

## Implementation of Gibbs Sampler

### Full conditional distributions

Gibbs sampler is the algorithm to sample each unknown variable based on their full conditional distributions. For each iteration of Gibbs sampler, we sample each parameter from their full conditional distributions and iterate through all the parameters. To present the full conditional distribution for all the parameters, we first re-express the model in matrix form.

$$\begin{aligned}\mathbf{y} &= \mathbf{1}\mu + \mathbf{Z}_g \mathbf{g} + \mathbf{Z}_h \mathbf{h} + (\mathbf{Z}_g \mathbf{b}) \# (\mathbf{Z}_h \mathbf{h}) + \boldsymbol{\varepsilon} \\ &= \mathbf{1}\mu + \mathbf{Z}_g \mathbf{L}_A \boldsymbol{\delta}_g + \mathbf{Z}_h \mathbf{L}_H \boldsymbol{\delta}_h + (\mathbf{Z}_g \mathbf{L}_A \boldsymbol{\delta}_b) \# [\mathbf{Z}_h \mathbf{L}_H \boldsymbol{\delta}_h] + \boldsymbol{\varepsilon}\end{aligned}$$

Where  $\#$  represents element-wise product between two vectors,  $\mathbf{y}$  is a vector of observed phenotypes,  $\mathbf{g}$  is a vector for the main genotype effects,  $\mathbf{Z}_g$  is the incidence matrix that relates each level of  $\mathbf{g}$  to  $\mathbf{y}$ ,  $\mathbf{h}$  is a vector for the main environment effects,  $\mathbf{Z}_h$  is the incidence matrix that related each level of  $\mathbf{h}$  to  $\mathbf{y}$ ,  $\mathbf{b}$  is the regression of the varieties on the main environmental effect. Instead of sampling  $\mathbf{b}$ ,  $\mathbf{g}$ ,  $\mathbf{h}$  directly, we transform them into an orthogonal basis as  $\boldsymbol{\delta}_g = \mathbf{L}_A^{-1} \mathbf{g}$ ,  $\boldsymbol{\delta}_b = \mathbf{L}_A^{-1} \mathbf{b}$  and  $\boldsymbol{\delta}_h = \mathbf{L}_H^{-1} \mathbf{h}$  where  $\mathbf{L}_A$  and  $\mathbf{L}_H$  are lower triangular matrix from the Cholesky decomposition of  $\mathbf{A}$  and  $\mathbf{H}$  ( $\mathbf{L}_A \mathbf{L}_A' = \mathbf{A}$ ,  $\mathbf{L}_H \mathbf{L}_H' = \mathbf{H}$ ). The  $\boldsymbol{\delta}_g$ ,  $\boldsymbol{\delta}_b$  and  $\boldsymbol{\delta}_h$  all follow IID Normal distribution:  $\boldsymbol{\delta}_g \sim N(\mathbf{0}, \mathbf{I}\sigma_g^2)$ ,  $\boldsymbol{\delta}_b \sim N(\mathbf{0}, \mathbf{I}\sigma_b^2)$  and  $\boldsymbol{\delta}_h \sim N(\mathbf{0}, \mathbf{I}\sigma_h^2)$ .

The overall mean  $\mu$  is assigned a flat prior. The scale parameters all follow a scaled inverse chi-square distribution:  $\sigma_\varepsilon^2 \sim \chi^{-2}(\nu_\varepsilon, S_\varepsilon^2)$ ,  $\sigma_g^2 \sim \chi^{-2}(\nu_g, S_g^2)$ ,  $\sigma_b^2 \sim \chi^{-2}(\nu_b, S_b^2)$ ,  $\sigma_h^2 \sim \chi^{-2}(\nu_h, S_h^2)$ . In the parameterization used we have  $E[\sigma^2] = \frac{\nu S^2}{\nu - 2}$ .

## General results for conjugate Normal distribution

We define a model with the general form  $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ , and with prior distributions  $\boldsymbol{\beta} \sim N(\mathbf{0}, \mathbf{I}\sigma_\beta^2)$ ,  $\boldsymbol{\varepsilon} \sim N(\mathbf{0}, \mathbf{I}\sigma_\varepsilon^2)$ ,  $\sigma_\varepsilon^2 \sim \chi^{-2}(\nu_\varepsilon, S_\varepsilon^2)$  and  $\sigma_\beta^2 \sim \chi^{-2}(\nu_\beta, S_\beta^2)$ . When sampling a location parameter  $\beta_i$  in the model, all other parameters will be treated as known. Let  $\tilde{\mathbf{y}} = \mathbf{y} - \mathbf{X}_{-i}\boldsymbol{\beta}_{-i} = \mathbf{X}_i\beta_i + \boldsymbol{\varepsilon}$ , where  $(-i)$  means all other columns (elements) except column (element)  $i$ . The conditional distribution of the new data vector  $\tilde{\mathbf{y}}$  will be  $p(\tilde{\mathbf{y}}|\mathbf{X}_i\beta_i) \sim N(\mathbf{X}_i\beta_i, \mathbf{I}\sigma_\varepsilon^2)$ . The full conditional posterior distribution for  $\beta_i$ ,  $\sigma_\varepsilon^2$  and  $\sigma_\beta^2$  will be:

$$p(\beta_i|\text{ELSE}) \sim N(C^{-1}r, C^{-1}), \text{ where } C = \frac{\mathbf{X}_i'\mathbf{X}_i}{\sigma_\varepsilon^2} + \frac{1}{\sigma_\beta^2} \text{ and } r = \frac{\mathbf{X}_i'\tilde{\mathbf{y}}}{\sigma_\varepsilon^2}$$

$$p(\sigma_\varepsilon^2|\text{ELSE}) \sim \chi^{-2}(n + \nu_\varepsilon, \frac{\nu_\varepsilon S_\varepsilon^2 + \boldsymbol{\varepsilon}'\boldsymbol{\varepsilon}}{n + \nu_\varepsilon}), \text{ where } n \text{ is the number of observations}$$

$$p(\sigma_\beta^2|\text{ELSE}) \sim \chi^{-2}(q_\beta + \nu_\beta, \frac{\nu_\beta S_\beta^2 + \boldsymbol{\beta}'\boldsymbol{\beta}}{q_\beta + \nu_\beta}), \text{ where } q_\beta \text{ is the number of levels in } \boldsymbol{\beta}.$$

Therefore, to sample a single location parameter from the conjugate Normal distribution, all we need to know is the incidence matrix  $\mathbf{X}_i$  and the form of  $C$ . We can use the above rules to express the full conditional distributions for the unknown parameters. In Table 1, we listed the  $\mathbf{X}_i$  and  $C$  for each of the location parameters for  $\mu$ ,  $\delta_{h_i}$ ,  $\delta_{b_i}$  and  $\delta_{g_i}$ .

Table 1 The  $\mathbf{X}_i$  and  $C$  needed to obtain the full conditional distributions for  $\mu$ ,  $\delta_{h_i}$ ,  $\delta_{b_i}$  and  $\delta_{g_i}$ .

Parameters	$\mathbf{X}_i$	$C$
$\mu$	$\mathbf{1}$	$\frac{\mathbf{1}'\mathbf{1}}{\sigma_\varepsilon^2}$

$\delta_{h_i}$	$\mathbf{Z}_h \mathbf{L}_{H_i} \# (\mathbf{Z}_g \mathbf{b} + \mathbf{1})$	$\frac{\mathbf{X}_i' \mathbf{X}_i}{\sigma_\varepsilon^2} + \frac{1}{\sigma_h^2}$
$\delta_{b_i}$	$(\mathbf{Z}_h \mathbf{h}) \# (\mathbf{Z}_g \mathbf{L}_{A_i})$	$\frac{\mathbf{X}_i' \mathbf{X}_i}{\sigma_\varepsilon^2} + \frac{1}{\sigma_h^2}$
$\delta_{g_i}$	$\mathbf{Z}_g \mathbf{L}_{A_i}$	$\frac{\mathbf{X}_i' \mathbf{X}_i}{\sigma_\varepsilon^2} + \frac{1}{\sigma_g^2}$

The full conditional distributions for the variance components will be:

$$p(\sigma_\varepsilon^2 | \text{ELSE}) \sim \chi^{-2}(n + \nu_\varepsilon, \frac{\nu_\varepsilon S_\varepsilon^2 + \boldsymbol{\varepsilon}' \boldsymbol{\varepsilon}}{n + \nu_\varepsilon}), \sigma_g^2 | \text{ELSE} \sim \chi^{-2}(q_g + \nu_g, \frac{\nu_g S_g^2 + \boldsymbol{\delta}_g' \boldsymbol{\delta}_g}{q_g + \nu_g}), \sigma_b^2 | \text{ELSE} \sim \chi^{-2}(q_g + \nu_b, \frac{\nu_b S_b^2 + \boldsymbol{\delta}_b' \boldsymbol{\delta}_b}{q_g + \nu_b})$$

and  $\sigma_h^2 | \text{ELSE} \sim \chi^{-2}(q_h + \nu_h, \frac{\nu_h S_h^2 + \boldsymbol{\delta}_h' \boldsymbol{\delta}_h}{q_h + \nu_h})$ , where  $q_g$  is the number of levels for the varieties, and  $q_h$  is the number of levels for the environments.

### Implementation of Gibbs Sampler

For each iteration of Gibbs Sampler, we sample each parameter from their full conditional distributions, the detailed implementation is as following:

1. Set initial values for all the parameters.
2. Set initial values for missing values of  $\mathbf{y}$  (denote here as  $\mathbf{y}_{NA}$ ).
3. Sample  $\delta_{h_i}$ , and update  $\boldsymbol{\varepsilon}$ . Do this for all  $i$  from 1 to  $q_h$ .
4. Sample  $\delta_{b_i}$  and update  $\boldsymbol{\varepsilon}$ ; sample  $\delta_{g_i}$  and update  $\boldsymbol{\varepsilon}$ . Do this for all  $i$  from 1 to  $q_g$ .
5. Sample  $\sigma_\varepsilon^2$ ,  $\sigma_h^2$ ,  $\sigma_b^2$  and  $\sigma_g^2$  from their full conditional distributions.
6. Sample  $\mu$  and update  $\boldsymbol{\varepsilon}$ .
7. Update  $\hat{\mathbf{y}} = \mathbf{y} - \boldsymbol{\varepsilon}$ .

8. Sample  $\mathbf{y}_{\text{NA}}$  from  $N(\hat{\mathbf{y}}_{\text{NA}}, \mathbf{I}\sigma_{\varepsilon}^2)$ . Update  $\mathbf{y}_{\text{NA}}$  and  $\boldsymbol{\varepsilon}$ .
9. Repeat steps 2 through 8 for the number of specified iterations.

## References

- Su, G., P. Madsen, M. S. Lund, D. Sorensen, I. R. Korsgaard, and J. Jensen. 2006.  
Bayesian analysis of the linear reaction norm model with unknown covariates.  
Journal Of Animal Science 84:1651-1657.