Risk Parity Portfolios under a Markov Regime-Switching Framework

Giorgio Costa & Roy H. Kwon

University of Toronto
Department of Mechanical and Industrial Engineering

MMF Symposium 2018, Blue Mountain, Ontario

January 19, 2018

Outline

What is this presentation about?

- Markets follow a cyclical behaviour - alternating between bullish and bearish states.
- If we hold a portfolio, how could we hedge against these cycles?
  - Could we take advantage of this information to improve our models?
- We will explore a typical asset allocation problem that incorporates the cyclical behaviour of markets via machine learning algorithms.

Outline

There are two terms to keep in mind throughout this presentation

- Hidden Markov Models (HMMs): Will be used to incorporate information about the market state into our portfolio
- Risk Parity: Will be used to solve our asset allocation problem.

Hidden Markov Models

- HMMs are popular in Machine Learning as both classification and clustering algorithms.
- What if we believe that asset returns are governed by hidden market states?
- Could we use observed market returns to discern between these states and incorporate these dynamics into a portfolio optimization problem?

Hidden Markov Models in Asset Allocation

- We can use HMMs to improve the estimated parameters used in portfolio optimization.
- Typical inputs are the asset expected returns and covariance matrix.
- Estimation errors in expected returns tend to have a significant impact during the optimization process.
- What effect would a HMM have in a portfolio that does not use expected returns?

Risk Parity Portfolios

- Risk parity seeks to construct a portfolio based on a risk-weighted basis.
- Does not require estimated returns as an input, improving stability.
- Each asset should have the same risk contribution.
- By design, the resulting portfolio is well-diversified.
- Risk parity is also known as ‘equal risk contribution’ (**ERC**).

Here is a summary of the key findings from our research

- We seek to improve a risk parity portfolio by identifying a regime-dependent risk measure.
- This would incorporate the market dynamics as an input to the optimization problem.
- We can also use the Markov model to dictate the portfolio rebalancing policy.
- Higher risk-adjusted returns can be achieved by reducing exposure to concentrated risk through this improved covariance estimate.

Regime-Switching Factor Model

A 2-state model can be explained by this factor model of asset returns

$$\boldsymbol{r} = I_1(\boldsymbol{\alpha}_1 + \boldsymbol{V}_1^\top \boldsymbol{f}_1 + \boldsymbol{\epsilon}_1) + I_2(\boldsymbol{\alpha}_2 + \boldsymbol{V}_2^\top \boldsymbol{f}_2 + \boldsymbol{\epsilon}_2).$$

- $\boldsymbol{r} \in \mathbb{R}^n$ are returns of the n assets.
- $I_i = 1$ when the current market state $s_t = i$ and $I_i = 0$ otherwise.
- $\boldsymbol{\alpha}_i \in \mathbb{R}^n$ is the vector of regression intercepts.
- $\boldsymbol{f}_i \sim \mathcal{N}(\boldsymbol{\phi}_i, \boldsymbol{F}_i) \in \mathbb{R}^m$ is the vector of factor returns.
- $\boldsymbol{V}_i \in \mathbb{R}^{m \times n}$ is the matrix of factor loadings.
- $\boldsymbol{\epsilon}_i \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{D}_i) \in \mathbb{R}^n$ is the vector of residuals.

From this factor model we can derive a **probability-weighted regime-dependent covariance matrix**.

Experimental Setup

- We use the Fama–French 3-factor model as the basis.
- We construct portfolios with 48 assets ($n = 48$).
- 4 different investment strategies were tested:
  - i Nominal ERC
  - ii Robust ERC
  - iii **2-State ERC**
  - iv 3-State ERC
- We use weekly asset and factor returns from 1985 to 2010.

Computational Experiments: Setup

We conducted two individual experiments

1. Fixed 6-month rebalance policy (22 investment periods)
2. Dynamic rebalance policy
   - We re-identify the regime every four weeks.
   - We identify the regime by inspection:
     - $\Rightarrow$ We check the last entry of the smoothed probabilities from our Markov model.
   - To mitigate estimation noise, rebalancing occurs only when a sustained change in regime is perceived:
     - $\Rightarrow$ A change in regime must be consistent for **two** consecutive 4-week periods.

Example of estimated regime changes for the out-of-sample test

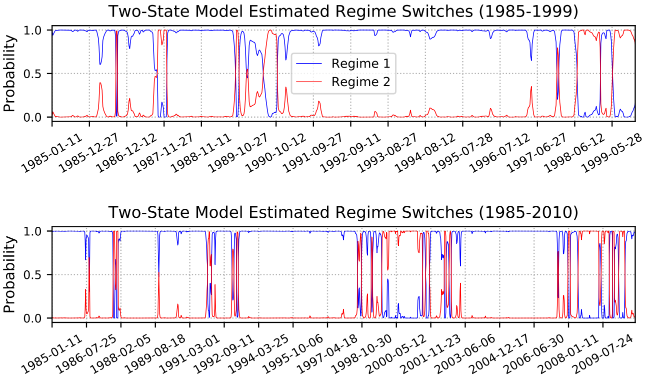

Figure: Estimated regime switches for test period 2000–2010. The probability of having a bullish (bearish) market is shown in blue (red). **Top**: regime switches estimated before the first investment period. **Bottom**: regime switches estimated after the last investment period.

Fixed-term rebalance policy

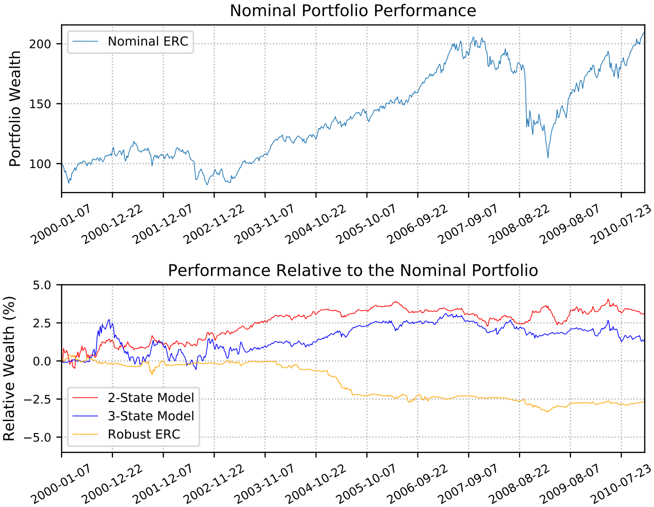

Figure: Evolution of wealth for test period 2000–2010 under a fixed-term rebalance policy. **Top**: evolution of wealth of nominal portfolio. **Bottom**: evolution of wealth relative to the nominal portfolio.

Summary of Results: Fixed-term rebalance policy

	Nominal ERC	Robust	**2-State**	3-State
Yearly Return	0.0695	0.0668	**0.0724**	0.0708
Sharpe Ratio	0.0511	0.0488	**0.0531**	0.0516
Turnover	0.0222	0.0826	**0.0722**	0.1866

Dynamic rebalance policy

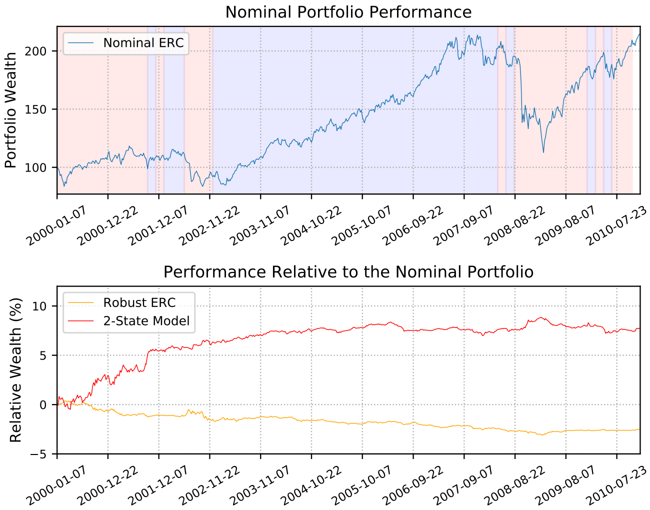

Figure: Evolution of wealth for test period 2000–2010 under a dynamic rebalance policy. **Top**: evolution of absolute wealth of nominal portfolio. **Bottom**: evolution of wealth relative to the nominal portfolio.

Summary of Results: Dynamic rebalance policy

	Nominal	Robust	**2-State**
Yearly Return	0.0719	0.0694	**0.0792**
Sharpe Ratio	0.0528	0.0508	**0.0579**
Turnover	0.0274	0.0516	**0.1699**

Key takeaways

- A regime-switching factor model allows for the estimation of regime-dependent covariance matrices.
- This model is able to capture and incorporate the market dynamics.
- We can use the Markov model to dictate the portfolio rebalancing policy.
- Higher risk-adjusted returns can be achieved by improving the quality of our estimated risk measure.
- The model developed is highly tractable and easily scalable.

Decomposing the risk measure

A decomposable risk measure can be used to derive the risk contribution per asset:

$$\sigma_p = \sqrt{\boldsymbol{x}^\top \boldsymbol{\Sigma} \boldsymbol{x}} = \sum_{i=1}^{n} \sigma_i,$$

$$\sigma_i = x_i \frac{\partial \sigma_p}{\partial x_i} = x_i \frac{(\boldsymbol{\Sigma} \boldsymbol{x})_i}{\sqrt{\boldsymbol{x}^\top \boldsymbol{\Sigma} \boldsymbol{x}}},$$

- $\boldsymbol{x} \in \mathbb{R}^n$ is the vector of asset weights.
- $\boldsymbol{\Sigma} \in \mathbb{R}^{n \times n}$ is the asset covariance matrix.
- σ_p is the portfolio standard deviation.
- σ_i is the risk contribution of asset i.

Risk Parity Portfolio Optimization

A typical risk parity formulation seeks to minimize the squared differences between the asset risk contributions:

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n}\left(x_i(\boldsymbol{\Sigma x})_i - x_j(\boldsymbol{\Sigma x})_j\right)^2 \\ \text{s.t.} \quad & \mathbf{1}^\top x = 1, \\ & x \geq 0. \end{aligned}$$

This yields a unique solution when short-selling is not allowed.

The general factor model of asset returns

$$\boldsymbol{r} = \boldsymbol{\alpha} + \boldsymbol{V}^\top \boldsymbol{f} + \boldsymbol{\epsilon}$$

- $\boldsymbol{r} \in \mathbb{R}^n$ are returns of the n assets.
- $\boldsymbol{\alpha} \in \mathbb{R}^n$ is the vector of regression intercepts.
- $\boldsymbol{f} \sim \mathcal{N}(\phi, \boldsymbol{F}) \in \mathbb{R}^m$ is the vector of factor returns.
- $\boldsymbol{V} \in \mathbb{R}^{m \times n}$ is the matrix of factor loadings.
- $\boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{D}) \in \mathbb{R}^n$ is the vector of residuals.

Accordingly, the expected returns and covariance matrix are

$$\boldsymbol{\mu} = \boldsymbol{\alpha} + \boldsymbol{V}^\top \phi,$$
$$\boldsymbol{\Sigma} = \boldsymbol{V}^\top \boldsymbol{F} \boldsymbol{V} + \boldsymbol{D}.$$

Suppose the factors behave differently under distinct market regimes

- A 2-state market is governed by the following Markov transition matrix

$$\Gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}$$

  - γ_{ij} is the probability of switching from state i to state j.

- Suppose this matrix dictates whether the asset returns are governed by

$$\boldsymbol{r} = \boldsymbol{\alpha}_1 + \boldsymbol{V}_1^\top \boldsymbol{f}_1 + \boldsymbol{\epsilon}_1,$$

or

$$\boldsymbol{r} = \boldsymbol{\alpha}_2 + \boldsymbol{V}_2^\top \boldsymbol{f}_2 + \boldsymbol{\epsilon}_2.$$

We can capture the market dynamics within a single factor model

- For a 2-state model, we have the factor model

$$\boldsymbol{r} = I_1(\boldsymbol{\alpha}_1 + \boldsymbol{V}_1^\top \boldsymbol{f}_1 + \epsilon_1) + I_2(\boldsymbol{\alpha}_2 + \boldsymbol{V}_2^\top \boldsymbol{f}_2 + \epsilon_2).$$

  - $I_i = 1$ when the current market state $s_t = i$ and $I_i = 0$ otherwise.

- Computing the covariance as $\mathrm{cov}(r_i, r_j)\ \forall i, j$, we obtain a probability-weighted regime-dependent covariance matrix

$$\begin{aligned}\boldsymbol{\Sigma}_{RS_i} = \quad & \gamma_{i1} \boldsymbol{V}_1^\top \boldsymbol{F}_1 \boldsymbol{V}_1 + \gamma_{i2} \boldsymbol{V}_2^\top \boldsymbol{F}_2 \boldsymbol{V}_2 + \gamma_{i1}(1 - \gamma_{i1}) \boldsymbol{V}_1^\top \phi_1 \phi_1^\top \boldsymbol{V}_1 \\ & + \gamma_{i2}(1 - \gamma_{i2}) \boldsymbol{V}_2^\top \phi_2 \phi_2^\top \boldsymbol{V}_2 - \gamma_{i1}\gamma_{i2} \boldsymbol{V}_1^\top \phi_1 \phi_2^\top \boldsymbol{V}_2 \\ & - \gamma_{i1}\gamma_{i2} \boldsymbol{V}_2^\top \phi_2 \phi_1^\top \boldsymbol{V}_1 + \gamma_{i1} \boldsymbol{D}_1 + \gamma_{i2} \boldsymbol{D}_2.\end{aligned}$$

  - $\gamma_{ij} = \mathbb{E}[I_j \mid s_t = i]$, i.e. the transition probability.

Estimation of regime switches between 1985 – 1999

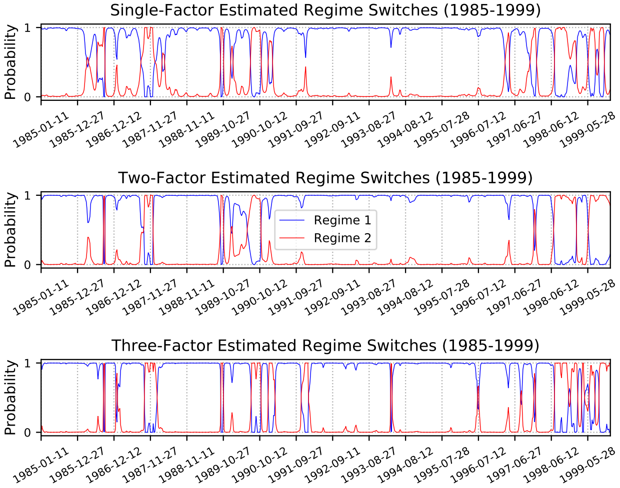

Figure: Estimated regime switches for the period of 1985–1999. The probability of having a bullish (bearish) market is shown in blue (red). **Top**: estimated switches based on the market factor. **Middle**: estimated switches based on the market and the 'high-minus-low' factors. **Bottom**: estimated switches based on all 3 Fama–French factors.