A Regime-Switching Framework for Mean–Variance Optimization

Giorgio Costa[1] and Roy H. Kwon[2]

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Abstract

We formulate a novel Markov regime-switching factor model to describe the cyclical nature of asset returns in modern financial markets. Maintaining a factor model structure allows us to easily derive the asset expected returns and their corresponding covariance matrix. By design, these two parameters are calibrated to better describe the properties of the different market regimes. In turn, these regime-dependent parameters serve as the inputs during mean–variance optimization, thereby constructing portfolios adapted to the current market environment. Through this formulation, the proposed model allows for the construction of large, realistic portfolios at no additional computational cost during optimization. Moreover, the viability of this model can be significantly improved by periodically rebalancing the portfolio, ensuring proper alignment between the estimated parameters and the transient market regimes. An out-of-sample computational experiment over a long investment horizon shows that the proposed regime-dependent portfolios are better aligned with the market environment, yielding a higher ex-post rate of return and lower volatility, even when compared against competing portfolios.

Keywords: Asset Allocation, Markov Regime-Switching, Factor Model, Mean–Variance Optimization, Robust Optimization.

[1]Department of Mechanical and Industrial Engineering, University of Toronto, 5 King's College Road, Toronto, Ontario M5S 3G8, Canada. Email: gcosta@mie.utoronto.ca.

[2]Department of Mechanical and Industrial Engineering, University of Toronto, 5 King's College Road, Toronto, Ontario M5S 3G8, Canada. Email: rkwon@mie.utoronto.ca. Correspondence should be addressed to: Roy H. Kwon.

1 Introduction

The application of mathematical optimization in portfolio construction has been prevalent in both academia and industry since the introduction of modern portfolio theory (MPT) in Markowitz (1952). The mean–variance optimization (MVO) problem introduced by MPT analyses the trade-off between risk and return to construct so-called optimal portfolios. These portfolios provide an optimal balance of risk and return relative to an investor's risk appetite. However, while MVO has received widespread attention as a quantitative tool in asset management, it has also been criticized for its susceptibility to estimation error in its input parameters, namely the asset expected returns and covariance matrix. The discrepancies between the estimated parameters and their ex-post realizations can severely hinder the performance of an 'optimal' portfolio, where its optimality was determined solely through these estimates. The sensitivity of MVO to its inputs has been extensively explored in literature, with important examples shown in Best and Grauer (1991), Merton (1980), and Broadie (1993). A widely referenced conclusion found in Chopra and Ziemba (1993) states estimation errors during optimization can lead to overconcentrated portfolios, arguing that errors in expected returns can have an impact ten times larger than errors in the covariance matrix.

The subject of parameter uncertainty and its effect during mathematical optimization is a discipline in its own right, and has found several applications in MVO. For example, a heuristic approach known as portfolio resampling (Michaud and Michaud, 2008) uses a Monte Carlo methods to mitigate the effect of estimation noise during optimization. More recently, robust optimization methods have been used to introduce uncertainty sets around the estimated parameters to explicitly capture this estimation error during optimization (Goldfarb and Iyengar, 2003; Tütüncü and Koenig, 2004), or even to introduce uncertainty regarding the choice of probability distribution governing the asset returns (Delage and Ye, 2010). Aside from robust techniques, another avenue to enhance portfolio optimization is to focus on the source of uncertainty: the input parameters and the methods by which these are estimated. One example of this is the application of shrinkage estimators to improve the quality of estimated parameters. Shrinkage can be a particularly useful tool during portfolio optimization when insufficient data is available to produce reliable estimates (Jorion, 1986; Ledoit and Wolf, 2003).

Following a similar rationale, the focus of this paper is to improve the estimation of input parameters for portfolio optimization. To achieve this, we introduce a novel regime-switching factor model of asset returns that will allow us to incorporate an additional dimension of risk into the estimated parameters. Before delving into this topic, we first proceed by discussing the financial application of generic factor models, followed by regime-switching models.

Factor models attempt to explicitly explain the behaviour of a random variable either through a single factor, such as the capital asset pricing model (Lintner, 1965; Mossin, 1966; Sharpe, 1964), or through a combination of multiple factors, such as the Fama–French three-factor model (Fama and French, 1993). Factor models have become popular in finance for the economic relevance of the factors and their ability to explain and quantify different sources of an asset's systematic risk. One important application of these models is to derive the estimated asset expected returns and covariance matrix, where the inherent properties of the factors are used to explain the asset returns and volatility.

Our intention is to maintain a traditional factor model structure that will allow us to naturally derive these two estimated parameters, while also introducing a regime-switching framework to capture the cyclical nature of financial markets. The original application of regime-switching in finance, proposed in Hamilton (1989), was to describe the abrupt changes observed in business cycles. Due to their similarities, this concept was naturally extended to financial markets to describe a transition from a period of stability and growth to periods of financial distress, typically known as bullish or bearish market regimes. Indeed, the topic of regime-switching in finance has been extensively explored in literature. Two important examples are Hamilton (2010) and Ang and Timmermann (2012), where the changes in regime are described through a Markov chain, a property that we exploit in the design of our proposed regime-switching factor model.

Through this regime-switching factor model, we can derive estimates of the expected returns and covariance matrix that implicitly incorporate a regime-dependency, allowing for a more faithful representation of the asset behaviour under the current market regime. This follows a similar framework to the one shown in Costa and Kwon (2018), where the underlying assumption is that the effect of the regime changes on the assets could be fully explained by the factors alone. However, the model in this paper differs by building on that assumption, and instead assume that both the idiosyncratic and systematic behaviour of the assets is regime-dependent. Subsequently, the regime-dependent asset expected returns and covariance matrix can be used to construct MVO portfolios that are better adapted to the current market environment.

The result is a single-period regime-dependent portfolio that is tailored to the current market regime. 'Single-period' denotes the static nature of traditional MVO, where we find an optimal solution today to be used for all subsequent time periods, without consideration of any time dependency observed by our input parameters or our decision variable. We favour a single-period approach over a dynamic programming framework to improve computational tractability and scalability, avoiding the 'curse of dimensionality' often encountered in dynamic programming (Bellman, 1961). In other words, a single-period approach allows us to efficiently find optimal portfolios with a large basket of assets. Furthermore, this static model is enhanced by the transient nature of a regime-switching model, reflecting the non-stationarity of the estimated asset parameters. This property can be further exploited through periodic re-estimation of parameters and rebalancing of the portfolio, allowing the portfolio to align itself with the market over a long investment horizon. This, in turn, is compatible with traditional asset management practice, where institutions rely on rebalancing policies with fixed calendar intervals, e.g., quarterly, semi-annually, or annually (Michaud and Michaud, 2008).

A brief outline of this paper follows. Section 2 introduces a Markov regime-switching factor model of asset returns. This factor model allows us to derive the regime-dependent expected return and covariance matrix, implicitly capturing the market dynamics. Section 3 proceeds to show how these parameters can be used in MVO and its variant, the minimum variance problem. In addition, we discuss the robust MVO formulation from Goldfarb and Iyengar (2003), which serves as a competing model for comparison against our proposed model. The results of a numerical experiment are presented in Section 4, where we implement and compare the different models under three different rebalancing policies. Finally, Section 5 summarizes the findings and contribution of this paper.

The experimental results in Section 4 highlight the advantages of a regime-dependent portfolio. Given that estimation errors in asset expected returns have a larger contribution than errors in the covariance matrix (Chopra and Ziemba, 1993), it comes as no surprise that any method that improves the quality of estimates can lead to improvements in out-of-sample MVO portfolio performance. With that said, our experimental results show that a regime-dependent portfolio is able to deliver higher risk-adjusted returns over a long investment horizon, consistently outperforming other competing portfolios. Another interesting result is given by the minimum variance portfolio, demonstrating that, even when expected returns are ignored, the proposed regime-switching framework is able to deliver a better estimate of the covariance matrix, highlighted by having lower ex-post volatility. This further corroborates the findings from Costa and Kwon (2018), where the use of a regime-switching framework was found to improve the performance of risk parity portfolios, which are also independent of asset expected returns.

The purpose of this paper is to introduce a novel regime-switching factor model that serves to seamlessly integrate the cyclic property of the market into single-period portfolio optimization while maintaining a highly tractable and scalable model. Moreover, we can overcome the undesired static nature intrinsic to MVO through periodic rebalancing, allowing the portfolio to adapt to current market conditions and further exploiting the benefits of the regime-switching framework. To the best of our knowledge, this regime-switching factor model and its application in MVO is a novel method.

2 Regime–switching factor model

In this section we introduce a Markov regime-switching factor model of asset returns. First, let us start with a generic factor model. Suppose that the random returns of n assets can be explained through a linear combination of m explanatory factors. Thus, we have that

$$\boldsymbol{r}_t = \boldsymbol{\mu} + \boldsymbol{V}^T \boldsymbol{f}_t + \boldsymbol{\epsilon}_t, \tag{1}$$

where $\boldsymbol{r}_t \in \mathbb{R}^n$ are the asset random returns at time t, $\boldsymbol{\mu} \in \mathbb{R}^n$ are the asset expected returns, $\boldsymbol{f}_t \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{F}) \in \mathbb{R}^m$ are the centered factor returns at time t, $\boldsymbol{V} \in \mathbb{R}^{m \times n}$ is the matrix of factor loadings, and $\boldsymbol{\epsilon}_t \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{D}) \in \mathbb{R}^n$ are the residual returns. $\boldsymbol{F} \in \mathbb{R}^{m \times m}$ and $\boldsymbol{D} \in \mathbb{R}^{n \times n}$ denote the factor covariance matrix and the diagonal matrix of residual variance, respectively. We note our choice to use *centered* factor returns, which serve to explain only the variability of the asset returns. Thus, the constant term in Equation (1) is equal to the asset expected returns. Stemming from this factor model, the asset covariance matrix, $\boldsymbol{\Sigma} \in \mathbb{R}^{n \times n}$, can be derived as

$$\boldsymbol{\Sigma} = \boldsymbol{V}^T \boldsymbol{F} \boldsymbol{V} + \boldsymbol{D}. \tag{2}$$

Thus, we have that $\boldsymbol{r}_t \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

This factor model assumes that the residual returns are independent of one another, i.e. $\text{cov}(\epsilon_t^i, \epsilon_t^j) = 0$ for $i \neq j$; and it also assumes that the residual returns are independent of the factor returns, i.e. $\text{cov}(\epsilon_t^i, f_t^j) = 0 \ \forall \ i, \ j$. However, this model does not assume the factors are independent of one another, i.e. the factor covariance matrix, $\boldsymbol{F}$, is not required to be a diagonal matrix. In practice, $\boldsymbol{F}$ must be positive semi-definite, i.e., $\boldsymbol{F} \succeq 0$.

In practice, the parameters $\boldsymbol{\mu}$, $\boldsymbol{V}$ and $\boldsymbol{D}$ are estimated by linear regression using historical asset and factor returns, while the parameter $\boldsymbol{F}$ is estimated directly from the factor returns. Through this process, one important assumption is made: that recent historical data can accurately estimate the future behaviour of the asset returns $\boldsymbol{r}_t$. Now, as an example, imagine we wish to calibrate our factor model at the onset of the financial crisis of 2008. Using recent historical data to do this may provide a false sense of stability given the bullish market observed prior to the crisis. Thus, the factor model in Equation (1) would fail to explain the abrupt change in behaviour of the asset returns.

Consider a market-dependent factor, such as the 'excess market return' factor from the Fama–French three-factor model[3] (Fama and French, 1993). We could describe this change in behaviour as a transition into a different market regime, where this transition can be described through a Markov process (Ang and Timmermann, 2012; Hamilton, 2010). Thus, we assume the factor itself is governed by a regime-dependent probability distribution that describes the observed factor returns.

As an example, assume two market regimes exist: bullish and bearish. We can use a discrete-time Markov chain to describe the probability that our state variable s_t will be in regime j at time t given that our previous state was $s_{t-1} = i$, independent of any further preceding states, i.e.,

$$P\{s_t = j \mid s_{t-1} = i, s_{t-2} = k, \cdots\} = P\{s_t = j \mid s_{t-1} = i\}.$$

[3] We note that all three factors in the Fama–French three-factor model are market-dependent, but we choose the 'excess market return' as the best example among the three factors to represent the market as a whole.

The state variable s_t is latent and corresponds to the market regime at time t. However, in practice, its probability distribution can be estimated from the observable market data through the Baum–Welch algorithm (Baum et al., 1970). This algorithm is a special case of expectation maximization, an iterative technique used to find the maximum likelihood. In the case of our two-regime example, the transition matrix governing the probability of switching from one regime to another is

$$\Gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}, \tag{3}$$

where γ_{ij} is the probability that regime i at time t will be followed by regime j at $t+1$. Thus, by definition, each row must sum up to one, i.e., $\sum_{j=1}^{2} \gamma_{ij} = 1$ for $i = 1, 2$.

We proceed in a similar fashion to Costa and Kwon (2018) and use this Markov process to explain the behaviour of the factors within the factor model in Equation (1). However, we note a fundamental difference between these two models. The original factor model in Costa and Kwon (2018) assumes that the factors alone can fully explain the regime-dependency of the assets,

$$\boldsymbol{r}_t = \boldsymbol{\alpha} + I_{1t}(\boldsymbol{V}_1^T \boldsymbol{g}_{1t}) + I_{2t}(\boldsymbol{V}_2^T \boldsymbol{g}_{2t}) + \boldsymbol{\epsilon}_t, \tag{4}$$

where $\boldsymbol{\alpha} \in \mathbb{R}^n$ is the intercept of regression, and $\boldsymbol{g}_{1t} \sim \mathcal{N}(\boldsymbol{\phi}_1, \boldsymbol{F}_1) \in \mathbb{R}^m$ and $\boldsymbol{g}_{2t} \sim \mathcal{N}(\boldsymbol{\phi}_2, \boldsymbol{F}_2) \in \mathbb{R}^m$ are the *non-centered* regime-dependent factors. In other words, this model assumes the asset returns under both bull and bear market regimes share the same intercept, as well as the same level of idiosyncratic risk. Taking the variance and covariance of the random returns in Equation (4) yields

$$\begin{aligned} \boldsymbol{Q}_{s_i} = {} & \gamma_{i1} \boldsymbol{V}_1^T \boldsymbol{F}_1 \boldsymbol{V}_1 + \gamma_{i2} \boldsymbol{V}_2^T \boldsymbol{F}_2 \boldsymbol{V}_2 + \gamma_{i1}(1-\gamma_{i1}) \boldsymbol{V}_1^T \boldsymbol{\phi}_1 \boldsymbol{\phi}_1^T \boldsymbol{V}_1 \\ & + \gamma_{i2}(1-\gamma_{i2}) \boldsymbol{V}_2^T \boldsymbol{\phi}_2 \boldsymbol{\phi}_2^T \boldsymbol{V}_2 - \gamma_{i1}\gamma_{i2} V_1^T \boldsymbol{\phi}_1 \boldsymbol{\phi}_2^T \boldsymbol{V}_2 - \gamma_{i1}\gamma_{i2} \boldsymbol{V}_2^T \boldsymbol{\phi}_2 \boldsymbol{\phi}_1^T \boldsymbol{V}_1 + \boldsymbol{D}, \end{aligned} \tag{5}$$

where $\boldsymbol{Q}_{s_i} \in \mathbb{R}^{n \times n}$ is the asset covariance matrix corresponding to regime i. We note our choice of symbol $\boldsymbol{Q}_{s_i}$ to differentiate the covariance term corresponding to Costa and Kwon (2018) from the subsequent covariance derivation in this paper, which is described later in this section. Most of the terms in Equation (5) arise naturally since the factors are not centered, and their regime-dependent expected value manifests through this derivation. Nevertheless, taking the covariance of the factor model in (4) may fail to fully explain an asset's shift in behaviour after a regime change. In particular, this model assumes that the diagonal matrix of residual variance $\boldsymbol{D}$, which represents the asset idiosyncratic volatility, is independent of the market regime.

Instead, we expand on this concept and assume that the idiosyncratic behaviour of the assets is also governed by the different market regimes. Thus, this paper proposes a novel factor model structure where the regime-dependency of the asset returns is described by the superposition of two individual factor models

$$\boldsymbol{r}_t = I_{1t}(\boldsymbol{\mu}_1 + \boldsymbol{V}_1^T \boldsymbol{f}_{1t} + \boldsymbol{\epsilon}_{1t}) + I_{2t}(\boldsymbol{\mu}_2 + \boldsymbol{V}_2^T \boldsymbol{f}_{2t} + \boldsymbol{\epsilon}_{2t}), \tag{6}$$

where each factor model explains the asset returns under a corresponding regime. The indicator variable $I_{it} = 1$ when the current regime $s_t = i$ and $I_{it} = 0$ otherwise. In addition, $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2 \in \mathbb{R}^n$ are the regime-dependent expected returns, $\boldsymbol{f}_{1t} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{F}_1) \in \mathbb{R}^m$ and $\boldsymbol{f}_{2t} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{F}_2) \in \mathbb{R}^m$ are the centered regime-dependent factors, and $\boldsymbol{\epsilon}_{1t} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{D}_1) \in \mathbb{R}^n$ and $\boldsymbol{\epsilon}_{2t} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{D}_2) \in \mathbb{R}^n$ are the regime-dependent residual returns. We will use Equation (6) as our regime-switching factor model for the remainder of this paper. The use of centered factors provides the added benefit of simplifying the subsequent derivation of the expectation, variance, and covariance of the random returns from Equation (6), yielding

$$\begin{aligned}
\boldsymbol{\mu}_{s_i} &= \gamma_{i1}\ \boldsymbol{\mu}_1 + \gamma_{i2}\ \boldsymbol{\mu}_2, \\
\boldsymbol{\Sigma}_{s_i} &= \gamma_{i1}(\boldsymbol{V}_1^T \boldsymbol{F}_1 \boldsymbol{V}_1 + \boldsymbol{D}_1) + \gamma_{i2}(\boldsymbol{V}_2^T \boldsymbol{F}_2 \boldsymbol{V}_2 + \boldsymbol{D}_2) \\
&\quad + \gamma_{i1}(1-\gamma_{i1})\boldsymbol{\mu}_1\boldsymbol{\mu}_1^T + \gamma_{i2}(1-\gamma_{i2})\boldsymbol{\mu}_2\boldsymbol{\mu}_2^T - \gamma_{i1}\gamma_{i2}(\boldsymbol{\mu}_1\boldsymbol{\mu}_2^T + \boldsymbol{\mu}_2\boldsymbol{\mu}_1^T),
\end{aligned} \tag{7}$$

where $\boldsymbol{\mu}_{s_i} \in \mathbb{R}^n$ and $\boldsymbol{\Sigma}_{s_i} \in \mathbb{R}^{n \times n}$ are the asset expected returns and covariance matrix corresponding to regime i, respectively. In addition, $\gamma_{ij} = \mathbb{E}[I_j \mid s_t = i]$, i.e. γ_{ij} is the probability of switching from regime i to regime j. These probabilities correspond to the two-state transition matrix in Equation (3).

The benefit of this model is that the cyclical property of the market is captured implicitly through the asset expected returns and covariance matrix in Equation (7). This, in turn, can serve as the input parameters during MVO at no additional computational cost, i.e., the formulation of this optimization problem is the same as the nominal one.

The parameters in Equation (7) lend themselves to single-period portfolio optimization. However, by design, these parameters are calibrated at time t to match the expectation of the current market regime, and do not take into consideration any future transitions from one regime to another. Thus, a limitation of this model is the lack of guarantees that our parameters will be properly aligned during any future time periods. Nevertheless, this limitation can be overcome through periodic re-estimation of the parameters, allowing them to re-align with the market. The frequency of re-estimation has a direct impact on the proper alignment of the estimated parameters. However, given the static property of portfolio optimization, this frequency must be reconciled with a realistic portfolio rebalancing policy. Later, in Section 4, we show that any sort of periodic parameter re-estimation and portfolio rebalancing suffices to observe the benefits of the proposed model.

As a final note, we discuss the number of regimes present in our model. Determining the ideal number of market regimes to use in this model is a difficult task. Our intention is not to suggest that a two-regime model is the most adequate, but rather it reflects our preference for a simplistic model. In fact, an investor with a predisposition towards a specific number of regimes can modify Equation (6) to accommodate additional market regimes, and re-derive the regime-dependent parameters in Equation (7) accordingly. Two examples of regime-switching models with additional regimes can be found in Ang and Bekaert (2002) and Guidolin and Timmermann (2007).

Increasing the number of regimes has both potential benefits and drawbacks. In theory, a larger number of regimes can give greater flexibility for the estimation algorithm to detect additional nuances within the raw market data, in particular if the investor believes more than two regimes exist. However, if the probability distributions that govern the different regimes are similar to each other, it may be difficult to discern between them in practice. This can lead to unstable and counter-intuitive results, and would hinder the subsequent estimation of parameters. Moreover, a larger number of regimes requires the estimation of additional regime-dependent parameters, potentially increasing the impact of estimation error during optimization. Some of these problems may be overcome by the use of predefined filters or restrictions to force the algorithm to discern between regimes. However, this requires an a priori assumption and would mean the regime estimation algorithm would no longer be fully data-driven. An analysis on the use of additional market regimes can be found in Costa and Kwon (2018), which reinforces our choice of having only two regimes. For the remainder of this paper, we assume a two-regime environment.

3 Investment models

This section presents the two investment models that will be tested under the regime-switching framework: MVO and minimum variance optimization. Starting with the framework presented in Markowitz (1952), the portfolio expected return and variance are

$$\begin{aligned} \mu_p &= \boldsymbol{\mu}^T \boldsymbol{x}, \\ \sigma_p^2 &= \boldsymbol{x}^T \boldsymbol{\Sigma} \boldsymbol{x}, \end{aligned} \tag{8}$$

where μ_p is the portfolio expected return, σ_p^2 is the portfolio variance, and $\boldsymbol{x} \in \mathbb{R}^n$ is the vector of asset weights, i.e. the proportion of wealth invested in each asset. Here, the portfolio variance is regarded as the financial risk measure. The problem of optimal asset allocation can be solved by finding a solution $\boldsymbol{x}^*$ that maximizes our utility subject to a set of prescribed constraints. Typical variants of this framework seek to maximize return while limiting our portfolio variance, or to attain an optimal compromise between risk and expected return. However, in this paper we limit ourselves to the version that attempts to minimize portfolio variance subject to a portfolio expected return constraint.

The next two subsections introduce the variants of MVO and minimum variance optimization that will be implemented during the computational experiment in Section 4. First, we present the basis, or *nominal*, investment models, followed by a brief introduction of a robust variant. Second, we present the regime-dependent investment models. Thus, we have two investment models, and three variants per model: (*i*) nominal, (*ii*) robust, and (*iii*) regime-dependent.

3.1 Nominal and robust models

The MVO model seeks to attain an optimal balance between risk and return. In the case of the nominal model, this takes a deterministic approach where the return and risk are perfectly described by the estimated parameters in Equation (8).

If we seek to minimize risk while subject to a target return, the MVO model can be written as

$$\begin{aligned} \min_{x} \quad & \boldsymbol{x}^T \boldsymbol{\Sigma} \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{\mu}^T \boldsymbol{x} \geq R, \\ & \mathbf{1}^T \boldsymbol{x} = 1, \end{aligned} \tag{9}$$

where the first constraint ensures we find portfolio a with an expected return that meets or exceeds a predetermined target, $R \in \mathbb{R}$, and the second constraint is the 'budget constraint', ensuring that the entirety of the available portfolio wealth is spent. We use the notation $\mathbf{1}$ to denote a column vector of appropriate size where all entries are equal to one. We will refer to this model as the nominal MVO problem for the remainder of this paper.

The MVO problem in Equation (9) is a typical example of a quadratic optimization problem with linear constraints. Given that the asset covariance matrix is positive semi-definite[4], this is a convex quadratic problem. This, in turn, implies the problem can be efficiently solved computationally and that any solution we find within the feasible set is a global optimal solution.

Selecting an appropriate target return R should be considered carefully. An excessively high target may lead to unreasonable portfolios with extreme long and short positions, or may even lead to infeasibility of the optimization problem itself. A reasonable approach to size the target return is to define it in terms of the asset expected returns $\boldsymbol{\mu}$. For example, we may demand to have a premium $\kappa \in \mathbb{R}$ above the average asset expected return, i.e.,

$$R = \left(\frac{1 + \kappa}{n} \right) \mathbf{1}^T \boldsymbol{\mu}. \tag{10}$$

The choice of an adequate value of κ retains the feasibility of the optimization problem while maintaining a realistic target return.

Next, we introduce the minimum variance model. This optimization model seeks to minimize the portfolio risk without imposing any conditions on the portfolio expected return. Thus, the only

[4]A useful property of covariance matrices in finance is their tendency to be positive definite or positive semi-definite.

input required is the asset covariance matrix. The nominal minimum variance problem is

$$\begin{aligned} \min_{x} \quad & \boldsymbol{x}^T \boldsymbol{\Sigma} \boldsymbol{x} \\ \text{s.t.} \quad & \mathbf{1}^T \boldsymbol{x} = 1. \end{aligned} \tag{11}$$

In essence, minimum variance is the same as MVO, except we relax the optimization model by removing the target return constraint, allowing the model to find the optimal portfolio with the lowest attainable variance.

Finally, we consider a robust variant of the MVO and minimum variance models. The robust optimization framework implemented in this paper is presented in detail in Goldfarb and Iyengar (2003), but is not described further in this paper for the sake of brevity. We chose this particular robust framework because it derives the uncertainty structure directly from a factor model. Specifically, these uncertainty structures are built around the estimated returns $\boldsymbol{\mu}$, factor loadings $\boldsymbol{V}$, and residual variance $\boldsymbol{D}$. As with the nominal models, the only difference between the robust MVO and robust minimum variance model is the inclusion of a target return constraint, subject to the appropriate uncertainty structure. The formulation of the robust optimization models is presented in detail in Appendix A. These robust variants will serve as an additional benchmark against the proposed regime-dependent models.

3.2 Regime-dependent models

Here, we introduce the regime-dependent investment models. As discussed Section 2, the cyclical nature of the market is fully captured through the estimated regime-dependent parameters $\boldsymbol{\mu}_{s_i}$ and $\boldsymbol{\Sigma}_{s_i}$ from Equation (7). Thus, the regime-dependent MVO variant can be easily attained by replacing the nominal input parameters with their regime-dependent counterparts, i.e.,

$$\begin{aligned} \min_{x} \quad & \boldsymbol{x}^T \boldsymbol{\Sigma}_{s_i} \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{\mu}_{s_i}^T \boldsymbol{x} \geq R_{s_i}, \\ & \mathbf{1}^T \boldsymbol{x} = 1, \end{aligned} \tag{12}$$

where the target return $R_{s_i} \in \mathbb{R}$ is based on the regime-dependent estimated returns $\boldsymbol{\mu}_{s_i}$ in a similar fashion to Equation (10). We will refer to this model as the regime-dependent MVO model.

We take a moment to highlight the importance of the target return in Equation 12. Similar to the nominal model, this constraint ensures the portfolio expected return satisfies a predetermined value. However, calculating the target return from the regime-dependent estimated returns $\boldsymbol{\mu}_{s_i}$ has another property: it sets a target return that is adapted to the current market environment. In other words, depending on the current market regime, the regime-dependent target return R_{s_i} will be higher or lower than its nominal counterpart, allowing the portfolio to have a more realistic expectation of an attainable return.

The final variant to discuss is the regime-dependent minimum variance model, which, as before, can be attained by removing the target return constraint. Thus, we have

$$\begin{aligned} \min_{x} \quad & \boldsymbol{x}^T \boldsymbol{\Sigma}_{s_i} \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{1}^T \boldsymbol{x} = 1. \end{aligned} \tag{13}$$

The reason we consider the minimum variance problem in this paper is to determine whether a regime-switching model can be beneficial even when estimated expected returns are not used as an input during portfolio optimization. The conclusion in Costa and Kwon (2018) indicates this is beneficial in risk parity portfolios, where expected returns are also not required. Thus, we wish to test whether this conclusion extends to other quantitative investment models.

Mathematically, the regime-dependent optimization models have the same structure as their nominal counterparts. This means we retain the same level of complexity as the nominal models: these models are quadratic and convex, guaranteeing a global optimal solution. From a mathematical optimization point of view, both the nominal and regime-dependent models are generic quadratic problems. However, if we look beyond the optimization framework, we can see that the models are fundamentally different given their input parameters. The regime-dependent models seek to minimize risk for a given market regime, thereby tailoring the resulting portfolio accordingly. This is in direct contrast to the nominal models, which yield a portfolio befitting a non-cyclical market.

We note that both investment models (MVO and minimum variance) and their three corresponding variants (nominal, robust, and regime-dependent) allow for the construction of portfolios with short positions, i.e., we do not enforce a 'long-only' constraint. The regime-switching model in this paper is only concerned with the estimated input parameters. Thus, any typical constraints on the wealth allocation variable $\boldsymbol{x}$ are allowed, such as limiting our exposure to certain industries or enforcing 'long-only' positions.

4 Computational experiment

In this section we perform a computational experiment to test the out-of-sample performance of the regime-dependent investment models relative to their nominal and robust counterparts. We chose the Fama–French three-factor model (Fama and French, 1993) as the basis to test the regime-switching framework due to its popularity as a trustworthy multiple factor model. However, we note that this framework is applicable to any model with market-dependent factors. The tractability and scalability of the proposed framework is tested by using a basket of 30 diverse assets. These assets consist of 30 different industries and will serve as the constituents of our optimal portfolios. Table 1 lists the 30 industries under consideration. The historical asset and factor monthly returns were were obtained from Kenneth R. French's data library (French, 2018).

Table 1: List of assets

Food Products	Tobacco	Beer and Liquor	Recreation	Printing
Household Products	Apparel	Healthcare	Chemicals	Textiles
Fabricated Products	Construction	Steel Works	Electrical Equip.	Automobiles
Aircraft, Ships, Rail Equip.	Mining	Coal	Oil and Gas	Utilities
Communication	Services	Business Equip.	Paper	Transportation
Restaurants and Hotels	Wholesale	Retail	Financials	Other

An overview of the experimental setup follows. We test the performance of both MVO and minimum variance models under each of the three variants described in Section 3: nominal, robust, and regime-dependent. Thus, we will construct and test six different portfolios. For the nominal and robust models, the asset expected returns and covariance matrix are estimated through an ordinary least squares regression stemming from the factor model in Equation (2), using two years of historical data (24 observations) immediately preceding the investment period. All estimated parameters are re-calibrated every time the portfolios are rebalanced. In addition, the robust portfolios are constructed as outlined in Appendix A using a confidence level of 0.95. The construction of the regime-dependent portfolios is explained below, and the rebalancing policy is explained thereafter.

The estimation of the regime-dependent parameters requires additional historical data to properly calibrate the Markov model. For this experiment, we use historical factor returns starting from Jan-1973, 30 years before the start of the out-of-sample test period. We apply the Baum–Welch algorithm (Baum et al., 1970) to find both the transition probabilities and the smoothed probabilities, with a practical implementation of this algorithm derived from Kritzman et al. (2012). We use the 'excess market return' factor as the sole input to the Baum–Welch algorithm. Assuming a two-regime model, the smoothed probabilities serve to differentiate between bullish and bearish historical periods. This data-driven approach allows us to select periods of historical factor and asset returns corresponding to each regime. For consistency, the most recent 24 observations corresponding to each regime are used to perform the regression. We note that these 48 observations stretch beyond the immediately preceding two years of historical data, and go as far back in time as necessary to ensure the availability of sufficient data for both regimes. We favour this approach to guarantee a consistent number of observations per regime during estimation, improving the stability of the regression model. The appropriate transition probabilities γ_{i1} and γ_{i2} are selected by identifying the 'current' regime, i.e., by inspecting the most recent observation of the smoothed probabilities and selecting the largest of the two. After the two regime-dependent regression models are calibrated, and together with the appropriate transition probabilities, the asset expected returns and covariance matrix are estimated as per Equation (7). This process is repeated every time the portfolios are rebalanced. Finally, we note that the the regime estimation window, which is originally 30 years long, expands every time we rebalance the portfolio, i.e., the Baum–Welch algorithm is applied over an increasingly long time period as the experiment moves forward through time.

The out-of-sample experiments are performed over the time period from Jan-2003 to Jun-2018. We conduct three individual experiments over this period, each with a different rebalancing policy, where the portfolios are rebalanced every three, six, or twelve months. Our choice of rebalancing frequency is motivated by practical choices made by investors (Michaud and Michaud, 2008). For all three experiments, the in-sample calibration window is rolled forward every time a portfolio is rebalanced in order to use the most recently available data. The experimental results are given in the following format. First, we present an example of the estimated regime changes. Next, we present the results of the MVO portfolios under the three different rebalancing policies. Finally, we present the results of the minimum variance portfolios.

All computations were performed on an Apple MacBook Pro computer (2.8 GHz Intel Core i7, 16 GB 2133 MHz DDR3 RAM) running OS X 'High Sierra'. All computations were performed using the Julia programming language (version 0.6.2) using the optimization modeling language JuMP (Dunning et al., 2017) with IPOPT (version 3.12.6) as the optimization solver.

An example of the estimated changes in regime from this experiment is presented in Figure 1, which plots the smoothed probabilities of a two-regime model prior to the start of the first investment period and the last investment period. The assumption that the factors within the factor model are subject to the market's cyclical behaviour is only valid if at least one of the factors is market-dependent. In our experiment, all three Fama–French factors are market-dependent. However, we assume the 'excess market return' factor conveys the latent market information more faithfully. Thus, the Baum–Welch algorithm is applied only on the observations corresponding to this factor. Using a single factor during the estimation process is less computationally expensive and tends to reduce the level of noise during the estimation process. Nevertheless, this data-driven process may sometimes result in unusual behaviour. For example, a bullish market regime is observed to dominate throughout the first estimate in Figure 1 (top plot). However, this same period (1973 to 2002) appears to be considered mostly bearish when we use data from 1973 to 2017 (bottom plot). Since, by design, we have imposed a two-regime model, then the observed probability distribution that governs a specific regime may change as new data becomes available, i.e., a period that was once considered to be bullish may be considered bearish if new data exacerbates the differences between the two regimes.

4.1 MVO experiments

The nominal, robust, and regime-dependent MVO portfolios were tested under three different rebalancing policies to observe whether the directional insight of the regime-switching framework is able to provide any significant benefit over both the nominal and robust portfolios.

The wealth evolution of the MVO portfolios is shown in Figure 2. Within the figure, a shaded or white background indicates whether the regime-switching portfolio was calibrated to align with

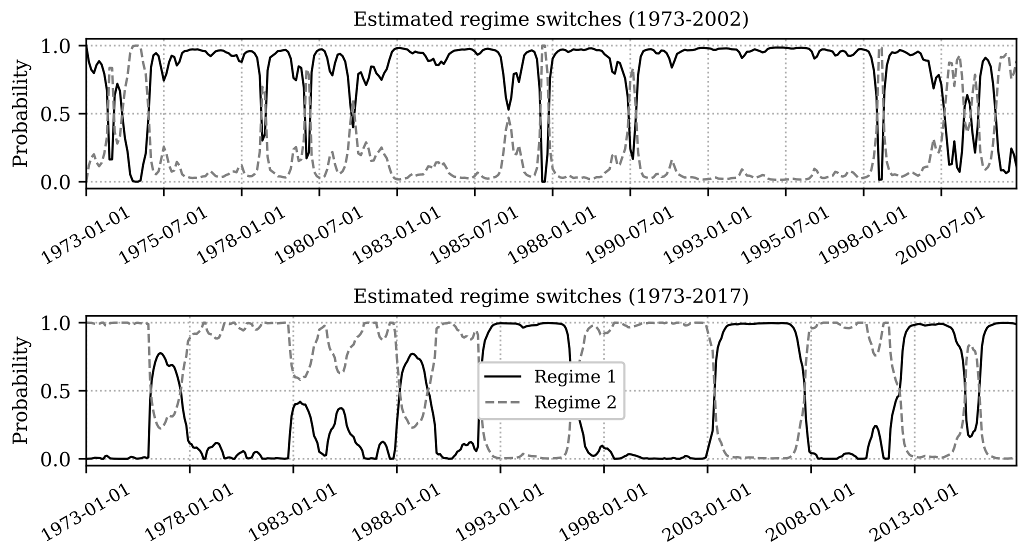

Figure 1: Estimated regime switches over the calibration window before the first investment period (top) and before the last investment period (bottom). Regime 1 corresponds to a bullish market.

a bearish or bullish market regime, respectively. We note that both the white and shaded periods are chosen on an ex-ante basis by the Baum–Welch algorithm, i.e., the model estimates the 'current' regime using only data from the calibration period during each rebalancing point, and is blind from any future data. The resulting estimated regimes align well with observed historical periods of market distress, such as the end of the early 2000's recession and the financial crisis of 2008.

The plots of the three-, six-, and twelve-month rebalancing policy in Figure 2 show that both the robust and regime-dependent portfolios are able to attain a higher terminal wealth than the nominal portfolio. However, upon closer inspection, only the regime-dependent portfolio is able to consistently outperform the nominal. In particular, the robust portfolio suffers a sharp downturn around the end of 2008, with the regime-dependent portfolio being able to remain above the nominal during this period. In addition, the robust portfolio exhibits a more volatile wealth path, oscillating above and below the path of the regime-dependent portfolio.

As discussed in Section 3, another attribute of the regime-dependent MVO model is the selection of a target return. The target return imposes a constraint before optimization takes place, with a lower target relaxing the optimization problem. For this experiment, we ask for a ten percent premium over the average asset expected return, i.e., $\kappa = 0.1$ in Equation (10). Figure 3 shows the difference in the target returns used during optimization of the regime-dependent and non-regime-dependent MVO portfolios, $R_{s_i} - R$, at each rebalancing point. The difference in target return allows us to observe the difference in predisposition of the portfolios towards the market each time the portfolios were optimized. From the three plots, it is possible to see that the regime-dependent portfolio has a lower target as it approaches the shaded regions, while it has a higher target as it approaches white regions.

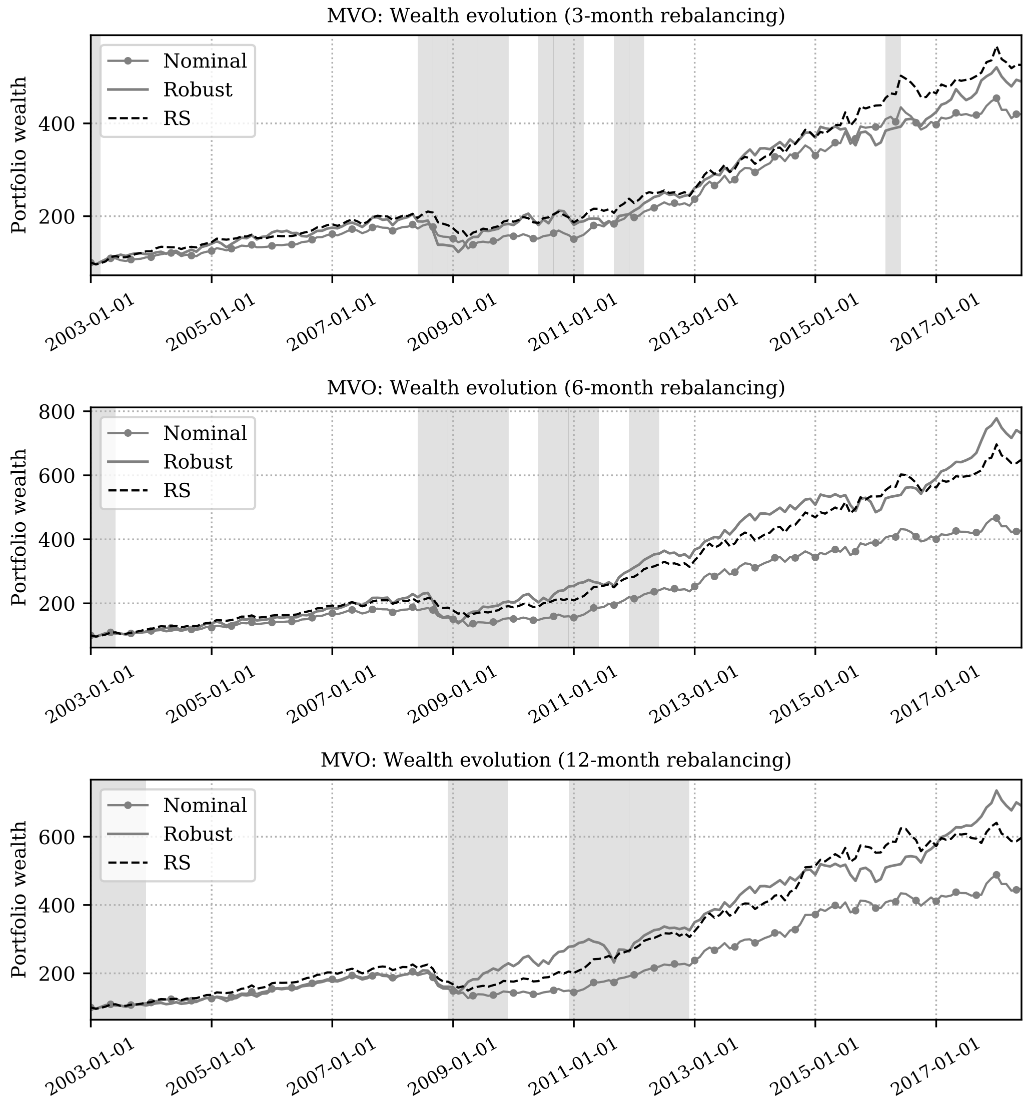

Figure 2: Wealth evolution of MVO portfolios. A shaded background indicates time periods where the regime-dependent portfolios were calibrated for a bearish market regime. A white background indicates calibration for a bullish market regime.

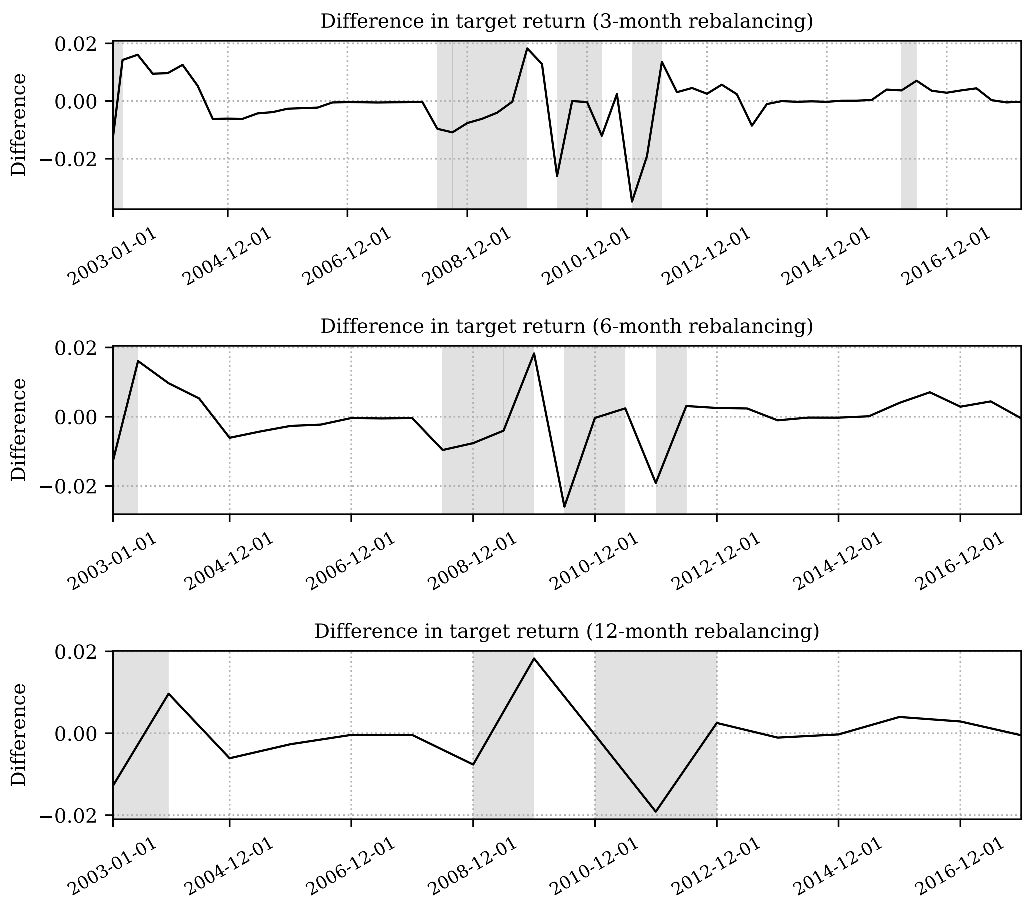

Figure 3: Difference in target return between regime-dependent and non-regime-dependent target returns.

Table 2 provides a summary of the portfolio performance throughout the entire investment horizon (2003–2018) for all three rebalancing policies. These results are calculated on an ex-post basis. From the table we can see that the regime-dependent MVO portfolio has a consistently better performance than the nominal portfolio in terms of both risk and reward. On the other hand, the robust MVO portfolio exhibits mixed results. The robust portfolio has a high rate of return, but maintains an equally high volatility, eroding its risk-adjusted returns. The risk-adjusted returns are given by the corresponding ex-post Sharpe ratios (Sharpe, 1994). With this in mind, we can see that the regime-dependent portfolio was able to deliver higher risk-adjusted returns over the entire investment horizon, consistently outperforming both the nominal and robust models. The average turnover rate, which we use to represent potential transaction costs, shows the regime-dependent portfolio is able to maintain a comparable turnover rate to its nominal counterpart throughout all three rebalancing policies.

Table 2: MVO summary of results

	3-month			6-month			12-month		
	Nom.	Rob.	RS	Nom.	Rob.	RS	Nom.	Rob.	RS
Ann. Return	0.098	0.109	0.114	0.099	0.138	0.129	0.102	0.134	0.123
Ann. Volatility	0.111	0.141	0.108	0.110	0.140	0.107	0.116	0.140	0.113
Sharpe Ratio	0.881	0.772	1.052	0.900	0.983	1.211	0.878	0.953	1.083
Avg. Turnover	1.138	1.785	1.314	1.602	1.880	1.698	2.230	1.724	2.114

4.2 Minimum variance experiments

The inclusion of the minimum variance portfolios as part of the computational experiments serve to test whether the regime-switching framework can have a measurable effect on investment models that do not necessitate estimated expected returns as an input parameter. In this sense, this computational experiment would also serve to further corroborate the findings in Costa and Kwon (2018), where a similar regime-switching framework is used in risk parity portfolios, which are also independent of estimated expected returns.

Figure 4 shows the wealth evolution of the minimum variance portfolios. As before, the shaded and white backgrounds indicate the predisposition of the regime-dependent portfolio to a bearish or bullish market, respectively. For all three rebalancing policies, the wealth path of the regime-dependent model steadily outpaces both its nominal and robust counterparts. This is particularly interesting because these results are fully driven by the estimated covariance matrices. In particular, the nominal and regime-dependent portfolios closely mimic each other during the first half of the investment horizon, and the regime-dependent model is only able to gain a measurable lead after the financial crisis of 2008. This effect is more pronounced as the rebalancing window increases. Moreover, the three plots suggest the robust minimum variance model have an excessively conservative outlook, yielding meager results.

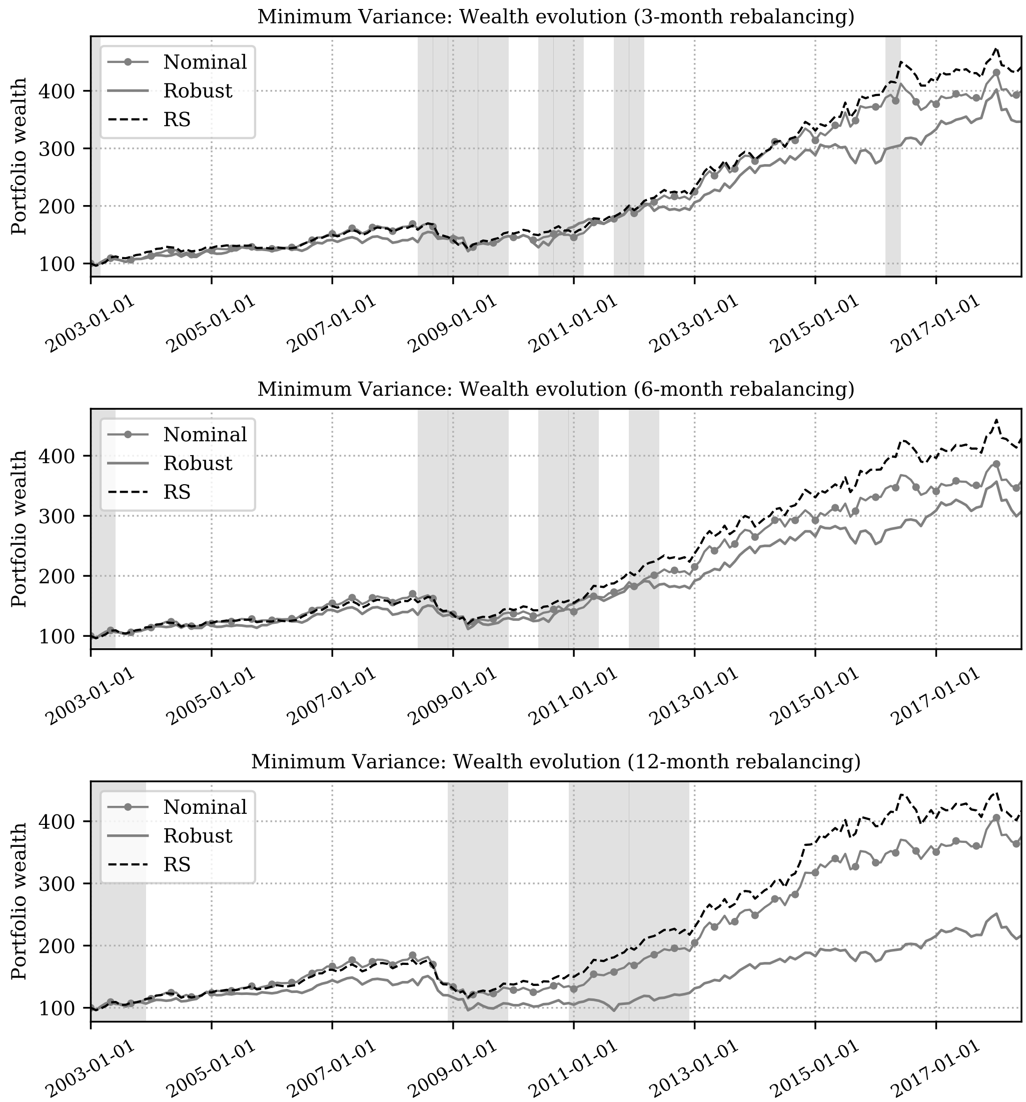

Figure 4: Wealth evolution of minimum variance portfolios. A shaded background indicates time periods where the regime-dependent portfolios were calibrated for a bearish market regime. A white background indicates calibration for a bullish market regime.

Finally, Table 3 provides a summary of the portfolio performance for all three rebalancing policies. By design, the rate of return attained by minimum variance portfolios is not driven by any target preset during optimization. Thus, it is difficult to attribute a higher rate of return directly to having better quality estimates of the covariance matrix. Nevertheless, this higher rate of return of the regime-dependent portfolio may be indirectly driven by the lower volatility observed in all three experiments, sheltering the portfolio from undesired downward movements. The ex-post Sharpe ratios reach the same conclusion as that from the MVO portfolios: the regime-dependent portfolio exhibits higher risk-adjusted returns under all three rebalancing policies while maintaining a similar average turnover rate as its nominal counterpart.

Table 3: Minimum variance summary of results

	3-month			6-month			12-month		
	Nom.	Rob.	RS	Nom.	Rob.	RS	Nom.	Rob.	RS
Ann. Return	0.094	0.084	0.101	0.086	0.075	0.099	0.090	0.051	0.097
Ann. Volatility	0.109	0.120	0.105	0.109	0.119	0.104	0.114	0.124	0.110
Sharpe Ratio	0.863	0.698	0.963	0.791	0.633	0.950	0.788	0.415	0.881
Avg. Turnover	0.940	1.191	1.013	1.355	1.622	1.376	1.917	1.831	1.767

5 Conclusion

This paper introduced a regime-switching factor model that reflects the cyclical nature of asset returns in modern financial markets. This model retains the benefits of a traditional factor model, namely the economic relevance of the factors and its ability to easily derive the asset expected returns and covariance matrix. Deriving these parameters through the regime-switching factor model implicitly captures the directional information from the market through a probability-weighted approach, adapting the parameters to the current market regime. In turn, using these parameters during MVO poises the resulting optimal portfolio to take advantage of this directional information, resulting in portfolios with reduced ex-post volatility and increased returns. The novelty of the proposed framework is the seamless integration of the regime-dependency of the asset returns with the static nature of MVO, reconciling it with the dynamism of the market through periodic portfolio rebalancing.

The regime-switching factor model can be easily derived from any generic model where at least one of the factors observes the cyclical behaviour of the market. Explaining the transition between market regimes through a discrete-time Markov chain allows us to retain a data-driven modeling structure where we can use historical observations to calibrate our model and estimate the regime-dependent asset expected returns and covariance matrix. Thus, this aligns well with a single-period portfolio optimization model like MVO. Since regime-dependency is already captured through the estimated parameters, we highlight that this framework comes at no additional computational cost during optimization.

The experimental results show that a regime-dependent MVO portfolio is able to steadily outperform its nominal counterpart over a long investment horizon, exhibiting lower volatility and higher returns. These results are consistent over three different rebalancing policies of varying lengths, showing that regime-dependent portfolios are quite resilient. Provided some instance of periodic rebalancing takes place, the portfolio is able to adapt itself to the market regime. Even if some regime misalignment exists through time between the portfolio and the market, the results show that this is outweighed by the benefit of having some partial alignment throughout the investment horizon. Moreover, the proposed model is also shown to be beneficial even when compared against a robust MVO portfolio. The regime-dependent portfolio is able to deliver higher risk-adjusted returns than both the competing nominal and robust portfolios. These results are sustained even when we do not apply a portfolio target return.

The application of this regime-switching factor model in portfolio optimization opens the door to future lines of research. Extrapolating from the experimental results, a possible improvement is to formulate a robust counterpart of the regime-dependent portfolios by introducing uncertainty structures as in Goldfarb and Iyengar (2003). Motivated by the experimental results in Costa and Kwon (2018), another exciting avenue of future research is the study of rebalancing control policies for single-period portfolio optimization models. This, in turn, could lend itself to develop a data-driven framework that identifies pertinent market signals, such as changes in regime, to trigger portfolio rebalancing.

Appendix A Robust optimization models

The robust MVO and robust minimum variance models discussed in this paper are an implementation of the framework introduced in Goldfarb and Iyengar (2003). The robust models attempt to mitigate the impact on optimization arising from parameter uncertainty. In turn, the parameter uncertainty is a natural occurrence in statistics from the estimation of parameters from raw data.

This robust framework is applicable to parameters estimated from factor models, where the uncertainty structure is based around the estimated regression coefficients. We place uncertainty sets around the intercept of regression (i.e., the asset expected returns), the factor loadings matrix, and the residual variance matrix

$$\begin{aligned} S_\mu = \quad & \left\{\boldsymbol{\mu} : \ \boldsymbol{\mu} = \boldsymbol{\mu}_0 + \boldsymbol{\xi}, \ |\xi_i| \leq \gamma_i, \ i = 1, ..., n\right\}, \\ S_D = \quad & \left\{\boldsymbol{D} : \ \boldsymbol{D} = \mathrm{diag}(\boldsymbol{d}), \ d_i \in [\underline{d}_i, \bar{d}_i], \ i = 1, ..., n\right\}, \\ S_V = \quad & \left\{\boldsymbol{V} : \ \boldsymbol{V} = \boldsymbol{V}_0 + \boldsymbol{W}, \ ||\boldsymbol{W}_i||_g \leq \rho_i, \ i = 1, ..., n\right\}, \end{aligned} \tag{14}$$

where d_i is the idiosyncratic risk of asset i, $\boldsymbol{W}_i$ is the i-th column of strength matrix $\boldsymbol{W}$ around $\boldsymbol{V}_0$ and $||\boldsymbol{W}_i||_g = \sqrt{\boldsymbol{W}_i^T \boldsymbol{G} \boldsymbol{W}_i}$ denotes an elliptic norm with respect to a symmetric, positive definite matrix $\boldsymbol{G}$.

For both the minimum variance model and our choice of the MVO model, the objective is to minimize variance. Applying the factor model structure, and considering the uncertainty sets from

Equation (14), we can present the following equivalence

$$\min_{x} \max_{V\in S_v,\ D\in S_d} \boldsymbol{x}^T\boldsymbol{\Sigma}\boldsymbol{x} \quad \Longleftrightarrow \quad \min_{x} \max_{V\in S_v,\ D\in S_d} \boldsymbol{x}^T\boldsymbol{V}^T\boldsymbol{F}\boldsymbol{V}\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{D}\boldsymbol{x}. \tag{15}$$

The nature of S_d shows that the residual portfolio variance has a trivial upper bound, i.e., $\boldsymbol{x}^T\boldsymbol{D}\boldsymbol{x} \leq \boldsymbol{x}^T\bar{\boldsymbol{D}}\boldsymbol{x}$, where $\bar{\boldsymbol{D}} = \text{diag}(\bar{\boldsymbol{d}})$. Therefore, we can rewrite Equation (15) as

$$\begin{array}{ll} \min\limits_{x,\delta} \max\limits_{V\in S_v} & \boldsymbol{x}^T\boldsymbol{V}^T\boldsymbol{F}\boldsymbol{V}\boldsymbol{x} + \delta \\ \text{s.t.} & \left|\left|\begin{bmatrix} 2\bar{\boldsymbol{D}}^{\frac{1}{2}}\boldsymbol{x} \\ 1-\delta \end{bmatrix}\right|\right| \leq 1+\delta. \end{array}$$

Reformulating the remainder of the objective function to incorporate S_V is not as straightforward. It is possible to reformulate the 'worst-case variance' as a collection of linear equality and inequality constraints, as well as restricted hyperbolic constraints. We have shown how to do this for the residual portfolio variance, and the same can be done for the systematic component of financial risk. The following lemma is taken from Goldfarb and Iyengar (2003), and is shown below for the reader's convenience.

Lemma 1 *Let $r, \nu > 0$, $\boldsymbol{y_0} = \boldsymbol{V}_0\boldsymbol{x}$, $\boldsymbol{y_0}, \boldsymbol{y} \in \mathbb{R}^m$ and $\boldsymbol{F}, \boldsymbol{G} \in \mathbb{R}^{m\times m}$ be positive definite matrices. Then the constraint*

$$\max_{\boldsymbol{y}:||\boldsymbol{y}||_g\leq r} ||\boldsymbol{y}_0 + \boldsymbol{y}||_f^2 \leq \nu,$$

where $||\boldsymbol{x}||_f : \boldsymbol{x} \to \sqrt{\boldsymbol{x}^T\boldsymbol{F}\boldsymbol{x}}$, is equivalent to either of the following:

(i) *There exist $\tau, \sigma \geq 0$, and $t \in \mathbb{R}^m_+$ that satisfy*

$$\begin{aligned} \nu &\geq \tau + \mathbf{1}^T t, \\ \sigma &\leq \frac{1}{\lambda_{\max}(H)}, \\ r^2 &\leq \sigma\tau, \\ w_k^2 &\leq (1-\sigma\lambda_k)t_k, \quad k = 1, ..., m, \end{aligned}$$

where $\boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^T$ is the spectral decomposition of $\boldsymbol{H} = \boldsymbol{G}^{-\frac{1}{2}}\boldsymbol{F}\boldsymbol{G}^{-\frac{1}{2}}, \boldsymbol{\Lambda} = \text{diag}(\lambda_i)$, $\boldsymbol{w} = \boldsymbol{Q}^T\boldsymbol{H}^{\frac{1}{2}}\boldsymbol{G}^{\frac{1}{2}}\boldsymbol{y}_0$, and $\lambda_{\max}(\boldsymbol{H})$ is the largest eigenvalue of $\boldsymbol{H}$.

(ii) *There exist $\tau \geq 0$, and $s \in \mathbb{R}^m_+$ that satisfy*

$$\begin{aligned} r^2 &\leq \tau(\nu - \mathbf{1}^T s), \\ u_k^2 &\leq (1-\tau\theta_k)s_k, \quad k = 1, ..., m, \\ \tau &\leq \frac{1}{\lambda_{\max}(K)}, \end{aligned}$$

where $\boldsymbol{P\Theta P}^T$ *is the spectral decomposition of* $\boldsymbol{K} = \boldsymbol{F}^{\frac{1}{2}}\boldsymbol{G}^{-1}\boldsymbol{F}^{\frac{1}{2}}, \boldsymbol{\Theta} = \mathrm{diag}(\theta_i)$, $\boldsymbol{u} = \boldsymbol{P}^T\boldsymbol{F}^{\frac{1}{2}}\boldsymbol{y}_0$, *and* $\lambda_{\max}(\boldsymbol{K})$ *is the largest eigenvalue of* $\boldsymbol{K}$.

Lemma 1 is proved by using the $\mathbb{S}$-procedure described in Boyd et al. (1994)[5]. The size of the uncertainty sets given in (14) are controlled by the joint confidence level ω, which is incorporated into the computation of $\boldsymbol{\rho}$ in Lemma 1. The reader can find the equation relating $\boldsymbol{\rho}$ to ω in (20) in Appendix A.1. A high joint confidence level implies we have a large uncertainty set, i.e. one demands robustness with respect to a very large set of parameter values. The typical choices of ω lie in the range 0.95–0.99.

Using part (ii) of Lemma 1, the nominal objective function is reformulated into its robust counterpart as follows,

$$
\begin{array}{lll}
\displaystyle\min_{x,\nu,\delta,\tau,s} & \nu+\delta & \\
\text{s.t.} & \left\|\begin{bmatrix} 2\bar{\boldsymbol{D}}^{\frac{1}{2}}\boldsymbol{x} \\ 1-\delta \end{bmatrix}\right\| & \leq 1+\delta \\
 & \left\|\begin{bmatrix} 2\boldsymbol{\rho}^T\boldsymbol{x} \\ \tau-\nu+\mathbf{1}^T\boldsymbol{s} \end{bmatrix}\right\| & \leq \tau+\nu-\mathbf{1}^T s \\
 & \left\|\begin{bmatrix} 2u_k \\ 1-\tau\theta_k-s_k \end{bmatrix}\right\| & \leq 1-\tau\theta_k+s_k, \quad k=1,...,m \\
 & \tau-\dfrac{1}{\lambda_{\max}(K)} & \leq 0 \\
 & \tau & \geq 0.
\end{array}
\tag{16}
$$

Next, we proceed to incorporate the uncertainty set around the estimated asset expected returns. This requires the introduction of the auxiliary variable $\boldsymbol{\psi} \in \mathbb{R}^n$, which will serve to take the absolute value of the decision variable $\boldsymbol{x}$. To summarize the robust MVO model, we include the reformulated objective function from Equation (16) and add to it the corresponding robust expected return

[5]For a detailed proof of Lemma 1 see Goldfarb and Iyengar (2003)

constraints, as well as the budget constraint. The complete problem is shown below

$$
\begin{array}{lll}
\min\limits_{x,\psi,\nu,\delta,\tau,s} & \nu+\delta & \\
\text{s.t.} & \left\| \begin{bmatrix} 2\bar{\boldsymbol{D}}^{\frac{1}{2}}\boldsymbol{x} \\ 1-\delta \end{bmatrix} \right\| & \leq 1+\delta \\
& \left\| \begin{bmatrix} 2\boldsymbol{\rho}^T\boldsymbol{x} \\ \tau-\nu+\mathbf{1}^T\boldsymbol{s} \end{bmatrix} \right\| & \leq \tau+\nu-\mathbf{1}^T\boldsymbol{s} \\
& \left\| \begin{bmatrix} 2u_k \\ 1-\tau\theta_k-s_k \end{bmatrix} \right\| & \leq 1-\tau\theta_k+s_k, \quad k=1,...,m \\
& \tau-\dfrac{1}{\lambda_{\max}(K)} & \leq 0 \\
& \boldsymbol{\mu}_0^T\boldsymbol{x}-\boldsymbol{\gamma}^T\boldsymbol{\psi} & \geq R \\
& \psi_i & \geq x_i, \qquad i=1,...,n \\
& \psi_i & \geq -x_i, \quad\; i=1,...,n \\
& \mathbf{1}^T\boldsymbol{x} & =1, \\
& \tau & \geq 0,
\end{array}
\tag{17}
$$

The robust minimum variance model is the same as this model, except it ignores the constraint pertaining to the target return and does not necessitate the auxiliary variable $\boldsymbol{\psi}$ or its corresponding constraints.

A.1 Robust parameter estimation

The parameters used in the robust optimization models arise from the procedure described in Goldfarb and Iyengar (2003). Let $\boldsymbol{r}_i \in \mathbb{R}^p$ be the vector of historical returns of asset i and let $\boldsymbol{B} = [\boldsymbol{f}^1\boldsymbol{f}^2...\boldsymbol{f}^p] \in \mathbb{R}^{m\times p}$ be the matrix of factor returns over all the periods $t = 1,...,p$. We describe the vector of asset returns in a different but equivalent form to Equation (1),

$$\boldsymbol{y}_i = \boldsymbol{A}\boldsymbol{X}_i + \epsilon_i, \tag{18}$$

where $\boldsymbol{y}_i = [r_i^1\ r_i^2\ ...\ r_i^p]^T$, $\boldsymbol{A} = [1\ B^T]$, $\boldsymbol{X}_i = [\mu_i\ V_{1i}\ V_{2i}\ ...\ V_{mi}]^T$ and $\boldsymbol{\epsilon}_i = [e_i^1\ e_i^2\ ...\ e_i^p]^T$. The least square estimate $\bar{\boldsymbol{X}}_i$ of the true parameter $\boldsymbol{X}_i$ is given by the solution to the equation

$$\bar{\boldsymbol{X}}_i = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{y}_i, \tag{19}$$

Substituting (18) into (19) yields

$$\bar{\boldsymbol{X}}_i - \boldsymbol{X}_i = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T\epsilon_i = \mathcal{N}\big(0,\ \sigma_i^2(\boldsymbol{A}^T\boldsymbol{A})^{-1}\big),$$

Although the true variance σ_i^2 is not known, in practice we replace σ_i^2 by $(m+1)s_i^2$, where s_i^2 is the unbiased estimate of σ_i^2. The unbiased estimate is given by

$$s_i^2 = \frac{||\boldsymbol{y}_i - \boldsymbol{A}\bar{\boldsymbol{X}}_i||}{p-m-1}$$

The resulting random variable

$$\mathcal{Y} = \frac{1}{\sigma_i^2}(\bar{\boldsymbol{X}}_i - \boldsymbol{X}_i)^T(\boldsymbol{A}^T\boldsymbol{A})(\bar{\boldsymbol{X}}_i - \boldsymbol{X}_i),$$

follows the F-distribution with $(m+1)$ degrees of freedom in the numerator and $(p-m-1)$ degrees of freedom in the denominator. By setting the ω-confidence region for sets $S_\mu(\omega)$ and $S_V(\omega)$, we obtain

$$\begin{aligned} S_\mu = &\quad \big\{\boldsymbol{\mu} :\ \boldsymbol{\mu} = \boldsymbol{\mu}_0 + \boldsymbol{\xi},\ |\xi_i| \le \gamma_i,\ i = 1, ..., n\big\}, \\ S_V = &\quad \big\{\boldsymbol{V} :\ \boldsymbol{V} = \boldsymbol{V}_0 + \boldsymbol{W},\ ||\boldsymbol{W}_i||_g \le \rho_i,\ i = 1, ..., n\big\}, \end{aligned}$$

where

$$\begin{aligned} &\boldsymbol{\mu}_0 = \bar{\boldsymbol{\mu}}, \\ &\gamma_i = \sqrt{(m+1)(\boldsymbol{A}^T\boldsymbol{A})^{-1}_{11} c_{m+1}(\omega) s_i^2}, \quad i = 1, ..., n, \end{aligned}$$

and

$$\begin{aligned} &\boldsymbol{V}_0 = \bar{\boldsymbol{V}}, \\ &\boldsymbol{G} = (\boldsymbol{Q}(\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{Q}^T)^{-1} = \boldsymbol{B}\boldsymbol{B}^T - \frac{1}{t}(\boldsymbol{B}\mathbf{1})(\boldsymbol{B}\mathbf{1})^T, \qquad (20) \\ &\rho_i = \sqrt{(m+1) c_{m+1}(\omega) s_i^2}, \quad i = 1, ..., n, \end{aligned}$$

where $c_J(\omega)$ is the ω-critical value. The upper bound of the variance of the residual is set to $\bar{\boldsymbol{D}} = \mathrm{diag}(s_i^2)$. Thus, all the necessary parameters to formulate a robust optimization problem can be extracted directly from raw market data.

Declarations of Interest

The authors report no conflicts of interest. This work was supported by a MITACS Accelerate grant.

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