Mechanical & Industrial Engineering
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A regime-switching framework for portfolio optimization

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Introduction

What is this presentation about?

- Markets follow a cyclical behaviour - alternating between bullish and bearish states.
- Can we hedge against these cycles?

  $\Rightarrow$ To do so, we must identify the parameters governing these cycles
- We will explore a typical asset allocation problem that incorporates the cyclical behaviour of markets via a simple machine learning algorithm.

Introduction

What tools will we need?

- Portfolio Optimization: We will use mean–variance optimization and risk parity for asset allocation.
- Hidden Markov Models: This will be used to incorporate information about the market state into our portfolio.
- Factor Models: This will serve to explain the rate of returns of our financial assets through simple linear regression.

Portfolio Optimization

What are optimal portfolios?

- We wish to select the best combination of assets (portfolio), as outlined by our objective.
- However, we are constrained by our budget, as well as other investment considerations.
- We wish to find this optimal portfolio out of all the possible portfolios being considered.

  $\Rightarrow$ Formulate an optimization problem!

Portfolio Optimization

Mean–Variance Optimization (MVO)

- We wish to have a portfolio with an expected rate of return R
- What is the lowest level of risk we can observe?

$$\begin{aligned} \min_{x} \quad & \boldsymbol{x}^T \boldsymbol{\Sigma} \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{\mu}^T \boldsymbol{x} \geq R, \\ & \mathbf{1}^T \boldsymbol{x} = 1, \end{aligned}$$

Portfolio Optimization

Mean–Variance Optimization (MVO)

- We wish to have a portfolio with an expected rate of return R

- What is the lowest level of risk we can observe?

$$
\begin{aligned}
\min_{x} \quad & \boldsymbol{x}^T \boldsymbol{\Sigma} \boldsymbol{x} && \Big\} \quad \text{Portfolio Risk} \\
\text{s.t.} \quad & \boldsymbol{\mu}^T \boldsymbol{x} \geq R && \Big\} \quad \text{Target Return} \\
& \mathbf{1}^T \boldsymbol{x} = 1 && \Big\} \quad \text{Budget}
\end{aligned}
$$

where

$\Rightarrow$ $\boldsymbol{x} \in \mathbb{R}^n$ is the vector of asset weights,

$\Rightarrow$ $\boldsymbol{\Sigma} \in \mathbb{R}^{n \times n}$ is the asset covariance matrix,

$\Rightarrow$ $\boldsymbol{\mu} \in \mathbb{R}^n$ are the asset expected returns,

Portfolio Optimization

Risk Parity

- Risk parity seeks to find portfolios based on a risk-weighted basis.
- Does not require estimated returns as an input, improving stability.
- Each asset should have the same risk contribution.
- By design, the resulting portfolio is well-diversified.

Portfolio Optimization

Decomposing the risk measure

- Our risk measure is the portfolio standard deviation: $\sigma_p = \sqrt{\boldsymbol{x}^T \boldsymbol{\Sigma} \boldsymbol{x}}$

- We can partition it to find individual contributions:

$$\sigma_p = \sqrt{\boldsymbol{x}^\top \boldsymbol{\Sigma} \boldsymbol{x}} = \sum_{i=1}^{n} RC_i,$$

$$RC_i = x_i \frac{\partial \sigma_p}{\partial x_i} = x_i \frac{(\boldsymbol{\Sigma} \boldsymbol{x})_i}{\sqrt{\boldsymbol{x}^\top \Sigma \boldsymbol{x}}},$$

$\Rightarrow$ RC_i is the risk contribution of asset i.

Portfolio Optimization

Finding risk parity portfolios

- We can achieve risk parity by taking a least-squares approach

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \sum_{i=1}^{n} \sum_{j=1}^{n} \left(x_i (\boldsymbol{\Sigma x})_i - x_j (\boldsymbol{\Sigma x})_j \right)^2 \\ \text{s.t.} \quad & \mathbf{1}^\top x = 1, \\ & x \geq 0. \end{aligned}$$

- If short selling is not allowed, we are guaranteed to find a unique solution.

Hidden Markov Models

What are Hidden Markov Models (HMMs)?

- What if we believe that asset returns are governed by hidden market regimes?
- Could we use observed market returns to discern between these regimes?
- We can encode regime-dependent information into our optimization inputs.

  $\Rightarrow$ Regime-dependent asset expected returns and covariance matrix.

Factor Models

Typical factor model of asset returns

$$\boldsymbol{r}_t = \boldsymbol{\mu} + \boldsymbol{V}^T \boldsymbol{f}_t + \boldsymbol{\epsilon}_t$$

- $\boldsymbol{r}_t \in \mathbb{R}^n$ are the returns of the n assets.
- $\boldsymbol{\mu} \in \mathbb{R}^n$ are the asset expected returns.
- $\boldsymbol{f}_t \sim \mathcal{N}(\mathbf{0}, \boldsymbol{F}) \in \mathbb{R}^m$ are the **centered** factors.
- $\boldsymbol{V} \in \mathbb{R}^{m \times n}$ is the matrix of factor loadings.
- $\boldsymbol{\epsilon}_t \sim \mathcal{N}(\mathbf{0}, \boldsymbol{D}) \in \mathbb{R}^n$ are the residuals.

Factor Models

Typical factor model of asset returns

- Expected returns:

$$\mathbb{E}[\boldsymbol{r}] = \boldsymbol{\mu} \quad \text{(the intercept of regression!)}$$

- Covariance matrix:

$$\mathrm{Cov}(r) = \boldsymbol{\Sigma} = \boldsymbol{V}^T \boldsymbol{F} \boldsymbol{V} + \boldsymbol{D}$$

Regime-Switching Factor Model

Two-state factor model of asset returns

$$\boldsymbol{r}_t = I_{1t}(\boldsymbol{\mu}_1 + \boldsymbol{V}_1^T \boldsymbol{f}_{1t} + \boldsymbol{\epsilon}_{1t}) + I_{2t}(\boldsymbol{\mu}_2 + \boldsymbol{V}_2^T \boldsymbol{f}_{2t} + \boldsymbol{\epsilon}_{2t})$$

- $\boldsymbol{r}_t \in \mathbb{R}^n$ are the returns of the n assets.
- $I_{it} = 1$ when the current market state $s_t = i$ and $I_{it} = 0$ otherwise.
- $\boldsymbol{\mu}_i \in \mathbb{R}^n$ are the regime-dependent asset expected returns.
- $\boldsymbol{f}_{it} \sim \mathcal{N}(\mathbf{0}_i, \boldsymbol{F}_i) \in \mathbb{R}^m$ are the regime-switching factors.
- $\boldsymbol{V}_i \in \mathbb{R}^{m \times n}$ is the matrix of regime-dependent factor loadings.
- $\boldsymbol{\epsilon}_{it} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{D}_i) \in \mathbb{R}^n$ are the regime-dependent residuals.

Regime-Switching Factor Model

Derive the regime-dependent portfolio parameters:

- Expected returns:

$$\mathbb{E}[r \mid s_t = i] = \boldsymbol{\mu}_{s_i} = \gamma_{i1}\ \boldsymbol{\mu}_1 + \gamma_{i2}\ \boldsymbol{\mu}_2$$

- Covariance matrix:

$$\begin{aligned}\boldsymbol{\Sigma}_{s_i} = {} & \gamma_{i1}(\boldsymbol{V}_1^T \boldsymbol{F}_1 \boldsymbol{V}_1 + \boldsymbol{D}_1) + \gamma_{i2}(\boldsymbol{V}_2^T \boldsymbol{F}_2 \boldsymbol{V}_2 + \boldsymbol{D}_2) \\ & + \gamma_{i1}(1 - \gamma_{i1})\boldsymbol{\mu}_1\boldsymbol{\mu}_1^T + \gamma_{i2}(1 - \gamma_{i2})\boldsymbol{\mu}_2\boldsymbol{\mu}_2^T - \gamma_{i1}\gamma_{i2}(\boldsymbol{\mu}_1\boldsymbol{\mu}_2^T + \boldsymbol{\mu}_2\boldsymbol{\mu}_1^T).\end{aligned}$$

$\Rightarrow$ γ_{ij} is the probability of transitioning from state i to state j,

$$\gamma_{ij} = \mathbb{E}[I_j \mid s_t = i]$$

Key Findings

This is a summary of the key findings

- We seek to create portfolios adapted to the current market environment.
- We can do this by calculating regime-dependent parameters through a regime-switching factor model.
- Using these parameters during optimization implicitly capture the cyclical nature of the market.

Computational Experiments

Experimental setup

- We use the Fama–French 3-factor model as the basis.
- We used historical asset data to estimate our parameters.
- We have **three** optimization models and **three** investment strategies

$\Rightarrow$ MVO	$\Rightarrow$ Nominal
$\Rightarrow$ Minimum variance	$\Rightarrow$ Robust
$\Rightarrow$ Risk parity	$\Rightarrow$ Regime-switching

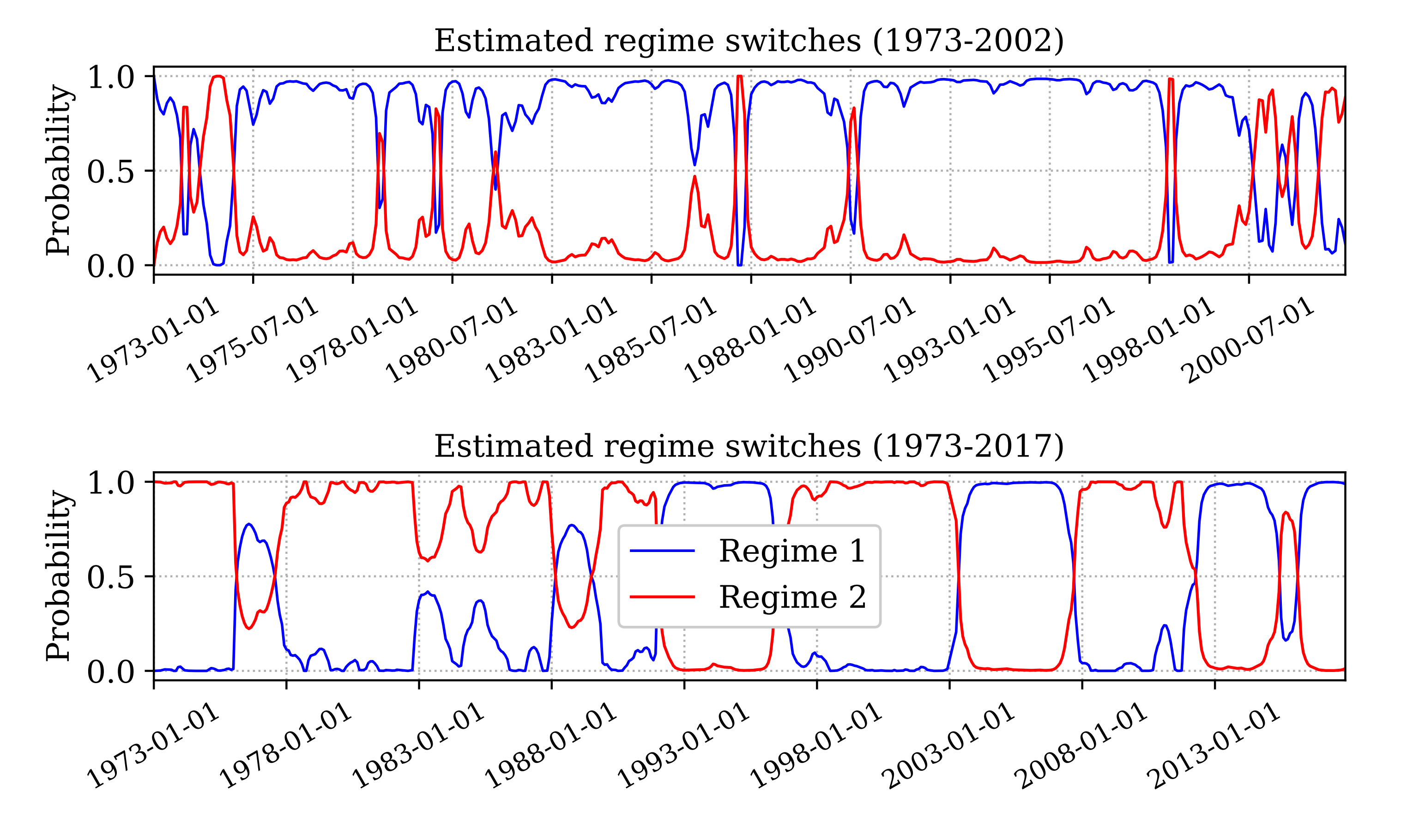
Computational Experiments

Experimental setup

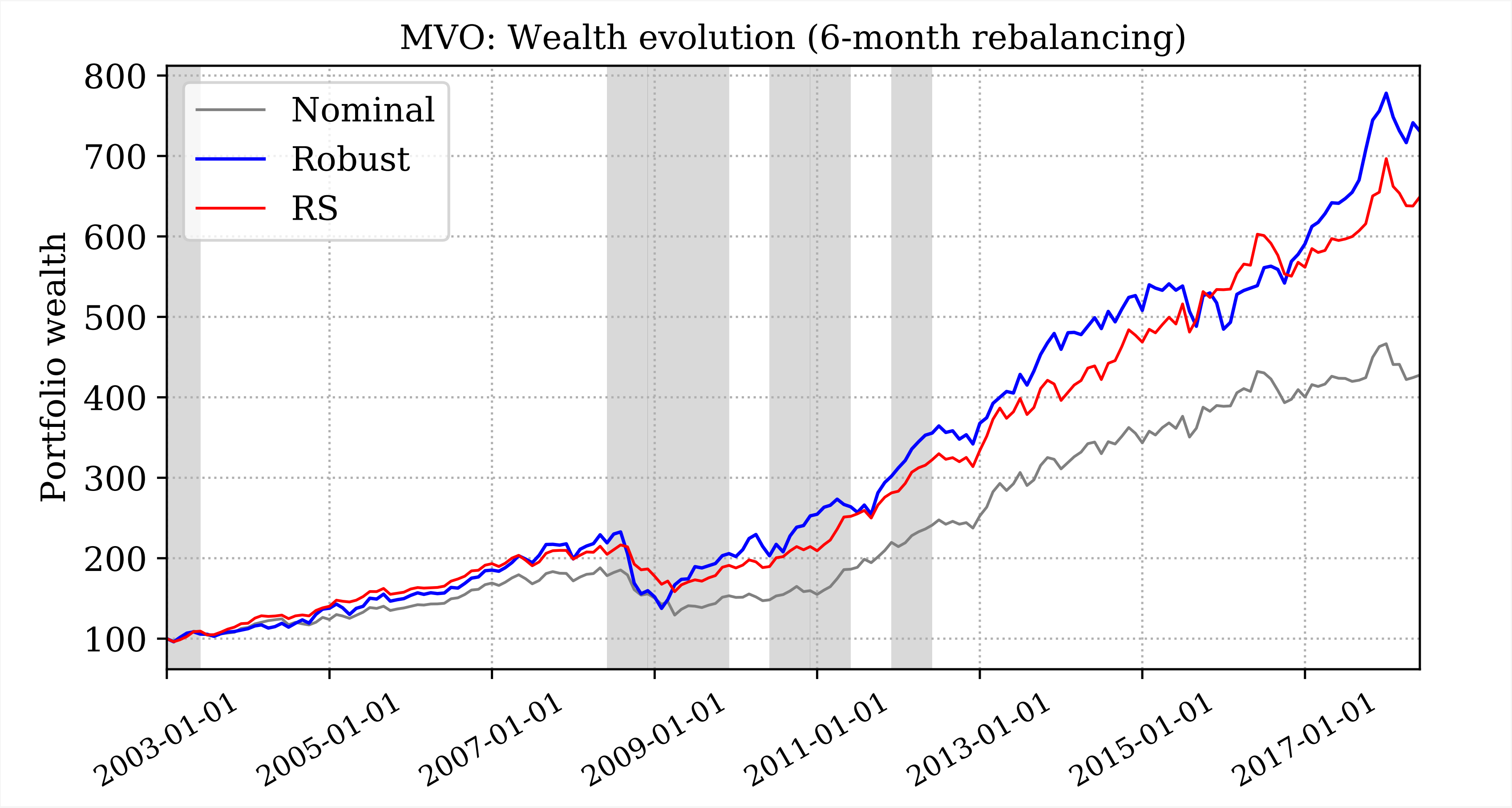
- We have 9 different portfolios (three per optimization model).
- We rebalance our portfolios every 6 months.
- Each time we rebalance, we re-calibrate our factor model and re-estimate the expected returns and covariance matrix.

Computational Experiments

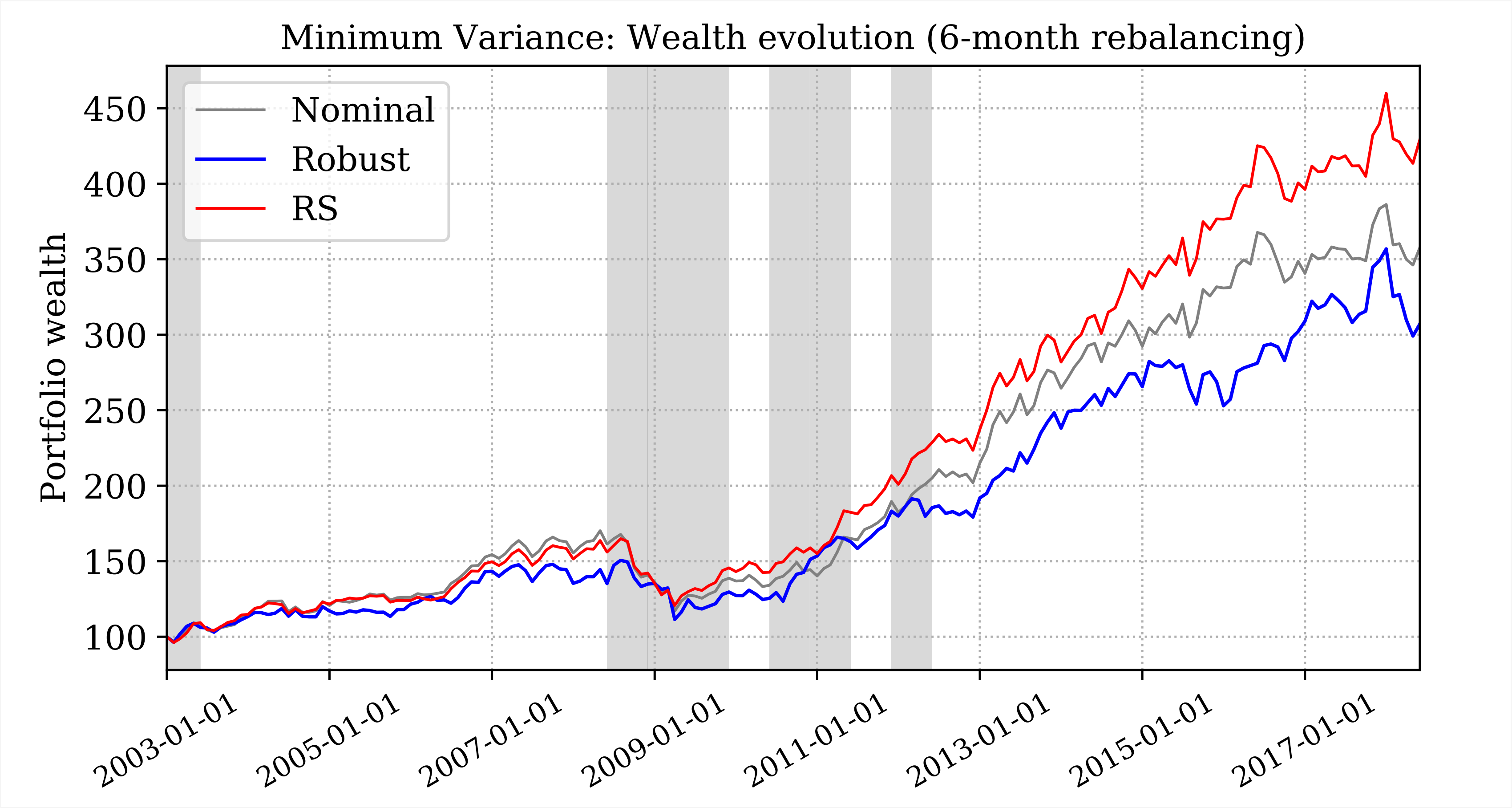
Example of estimated regime changes for the out-of-sample test



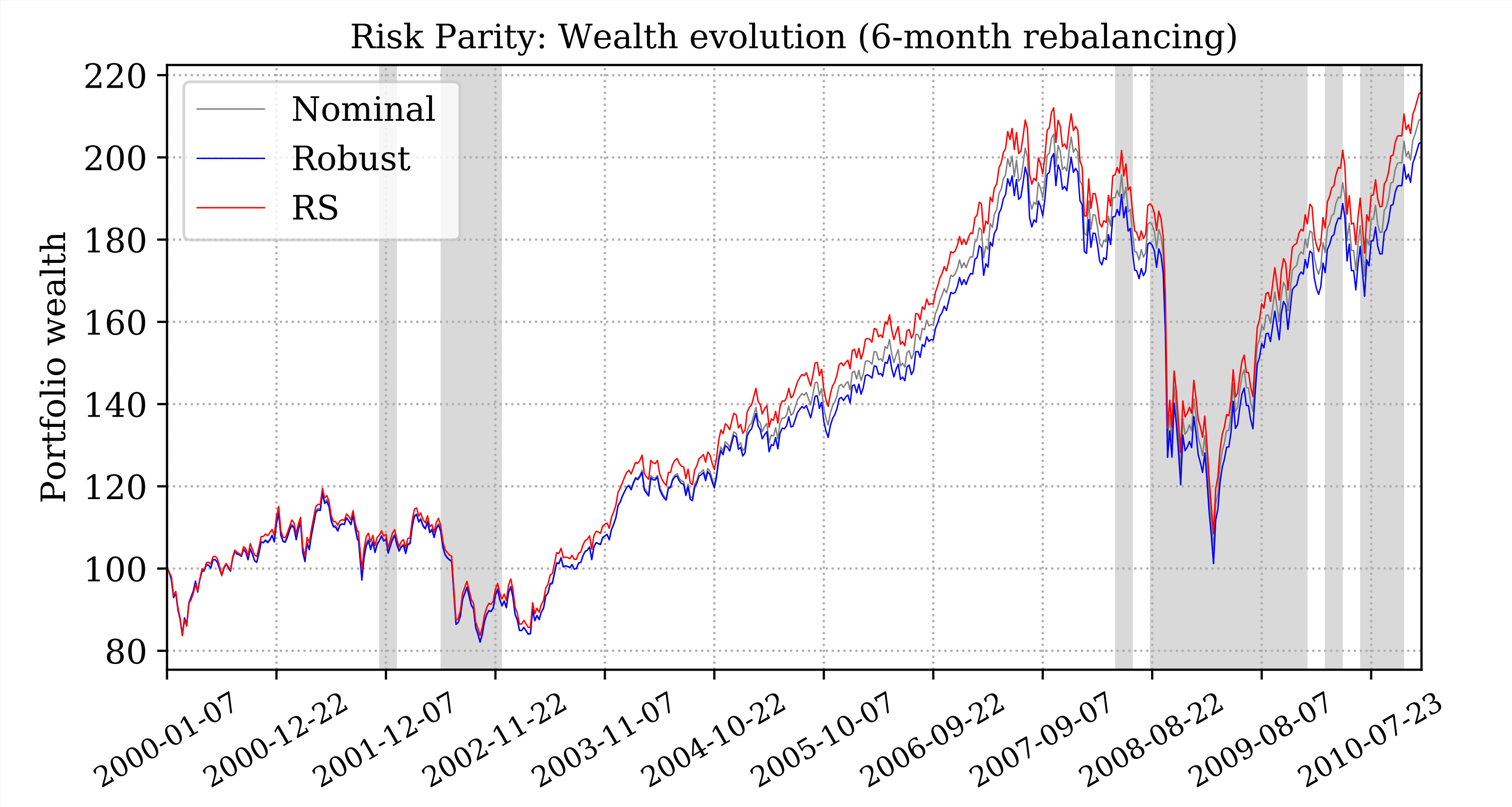
Computational Experiments: MVO



Computational Experiments: Minimum Variance



Computational Experiments: Risk Parity



Computational Experiments

Summary of Results

	MVO			Minimum Variance			Risk Parity		
	Nominal	Robust	**RS**	Nominal	Robust	**RS**	Nominal	Robust	**RS**
Ann. Return	0.099	0.138	**0.129**	0.086	0.075	**0.099**	0.070	0.069	**0.072**
Sharpe Ratio	0.900	0.983	**1.211**	0.791	0.633	**0.950**	0.511	0.488	**0.530**
Avg. Turnover	1.602	1.880	**1.698**	1.355	1.622	**1.376**	0.022	0.083	**0.072**

Conclusion

- We use a data-driven approach that can be built onto a generic factor model.
- A regime-switching factor model encodes the cyclical behaviour of the market into our estimated parameters:

  $\Rightarrow$ Regime-dependent expected returns,

  $\Rightarrow$ Regime-dependent covariance matrix.

- The resulting portfolios are tailored to the current market environment.
- We can attain higher risk-adjusted returns.