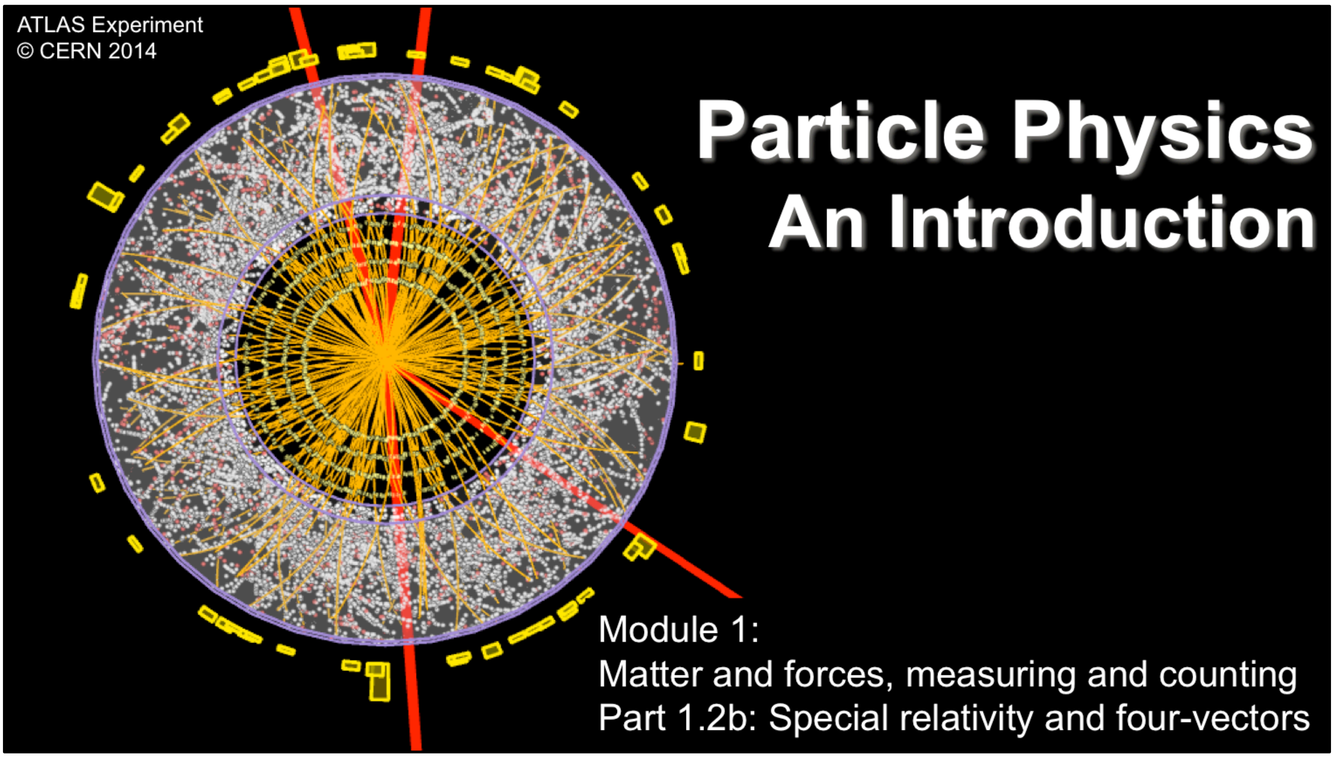


For those who have forgotten about special relativity and the notation of four-vectors, we will quickly remind you about these essential tools.

A ray of light connects two events, $x_1^\mu = (ct_1, \vec{x}_1)$ et $x_2^\mu = (ct_2, \vec{x}_2)$:

$$\left((x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2\right)^{\frac{1}{2}} = c\,(t_1 - t_2)$$

Consequence: the length of a four-vector is independent of the inertial frame:

$$s = c^2t^2 - x^2 - y^2 - z^2 = \text{invariant}$$

Lorentz transformations = rotations and translations in space-time which leave s invariant.

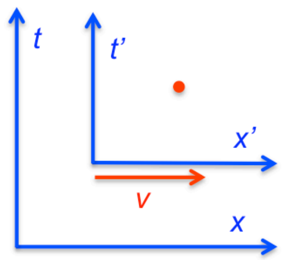
- This course deals with phenomena at high energy, i.e. at velocities which require a relativistic treatment. The fundamental principles of special relativity (based on experimental findings) are:
  - The speed of light *c* is the same constant in every inertial frame. This means that the Galilean way to add velocities is not valid at high speeds.
  - This way the speed of light is a maximum velocity, which cannot be exceeded in any reference frame.
- Motion is described in a four-dimensional vector space, called space-time. An event in space-time, localized by Cartesian coordinates (ct,x) is described by a four-vector x^μ.
- The metric of space-time is defined by the speed of light *c*, an invariant constant in every inertial frame.

- Suppose that a light ray connects two events (ct_1,x_1) and (ct_2,x_2).
- The distance between these two points in space is always proportional to the propagation time $(t_1 - t_2)$ of the light.
- Since the proportionality constant, the speed of light, is the same in every inertial frame, it follows that the norm *s* of a four-vector is also constant.
- Lorentz transformations describe the rotations and translations of space-time which are compatible with these principles.

Example:

- Frame $(ct', \vec{x}')$ moves with $\vec{v} = \beta c \hat{x}$ in direction $\hat{x}$ with respect to the 'laboratory" $(ct, \vec{x})$
- Relativistic factor $\gamma = 1/\sqrt{1-\beta^2}$
- Coordinates in the moving frame:

$$
\begin{aligned}
t' &= \gamma\left(t - \frac{\beta}{c}x\right) \\
x' &= \gamma\,(x - \beta c t) \\
y' &= y \\
z' &= z
\end{aligned}
$$



Let us specify Lorentz transformations by a simple example:

- Let *(ct,x)* and *(ct',x')* be two reference frames, the first one at rest in the laboratory, the second one moving relative to the first one at constant velocity *v* in the direction of the *x* axis, which is parallel in the two frames.
- Which are the coordinates of one point *(ct',x')* in the moving frame expressed in the coordinates of the laboratory frame? The answer is given by the Lorentz transformation, which relates *t'* and *x'* to *t* and *x* by the relative velocity $\beta = v/c$ and the relativistic factor γ.
- The transformation corresponds to a rotation in space-time which leaves the norm of four-vectors like x_μ invariant.
- The transformation involves only the time coordinate *t* and the spatial coordinate *x* along the direction of relative motion. The coordinates orthogonal to the relative motion, *y* and *z,* are left untouched.
- In the nonrelativistic limit, $v \ll c$, i.e. $\beta \ll 1$ and $\gamma \rightarrow 1$, , the Lorentz transformations reproduce the Galilei transformations, where $t=t'$ and $x'=x-vt$.

- Contravariant four-vectors transform like space-time x^μ:

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$$

- Example: energy-momentum p_μ:

$$p^\mu = (p^0, p^1, p^2, p^3) = (\frac{E}{c}, p_x, p_y, p_z)$$

- Contra- and covariant four-vectors related by metric tensor:

$$x_\mu = (x_0, x_1, x_2, x_3) = (ct, -x, -y, -z) \quad ; \quad p_\mu = (p_0, p_1, p_2, p_3) = (\frac{E}{c}, -p_x, -p_y, -p_z)$$

$$x_\mu = g_{\mu\nu} x^\nu \quad ; \quad g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- Four-vector notation unites time and space coordinates in a single vector.
- Four-vectors transforming like the four-vector x^μ of space-time under Lorentz transformations are called contravariant.
- An important example is the contravariant energy-momentum vector p^μ.
- The norm of a four-vector is defined via the scalar product between a contravariant four-vector and its covariant form. The two are related by the metric tensor $g_{\mu\upsilon}$ as shown.
- All scalar products between four-vectors are invariant under Lorentz transformation.

- Scalar product between four-vectors:

$$
\begin{aligned}
x_\mu x^\mu &= c^2t^2 - x^2 - y^2 - z^2 = c^2t^2 - \vec{x}^2 \\
p_\mu p^\mu &= \frac{E^2}{c^2} - p_x^2 - p_y^2 - p_z^2 = E^2/c^2 - \vec{p}^2 = m^2c^2 \\
p_\mu x^\mu &= Et - p_x x - p_y y - p_z z = Et - \vec{p}\vec{x}
\end{aligned}
$$

- Scalars under Lorentz transformations, invariant when changing between inertial frames

- System of "natural" units:

$$
\begin{aligned}
x^\mu = (x^0, x^1, x^2, x^3) = (t, x, y, z) \quad &; \quad p^\mu = (p^0, p^1, p^2, p3) = (E, p_x, p_y, p_z) \\
x_\mu x^\mu = t^2 - x^2 - y^2 - z^2 = t^2 - \vec{x}^2 \quad &; \quad p_\mu p^\mu = E^2 - p_x^2 - p_y^2 - p_z^2 = E^2 - \vec{p}^2 = m^2
\end{aligned}
$$

- More generally, the scalar product is thus defined as the product between a co- and a contravariant four-vector. When the same Greek index shows up in the two, implicit summation over this index from 0 to 3 is assumed.
- The first example repeats the norm (squared) of the space-time four-vector. The second example shows the norm of the energy-momentum four-vector, which is the square of the invariant mass. The third example is the scalar product between the space-time and the energy-momentum four-vector. This product shows up in the wave function of particles.
- The scalar products between four-vectors are all indeed scalars under Lorentz transformation, invariant when one changes from one inertial frame to another.
- The notation simplifies tremendously when one adopts the system of natural units, since the ubiquitous speed of light disappears.

- Four-vector of electromagnetic current density:

$$j^{\mu} = (\rho, \vec{j})$$

  - ρ: charge density
  - $\vec{j}$: current density

- Four-vector of electromagnetic potential:

$$A^{\mu} = (V, \vec{A})$$

  - V: scalar electric potential
  - $\vec{A}$: magnetic vector potential

Two other important examples of four-vectors are:

- The four-vector of the electromagnetic current density j_{μ}, which has the charge density ρ as the time-like component, and the vector of current density j as the space-like component.
- The four-vector of the electromagnetic potential A_{μ}, which has the electric potential V as the time-like component and the magnetic vector potential A as the space-like component.

As an example we calculate here the total available energy in the collision of a 200 GeV electron with a proton at rest in the laboratory frame. And we confront the result with a collision in the center-of-mass frame, where a 200 GV electron collides with a 200 GeV proton.