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A Robust Framework for Risk Parity Portfolios

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Abstract We propose a robust formulation of the traditional risk parity problem by introducing an uncertainty structure specifically tailored to capture the intricacies of risk parity. Typical minimum variance portfolios attempt to introduce robustness by computing the worst-case estimate of the risk measure, which is not intuitive for risk parity. Instead, our motivation is to shield the risk parity portfolio against noise in the estimated asset risk contributions. We do this by introducing a novel robust risk parity model that places an uncertainty set around the assets marginal risk contributions, and we provide a tractable formulation that is able to retain the same level of complexity as the original problem. We provide a general procedure by which to create the uncertainty structure around the vector of marginal risk contributions, and we elaborate on the specific procedure to construct a robust risk parity portfolio through a factor model of asset returns. Computational experiments show that the robust formulation yields a higher risk-adjusted rate of return than the nominal model while maintaining a sufficiently risk-diverse portfolio.

Keywords Robust Optimization · Risk Parity · Asset Allocation · Uncertainty · Factor Model

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1 Introduction

Optimal portfolio construction has become an academic discipline since the introduction of Modern Portfolio Theory in Markowitz (1952). Mean-variance optimization (MVO) has the power to yield portfolios that minimize risk while maintaining a target expected return. However, in practice, these 'optimal' portfolios suffer from considerable drawbacks. First, optimal solutions often yield over-concentrated and counter-intuitive portfolios (Chopra and Ziemba, 1993). The second issue pertains to the sensitivity of the solution to estimation error from input parameters, specifically the asset expected returns and covariance matrix. This sensitivity has been widely explored in literature (Best and Grauer, 1991; Broadie, 1993; Merton, 1980). In particular, Chopra and Ziemba (1993) found that estimation errors in the expected returns can have an impact ten times larger than errors in the covariance matrix in MVO. Thus, robust portfolio optimization methods, such as resampling (Michaud and Michaud, 2007), were introduced in an effort to mitigate the impact of estimation error. More recent academic publications attempt to capture estimation error by introducing uncertainty sets around noisy parameters that bound the 'true' parameters within a given radius of their corresponding estimates (Delage and Ye, 2010; Goldfarb and Iyengar, 2003; Tütüncü and Koenig, 2004).

A modern attempt to mitigate estimation error and create diversified portfolios is the introduction of risk parity portfolios, also referred to as equal risk contribution portfolios. These portfolios do not set targets on the expected portfolio return or risk, but instead yield portfolios where resources are allocated based solely on the risk measure such that the risk contribution of each asset is the same (Maillard et al, 2010). Thus, by design, the construction of a risk parity portfolio does not require the estimation of expected returns, thereby reducing the most significant source of estimation noise. Moreover, since every asset must contribute the same level of risk, the resulting portfolios tend to be well diversified.

Our proposed robust formulation deviates from this approach, and instead proposes a model specifically designed for risk parity that is not concerned with uncertainty on the overall portfolio risk, but instead builds an uncertainty structure around the assets' marginal risk contributions. The risk measure we consider is the asset covariance matrix, which is the most typical risk measure used when studying marginal risk contributions. Alternative risk measures where marginal risk contributions are a concern are discussed in Ji and Lejeune (2018) and Mausser and Romanko (2018). We note that the robust risk parity model presented in this paper is able to accommodate any generic form of uncertainty around the covariance matrix. Computational results show this robust model is able to improve both portfolio returns and risk-adjusted returns relative to the nominal[1] model. Hereafter, we provide a description of each section in this paper.

Section 2 briefly introduces the nominal risk parity non-linear program (NLP) from Maillard et al (2010). We then propose a novel yet equivalent version of this NLP that reduces its complexity by reformulating it as a quadratically-constrained linear program (QCLP). We justify our preference for this QCLP by conducting a brief computational performance test.

[1]The authors note the use of the word *nominal* throughout this paper in the context of the Operations Research discipline, where it serves to differentiate the basis model from its *robust* counterpart.

In Section 3 we develop an uncertainty set specifically designed to capture the intricacies of risk parity. Unlike traditional MVO portfolios, risk parity is not concerned with minimizing risk. Thus, our focus is not to create an uncertainty set to capture the worst-case estimate of the risk measure, but instead to capture estimation error that would directly impact the risk distribution among the assets. This is achieved by quantifying the uncertainty surrounding the estimated marginal risk contributions. The uncertainty structure can be derived from any estimate of the covariance matrix that has upper and lower bounds. An example of such an uncertainty set is derived from a factor model of asset returns. Proceeding as in Goldfarb and Iyengar (2003), we use the estimation error of the regression coefficients to formulate the upper and lower bounds of the asset covariance matrix.

Section 4 presents the robust risk parity optimization problem. We begin by introducing a second-order cone program (SOCP) reformulation of the nominal problem as in Mausser and Romanko (2014). The rationale for selecting a SOCP formulation is that it allows for a tractable introduction of robustness while still maintaining the same level of complexity as the original problem (i.e., the robust problem is also a SOCP).

The robust instance of most MVO models is attained when the worst-case estimate of the risk measure is assumed (Delage and Ye, 2010; Tütüncü and Koenig, 2004). In the case of portfolios where short positions are not allowed, a robust 'worst-case' variance is attained simply by assuming the upper bound estimate of the covariance matrix. This shields a portfolio whose objective is to minimize risk from estimation error in the risk measure. This same approach can be directly applied to risk parity, as proposed in Kapsos et al (2017), where a robust model is formulated by taking the worst-case estimate of the covariance matrix subject to a risk diversification constraint. The computational results show that this is a reliable method to mitigate the effects of uncertainty.

However, we attempt to deviate from traditional robust MVO techniques that tend to be satisfied when the worst-case estimate of the covariance matrix is assumed. The aforementioned techniques may be unable to fully capture the intricacies behind a risk parity portfolio, where overall portfolio risk is not a concern, but where we seek to correctly estimate the risk contribution per asset in order to allocate wealth accordingly. Thus, we propose a model that relaxes the 'worst-case' variance assumption by focusing not on the overall risk of the portfolio, but rather on the more relevant marginal risk contributions.

A computational experiment is presented in Section 5 to test the performance of the robust risk parity portfolio when compared against both the nominal and worst-case variance portfolios. The results show that our robust SOCP is able to outperform its nominal counterpart, in particular during periods of market distress, by attaining both higher returns and risk-adjusted returns.

The improved performance can be explained by the design of the robust problem. First, this formulation places a restriction on assets with a high level of noise in their marginal risk contribution, thereby reducing the exposure to noisier assets. Second, it implicitly reduces our exposure to assets with larger marginal risk contributions. These two characteristics of our robust risk parity problem ensures the resulting portfolios are shielded against errors in the allocation of risk among its constituents. In general, risk parity yields highly diverse portfolios, where diversification itself can lead to higher portfolio returns (Booth and Fama, 1992). Thus, this robust risk parity portfolio is able to reduce exposure to riskier assets while maintaining a sufficient degree of diversification from a risk contribution perspective.

2 Risk Parity Portfolios

The principle of risk parity portfolios is to determine resource allocation by distributing wealth in such a way that each asset has an equal contribution to the overall portfolio risk. Fundamentally, the concept mirrors the classic '$1/n$' portfolio, where wealth is distributed equally among all assets. For this reason, risk parity portfolios are sometimes referred to as ERC portfolios.

The distribution of wealth by equal risk contribution is entirely dependent on the measure of risk selected. The typical choice is to use portfolio variance, which aligns with the framework of MVO presented in Markowitz (1952). For a portfolio with n assets, the expected return and variance are given by

$$\begin{aligned}\mu_p &= \boldsymbol{\mu}^\top \boldsymbol{x},\\ \sigma_p^2 &= \boldsymbol{x}^\top \boldsymbol{\Sigma} \boldsymbol{x},\end{aligned}$$

where μ_p is the portfolio expected return, $\boldsymbol{\mu} \in \mathbb{R}^n$ is the vector of asset expected returns, $\boldsymbol{x} \in \mathbb{R}^n$ is the vector of asset weights (i.e. the proportion of wealth invested in each asset), σ_p^2 is the portfolio variance, and $\boldsymbol{\Sigma} \in \mathbb{R}^{n\times n}$ is the asset covariance matrix. The individual risk contribution of each asset can be derived by an Euler decomposition of the portfolio standard deviation. As shown in Maillard et al (2010), the risk contribution per asset is given by

$$\begin{aligned}\sigma_p &= \sqrt{\boldsymbol{x}^\top \boldsymbol{\Sigma} \boldsymbol{x}} = \sum_{i=1}^{n} \sigma_i,\\ \sigma_i &= x_i \frac{\partial \sigma_p}{\partial x_i} \quad = x_i \frac{(\boldsymbol{\Sigma}\boldsymbol{x})_i}{\sqrt{\boldsymbol{x}^\top \boldsymbol{\Sigma} \boldsymbol{x}}},\end{aligned}$$

where x_i is the proportion of wealth allocated to asset i (i.e. the weight of asset i), $\partial\sigma_p/\partial x_i$ is the marginal risk contribution of asset i, and σ_i is the individual risk contribution of asset i. The use of any alternative decomposable risk measure is also acceptable. For the remainder of this paper, the sole risk measure we consider is the asset covariance matrix.

It is common to impose a set of restrictions on the weight allocation variable, $\boldsymbol{x}$. These may include allocation limits for certain industries or assets, or cardinality constraints to limit the size of a portfolio. Thus, we consider a convex set $\mathcal{X}$ of acceptable weights such that $x \in \mathcal{X}$. As discussed in Maillard et al (2010), the allowance of short positions in risk parity portfolios may impact the tractability of the model, where multiple solutions may exist for the same basket of assets. On the other hand, long-only portfolios guarantee the existence of a unique solution. This convex set is the following

$$\mathcal{X} = \left\{ \boldsymbol{x} \in \mathbb{R}^n : \ \mathbf{1}^\top \boldsymbol{x} = 1; \ \boldsymbol{x} \geq 0 \right\},$$

where the first restriction ensures that all available wealth is distributed among the assets, and the second restriction enforces the long-only nature of unique risk parity portfolios. We use the notation **1** to denote a column vector of appropriate size where all entries are equal to one.

Additional restrictions may be imposed to formulate a generalized version of the risk parity problem, also known as risk budgeting (Bruder et al, 2012; Kapsos et al, 2017). This method has gained popularity due to the additional benefit of allowing the user to limit its concentration of risk to a particular asset or industry. Ji and Lejeune (2018) propose a stochastic risk budgeting multi-portfolio optimization model that imposes constraints on the marginal risk contribution of each asset, as well as using semi-deviation as the risk measure.

The objective of a nominal risk parity model is to find a portfolio where $\sigma_i = \sigma_j \ \forall\, i, j$. As presented in Maillard et al (2010), this objective can be attained by minimizing the squared differences in risk contribution. Thus, the optimization model can be written as

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} \big(x_i(\boldsymbol{\Sigma x})_i - x_j(\boldsymbol{\Sigma x})_j\big)^2 \\ \text{s.t.} \quad & \boldsymbol{x} \in \mathcal{X}. \end{aligned} \tag{1}$$

The objective function in (1) is a fourth-degree polynomial. A reformulation of this problem is presented in Bai et al (2016), which provides an equivalent yet numerically more efficient version of this problem. However, this formulation maintains the same degree of non-linearity in the objective function. The NLP proposed in Bai et al (2016) can be simplified further and expressed as a QCLP by adding only two auxiliary variables, $2n$ quadratic constraints, and one linear constraint. Although this approach increases the number of constraints, it allows us to reduce the complexity of the objective function. We take this opportunity to propose a novel formulation by which to construct risk parity portfolios,

$$\begin{aligned} \min_{\boldsymbol{x},\theta,\zeta} \quad & \zeta \\ \text{s.t.} \quad & \zeta \geq \theta - x_i(\boldsymbol{\Sigma x})_i, \quad i = 1, ..., n, \\ & \zeta \geq x_i(\boldsymbol{\Sigma x})_i - \theta, \quad i = 1, ..., n, \\ & \boldsymbol{x} \in \mathcal{X}, \end{aligned} \tag{2}$$

where $\theta,\ \zeta \in \mathbb{R}$ are auxiliary variables. As with the original problem in (1), the QCLP in (2) is non-convex but is numerically efficient. A more in depth analysis of the benefits of this approach over equivalent convex formulations can be found in Maillard et al (2010) and Bai et al (2016). We would like to highlight that, at optimality, the value of ζ is zero, tightening the $2n$ quadratic constraints and yielding

$$\begin{aligned} 0 &= \theta - x_i(\boldsymbol{\Sigma x})_i, \quad i = 1, ..., n, \\ x_i(\boldsymbol{\Sigma x})_i &= x_j(\boldsymbol{\Sigma x})_j \qquad \forall\ i,\ j. \end{aligned}$$

A test to compare the performance of these two models was conducted with randomly generated data. We generated 100 independent, symmetric positive definite matrices, each with 200 rows and 200 columns, mimicking large portfolios with 200 assets each. Performance was judged on the basis of computational runtime and on the resulting coefficient of variation (CV) of the risk contributions.

The CV is calculated by dividing the standard deviation of the risk contributions by their average, i.e.,

$$\mathrm{CV} = \frac{\mathrm{SD}(\boldsymbol{x} \odot (\boldsymbol{\Sigma x}))}{\frac{1}{n}\boldsymbol{x}^\top \boldsymbol{\Sigma x}}, \tag{3}$$

where '$\odot$' is the element-wise multiplication operator and SD($\cdot$) would compute the standard deviation of the corresponding vector. In theory, an optimal solution should yield a CV of zero. Table 1 shows the average performance per trial.

Table 1 Average computational performance of nominal risk parity problems

	NLP: Problem (1)	QCLP: Problem (2)
Run-time (s)	65.59	1.81
CV	6.74e-12	8.17e-14

This is a non-exhaustive experiment and does not compare other equivalent risk parity formulations; it merely serves to justify our choice of a nominal risk parity model. A more exhaustive performance comparison between equivalent risk parity formulations is described in Mausser and Romanko (2014).

Beyond our numerical results, we highlight that the benefit of the formulation proposed in (2) is the reduction in complexity of the problem. Thus, this paper will consider the QCLP in (2) as the nominal risk parity model during subsequent computational experiments, and we will use it as a benchmark for comparison against the robust risk parity formulation.

3 Uncertainty Structure

In this section we discuss the uncertainty structure used to construct the robust formulation of the risk parity problem. The only estimated parameter involved in the construction of these portfolios is the measure of risk. Thus, in our case, we are only concerned with uncertainty in the asset covariance matrix. We construct the uncertainty set in a straightforward fashion by placing a set of box constraints on each entry of the covariance matrix Σ_{ij}. The estimation of an uncertainty set based on a factor model of asset returns is discussed later in Section 3.1.

Let $\boldsymbol{\Sigma}$ be the true (but uncertain) asset covariance matrix. To have a well-behaved problem, we typically require that the covariance matrix is positive semi-definite (PSD), i.e., $\boldsymbol{\Sigma} \succeq 0$. Next, we define the set of box constraints as

$$\underline{\Sigma}_{ij} \leq \Sigma_{ij} \leq \overline{\Sigma}_{ij} \quad \forall\ i,\ j$$

where both $\underline{\Sigma}_{ij}$ and $\overline{\Sigma}_{ij}$ are defined by the user depending on their choice of parameter estimation technique and the available raw data. Examples of uncertainty bounds on the covariance matrix can be found in Delage and Ye (2010) and Lobo and Boyd (2000).

For the case of long-only minimum variance portfolios, it has been widely researched that a robust formulation can be achieved simply by taking the upper bound of the covariance uncertainty structure, i.e., simply assuming the worst-case estimate of the covariance matrix (Tütüncü and Koenig, 2004). Thus, setting $\Sigma_{ij} = \overline{\Sigma}_{ij}$ provides robustness when we seek to minimize variance, but it also selects the most extreme corner within the uncertainty set when constructing risk parity portfolios. The objective of risk parity is to equalize the risk contribution per asset, not to minimize risk. Therefore, shielding against estimation errors in the overall portfolio risk is not necessarily relevant in this case.

The set of box constraints can be expressed as the uncertainty set

$$\mathcal{U}_{\Sigma} = \left\{ \boldsymbol{\Sigma} \in \mathbb{R}^{n \times n} : \ \boldsymbol{\Sigma} = \boldsymbol{\Sigma}^0 + \boldsymbol{\Sigma}^{\Delta}, \ \underline{\Sigma}_{ij} \leq (\Sigma^0_{ij} + \Sigma^{\Delta}_{ij}) \leq \overline{\Sigma}_{ij} \right\}, \tag{4}$$

where $\boldsymbol{\Sigma}^0 \in \mathbb{R}^{n \times n}$ is the nominal covariance matrix estimated from raw data and $\boldsymbol{\Sigma}^{\Delta} \in \mathbb{R}^{n \times n}$ is the perturbation on the estimate. Modeling the perturbation in this fashion implicitly assumes that the error term in each element of the covariance matrix is independent.

In the case of where the covariance matrix is bounded by fixed upper and lower limits, a simple and tractable way to size the perturbation term is by proceeding as in Tütüncü and Koenig (2004) and let it equal the difference between the nominal and the worst-case variance, $\boldsymbol{\Sigma}^{\Delta} = \overline{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}^0$. Later on we describe a scenario where the bounds are derived from the standard error arising from the regression coefficients in a factor model.

We seek to create a robust portfolio that will reduce our exposure to assets with a higher degree of error in their estimated risk contribution. The perturbation on the covariance matrix is used to prepare an ellipsoidal uncertainty set around the vector of marginal risk contributions (MRCs), $\partial \sigma_p / \partial x_i = (\boldsymbol{\Sigma x})_i / \sqrt{\boldsymbol{x}^{\top} \boldsymbol{\Sigma x}}$, where we can consider solely the numerator since the denominator is consistent for all assets. The uncertainty set around the MRCs is

$$\begin{aligned} \mathcal{U}_{\mathrm{MRC}} = \big\{ (\boldsymbol{\Sigma x}) \in \mathbb{R}^n : \ (\boldsymbol{\Sigma x})_i = (\boldsymbol{\Sigma}^0 \boldsymbol{x})_i + (\boldsymbol{\Sigma}^{\Delta} \boldsymbol{x})_i, \\ i = 1, ..., n, ||\boldsymbol{\Sigma}^{\Delta} \boldsymbol{x}||_2 \leq \sqrt{n}\ y \big\}. \end{aligned} \tag{5}$$

The size of the error in the MRCs is bounded by some parameter $y \in \mathbb{R}$ scaled by the total number of assets n. Thus, y can be conceptualized as the average error in the MRCs.

3.1 Factor Models

The uncertainty set in (5) can be derived from any generic set of bounds on the covariance matrix. As an example, here we demonstrate how to derive the MRC uncertainty set from a factor model of asset returns. Suppose that the asset returns $\boldsymbol{r} \in \mathbb{R}^n$ can be explained through a combination of m explanatory factors. Thus, by ordinary least squares regression, we have that

$$\boldsymbol{r} = \boldsymbol{\alpha} + \boldsymbol{V}^{\top} \boldsymbol{f} + \boldsymbol{\epsilon}, \tag{6}$$

where $\boldsymbol{\alpha} \in \mathbb{R}^n$ is the vector of regression intercepts, $\boldsymbol{f} \sim \mathcal{N}(\boldsymbol{\phi}, \boldsymbol{F}) \in \mathbb{R}^m$ is the vector of factor returns, $\boldsymbol{V} \in \mathbb{R}^{m \times n}$ is the matrix of factor loadings, and $\boldsymbol{\epsilon} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{D}) \in \mathbb{R}^n$ is the vector of residual returns. $\boldsymbol{F} \in \mathbb{R}^{m \times m}$ and $\boldsymbol{D} \in \mathbb{R}^{n \times n}$

denote the factor covariance matrix and the diagonal matrix of residual variance, respectively.

This model assumes that the residual returns are independent of one another, i.e. $\mathrm{cov}(\epsilon_i, \epsilon_j) = 0$ for $i \neq j$; and it also assumes that the residual returns are independent of the factor returns, i.e. $\mathrm{cov}(\epsilon_i, f_j) = 0 \; \forall \; i, j$. Moreover, we note that this model does not assume the factors are independent of one another, i.e. the factor covariance matrix, $\boldsymbol{F}$, is not required to be a diagonal matrix. However, in practice we may need $\boldsymbol{F} \succeq 0$.

Stemming from the factor model in (6), the asset parameters may be estimated as

$$\begin{aligned} \boldsymbol{\mu} &= \boldsymbol{\alpha} + \boldsymbol{V}^\top \boldsymbol{\phi}, \\ \boldsymbol{\Sigma} &= \boldsymbol{V}^\top \boldsymbol{F} \boldsymbol{V} + \boldsymbol{D}, \end{aligned} \tag{7}$$

where, as before, $\boldsymbol{\mu} \in \mathbb{R}^n$ is the vector of expected returns and $\boldsymbol{\Sigma} \in \mathbb{R}^{n \times n}$ is the covariance matrix. This, in turn, implies $\boldsymbol{r} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

We proceed as in Goldfarb and Iyengar (2003) and assume that the estimated factor covariance matrix $\boldsymbol{F}$ is stable and known exactly. Moreover, the eigenvalues of the residual matrix $\boldsymbol{D}$ are typically much smaller than those of the matrix $\boldsymbol{V}^\top \boldsymbol{F} \boldsymbol{V}$. Thus, the latter is considered a good low-rank approximation to the asset covariance matrix. With this in mind, we focus on developing an uncertainty structure based on the estimation errors surrounding the matrix of factor loadings $\boldsymbol{V}$.

The calculation of the standard error on the factor loadings is described here. Suppose that the raw data of the asset returns and factor returns in (6) are $\{\boldsymbol{r}^t \in \mathbb{R}^n : t = 1, ..., p\}$ and $\{\boldsymbol{f}^t \in \mathbb{R}^m : t = 1, ..., p\}$, respectively, for p observations. The generic linear model can be described by

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{\beta} + \boldsymbol{\epsilon},$$

where

$$\begin{aligned} \boldsymbol{y} &= [\boldsymbol{r}^1 \; \boldsymbol{r}^2 \; ... \; \boldsymbol{r}^p]^\top \in \mathbb{R}^{p \times n}, & \boldsymbol{B} &= [\boldsymbol{f}^1 \; \boldsymbol{f}^2 \; ... \; \boldsymbol{f}^p] \in \mathbb{R}^{m \times p}, \\ \boldsymbol{A} &= [\boldsymbol{1} \; \boldsymbol{B}^\top] \in \mathbb{R}^{p \times (m+1)}, & \boldsymbol{\beta} &= [\boldsymbol{\alpha} \; \boldsymbol{V}^\top]^\top \in \mathbb{R}^{(m+1) \times n}. \end{aligned}$$

If the matrix $\boldsymbol{A}$ has full column rank $m+1$, then the least squares estimates $\widehat{\boldsymbol{\beta}}$ of the true parameter $\boldsymbol{\beta}$ is obtained by

$$\widehat{\boldsymbol{\beta}} = (\boldsymbol{A}^\top \boldsymbol{A})^{-1} \boldsymbol{A}^\top \boldsymbol{y}.$$

For a single asset i, the error term on its estimated regression coefficients $\widehat{\boldsymbol{\beta}}_i \in \mathbb{R}^{(m+1)}$ is

$$\widehat{\boldsymbol{\beta}}_i - \boldsymbol{\beta}_i = (\boldsymbol{A}^\top \boldsymbol{A})^{-1} \boldsymbol{A}^\top \boldsymbol{\epsilon}_i,$$

where the variance of the error is given by $\sigma^2_{\epsilon_i} (\boldsymbol{A}^\top \boldsymbol{A})^{-1}$. The term $\sigma^2_{\epsilon_i}$ is the true variance of the residual corresponding to asset i, i.e., it corresponds to the diagonal entries of the matrix $\boldsymbol{D}$. The unbiased estimate of $\sigma^2_{\epsilon_i}$ is then found by

$$s_i^2 = \frac{||\boldsymbol{y}_i - \boldsymbol{A}\widehat{\boldsymbol{\beta}}_i||_2^2}{p - m - 1},$$

where $||\cdot||_2$ is the Euclidean norm operator. Finally, the variance of the regression coefficients corresponding to asset i is

$$\mathrm{Var}(\widehat{\boldsymbol{\beta}}_i) = s_i^2(\boldsymbol{A}^\top \boldsymbol{A})^{-1},$$

where the standard error of the factor loadings can be easily computed by finding the square root of the diagonal of $\mathrm{Var}(\widehat{\boldsymbol{\beta}}_i)$.

An uncertainty set on the factor loadings can be constructed by using the standard errors to bound the perturbation. Let $\boldsymbol{V}$ be the true (but uncertain) matrix of factor loadings, and let it belong to the uncertainty set

$$\mathcal{U}_V = \left\{ \boldsymbol{V} \in \mathbb{R}^{m\times n} :\ \boldsymbol{V} = \boldsymbol{V}^0 + \boldsymbol{V}^\Delta, -\mathrm{SE}(V_{ij}^0) \leq V_{ij}^\Delta \leq \mathrm{SE}(V_{ij}^0) \right\} \tag{8}$$

where $\boldsymbol{V}^0 \in \mathbb{R}^{m\times n}$ is the least squares estimate of $\boldsymbol{V}$, and $\mathrm{SE}(V_{ij}^0)$ denotes the standard error of the estimated factor loading of factor i corresponding to asset j. Estimating a worst-case covariance matrix from a factor model with uncertain parameters is a difficult process. A closed-form solution is not possible due to the nature of the factor covariance matrix $\boldsymbol{F}$, where positive and negative correlation between the factors may influence the size and direction of the perturbation in order to attain a worst-case variance. We approach this issue by formulating a simple mathematical program to find the estimate of the factor loadings that maximizes the sum of all elements in the covariance matrix under the constraints given by $\mathcal{U}_V$,

$$\boldsymbol{V}^* = \underset{\boldsymbol{V}\in\ \mathcal{U}_V}{\arg\max} \quad \boldsymbol{1}^\top(\boldsymbol{V}^\top \boldsymbol{F}\boldsymbol{V} + \boldsymbol{D})\boldsymbol{1}.$$

We can then use the nominal and worst-case estimates of the covariance matrix to recover the corresponding perturbation $\boldsymbol{\Sigma}^\Delta$, which we can then use to construct the MRC uncertainty set

$$\boldsymbol{V}^{*\top}\boldsymbol{F}\boldsymbol{V}^* + \boldsymbol{D} = \overline{\boldsymbol{\Sigma}} \quad \Rightarrow \quad \boldsymbol{\Sigma}^\Delta = \overline{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}^0.$$

4 Robust Risk Parity Portfolios

In this section we use the uncertainty set of the MRCs in (5) to construct a robust risk parity optimization model. To accommodate the uncertainty set while still maintaining the tractability of the original model, we introduce a second-order cone program (SOCP) reformulation of the risk parity problem.

As shown in Mausser and Romanko (2014), the nominal risk parity problem can be written as a SOCP as follows

$$\min_{\boldsymbol{x},\boldsymbol{z},t,p} \quad p - t \tag{9a}$$

$$\text{s.t.} \quad z_i = (\boldsymbol{\Sigma}\boldsymbol{x})_i, \quad i = 1, ..., n \tag{9b}$$

$$\left|\left| \begin{bmatrix} 2t \\ x_i - z_i \end{bmatrix} \right|\right|_2 \leq x_i + z_i, \quad i = 1, ..., n \tag{9c}$$

$$\left|\left| (\boldsymbol{\Sigma})^{1/2}\boldsymbol{x} \right|\right|_2 \leq \sqrt{n}\ p, \tag{9d}$$

$$\boldsymbol{z},\ p,\ t \geq 0, \tag{9e}$$

$$\boldsymbol{x} \in \mathcal{X}, \tag{9f}$$

where $p,\ t \in \mathbb{R}$, and $\boldsymbol{z} \in \mathbb{R}^n$ are auxiliary variables, and the constraint (9c) is equivalent to the hyperbolic constraint $x_i z_i \geq t^2$ for $x_i \geq 0$ and $z_i \geq 0$. The objective function in (9a) is equivalent to

$$\sqrt{\frac{\boldsymbol{x}^\top \boldsymbol{\Sigma} \boldsymbol{x}}{n}} - \sqrt{\min_{1 \leq i \leq n} \{x_i (\boldsymbol{\Sigma} \boldsymbol{x})_i\}},$$

where the square roots are simply a construct of the SOCP reformulation. By design, the objective function is zero at optimiality and otherwise positive. Thus, optimality is attained when the smallest risk contribution is equal to the average risk contribution of the portfolio.

We construct a robust counterpart to (9) by focusing on the marginal risk contribution per asset, $\partial \sigma_p / \partial x_i = (\boldsymbol{\Sigma} \boldsymbol{x})_i / \sqrt{\boldsymbol{x}^\top \boldsymbol{\Sigma} \boldsymbol{x}}$, where we can ignore the denominator since it is the same for all assets and focus solely on the numerator.

Robustness can be introduced by realizing that constraint (9b) can be relaxed to $z_i \leq (\boldsymbol{\Sigma} \boldsymbol{x})_i$, as it will become tight at optimality. The relaxation of this constraint allows us to proceed in a similar fashion to Bertsimas and Sim (2006), where we can introduce the uncertainty set into this constraint as an error term to penalize the MRCs. The robust risk parity problem is as follows,

$$\begin{aligned}
\min_{\boldsymbol{x}, \boldsymbol{z}, t, p, y} \quad & p - t && \text{(10a)} \\
\text{s.t.} \quad & \left\| \boldsymbol{\Sigma}^{\Delta} \boldsymbol{x} \right\|_2 \leq \sqrt{n}\, y, && \text{(10b)} \\
& \Omega y \leq (\ \boldsymbol{\Sigma}^0 \boldsymbol{x}\ )_i - z_i, \quad i = 1, ..., n && \text{(10c)} \\
& \left\| \begin{bmatrix} 2t \\ x_i - z_i \end{bmatrix} \right\|_2 \leq x_i + z_i, \quad i = 1, ..., n && \text{(10d)} \\
& \left\| (\boldsymbol{\Sigma}^0)^{1/2}\ \boldsymbol{x} \right\|_2 \leq \sqrt{n}\, p, && \text{(10e)} \\
& \boldsymbol{z},\ p,\ t \geq 0, && \text{(10f)} \\
& \boldsymbol{x} \in \mathcal{X}. && \text{(10g)}
\end{aligned}$$

Constraint (10b) introduces a new auxiliary variable $y \in \mathbb{R}$ that serves to model the average error in the MRC as a second-order cone constraint (SOCC). This error term is then introduced into constraint (10c) as a penalty on the equivalence constraint between each MRC and its corresponding variable z_i.

Conceptually, the robust problem will attempt to reduce the size of the error term in constraint (10c) as it becomes tight. This will not only diminish our exposure to assets with larger estimation error in their MRCs, but it also yields a set of error-adjusted MRCs in the form of the vector of variables $\boldsymbol{z}$. The subsequent attempt to equalize risk contributions is based on $\boldsymbol{z}$, implicitly reducing our exposure to assets with larger MRCs and further shielding the portfolio against riskier assets.

Introducing robustness in this fashion is preferred for two reasons. First, introducing individual error terms for the MRC of each asset would not be able to capture the intricacies of the risk contributions, where the MRC of asset i is not only dependent on itself, but also on its interaction with asset j as measured by their covariance $\sigma_{ij}\ \forall\ j \neq i$. Thus, formulating this as a SOCC ensures these dependencies are captured. Second, this formulation allows us to model the error term as a single SOCC, thereby avoiding a convoluted set of constraints that would otherwise increase the computational complexity of the problem.

Constraint (10c) also introduces the parameter $\Omega \in \mathbb{R}_+$, which serves to scale the sensitivity of the MRCs to the error term. A larger value of Ω corresponds to a more conservative outlook. The parameter Ω can be sized to provide a probabilistic guarantee on the results as shown in Bertsimas and Sim (2006). However, we prefer to proceed with a data-driven approach to determine an appropriate value of Ω, which is based on the relative size between our nominal covariance matrix $\boldsymbol{\Sigma}^0$ and its corresponding perturbation $\boldsymbol{\Sigma}^\Delta$,

$$\Omega = \ \omega \ \frac{||\boldsymbol{\Sigma}^\Delta||_F}{||\boldsymbol{\Sigma}^0||_F},$$

where $||\cdot||_F$ is the Frobenius norm operator, and $\omega \in \mathbb{R}_+$ is a positive parameter that scales Ω based on the ratio of the nominal covariance matrix to its perturbation. Intuitively, setting $\omega = 0$ will revert the robust SOCP to its nominal version. On the other hand, an excessively large value of ω would be prohibitively conservative, rendering the problem infeasible. Thus, to guarantee feasibility, we must have that

$$0 \le \omega < \sqrt{n} \cdot \frac{(\boldsymbol{\Sigma}^0\boldsymbol{x})_i}{||\boldsymbol{\Sigma}^\Delta\boldsymbol{x}||_2} \cdot \frac{||\boldsymbol{\Sigma}^0||_F}{||\boldsymbol{\Sigma}^\Delta||_F}, \quad i = 1, ..., n.$$

The value of ω can be used to determine the level of robustness according to the risk appetite of the user. In other words, ω can be described as a risk aversion parameter. The subsequent computational experiment will test robust portfolios with different values of ω.

5 Computational Experiment

In this section we perform a computational experiment designed to test the out-of-sample performance of the proposed robust risk parity SOCP. The covariance matrix and its uncertainty set are built through a factor model. The Fama–French three-factor model (Fama and French, 1993) was selected as the basis of the experiment to show how a well-known multi-factor model would behave under the proposed robust framework.

An explanation of the experimental setup follows. The experiment considers a pool of 74 U.S. stocks belonging to the S&P 500 index, with stocks from each of the eleven *Global Industry Classification Standard* (GICS) sectors. A list of these stocks is presented in Table 2. The historical stock prices were obtained from Quandl.com (2017). The Fama–French three-factor model is used to estimate the covariance matrix, with the historical factor returns obtained from Professor Kenneth R. French's website (French, 2016). The experiment is performed using weekly historical data from 01-Jan-1995 to 31-Dec-2016, with the first five-year period used to perform the initial model calibration (i.e., the out-of-sample period begins in 01-Jan-2000). Thereafter, the portfolios are rebalanced every six months, re-calibrating the parameters using data from the preceding five-year period. The length of this time series introduces a survivorship bias to our results as we only consider stocks with sufficient historical data. However, we expect this bias to have a similar effect on all portfolios. Thus, it is the relative performance between these that is of main interest.

The experiment consists of 300 independent trials, where a sample set of 15 assets is drawn at random each time. During each trial, the 15 assets are used to construct optimal portfolios under three different models (the constituents are the same for all portfolios per trial). First, we consider a portfolio using the nominal covariance estimate $\boldsymbol{\Sigma}^0$ and apply it to the QCLP in (2). The second portfolio is constructed by assuming the 'worst-case' variance $\overline{\boldsymbol{\Sigma}}$ also using the QCLP in (2). Finally, we construct four examples of robust portfolios under the SOCP in (10). Each of the robust portfolios corresponds to increasingly conservative outlooks (i.e., increasing values of the risk aversion parameter ω).

Table 2 List of assets

GICS Sector	**Company Tickers**						
Consumer Disc.	DIS	F	GPS	GT	HAS	MCD	NKE
Consumer Staples	CL	KO	KR	MO	PEP	PG	WMT
Financials	AFL	AON	AXP	BAC	C	JPM	WFC
Healthcare	BMY	BIIB	HUM	JNJ	LLY	MRK	PFE
Industrials	BA	CAT	CSX	FDX	GE	LMT	MMM
Energy	COP	CVX	HAL	MRO	OXY	SLB	XOM
Information Tech.	AAPL	CSCO	HPQ	IBM	ORCL	QCOM	TXN
Materials	DD	DOW	IP	MOS	NEM	PPG	PX
Utilities	AEP	CNP	D	DTE	DUK	ED	ETR
Telecommunications	CTL	S	T	VZ			
Real Estate	HCP	HST	KIM	REG	SPG	UDR	WY

This experiment was performed on an Apple MacBook Pro computer (2.4 GHz Intel Core i5, 8 GB 1600 MHz DDR3 RAM) running OS X 'Sierra'. All computations were performed using the Julia programming language (version 0.6.1) using the optimization modeling language JuMP (Dunning et al, 2017) with IPOPT (version 3.12.6) as the optimization solver for the QCLP and MOSEK (version 8.0.1.30) as the solver for the SOCP.

The portfolio performance, measured by its wealth evolution, is presented in Figure 1. Here we can see how the wealth of three of our portfolios evolved through time. The top plot shows the average wealth of the nominal, 'worst-case' variance, and robust risk parity portfolios. The average wealth is calculated by taking the mean of the 300 trials at each discrete point in time corresponding to each portfolio. This instance of the robust portfolio is constructed with $\omega = 2.5$. The error bars display the standard deviation corresponding to the distribution of the 300 trials. The bottom plot shows the wealth of the 'worst-case' variance and robust portfolios relative to the nominal, i.e., $(W_i^t / W_{\text{nom}}^t - 1) \times 100$ for the wealth W of portfolio i at each point in time t.

The results show that our robust portfolio behaves as would be expected, outperforming the nominal during bear market periods. This is exemplified by the behaviour observed during the early 2000s as well as the spike seen during the

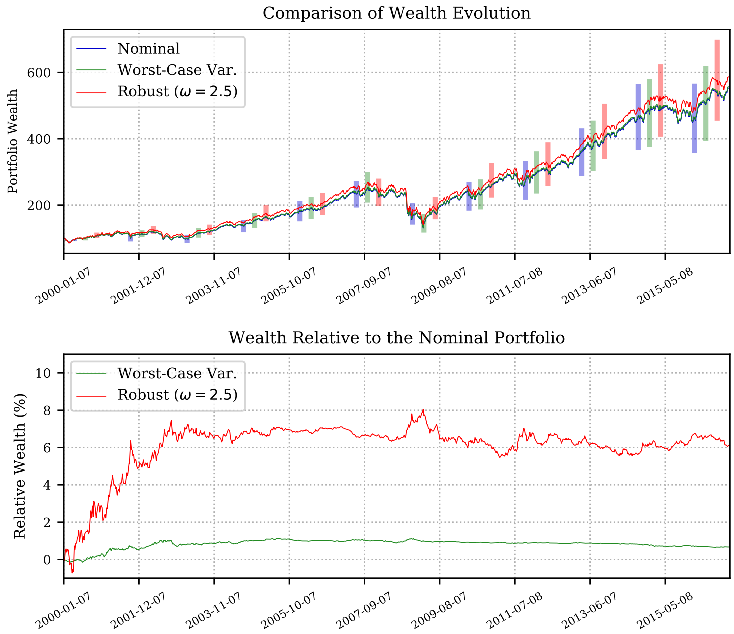

Fig. 1 Wealth evolution of 300 individual trials with 15 assets per portfolio over the period 2000–2016. Top: Average wealth of portfolios with the standard deviation over the 300 test runs shown as error bars. Bottom: Average wealth relative to the nominal portfolio.

recent financial crisis of 2008. Moreover, the robust portfolio appears to be able to maintain its relative advantage throughout bull market periods as well. On the other hand, the ‘worst-case’ variance portfolio closely mimics the nominal, having a marginally better performance but not displaying a meaningful behaviour during different market cycles.

Next, we evaluate the performance of robust portfolios for different values of ω. Figure 2 shows the average wealth evolution of four different robust portfolios relative to the nominal. The plot of the robust portfolio with $\omega = 2.5$ is the same as in Figure 1.

The results suggest that taking an increasingly conservative stance is able to significantly improve the relative performance of the robust portfolios. A robust portfolio with $\omega = 3.5$ is able to attain a peak advantage of over thirteen percent over the nominal during the financial crisis of 2008. It is worthwhile to note that the upward trend seen during bearish periods does not necessarily imply that the robust portfolios had a positive return, but rather that they were better shielded against the losses observed by the nominal portfolio.

A summary of results is presented in Table 3. The average yearly returns are calculated from the 300 trials throughout the entire out-of-sample test period. The ex-post Sharpe ratio (Sharpe, 1994) corresponds to the realized portfolio excess returns adjusted by their observed volatility, computed for each individual trial throughout the entire investment horizon. The average Sharpe Ratio of the 300 trials is provided in the table. The turnover rate considers the average period-over-period absolute change in the asset weights per trial, with the average of

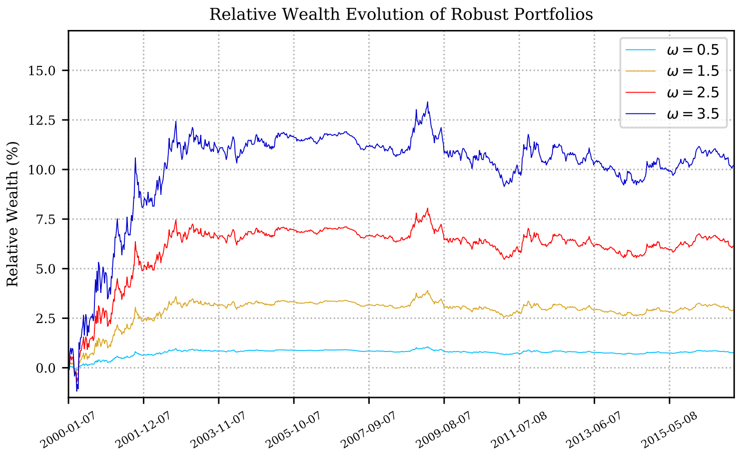

Fig. 2 Average wealth evolution of robust portfolios relative to the nominal portfolio for several values of ω.

the 300 trials given in the table. The turnover rate serves to exemplify potential transaction costs.

We also provide the CV of the risk contributions, calculated in the same fashion as in (3). The CV serves to explain the compromise between equally partitioning the portfolio risk and reducing our exposure to assets with noisy MRCs. The CV is calculated per rebalance period for each individual trial, and for each of the six portfolios. In all cases we compute the CV relative to the nominal risk measure $\boldsymbol{\Sigma}^0$. An idealized risk parity portfolio should have a CV of zero. The CV given in this table is the average of all trials for all rebalance periods per portfolio.

Table 3 Summary of results

	Nominal	Worst-Case Var.	Robust			
			$\omega = 0.5$	$\omega = 1.5$	$\omega = 2.5$	$\omega = 3.5$
Yearly Return	0.1054	0.1058	0.1059	0.1073	0.1093	0.1118
Sharpe Ratio	0.0805	0.0808	0.0812	0.0829	0.0853	0.0880
Turnover	0.0450	0.0442	0.0474	0.0540	0.0641	0.0797
CV	2e-11	0.0384	0.0221	0.0784	0.1593	0.2744

From Table 3 we can see that, on average, robust portfolios not only have higher returns, but are also able to maintain or reduce their volatility, yielding higher risk-adjusted returns relative to the nominal portfolio. The improved performance can be attributed to a reduced exposure to riskier assets with large estimation errors, which tend to drive portfolio losses. Mitigating portfolio losses yields an improved long-run rate of return while maintaining a lower level of volatility. Moreover, we are able preserve a sufficient degree of diversity from a risk parity perspective.

The improved performance comes at the cost of a higher turnover rate, presumably increasing the cost of management. Nevertheless, it is worthwhile to note that nominal risk parity portfolios are generally very stable in their period-over-period wealth reallocation. Thus, turnover rates are traditionally very low when compared to a broader range of optimal portfolios. Unsurprisingly, the CV of robust portfolios tends to increase as we become more conservative. It is not possible to attain an equal risk contribution per asset as we limit our exposure to the assets with noisier and larger MRCs. Nevertheless, the CV of the risk contributions is generally low even as ω is increased, exemplifying that these robust portfolios are still able to attain a diverse portfolio from a risk contribution perspective.

6 Conclusion

In this paper we introduced a robust formulation of a risk parity portfolio. Unlike MVO portfolios, we do not seek to minimize risk. Traditional techniques in portfolio optimization, where robustness is guaranteed by taking the 'worst-case' estimate of the risk measure, are not applicable when our objective is to have an equalized risk contribution per asset. Instead, due to the nature of our objective, we are most concerned with correctly estimating the assets' risk contributions. Our proposed framework was developed specifically to address uncertainty in risk parity. Thus, by design, this robust problem seeks to shield the portfolio from overexposure to assets with a high degree of estimation error in their MRCs, as well as reduce our exposure to riskier assets with larger MRCs. Intuitively, we define an uncertainty set on the vector of MRCs to attempt to reduce our exposure to these assets. This method is able to attain both a higher rate of return and a higher risk-adjusted rate of return, both of which can be attributed to a reduced exposure to riskier assets, which tend to drive portfolio losses.

Moreover, the proposed robust formulation relies on the insertion of a single new constraint upon the original SOCP, which, by design, maintains the same level of complexity as the original problem. In other words, our robust SOCP is equally as tractable and efficient as the original risk parity problem.

The uncertainty set on the MRCs can be derived from any generic set of error bounds on the covariance matrix, and we showed how to derive the uncertainty set from the standard error of regression coefficients from a factor model of asset returns. Factor models are more difficult to handle, requiring an additional level of computational cost to create the uncertainty structure. Nevertheless, we are able to show that it is possible to use a factor model to construct a viable robust portfolio, as shown in the computational experiment in which we used the Fama–French three-factor model.

The experimental results show that robust risk parity portfolios are able to outperform their nominal counterpart throughout a long investment horizon. As would be expected, the robust portfolios were able to realize a significant advantage during periods of market distress. More surprisingly, the robust portfolios were able to maintain its relative advantage during bullish periods. Robust portfolios with an increasing risk aversion ω were tested to visualize how this user-defined parameter affects the resulting optimal portfolios. Aside from an improved rate of return, robust portfolios are also able to attain a higher-than-nominal risk-adjusted rate of return, which is explained by our reduced exposure to noisy assets. However, this enhanced performance comes at the cost of a higher-than-nominal turnover rate, which, depending on context, may be acceptable to a user.

This robust formulation opens the door to additional applications, such as using it to construct robust portfolios under a risk budgeting approach (Bruder et al, 2012; Haugh et al, 2017; Kapsos et al, 2017). This could be the subject of future research.

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