TRANSLATION OF LEONHARD EULER'S: GENERAL PRINCIPLES OF THE MOTION OF FLUIDS

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This is an adaptation by U. Frisch of an English translation by Thomas E. Burton of Euler's memoir 'Principes généraux du mouvement des fluides' (Euler, 1775b). Burton's translation appeared in *Fluid Dynamics* **34** (1999) pp. 801–82, Springer and is here adapted by permission. A detailed presentation of Euler's published work can be found in Truesdell, 1954. Euler's work is discussed also in the perspective of eighteenth century fluid dynamics research by Darrigol and Frisch, 2008. Explanatory footnotes have been supplied where necessary by G.K. Mikhailov and a few more by U. Frisch and O. Darrigol. Euler's memoir had neither footnotes nor a list of references.

- 1. Having established in my previous Memoir¹ the principles of fluid equilibrium in their most general form, regarding both the diverse nature of fluids and the forces that act upon them, I now propose to deal with the motion of fluids in the same way and to seek out the general principles on which the entire science of fluid motion is based. It will readily be understood that this is a much more difficult undertaking and involves studies of incomparably greater depth. Nevertheless, I hope to arrive at an equally successful conclusion, so that, if difficulties remain, they will pertain not to Mechanics but purely to Analysis, this science not yet having been brought to the degree of perfection necessary to develop analytical equations [formules]² that embody the principles of fluid motion.
- The task, then, is to discover the principles by means of which the motion of a fluid can be determined, whatever its state and whatever the forces to which it is subjected. To this end, we shall examine in detail all the elements which form the subject of our research and contain quantities both known and unknown. First of all, the nature of the fluid is assumed to be known, in which case it is necessary to consider its various forms since it may be compressible or incompressible. If it is not compressible, then there are two possibilities: either the entire mass is composed of homogeneous parts, whose density is everywhere and always the same, or it is composed of heterogeneous parts and in this case it is necessary to know the density of each component and the proportions of the mixture. If the fluid is compressible and its density is variable, we must know the law according to which its elasticity depends on the

density and whether the elasticity depends only on the density or also on some other property, such as heat,⁴ which is proper to each particle of fluid, at least for each instant of time.

- 3. It must also be assumed that the state of the fluid at a certain moment of time is known and I shall call this the initial state [état primitif] of the fluid. As this state is quasi-arbitrary, it is necessary, first of all, to know the distribution of the particles of which the fluid is composed and, unless in the initial state the fluid is at rest, the motion impressed upon them. However, the initial motion is not entirely arbitrary since both the continuity and the impenetrability of the fluid impose a certain limitation which I shall investigate below. Often, however, nothing is known of the initial state, for example when it is a question of determining the motion of a river, and then it is usually only possible to seek the steady state at which the fluid finally arrives, thereafter undergoing no further changes. Now, neither this circumstance nor the initial state in any way affect the investigation to be made and the calculations will always be the same. It is only in the integrations that they need to be taken into account for the purpose of determining the constants which every integration involves.
- 4. Thirdly, the data must include the external forces to which the fluid is subjected. I shall call these forces external to distinguish them from the internal forces which the fluid particles exert on each other and which will constitute the main topic for subsequent investigation. Thus, it could be assumed that the fluid is not exposed to any external force, unless it be natural gravity which is everywhere considered to be constant in magnitude and to act in the same direction. However, to generalize the investigation, I shall consider the fluid to be acted upon by forces which may be directed towards one or more centers or obey some other law with respect

¹ Euler, 1755a

 $^{^2}$ Bracketed words are from the original eighteenth century French text.

³ By elasticity [élasticité] Euler means that property of a fluid which is expressed in the creation of internal pressure and therefore uses the term on an equal footing with the term "pressure" (see § 5 below).

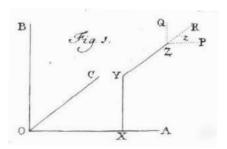
⁴ Essentially, heat [chaleur] should be taken to mean temperature.

to both magnitude and direction. As far as these forces are concerned, only their accelerating action is directly known, irrespective of the masses upon which they act. Accordingly, I shall introduce into the calculations only the accelerative forces, from which it will be easy to obtain the true motive forces by multiplying in each case the accelerative forces by the masses to which they are applied.⁵

5. Let us now turn to those elements which contain that which is unknown. In order properly to understand the motion that will be imparted to the fluid it is necessary to determine, for each instant and for each point, both the motion and the pressure [pression] of the fluid situated there. And if the fluid is compressible, it is also necessary to determine the density, knowing the above-mentioned other property which, together with the density, makes it possible to determine the elasticity. The latter, being counterbalanced by the fluid pressure, must be considered equal to that pressure, exactly as in the case of equilibrium, where I have developed these ideas more thoroughly. 6 Clearly, then, the number of quantities which enter into the study of fluid motion is much greater than in the case of equilibrium, since it is necessary to introduce letters which denote the motion of each particle and all these quantities may vary with time. Thus, in addition to the letters which determine the location of each conceivable point in the fluid, another is required which denotes the time already elapsed and which, by virtue of its variability, can be applied to any given time.

6. Suppose (Fig. 1) that from the initial state a time t has elapsed and that the fluid is now in a state of motion which is to be determined. Whatever the volume that the fluid now occupies, I begin by considering any point Z in the fluid mass and in order to introduce the location of this point Z into the calculations I relate it to three fixed axes, OA, OB and OC, mutually perpendicular at the point O and having a given position. Let the two axes OA and OB lie in the plane represented by the page and let the third OC

be perpendicular to it. Then from the point Z we draw a perpendicular ZY to the plane AOB and from the point Y a normal YX to the axis OA to obtain three coordinates: OX = x, XY = y and YZ = z parallel to our three axes. For each point in the fluid mass, these three coordinates x, y and z will have specific values and by successively giving these three coordinates all possible values, both positive and negative, we can run through all the points of infinite space, including those lying in the volume occupied by the fluid at each instant of time.



Secondly, I shall consider the accelerative forces which act at a given moment on the fluid particle located at Z. Now, whatever these forces may be, they can always be reduced to three acting in the three directions ZP. ZQ and ZR parallel to our three axes 0A, OB and OC. Taking the accelerative force of natural gravity⁸ as the unit, we let P, Q and R be the accelerative forces acting on the point Z in the directions ZP, ZQ and ZR, the letters P, Q and R denoting abstract numbers [nombres absolus. 9 If unchanging forces always act at the same point in space Z, the quantities P, Q and R will be expressed by certain functions of the three coordinates x, yand z, However, if the forces also vary with time t, these functions will likewise contain time t. I shall assume that these functions are known, since the acting forces must be included among the known quantities, whether they depend only on the variables x, y, z or also on time t.

8. Let r now express the heat at the point Z or that other property which, in addition to the density, influences the elasticity in the case of a compressible fluid. The quantity r must also be considered to be a function of the three variables x, y, z and time t, since it might vary with time t at the same point Z in space. Thus, this function may be regarded as being known. Moreover, let the present density of the fluid particle located at Z be equal to q. As the unit of density I shall take the density of a certain homogeneous substance which I shall use to measure pressures in

⁵ Newton distinguishes between the "accelerative" and "motive" aspects of a force, the former being "a measure proportional to the velocity which it generates" and the latter "a measure proportional to the quantity of motion which it generates in a given time". Thus, the "accelerative force" is the ratio of the acting force to the mass of the particle on which it acts, i.e. the acceleration which it imparts, and the "motive force" is that which, strictly speaking, we now understand by force. The neutral term "acting forces" [forces sollicitantes], not used by Newton, was widely employed by Euler, starting with his well-known "Mechanics" (Euler, 1736).

⁶ Cf. Euler, 1755a.

⁷ In the original publication all figures are on the fourth table following the end (on p. 402) of the part of the volume dedicated to the Mathematics Class. As was the rule at the time figures are devoid of captions.

⁸ The acceleration of gravity is intended.

 $^{^{9}}$ The non-dimensionality of the values of P, Q and R is emphasized.

 $^{^{10}}$ Euler is confining himself to the consideration of fluid motion in a given temperature field.

terms of heights, as explained at greater length in my memoir on the equilibrium of fluids. Let, moreover, the present value of the fluid pressure at the point Z, expressed in terms of height, be equal to p, which will thus also denote the elasticity. Since the nature of the fluid is assumed to be known, we will know the relation between the height p and the quantities q and r. Thus, p and q will likewise be functions, albeit unknown, of the four variables x, y, z and t; however if the fluid is not compressible, t the pressure t will be independent of the density t and the other property t and t will not enter into consideration at all.

Finally, whatever the motion corresponding at a given time to the fluid element located at the point Z, it too can be decomposed in the directions ZP, ZQ and ZR parallel to our three axes. Thus, let u, v and w be the velocities of this motion decomposed in the three directions ZP, ZQ and ZR. It is then obvious that these three quantities must also be considered to be functions of the four variables x, y, z and t. Indeed, having found the nature of these functions, if the time t is assumed to be constant, then by varying the coordinates x, y and zthe three velocities u, v and w and hence the true motion imparted to each element of the fluid at a given time will be known. If, the coordinates x, y and z are assumed to be constant and only the time t is considered to be variable, we shall find the motion not of some particular element of the fluid but of all the elements that pass successively through the same point Z: in other words. at each moment of time the motion of that fluid element which is then located at the point Z will be known.

10. Let us consider what path will be described by a fluid element now at Z during the infinitely small¹⁴ time dt; or the point at which it will be an instant later.¹⁵ If

we express the distance as the product of velocity and time, a fluid element currently at Z will travel a distance udt in the direction ZP, a distance vdt in the direction ZQ and a distance wdt in the direction ZR. Therefore, if we set

$$ZP = udt$$
, $ZQ = vdt$, and $ZR = wdt$

and from these three sides complete the construction of the parallelepiped, then the corner opposite the point Z will represent the point at which the fluid element in question will be after the time dt and the diagonal of the parallelepiped, which is equal to $dt\sqrt{(uu+vv+ww)}$ will give the true path described. Consequently, the velocity of this true motion will be equal to $\sqrt{(uu+vv+ww)}$ and the direction can easily be determined from the sides of the parallelepiped since it will be inclined to the plane AOB at an angle whose sine is equal to

$$\frac{w}{\sqrt{(uu+vv+ww)}} \; ,$$

to the plane AOC at an angle whose sine is equal to

$$\frac{v}{\sqrt{(uu+vv+ww)}} \; ,$$

and, finally, to the plane BOC at an angle whose sine is equal to

$$\frac{u}{\sqrt{(uu+vv+ww)}} \ .$$

11. Having determined the motion of a fluid element which at a given instant is located at the point Z, let us now also examine that of some other infinitely close element located at the point z with the coordinates x+dx, y+dy and z+dz. The three velocities of this element in the direction of the three axes can thus be expressed by u, v, w after substituting in those quantities x+dx, y+dy and z+dz or after adding to them their differentials while assuming the time t to be constant. Thus, when x+dx is substituted for x, the increments of u, v and w will be: t

$$dx\left(\frac{du}{dx}\right), \quad dx\left(\frac{dv}{dx}\right), \quad dx\left(\frac{dw}{dx}\right),$$

and when y + dy is substituted for y, the increments will be:

$$dy\left(\frac{du}{dy}\right), \quad dy\left(\frac{dv}{dy}\right), \quad dy\left(\frac{dw}{dy}\right),$$

Clearly, for Euler the density q is non-dimensional, being divided by the constant density ρ_0 of a certain auxiliary fluid: $q = \rho/\rho_0$. Euler defines the pressure in the fluid as the height p of a column of this same homogeneous auxiliary fluid. Thus, for Euler pressure is measured by a quantity with the dimension of length – the ratio of the acting pressure to the constant quantity $\rho_0 g$ (where g is the acceleration of gravity). For further details see Euler, 1755a.

 $^{^{\}rm 12}$ That is, the "equation of state" of the moving medium is assumed to be known.

¹³ The 1757 printed version of the memoir has "not incompressible" [pas incompressible], but a handwritten copy of the manuscript dated 1755, henceforth cited as Euler, 1755c has "not compressible" [pas compressible] which is obviously the correct form.

¹⁴ The differential operator d, now denoted using roman fonts, was at the time of Euler italicized: we shall follow his usage.

¹⁵ The intuitive derivation of the equations of motion and continuity of an ideal (inviscid and non-heat-conducting) compressible fluid proposed by Euler is valid provided that the functions in question have bounded derivatives, up to and including the second. The modern derivation of these equations, based on the integral laws of conservation of mass and momentum of the fluid particles and the use of the Gauss theorem, is free of this limitation.

¹⁶ In the 1757 printed version, which we here follow, we usually find the old notation xx rather than x^2 for the square of the quantity x and $\sqrt{(\ldots)}$ rather than $\sqrt{\ldots}$ for the square root of an expression. The manuscript Euler, 1755c, which is not in Euler's hand, uses modern notation.

 $^{^{17}}$ Rather than the now customary notation for partial derivatives using the symbol $\partial,$ Euler employs only the symbol d but encloses the expressions for partial derivatives in round brackets.

and the same will apply to the variation of z. Then, the three velocities of the fluid element currently located at z will be:

in the direction OA

$$u + dx \left(\frac{du}{dx}\right) + dy \left(\frac{du}{dy}\right) + dz \left(\frac{du}{dz}\right) ,$$

in the direction OB

$$v + dx \left(\frac{dv}{dx}\right) + dy \left(\frac{dv}{dy}\right) + dz \left(\frac{dv}{dz}\right) ,$$

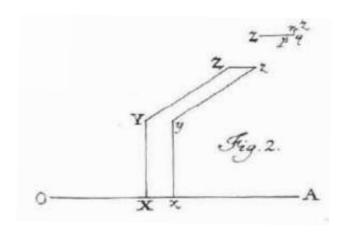
in the direction OC

$$w + dx \left(\frac{dw}{dx}\right) + dy \left(\frac{dw}{dy}\right) + dz \left(\frac{dw}{dz}\right)$$
.

12. These are the velocities corresponding to a fluid element at the point z, which is infinitely close to the point Z and whose position is determined by the three coordinates x + dx, y + dy and z + dz. Thus, if we choose a point Z (Fig. 2) such that only x changes by dx, the other two coordinates y and z remaining the same as for the point Z, the three velocities of the fluid element located at this point z will be:

$$u + dx \left(\frac{du}{dx}\right), \quad v + dx \left(\frac{dv}{dx}\right), \quad w + dx \left(\frac{dw}{dx}\right).$$

These velocities will transport the element in the time dt to another point z' whose position must be determined relative to the point Z', namely the point to which the fluid element which was at Z is transported in the same time dt and whose position was determined above (see § 10). For determining this point z', I note that if the velocities of the point z were exactly the same as those of Z, then the point z' would fall at the point p, 18 such that the distance Z'p would be equal and parallel to the distance Zz. Since, by hypothesis, Zz is parallel to the OA axis and equal to dx, the segment Z'p will also be equal to dx and parallel to the OA axis.



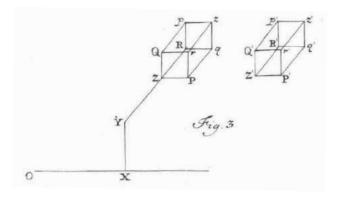
13. Now, since the velocity along OA is not u but $u + dx \left(\frac{du}{dx}\right)$, this velocity increment will transport the element in question from p to q in the direction Z'p, such that $pq = dtdx \left(\frac{du}{dx}\right)$: this element would thus be at q, if the other two velocities were equal to v and w. However, since the velocity along the OB axis is $v + dx \left(\frac{dv}{dx}\right)$, this increment will transport our element from q to r, through the distance $qr = dtdx\left(\frac{dv}{dx}\right)$, and parallel to the axis OB. Finally, the increment $dx\left(\frac{dw}{dx}\right)$ of the velocity w will transport the element from r to z'through the infinitesimal distance $[particule\ d'espace]^{19}$ $rz' = dtdx\left(\frac{dw}{dx}\right)$, and parallel to the third axis OC. From this I conclude that the fluid element which occupied the small linear segment $\mathbf{Z}z$ would be transported in the time dt to the segment Z'z', inclined at an infinitely small angle to the OA axis, whose length by virtue of the fact that $Z'q = dx \left(1 + dt \left(\frac{du}{dx}\right)\right)$ will be

$$dx\sqrt{\left(\left(1+dt\left(\frac{du}{dx}\right)\right)^2+dt^2\left(\frac{dv}{dx}\right)^2+dt^2\left(\frac{dw}{dx}\right)^2\right)}.$$

Thus, neglecting the terms that contain the square of dt, the length Z'z' will not differ from Z'q and we shall have: $Z'z' = dx\left(1 + dt\left(\frac{du}{dx}\right)\right)$. For the inclination of this line to the OA axis, it will suffice to note that it is an infinitely small quantity of the first order and can be expressed as αdt .

 $^{^{18}}$ Euler frequently uses the same notation for different quantities. Thus, both here and later on, the letters p and q, which in this article are mainly employed to denote pressure and density, are used to denote certain auxiliary points.

¹⁹ The 1757 printed version of the memoir has "through the particle" [par la particule], but Euler 1755c has "through the particle of distance" [par la particule d'espace].



14. If the small segment Zz had been taken equal to dy and parallel to the OB axis, by the same reasoning it could have been shown that the fluid which occupied that segment would have been transported to another segment $Z'z' = dy\left(1 + dt\left(\frac{dv}{dy}\right)\right)$, and which would have been inclined to the OB axis at an infinitely small angle. And if we had taken the segment Zz = dz, and parallel to the third axis OC, the fluid which occupied it would have been transported to another segment $Z'z' = dz\left(1 + dt\left(\frac{dw}{dz}\right)\right)$, and which would have been inclined to the OC axis at an infittely small angle. Thus, if we consider a rectangular parallelepiped ZPQRzpqr (Fig. 3) formed by the three sides ZP = dx, ZQ = dy, ZR = dz, the fluid occupying that volume would be transported in the time dt to fill a volume Z'P'Q'R'z'p'q'r' differing infinitely slightly from a rectangular parallelepiped whose three sides would be

$$Z'P' = dx \left(1 + dt \left(\frac{du}{dx} \right) \right);$$

$$Z'Q' = dy \left(1 + dt \left(\frac{dv}{dy} \right) \right);$$

$$Z'R' = dz \left(1 + dt \left(\frac{dw}{dz} \right) \right).$$

Since the sides ZP, ZQ, ZR go over into Z'P', Z'Q', Z'R', there is no doubt that the fluid contained in the first volume will be transported into the other in the time dt.

15. We can now judge whether'the volume of fluid occupying the parallelepiped Zz has increased or decreased in the time dt. For this we need only to find the volume or the capacity of each of these two solids. Since the first is a parallelepiped formed by the sides dx, dy, dz, its volume is equal to dxdydz. As for the other, whose plane angles differ infinitely slightly from a right angle, I note that its volume can also be found by multiplying its three sides, since the error due to the infinitesimal distortion of the angles will enter into terms which contain the square of the time element dt and can therefore be neglected. Thus, the volume Z'z' can be represented by

the expression:

$$dxdydz\left(1+dt\left(\frac{du}{dx}\right)+dt\left(\frac{dv}{dy}\right)+dt\left(\frac{dw}{dz}\right)\right)$$
.

Anyone still harboring doubts about the reasonableness of this conclusion need only consult my Latin paper *Principia motus fluidorurn* in which I calculate this volume without neglecting anything.²⁰

16. Thus, if the fluid is not compressible, these two volumes should be equal, since the mass occupying the volume Zz would not fit into either a larger or a smaller volume. However, since I propose to examine the problem in the most general possible form and have denoted the density at Z by q, considering q to be a function of the three coordinates and time, I note that to find the density at Z' it will first be necessary to increase the time t by its differential dt; then, as the point Z' is different from Z, the quantities x, y, z will have to be increased by the small increments udt, vdt, wdt; whence the density at Z' will be:

$$q + dt \left(\frac{dq}{dt}\right) + udt \left(\frac{dq}{dx}\right) + vdt \left(\frac{dq}{dy}\right) + wdt \left(\frac{dq}{dz}\right)$$

and since the density is inversely proportional to the volume, this quantity will be to q as dxdydz to

$$dxdydz\left(1+dt\left(\frac{du}{dx}\right)+dt\left(\frac{dv}{dy}\right)+dt\left(\frac{dw}{dz}\right)\right)$$
 .

Thus, dividing by dt, we find that consideration of the density leads to the following equation:

$$\left(\frac{dq}{dt}\right) + u\left(\frac{dq}{dx}\right) + v\left(\frac{dq}{dy}\right) + w\left(\frac{dq}{dz}\right) + q\left(\frac{du}{dx}\right) + q\left(\frac{du}{dz}\right) + q\left(\frac{dw}{dz}\right) + q\left(\frac{dw}{dz}\right) = 0.$$

17. Here, then, is a very remarkable condition which already establishes a certain relation between the three velocities u, v and w and the fluid density q. Now this equation can be reduced to a simpler form. Thus, $u\left(\frac{dq}{dx}\right)$ is no different from $\left(u\frac{dq}{dx}\right)$ since this form of expression must be taken to mean that in differentiating q only the quantity x is taken to be a variable, and similarly $q\left(\frac{du}{dx}\right) = \left(q\frac{du}{dx}\right)$; from which it follows that

$$q\left(\frac{du}{dx}\right) + u\left(\frac{dq}{dx}\right) = \left(\frac{udq + qdu}{dx}\right) = \left(\frac{d.qu}{dx}\right)$$
,

²⁰ See Euler, 1756–1757. This memoir was originally entitled *De motu fluidorum in genere*, but the final title was already used here.

²¹ In Euler's subsequent exposition the use of round brackets goes beyond the scope of simple partial derivative notation, but the meaning of the operations is still clear, in Euler's notation d.qu = d(qu), etc.

the differential of the product qu being so understood that only the quantity x is regarded as a variable. Accordingly, the equation obtained can be reduced to the following:

$$\left(\frac{dq}{dt}\right) + \left(\frac{d.qu}{dx}\right) + \left(\frac{d.qv}{dy}\right) + \left(\frac{d.qw}{dz}\right) = 0.$$

If the fluid was not compressible, the density q would be the same at both Z and Z' and for this case we would have the equation:

$$\left(\frac{du}{dx}\right) + \left(\frac{dv}{dy}\right) + \left(\frac{dw}{dz}\right) = 0 ,$$

which is also that on which I based my Latin memoir mentioned above. $^{22}\,$

This equation, obtained by considering the con-**18**. tinuity of the fluid, already contains a certain relation which must exist between the quantities u, v, w and q. The other relations must be obtained by considering the forces to which each fluid particle is subjected. Thus, in addition to the accelerative forces²³ P, Q, R, which act on the fluid at Z, the fluid is also subjected to the pressure [pression] exerted from all sides on the fluid element contained at Z. Combining these two forces, we obtain three accelerative forces in the direction of the three axes. Since the accelerations themselves can be determined by considering the velocities u, v and w, we can derive three equations which, together with that which we have just found, will contain everything that relates to the motion of fluids, so that we shall then have the general and complete laws of the entire science of fluid motion.

19. In order to find the accelerations undergone by a fluid element at Z, we need only compare the velocities u, v, w which currently correspond to the point Z with the velocities corresponding to the point Z' after the lapse of the time dt. Thus, a double change takes place: with respect to the coordinates x, y, z, which receive the increments udt, vdt, wdt, as well as with respect to time, which increases by dt. Hence it follows that the three velocities at the point Z' are:

in the direction OA

$$u + dt \left(\frac{du}{dt}\right) + udt \left(\frac{du}{dx}\right) + vdt \left(\frac{du}{dy}\right) + wdt \left(\frac{du}{dz}\right) ,$$

in the direction OB

$$v + dt \left(\frac{dv}{dt}\right) + udt \left(\frac{dv}{dx}\right) + vdt \left(\frac{dv}{dy}\right) + wdt \left(\frac{dv}{dz}\right) ,$$

in the direction OC

$$w + dt \left(\frac{dw}{dt}\right) + udt \left(\frac{dw}{dx}\right) + vdt \left(\frac{dw}{dy}\right) + wdt \left(\frac{dw}{dz}\right) ,$$

and hence the accelerations, expressed in terms of the velocity increments divided by the time element dt, will be:

in the direction OA

$$\left(\frac{du}{dt}\right) + u\left(\frac{du}{dx}\right) + v\left(\frac{du}{dy}\right) + w\left(\frac{du}{dz}\right) ,$$

in the direction OB

$$\left(\frac{dv}{dt}\right) + u\left(\frac{dv}{dx}\right) + v\left(\frac{dv}{dy}\right) + w\left(\frac{dv}{dz}\right)$$
,

in the direction OC

$$\left(\frac{dw}{dt}\right) + u\left(\frac{dw}{dx}\right) + v\left(\frac{dw}{dy}\right) + w\left(\frac{dw}{dz}\right)$$
.

20. We will now seek the accelerative forces acting in these same directions due to the pressure exerted by the fluid on the parallelepiped Zz, whose volume is equal to dxdydz, the mass of the fluid occupying that volume thus being equal to qdxdydz. Since the pressure at the point Z is expressed in terms of the height p, the motive force acting on the face ZQRp is equal to pdxdydz. For the opposite face zqrP with the area dydz, the height p is increased by its differential $dx\left(\frac{dp}{dx}\right)$, obtained on the assumption that only x is variable. Accordingly, this fluid mass Zz is driven in the direction AO by the motive force $dxdydz\left(\frac{dp}{dx}\right)$ or by the accelerative force $\frac{1}{q}\left(\frac{dp}{dx}\right)$. Similarly, we find that the fluid mass Zz is subjected to the action of the accelerative force $\frac{1}{q}\left(\frac{dp}{dz}\right)$ in the direction BO and to that of the accelerative force $\frac{1}{q}\left(\frac{dp}{dz}\right)$ in the

direction CO. To these forces we add the given forces P, Q, R, and the total accelerative forces will be:

in the direction OA:
$$P - \frac{1}{q} \left(\frac{dp}{dx} \right)$$

in the direction OB: $Q - \frac{1}{q} \left(\frac{dp}{dy} \right)$
in the direction OC: $R - \frac{1}{q} \left(\frac{dp}{dz} \right)$.

21. Thus, it only remains to equate these accelerative forces with the actual accelerations which we have just

²² See Euler, 1756–1757.

 $^{^{23}}$ Concerning the concept of "accelerative" (body) forces, see footnote 5.

found. We then obtain the following three equations:²⁴

$$\begin{split} \mathbf{P} &- \frac{1}{q} \left(\frac{dp}{dx} \right) = \left(\frac{d\,u}{dt} \right) + u \left(\frac{d\,u}{dx} \right) + v \left(\frac{d\,u}{dy} \right) + w \left(\frac{d\,u}{dz} \right) \\ \mathbf{Q} &- \frac{1}{q} \left(\frac{dp}{dy} \right) = \left(\frac{d\,v}{dt} \right) + u \left(\frac{d\,v}{dx} \right) + v \left(\frac{d\,v}{dy} \right) + w \left(\frac{d\,v}{dz} \right) \\ \mathbf{R} &- \frac{1}{q} \left(\frac{dp}{dz} \right) = \left(\frac{dw}{dt} \right) + u \left(\frac{dw}{dx} \right) + v \left(\frac{dw}{dy} \right) + w \left(\frac{dw}{dz} \right). \end{split}$$

If we add to these three equations, first, that obtained from considering the continuity of the fluid, namely

$$\left(\frac{dq}{dt}\right) + \left(\frac{d.qu}{dx}\right) + \left(\frac{d.qv}{dy}\right) + \left(\frac{d.qw}{dz}\right) = 0 ,$$

and then the equation²⁵ which gives the relation between the elasticity p, the density q and the other property r which, in addition to the density q influences the elasticity p, we shall have five equations encompassing the entire Theory of the motion of fluids.

22. Whatever be the nature of the forces P, Q, R, provided that they are real, it should be noted that Pdx + Qdy + Rdz is always a total $[r\acute{e}el]$ differential of a certain finite and determinate quantity, 26 assuming the three coordinates x, y and z to be variables. Thus, we will always have:

$$\left(\frac{d\mathbf{P}}{dy}\right) = \left(\frac{d\mathbf{Q}}{dx}\right); \quad \left(\frac{d\mathbf{P}}{dz}\right) = \left(\frac{d\mathbf{R}}{dx}\right); \quad \left(\frac{d\mathbf{Q}}{dz}\right) = \left(\frac{d\mathbf{R}}{dy}\right) \; ,$$

and if we set this finite quantity equal to S, then, we have

$$dS = Pdr + Qdy + Rdz,$$

assuming the time t to be constant for the case in which the forces P, Q, R also vary with time at the same points. The quantity S expresses what I shall call the effort [l'effort] of the acting forces²⁷ and is equal to the sum of the integrals of each force multiplied by the elementary interval in the direction of that force

Despite the outward resemblance between Euler's equations and modern notation, they have been written here in dimensionless form. As mentioned above, the pressure p is measured as the ratio of the acting pressure to the specific weight $\rho_0 g$ of a certain homogeneous auxiliary fluid, the density q is dimensionless $(q=\rho/\rho_0)$, the components of the body forces have been divided by the acceleration of gravity g, the transition from the Eulerian velocities u, v, w to the real velocities U, V, W is effected by means of a transformation of the form $u\mapsto U/\sqrt{g}$ and the transition from Eulerian time to real time by means of the transformation $t\mapsto T\sqrt{g}$. (For further details concerning Euler's system of physical units, see Mikhailov, 1999.)

or by the small distance through which it would drag a body subjected to its action. This notion of effort is of the utmost importance for the entire theory of both equilibrium and motion, since it makes it possible to see that the sum of all the efforts is always a maximum or a minimum. This excellent property fits in admirably with the splendid principle of least action whose discovery we owe to our illustrious President, Mr. de Maupertuis.²⁸

- 23. The equations just obtained contain four variables x, y, z and t which are absolutely independent of each other since the variability of the first three extends to all elements of the fluid and that of the fourth to all times. Therefore, for the equations to continue to hold, the other variables u, v, w, p and q must be certain functions of the former. For although a differential equation with two variables²⁹ is always possible,³⁰ we know that a differential equation containing three or more variables is possible only under certain conditions, by virtue of which a certain relationship must exist between the terms of the equation. Therefore, before we can begin solving the equations, we need to know what sort of functions of x, y, z and t must be used to express the values of u, v, w, p and q in order for these same equations to be possible.
- **24.** We now multiply the first of the three equations obtained by dx, the second by dy and the third by dz, and since $dx\left(\frac{dp}{dx}\right) + dy\left(\frac{dp}{dy}\right) + dz\left(\frac{dp}{dz}\right)$ represents the differential of p, assuming only time t to be constant, we obtain³¹

$$\begin{split} d\mathbf{S} \; - \; & \frac{dp}{q} = \\ & + \; dx \left(\frac{du}{dt} \right) + u dx \left(\frac{du}{dx} \right) + v dx \left(\frac{du}{dy} \right) + w dx \left(\frac{du}{dz} \right) \\ & + \; dy \left(\frac{dv}{dt} \right) + u dy \left(\frac{dv}{dx} \right) + v dy \left(\frac{dv}{dy} \right) + w dy \left(\frac{dv}{dz} \right) \\ & + \; dz \left(\frac{dw}{dt} \right) + u dz \left(\frac{dw}{dx} \right) + v dz \left(\frac{dw}{dy} \right) + w dz \left(\frac{dw}{dz} \right). \end{split}$$

It is now a question of finding the integral of this equation in which time is assumed to be constant. It should be noted that this single equation contains the three equations of which it is composed and that as soon as it is satisfied the conditions of all three will be fulfilled. Thus, if the expression $d\mathbf{S} - \frac{dp}{q}$ is equal to the three lines, where x, y and z are variables, the portion

 $^{^{25}}$ What we now call the equation of state.

²⁶ Euler is thinking here of real body forces possessing a potential (more correctly, a force function). By "finite" quantities (functions) Euler means quantities that do not contain differentials.

²⁷ Euler's "effort" is equivalent to the modern notion of potential.

 $^{^{28}}$ Maupertuis was president of the Berlin Academy at the time.

²⁹ Here, by variable Euler understands both independent variables and their functions.

 $^{^{30}}$ We would now say "soluble".

³¹ The first term on the r.h.s. is correct in the manuscript Euler, 1755c but misprinted as $dz\left(\frac{du}{dt}\right)$ in the printed version.

of $dS - \frac{dp}{q}$ due to the variability of x alone, namely $Pdx - \frac{dx}{q} \left(\frac{dp}{dx}\right)$ must necessarily be equal to the first line, and similarly for the other two. The terms $\left(\frac{du}{dt}\right)$, $\left(\frac{dv}{dt}\right)$, and $\left(\frac{dw}{dt}\right)$, found by assuming the variability of time t, since they denote certain finite functions, do not prevent time t from now being taken to be constant.

25. Suppose that this equation has already been solved and the quantities u, v, w, q and p have been found as certain finite functions of x, y, z and t. The substitution of these functions in the differential equation, with time t assumed constant, yields an identity. Since after this substitution we will have three types of terms, the first associated with dx, the second with dy and the third with dz, the identity leads us to three equations whence it is dear that although only one differential equation is being considered, it actually has the force of three and determines three of our unknowns. What is also clear is that a differential equation with three variables, such as Ldx + Mdy + Ndz = 0, cannot be solved unless a certain relationship exists between the quantities L, M and N. However, since very little work has yet been done on solving these three-variable equations, we cannot hope to obtain a complete solution of our equation until the limits of Analysis have been extended much further.

26. The best approach would therefore be to ponder well on the particular solutions of our differential equation that we are in a position to obtain, as this would enable us to judge which path to follow in order to arrive at a complete solution. I have already pointed out³² that where the density q is assumed to be constant a very elegant solution can be obtained when the velocities u, v and w arc such that the differential expression [formule] udx + vdy + wdz can be integrated. Suppose, then, that W is that integral, being any function of x, y, z and time t, and that its differentiation, also including t as a variable, gives

$$dW = udx + vdy + wdz + \Pi dt .$$

Then the quantities u, v, w and Π will be related as follows:³³

27. Using these equalities, we can reduce our differential equation to the following form:

$$\begin{split} d\mathbf{S} \; - \; & \frac{dp}{q} = \\ & + \, dx \left(\frac{d\Pi}{dx} \right) + u dx \left(\frac{du}{dx} \right) + v dx \left(\frac{du}{dy} \right) + w dx \left(\frac{du}{dz} \right) \\ & + \, dy \left(\frac{d\Pi}{dy} \right) + u dy \left(\frac{dv}{dx} \right) + v dy \left(\frac{dv}{dy} \right) + w dy \left(\frac{dv}{dz} \right) \\ & + \, dz \left(\frac{d\Pi}{dz} \right) + u dz \left(\frac{dw}{dx} \right) + v dz \left(\frac{dw}{dy} \right) + w dz \left(\frac{dw}{dz} \right). \end{split}$$

Since here time t is assumed to be constant, using the same hypothesis we will have

$$dx\left(\frac{d\Pi}{dx}\right) + dy\left(\frac{d\Pi}{dy}\right) + dz\left(\frac{d\Pi}{dz}\right) = d\Pi$$
$$dx\left(\frac{du}{dx}\right) + dy\left(\frac{du}{dy}\right) + dz\left(\frac{du}{dz}\right) = du$$

Thus, our equation will become

$$dS - \frac{dp}{q} = d\Pi = udu + vdv + wdw ,$$

or

$$dp = q (dS - d\Pi - udu - vdv - wdw) .$$

Hence, if the density of the fluid is everywhere the same, or q = g, as a result of integration we obtain:³⁴

$$p = g \left(C + S - \Pi - \frac{1}{2}uu - \frac{1}{2}vv - \frac{1}{2}ww \right)$$
.

28. For brevity, let us set

$$C + S - \Pi - \frac{1}{2}uu - \frac{1}{2}vv - \frac{1}{2}ww = V$$
,

where it should be noted that the constant C may well contain the time t, since it is considered to be constant in this integration and, as dp = qdV, it is clear that the hypothesis

$$dW = udx + vdy + wdz + \Pi dt,$$

also makes our differential equation possible, when the elasticity p depends in any way on the density q only or q is any function of p. It will also become possible if the fluid is not compressible but the density q varies in such a way that it is an arbitrary function of the quantity

³² Euler, 1756–1757: §§ 60–67.

³³ In modem terminology, the function introduced by Euler W = W(x,y,z,t) is the velocity potential; here, the equality of the cross derivatives of W with respect to the coordinates (condition of integrability of dW) is the condition of absence of vorticity.

³⁴ The subsequent equation, which generalizes the Bernoulli integral, is usually associated with the names of Cauchy and Lagrange.

V. And in general, if the elasticity p depends both on the density q and on some other quantity represented by the letter r, the hypothesis may also be satisfied provided that r is a function of V. In all these cases, for the motion to exist under this hypothesis it is also necessary for the following condition to be satisfied:

$$\left(\frac{dq}{dt}\right) + \left(\frac{d \cdot qu}{dx}\right) + \left(\frac{d \cdot qv}{dy}\right) + \left(\frac{d \cdot qw}{dz}\right) = 0.$$

29. This hypothesis is so general that it seems that there is not a single case that is not included and hence that, generally speaking, the equation dp = qdV, together with the other equations which present hardly any difficulty, incorporates all the foundations of the Theory of the motion of fluids. Thus, I concerned myself exclusively with this case in my Latin memoir on the laws of fluid motion³⁵ in which I considered incompressible fluids only and showed that all the cases previously considered, in which the fluid moves through pipes of arbitrary shapes, are contained in this supposition and that the velocities u, v and w are always such that the differential expression udx + vdy + wdz is integrable. However, I have since noted that there are also cases, even when the fluid is incompressible and everywhere homogeneous, in which this condition does not hold, which is enough to convince me that the solution I have just given is only a particular one.³⁶

30. To give an example of a real motion which would be perfectly consistent with all the equations that follow from the laws of Mechanics, but without the expression udx + vdy + wdz being integrable, let us assume that the fluid is incompressible and everywhere homogeneous, i.e. that q is constant and equal to g, and that there are no forces acting on the fluid, so that P = 0, Q = 0 and R = 0. Then, let w = 0, v = Zx and u = -Zy, where Z denotes any function of $\sqrt{(xx + yy)}$. It is now obvious that the expression udx + vdy + wdz, which takes the form -Zydx + Zxdy, is integrable only in the case $Z = \frac{1}{xx + yy}$. However, these values³⁷ satisfy all our formulas so that the possibility of this motion cannot be questioned. Since Z is a function of $\sqrt{(xx + yy)}$, its differential will have the form dZ = Lxdx + Lydy, where L will again be any function of $\sqrt{(xx + yy)}$.

$$\begin{pmatrix} \frac{du}{dt} \end{pmatrix} = 0; \qquad \begin{pmatrix} \frac{dv}{dt} \end{pmatrix} = 0; \qquad \begin{pmatrix} \frac{dw}{dt} \end{pmatrix} = 0; \\
\begin{pmatrix} \frac{du}{dx} \end{pmatrix} = -Lxy; \qquad \begin{pmatrix} \frac{dv}{dx} \end{pmatrix} = Z + Lxx; \qquad \begin{pmatrix} \frac{dw}{dx} \end{pmatrix} = 0; \\
\begin{pmatrix} \frac{du}{dy} \end{pmatrix} = -Z - Lyy; \qquad \begin{pmatrix} \frac{dv}{dy} \end{pmatrix} = Lxy; \qquad \begin{pmatrix} \frac{dw}{dy} \end{pmatrix} = 0; \\
\begin{pmatrix} \frac{du}{dz} \end{pmatrix} = 0; \qquad \begin{pmatrix} \frac{dv}{dz} \end{pmatrix} = 0; \qquad \begin{pmatrix} \frac{dw}{dz} \end{pmatrix} = 0;$$

and since dS = O, assuming time t to be constant, we have the following differential equation:

$$\frac{dp}{g} = LZxyydx - ZZxdx$$

$$- LZxyydx - ZZydy - LZxxydy + LZxxydy$$

$$= -ZZ(xdx + ydy).$$

Consequently dp = gZZ(xdx + ydy), since Z is assumed to be a function of $\sqrt{(xx + yy)}$, this equation will definitely be possible and will yield the integral $p = g\int ZZ(xdx + ydy)$. We see that the differential equation would also be possible if the fluid were subjected to the action of certain arbitrary forces P, Q, R, provided that the expression Pdx + Qdy + Rdz was a total [possible] differential equal to dS, since then $p = gS + g\int ZZ(xdx + ydy)$.

32. As these values u = -Zy, v = Zx and w = 0 satisfy our differential equation, they can also be seen to satisfy the condition contained in the equation:³⁸

$$\left(\frac{dq}{dt}\right) + \left(\frac{d \cdot qu}{dx}\right) + \left(\frac{d \cdot qv}{dy}\right) + \left(\frac{d \cdot qw}{dz}\right) = 0.$$

By virtue of the fact that q = g, this equation goes over into

$$-gLxy + gLxy = 0$$

which, being an identity, satisfies the required conditions. Thus, it is quite possible for a fluid to have a motion such that the velocities of each of its elements are $u=-\mathrm{Z}y,\ v=Zx$ and w=0, although the differential expression udx+vdy+wdz is not possible;³⁹ this confirms that there are cases in which fluid motion is possible without this condition, which seemed general, being fulfilled. Thus, the assumption that the differential expression udx+vdy+wdz is possible yields

³¹. Using these values of u, v and w, we obtain:

³⁵ See Euler, 1756–1757.

³⁶ Here, Euler recognizes that his previous memoir on fluid motion was too restricted, in so far as it ignored what we now call vorticity.

³⁷ The corresponding values of u, v and w.

³⁸ Strictly speaking, it cannot be said that the values of u, v and w assumed in § 30 also satisfy the equation of motion from § 31; in reality, this equation determines the corresponding pressure p = p(s) ($s = \sqrt{(xx+yy)}$, the continuity equation being satisfied irrespective of the equations of motion.

³⁹ That is a total differential.

only a particular solution of the equations we have found.

- Clearly, the motion corresponding to this case reduces to a rotational motion about the axis OC and since what has been said about the axis OC can be applied to any other fixed axis, we may conclude that it is possible for a fluid acted upon by any forces whose effort⁴⁰ is equal to S to have a motion about a fixed axis such that the rotational velocities are proportional to any function of the distance to that axis. Thus, if the distance to that axis is denoted by s and the rotation velocity at that distance by 8^{41} , then since xx + yy = ssand ZZss = 88, the pressure there will be expressed by the height $p = gS + g \int \frac{88 ds}{s}$. Thus, such a motion, which corresponds to that of a vortex [tourbillon], is just as possible as those contained in the expression udx + vdy + wdz when the latter is integrable. No doubt there is an infinity of other motions, which satisfying our equations, are also equally possible.
- **34**. Let us now return to our general formulas and, since they are somewhat too complicated, introduce, for greater conciseness, the notation:

$$\left(\frac{d \, u}{dt} \right) + u \left(\frac{d \, u}{dx} \right) + v \left(\frac{d \, u}{dy} \right) + w \left(\frac{d \, u}{dz} \right) \; = \; \; \mathbf{X}$$

$$\left(\frac{d \, v}{dt} \right) + u \left(\frac{d \, v}{dx} \right) + v \left(\frac{d \, v}{dy} \right) + w \left(\frac{d \, v}{dz} \right) \; = \; \; \mathbf{Y}$$

$$\left(\frac{d w}{dt} \right) + u \left(\frac{d w}{dx} \right) + v \left(\frac{d w}{dy} \right) + w \left(\frac{d w}{dz} \right) \; = \; \; \mathbf{Z} \; .$$

Whatever the nature of the three accelerative forces P, Q and R, granted that 42 dS = Pdx + Qdy + Rdz, the differential equation

$$\frac{dp}{q} = (P - X) dx + (Q - Y) dy + (R - Z) dz,$$

in which t is assumed to be constant must be satisfied. Moreover, the continuity of the fluid requires that:

$$\left(\frac{dq}{dt}\right) + \left(\frac{d.qu}{dx}\right) + \left(\frac{d.qv}{dy}\right) + \left(\frac{d.qw}{dz}\right) = 0.$$

In whatever manner these two equations are satisfied, there will always be a motion which can actually take place in the fluid.

35. If the fluid is everywhere incompressible and homogeneous, i.e. the density q is constant and equal to q,

See footnote 27.

then, clearly, the differential equation cannot be satisfied unless the differential

$$(P - X)dx + (Q - Y)dy + (R - Z)dz$$
,

is possible or total, i.e. unless it can be obtained as a result of the actual differentiation of some finite function of the variables x, y and z, which may also contain the time t, although in the differentiation the latter is assumed to be constant. It is also obvious that this differential expression must be soluble or total when the fluid is compressible and the density q is expressed in terms of any function of the elasticity p. In both cases, if we denote by V the finite quantity whose differential has the form:

$$dV = (P - X)dx + (Q - Y)dy + (R - Z)dz,$$

our differential equation will yield either $\frac{p}{g} = V$ or $\int \frac{dp}{q} = V$. In addition, however, for the motion to be possible the other condition derived from the continuity must also be fulfilled.

36. If the fluid is not compressible, but its density q is variable and can be expressed in terms of any function of position, i.e. of the three coordinates x, y, z and time t, it is not sufficient for the expression

$$(P - X)dx + (Q - Y)dy + (R - Z)dz = dV,$$

to be integrable; in addition, the integral V must be a function of q. Since $\frac{dp}{q}=d\mathrm{V}$ or $dp=qd\mathrm{V}$, it is clear that the pressure p cannot have a definite value unless the expression $qd\mathrm{V}$ can be integrated. However, it should also be noted that in this case it is not necessary that the expression

$$(P - X)dx + (Q - Y)dy + (R - Z)dz$$

be integrable, only that on being multiplied by a certain function U it becomes integrable. Thus, let

$$U(P - X)dx + U(Q - Y)dy + U(R - Z)dz = dW,$$

since $\frac{dp}{q} = \frac{dW}{U}$, or $dp = \frac{qdW}{U}$ for this equation to be possible it is sufficient that W be a function of $\frac{q}{U}$, or that W be a function of zero dimension of the quantities q and U.⁴³

37. In general, however the elasticity p depends on the density q or on some other property denoted by r which is any function of the coordinates x, y, z that could also contain time t, it is clear from our equation $q = \frac{dp}{dV}$ that the differential dp must always be divisible by dV,

⁴¹ In modern terms Z is the angular velocity at the radial distance s and \aleph = Zs is the tangential velocity.

 $^{^{42}}$ Here, Euler assumes that all real body forces have a potential $\mathbf{S}=\mathbf{S}(x,y,z).$

 $^{^{43}}$ This latter expression is equivalent, in 18th century terminology, to the condition that W should depend only on the ratio q/U.

where dV denotes not so much a total differential than the expression

$$(P - X)dx + (Q - V)dy + (R - Z)dz$$
,

and this so much that, as a result of division the differentials dx, dy and dz are entirely eliminated from the calculations, because both p and q must always be expressed in terms of finite functions of the variables x, y and z, without their differentials entering into these functions. Now this could not be so unless there were a function U, multiplication by which rendered the expression dV integrable: indeed, setting $\int U dV = W$, clearly, p must be a function of W in order for the expression $\frac{dp}{dV}$ to take a definite value corresponding to the density q.

38. Since U dV = dW, we have $q = \frac{\mathrm{U} dp}{dW}$. Consequently, if we choose W to be any function of the coordinates x, y and z, which contains time t among the constants, and if we set p equal to any function of W, namely⁴⁴ $p = \varphi$, W, and $dp = dW.\varphi'$, W, we will have $q = \mathrm{U}.\varphi'$, W, whence $\mathrm{U} = \frac{q}{\varphi',\mathrm{W}}$. Thus, in however way the density q is expressed in terms of the elasticity p and some other function r of the coordinates x, y and z, we obtain the value $\mathrm{U} = \frac{q}{\varphi',\mathrm{W}}$ and, consequently, the value $d\mathrm{V} = \frac{dW.\varphi',\mathrm{W}}{q}$, which then gives us the following equation:

$$(\mathbf{P} - \mathbf{X})dx + (\mathbf{Q} - \mathbf{V})dy + (\mathbf{R} - \mathbf{Z})dz = \frac{d\mathbf{W}.\varphi', \mathbf{W}}{q} = \frac{dp}{q} \; .$$

This will yield the values of X, Y, Z, from which we must then look for the values of the velocities u, v and w: and when the latter also satisfy the continuity condition, we shall have a case of possible motion of the fluid.

39. The question of the nature of the expression (P-X)dx+(Q-Y)dy+(R-Z)dz then reduces to the following. When the density q is constant or depends only on the elasticity p, this expression must be absolutely integrable and to this end one must determine suitable values of the three velocities u, v and w. When the density q depends on a given function of place and time, ⁴⁵ the expression must be such that it becomes integrable on multiplication by some given function U. In both cases, then, the velocities u, v and w must be such that the equation

$$(P - X)dx + (Q - Y)dy + (R - Z)dz = 0$$

be soluble; 46 and we know the conditions under which a differential equation with three variables is soluble; hav-

- These are the conditions which restrict the functions expressing the three velocities u, v and w, and the study of the motion of fluids reduces to determining, in general form, the nature of those functions such that the conditions of our differential equation and of continuity be fulfilled. Since the quantities X, Y and Z depend not only on the velocities u, v and w themselves but also on their variability with respect to each of the coordinates x, y and z and, moreover, on time t, this study would appear to be the most far-reaching of those to be encountered in the field of Analysis, and if we are unable to achieve a complete understanding of the motion of fluids, it is not Mechanics or the inadequacy of the known laws of motion but Analysis itself that is to blame, given that the entire Theory of the motion of fluids has just been reduced to the solution of analytical equations.
- 41. Since a general solution must be deemed impossible due to the shortcomings of Analysis, we must content ourselves with the consideration of certain particular cases, especially as the study of several cases seems to be the only means of perfecting our knowledge. Now the simplest case imaginable is, no doubt, that in which the three velocities u, v and w are set equal to zero, namely the case in which the fluid remains at perfect rest and which I dealt with in my previous Memoir.⁴⁷ The formulas we have obtained for motion in general also include the case of equilibrium, since when X = 0, Y = 0 and Z = 0 we have: $\frac{dp}{q} = Pdx + Qdy + Rdz$, and $\left(\frac{dq}{dt}\right) = 0$, from which it follows, first of all, that the density q cannot depend on time t, i.e. should remain always the same at the same place. Furthermore, the forces P, Q and R must be such that the differential expression Pdx + Qdy + Rdz either is integrable, when q is constant or depends only on the elasticity p, or becomes integrable upon being multiplied by a suitable function.
- 42. In my Memoir on fluid equilibrium⁴⁸ I only considered cases of the acting forces P, Q, R for which the differential expression Pdx + Qdy + Rdz is integrable, since this seemed to be the only case that could occur in Nature. In fact, if the density q is either constant or depends only on the pressure p, the fluid could never be in equilibrium unless this condition relating to the acting forces is satisfied. However, if it were possible for

ing satisfied these conditions, it remains to satisfy that imposed by continuity.

⁴⁴ For representing the functional dependence, now denoted f(x), Euler used the notation f, x or f: x. For example Euler's φ , W and φ' , W would now be denoted $\varphi(W)$ and $\varphi'(W)$. In Euler, 1755c, the comma is omitted.

⁴⁵ The function r.

⁴⁶ Indeed, if $\Phi(x,y,z) = \text{Cnst.}$ is the general integral of this equa-

tion, then the form (P-X)dx+(Q-Y)dy+(R-Z)dz must vanish whenever the differential $d\Phi$ vanishes; hence the two forms are proportional, which means that there exists an integrating factor for the first form.

 $^{^{\}rm 47}$ See Euler, 1755a.

⁴⁸ Euler, 1755a.

the acting forces to obey some other law, there could be equilibrium provided that the forces were such that there existed some function U which when multiplied by the expression Pdx + Qdy + Rdz made that expression integrable, or, equivalently, provided that the differential equation Pdx + Qdy + Rdz = 0 were integrable; for then if the density q is equated to this function U or to the product of this function U and some arbitrary function of the elasticity p, equilibrium may also exist. However, since these cases may not be possible, I shall not consider them in greater detail.

43. After the case of equilibrium, the simplest state that could exist in a fluid is that in which the entire fluid is in uniform motion in the same direction. Let us see, then, how this state is described by our two formulas. In this case, the three velocities being constant, we set u = a, v = b and w = c; we have X = 0, Y = 0 and Z = 0. Then our two equations assume the form:

$$\begin{split} \frac{dp}{q} &= \mathbf{P} dx + \mathbf{Q} dy + \mathbf{R} dz \ , \\ \left(\frac{dq}{dt}\right) + a \left(\frac{dq}{dx}\right) + b \left(\frac{dq}{dy}\right) + c \left(\frac{dq}{dz}\right) = 0 \ , \end{split}$$

and hence it is dear that if the density q is constant, the condition of the second equation is satisfied; however, the first equation cannot be satisfied unless the expression Pdx + Qdy + Rdz admits integration, just as if the fluid were at rest. Of course, such motion can have no effect on the pressure.

44. If, however, the density q is not constant, let us first see what function of x, y, z and t it must be for the second equation to be satisfied. This leads us to the curious analytical question of what function of the variables x, y, z and t must be taken for q in order that:

$$\left(\frac{dq}{dt}\right) + a\left(\frac{dq}{dx}\right) + b\left(\frac{dq}{dy}\right) + c\left(\frac{dq}{dz}\right) = 0 \ .$$

This would appear to be very difficult to answer if formulated in its broadest possible form. However, since when $a=0,\,b=0,\,c=0$ the quantity q is any function of $x,\,y,\,z$ that does not contain time t, if we reduce this case⁴⁹ to that of rest by imposing on the volume an equal and opposite motion, then, clearly, after time t the coordinates $x,\,y$ and z will be transformed by the change into $x-at,\,y-bt,\,z-ct$. From this we conclude that our equation will be satisfied if as q we take any function of the three quantities $x-at,\,y-bt,\,z-ct$. And in fact it is easy to see that such a function satisfies the equation, since

$$dq = L(dx - adt) + M(dy - bdt) + N(dz - cdt)$$
,

and, consequently,

- **45**. Now, as I have already noted, in order to satisfy the first equation it is necessary that after multiplication by some function U the differential expression Pdx + Qdy + Rdz be integrable. Therefore let $\int U(Pdx + Qdy + Rdz) = W$, where the constant of integration also in some way contains time t. Clearly, the expression Pdx + Qdy + Rdz will also be integrable if it is multiplied by Uf, W, 51 where U and W are known functions, since the acting forces are assumed to be known. Thus, if q does not depend on p, then necessarily q = Uf, W, whence the function of the three quantities x - at, y - bt and z - ct must be so determined that it can be reduced to the form Uf, W. If, however, gdepends only on p, the expression Pdx + Qdy + Rdzmust be absolutely integrable or U = 1; then, since p will be found in the form of a function of W, the density q will likewise be a function of W, which must also be a function of the quantities x - at, y - bt and z - ct, and from this we can deduce the nature of this function.
- However, it can be seen that, in general, the pressure p must always be a function of W, since otherwise the density could not be a finite function. Therefore let p=f, W and dp=dW.f', W; then, by virtue of the fact that $Pdx+Qdy+Rdz=\frac{dW}{U}$, we obtain q = Uf', W. Consequently, this case could not arise unless the density q was proportional to the product of the quantity U and a function of the pressure p or to the product of the quantity $U\varphi$, W and any function of p, where φ , W is used to denote a given function of W. For example, let $q = ppU\varphi$, W; we then have f', $W = \frac{d(f,W)}{dW} = (f,W)^2\varphi$, W, f'^{52} whence we find that the unknown function f, W is composed of W, for in this example we have $\frac{1}{f,W} = -\int dW$, $\varphi W = \frac{1}{p}$ and hence pcan be expressed in terms of W and thus, the quantity qwill also be known. When the latter can be reduced to the form of a function of x - at, y - bt and z - ct, the assumed state of the fluid will be possible and we shall know the pressure and the density at any time and at any point.
 - 47. An example 53 will throw more light on these

⁴⁹ The case of motion.

⁵⁰ Here Euler performs a Galilean transformation.

⁵¹ The equivalent modern notation would be Uf(W), cf. footnote 44.

⁵² The equivalent modern form would be $f'(W) = df(W)/dW = f^2(W)\omega(W)$.

⁵³ In this example forces are considered which do not derive from a potential and the integrating factor U is found for these forces.

operations which, as they are not yet sufficiently familiar, might appear overly obscure. Thus, let P=y, Q=-x and R=0; since $\frac{dp}{q}=ydx-xdy$, we obtain $U=\frac{1}{yy}$ and $W=\frac{x}{y}+T$, where T is any function of time t. Moreover, let $q=\frac{pp}{yy}$; since $\frac{dp}{pp}=\frac{ydx-xdy}{yy}$, we shall obtain $\frac{1}{p}=\Theta-\frac{x}{y}$, and $p=\frac{y}{\Theta y-x}$, where the constant Θ also contains time t. As a result, we have $q=\frac{1}{(\Theta y-x)^2}$, and this expression must be a function of x-at and y-bt, since z does not enter into it and this is only possible when $\Theta=\frac{a}{b}$; we then have $q=\frac{bb}{(ay-bx)^2}$, and $p=\frac{by}{ay-bx}$. Thus, neither the pressure nor the density depend on time and at a given point will be always the same. This example shows how the calculations should be performed in other cases that might be imagined.

48. Having dealt with this case in which the three velocities are constant, let us now assume that two velocities u and v vanish, which corresponds to the case in which all the fluid particles move in the direction of the OA axis, so that the trajectory described by each is a straight line⁵⁴ parallel to the OA axis; this case differs from the previous one, since the velocity u is assumed to vary with respect to both place and time. Since

$$X = \left(\frac{du}{dt}\right) + u\left(\frac{du}{dx}\right); \quad Y = 0; \quad Z = 0,$$

our two equations will take the form:

$$\frac{dp}{q} = Pdx + Qdy + Rdz - dx \left(\frac{du}{dt}\right) - udx \left(\frac{du}{dx}\right) ,$$
 &
$$\left(\frac{dq}{dt}\right) + \left(\frac{d \cdot qu}{dx}\right) = 0 .$$

This latter equation tells us, first of all, that the expression qdx - qudt must be integrable, the quantities y and z being considered constant with respect to this integration. Thus, the product of q and dx - udt must be a total differential, i.e. must be integrable.

49. If the density of the fluid is everywhere and always the same, i.e. if q is a constant equal to g, then, since $\left(\frac{du}{dx}\right) = 0$, it is dear that the velocity u must be independent of the variable x. Let u be any function of the two coordinates y, z and time t. Then our differential equation will take the form:

$$\frac{dp}{q} = Pdx + Qdy + Rdz - dx \left(\frac{du}{dt}\right) ,$$

where time t is assumed to be constant; thus, this expression must be integrable. Accordingly, if the

expression Pdx + Qdy + Rdz obtained from considering the acting forces is integrable in itself, then $dx\left(\frac{du}{dt}\right)$ must also be integrable. The expression $\left(\frac{du}{dt}\right)$ does not contain x, but if it were to contain y and z, the expression $dx\left(\frac{du}{dt}\right)$ could not be integrable. Thus, $\left(\frac{du}{dt}\right)$ must not contain y and z. Let Z be any function of y and z, and T any function of time t only; then the quantity u=Z+T will satisfy this condition, whence by virtue of the fact that Pdx+Qdy+Rdz=dV and $\left(\frac{du}{dt}\right)=\left(\frac{dT}{dt}\right)$, we obtain the following integral: $\frac{p}{q}=V-x\left(\frac{dT}{dt}\right)+C$ nst.

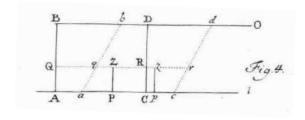
- **50.** In further clarification of this case, it should be noted that each fluid particle Z moves exclusively in the direction ZP parallel to the ZA axis and hence the motion of each fluid element will describe a straight line parallel to that axis, so that for the same element there is no change in the value of the two coordinates y and z. Thus, the motion of each particle will either be uniform or will vary with time in such a way that at each instant all the particles undergo the same changes in their motions, which is obvious from the equation u = Z + T. As to the state of pressure, given that we have $p = gV gx\left(\frac{dT}{dt}\right) + \text{Cnst.}$ where the constant has any dependence on time t, it depends not only on the effort⁵⁵ V but also on the change of velocity undergone by each element of the fluid; and, moreover, it may vary in any way with time.
- This case provides me with an opportunity to deal with certain questions which naturally arise and whose clarification is of the utmost importance for the theory of both fluid equilibrium and fluid motion. First of all, surprisingly, a change in the velocity of the fluid can occur without the acting forces P, Q, R helping to produce it. Since such a change could take place even when the acting forces vanish, it is reasonable to inquire how it is produced. Next, it also seems paradoxical that the pressure can vary arbitrarily at any instant, and that irrespective of the aforesaid change to which the motion is subjected. The latter difficulty remains even in the state of equilibrium. Thus, letting the three velocities u, v, w vanish, for incompressible fluids we have the integral $\frac{p}{g} = V + Cnst.$, where the constant may contain the time t in any way.
- 52. To understand this more clearly, one need only imagine a certain mass enclosed in a vessel. Clearly, the state of pressure depends not only on the acting forces but also on any extraneous forces which might be exerted on the vessel. For, even if there were no acting forces, by means of a piston applied to the fluid one could successively produce every possible state of pressure without the equilibrium being disturbed. This is precisely what we can conclude from our formula,

⁵⁴ This is the case of so-called shear flow.

which in this case shows that $\frac{p}{g}$ is a function of time t. From this we see that the state of pressure may vary at any instant, irrespective of the equilibrium. However, if for each instant of time the pressure at any point is known, then the pressures at all the other points can be determined, and since the force applied to the piston might now increase and now decrease, the calculations must reflect all these possible changes. The same variability should also be observed when the fluid is subjected to the action of arbitrary accelerative forces, so that at each instant the state of pressure is indeterminate and depends on the force then acting on the piston.

Here, then, is a vital difference between the **53**. accelerative forces, which act on all the elements of the fluid, and the force of a piston that presses on the fluid. Only the accelerative forces enter into our differential equation, while the piston force enters into the calculations only after integration and only affects the constant of integration. Consequently, in each case the constant must be so determined that at the point at which the piston acts the pressure is exactly equal to the force driving the piston at each instant, and it is for this reason that the constant contains time, so that it can be varied with time at will, as the circumstances require. This variability can always be represented by the action of a piston since, whatever the nature of the case considered, for it to be determined it must always be assumed that at one point at least in the fluid the pressure is known at every instant, and it is precisely this which makes it possible to determine the constant introduced into the calculations through the integration of our differential equation.

54. However, in our case of the motion considered in § 49, let us also assume that the accelerative forces vanish, i.e. that V=0, and to make this case perfectly determinate, let us assume that $u=a+\alpha y+\beta t.^{56}$ Then the equation for the pressure will take the form $\frac{p}{g}=\mathrm{Cnst.}-\beta x.$ Let us assume, moreover, that this constant is equal to $\gamma+\delta t$, so that $\frac{p}{g}=\gamma+\delta t-\beta x$, and let us see under what conditions this motion can take place. Since each fluid element moves in the direction of the OA axis, the motion could only take place in a cylindrical pipe laid in the same direction. Let ABIO (Fig. 4) be that pipe and initially, at t=0, let the fluid occupy the portion ABCD bounded by cross sections



AB and CD perpendicular to the pipe. We will reckon the abscissas from the point A along the straight line AI and let the pressure p be equal to γg everywhere along the base AB and to $\gamma g - \beta g$.AC along the other base CD. In the interior of the fluid, however, at any point Z with the coordinates AP = x, PZ = y, the pressure will be equal to $\gamma g - \beta gx$. Consequently, it is impossible to consider the fluid in the pipe beyond CD, taking $AC = \frac{\gamma}{\beta}$, so that the pressure at CD does not become negative.

Let us set for this determinate fluid mass the **55**. length AC = b and the width AB = CD = c, the height not entering into consideration since neither the velocities nor the pressures depend on the third coordinate z; when $\gamma = \beta b$, in the initial state ABCD the pressure is equal to βbg on the base AB and zero on the base CD, while at any point Z it is equal to $\beta g(b-x) = \beta g$.CP. We will assume that in this state the fluid has a motion in the direction of the pipe such that the velocity on the line AC is equal to a and that on the line BD equal to $a + \alpha c$, while on any line QR parallel to the direction of the pipe it is equal to $a + \alpha y$, where AQ = CR = y. Thus, we believe that something has caused this motion to be impressed on the fluid and that, at the initial instant, the surface AB is subjected to the said force βbg , exerted by means of a piston, while the other base CD is not subjected to any pressure. However, at subsequent moments of time the forces acting on the end faces could vary arbitrarily. Now this variability is determined by the hypotheses we have just established. Therefore let us see how by virtue of these hypotheses the motion of the fluid will be continued.

56. After the lapse of a time t, all the fluid elements on the line QR will have a velocity in that same direction equal to $a + \alpha y + \beta t$, as a result of which in the time dt they will travel a distance $(a + \alpha y + \beta t)dt$; thus, from the beginning of the motion they will have traveled a distance $at + \alpha yt + \frac{1}{2}\beta tt$; and the alignment of fluid particles⁵⁷ initially at QR will now have advanced to qr, having traversed the distance $Qq = at + \alpha yt + \frac{1}{2}\beta tt$.

 $^{^{56}}$ In Euler, 1755b the symbol β is used in the r.h.s. of this equation; in the printed version it is replaced by a symbol resembling a capital C with curled ends.

⁵⁷ Euler uses "filée du fluide" where "filée" is a somewhat poetic variant of "file" (alignment, file) or "fil" (thread); this is just a line of fluid elements and not what is now called a fillet of fluid, the latter having also an infinitesimal width, a concept introduced by Euler, 1745 (see Grimberg, Pauls and Frisch, 2008).

Thus, the thread AC will have arrived at ac, having traveled a distance $Aa = at + \frac{1}{2}\beta tt$, while the thread BD will have arrived at bd, having traveled a distance $Bb = at + \alpha ct + \frac{1}{2}\beta tt$, so that the fluid mass will now be bounded by the faces ab and cd, which are straight but inclined to the direction of the pipe. The pressure on the face ab at q must now be $g(\beta b + \delta t - \beta .Qq) = g(\beta b + \delta t - \beta at - \alpha \beta yt - \frac{1}{2}\beta \beta tt)$, and on the face cd at r it must now be $g(\beta b + \delta t - \beta .Qr) = g(\delta t - \beta at - \alpha \beta yt - \frac{1}{2}\beta \beta tt)$. Thus, we need to visualize pistons which act with these forces on the two end faces ab and cd, and since the pressures are not the same over the entire length of these faces, the pistons must be imagined as being flexible and pliable enough to exert such pressures.

57. This motion would remain the same if in integrating the pressure p we were to take any function of t instead of δt , but then the state of pressure in the fluid mass would be different at each instant of time, even though the assumed motion of the fluid itself would not be affected in any way. Thus, let us set $\delta t = \beta at + \alpha \beta ct + \frac{1}{2}\beta \beta tt$; after a time t the pressure at any point q on the face ab will be $g[\beta b + \alpha \beta (c - y)t]$, and at any point z on the line qr it will be equal to $g[\beta b + \alpha \beta (c - y)t - \beta qz]$; therefore the pressure at the other end r will be $\alpha\beta g(c-y)t$. Hence, on the face ab the pressure will be equal to $\beta q(b + \alpha ct)$ at a and to βqb at b, while on the other face cd the pressure will be equal to $\alpha\beta qct$ at c and to zero at d. Moreover, each thread QR will move in its own direction with uniform acceleration, i.e. will receive equal increments of velocity in equal times. The study of this particular case could serve to elucidate the calculations to be made in all other cases.

58. Let us now return to the case proposed (§ 48) and assume the density q to be constant and equal to g, while making the forces P, Q, R such that the fluid could never be in equilibrium. To this end, let P=0, $Q=-\frac{x}{a}$ and $R=-\frac{x}{a}$ and let $u=b+\frac{(y+z)t}{a}$, so that we have $\left(\frac{du}{dx}\right)=0$ and $\frac{dp}{g}=-\frac{xdy+xdz}{a}-\frac{ydx+zdx}{a}$, where the constant may contain time in any way. Thus, it is not possible for the entire fluid mass ever to remain at rest, since even if we set b=0 in order to have the fluid at rest at the outset when t=0, immediately after that first instant it would be agitated and only the elements for which y=0 or z=0 or y+z=0 would remain at rest; all the others would be set in motion either forward or backward, depending on whether y+z was positive or negative. It is also easy to determine the pressures required to maintain the assumed motion.

59. Let, however, the density be no longer constant but variable, i.e. let the fluid be compressible. Then in order for the expression qdx - qudt to be a total differential we can take for u any function of the variables x, y, z and t. Here, since only x and t are regarded as variable, while y and z are taken constant, it will always be possible to assign a quantity s such that s(dx - udt) is integrable. Let S be that integral; then this condition will be satisfied if we take q = sf : S, ⁵⁹ Furthermore, it is now necessary that the following differential be integrable:

$$\frac{dp}{q} = Pdx + Qdy + Rdz - dx\left(\frac{du}{dt}\right) - udx\left(\frac{du}{dx}\right).$$

Note that if the forces P, Q, R were to vanish, the pressure p would become a function of x and t and hence the quantity $q\left(\left(\frac{du}{dt}\right) + u\left(\frac{du}{dx}\right)\right)$ would only involve the two variables x and t, from which the nature of the function u must be determined, insofar as it involves y and z.

60. Although I have assumed that v=0 and w=0, these formulas cover all the cases in which all the fluid particles always move in the same direction, the only requirement being that the OA axis be taken in that direction. Therefore we will also be able to solve our equations when the direction of motion is inclined to the three axes, which cannot fail to throw further light on the analysis. To this end, let us consider the true velocity of any fluid particle Z and let that velocity be equal to \mathfrak{G} , and since its direction is given with respect to the three axes, the velocity components will hold certain ratios to it. Let $u=\alpha\mathfrak{G}$, $v=\beta\mathfrak{G}$ and $w=\gamma\mathfrak{G}$; setting $d\mathfrak{G}=\mathrm{K}dt+\mathrm{L}dx+\mathrm{M}dy+\mathrm{N}dz$, we shall have

$$X = \alpha K + \alpha \alpha L + \alpha \beta M + \alpha \gamma N$$

$$Y = \beta K + \alpha \beta L + \beta \beta M + \beta \gamma N$$

$$Z = \gamma K + \alpha \gamma L + \beta \gamma M + \gamma \gamma N$$

Consequently, if, for conciseness, we write $K + \alpha L + \beta M + \gamma N = O$, having X = aO, $Y = \beta O$, $Z = \gamma O$, our equations will take the form:

$$\frac{dp}{q} = Pdx + Qdy + Rdz - O(\alpha dx + \beta dy + \gamma dz)$$

$$\left(\frac{dq}{dt}\right) + \alpha \left(\frac{d \cdot q \forall}{dx}\right) + \beta \left(\frac{d \cdot q \forall}{dy}\right) + \gamma \left(\frac{d \cdot q \forall}{dz}\right) = 0.$$

61. First, let the density q = g. As we have seen in § 44, in order to satisfy the equation $\alpha \left(\frac{d \aleph}{dx} \right) + \beta \left(\frac{d \aleph}{dy} \right) + \gamma \left(\frac{d \aleph}{dz} \right) = 0$ the quantity \aleph must be any function of the

⁵⁸ In the printed version the two fractions in the r.h.s. have a minus instead of a correct plus in the numerator; in the manuscript Euler, 1755c, the handwritten notation is ambiguous.

⁵⁹ This equation would now be written q = sf(S).

quantities $\alpha y - \beta x$ and $\alpha z - \gamma x$ or $\beta z - \gamma y$ and, in addition, may in an arbitrary way contain time t. Thus, let \eth be any function of the quantities $\alpha y - \beta x$, $\alpha z - \gamma x$, and t, since the expression $\beta z - \gamma y$ has already been formed from the other two. From this it is easy to see that at each instant the velocity of particles on the same straight line parallel to the direction of motion will be everywhere the same, just as the nature of the hypothesis requires. Hence the differential of \eth will have the following form:

$$d\mathbf{\delta} = \mathbf{F}dt + \mathbf{G}(\alpha dy - \beta dx) + \mathbf{H}(\alpha dz - \gamma dx) ,$$

so that K = F; L = $-\beta$ G - γ H; M = α G; and N = α H. Consequently, O = F is a function of $\alpha y - \beta x$, $\alpha z - \gamma x$ and of t. Hence the differential equation, which remains to be solved, will be:

$$\frac{dp}{q} = Pdx = Qdy + Rdz - F(\alpha dx + \beta dy + \gamma dz).$$

- **62**. The time t being here assumed constant, if the expression Pdx + Qdy + Rdz = dV is integrable in itself, the other part of the equation $F(\alpha dx + \beta dy + \gamma dz)$ must be likewise, and this could not be so unless F were a function of $\alpha x + \beta y + \gamma z$ and of time t. In addition, however, F must must also be a function of the quantities $\alpha y - \beta x$, $\alpha z - \gamma x$ and time t; consequently, since the expression $\alpha x + \beta y + \gamma z$ cannot be formed from the expressions $\alpha y - \beta x$ and $\alpha z - \gamma x$, it is clear that the quantity F must be a function of time tonly. Consequently, the velocity **\(\delta\)** will have the form $\delta = Z + T$, where Z denotes an arbitrary function of the two quantities $\alpha y - \beta x$ and $\alpha z - \gamma x$ that does not contain time t, while T is an arbitrary function of time t only, so that dT = Fdt. Hence the integral of our differential equation will be $\frac{p}{g} = V - F(\alpha x + \beta y + \gamma z) + Cnst.$ where the constant may contain time t in an arbitrary way. Together with the relation $\aleph = \mathbb{Z} + \mathbb{T}$, this integral contains everything relating to the motion in the case in question.
- **63**. But if the density q is not constant, it will be important to obtain the solution of the following equation:

$$\left(\frac{dq}{dt}\right) + \alpha \left(\frac{d.q \aleph}{dx}\right) + \beta \left(\frac{d.q \aleph}{dy}\right) + \gamma \left(\frac{d.q \aleph}{dz}\right) = 0 \; .$$

However difficult this may appear, reduction to the previous case shows that the velocity $\mbox{\bf 8}$ can be an arbitrary function of the four variables x, y, z and t, while the value of q must be determined as follows. Let us consider, generally, an expression

$$s(ldx + mdy + ndz - \aleph dt) = dS$$
,

which has become integrable after multiplication by s,

and let q = sf : S; then, if we set $d.f : S = dS.f' : S,^{60}$ our expression will take the form

$$\begin{split} &f: \mathbf{S}\left(\frac{ds}{dt}\right) - sf': \mathbf{S}.s \aleph \\ &+ \ \alpha sf: \mathbf{S}\left(\frac{d\aleph}{dx}\right) + \alpha \aleph f: \mathbf{S}\left(\frac{ds}{dx}\right) + \alpha \aleph sf': \mathbf{S}.ls \\ &+ \ \beta sf: \mathbf{S}\left(\frac{d\aleph}{dy}\right) + \beta \aleph f: \mathbf{S}\left(\frac{ds}{dy}\right) + \beta \aleph sf': \mathbf{S}.ms \\ &+ \ \gamma sf: \mathbf{S}\left(\frac{d\aleph}{dz}\right) + \gamma \aleph f: \mathbf{S}\left(\frac{ds}{dz}\right) + \gamma \aleph sf': \mathbf{S}.ns \end{split}$$

which must be equal to zero.

64. First of all, we equate to zero the terms containing f': S, as a result of which we obtain $1 = \alpha l + \beta m + \gamma n$; after division by f': S the remaining terms give

$$\left(\frac{ds}{dt}\right) + \alpha \left(\frac{d.s \mathbf{d}}{dx}\right) + \beta \left(\frac{d.s \mathbf{d}}{dy}\right) + \gamma \left(\frac{d.s \mathbf{d}}{dz}\right) = 0 \;,$$

which is indeed similar to the expression proposed; however, it should be noted that the integrability of the quantity dS is conditioned by:

whence we obtain: $\left(\frac{ds}{dt}\right)(1-\alpha l-\beta m-\gamma n)=0,^{61}$ which is consistent with the previous condition. Thus, provided that $\alpha l+\beta m+\gamma n=1$, and s is a function such that $s(ldx+mdy+ndz-\aleph dt)=d\mathbb{S}$, or integrable, our equation will be satisfied if we take q=sf: \mathbb{S} , or $\frac{q}{s}$ equal to any function of \mathbb{S} . The quantities l,m and n do not have to be constant, but then the following must hold

$$\alpha \left(\frac{dl}{dt}\right) + \beta \left(\frac{dm}{dt}\right) + \gamma \left(\frac{dn}{dt}\right) = 0 ,$$

a condition already contained in the equation $1 = \alpha l + \beta m + \gamma n$.

65. In addition, l, m and n must be functions such that the differential equation $ldx + mdy + ndz - \mathbf{8} dt = 0$ becomes possible, since without this condition it would be impossible to find a multiplier s which made the equation integrable. Thus, if we arbitrarily choose some value for l, the values of m and n will be already determined and we can avoid having to find them. We

⁶⁰ Here, q = sf : S and d.f : S = dS.f' : S would now be denoted q = sf(S) and df(S) = dSf'(S), respectively.

⁶¹ The r.h.s. = 0 is missing both in the printed version and in Euler, 1755c.

will set $\alpha l=1$ or $l=\frac{1}{\alpha}$; then, necessarily, $\beta m+\gamma n=0$ and it remains only to find the factor s for which the expression $s\left(\frac{dx}{\alpha}-\aleph dt\right)$ is integrable, the two quantities y and z being regarded as constants. Thus, let $S=\int s\left(\frac{dx}{\alpha}-\aleph dt\right)$, so that y and z are contained in S as constants; we can now take q=sf:S, which gives us the same solution as if we had changed the position of the three axes so much that one of them coincided with the direction of motion of all the fluid elements. Hence we see that this apparent restriction in no way diminishes the generality of the solution.

66. In the same way it would be possible to study several other particular cases of sometimes greater and sometimes lesser scope, but we would not find a case more general than that in which the three velocities u, v and w are such that the expression udx + vdy + wdz becomes integrable. Let S be an integral which also contains time t and let its total differential be $dS = udx + vdy + wdz + \Pi dt$. Since we have

we shall have

$$\begin{split} \mathbf{X} &= \left(\frac{d\mathbf{\Pi}}{dx}\right) + u\left(\frac{d\,u}{dx}\right) + v\left(\frac{d\,v}{dx}\right) + w\left(\frac{d\,w}{dx}\right) \\ \mathbf{Y} &= \left(\frac{d\mathbf{\Pi}}{dy}\right) + u\left(\frac{d\,u}{dy}\right) + v\left(\frac{d\,v}{dy}\right) + w\left(\frac{d\,w}{dy}\right) \\ \mathbf{Z} &= \left(\frac{d\mathbf{\Pi}}{dz}\right) + u\left(\frac{d\,u}{dz}\right) + v\left(\frac{d\,v}{dz}\right) + w\left(\frac{d\,w}{dz}\right) \end{split}$$

and our differential equation now becomes:

$$\frac{dp}{dt} = Pdx + Qdy + Rdz - d\Pi - udu - vdv - wdw$$

(the last member of which is absolutely integrable), while the other equation remains as before:

$$\left(\frac{dq}{dt}\right) + \left(\frac{d.qu}{dx}\right) + \left(\frac{d.qv}{dy}\right) + \left(\frac{d.qw}{dz}\right) = 0 \; .$$

67. Thus, everything reduces to finding suitable values for the three velocities u, v and w that satisfy our

two equations, which contain everything we know about the motion of fluids. For if these three velocities are known, we can determine the trajectory described by each element of the fluid in its motion. Let us consider a particle which at a given instant is located at the point Z; for finding the trajectory which it has already described and which it has yet to describe, since its three velocities u, v and w are assumed to be known, for its position at the next instant we have dx = udt, dy = vdt and dz = wdt. Eliminating time t from these three equations, we obtain two more equations in the three coordinates x, y and z which will determine the unknown trajectory of the fluid element now at Z and, in general, we shall know the path which each particle has traveled and has yet to travel.

68. The determination of these trajectories is of the utmost importance and should be used to apply the Theory to each case considered. If the shape of the vessel in which the fluid moves is given, the fluid particles which touch the surface of the vessel must necessarily follow its direction; therefore the velocities u, v and wmust be such that the trajectories derived therefrom lie on that same surface. 63 This makes it quite clear how far removed we are from a complete understanding of the motion of fluids and that my exposition is no more than a mere beginning. Nevertheless, everything that the Theory of Fluids contains is embodied in the two equations formulated above (§ 34), so that it is not the laws of Mechanics that we lack in order to pursue this research but only the Analysis, which has not yet been sufficiently developed for this purpose. It is therefore clearly apparent what discoveries we still need to make in this branch of Science before we can arrive at a more perfect Theory of the motion of fluids.

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⁶² In §§ 30–33 above, Euler has already pointed out the possibility and given examples of non-potential fluid flows. Truesdell, 1954 considers that Euler based § 66 of his memoir on his previous work (Euler, 1756–1757) which was completed before he had discovered the existence of non-potential flows. This seems all the more likely in that, as Truesdell points out, Euler here denotes the velocity potential not by W, as in § 26, but by S, as in his earlier study.

⁶³ Here, Euler is drawing attention to the fact that in order to calculate the motion of a fluid, in addition to the equations of motion, continuity and state and the initial conditions, we also need the boundary conditions, namely the vanishing of the normal component of the velocity.

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