Mechanical & Industrial Engineering
UNIVERSITY OF TORONTO

Generalized Risk Parity Portfolio Optimization: An ADMM Approach

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What is this presentation about?

- We wish to construct an optimal portfolio

  $\Rightarrow$ High return

  $\Rightarrow$ Low risk

  $\Rightarrow$ Well-diversified

  $\Rightarrow$ Flexibility for the investor

Introduction

What is this presentation about?

- We wish to construct an optimal portfolio

  $\Rightarrow$ High return

  $\Rightarrow$ Low risk

  $\Rightarrow$ Well-diversified

  $\Rightarrow$ Flexibility for the investor

- Meeting these criteria is **difficult**

  $\Rightarrow$ It may lead to non-convex problems

Introduction

Could we design a model that:

- **Meets the investor's criteria**
- **Addresses non-convexity**

Portfolio Optimization

Assets and Portfolios

- An asset i has some expected return μ_i and variance (risk) σ_i^2
- A portfolio $\boldsymbol{x} \in \mathbb{R}^n$ is a collection of n financial assets

  $\Rightarrow$ x_i is our weight on asset i

- The relevant measures of risk and return are

Assets	Portfolio
$\Rightarrow$ **Return**: $\boldsymbol{\mu} \in \mathbb{R}^n$	$\Rightarrow$ **Return**: $\mu_p = \boldsymbol{\mu}^T \boldsymbol{x}$
$\Rightarrow$ **Risk**: $\boldsymbol{\Sigma} \in \mathbb{R}^{n \times n}$	$\Rightarrow$ **Risk**: $\sigma_p^2 = \boldsymbol{x}^T \boldsymbol{\Sigma} \boldsymbol{x}$

Portfolio Optimization

Mean–Variance Optimization (MVO)

- Introduced by Markowitz (1952),

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{x}^T \boldsymbol{\Sigma} \boldsymbol{x} - \lambda \boldsymbol{\mu}^T \boldsymbol{x} && \text{Min. risk and max. return} \\ \text{s.t.} \quad & \mathbf{1}^T \boldsymbol{x} = 1 && \text{Budget constraint} \\ & (\boldsymbol{x} \geq 0) && \text{Disallow short sales (optional)} \end{aligned}$$

- Weaknesses:

  $\Rightarrow$ May lead to over-concentrated portfolios

  $\Rightarrow$ The estimated parameter μ is very **noisy**

What is risk parity?

- Risk parity seeks to find portfolios based on a risk-weighted basis
- Does not require estimated returns as an input, improving stability
- Each asset contributes the same level of risk
- The resulting portfolio is well-diversified

Measuring the risk contribution per asset

- Decompose the portfolio variance

$$\sigma_p^2 = \boldsymbol{x}^T \boldsymbol{\Sigma} \boldsymbol{x} = \sum_{i=1}^{n} x_i (\boldsymbol{\Sigma} \boldsymbol{x})_i$$

$\Rightarrow$ $x_i(\boldsymbol{\Sigma}\boldsymbol{x})_i$ is the individual **risk contribution** of asset i

Measuring the risk contribution per asset

- Decompose the portfolio variance

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$\Rightarrow x_i(\boldsymbol{\Sigma}\boldsymbol{x})_i$ is the individual risk contribution of asset i

- **Risk parity**: Take a least-squares approach

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \sum_{i=1}^{n} \sum_{j=1}^{n} \left(x_i (\boldsymbol{\Sigma}\boldsymbol{x})_i - x_j (\boldsymbol{\Sigma}\boldsymbol{x})_j \right)^2 \\ \text{s.t.} \quad & \mathbf{1}^T \boldsymbol{x} = 1 \\ & \boldsymbol{x} \geq \mathbf{0} \end{aligned}$$

Non-convexity of risk parity

- **Problem**: The objective is non-convex

  $\Rightarrow$ In standard quadratic notation: $x_i(\boldsymbol{\Sigma}\boldsymbol{x})_i = \boldsymbol{x}^T \boldsymbol{A}_i \boldsymbol{x}$

  $\Rightarrow$ $\boldsymbol{A}_i \in \mathbb{R}^{n \times n}$ captures the individual risk contribution of asset i

  $\Rightarrow$ The matrices $\boldsymbol{A}_i$ are indefinite

Risk Parity

Non-convexity of risk parity

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  ⇒ $\boldsymbol{A}_i \in \mathbb{R}^{n \times n}$ captures the individual risk contribution of asset i

  ⇒ The matrices $\boldsymbol{A}_i$ are indefinite

- **Solution**: Disallow short selling

  ⇒ This limits the investor's possibilities

Formulate a new optimization problem

Formulate a new optimization problem

- We seek a desirable portfolio

  $\Rightarrow$ Minimize risk and maximize return

  $\Rightarrow$ Risk-based diversification

  $\Rightarrow$ Short selling allowed

Generalized Risk Parity

Formulate a new optimization problem

- We seek a desirable portfolio

  ⇒ Minimize risk and maximize return

  ⇒ Risk-based diversification

  ⇒ Short selling allowed

- Generalized Risk Parity (GRP)

$$
\begin{aligned}
\min_{\boldsymbol{x},\ \theta} \quad & \boldsymbol{x}^T \boldsymbol{\Sigma} \boldsymbol{x} - \lambda \boldsymbol{\mu}^T \boldsymbol{x} \\
\text{s.t.} \quad & (1+c)\theta - \boldsymbol{x}^T \boldsymbol{A}_i \boldsymbol{x} \geq 0, \quad i = 1, ..., n \\
& \boldsymbol{x}^T \boldsymbol{A}_i \boldsymbol{x} - (1-c)\theta \geq 0, \quad i = 1, ..., n \\
& \mathbf{1}^T \boldsymbol{x} = 1
\end{aligned}
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Generalized Risk Parity

A closer look at the risk diversification constraints

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(1+c)\theta - \boldsymbol{x}^T \boldsymbol{A}_i \boldsymbol{x} &\geq 0, \quad i = 1, ..., n \\
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- $\theta \in \mathbb{R}$ is an auxiliary variable
- $c \in \mathbb{R}_+$ is a user-defined risk diversification parameter

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- $\theta \in \mathbb{R}$ is an auxiliary variable.
- $c \in \mathbb{R}_+$ is a user-defined risk diversification parameter.

  $\Rightarrow c = 0$ enforces perfect risk parity

  $\Rightarrow c > 1$ reverts to MVO

Adding robustness to the portfolio return

- μ is a **noisy** estimate

Generalized Risk Parity

Adding robustness to the portfolio return

- μ is a **noisy** estimate

- Robust GRP

$$
\begin{aligned}
\min_{\boldsymbol{x},\ \theta} \quad & \boldsymbol{x}^T \boldsymbol{\Sigma} \boldsymbol{x} - \lambda \left(\boldsymbol{\mu}^T \boldsymbol{x} - \omega \|\boldsymbol{\Omega}^{1/2} \boldsymbol{x}\|_2 \right) \\
\text{s.t.} \quad & (1+c)\theta - \boldsymbol{x}^T \boldsymbol{A}_i \boldsymbol{x} \geq 0, \quad i = 1, ..., n \\
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- $\boldsymbol{\Omega} \in \mathbb{R}^{n \times n}$ and $\omega \in \mathbb{R}_+$ quantify the estimation error around $\boldsymbol{\mu}$

The issue of non-convexity still remains

$$(1+c)\theta - \boldsymbol{x}^T \boldsymbol{A}_i \boldsymbol{x} \geq 0, \quad i = 1, ..., n$$
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- **Relax the problem into a Semidefinite Program (SDP)**

SDP Relaxation

Semidefinite relaxation

- Introduce a new variable $\boldsymbol{X} \in \mathbb{R}^{n\times n}$

  $\Rightarrow$ **Non-convex**: $\boldsymbol{X} = \boldsymbol{x}\boldsymbol{x}^T$

SDP Relaxation

Semidefinite relaxation

- Introduce a new variable $\boldsymbol{X} \in \mathbb{R}^{n\times n}$

  $\Rightarrow$ Non-convex: $\boldsymbol{X} = \boldsymbol{x}\boldsymbol{x}^T$

  $\Rightarrow$ **Convex relaxation**: $\boldsymbol{X} \succeq \boldsymbol{x}\boldsymbol{x}^T$

$$\begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1n} \\ X_{21} & X_{22} & \cdots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & \cdots & X_{nn} \end{bmatrix} \succeq \begin{bmatrix} x_1^2 & x_1x_2 & \cdots & x_1x_n \\ x_2x_1 & x_2^2 & \cdots & x_2x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_nx_1 & x_nx_2 & \cdots & x_n^2 \end{bmatrix}$$

SDP Relaxation

Semidefinite relaxation of the problem

- Introduce a new variable $\boldsymbol{X} \in \mathbb{R}^{n \times n}$

  $\Rightarrow$ Non-convex: $\boldsymbol{X} = \boldsymbol{x}\boldsymbol{x}^T$

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- Taking the Schur complement, $\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{X} & \boldsymbol{x} \\ \boldsymbol{x}^{\boldsymbol{T}} & 1 \end{bmatrix} \succeq 0$.

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- Taking the Schur complement, $\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{X} & \boldsymbol{x} \\ \boldsymbol{x}^{\boldsymbol{T}} & 1 \end{bmatrix} \succeq 0$.

- Align the input parameters with the dimensions of $\boldsymbol{Y}$

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{\Sigma} & -\frac{\lambda}{2}\boldsymbol{\mu} \\ -\frac{\lambda}{2}\boldsymbol{\mu}^{\boldsymbol{T}} & \boldsymbol{0} \end{bmatrix}, \quad \boldsymbol{B_i} = \begin{bmatrix} \boldsymbol{A}_i & \boldsymbol{0} \\ \boldsymbol{0}^{\boldsymbol{T}} & 0 \end{bmatrix} \text{ for } i = 1, ..., n.$$

SDP Relaxation

Relax the GRP model into a SDP

Original $\Rightarrow$

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SDP $\Rightarrow$

$$
\begin{aligned}
\min_{\boldsymbol{Y},\ \theta} \quad & \mathrm{Tr}(\boldsymbol{Q}\boldsymbol{Y}) + \lambda\omega \|\boldsymbol{\Omega}^{1/2} \mathbf{Y}_{1:n,n+1}\|_2 \\
\text{s.t.} \quad & (1+c)\theta - \mathrm{Tr}(\boldsymbol{B}_i \boldsymbol{Y}) \geq 0, \quad i = 1, ..., n \\
& \mathrm{Tr}(\boldsymbol{B}_i \boldsymbol{Y}) - (1-c)\theta \geq 0, \quad i = 1, ..., n \\
& \sum_{i=1}^{n} Y_{i,n+1} = 1 \\
& Y_{n+1,n+1} = 1 \\
& \boldsymbol{Y} \succeq 0
\end{aligned}
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SDP Relaxation

Relax the GRP model into a SDP

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$$\begin{aligned} \min_{\boldsymbol{x},\,\theta} \quad & \boldsymbol{x}^T \boldsymbol{\Sigma} \boldsymbol{x} - \lambda \left(\boldsymbol{\mu}^T \boldsymbol{x} - \omega \|\boldsymbol{\Omega}^{1/2} \boldsymbol{x}\|_2 \right) \\ \text{s.t.} \quad & (1+c)\theta - \boldsymbol{x}^T \boldsymbol{A}_i \boldsymbol{x} \geq 0, \quad i = 1, \ldots, n \\ & \boldsymbol{x}^T \boldsymbol{A}_i \boldsymbol{x} - (1-c)\theta \geq 0, \quad i = 1, \ldots, n \\ & \mathbf{1}^T \boldsymbol{x} = 1 \end{aligned}$$

SDP $\Rightarrow$

$$\begin{aligned} \min_{\boldsymbol{Y},\,\theta} \quad & \underbrace{\mathrm{Tr}(\boldsymbol{Q}\boldsymbol{Y}) + \lambda\omega \|\boldsymbol{\Omega}^{1/2} \mathbf{Y}_{1:n,n+1}\|_2}_{f(\boldsymbol{Y})} \\ \text{s.t.} \quad & \left. \begin{aligned} & (1+c)\theta - \mathrm{Tr}(\boldsymbol{B}_i \boldsymbol{Y}) \geq 0, \quad i = 1, \ldots, n \\ & \mathrm{Tr}(\boldsymbol{B}_i \boldsymbol{Y}) - (1-c)\theta \geq 0, \quad i = 1, \ldots, n \\ & \sum_{i=1}^{n} Y_{i,n+1} = 1 \\ & Y_{n+1,n+1} = 1 \\ & \boldsymbol{Y} \succeq 0 \end{aligned} \right\} \mathcal{S} \end{aligned}$$

SDP Relaxation

How do we solve the original problem?

- Recall $\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{X} & \boldsymbol{x} \\ \boldsymbol{x}^T & 1 \end{bmatrix} \succeq \mathbf{0}$
- If $\boldsymbol{X} = \boldsymbol{x}\boldsymbol{x}^T$, we recover the original problem

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- This is equivalent to a rank-1 constraint

$$\boldsymbol{X} = \boldsymbol{x}\boldsymbol{x}^T \iff \operatorname{rank}(\boldsymbol{Y}) = 1$$

SDP Relaxation

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$$\boldsymbol{X} = \boldsymbol{x}\boldsymbol{x}^T \iff \operatorname{rank}(\boldsymbol{Y}) = 1$$

- **Idea: Approximate the rank-1 condition**

Approximate the rank-1 condition

- Use the Alternating Direction Method of Multipliers (ADMM)
- Transfer the rank-1 requirement to a new variable $\boldsymbol{Z}$

$\Rightarrow \boldsymbol{Z} = \boldsymbol{Y}$

$\Rightarrow \mathrm{rank}(\boldsymbol{Z}) = 1$

Sequential Tightening

Approximate the rank-1 condition

- Use the Alternating Direction Method of Multipliers (ADMM)
- Transfer the rank-1 requirement to a new variable Z

  $\Rightarrow \boldsymbol{Z} = \boldsymbol{Y}$

  $\Rightarrow \operatorname{rank}(\boldsymbol{Z}) = 1$

- Reformulate as an Augmented Lagrangian

$$L(\boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{\Lambda}) = f(\boldsymbol{Y}) + \frac{\rho}{2} \left\| \boldsymbol{Y} - \left(\boldsymbol{Z} - \frac{1}{\rho} \boldsymbol{\Lambda} \right) \right\|_F^2$$

  $\Rightarrow \rho \in \mathbb{R}_+$ is a tuning parameter

  $\Rightarrow \boldsymbol{\Lambda} \in \mathbb{R}^{(n+1)\times(n+1)}$ is the dual variable of the constraint $\boldsymbol{Z} = \boldsymbol{Y}$

ADMM algorithm

- Iterate through the steps:

  1) Convex $\boldsymbol{Y}$-minimization: $$\boldsymbol{Y}^{k+1} = \underset{\boldsymbol{Y},\theta \,\in\, \mathcal{S}}{\operatorname{argmin}} \, L(\boldsymbol{Y}, \boldsymbol{Z}^k, \boldsymbol{\Lambda}^k)$$

Sequential Tightening

ADMM algorithm

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  $\mathbf{1})$ Convex $\boldsymbol{Y}$-minimization: $\quad \boldsymbol{Y}^{k+1} = \underset{\boldsymbol{Y},\theta\, \in\, \mathcal{S}}{\operatorname{argmin}}\, L(\boldsymbol{Y}, \boldsymbol{Z}^k, \boldsymbol{\Lambda}^k)$

  $\mathbf{2})$ Non-convex $\boldsymbol{Z}$-minimization: $\quad \boldsymbol{Z}^{k+1} = \underset{\operatorname{rank}(\boldsymbol{Z})=1}{\operatorname{argmin}}\, L(\boldsymbol{Y}^{k+1}, \boldsymbol{Z}, \boldsymbol{\Lambda}^k)$

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  2) Non-convex $\boldsymbol{Z}$-minimization: $\boldsymbol{Z}^{k+1} = \underset{\operatorname{rank}(\boldsymbol{Z})=1}{\operatorname{argmin}}\, L(\boldsymbol{Y}^{k+1}, \boldsymbol{Z}, \boldsymbol{\Lambda}^k)$

  3) Dual variable $\boldsymbol{\Lambda}$-update: $\boldsymbol{\Lambda}^{k+1} = \boldsymbol{\Lambda}^k + \rho(\boldsymbol{Y}^{k+1} - \boldsymbol{Z}^{k+1})$

Sequential Tightening

ADMM algorithm

- Iterate through the steps:

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- We can efficiently solve the **non-convex** step

Sequential Tightening

ADMM algorithm

- Iterate through the steps:

$$\begin{aligned} &1)\ \text{Convex } \boldsymbol{Y}\text{-minimization:} && \boldsymbol{Y}^{k+1} = \underset{\boldsymbol{Y},\theta\,\in\,\mathcal{S}}{\operatorname{argmin}}\, L(\boldsymbol{Y}, \boldsymbol{Z}^k, \boldsymbol{\Lambda}^k) \\ &2)\ \text{Non-convex } \boldsymbol{Z}\text{-minimization:} && \boldsymbol{Z}^{k+1} = \underset{\operatorname{rank}(\boldsymbol{Z})=1}{\operatorname{argmin}}\, L(\boldsymbol{Y}^{k+1}, \boldsymbol{Z}, \boldsymbol{\Lambda}^k) \\ &3)\ \text{Dual variable } \Lambda\text{-update:} && \boldsymbol{\Lambda}^{k+1} = \boldsymbol{\Lambda}^k + \rho(\boldsymbol{Y}^{k+1} - \boldsymbol{Z}^{k+1}) \end{aligned}$$

- We can efficiently solve the **non-convex** step

$\Rightarrow$ Singular Value Decomposition of $\boldsymbol{Y}^{k+1} + \frac{1}{\rho}\boldsymbol{\Lambda}^k$

$$\boldsymbol{Z}^{k+1} = s_1 \boldsymbol{v}_1 \boldsymbol{v}_1^T,$$

$\Rightarrow$ $s_1 \in \mathbb{R}$ and $\boldsymbol{v}_1 \in \mathbb{R}^{n+1}$ are the top singular value and vector

Sequential Tightening

ADMM algorithm

- As we iterate, we close the distance between $\boldsymbol{Y}$ and $\boldsymbol{Z}$
- $\boldsymbol{Z}$ is the closest rank-1 approximation of $\boldsymbol{Y}$

Sequential Tightening

ADMM algorithm

- As we iterate, we close the distance between Y and Z
- Z is the closest rank-1 approximation of Y
- At convergence, $\mathrm{rank}(\boldsymbol{Y}) = \mathrm{rank}(\boldsymbol{Z}) = 1$

  $\Rightarrow \boldsymbol{X} = \boldsymbol{x}\boldsymbol{x}^T$

  $\Rightarrow$ We solve the original problem

Sequential Tightening

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- As we iterate, we close the distance between $\boldsymbol{Y}$ and $\boldsymbol{Z}$
- $\boldsymbol{Z}$ is the closest rank-1 approximation of $\boldsymbol{Y}$
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- By tightening the lower bound, we can attain feasibility

  $\Rightarrow$ We converge to a highly quality local optimum

Numerical Experiments

Experimental setup

- Two experiments with $n = 33$ and $n = 50$
- Data
  - $\Rightarrow$ U.S. stocks belonging to the S&P 500 index
  - $\Rightarrow$ Weekly rates of return from 01-Jan-2007 to 31-Dec-2009

Numerical Experiments

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- Two experiments with $n = 33$ and $n = 50$
- Data
  - ⇒ U.S. stocks belonging to the S&P 500 index
  - ⇒ Weekly rates of return from 01-Jan-2007 to 31-Dec-2009
- Competing models
  - ⇒ Robust MVO
  - ⇒ SDP relaxation of GRP
  - ⇒ ADMM
  - ⇒ Non-convex GRP
  - ⇒ Non-convex GRP (warm)

Numerical Experiments

Points to keep in mind

- Measures of performance

  $\Rightarrow$ Objective value

  $\Rightarrow$ Coefficient of variation of the asset risk contributions

  $\Rightarrow$ Ex-post "c" parameter

  $\Rightarrow$ Runtime

Numerical Experiments

Points to keep in mind

- Measures of performance

  $\Rightarrow$ **Objective value** $\}$ Lower is better

  $\Rightarrow$ Coefficient of variation of the asset risk contributions

  $\Rightarrow$ Ex-post "c" parameter

  $\Rightarrow$ Runtime

- **Note**: We ignore the Augmented Lagrangian terms

Numerical Experiments

Points to keep in mind

- Measures of performance
  - $\Rightarrow$ Objective value
  - $\Rightarrow$ **Coefficient of Variation (CV)** $\}$ Lower = more diversification
  - $\Rightarrow$ Ex-post "c" parameter
  - $\Rightarrow$ Runtime

Numerical Experiments

Points to keep in mind

- Measures of performance

  $\Rightarrow$ Objective value

  $\Rightarrow$ Coefficient of Variation (CV)

  $\Rightarrow$ **Ex-post "c" parameter** $\}$ should approximate the user-defined "c"

  $\Rightarrow$ Runtime

- **Note**: c is the percentage value the asset risk contributions can deviate from their midpoint

Numerical Experiments

Points to keep in mind

- Measures of performance

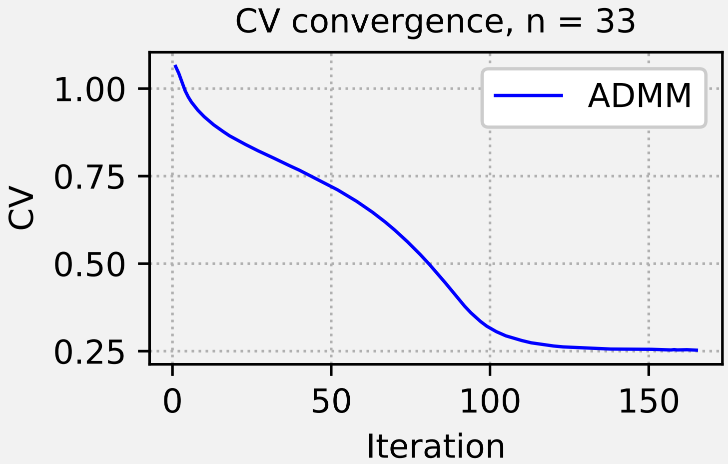
  $\Rightarrow$ Objective value

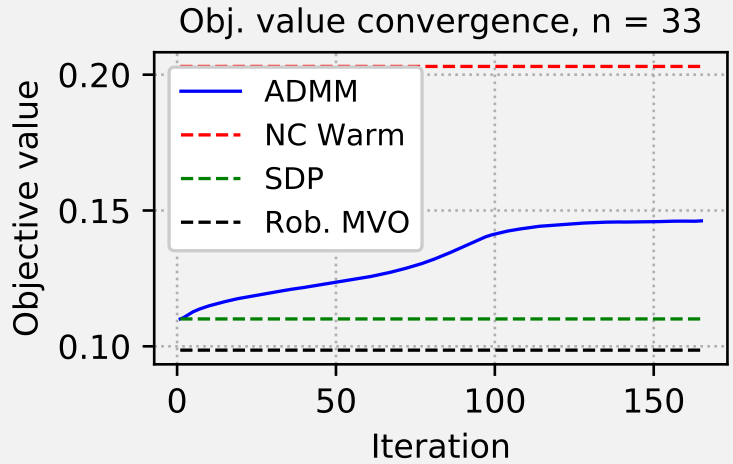
  $\Rightarrow$ Coefficient of Variation (CV)

  $\Rightarrow$ Ex-post "c" parameter

  $\Rightarrow$ **Runtime** $\}$ Lower is better

Coefficient of variation and objective value convergence





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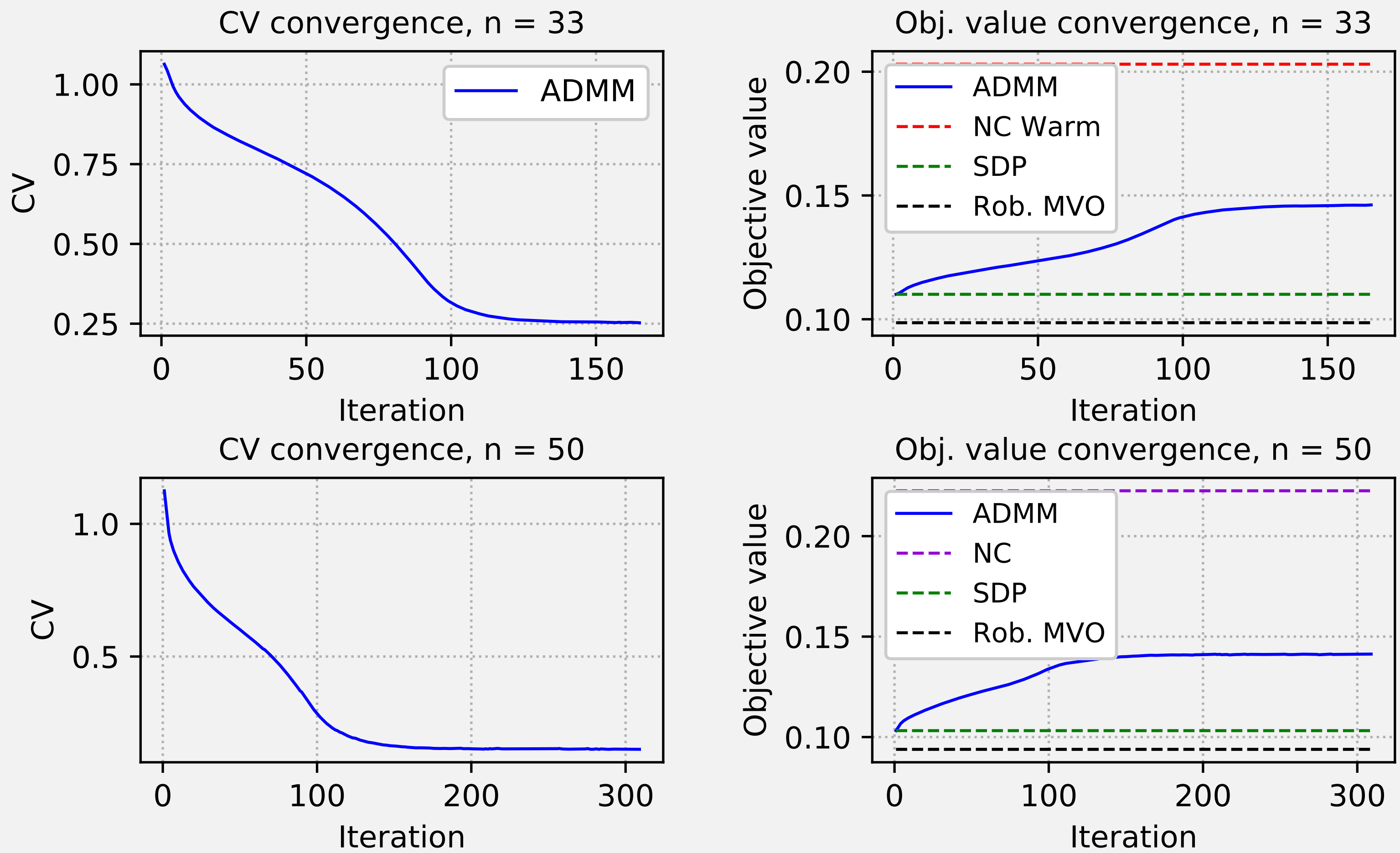

Summary of results

	$n = 33$, $c = 0.25$, $\lambda = 0.1$				
	MVO	SDP	Non-Convex	NC (Warm)	**ADMM**
Obj. Value	0.10	0.11	0.536	0.203	0.146
CV	1.72	1.06	0.258	0.253	0.253
Ex-post c	1.87	11.9	0.250	0.250	0.259
Runtime (sec)	0.03	0.03	0.049	0.061	103.1

	$n = 50$, $c = 0.15$, $\lambda = 0.1$				
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Obj. Value	0.09	0.10	0.223	0.259	0.141
CV	1.75	1.12	0.150	0.150	0.150
Ex-post c	1.99	6.93	0.151	0.151	0.155
Runtime (sec)	0.04	0.25	0.101	0.103	1,376.3

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	$n = 33$, $c = 0.25$, $\lambda = 0.1$				
	MVO	SDP	Non-Convex	NC (Warm)	**ADMM**
Obj. Value	0.10	0.11	0.536	0.203	0.146
CV	1.72	1.06	0.258	0.253	0.253
Ex-post c	1.87	11.9	0.250	0.250	0.259
Runtime (sec)	0.03	0.03	0.049	**0.061**	**103.1**

	$n = 50$, $c = 0.15$, $\lambda = 0.1$				
	MVO	SDP	Non-Convex	NC (Warm)	**ADMM**
Obj. Value	0.09	0.10	0.223	0.259	0.141
CV	1.75	1.12	0.150	0.150	0.150
Ex-post c	1.99	6.93	0.151	0.151	0.155
Runtime (sec)	0.04	0.25	**0.101**	0.103	**1,376.3**

Conclusion

What was our goal?

- We wanted to address

  $\Rightarrow$ Risk–return profile

  $\Rightarrow$ Risk-based diversification

  $\Rightarrow$ Short selling (flexibility)

Conclusion

What was our goal?

- We wanted to address

  $\Rightarrow$ Risk–return profile

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  $\Rightarrow$ Short selling (flexibility)

- Meeting these criteria is difficult

  $\Rightarrow$ We have a **non-convex** problem

Conclusion

Our contribution

- **Proposed a generalized risk parity model**
- **Addressed the non-convexity of risk parity**

  $\Rightarrow$ **Imposed a rank-1 constraint via ADMM**