INSTABILITY OF THE FLOW OVER A ROTATING DISK WITH VISCOSITY GRADIENT: HIGHER ORDER PREDICTOR-CORRECTOR METHODS

Fernando Andrecioli
Department of Mechanical Engineering – SEM/EESC/USP
PO Box 359 , 13566-590 São Carlos, SP, Brazil
fandrecioli@yahoo.com.br

Norberto Mangiavacchi
Mechanical Engineering Department – State University of Rio de Janeiro, UERJ,
Rua São Francisco Xavier 524, 20550-900 Rio de Janeiro, RJ, Brazil
norberto@uerj.br

J. Pontes
Metallurgy and Materials Engineering Department – EE/COPPE/UFRJ
PO Box 68505, 21945-970 Rio de Janeiro RJ, Brazil
jopontes@.ufrj.br

Antônio Castelo Filho
Institute of Mathematical and Computational Sciences USP – S. Carlos,
PO Box 668, 13560-161 S.Carlos, SP, Brazil
castelo@icmc.sc.usp.br

Abstract. *T*he boundary layer over a rotating disk has been utilized as a prototype of 3-D boundary layers for the study of cross-flow instabilities. This type of flow is encountered, for instance, in the boundary layer over swept wings. It has been shown (Pontes et. al. 2002) that the neutral stability curves for the boundary layer over a rotating disk in the presence of a viscosity gradient normal to the disk wall, are displaced in the sense of increased instability when the viscosity close to the wall is larger than the bulk viscosity. In order to complete the stability analysis using moderate computational resources, it is necessary to compute the neutral-stability curves defining the region of instability in an efficient way.

The numerical method used to determine the neutral-curve is a numerical continuation method know as predictor-corrector method. This numerical method is an interesting way to solve the problem of finding the hydrodynamic stability curves of a boundary layer over a rotating disk because it demands computational costs much smaller than other kinds of numerical methods. In this work, the first order predictor-corrector method employed for the computation of the neutral curves is discussed. The technique is then extended to include high order predictor methods.

Keywords:*Rotating Disk Flow, Electrochemical Instabilities, Hydrodynamic Stability, Turbulence*

1. Introduction

The boundary layer over a rotating disk has been utilized as a prototype of 3-D boundary layers for the study of cross-flow instabilities. This type of flow is encountered, for instance, in the boundary layer over swept wings. The existence of a hydrodynamic instability in rotating disk flow has been the object of a number of investigations, both experimental (Faller, 1991), (Lingwood, 1995), (Wilkinson and Malik, 1985), and theoretical (Lingwood, 1995), (Malik, 1981), (Malik, 1986), in the case of fluids with uniform viscosity. The main result shows that the steady flow becomes unstable beyond a certain non-dimensional distance from the axis of rotation. This distance is the *Reynolds number* of the problem, defined by:

$$R = r\left(\frac{\Omega}{\nu(\infty)}\right)^{1/2} \tag{1}$$

where r is the radial distance from the axis, Ω and $\nu(\infty)$ are the angular velocity and the bulk viscosity far from the disk surface. In this work we are concerned with a linear stability analysis to investigate the influence of a viscosity gradient along the axial direction, on the stability of the boundary layer developed close to the rotating disk. In order to overcome the difficulties of evaluating the steady hydrodynamic and concentration fields and performing a stability analysis of the coupled fields we shall adopt the strategy of assuming a steady viscosity profile, dependent on the axial direction.

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The investigation of the stability of laminar flows can be performed decomposing the motion into the base flow (von Karman, 1921) whose stability is to be examined, and a superimposed perturbation motion (Schlichting, 1968). In most cases it is assumed that the perturbation quantities are small compared to those of the base flow. The base flow is a solution of the Navier-Stokes equations. The superimposed perturbation is assumed to be small in the sense that all quadratic terms of the perturbation can be neglected compared to the linear terms. Subtracting the Navier-Stokes equations of the base flow from the Navier-Stokes for the flow with perturbations, and ignoring all terms quadratic in the perturbation velocities, a linearized set of equations for the evolution of the perturbation quantities is obtained.

The perturbation is considered to be constituted of single partial perturbations or modes. At this stage we assume that the perturbation variables are separable and look for a solution in the form:

$$\begin{pmatrix} \tilde{v}_r \\ \tilde{v}_\theta \\ \tilde{v}_z \\ \tilde{p} \end{pmatrix} = \begin{pmatrix} f(\xi) \\ g(\xi) \\ h(\xi) \\ \pi(\xi) \end{pmatrix} \exp[i(\alpha r + \beta R\theta - \omega t)] \tag{2}$$

where ω is a complex number, with $\Re(\omega)$ and $\Im(\omega)$ being, respectively, the frequency and the rate of growth of the perturbation. Parameters α and β are the components of the perturbation wave-vector along the radial and azimuthal directions.

Inserting the form of the perturbation, an ordinary differential equation is found for the amplitude of the perturbation. The stability analysis of a laminar flow is now an eigenvalue problem of the perturbation differential equation, with appropriate boundary conditions.

The linearized momentum equations for the perturbations reduce to Eqs. 2.20–2.21 given by Malik (1986), in the case of constant viscosity fluids. For the case of a viscosity profile can be rewritten in the form:

$$\begin{pmatrix} a_4D^4 + a_3D^3 + a_2D^2 + a_1D + a_0; & b_1D + b_0 \\ c_1D + c_0; & d_2D^2 + d_1D + d_0 \end{pmatrix} \begin{pmatrix} h \\ \eta \end{pmatrix} = \omega \begin{pmatrix} q_2D^2 + q_0; & 0 \\ 0; & s_0 \end{pmatrix} \begin{pmatrix} h \\ \eta \end{pmatrix} \tag{3}$$

where a_0, $\cdot$, a_4, b_0, b_1, c_0, c_1, d_0, d_1, d_2, q_0, q_1, q_2, and s_0 are known complex functions of the base velocity and viscosity profiles, and of parameters α, $beta$, and R (Pontes et al. 2002).

Eq. (3) defines a generalized eigenvalue/eigenfunction problem. The eigenfunctions h and η are the normal modes of the model, the imaginary and real parts of each eigenvalue ω being, respectively, the rate of growth and the angular velocity of the perturbation relative to the angular velocity of the disk. The generalized eigenvalue/eigenfunction problem is solved numerically. Space derivatives are represented by second-order discrete formulæ, transforming the original problem in a generalized eigenvalue/eigenvector problem.

For a given viscosity profile the parameter space of the problem contains *three* variables, the Reynolds number R and the perturbation wave-vector components α and β.

The results are presented in the form of neutral stability curves ($\Im(\omega) = 0$) for the case of constant viscosity fluids. The neutral curves are plotted in the $\beta \times R$ and $\alpha \times R$ planes, for specified values of $\Re(\omega) = \omega_r$.

Building the neutral curves requires finding the set of points $c(s) = (\alpha(s), \beta(s), R(s))$ that satisfy $F\,(c(s)) = 0$, where $F : R^3 -> R^2$ is given by $F = (\Im(\omega), \Re(\omega) - \omega_r)^T$. The neutral curves are built using a Predictor-Corrector Continuation method described in the following sections. Here, we will give an overview of the solution method:

- The perturbation frequency ω_r is specified and an initial point c_0, in the parameters space α, β, R is given. This point is not necessarily on the neutral curve;
- This initial point is corrected using an inexact Newton iteration given by

$$c_i^{n+1} = c_i^n - F'(c_i^0)^+ F(c_i^n) \tag{4}$$

  where $F'(v_0)^+$ is the pseudo-inverse of Moore-Pen rose of the Jacobian of F. The Jacobian is computed numerically, using a finite difference approximation.

- To obtain an initial estimate of the next point over the curve, a Predictor step is employed, based on the first order Euler method:

$$c_{i+1}^0 = c_i + h\ t(F'(c_i)) \tag{5}$$

  where h is a suitable step size, and $t(F'(c_i))$ is the tangent vector to curve $c(s)$.

- The value c_{i+1}^0 is corrected in a Corrector step using Eq. (4) iteratively until a satisfactorily converged value is obtained.
- The generalized eigenvalue/eigenfunction problem required to evaluate $F(c(s))$ is solved numerically, using an Inverse Power Method double precision *zgipm* routine for complex generalized non-symmetric eigenproblems, that takes advantage of the sparsity of the coefficient matrices.

2. Predictor corrector methods

Before we talk about the predictor corrector methods, let us describe the continuation methods or homotopy methods, regarded as forerunners of the PC methods (Stoer, 1993), (Eugene, 1990). Suppose one wishes to obtain the solution of a nonlinear system of n equations in n variables and a priori knowledge about concerning zero points is not available.

The iterative method employed to solve the system can fail, because poor starting values are likely to be chosen. As a possible remedy, one defines a homotopy $F : \mathbf{R}^{n+1} \to \mathbf{R}^n$ and attempt to trace an implicitly defined curve $c(s) \in F^{-1}(0)$ from a starting point to a solution point. the arclength parameter is consider a natural parameter for the curve.

The curve c, parametrized with respect to arclength s, may be regarded as the solution of an initial value problem which is obtained by differentiating the equation

$$F(c(s)) = 0 \tag{6}$$

with respect to s:

$$(c)c = 0, \qquad ||c|| = 1, \qquad c(0) = c_0. \tag{7}$$

Equation(7) seems to be a more complicated problem to solve than Eq(6). In the fact, but the solution curve c consists of zero points of F , and as such it enjoys powerful local contractive properties with respect to iterative methods.

The general idea of PC methods is to numerically trace the curve c by generating a sequence of points along the curve satisfying a chosen tolerance criterion. This sequence is obtained by numerically integrating Eq(7) very coarsely, and then locally use an iterative method for solving Eq(6) as a stabilizer.

2.1. Predictor

The predictor step corresponds to numerically integrate Eq(7). Its scope is to orient the search of points along the curve. Generally, the the most used kind of predictor is know as the Euler predictor and take the form

$$v_{i+1} = u_i + ht(F'(u_i)), \tag{8}$$

where $h > 0$ represents a "step size" and $t(F'(ui)) \in \mathbf{R}^{n+1}$ represents the unique tangent vector induced by F' satisfying three conditions:

$$F't = 0 \tag{9}$$

$$||t|| = 1 \tag{10}$$

$$\det \begin{pmatrix} F' \\ t^t \end{pmatrix} > 0. \tag{11}$$

Later it will be discussed in detail how h is to be chosen and how t is then obtained. There are other kinds of predictors and they are called high order predictors and can be obtained by interpolating the points belonging to the curve c. The Euler predictor can be consider as a first order predictor. To make a robust and efficient predictor corrector method we have to develop an effective step size adaptation and an efficient incorporation of high order predictors.

2.2. Corrector

A straightforward way to solve the problem Eq(6) in the corrector step is to use a Newton type method. As it is well know, the Newton's methods takes the form

$$x_{i+1} = x_i - F'(x_i)_{-1}F(x_i) \tag{12}$$

but the problem is that the Jacobian F' is not a square matrix and cannot be inverted. Hence, Newton's method has to be modified. At this point, we introduce a suitable right inverse of F' denoted by F'^+ to replace the inverse Jacobian matrix in the Newton formula. Such a suitable right inverse is provided by the Moore-Penrose inverse defined as

$$F^+ = F^t(FF^t)^{-1} \tag{13}$$

where F is a $n \times (n+1)$ matrix with maximal rank.

Now we need to calculate the tangent vector and the Moore-Penrose inverse. To obtain $t(F')$ and F'^+, we use a QR decomposition of F', which is given by

$$F'^t = Q \begin{pmatrix} R \\ 0^t \end{pmatrix} \tag{14}$$

where Q is an $(n+1) \times (n+1)$ orthogonal matrix and R is a nonsingular $n \times n$ upper triangular matrix.

To determining the Moore-Penrose inverse, we have $F'^+ = F'^t(F'F'^t)^{-1}$, and from $F'^t = Q \begin{pmatrix} R \\ 0^t \end{pmatrix}$ and $F' = (R^t, 0)Q^t$ we obtain

$$F'^+ = Q \begin{pmatrix} (R^t)^{-1} \\ 0^t \end{pmatrix} \tag{15}$$

As is usual in solving linear systems of equations, we do not invert R^t, but we rather calculate $w = F'^{+}b$ by forward solving $R^t y = b$.

For the tangent vector, if t denotes the last column of Q, then $F't = 0$ and $||t|| = 1$. Note that $(F^t, t) = Q \begin{pmatrix} R & 0 \\ 0^t & 1 \end{pmatrix}$ implies $\det \begin{pmatrix} F' \\ t^t \end{pmatrix} = \det(F^t, t) = \det Q \det R$.

Now, sign $\det R$ is easily determined. The sign of $\det Q$ is usually easily obtained as well. For example, if Givens rotations are used, it is equal to unity. If Householder reflections are used, each reflection changes the sign, so sign $\det Q = (-1)^p$ where p is the number of reflections which are involved in the factorization of F^t by Householder's method.

To obtain the QR decomposition of F' we use the Givens rotations methods acting on two co-ordinates described by a matrix $G = \begin{pmatrix} s_1 & s_2 \\ -s_2 & s_1 \end{pmatrix}$ such that $s_1^2 + s_2^2 = 1$

2.3. Step-length Adaptations for the predictor

To obtain a PC method more powerful and able to build curves more complicated with affordable computational costs, we need to incorporate an automatic strategy for controlling the step length in the method. This is due to the fact that the Newton's methods do not assure convergence to the solution of the curve and then we need to assure that the point given by the predictor step is sufficiently close to the curve. The basic idea behind this strategy is to observe the performance of the corrector procedure and then to adapt the step length $h > 0$ accordingly. This step length strategy is based upon a posteriori estimates of the performance of the corrector process in order to answer the following question: given the manner in which the corrector process was performed, which step length would have been the "best" for the last predictor step. This "ideal" step length is determined via asymptotic estimates, therefore it receives the name Steplength Adaptation by Asymptotic Expansion, and it is then taken as the step length for the next predictor step. This strategy depends primarily upon two factors: the particular PC method being utilized and the criteria used in deciding what performance is considered "best".

An intuitive criteria to govern the step length can be the distance to the curve approximated by the formula

$$\delta(u, h) = ||F'(v(h))^{+} F(v(h))|| \tag{16}$$

where $v(h)$ is the point given by the predictor step with step length h, $F'(v(h))^{+}$ and $F(v(h))$ is the Moore-Penrose inverse of the Jacobian matrix and the function value evaluated in the point $v(h)$ respectively and u is the last calculated point belonging to the curve. Another useful criteria is the contraction rate of the corrector process.defined as the quotient of the first two successive Newtons steps

$$\kappa(u, h) = \frac{\| H'(v(h))^{+} H(w(h)) \|}{\| H'(v(h))^{+} H(v(h)) \|} \tag{17}$$

where $w(h)$ is the point given by the corrector process starting in the point $v(h)$. Since Newton's method is locally quadratically convergent, it is clear that $\kappa(u, h)$ will decrease if h decreases and hence $v(h)$ approaches $F^{-1}(0)$.

To estimate the step length for the next predictor process we make use of the approximated formulae

$$\tilde{h} = h \sqrt{\frac{\tilde{\kappa}}{\kappa(u, h)}} \tag{18}$$

$$\tilde{h} = h \sqrt{\frac{\tilde{\delta}}{\delta(u, h)}} \tag{19}$$

where $\tilde{h}$ represents the step length to the next predictor step and $\tilde{\kappa}$ and $\tilde{\delta}$ represent the nominal contraction rate and distance of the curve respectively. The choice of $\tilde{\kappa}$ and $\tilde{\delta}$ will generally depend upon the nature of the problem at hand, and on the desired reliability with which we want to traverse $F^{-1}(0)$.

Supposing that an initial point u belonging to the curve is given, we can predict the point u_n with an initial step length h and calculate the contraction rate and the distance to the curve as function of the point u_n and the step length h. The corrector process is then employed and a new point u_{n+1} belonging to the curve is determined. The deceleration factor f receives the largest value between $\frac{\kappa(u,h)}{\tilde{\kappa}}$ and $\frac{\delta(u,h)}{\tilde{\delta}}$ and hence we have $\tilde{h} = \frac{h}{f}$. It can happen that the corrector process, starting from the point given by the predictor step with step length h, does not converge to the curve; then the predictor step is applied again at the same point, however, with a new step length smaller than the last. This procedure is repeated until the corrector process converges. When the corrector process converges, the step length for the next predictor step is given by the maximum value between $h_{\max}$ and $h \times f$ and this permits to increase or to decrease the step length according to the part of the curve that is being traced.

2.4. High order predictors

As it was seen previously, in Euler predictor we trace the curve by tangent lines. In the case of high order predictors we trace the curve by interpolating the already determined points by polynomials. Using this approach, the computational time spent can be reduced because the predicted point is closer to the curve than the point obtained by

the Euler formula. We used the interpolating polynomial which uses the tangents t gave by the Hermite formula which provide more precision than the Newton formula. In principle, we can use as many points as we wish, hence producing a polynomial with arbitrary degree n. However an excessive quantity of points implies in a polynomial with elevated degree inducing instability in the solution of the problem.

In order to use the interpolating polynomial P_q, we need to express it in terms of a suitable parameter ξ. The arclength parameter s would be ideal to use, however it gives cause for additional complexity of obtaining precise numerical approximations of the arclength s_i, such that $c(s_i) = u_i$. We therefore use a local parametrization ξ induced by the current approximation $t \approx t(F'(u_n))$, and $\| t \| = 1$. This local parametrization $c(\xi)$ is defied as the locally unique solution of the system

$$F(u) = 0 \tag{20}$$

$$t \cdot (u_n + \xi t - u) = 0 \tag{21}$$

it follows immediately that $c(\xi_i) = u_i \quad$ were $\quad \xi_i = t \cdot (u_i - u_n)$ and differentiating Eq(20) and Eq(21) with respect to ξ yields

$$\frac{dc(\xi)}{d\xi} = \frac{\mathring{c}(s)}{t \cdot \mathring{c}(s)}. \tag{22}$$

The Hermite formula for the interpolating polynomial is given by

$$P(h) = \sum_{i=0}^{n} a_i (h - x_0)^i \tag{23}$$

where $\quad a_i = f[x_0, x_1, ..., x_i], \quad (h - x_0)^i = (h - x_0)(h - x_1)...(h - x_{i-1}), \quad (h - x_0)^0 = 1$ and the terms a_i can be obtained from the table below of divided differences

$$\begin{array}{cccccc} x_0 & f[x_0] & & & & \\ x_1 & f[x_1] & f[x_0, x_1] & & & \\ x_2 & f[x_2] & f[x_1, x_2] & f[x_0, x_1, x_2] & & \\ x_3 & f[x_3] & f[x_2, x_3] & f[x_1, x_2, x_3] & f[x_0, x_1, x_2, x_3] & \\ ... & ... & ... & ... & ... & ... \end{array} \tag{24}$$

where the entries of the tables are given by

$$x_i = \xi_i = t \cdot (u_i - u_n) \tag{25}$$

$$f[x_i] = u_i \tag{26}$$

$$f[x_i, ..., x_{i+k}] = \frac{t_i}{t \cdot t_i} \qquad if x_i = x_{i+k} \tag{27}$$

$$f[x_i, ..., x_{i+k}] = \frac{f[x_{i+1}, ..., x_{i+k}] - f[x_i, ..., x_{i+k-1}]}{x_{i+k} - x_i} \qquad otherwise. \tag{28}$$

In this way, a predictor of degree n takes the form

$$v = f[x_0] + f[x_0, x_1](h - x_0) + f[x_0, x_1, x_2](h - x_0)(h - x_1) + ... + f[x_0, x_1, ..., x_n](h - x_0)...(h - x_{n-1}). \tag{29}$$

If the PC method presents a high order predictor, the step length strategies can involve controlling the polynomial degree. The step-length control is essential in the strategies involving a high order predictor, but the control of the polynomial degree is not. Depending on the problem at hands, good results can be obtained using the step length adaptation by asymptotic expansion to high order predictors. Figures (1), (2), and (3) show various results obtained employing different predictors to trace a circle with a unity radius. Note that using a high order predictor we can trace a larger distance along the curve than the Euler predictor for identical entry parameters. This can be attribute to the fact that the error performed by the high order predictor, see Fig.(1), is smaller than the Euler predictor, hence the high order predictor allows a larger step length and can "walk" a larger distance than the first order predictor. The results obtained for each kind of predictor to solve the problem $F(x, y) = x^2 + y^2 = 0$ are presented in Tab.(1)

Table 1. Distance traversed along the curve defined implicitly by $F(x, y) = x^2 + y^2 = 0$ for each kind of predictor

Kind of predictor	First point (x,y)	Last point (x,y)	distance
Euler Predictor	(0,1)	(0.102310594,-0.994756691)	4.81483
2nd Order Predictor	(0,1)	(0.989908831,-0.141702288)	6.14100
3rd Order Predictor	(0,1)	(0.958164011,-0.286227576)	5.99289
4th Order Predictor	(0,1)	(-0.257910782,-0.966168737)	4.97324
5th Order Predictor	(0,1)	(0.060809271,-0.998153345)	4.77317

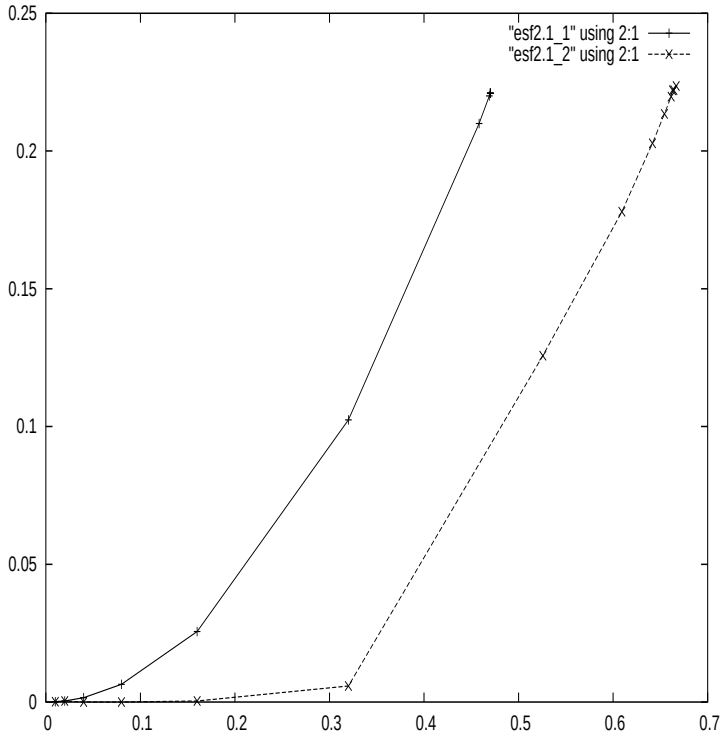

Figure 1: Performed error by the Euler and 3^{rd} order predictors. Note that the step length of the 3^{rt} order predictor is about 25% larger than the Euler predictor.

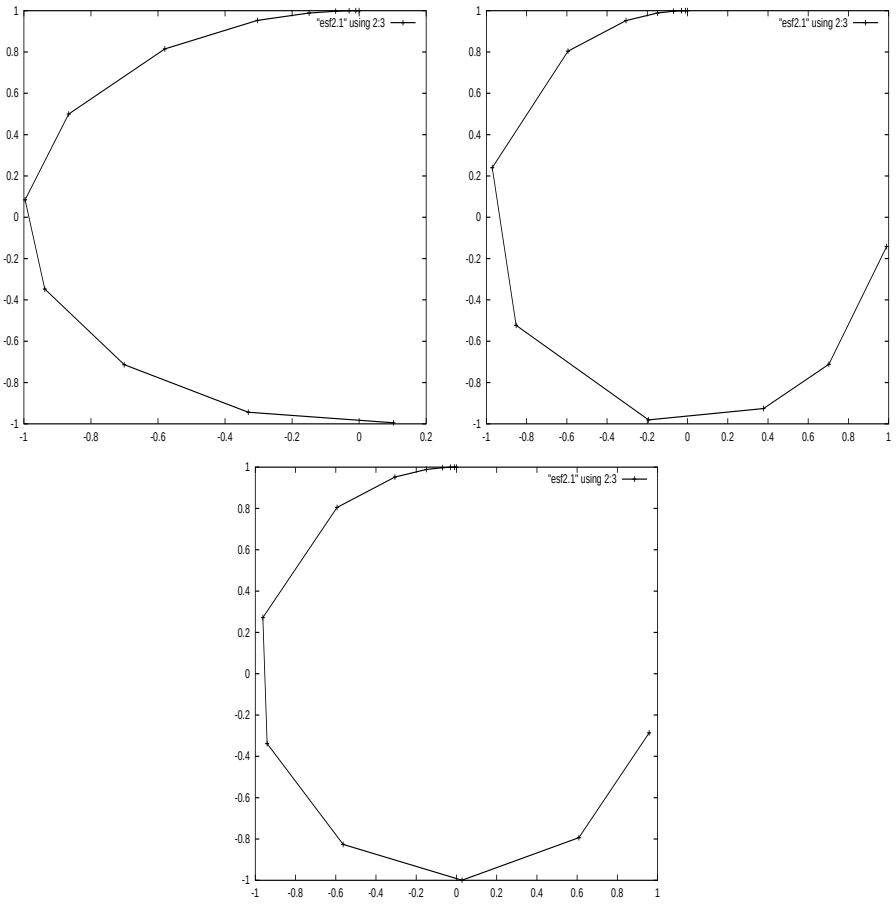

Figure 2: From left to right, curves obtained with 1^{st}, 2^{nd} and 3^{rd} orders predictors respectively.

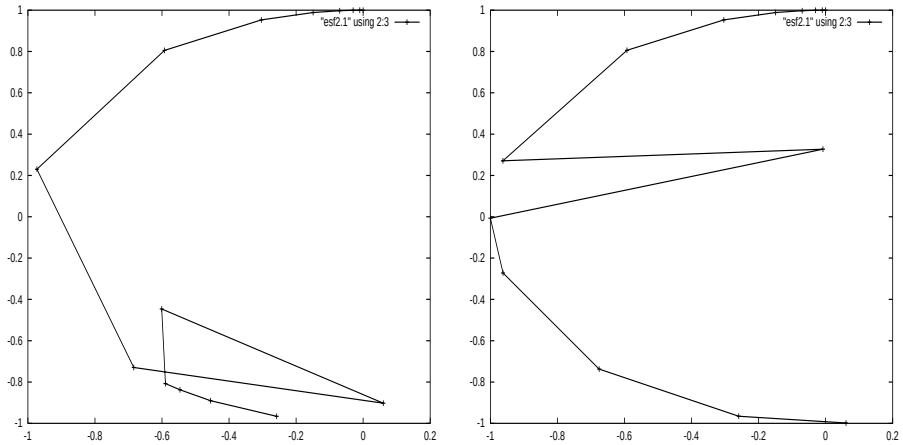

Figure 3: From left to right, curves obtained with 4^{st} and 5^{sd} orders predictors respectively.

As was expected, in Fig.(2) we can note that the second order predictor gets the best performance to trace the curve because its interpolating has the same degree of the problem. Introducing a high order predictor we get a traversing distance around 20% larger than a first order predictor. In the Fig.(3) we can note that predictors with degree of interpolating polynomial higher than degree three do not present a good performance. We remember that for all kind of predictors showed here we used the same problem parameters.

These kind of errors, showed in Fig.(3), observed when the degree of the polynomial exceeds the ideal degree of interpolation, become worse when we have problems like $F(x, y) = x^i + y^j = 0$ with i and j assuming elevated values. When this occurs it is better to take a predictor with lower degree of interpolation.

3. Conclusions

We studied Predictor-Corrector methods for the construction of the neutral stability curves based on the hydrodynamic stability of rotating disk flows to small disturbances, where the fluid viscosity varies along the axis of the rotating and presented the linear equations governing the evolution of spiral perturbations imposed to the steady flow. These equations reduce to those presented by Malik (1986) in the case of constant viscosity fluids. Comparison of our results for constant viscosity flows, concerning the coordinates of the minimum of the neutral curve for stationary disturbances with results existing in the literature indicate good agreement and provide validation of our numerical code.

The numerical method used to determine the neutral-curve is a numerical continuation method know as predictor-corrector method. This numerical method is an interesting way to solve the problem of finding the hydrodynamic stability curves of a boundary layer over a rotating disk because it requires a much smaller computational costs than other approaches.

The predictor-corrector method employed for the computation of the neutral curves is discussed. For the Euler predictor, strategies for step adaptation are discussed. The technique is then extended to include high order predictor methods. The numerical results, conducted on simple problems, show that second and third order predictors can lead to a 25 % increase of the marching step with respect to a Euler step, therefore reducing the computational cost associated to compute a fixed archlength of the stability curves. However, numerical experiments show that fourth and fifth order predictors are prone to instabilities, producing in some occasions very bad predictions.

4. Acknowledgments

We gratefully acknowledge support given by Fapesp (grants #00/03385-0 and 02/01207-3), and CNPq (grant 460473/2001-8).

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