice is to use portfolio variance, which aligns with the framework of MVO presented in Markowitz (1952). For a portfolio with n assets, the expected return and variance are given by

$$\mu_p = \boldsymbol{\mu}^\top \boldsymbol{x},$$

$$\sigma_p^2 = \boldsymbol{x}^\top \boldsymbol{\Sigma} \boldsymbol{x},$$

where μ_p is the portfolio expected return, $\boldsymbol{\mu} \in \mathbb{R}^n$ is the vector of asset expected returns, $\boldsymbol{x} \in \mathbb{R}^n$ is the vector of asset weights (i.e. the proportion of wealth invested in each asset), σ_p^2 is the portfolio variance, and $\boldsymbol{\Sigma} \in \mathbb{R}^{n \times n}$ is the asset covariance matrix. The individual risk contribution of each asset can be derived by an Euler decomposition of the portfolio standard deviation. As shown in Maillard et al. (2010), the risk contribution per asset is given by

$$\sigma_p = \sqrt{\boldsymbol{x}^\top \boldsymbol{\Sigma} \boldsymbol{x}} = \sum_{i=1}^{n} x_i \frac{\partial \sigma_p}{\partial x_i} = \sum_{i=1}^{n} x_i \frac{(\boldsymbol{\Sigma} \boldsymbol{x})_i}{\sqrt{\boldsymbol{x}^\top \boldsymbol{\Sigma} \boldsymbol{x}}}, \tag{1}$$

where $\partial \sigma_p / \partial x_i$ is the marginal contribution of asset i towards the portfolio standard deviation. The sum of the partitioned standard deviation in (1) has a common denominator that is consistent for all parts. Moreover, this denominator is equal to the portfolio standard deviation, allowing us to express the portfolio variance as follows

$$\sigma_p^2 = \boldsymbol{x}^T \boldsymbol{\Sigma} \boldsymbol{x} = \sum_{i=1}^{n} x_i (\boldsymbol{\Sigma} \boldsymbol{x})_i = \sum_{i=1}^{n} R_i, \tag{2}$$

where $R_i = x_i(\boldsymbol{\Sigma}\boldsymbol{x})_i$ is the individual risk contribution of asset i to the overall portfolio variance. The use of any alternative decomposable risk measure is also acceptable. For the remainder of this paper, the sole risk measure we consider is the portfolio variance. As such, we refer to R_i and $(\boldsymbol{\Sigma}\boldsymbol{x})_i$ as the risk contribution and marginal risk contribution of asset i, respectively.

It is common to impose a set of restrictions on the weight allocation variable, $\boldsymbol{x}$. These may include allocation limits for certain industries or assets, or cardinality constraints to limit the size of a portfolio. Thus, we consider a convex set $\mathcal{X}$ of acceptable weights such that $x \in \mathcal{X}$. As discussed in Maillard et al. (2010), the allowance of short positions in risk parity portfolios may impact the tractability of the model, where multiple optimal solutions may exist for the same basket of assets.

These multiple solutions are all considered global since they comply with the risk parity condition. However, this means uniqueness is not guaranteed unless we specify a secondary objective, such as finding the risk parity problem with the lowest variance or highest return. In general, allowing for short sales fundamentally changes the risk parity problem and requires other considerations outside of the scope of this paper. Additional references can be found in Bai et al. (2016), Haugh et al. (2017), and Costa and Kwon (in press).

On the other hand, long-only portfolios guarantee the existence of a unique solution. This convex set is the following

$$\mathcal{X} = \left\{ \boldsymbol{x} \in \mathbb{R}^n : \ \mathbf{1}^\top \boldsymbol{x} = 1; \ \boldsymbol{x} \geq 0 \right\},$$

where the first restriction ensures that all available wealth is distributed among the assets, and the second restriction enforces the long-only nature of unique risk parity portfolios. We use the notation $\mathbf{1}$ to denote a column vector of appropriate size where all elements are equal to one.

Additional restrictions may be imposed to formulate a generalized version of the risk parity problem, also known as risk budgeting (Bruder et al., 2012; Kapsos et al., 2018). This method has gained popularity due to the additional benefit of allowing the user to limit its concentration of risk to a particular asset or industry. Ji and Lejeune (2018) propose a stochastic risk budgeting multi-portfolio optimization model that imposes constraints on the marginal risk contribution of each asset, as well as using semi-deviation as the risk measure.

The objective of a nominal risk parity model is to find a portfolio where $R_i = R_j \ \forall \ i, j$. As presented in Maillard et al. (2010), this objective can be attained by minimizing the squared differences in risk contribution. Thus, the optimization model can be written as

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \sum_{i=1}^{n} \sum_{j=1}^{n} \left(x_i (\boldsymbol{\Sigma x})_i - x_j (\boldsymbol{\Sigma x})_j \right)^2 \\ \text{s.t.} \quad & \boldsymbol{x} \in \mathcal{X}. \end{aligned} \tag{3}$$

The objective function in (3) is a fourth-degree polynomial. A reformulation of this problem is presented in Bai et al. (2016), which provides an equivalent yet numerically more efficient version of this problem. However, this formulation maintains the same degree of non-linearity in the objective function. The NLP proposed in Bai et al. (2016) can be simplified further and expressed as a QCLP by adding only two auxiliary variables, $2n$ quadratic constraints, and one linear constraint. Although this approach increases the number of constraints, it allows us to reduce the complexity of the objective function. We take this opportunity to propose a novel formulation by which to construct risk parity portfolios,

$$\begin{aligned} \min_{\boldsymbol{x},\theta,\zeta} \quad & \zeta \\ \text{s.t.} \quad & \zeta \geq \theta - x_i(\boldsymbol{\Sigma x})_i, \quad i = 1, ..., n, \\ & \zeta \geq x_i(\boldsymbol{\Sigma x})_i - \theta, \quad i = 1, ..., n, \\ & \boldsymbol{x} \in \mathcal{X}, \end{aligned} \tag{4}$$

where $\theta,\ \zeta \in \mathbb{R}$ are auxiliary variables. As with the original problem in (3), the QCLP in (4) is non-convex but is numerically efficient. A more in depth analysis of the benefits of this approach over equivalent convex formulations can be found in Maillard et al. (2010) and Bai et al. (2016). We would like to highlight that, at optimality, the value of ζ is zero, tightening the $2n$ quadratic constraints and yielding

$$0 = \theta - x_i(\boldsymbol{\Sigma x})_i, \quad i = 1, ..., n,$$

$$x_i(\boldsymbol{\Sigma x})_i = x_j(\boldsymbol{\Sigma x})_j \qquad \forall\ i,\ j.$$

A test to compare the performance of these two models was conducted with randomly generated data. We generated 100 independent, symmetric positive definite matrices, each with 200 rows and 200 columns, mimicking large portfolios with 200 assets each. Performance was judged on the basis of computational run-time and on the resulting coefficient of variation (CV) of the risk contributions. The CV is calculated by dividing the standard deviation of the risk contributions by their average, i.e.,

$$\text{CV} = \frac{\text{SD}(\boldsymbol{x} \odot (\boldsymbol{\Sigma x}))}{\frac{1}{n}\boldsymbol{x}^\top \boldsymbol{\Sigma x}}, \tag{5}$$

where '$\odot$' is the element-wise multiplication operator and $\mathrm{SD}(\cdot)$ would compute the standard deviation of the corresponding vector. In theory, an optimal solution should yield a CV of zero. Table 1 shows the average performance per trial. This is a non-exhaustive experiment and does not compare

Table 1: Average computational performance of nominal risk parity problems

	NLP: Problem (3)	QCLP: Problem (4)
Run-time (s)	65.59	1.81
CV	6.74e-12	8.17e-14

other equivalent risk parity formulations; it merely serves to justify our choice of a nominal risk parity model. A more exhaustive performance comparison between equivalent risk parity formulations is described in Mausser and Romanko (2014).

Beyond our numerical results, we highlight that the benefit of the formulation proposed in (4) is the reduction in complexity of the problem. Thus, this paper will consider the QCLP in (4) as the nominal risk parity model during subsequent computational experiments, and we will use it as a benchmark for comparison against the robust risk parity formulation.

3 Uncertainty Structure

In this section we discuss the uncertainty structure used to construct the robust formulation of the risk parity problem. The only estimated parameter involved in the construction of these portfolios is the measure of risk. Thus, in our case, we are only concerned with uncertainty in the asset covariance matrix. We construct the uncertainty set in a straightforward fashion by placing a set of box constraints on each element of the covariance matrix Σ_{ij}. The estimation of an uncertainty set based on a factor model of asset returns is discussed later in Section 3.1.

Let $\mathbf{\Sigma}$ be the true (but uncertain) asset covariance matrix. To have a well-behaved problem, we typically require that the covariance matrix is positive semi-definite (PSD), i.e., $\mathbf{\Sigma} \succeq 0$. Next, we define the uncertainty set of box constraints on the covariance matrix as

$$\mathcal{U}_{\mathbf{\Sigma}} = \left\{ \mathbf{\Sigma} \in \mathbb{R}^{n \times n} : \ \underline{\Sigma}_{ij} \leq \Sigma_{ij} \leq \overline{\Sigma}_{ij} \right\}, \tag{6}$$

where both $\underline{\Sigma}_{ij}$ and $\overline{\Sigma}_{ij}$ are defined by the user depending on their choice of parameter estimation technique and the available raw data. Examples of uncertainty bounds on the covariance matrix can be found in Delage and Ye (2010) and Lobo and Boyd (2000).

For the case of long-only minimum variance portfolios, it has been widely researched that a robust formulation can be achieved simply by taking the upper bound of the covariance uncertainty structure, i.e., simply assuming the worst-case estimate of the covariance matrix (Tütüncü and Koenig, 2004). Thus, setting $\Sigma_{ij} = \overline{\Sigma}_{ij}$ provides robustness when we seek to minimize variance, but it also selects the most extreme corner within the uncertainty set when constructing risk parity portfolios. The objective of risk parity is to equalize the risk contribution per asset, not to minimize risk. Therefore, shielding against estimation errors in the overall portfolio risk is not necessarily relevant in this case.

The set of box constraints on the covariance matrix in (6) can be expressed as a perturbation on the nominal estimate as follows

$$\mathcal{U}_{\mathbf{\Sigma}} = \left\{ \mathbf{\Sigma} \mid \exists \boldsymbol{\delta} \in \mathbb{R}^{n \times n} : \ \mathbf{\Sigma} = \mathbf{\Sigma}^0 + \boldsymbol{\delta} \odot \mathbf{\Sigma}^{\Delta}, \ \underline{\Sigma}_{ij} \leq (\Sigma^0_{ij} + \delta_{ij}\Sigma^{\Delta}_{ij}) \leq \overline{\Sigma}_{ij}, \ -1 \leq \delta_{ij} \leq 1 \right\}, \tag{7}$$

where $\mathbf{\Sigma}^0 \in \mathbb{R}^{n \times n}$ is the nominal covariance matrix estimated from raw data, $\boldsymbol{\delta} \in \mathbb{R}^{n \times n}$ are independent and identically distributed random variables with mean zero and serve to define the independent perturbations on each element Σ^0_{ij}, and $\mathbf{\Sigma}^{\Delta} \in \mathbb{R}^{n \times n}$ is a constant that appropriately scales the perturbation on the nominal estimate.

In the case of where the covariance matrix is bounded by fixed upper and lower limits, a simple and tractable way to size the perturbation term is by proceeding as in Tütüncü and Koenig (2004) and let it equal the difference between the nominal and the worst-case variance, $\mathbf{\Sigma}^{\Delta} = \overline{\mathbf{\Sigma}} - \mathbf{\Sigma}^0$.

Later on we describe a scenario where the bounds are derived from the standard error arising from the regression coefficients in a factor model.

We seek to create a robust portfolio that will reduce our exposure to assets with a higher degree of error in their estimated risk contribution per asset. As defined in (2), the risk contribution of asset i, R_i, is defined as the product of the marginal risk contribution multiplied by our decision variable, i.e., $x_i(\boldsymbol{\Sigma x})_{\boldsymbol{i}}$. Since the estimation error is intrinsic to the covariance matrix, we will introduce robustness into our risk parity model by targeting the marginal risk contributions. This procedure is explained in Section 4. For now, we proceed to show one example of how to estimate the size of the covariance perturbation, $\boldsymbol{\Sigma}^{\Delta}$, using a factor model of asset returns.

3.1 Factor Models

The size of the covariance perturbation from (7), $\boldsymbol{\Sigma}^{\Delta}$, can be derived from any generic set of bounds on the covariance matrix. As an example, this section demonstrates how to derive $\boldsymbol{\Sigma}^{\Delta}$ from a factor model of asset returns. Suppose that the asset returns $\boldsymbol{r} \in \mathbb{R}^n$ can be explained through a combination of m explanatory factors. Thus, by ordinary least squares regression, we have that

$$\boldsymbol{r} = \boldsymbol{\alpha} + \boldsymbol{V}^{\top}\boldsymbol{f} + \boldsymbol{\epsilon}, \tag{8}$$

where $\boldsymbol{\alpha} \in \mathbb{R}^n$ is the vector of regression intercepts, $\boldsymbol{f} \sim \mathcal{N}(\boldsymbol{\phi}, \boldsymbol{F}) \in \mathbb{R}^m$ is the vector of factor returns, $\boldsymbol{V} \in \mathbb{R}^{m\times n}$ is the matrix of factor loadings, and $\boldsymbol{\epsilon} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{D}) \in \mathbb{R}^n$ is the vector of residual returns. $\boldsymbol{F} \in \mathbb{R}^{m\times m}$ and $\boldsymbol{D} \in \mathbb{R}^{n\times n}$ denote the factor covariance matrix and the diagonal matrix of residual variance, respectively.

This model assumes that the residual returns are independent of one another, i.e. $\mathrm{cov}(\epsilon_i, \epsilon_j) = 0$ for $i \neq j$; and it also assumes that the residual returns are independent of the factor returns, i.e. $\mathrm{cov}(\epsilon_i, f_j) = 0 \; \forall \, i, j$. Moreover, we note that this model does not assume the factors are independent of one another, i.e. the factor covariance matrix, $\boldsymbol{F}$, is not required to be a diagonal matrix. However, in practice we may need $\boldsymbol{F} \succeq 0$.

Stemming from the factor model in (8), the asset parameters may be estimated as

$$\begin{aligned} \boldsymbol{\mu} &= \boldsymbol{\alpha} + \boldsymbol{V}^\top \boldsymbol{\phi}, \\ \boldsymbol{\Sigma} &= \boldsymbol{V}^\top \boldsymbol{F} \boldsymbol{V} + \boldsymbol{D}, \end{aligned} \tag{9}$$

where, as before, $\boldsymbol{\mu} \in \mathbb{R}^n$ is the vector of expected returns and $\boldsymbol{\Sigma} \in \mathbb{R}^{n\times n}$ is the covariance matrix. This, in turn, implies $\boldsymbol{r} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

We proceed as in Goldfarb and Iyengar (2003) and assume that the estimated factor covariance matrix $\boldsymbol{F}$ is stable and known exactly. Moreover, the eigenvalues of the residual matrix $\boldsymbol{D}$ are typically much smaller than those of the matrix $\boldsymbol{V}^\top \boldsymbol{F} \boldsymbol{V}$. Thus, the latter is considered a good low-rank approximation to the asset covariance matrix. With this in mind, we focus on developing an uncertainty structure based on the estimation errors surrounding the matrix of factor loadings $\boldsymbol{V}$.

The calculation of the standard error on the factor loadings is described here. Suppose that the raw data of the asset returns and factor returns in (8) are $\{\boldsymbol{r}^t \in \mathbb{R}^n : t = 1, ..., p\}$ and $\{\boldsymbol{f}^t \in \mathbb{R}^m : t = 1, ..., p\}$, respectively, for p observations. The generic linear model can be described by

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{\beta} + \boldsymbol{\epsilon},$$

where

$$\boldsymbol{y} = [\boldsymbol{r}^1 \ \boldsymbol{r}^2 \ ... \ \boldsymbol{r}^p]^\top \in \mathbb{R}^{p\times n}, \qquad \boldsymbol{B} = [\boldsymbol{f}^1 \ \boldsymbol{f}^2 \ ... \ \boldsymbol{f}^p] \in \mathbb{R}^{m\times p},$$
$$\boldsymbol{A} = [\boldsymbol{1} \ \boldsymbol{B}^\top] \in \mathbb{R}^{p\times(m+1)}, \qquad \boldsymbol{\beta} = [\boldsymbol{\alpha} \ \boldsymbol{V}^\top]^\top \in \mathbb{R}^{(m+1)\times n}.$$

If the matrix $\boldsymbol{A}$ has full column rank $m+1$, then the least squares estimates $\widehat{\boldsymbol{\beta}}$ of the true parameter $\boldsymbol{\beta}$ is obtained by

$$\widehat{\boldsymbol{\beta}} = (\boldsymbol{A}^\top \boldsymbol{A})^{-1} \boldsymbol{A}^\top \boldsymbol{y}.$$

For a single asset i, the error term on its estimated regression coefficients $\widehat{\boldsymbol{\beta}}_i \in \mathbb{R}^{(m+1)}$ is

$$\widehat{\boldsymbol{\beta}}_i - \boldsymbol{\beta}_i = (\boldsymbol{A}^\top \boldsymbol{A})^{-1} \boldsymbol{A}^\top \boldsymbol{\epsilon}_i,$$

where the variance of the error is given by $\sigma^2_{\epsilon_i}(\boldsymbol{A}^\top \boldsymbol{A})^{-1}$. The term $\sigma^2_{\epsilon_i}$ is the true variance of the residual corresponding to asset i, i.e., it corresponds to the diagonal elements of the matrix $\boldsymbol{D}$. The unbiased estimate of $\sigma^2_{\epsilon_i}$ is then found by

$$s_i^2 = \frac{\|\boldsymbol{y}_i - \boldsymbol{A}\widehat{\boldsymbol{\beta}}_i\|_2^2}{p - m - 1},$$

where $\|\cdot\|_2$ is the Euclidean norm operator. Finally, the variance of the regression coefficients corresponding to asset i is

$$\mathrm{Var}(\widehat{\boldsymbol{\beta}}_i) = s_i^2 (\boldsymbol{A}^\top \boldsymbol{A})^{-1},$$

where the standard error of the factor loadings can be easily computed by finding the square root of the diagonal of $\mathrm{Var}(\widehat{\boldsymbol{\beta}}_i)$.

An uncertainty set on the factor loadings can be constructed by using the standard errors to bound the perturbation. Let $\boldsymbol{V}$ be the true (but unknown) matrix of factor loadings, and let it belong to the uncertainty set

$$\mathcal{U}_V = \left\{ \boldsymbol{V} \mid \exists \boldsymbol{\gamma} \in \mathbb{R}^{m \times n} : \ \boldsymbol{V} = \boldsymbol{V}^0 + \boldsymbol{\gamma} \odot \boldsymbol{V}^\Delta, -\mathrm{SE}(V_{ij}^0) \leq V_{ij}^\Delta \leq \mathrm{SE}(V_{ij}^0), \ -1 \leq \gamma_{ij} \leq 1 \right\} \tag{10}$$

where $\boldsymbol{V}^0 \in \mathbb{R}^{m \times n}$ is the least squares estimate of $\boldsymbol{V}$, $\boldsymbol{\gamma} \in \mathbb{R}^{m \times n}$ are independent and identically distributed random variables with mean zero and serve to define the independent perturbations on each element V_{ij}^0, and $\mathrm{SE}(V_{ij}^0)$ denotes the standard error of the estimated factor loading of factor i corresponding to asset j. Estimating a worst-case covariance matrix from a factor model with uncertain parameters is a difficult process. A closed-form solution is not possible due to the nature of the factor covariance matrix $\boldsymbol{F}$, where positive and negative correlation between the factors may influence the size and direction of the perturbation in order to attain a worst-case variance. We approach this issue by formulating a simple mathematical program to find the estimate of the factor loadings that maximizes the sum of all elements in the covariance matrix under the constraints given by $\mathcal{U}_V$,

$$\boldsymbol{V}^* = \underset{\boldsymbol{V} \in \, \mathcal{U}_V}{\arg\max} \quad \mathbf{1}^\top (\boldsymbol{V}^\top \boldsymbol{F} \boldsymbol{V} + \boldsymbol{D}) \mathbf{1}.$$

We can then use the nominal and worst-case estimates of the covariance matrix to recover the corresponding perturbation $\boldsymbol{\Sigma}^\Delta$,

$$\boldsymbol{V}^{*\top} \boldsymbol{F} \boldsymbol{V}^* + \boldsymbol{D} = \overline{\boldsymbol{\Sigma}} \quad \Rightarrow \quad \boldsymbol{\Sigma}^\Delta = \overline{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}^0.$$

We will this perturbation to formulate the robust risk parity framework in the next section.

4 Robust Risk Parity Portfolios

In this section we use the uncertainty set of the covariance matrix in (7) to construct a robust optimization model specifically tailored towards risk parity. To do so, we begin by reformulating the risk parity problem as a SOCP. As shown in Mausser and Romanko (2014), the nominal risk parity problem can be written as a SOCP as follows

$$\min_{\boldsymbol{x},\boldsymbol{z},t,p} \quad p - t \tag{11a}$$

$$\text{s.t.} \quad z_i = (\boldsymbol{\Sigma}\boldsymbol{x})_i, \quad i = 1, ..., n \tag{11b}$$

$$\left\| \begin{bmatrix} 2t \\ x_i - z_i \end{bmatrix} \right\|_2 \leq x_i + z_i, \quad i = 1, ..., n \tag{11c}$$

$$\left\| (\boldsymbol{\Sigma})^{1/2} \boldsymbol{x} \right\|_2 \leq \sqrt{n}\, p, \tag{11d}$$

$$\boldsymbol{z},\ p,\ t \geq 0, \tag{11e}$$

$$\boldsymbol{x} \in \mathcal{X}, \tag{11f}$$

where $p,\ t \in \mathbb{R}$, and $\boldsymbol{z} \in \mathbb{R}^n$ are auxiliary variables, and the constraint (11c) is equivalent to the hyperbolic constraint $x_i z_i \geq t^2$ for $x_i \geq 0$ and $z_i \geq 0$. The objective function in (11a) is equivalent to

$$\sqrt{\frac{\boldsymbol{x}^\top \boldsymbol{\Sigma} \boldsymbol{x}}{n}} - \sqrt{\min_{1 \leq i \leq n} \{x_i(\boldsymbol{\Sigma}\boldsymbol{x})_i\}},$$

where the square roots are simply a construct of the SOCP reformulation. By design, the objective function is zero at optimality and otherwise positive. Thus, optimality is attained when the smallest risk contribution is equal to the average risk contribution of the portfolio.

The reason we reformulate the problem as a SOCP is because this greatly simplifies the complexity of introducing robustness. Generally, risk parity optimization models such as (3) and (4) are non-convex[2]. However, by definition, the SOCP in (11) is convex. Moreover, this risk parity SOCP explicitly defines the overall portfolio risk in constraint (11d) and partitions the asset risk contributions into the corresponding marginal components in constraint (11b). The former is a single second-order cone constraint (SOCC), while the latter is a collection of n linear constraints. Since $\boldsymbol{\Sigma} \in \mathcal{U}_{\boldsymbol{\Sigma}}$, we introduce robustness by targeting constraints (11b) and (11d).

Constraint (11d) pertains to the minimization of the average portfolio risk, which is fundamentally equivalent to minimizing the total portfolio risk. As such, we introduce robustness in a similar fashion to Tütüncü and Koenig (2004) for long-only portfolios and take the worst-case instance of the covariance matrix as the input parameter, i.e.,

$$\left\| (\boldsymbol{\Sigma})^{1/2} \boldsymbol{x} \right\|_2 \leq \sqrt{n}\ p \quad \Rightarrow \quad \left\| (\boldsymbol{\Sigma}^0 + \boldsymbol{\Sigma}^\Delta)^{1/2} \boldsymbol{x} \right\|_2 \leq \sqrt{n}\ p.$$

On the other hand, constraint (11b) pertains to the marginal risk contribution per asset, $(\boldsymbol{\Sigma}\boldsymbol{x})_i$. With that in mind, we introduce robustness by realizing that constraint (11b) can be relaxed to $z_i \leq (\boldsymbol{\Sigma}\boldsymbol{x})_i$, as it will become tight at optimality. The relaxation of this constraint allows us to proceed in a similar fashion to Bertsimas and Sim (2006), where we introduce the uncertainty set

[2] Risk parity optimization is a non-convex activity when we optimize over the quadratic expression of the asset risk contributions, $x_i(\boldsymbol{\Sigma}\boldsymbol{x})_i$, where we have indefinite Hessian matrices for $i = 1, ...n$. However, non-convex risk parity model are well-defined for portfolios $\boldsymbol{x} \in \mathcal{X}$, where a unique global solution is guaranteed to exist (Bai et al., 2016).

into this constraint as an error term to penalize the marginal risk contributions,

$$z_i \leq (\boldsymbol{\Sigma x})_i \quad \Rightarrow \quad \Omega y \leq (\ \boldsymbol{\Sigma}^0 \boldsymbol{x}\ )_i - z_i,$$

where $\Omega \in \mathbb{R}_+$ is a penalty parameter and $y \in \mathbb{R}_+$ is an auxiliary variable that allows us to model the error of the marginal risk contributions as a SOCC,

$$\sqrt{n}\ y \geq \left\| \boldsymbol{\Sigma}^{\Delta} \boldsymbol{x} \right\|_2 .$$

By design, this definition of the auxiliary variable y imposes the same penalty term on all n marginal risk contributions constraints. We impose the penalty term in this fashion because the marginal risk contributions are fundamentally linked by the vector of asset weights, $\boldsymbol{x}$. Thus, all marginal risk contributions should be equally penalized. After addressing the constraints pertinent to $\boldsymbol{\Sigma} \in \mathcal{U}_{\boldsymbol{\Sigma}}$, we now introduce the robust risk parity problem,

$$\begin{aligned}
\min_{\boldsymbol{x},\boldsymbol{z},t,p,y} \quad & p - t && \text{(12a)}\\
\text{s.t.} \quad & \left\| \boldsymbol{\Sigma}^{\Delta} \boldsymbol{x} \right\|_2 \leq \sqrt{n}\ y, && \text{(12b)}\\
& \Omega y \leq (\ \boldsymbol{\Sigma}^0 \boldsymbol{x}\ )_i - z_i, \quad i = 1, ..., n && \text{(12c)}\\
& \left\| \begin{bmatrix} 2t \\ x_i - z_i \end{bmatrix} \right\|_2 \leq x_i + z_i, \quad i = 1, ..., n && \text{(12d)}\\
& \left\| (\boldsymbol{\Sigma}^0 + \boldsymbol{\Sigma}^{\Delta})^{1/2}\ \boldsymbol{x} \right\|_2 \leq \sqrt{n}\ p, && \text{(12e)}\\
& \boldsymbol{z},\ p,\ t,\ y \geq 0, && \text{(12f)}\\
& \boldsymbol{x} \in \mathcal{X}. && \text{(12g)}
\end{aligned}$$

Conceptually, the robust problem will attempt to reduce the size of the error term in constraint (12c) as it becomes tight. This will not only diminish our exposure to assets with larger estimation error in their marginal risk contributions, but it also yields a set of error-adjusted marginal risk contributions in the form of the vector of variables $\boldsymbol{z}$. The subsequent attempt to equalize risk contributions is based on $\boldsymbol{z}$, implicitly reducing our exposure to assets with larger marginal risk contributions and further shielding the portfolio against riskier assets. Additionally, constraint (12e) ensures we are still safeguarding against the worst-case instance of our risk measure in a similar fashion to conventional robust portfolio optimization models.

Introducing robustness in this fashion is preferred for two reasons. First, introducing individual error terms for the marginal risk contribution of each asset would not be able to capture the intricacies of the risk contributions, where the marginal risk contribution of asset i is not only dependent on itself, but also on its interaction with asset j as measured by their covariance $\sigma_{ij}\ \forall\ j \neq i$. Thus, formulating this as a SOCC ensures these dependencies are captured. Second, this formulation allows us to model the error term as a single SOCC, thereby avoiding a convoluted set of constraints that would otherwise increase the computational complexity of the problem.

As we previously discussed, constraint (12c) also introduces the penalty parameter Ω that scales the sensitivity of the marginal risk contributions to the error term. A larger value of Ω corresponds to a more conservative outlook. The parameter Ω can be sized to provide a probabilistic guarantee on the results as shown in Bertsimas and Sim (2006). However, we prefer to proceed with a data-driven approach to determine an appropriate value of Ω, which is based on the relative size between our nominal covariance matrix $\boldsymbol{\Sigma}^0$ and its corresponding perturbation $\boldsymbol{\Sigma}^\Delta$,

$$\Omega = \ \omega \ \frac{\|\boldsymbol{\Sigma}^\Delta\|_F}{\|\boldsymbol{\Sigma}^0\|_F},$$

where $\|\cdot\|_F$ is the Frobenius norm operator, and $\omega \in \mathbb{R}_+$ is a positive parameter that scales Ω based on the ratio of the nominal covariance matrix to its perturbation. Intuitively, setting $\omega = 0$ will revert the robust SOCP to its nominal version. On the other hand, an excessively large value of ω would be prohibitively conservative, rendering the problem infeasible. Thus, to guarantee feasibility, we must have that

$$0 \leq \omega < \sqrt{n} \cdot \frac{(\boldsymbol{\Sigma}^0 \boldsymbol{x})_i}{\|\boldsymbol{\Sigma}^\Delta \boldsymbol{x}\|_2} \cdot \frac{\|\boldsymbol{\Sigma}^0\|_F}{\|\boldsymbol{\Sigma}^\Delta\|_F}, \quad i = 1, ..., n.$$

The value of ω can be used to determine the level of robustness according to the risk appetite of the user. In other words, ω can be described as a risk aversion parameter. The subsequent computational experiment will test robust portfolios with different values of ω.

5 Computational Experiments

This section presents two computational experiments designed to test the out-of-sample performance of the proposed robust risk parity SOCP. The first experiment tests multiple portfolios each with $n = 25$ assets and studies six robust portfolios with varying degrees of robustness, testing different values of the parameter ω. The second experiment tests the impact of robustness for portfolios of different sizes, testing different number of assets n.

Both experiments were performed on an Apple MacBook Pro computer (2.8 GHz Intel Core i7, 16 GB 2133 MHz DDR3 RAM) running OS X 'Catalina'. All computations were performed using the Julia programming language (version 1.1.0) using the optimization modelling language JuMP (Dunning et al., 2017) with IPOPT (version 3.12.11) as the optimization solver for the QCLP and MOSEK (version 9.0.1) as the solver for the SOCP.

The general methodology is the same for both experiments, both described below. The nominal estimate of the covariance matrix, $\boldsymbol{\Sigma}^0$ and its maximum absolute perturbation, $\boldsymbol{\Sigma}^\Delta$, are built using a factor model as shown in Section 3.1. The Fama–French three-factor model (Fama and French, 1993) was selected as the base parameter estimation model for both experiments. The Fama–French model was purposely chosen to show how a well-known multi-factor model would behave under our proposed robust framework. The historical factor returns were obtained from Kenneth R. French's website (French, 2016).

For both experiments, the assets used for portfolio construction are selected from a universe of

250 diverse stocks regularly traded in major U.S. exchanges. A list of these stocks is shown in Table 2, with representative stocks from each of the eleven *Global Industry Classification Standard* (GICS) sectors. The historical stock prices were obtained from Quandl.com (2017). The experiment is performed using weekly historical data from 01-Jan-1995 to 31-Dec-2016, with the first five-year period used to perform the initial factor model calibration and subsequent parameter estimation (i.e., the out-of-sample period begins in 01-Jan-2000), using the Fama–French model for parameter estimation. Thereafter, the portfolios are rebalanced every six months, re-estimating the parameters using data from the preceding five-year period. The length of this time series introduces a survivorship bias to our results as we only consider stocks with sufficient historical data. However, we expect this bias to have a similar effect on all portfolios. Thus, it is the relative performance between these that is of main interest.

Both experiments consist of 1,000 independent trials, where a basket of n assets is randomly drawn from the universe of 250 assets at the beginning of each trial. This basket of n assets is held constant throughout the duration of the trial. Using the rolling window approach described in the previous paragraph, we estimate our parameters and optimize our portfolios using three different optimization models to construct our optimal portfolios. The three optimization models are described below.

- Nominal Model: The QCLP from (4), using the nominal covariance estimate, $\mathbf{\Sigma}^0$, as the sole input parameter.
- Worst-Case Model: This model is also based on the QCLP from (4), but it considers the 'worst-case' instance of the portfolio variance, $\overline{\mathbf{\Sigma}}$, as the sole input parameter.
- Robust Model: The robust SOCP presented in (12), which takes $\mathbf{\Sigma}^0$ and $\mathbf{\Sigma}^\Delta$ as the inputs derived from data, and it also takes the user-defined value ω. We will test multiple values of ω, with each producing its own robust portfolio. Increasing the value of ω yields robust portfolios with increasingly conservative outlooks.

Once a trial is complete, we repeat the process by randomly drawing a new set of n assets and

Table 2: List of assets

GICS Sector	Company Tickers							
Energy	XOM	HAL	OXY	MRO	SLB	CVX	HES	APA	COG
	COP	DVN	EOG	AE	APC	BP	CKH	EGN	
Materials	MOS	NEM	IP	IFF	NUE	PPG	APD	AVY	BLL
	ECL	FMC	SEE	SHW	VMC	AP	BMS	CCK	CRS
	EMN	FUL	GLT	MLM	OLN	SON	TG		
Industrials	BA	CAT	GE	DE	HON	LMT	GD	CSX	CMI
	DOV	ETN	EFX	FDX	ABM	AIR	ALG	AME	AOS
	BGG	CSL	MMM	HNI	NPK	RHI	SPA	SSD	
Cons. Disc.	MCD	F	GPS	HRB	NKE	AZO	HOG	HAS	HD
	JWN	TGT	AAN	CBRL	BKS	ETH	GT	LOW	NWL
	PZZA	RCL	WWW						
Cons. Staples	WMT	KO	KR	HSY	CL	CLX	WBA	COST	MO
	CPB	GIS	MKC	PEP	PG	ALCO	FARM	FLO	CAG
	CASY	LANC	ODC	TSN	UVV	STZ	WMK	TR	
Healthcare	BMY	PFE	JNJ	BAX	CVS	CI	DHR	HUM	ABT
	AGN	AMGN	BDX	BSX	LLY	BIO	CAH	HAE	HRC
	LH	MCK	MRK	OMI	PKI	STE	SYK	TMO	
Financials	JPM	BAC	AON	AFL	WFC	AXP	BK	BEN	MS
	AFG	ALL	BANF	BXS	CMA	EV	KEY	LM	MSL
	PNC	STT	WTM	TMP					
IT	HPQ	IBM	MSFT	INTC	ADSK	ADP	CSCO	GLW	ORCL
	PAYX	TXN	PLT	XRX	ROG	TSS			
Comm. Serv.	DIS	VZ	S	EA	CTL	T	IPG	OMC	NYT
	VOD	CBB	RDI	SSP	MDP				
Utilities	CNP	DTE	DUK	ED	AEP	D	ETR	ATO	ES
	EXC	NEE	PEG	SO	AVA	AWR	BKH	CMS	CPK
	GXP	NFG	NJR	OGE	PNM	PNW	PPL	SWX	UGI
	XEL	SJW							
Real Estate	REG	HCP	UDR	DRE	AIV	WY	KIM	MAA	AVB
	EQR	HST	PSA	SPG	VNO	ADC	ALX	BDN	BFS
	CLI	CPT	CTO	CUZ	EGP	ELS	GTY	JOE	LXP
	MAC	WRI							

creating the corresponding optimal portfolios using the aforementioned rolling window approach. This process is repeated for each of the 1,000 trials.

5.1 Experiment with varying degrees of robustness

Our first experiment analyzes portfolios with $n = 25$ assets. There are eight portfolios in this experiment: one nominal portfolio, one 'worst-case' portfolio, and six robust portfolios with $\omega = 0.5, 1.0, 1.5, 2.0, 2.5, 3.0$. The experiment consists of 1,000 trials. Each trial begins by randomly drawing 25 assets from those listed in Table 2. We then construct eight portfolios using these 25 assets. The necessary parameters are estimated using the Fama–French model and the eight portfolios are built using the corresponding optimization models. The trial is conducted using a rolling window where parameters are re-estimated and portfolios are re-optimized every six months over the 17-year investment horizon. We then repeat this process for every subsequent trial, randomly drawing a new set of 25 constituent assets at the beginning of every trial. The results in this section are shown for the average and the observed standard deviation of the 1,000 trials.

The portfolio performance, measured by the wealth evolution, is presented in Figure 1. This figure shows both the absolute and relative wealth of three of our eight portfolios. Only three portfolios are shown in this figure for the sake of clarity (the remaining portfolios are discussed later on). The top plot in Figure 1 shows the average wealth evolution of the nominal, 'worst-case', and robust ($\omega = 2.0$) portfolios. The average wealth is calculated by taking the average value of the 1,000 trials at each discrete point in time corresponding to each portfolio. The error bars display the standard deviation corresponding to the distribution of the 1,000 trials. The bottom plot shows the wealth of the 'worst-case' and robust portfolios relative to the nominal, i.e., $(W_i^t / W_{\mathrm{Nom}}^t - 1) \times 100$ for the wealth W_i^t of portfolio i at each point in time t.

The results show that our robust portfolio with $\omega = 2.0$ behaves as expected, outperforming the nominal during bear market periods. This is exemplified in the bottom plot of Figure 1 by the relative gains made by the robust portfolio during the recession of the early 2000s, as well as the spike observed during the recent financial crisis of 2008. Moreover, the robust portfolio appears to

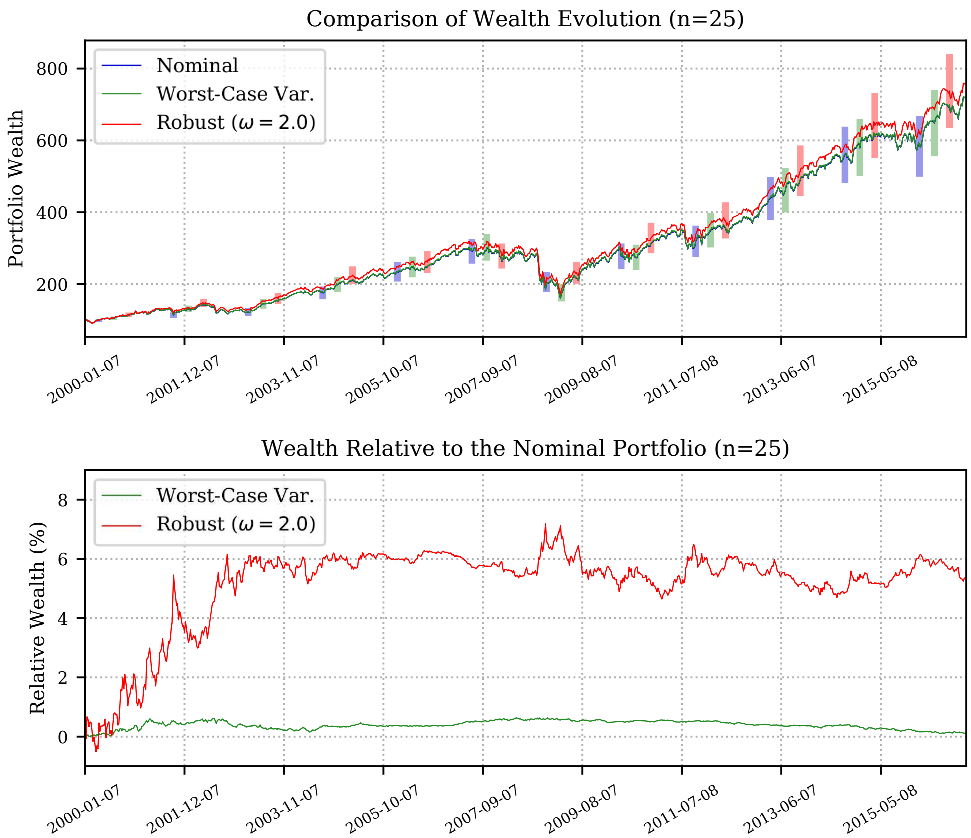

Figure 1: Wealth evolution of 1,000 individual trials with 25 assets per portfolio over the period 2000–2016. Top: Average wealth of portfolios over the 1,000 trials with the standard deviation shown as error bars. Bottom: Average wealth relative to the nominal portfolio.

be able to maintain its relative advantage throughout bull market periods as well. On the other hand, the 'worst-case' variance portfolio closely mimics the nominal, having a marginally better performance but not displaying a meaningful behaviour during different market cycles.

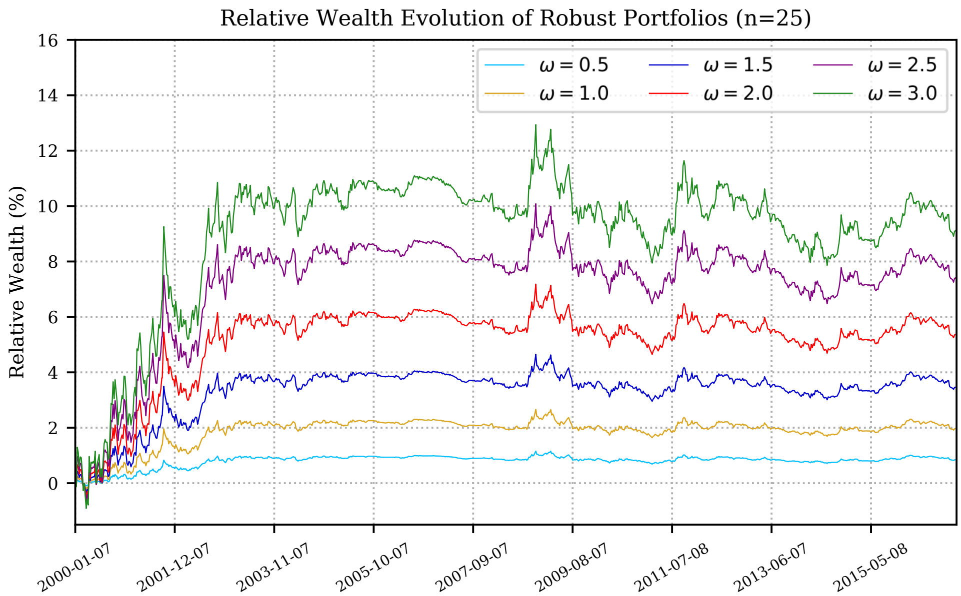

Figure 2: Average wealth evolution of robust portfolios relative to the nominal portfolio for several values of ω.

Next, we discuss the performance of our six robust portfolios for different values of ω. The purpose is to analyze the out-of-sample effect of varying the degree of robustness in the portfolios. A larger value of the parameter ω corresponds to a more conservative portfolio. Figure 2 shows the average wealth evolution of the robust portfolios relative to the nominal. The plot of the robust portfolio with $\omega = 2.0$ is the same as in Figure 1. Figure 2 suggests that taking an increasingly conservative stance can significantly improve the relative performance of the robust portfolios. A robust portfolio with $\omega = 3.0$ is able to attain a peak advantage of 12.94% over the nominal during the financial crisis of 2008. It is worthwhile to note that the upward trend seen during bearish periods does not necessarily imply that the robust portfolios had a positive return, but rather that they were better shielded against the losses observed by the nominal portfolio. Moreover, the volatility observed on a relative wealth plot, such as Figure 2, is the result of the relative difference in wealth evolution between the nominal portfolio and its robust counterpart. In other words, the relative wealth plots may appear volatile even when the robust portfolio has a low volatility if this also coincides with a period of high volatility of the underlying nominal portfolio.

A summary of the financial performance of all eight portfolios is presented in Table 3. The results shown in this table are calculated as follows. The portfolio excess returns are calculated by finding the weekly return of each portfolio from each individual trial and subtracting the corresponding weekly risk-free rate. We then use the time series of observed weekly excess returns to calculate the average excess return, volatility and ex-post Sharpe ratio (Sharpe, 1994) for each portfolio per trial. Finally, we annualized these weekly values and we calculated the average and standard deviation over the 1,000 trials. For clarity, the row in Table 3 labelled 'Ann. Volatility' corresponds to the average annualized volatility experienced by the portfolios over the 1,000 trials; the row labelled 'σ_{Vol}' is the corresponding standard deviation of measured volatility over the 1,000 trials. Finally, the turnover rate measures the period-over-period absolute change in the asset weights per trial, with the average over the 1,000 trials labelled as the 'Turnover Rate' and its corresponding standard deviation as 'σ_{TR}'. The turnover rate serves to exemplify potential transaction costs.

Table 3: Summary of financial performance of 1,000 trials with $n = 25$.

	Nominal	Worst-Case	Robust					
			$\omega = 0.5$	$\omega = 1.0$	$\omega = 1.5$	$\omega = 2.0$	$\omega = 2.5$	$\omega = 3.0$
Ann. Ex. Return (%)	9.76	9.77	9.81	9.88	9.96	10.07	10.18	10.27
σ_{Ret} (%)	0.84	0.83	0.84	0.84	0.84	0.85	0.85	0.86
Ann. Volatility (%)	16.19	16.20	16.09	15.98	15.86	15.73	15.60	15.49
σ_{Vol} (%)	0.94	0.95	0.94	0.94	0.94	0.94	0.95	0.97
Sharpe Ratio (%)	60.49	60.51	61.17	62.01	63.04	64.24	65.48	66.57
σ_{Sharpe} (%)	6.45	6.40	6.48	6.52	6.60	6.71	6.77	6.93
Turnover Rate (%)	5.13	4.97	5.58	6.20	7.04	8.22	9.88	11.94
σ_{TR} (%)	3.17	3.03	3.42	3.81	4.50	5.71	7.68	9.83

The results in Table 3 show a clear trend when comparing the robust portfolios against the nominal. As the level of robustness is increased, the realized excess return increases while the volatility decreases. In turn, this translates into an increasing ex-post Sharpe ratio as the level of robustness increases. The design of the robust optimization model in (12) is meant to reduce our exposure to assets with large ex-ante errors in their estimated risk contributions. Thus, our results suggest that a reduced exposure to such assets helps to mitigate portfolio losses. In general, mitigating portfolio losses yields an improved long-run rate of return while maintaining a lower level of volatility. Juxtaposing the robust portfolios against the worst-case portfolio, we can see that their difference in methodology may also affect the ex-post performance. Given that the worst-case portfolio applies the nominal methodology on the worst-case estimate of the covariance matrix, the resulting worst-case portfolio has a similar behaviour to the nominal itself. This can be observed in the relative wealth evolution in the bottom plot of Figure 1, as well as summarized in the ex-post Sharpe ratio in Table 3. Nevertheless, our results show that the standard deviation of the excess return and volatility, σ_{Ret} and σ_{Vol}, are relatively small and quite stable for all portfolios, even as the level of robustness increased for the robust portfolios. Finally, the turnover rate of our robust portfolios increased as the level of robustness increased. This suggests that penalizing our positions on assets with noisier marginal risk contributions resulted in greater period-over-period changes to our optimal weights. It is worthwhile to note that nominal risk parity portfolios are generally very stable in their period-over-period wealth reallocation. Thus, risk parity turnover rates are traditionally lower when compared to a broader range of optimal portfolios, such as mean–variance optimization (Chaves et al., 2011).

Overall, our results suggest that the financial performance of a risk parity portfolio can be enhanced through the proposed robust framework. The computational examples shown rely on the Fama–French three-factor model for the estimation of the nominal covariance matrix and its uncertainty structure. Thus, a more elaborate factor-based analysis could be conducted to identify the source of performance stemming from robustness. However, given that the application of factor models in this paper is more illustrative than prescriptive, we refrain from overextending our numerical experiments in this direction. Furthermore, given the financial considerations of our experimental setup, the experiments evaluate the impact of robustness from an ex-post perspective. However, we note that a sensitivity analysis on the effect of robustness could be performed through simulation and resampling. An adequate implementation of such an approach can be found in Jorion (1992) and Kim et al. (2016). For the remainder of this section, our experiments continue to discuss the financial and statistical merits of our robust portfolios stemming from their consistency during the 1,000 out-of-sample trials.

An analysis of the Sharpe ratio in indicates that the observed differences between the nominal and robust portfolios is statistically significant. A summary of this analysis is shown in Table 4. The first row of this table indicates the total number of trials for which the Sharpe ratio of a given portfolio was greater than the nominal portfolio. For example, from the first row we can see that the Sharpe ratio of the robust portfolio with $\omega = 2.0$ was greater than its nominal counterpart during 998 out of 1,000 trials. Similarly, by looking at Table 4, we can see that the robust portfolios outperformed the nominal in almost every trial. On the other hand, the worst-case portfolio had a superior performance in only 542 times out of 1,000. The second row of this table presents the t-statistic for a paired sample t-test. We performed this analysis by calculating the pair-wise difference in the ex-post Sharpe ratio between the nominal portfolio and all other competing portfolios over each of the 1,000 trials. Thus, for this number of trials, we have 999 degrees of freedom. The t-statistics in Table 4 imply that we have more than 99.9% confidence that the Sharpe ratio of the robust portfolios is greater than the nominal. However, we can only say the same of the worst-case portfolio with less than 90% confidence.

Table 4: Sharpe ratio analysis of 1,000 trials with $n = 25$.

	Worst-Case	Robust					
		$\omega = 0.5$	$\omega = 1.0$	$\omega = 1.5$	$\omega = 2.0$	$\omega = 2.5$	$\omega = 3.0$
# of trials greater than nominal	542	1000	1000	999	998	997	988
t-statistic	1.247	73.68	70.51	68.07	68.61	72.20	69.77

Next, we proceed to discuss the difference in risk concentration between the portfolios. The results in Table 5 serve to quantify our trade-off between risk parity and robustness. This 'cost' of robustness can be measured by our deviation away from perfect risk diversification (i.e., risk parity). We define perfect risk diversification in terms of the nominal covariance matrix, $\mathbf{\Sigma}^0$. As such, our measures of risk concentration are defined solely in terms of $\boldsymbol{\Sigma}^0$ to allow for a fair comparison between our nominal portfolio and its robust counterparts. In line with our previous results, we report both the average values and the standard deviation of our risk concentration measures. Since each individual portfolio is re-optimized and rebalanced 34 times per trial, the values in Table 5 correspond to 34,000 observations for each portfolio. We use the following three measures of risk concentration measures:

- Coefficient of Variation (CV): The CV was outlined in (5) in Section 2. However, the equation for the CV is shown below for the reader's convenience,

$$\mathrm{CV} = \frac{\mathrm{SD}(\boldsymbol{x} \odot (\boldsymbol{\Sigma}^0 \boldsymbol{x}))}{\frac{1}{n}\boldsymbol{x}^\top \boldsymbol{\Sigma}^0 \boldsymbol{x}}.$$

  If we have perfect risk diversification, the CV of the portfolio is equal to zero.

- Highest Risk Contribution (HRC): As its name suggests, the HRC is mathematically defined as the ratio of the highest individual risk contribution over the total portfolio variance,

$$HRC = \max_i \frac{x_i (\boldsymbol{\Sigma}^0 \boldsymbol{x})_i}{\boldsymbol{x}^\top \boldsymbol{\Sigma}^0 \boldsymbol{x}}.$$

  The value of the HRC ranges between $1/n$ for the risk parity portfolio and 1 for a fully concentrated portfolio.

- Herfindahl Index (H): The H-index is defined as follows

$$H = \sum_{i=1}^{n} \left(\frac{x_i (\boldsymbol{\Sigma}^0 \boldsymbol{x})_i}{\boldsymbol{x}^\top \boldsymbol{\Sigma}^0 \boldsymbol{x}} \right)^2.$$

  The value of the H-index also ranges between $1/n$ for the risk parity portfolio and 1 for a fully concentrated portfolio.

The results in Table 5 show that, as robustness increases, we deviate further away from perfect risk diversification. This holds for all three measures of risk concentration. This is a natural result,

Table 5: Summary of risk concentration measures of 1,000 trials with $n = 25$.

	Nominal	Worst-Case	Robust					
			$\omega = 0.5$	$\omega = 1.0$	$\omega = 1.5$	$\omega = 2.0$	$\omega = 2.5$	$\omega = 3.0$
CV	0.000	0.054	0.035	0.080	0.139	0.219	0.322	0.448
σ_{CV}	0.000	0.029	0.023	0.060	0.121	0.208	0.319	0.437
HRC	0.040	0.044	0.044	0.049	0.056	0.066	0.080	0.097
σ_{HRC}	0.000	0.002	0.003	0.008	0.016	0.029	0.047	0.068
H	0.040	0.040	0.040	0.040	0.041	0.044	0.048	0.055
σ_H	0.000	0.000	0.000	0.001	0.003	0.010	0.023	0.040

as it becomes increasingly difficult to attain equalized asset risk contributions while simultaneously limiting our exposure to assets with noisier marginal risk contributions. Nevertheless, we are able preserve a sufficient degree of diversity even when we assess the risk contributions derived from the nominal estimate of the covariance matrix[3]. This is particularly true of the robust portfolios with low ω values, where the average H and HRC values are similar to the nominal portfolio. However, risk begins to concentrate as we become increasingly conservative. Overall, the results suggest that the proposed robust framework is still able to attain a risk-diverse portfolio, particularly for low ω values.

5.2 Experiment with different portfolio sizes

Our second experiment compares portfolios of different sizes with $n = 15, 25, 50, 75, 100$. As before, we performed 1,000 trials for each value of n (with the results for $n = 25$ being the same as the previous experiment). For each trial, we construct four portfolios: one nominal portfolio, one 'worst-case' portfolio, and two robust portfolios with $\omega = 1.0, 2.0$.

The portfolio performance, measured by the wealth evolution, is shown in Figure 3. There are five subplots in this figure corresponding to the portfolios of different sizes. After taking the average over the 1,000 trials, the five plots show that the highest and second-highest terminal wealth is attained by the two robust portfolios. The first three plots show that the robust portfolios with

[3]When we use the nominal covariance matrix, $\boldsymbol{\Sigma}^0$, as the risk measure, only one perfect risk-diverse portfolio exists: the nominal risk parity portfolio. Any other portfolio $\boldsymbol{x} \in \mathbb{R}^n$ that differs from the nominal will have a CV greater than zero, as well as H and HRC values greater than $1/n$. Thus, when risk is measured using $\boldsymbol{\Sigma}^0$, none of the other competing models can attain perfect risk diversification.

$\omega = 2.0$ have the highest return. However, the last two plots, where the portfolios have 75 and 100 assets, show that the robust portfolios with $\omega = 1.0$ have the highest terminal wealth.

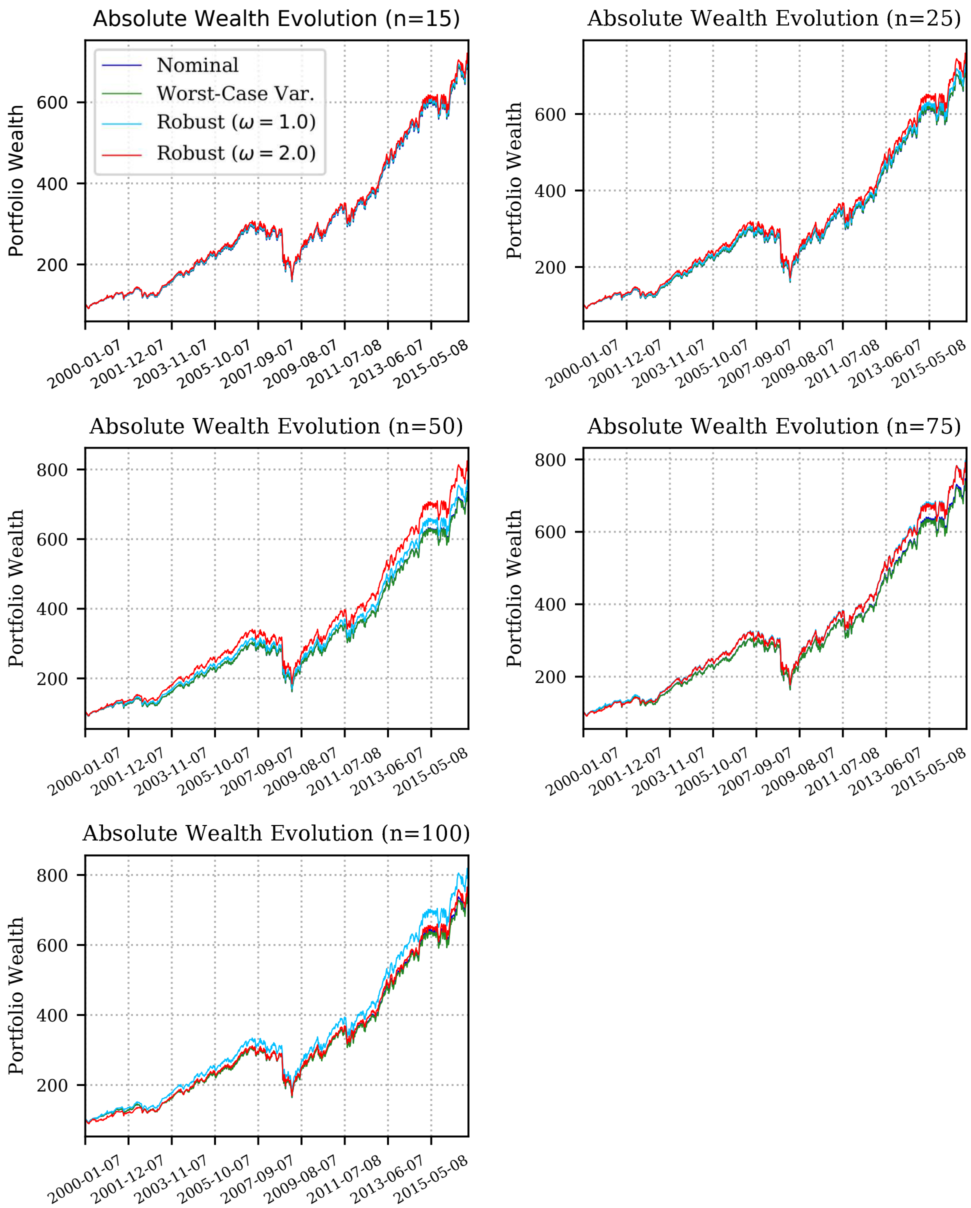

Figure 3: Wealth evolution of portfolios of different sizes. There were 1,000 trials conducted for each portfolio size n over the period 2000–2016. The plots show the average wealth evolution of the 1,000 trials.

This same results can be observed in greater detail in Figure 4. Here we can see the portfolio performance relative to the nominal. The robust portfolios in the first three plots ($n = 15, 25, 50$)

exhibit a similar pattern, except the increase in relative wealth is magnified as the size increases. This ceases to be the case when we inspect the plots with $n = 75, 100$, where we can see that the more conservative robust portfolios experience a significant drop in relative wealth during the beginning of the experiment. Nevertheless, the robust portfolios with $\omega = 1.0$ demonstrate greater consistency between the different portfolio sizes. Indeed, the robust portfolios with $\omega = 1.0$ not only consistently outperform the nominal and worst-case portfolios for every value of n, but this increase in relative wealth is magnified as the portfolio size increases from $n = 15$ to $n = 100$.

Table 6 presents the results corresponding to the 1,000 trials for each value of n. These results are calculated in the same fashion as our previous experiment. Evaluating the average portfolio performance over the course of the entire investment horizon shows that the robust portfolios outperform the nominal in all three categories: excess return, volatility and Sharpe ratio. As before, the turnover rate, however, is larger for the robust portfolios. In particular, the rebalancing of the robust portfolios is more drastic as the number of assets increases. This is not surprising since, as n increases, we have greater choice for our asset allocation strategy, as well as having more noise from the increase in estimated parameters. Moreover, while the robust portfolios are consistently able to financially outperform the nominal, the performance of the worst-case portfolio worsens as the portfolio size increases. When $n \geq 50$, the worst-case portfolio fails to surpass the nominal. Finally, it is worthwhile to note that the performance of the nominal portfolio improves as the number of assets increases. This improvement with size is consistent but not steady. The portfolios benefit from greater diversification between more assets, but this benefit diminishes as n increases.

A statistical analysis of the Sharpe ratio is shown in Table 7. Similar to the previous experiment, we find that the Sharpe ratio of the robust portfolios exceeds that of the nominal portfolio in almost every single trial. In fact, for $n \geq 50$, the Sharpe ratio of the robust portfolio with $\omega = 1.0$ exceeds the nominal in every single trial. Moreover, the t-statistic of the difference in the Sharpe ratio between robust and nominal portfolios suggests that the robust portfolios have a better risk-adjusted performance with more than 99.9% confidence for over all values of n. However, we note that the Sharpe ratio of the robust portfolios with $\omega = 2.0$ begins to deteriorate when $n \geq 50$. Finally, in

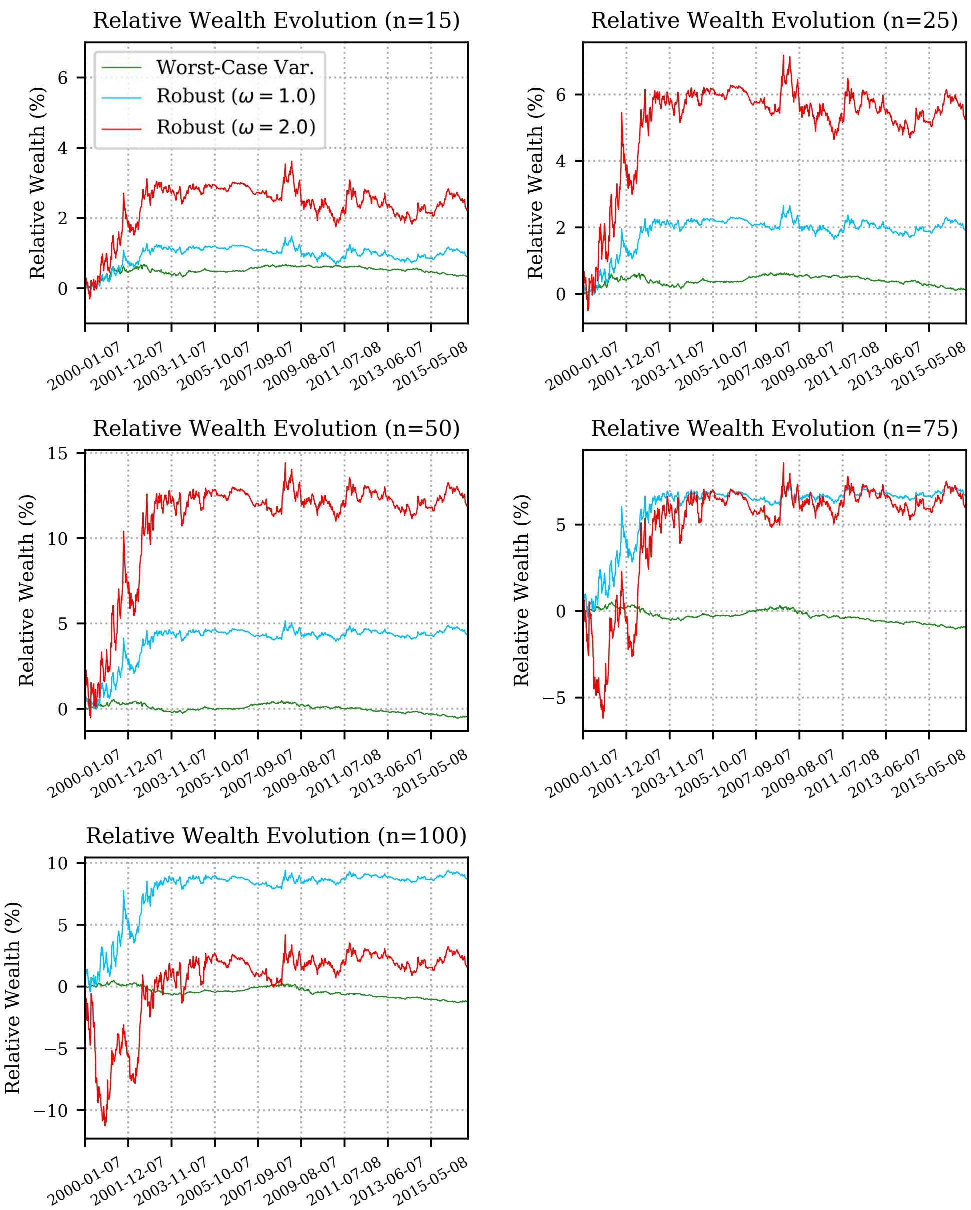

Figure 4: Wealth evolution relative to the nominal of portfolios of different sizes. There were 1,000 trials conducted for each portfolio size n over the period 2000–2016. The plots show the average wealth evolution of the 1,000 trials relative to the nominal portfolio.

Table 6: Summary of financial performance of 1,000 trials with $n = 15,\ 25,\ 50,\ 75,\ 100$.

	$n=15$				$n=25$			
	Nom.	WC	Robust		Nom.	WC	Robust	
			$\omega=1.0$	$\omega=2.0$			$\omega=1.0$	$\omega=2.0$
Ann. Ex. Return (%)	9.59	9.61	9.65	9.73	9.76	9.77	9.88	10.07
σ_{Ret} (%)	1.08	1.08	1.08	1.08	0.84	0.83	0.84	0.85
Ann. Volatility (%)	16.86	16.85	16.71	16.54	16.19	16.20	15.98	15.73
σ_{Vol} (%)	1.27	1.28	1.27	1.27	0.94	0.95	0.94	0.94
Sharpe Ratio (%)	57.27	57.44	58.10	59.19	60.49	60.51	62.01	64.24
σ_{Sharpe} (%)	8.19	8.20	8.25	8.36	6.45	6.40	6.52	6.71
Turnover Rate (%)	4.61	4.56	5.24	6.21	5.13	4.97	6.20	8.22
σ_{TR} (%)	3.02	2.96	3.39	4.10	3.17	3.03	3.81	5.71

	$n=50$				$n=75$			
	Nom.	WC	Robust		Nom.	WC	Robust	
			$\omega=1.0$	$\omega=2.0$			$\omega=1.0$	$\omega=2.0$
Ann. Ex. Return (%)	9.92	9.89	10.17	10.58	10.01	9.96	10.40	10.34
σ_{Ret} (%)	0.61	0.59	0.62	0.68	0.43	0.41	0.46	0.72
Ann. Volatility (%)	15.68	15.73	15.35	15.09	15.42	15.50	15.00	14.95
σ_{Vol} (%)	0.65	0.65	0.66	0.69	0.5	0.5	0.52	0.57
Sharpe Ratio (%)	63.41	63.03	66.43	70.32	65.02	64.34	69.42	69.31
σ_{Sharpe} (%)	4.97	4.84	5.28	5.79	3.78	3.60	4.24	5.79
Turnover Rate (%)	5.80	5.46	8.01	14.53	6.18	5.72	9.65	20.32
σ_{TR} (%)	3.42	3.17	5.18	14.62	3.58	3.26	7.10	21.64

	$n=100$			
	Nom.	WC	Robust	
			$\omega=1.0$	$\omega=2.0$
Ann. Ex. Return (%)	10.07	10.00	10.57	10.15
σ_{Ret} (%)	0.39	0.36	0.43	0.68
Ann. Volatility (%)	15.32	15.42	14.82	14.92
σ_{Vol} (%)	0.40	0.40	0.42	0.53
Sharpe Ratio (%)	65.79	64.92	71.35	68.15
σ_{Sharpe} (%)	3.27	3.02	3.87	5.72
Turnover Rate (%)	6.41	5.86	11.28	24.44
σ_{TR} (%)	3.68	3.31	9.53	25.64

line with other results, the Sharpe ratio of the worst-case portfolio worsens as size increases. In fact, when $n \geq 50$, we can see that the t-statistic suggests that the Sharpe ratio is actually lower for the worst-case portfolio when compared against the nominal with more than 99.9% confidence.

Table 7: Sharpe ratio analysis of 1,000 trials with $n = 15,\ 25,\ 50,\ 75,\ 100$.

	$n=15$			$n=25$		
	WC	Robust		WC	Robust	
		$\omega = 1.0$	$\omega = 2.0$		$\omega = 1.0$	$\omega = 2.0$
# of trials greater than nominal	681	983	983	542	1000	998
t-statistic	13.50	50.85	48.62	1.496	69.40	66.14
	$n=50$			$n=75$		
	WC	Robust		WC	Robust	
		$\omega = 1.0$	$\omega = 2.0$		$\omega = 1.0$	$\omega = 2.0$
# of trials greater than nominal	229	1000	988	87	1000	839
t-statistic	-24.29	92.61	82.42	-38.35	107.20	30.91
	$n=100$					
	WC	Robust				
		$\omega = 1.0$	$\omega = 2.0$			
# of trials greater than nominal	26	1000	697			
t-statistic	-46.68	123.92	16.66			

The risk concentration measures are presented in Table 8. Unsurprisingly, all portfolios deviate away from perfect risk diversification as the number of assets increases. The only exception is the nominal portfolio, which serves as the benchmark for perfect risk diversification given that all risk concentration measures are calculated using $\mathbf{\Sigma}^0$ as the risk measure. With that said, the worst-case portfolios deviate the least from risk parity out of the three competing portfolios. This falls in line with our previous findings, where the cost of robustness manifests by deviating away from risk parity and increasing the turnover rate. In turn, this reflects our aversion to assets with noisier marginal risk contributions, limiting our capacity to attain perfect risk diversification.

The results of this experiment suggest that our proposed robust framework for risk parity portfolios can lead to higher out-of-sample risk-adjusted returns while maintaining a sufficiently risk-diverse portfolio. The trade-off between performance and risk diversification is expected of any investment

Table 8: Summary of risk concentration measures of 1,000 trials with $n = 15,\ 25,\ 50,\ 75,\ 100$.

	$n = 15$				$n = 25$			
	Nom.	WC	Robust		Nom.	WC	Robust	
			$\omega = 1.0$	$\omega = 2.0$			$\omega = 1.0$	$\omega = 2.0$
CV	0.000	0.045	0.054	0.131	0.000	0.054	0.080	0.219
σ_{CV}	0.000	0.028	0.035	0.100	0.000	0.029	0.060	0.208
HRC	0.067	0.071	0.075	0.088	0.040	0.044	0.049	0.066
σ_{HRC}	0.000	0.003	0.006	0.019	0.000	0.002	0.008	0.029
H	0.067	0.067	0.067	0.068	0.040	0.040	0.040	0.044
σ_H	0.000	0.000	0.000	0.004	0.000	0.000	0.001	0.010

	$n = 50$				$n = 75$			
	Nom.	WC	Robust		Nom.	WC	Robust	
			$\omega = 1.0$	$\omega = 2.0$			$\omega = 1.0$	$\omega = 2.0$
CV	0.000	0.065	0.142	0.528	0.000	0.071	0.209	0.996
σ_{CV}	0.000	0.032	0.151	0.782	0.000	0.033	0.265	1.646
HRC	0.020	0.022	0.031	0.068	0.013	0.015	0.027	0.103
σ_{HRC}	0.000	0.001	0.014	0.097	0.000	0.001	0.019	0.183
H	0.020	0.020	0.021	0.037	0.013	0.013	0.015	0.062
σ_H	0.000	0.000	0.002	0.064	0.000	0.000	0.004	0.137

	$n = 100$			
	Nom.	WC	Robust	
			$\omega = 1.0$	$\omega = 2.0$
CV	0.000	0.074	0.281	1.370
σ_{CV}	0.000	0.033	0.398	2.221
HRC	0.010	0.011	0.026	0.121
σ_{HRC}	0.000	0.001	0.025	0.216
H	0.010	0.010	0.012	0.077
σ_H	0.000	0.000	0.007	0.169

model that differs from the nominal model[4]. Moreover, our results suggest that simply taking the worst-case estimate of the covariance matrix does not deliver the robust behaviour expected during bear market periods. Our analysis of portfolios of different sizes shows that, for mild levels of robustness (e.g., $\omega = 1.0$), we observe this robust behaviour, with our robust portfolios making most of their relative gains during periods of financial distress. Moreover, the robust portfolios are able to do while maintaining a good degree of risk-based diversification relative to the nominal.

6 Conclusion

In this paper we introduced a robust formulation of a risk parity portfolio. Unlike MVO portfolios, we do not seek to minimize risk. Traditional techniques in portfolio optimization, where robustness is guaranteed by taking the 'worst-case' estimate of the risk measure, are not applicable when our objective is to have an equalized risk contribution per asset. Instead, due to the nature of our objective, we are most concerned with correctly estimating the assets' risk contributions. Our proposed framework was developed specifically to address uncertainty in risk parity. Thus, by design, this robust problem seeks to shield the portfolio from overexposure to assets with a high degree of estimation error in their marginal risk contributions, as well as reduce our exposure to riskier assets with larger marginal risk contributions. Intuitively, we define the uncertainty set on our estimated risk measure, namely the asset covariance matrix. We then formulate the risk parity optimization problem in such a way that it explicitly measures the asset marginal risk contributions, allowing us to introduce robustness such that we limit our exposure to assets with noisier marginal risk contributions. Our proposed framework is able to attain both a higher rate of return and a higher risk-adjusted rate of return, both of which can be attributed to a reduced exposure to riskier assets, which tend to drive portfolio losses.

Moreover, the proposed robust formulation relies on the insertion of a single new constraint upon the original SOCP, which, by design, maintains the same level of complexity as the original

[4] We note that only the nominal model can attain perfect risk diversification when we measure risk concentration using $\mathbf{\Sigma}^0$. By definition, all other competing models cannot attain risk parity unless they converge to the exact same portfolio as the nominal.

problem. In other words, our robust SOCP is equally as tractable and efficient as the original risk parity problem.

The uncertainty set on the asset covariance matrix can be derived from any generic method that yields a set of box constraints on the covariance matrix, and we showed one example of how to derive this uncertainty set from the standard error of regression coefficients from a factor model of asset returns. Factor models are more difficult to handle, requiring an additional level of computational cost to create the uncertainty structure. Nevertheless, we are able to show that it is possible to use a factor model to construct a viable robust portfolio, as shown in the computational experiment in which we used the Fama–French three-factor model.

The experimental results show that robust risk parity portfolios are able to outperform their nominal counterpart throughout a long investment horizon. As would be expected, the robust portfolios were able to realize a significant advantage during periods of market distress. More surprisingly, the robust portfolios were able to maintain its relative advantage during bullish periods. Robust portfolios with an increasing risk aversion ω were tested to visualize how this user-defined parameter affects the resulting optimal portfolios. Aside from an improved rate of return, robust portfolios are also able to attain a higher-than-nominal risk-adjusted rate of return, which is explained by our reduced exposure to noisy assets. However, the trade-off for this enhanced financial performance, which we refer to as the cost of robustness, manifests in two manners: (i) by deviating away from perfect risk diversification, and (ii) by increasing our period-over-period turnover rate. This cost of robustness is the result of our aversion to assets with noisier marginal risk contributions, limiting our capacity to attain perfect risk diversification.

This robust formulation opens the door to additional applications, such as using it to construct robust portfolios under a risk budgeting approach (Bruder et al., 2012; Haugh et al., 2017; Kapsos et al., 2018). This could be the subject of future research.

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