The Great Hunt for Small Subsystem Codes

Gregory Crosswhite and Dave Bacon

Stabilizer Codes

$$\mathcal{H} = \boxed{\mathcal{S} \otimes \mathcal{L}}$$

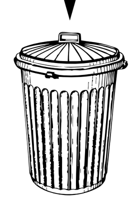
Stabilizers

$S_1, S_2, S_3, S_4, \ldots$

Logical Qubits

$L_1, L_2, L_3, L_4, \ldots$

In a stabilizer code, the Hilbert space is split into a tensor product of the subspace spanned by the stabilizers and the subspace spanned by the gauge qubits.



Any stabilizer code can be made into a subsystem code simply by taking some of the logical qubits and declaring that you no longer care about them; they become "gauge qubits".

Subsystem Codes

$$\mathcal{H} = \boxed{\mathcal{S} \otimes \mathcal{G} \otimes \mathcal{L}}$$

Stabilizers

$S_1, S_2, S_3, S_4, \ldots$

Gauge Qubits

$G_1, G_2, G_3, G_4, \ldots$

Logical Qubits

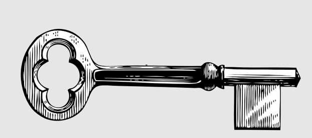
$L_1, L_2, L_3, L_4, \ldots$

In a subsystem code, the Hilbert space is split into a tensor product of the stabilizers, gauge qubits (whose state we do not care about), and logical qubits which encode the desired quantum information.

Why throw away some of the qubits? For many reasons, including some or all of the following:

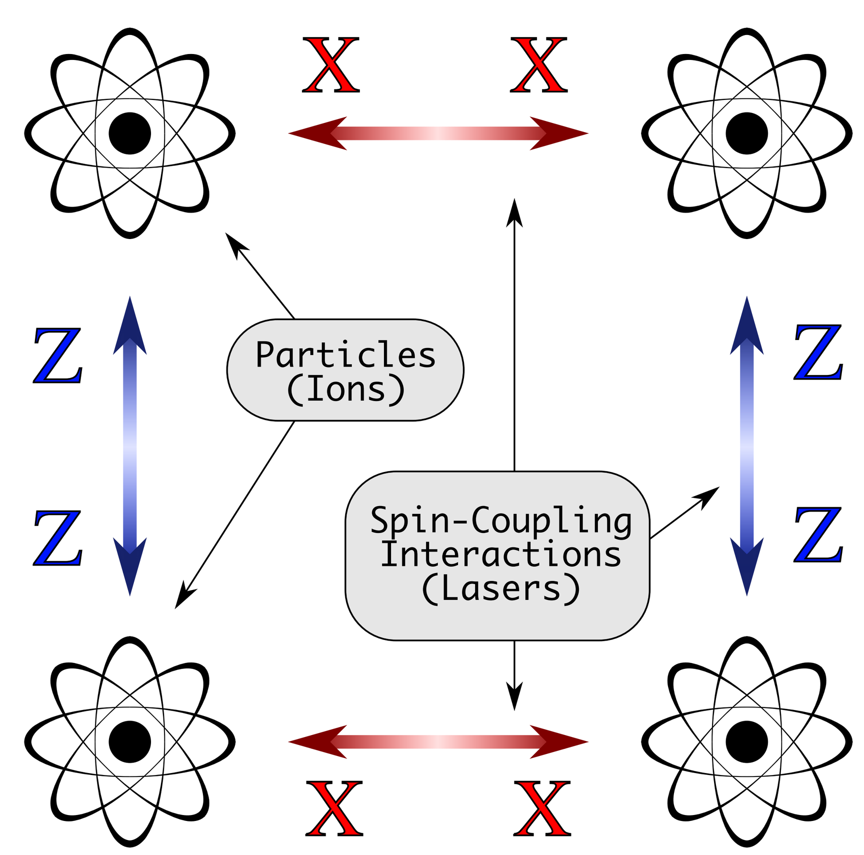
- Having qubits you don't care about makes the error recovery process simpler.
- You can choose to focus on only the qubits which have large distance.
- Because the gauge qubits are not self-correcting, but the logical qubits are.

KEY REASONS TO CARE:



- Any set of operators generates a subsystem code!
- Conversely, the generators of a code can themselves all be written in terms of products of stabilizers and *gauge* qubits.
- Therefore, any Hamiltonian can be rewritten in terms of stabilizers and gauge qubits.
- Ergo, the logical qubits generated by a Hamiltonian (if any) are encoded in the ground state!

Ising-Interaction Graphs



Ising interaction graphs are graphs with quantum particles at the vertices and 2-body spin-coupling interactions (XX, XY, ZZ, ZX, etc.) on the edges. These are models of very realistic physical systems that can be built with increasing skill by experimentalists, such as by using ion traps for the particles at the vertices and lasers to create the interactions on the edges.

As discussed earlier, any set of interactions generates a subsystem code. Below we compute the code for the 4-qubit system on the left.

Subsystem Example

$$H = \underbrace{\begin{array}{cc} X & X \\ I & I \end{array} + \begin{array}{cc} I & I \\ X & X \end{array} + \begin{array}{cc} Z & I \\ Z & I \end{array} + \begin{array}{cc} I & Z \\ I & Z \end{array}}$$

The stabilizers and gauge qubits are generated from the terms in the hamiltonian.

By contrast, the logical qubits are born out of the "leftover" Hilbert space outside that spanned by the stabilizers and gauge qubits.

Stabilizers

$\begin{array}{cc} X & X \\ X & X \end{array}$ and $\begin{array}{cc} Z & Z \\ Z & Z \end{array}$

S_1 S_2

Gauge Qubit

$\begin{array}{cc} I & I \\ X & X \end{array}$ $\begin{array}{cc} Z & I \\ Y & X \end{array}$ $\begin{array}{cc} Z & I \\ Z & I \end{array}$

$\overline{X}_G$ $\overline{Y}_G$ $\overline{Z}_G$

Logical Qubit

$\begin{array}{cc} I & I \\ X & X \end{array}$ $\begin{array}{cc} Z & I \\ Y & X \end{array}$ $\begin{array}{cc} Z & I \\ Z & I \end{array}$

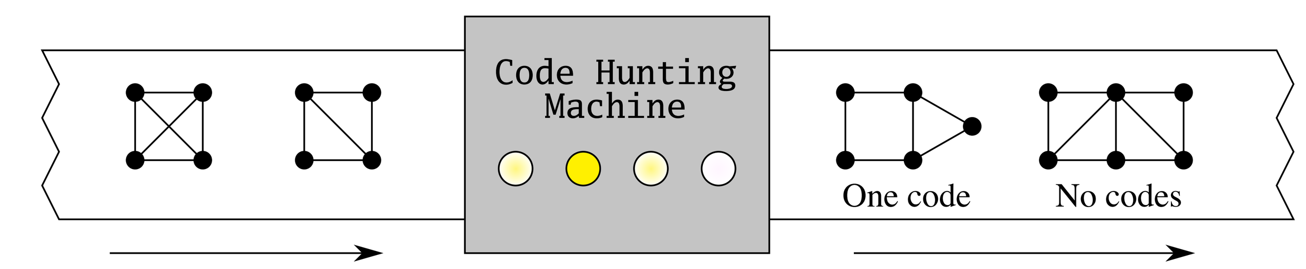
$\overline{X}_L$ $\overline{Y}_L$ $\overline{Z}_L$

As promised, the Hamiltonian can be rewritten in terms of gauge qubits operators and stabilizers. Since no logical operators appear in the hamiltonian, this means that the logical qubits is encoded in each eigenstate of the Hamiltonian -- including in particular the ground state!

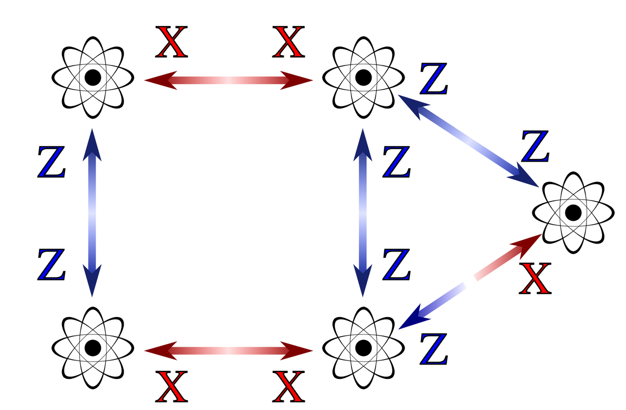
$$H = \overline{X}_G \cdot (1 + S_1) + \overline{Z}_G \cdot (1 + S_2)$$

Automated Searching

We have written a code which exhaustively searches through all of the physical systems with interactions given by a particular graph, and reports all of the "interesting" subsystem codes that it sees.



The 5-Qubit Error-Detecting Code



Stabilizers

$\begin{array}{ccc} Z & Z & \\ & & I \\ Z & Z & \end{array}$ $\begin{array}{ccc} Y & Y & \\ & & Y \\ Y & Y & \end{array}$

Gauge Qubit 1

$\begin{array}{ccc} I & I & \\ & & I \\ X & X & \end{array}$ $\begin{array}{ccc} Z & I & \\ & & I \\ Z & I & \end{array}$

$\overline{X}_{G_1}$ $\overline{Z}_{G_1}$

Gauge Qubit 2

$\begin{array}{ccc} X & X & \\ & & I \\ X & X & \end{array}$ $\begin{array}{ccc} Z & I & \\ & & X \\ Z & Z & \end{array}$

$\overline{X}_{G_2}$ $\overline{Z}_{G_2}$

Logical Qubit

$\begin{array}{ccc} X & I & \\ & & I \\ X & I & \end{array}$ $\begin{array}{ccc} Z & Z & \\ & & I \\ I & I & \end{array}$

$\overline{X}_L$ $\overline{Z}_L$

By exhaustive search, we have shown that this code is the *only* 5-qubit Ising graph code that has error-detecting properties!

We have found many others, including a large number of error-correcting codes in systems of 9 through 13 physical qubits.

Future Work

- More analysis to better understand the found codes.
- Improved filtering out of equivalent codes.
- Continuing to sweep through larger graphs.
- Expanding the search to systems with 3-body interactions that are also experimentally buildable.