University of Utrecht
Mathematics Department
Introduction to Complex Systems
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DEADLINE: 10 November!!!!

Project 1

PART I: *Diploid* Cellular Automata

1.1 Framework and definition of *diploid* CA

In elementary cellular automata (ECA) all cells are updated synchronously so that the state of each cell is updated according to the state of the cell itself and to that of the two neighboring cells. The set of these three cells will be called *neighborhood* of a given cell. More precisely, given a *local rule* f, we denote by $F : X_n \to X_n$ the map defined by letting

$$(F(x))_i = f(x_{i-1}, x_i, x_{i+1})$$

for any cell i. The *Elementary Cellular Automata* (ECA) associated with the local rule f is the collection of all the sequences of configurations $(x^t)_{t\in\mathbb{N}}$ obtained by applying the map F iteratively, namely, such that $x^t = F(x^{t-1})$. The particular sequence $(x^t)_{t\in\mathbb{N}}$ such that $x^0 = x \in X_n$ is called *trajectory* of the cellular automaton associated with the *initial condition* x.

Each of the possible 256 local rules f is identified by the integer number $W \in \{0, \dots, 255\}$ such that

$$W = f(0,0,0) \cdot 2^0 + f(0,0,1) \cdot 2^1 + f(0,1,0) \cdot 2^2 + f(0,1,1) \cdot 2^3 + \cdots + f(1,1,1) \cdot 2^7$$

that is to say the collection of the digits

$$f(1,1,1)f(1,1,0)f(1,0,1)f(1,0,0)f(0,1,1)f(0,1,0)f(0,0,1)f(0,0,0)$$

is the binary representation of the number W.

Some examples. The rule 0 is called the *null* rule and associates the state 0 to any configuration in the neighborhood. The rule 22 associates the state 0 to any configuration in the neighborhood but for the three local configurations in which one single 1 is present in the neighborhood (001, 010, and 100) to which it associates 1. The rule 150 associates the state 0 to any configuration in the neighborhood but for the four local configurations in which an odd number of 1's is present in the neighborhood (001, 010, 100, and 111) to which it associates 1. The rule 204 is called the *identity* and associates to any configuration in the neighborhood the state of the cell at the center (namely, the cell that one is going to update). The rule 224 associates the state 1 to any configuration in the neighborhood but for the local configuration 000 to which it associates 0. The rule 232 is called the *majority rule* and associates to any configuration in the neighborhood the majority state, namely 0 to 000, 001, 010, and 100 and 1 to the others. The rule 255 associates the state 1 to any configuration in the neighborhood. Finally, note that all the rules represented by an even number associated to the local configuration 000 the states 0.

In this context a Probabilistic Cellular Automata, called *probabilistic* or *stochastic* ECA, is a Markov chain $(\xi^t)_{t\in\mathbb{N}}$ on the configuration space X_n with transition matrix

$$p(x,y) = \prod_{i\in\mathbb{L}_n} p_i(y_i|x) \;\text{ with }\; p_i(y_i|x) = y_i\phi(x_{i-1},x_i,x_{i+1}) + (1-y_i)[1-\phi(x_{i-1},x_i,x_{i+1})] \tag{1}$$

where ϕ has to be interpreted as the probability to set the cell to 1 given the neighborhood $x_{i-1}x_ix_{i+1}$ and, similarly, $1-\phi$ the probability select 0. We denote by P_x the probability associated with the process started at $x \in X_n$. We shall denote by $\mu_t^x(y) = P_x(\xi^t = y)$ the probability that the chain started at x will be in the configuration y at time t. Abusing the notation, $\mu_t^x(Y) = P_x(\xi^t \in Y)$ will denote the probability that the chain started at x will be in the set of configurations $Y \subset X_n$ at time t.

An important class of stochastic ECA is made of those models obtained by randomly mixing two of the 256 elementary cellular automata. More precisely, given $\lambda \in (0,1)$ and picked two local rules $f_1 \neq f_2$, the stochastic ECA defined by

$$\phi = (1-\lambda)f_1 + \lambda f_2 \tag{2}$$

is called a *diploid* ECA. Note that in the limiting cases $\lambda = 0, 1$ or $f_1 = f_2$ a (deterministic) ECA is recovered.

It is important to note that the time evolution of the diploid ECA can be described as follows: at time t for each cell i one chooses either the rule f_1 with probability $1-\lambda$ or the rule f_2 with probability λ and performs the updating based on the neighborhood configuration at time $t-1$. Indeed, with this algorithm the probability to set the cell to 1 a time t is 0 if $f_1 = 0$ and $f_2 = 0$ (where the local rules are computed in the neighborhood configuration at time $t-1$), $1-\lambda$ if $f_1 = 1$ and $f_2 = 0$, λ if $f_1 = 0$ and $f_2 = 1$, 1 if $f_1 = 1$ and $f_2 = 1$, which is coherent with the definition (2).

1.2 The project

We start trying to simulate diploid ECA of the type:

$$\phi = (1-\lambda)f_1 + \lambda f_2$$

Here, we focus on models in which f_1 is the **null rule** (i.e. rule 0) and f_2 is any other rule in the following list:

$$\begin{aligned}\mathcal{F}_2 \;:=\{\; & 18, 22, 26, 28, 30, 50, 54, 58, 60, 62, 78, 90, 94, 110,\\ & 122, 126, 146, 150, 154, 156, 158, 178, 182, 186, 188,\\ & 190, 202, 206, 218, 234, 238, 250, 254\}\end{aligned}$$

1.3 Simulation Studies

- choose two rules f_2 from $\mathcal{F}_2$ and repeat the simulation study explained below for each of them.
- The number of cells is set to $n = 10^4$.
- As initial condition a configuration in which cells are populated with zeros or ones with equal probability
- The diploid is let evolve for the time $T = 5 \cdot 10^3$.
- In the final configuration the fraction of cells set to 1, called *density*, is computed.
- **QUESTION:** is this number an estimate of the averaged density along an infinite long run of the diploid in the infinite volume limit $n \to \infty$?
- Try to estimate the curve of the density in function of the parameter λ, trying to be more accurate around the points where you notice a change of behavior. For instance increase λ by step of 0.05 and then by steps of 0.01 near the sudden change of density.
- **QUESTION:** critically study the behaviour of the density.
- **QUESTION:** is there any *abrupt* change of behavior?
- **QUESTION:** do you think, from the indication of your plot that the system has a phase transition in λ (e.g. first or second order)?
- **QUESTION:** if yes, is there any indication of the presence of any *critical* λ?
- **QUESTION:** if you look carefully at the list $\mathcal{F}$, you might notice that it contains only **even** rules. What do you think would happen heuristically in case of an **odd** f_2 rule?

1.4 Theoretical study (f_2 is rule 204)

Consider the diploid with f_2 being the rule 255. Each cell is updated independently on the others and also on the past. Hence, the evolution of a cell amounts to be a sequence of Bernoulli variables with parameter λ. Thus, the invariant measure of the chain is product and for each cell it is equal to $\pi_i(0) = 1 - \lambda$ and $\pi_i(1) = \lambda$.

Consider now the diploid with f_2 being the identity (rule 204). For this model the configuration 0 is a *fixed point* (also called *stationary state*), in the sense that $p(0, x) = 0$ for any $x \in X_n$. The cells are updated independently one from each other but not on the past. The chain can be described as a collection of n single cell Markov chains evolving with the following transition matrix

$$p_i(0,0) = 1,\ p_i(0,1) = 0,\ p_i(1,0) = 1 - \lambda,\ p_i(1,1) = \lambda.$$

for any cell i. The stationary measure is product and the single cell stationary measure is concentrated on 0, namely, $\pi_i(0) = 1$ and $\pi_i(1) = 0$.

- **QUESTION:** For a single cell started at 1, calculate the probability that its state is 0 at time t . For the all chain show that $\mu_t^1(0) = (1-\lambda^t)^n$.

- **QUESTION:** Try to obtain the probability of reaching at time t the configuration with all zeros if started in the configuration with all ones ($\mu_t^1(0)$) of the previous point by using the *multinomial theorem*. In fact, one can rewrite the probability as sum over all the ways in which 1's are removed once we know that $\sum_{k=1}^t s_k = n, where\ s_k$ is the number of ones removed at time k.

- **QUESTION:** Now, suppose to compute this probability on a time scale *diverging logarithmically* with n, namely, take $t = \alpha \log n$, for some $\alpha > 0$. Show that, for any $\lambda \in (0,1)$, if $\alpha < -1/\log\lambda$ then $\mu_t^1(0)$ tends to 0 in the infinite volume limit, whereas it tends to 1 if $\alpha > -1/\log\lambda$.

- For this simple model you have showed that: fixed λ on a time scale smaller than $(-1/\log\lambda)\log n$ the chain started at 1 stays in 1. On a time scale larger than $(-1/\log\lambda)\log n$ it converges to the configuration 0. Note that in this model this happens for any value of λ. This suggests that in the other models, something similar is observed but only for λ small enough.

- **QUESTION:** Despite rule 204 is not in the list $\mathcal{F}_2$, does this heuristics shed some light on what it is going on? Come back to the simulation studies you have performed. The previous observation is consistent with what you have observed?

PART II

You have to choose and solve **only one** of the following two projects: Asynchronous CA or Explicit solution of a CA described in the following sections.

2.1 Asynchronous CA

This part of the project would like to investigate the robustness of CA to *asynchronous updating* via the so called $\alpha - asynchronous$ dynamics. In this dynamics we apply a local rule f with a probability α (the *synchrony rate*) or we keep the same state with a probability $1-\alpha$, independently for each cell. Note that by taking $\alpha = 1$, we recover the synchronous case (i.e. parallel updating) and as α is decreased, the update rule becomes more *asynchronous*.

In the context of ECA, an $\alpha - asynchronous$ dynamics can be written as a diploid ECA as:

$$\phi = \alpha f_1 + (1-\alpha) f_2 \tag{3}$$

with f_2 being the rule 204 (identity rule!).

- **QUESTION:** consider now the thee $\alpha - asynchronous$ dynamics, when $f_1 \in \{6, 50, 178\}$. With simulation parameters similar to the Simulation Studies, draw (part of) the space time diagrams with $\alpha = 0.25$, $\alpha = 0.50$ and $\alpha = 0.75$, starting from a random initial configuration. What do you observe? How does qualitatively change the approach to the equilibrium as α increases?

- **QUESTION:** try to estimate the curve of the equilibrium density in function of the parameter α, for $f_1 \in \{6, 50\}$. Are there any critical value of α? Try to give an heuristic explanation of what you observe, underline the differences between the two ECA.

2.2 Explicit solution of a CA

This part of the project would like to investigate the solution of a ECA in terms of its initial condition. In particular we will focus on ECA 172.

- **QUESTION:** Starting from a configuration in which cells are populated with zeros or ones with equal probability, draw part of a time-space diagram and try to assess to which Wolfram class the ECA 172 belongs.

- **QUESTION:** Try to inspect the fate of clusters of two or more zeroes and of isolated zeros.

- We consider now an infinite ECA 172, i.e., defined in all $\mathbb{Z}$. We denote with x the initial condition $x \in \{0,1\}^{\mathbb{Z}}$ and we study $[F^n(x)]_0$, i.e., the value at the origin after n iteration of the CA. One can indeed proves that:

$$[F^n(x)]_0 = \bar{x}_{-2}\bar{x}_{-1}x_0 + \left(\prod_{i=-2}^{n-3}(1-\bar{x}_i\bar{x}_{i+1})\right)(\bar{x}_{n-2}x_{n-1} - x_{n-2}x_n) \tag{4}$$

where we used the notation $\bar{x}_i := 1 - x_i$, for $i \in \mathbb{Z}$.

- **QUESTION:** Derive the solution for $[F^n(x)]_j$ for any $j \in \mathbb{Z}$.
- The initial configuration is now a random configuration, sampled from a Bernoulli distribution with parameter p. More precisely, let $x_j = X_j$ for $j \in \mathbb{Z}$, where X_j are independent and identically distributed random variables such that $\mathbb{P}(X_j = 1) = q$, where $q \in (0, 1)$.
- It can be proven (**BONUS**: prove it, but it is not trivial!!!) that the expectation with respect to the Bernoulli product measure is:

$$\mathbb{E}\left(\prod_{i=1}^{n-1}(1 - \bar{X}_i\bar{X}_{i+1})\right) = \frac{q}{\lambda_2 - \lambda_1}(\gamma_1\lambda_1^{n-2} + \gamma_2\lambda_2^{n-2}) \tag{5}$$

with

$$\lambda_{1,2} := \frac{1}{2}q \pm \frac{1}{2}\sqrt{q(4-3q)},$$

$$\gamma_{1,2} := \left(\frac{q}{2} - 1\right)\sqrt{q(4-3q)} \pm \left(\frac{q^2}{2} - 1\right)$$

- **QUESTION:** By using (5), calculate $\mathbb{E}([F^n(X)]_0)$ and $\mathbb{E}([F^n(X)]_j)$. Are they equal?
- **QUESTION:** Calculate the limit:

$$\lim_{n\to\infty} \mathbb{E}([F^n(X)]_j)$$

and try to relate it to the asymptotic density of ones of ECA 172. If you look back at the space-time diagram you have drawn, what you can say about the asymptotic configuration?

- **CURIOSITY:** In case $q = \frac{1}{2}$,

$$\mathbb{E}([F^n(X)]_j) = \frac{1}{8} + \frac{\mathcal{F}_{n+3}}{2^{n+2}}$$

where $\mathcal{F}_n$ is the n-th Fibonacci number.