Mechanical & Industrial Engineering
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Data-driven distributionally robust risk parity portfolio optimization

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What is this presentation about?

- We will discuss optimal asset allocation strategies.
- In particular, we discuss a strategy known as risk parity.
- We rely on estimated inputs, making us susceptible to estimation error.
- We introduce distributional robustness to mitigate the impact of uncertainty.
- The user is able to define their preferred statistical measure for ‘robustness’.
- We present a novel algorithm to solve the resulting problem.

Optimal asset allocation

- **Mean–variance optimization (MVO)**: Construct optimal portfolios as a trade-off between risk and expected return.

- However, MVO is susceptible to estimation errors in both parametric measures of return and risk.
    - In particular the measure of return is considered to be quite unreliable.

Optimal asset allocation

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- However, MVO is susceptible to estimation errors in both parametric measures of return and risk.
  - In particular the measure of return is considered to be quite unreliable.
- Alternative: **Risk parity**
  - Equalizes the risk contributions of each asset
  - Does not require a return measure.
  - However, it is still susceptible to estimation errors in the risk measure.

Distributional robustness

Introduce robustness at the parameter estimation step

- We use a discrete probability distribution to model the 'weights' associated with each scenario in our dataset.
  - E.g., consider a simple estimate of the expected value and variance of a <u>discrete</u> random variable:

$$\mu = \sum_{i=1}^{n} p_i \cdot x_i$$

$$\sigma^2 = \sum_{i=1}^{n} p_i \cdot (x_i - \mu)^2$$

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  - If we have raw data, we assume scenarios are equally likely, $p_i = 1/n$
  - What if we break away from this assumption?

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  - We minimize the objective by changing our asset weights to attain risk parity.
  - We can maximize the objective by using the discrete probabilities as adversarial variables

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- We have a minimax problem.

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  - The axioms of probability
  - Some statistical distance from our nominal distribution
  - Naturally, the nominal is the ‘equally likely’ distribution

Distributionally robust risk parity

Statistical distance measures

- As long as the statistical distance measure is both convex and bounded, the user can choose their preferred measure.
- We exemplify this through the following three measures
  - Jensen–Shannon (JS) divergence
  - Hellinger (H) distance
  - Total variation (TV) distance
- Set the distance proportional to an investor's confidence level.

Projected gradient descent–ascent algorithm

- We can solve our minimax problem through a projected gradient descent–ascent (PGDA) algorithm.
  - Iteratively descend in the asset weight space, ascend in the probability space.
  - A projection (or similar approach) is necessary due to the constraints.
  - We must define several parameters in both directions.
  - Problem: slow convergence and vanishing gradients in both directions.

Projected gradient ascent with sequential convex programming

- We propose a projected gradient ascent algorithm grounded on sequential convex programming (PGA–SCP).
  - Iteratively ascend in the probability space.
  - Solve for the corresponding risk parity portfolio in the asset weight space after every iteration.
  - In other words, we repeatedly solve the convex risk parity problem.
  - The resulting algorithm is much faster and stable

Comparison between PGDA and PGA–SCP

- Data:
  - Synthetic asset return data
  - We have 200 assets ($n = 200$) and 5,000 observations ($T = 5,000$)
  - The investor confidence is set to 30% ($\delta = 0.3$)

Numerical experiments

Comparison between PGDA and PGA–SCP

	$n = 200$, $T = 5,000$, $\delta = 0.3$					
	JS		Hellinger		TV	
	PGDA	PGA–SCP	PGDA	PGA–SCP	PGDA	PGA–SCP
Time (s)	**157**	**79.9**	569	99.6	235	128
Iterations	59	33	132	28	48	37
Var. ($\times 10^4$)	3.97	9.06	3.67	9.98	0.71	12.9

- The runtime of the proposed PGA–SCP is considerably lower.

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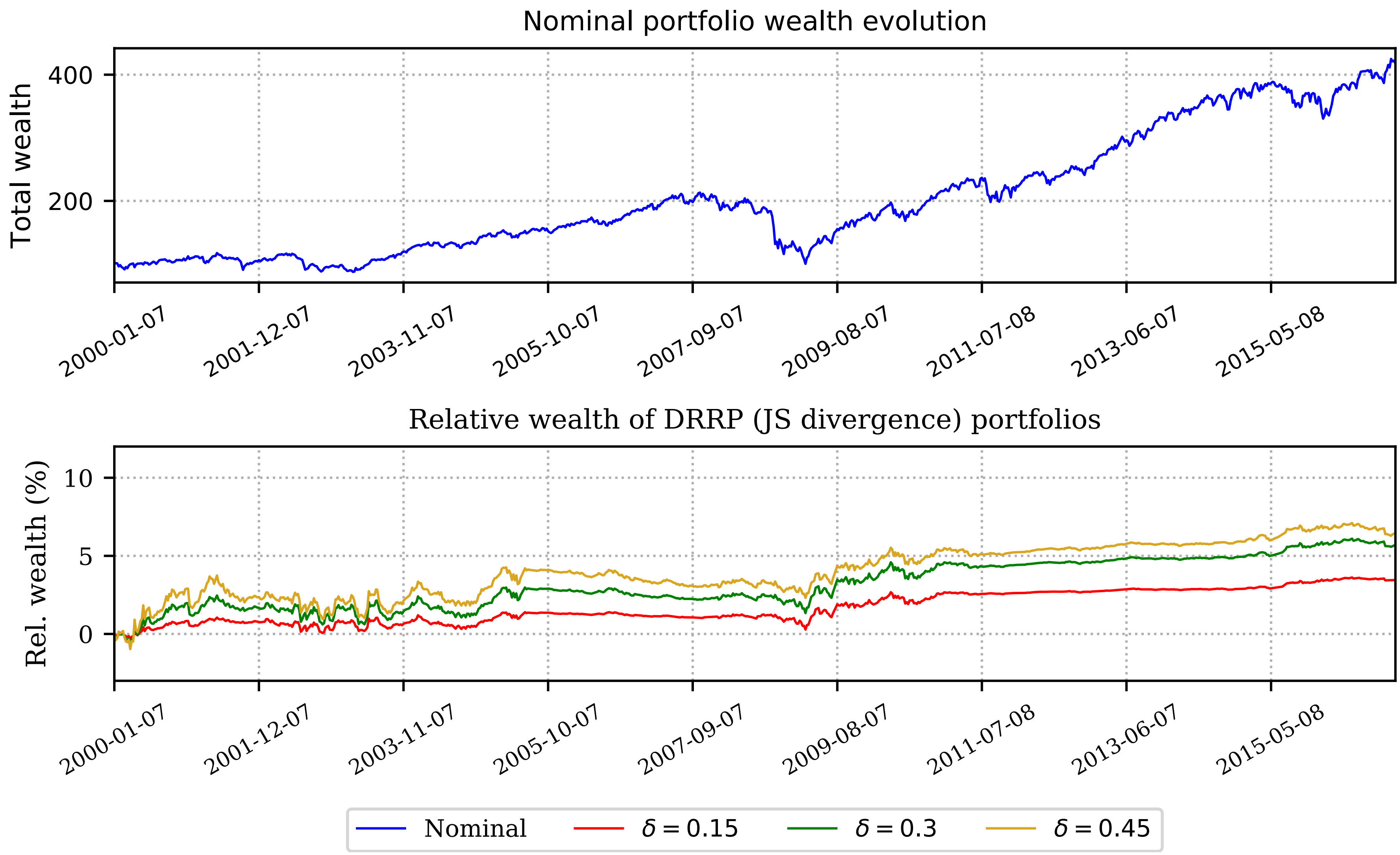
- The PGDA algorithm converges to a sub-optimal solution.

Out-of-sample experimental setup

- Data:
  - 30 industry portfolios (from Kenneth French's data library).
  - Weekly asset return data from 1998 until 2016.
  - Two years worth of data for calibration.

Numerical experiments



Numerical experiments

Summary of financial performance between 2000–2016

	Nom.	JS			Hellinger			TV		
$\delta =$		0.15	0.3	0.45	0.15	0.3	0.45	0.15	0.3	0.45
Ann. Ex. Return (%)	6.64	6.84	6.96	7.01	6.85	6.97	7.02	6.95	6.98	7.01
Ann. Volatility (%)	17.0	17.2	17.2	17.2	17.2	17.2	17.2	17.2	17.2	17.2
Sharpe Ratio (%)	**39.0**	39.8	**40.5**	40.7	39.9	**40.5**	40.8	40.4	**40.6**	40.7
Avg. Turnover (%)	10.0	12.6	15.1	16.7	12.8	15.2	16.8	16.0	17.2	18.0

- Robust portfolios have a higher Sharpe ratio than the nominal.
- Performance is similar between robust portfolios with the same δ.

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- The Sharpe ratio increases with robustness

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- Not surprisingly, turnover also increases with robustness

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  - The user can choose their preferred statistical distance measure.
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  - The algorithm can tackle other portfolio selection problems.
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- For more information:

  Costa, G. and Kwon, R. H. (2020). Data-driven distributionally robust risk parity portfolio optimization. *Available at SSRN 3709680*

References

[1] Costa, G. and Kwon, R. H. (2020). Data-driven distributionally robust risk parity portfolio optimization. *Available at SSRN 3709680*.