

Relativistic Quantum Field Theory Revision Notes

The purpose of this set of notes is to remind you of things that you were exposed to last year in *Quantum Theory*. There are also one or two new topics, such as the covariant formulation of Maxwell's equations, which you are advised to work through.

A Review of Special Relativity

Four-Vector Notation

The coordinates of an object or ‘event’ in four-dimensional space-time, Minkowski space, form a *contravariant* four-vector whose components have ‘upper’ indices

$$x^\mu \equiv (x^0, x^1, x^2, x^3) \equiv (ct, \underline{x});$$

similarly, we define a *covariant* four-vector whose components have ‘lower’ indices

$$x_\mu \equiv (x_0, x_1, x_2, x_3) \equiv (ct, -\underline{x}).$$

A general four-vector a^μ is defined in the same way,

$$\begin{aligned} a^\mu &\equiv (a^0, a^1, a^2, a^3) \equiv (a^0, \underline{a}) \\ a_\mu &\equiv (a_0, a_1, a_2, a_3) \equiv (a^0, -\underline{a}), \end{aligned}$$

so that $a^0 = a_0$ and $a^i = -a_i$, $i = 1, 2, 3$. Upper and lower indices are related by the *metric tensor* $g^{\mu\nu}$:

$$a^\mu = g^{\mu\nu} a_\nu \quad a_\mu = g_{\mu\nu} a^\nu$$

where

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad g_\mu^\nu = g^\mu_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The *scalar product* in Minkowski space is defined, for general 4-vectors a^μ and b^μ by

$$\begin{aligned} a \cdot b &\equiv a^\mu b_\mu = a_\mu b^\mu = a_\mu b_\nu g^{\mu\nu} = a^\mu b^\nu g_{\mu\nu} \\ &= a^0 b^0 - \underline{a} \cdot \underline{b}, \end{aligned}$$

where \underline{a} and \underline{b} are ordinary 3-vectors.

Lorentz transformations

Lorentz transformations are linear transformations on 4-vectors which leave invariant this scalar product

$$a'^\mu = \Lambda^\mu_\nu a^\nu, \quad \text{e.g.,} \quad x'^\mu = \Lambda^\mu_\nu x^\nu.$$

Strictly, these are *homogeneous* Lorentz transformations — translations are not included. They form a group known as the *homogeneous Lorentz group*.

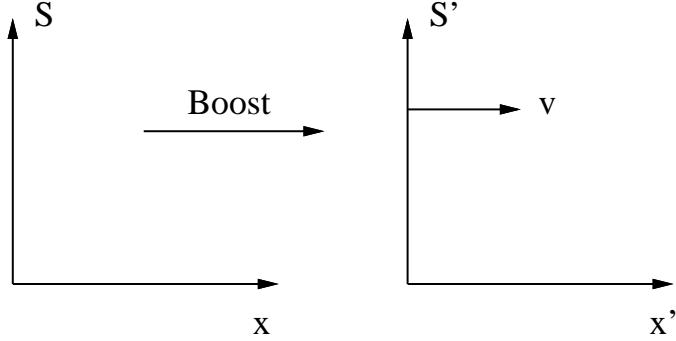
The ‘standard’ Lorentz transformation is a ‘boost’ along the x direction

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \cosh \omega & -\sinh \omega & 0 & 0 \\ -\sinh \omega & \cosh \omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where

$$\begin{aligned} \tanh \omega &\equiv \beta \equiv \frac{v}{c} \\ \cosh \omega &\equiv \gamma = (1 - \beta^2)^{-1/2} = \left(1 - \left(\frac{v}{c}\right)^2\right)^{-1/2} \\ \sinh \omega &= \gamma \beta. \end{aligned}$$

Hence $ct' = \gamma(ct - (v/c)x)$ and $x' = \gamma(x - vt)$ as usual, relating the time and space co-ordinates of a given event in two inertial frames in relative motion: Note also that the



group of homogeneous Lorentz transformations includes ordinary 3-dimensional rotations. In fact, such rotations form a subgroup known as the *rotation group*. A familiar example is the rotation of Cartesian co-ordinates about the z -axis, corresponding to the Lorentz transformation

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \phi & \sin \phi & 0 \\ 0 & -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The 3×3 submatrix relates the co-ordinates of a point P in one basis to the co-ordinates of the same point in a rotated basis.

Invariance of the scalar product means that

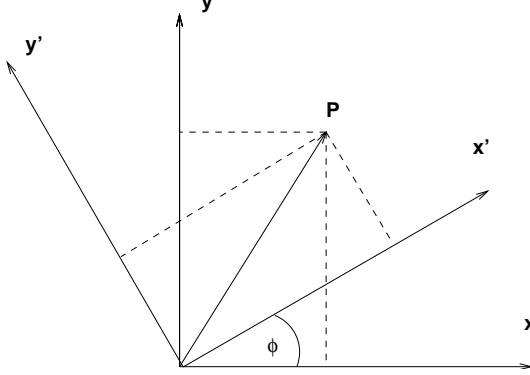
$$a'^\mu b'^\nu g_{\mu\nu} = \Lambda^\mu{}_\rho a^\rho \Lambda^\nu{}_\sigma b^\sigma g_{\mu\nu} = a^\rho b^\sigma g_{\rho\sigma},$$

for arbitrary 4-vectors a^μ and b^μ , which implies that

$$\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma g_{\mu\nu} = (\Lambda^T)^\mu{}_\rho g_{\mu\nu} \Lambda^\nu{}_\sigma = g_{\rho\sigma}. \quad (1)$$

In matrix notation this reads $\Lambda^T g \Lambda = g$ so that taking the determinant yields

$$\det(\Lambda^T g \Lambda) = \det \Lambda^T \det g \det \Lambda = \det g \Rightarrow (\det \Lambda)^2 = 1,$$



since the determinant is real, we conclude that

$$\det \Lambda = \pm 1 .$$

Transformations with $\det \Lambda = +1$ are called *proper Lorentz transformations*; those with $\det \Lambda = -1$ are called *improper Lorentz transformations*.

If we set the free indices $\rho = \sigma = 0$ in eqn. 1, we find that

$$\Lambda^{\mu}_0 \Lambda^{\nu}_0 g_{\mu\nu} = g_{00} = 1,$$

which we can rewrite as

$$(\Lambda^0_0)^2 = 1 + \sum_{i=1}^3 (\Lambda^i_0)^2 \geq 1.$$

We conclude that

$$\Lambda^0_0 \geq 1 \quad \text{or} \quad \Lambda^0_0 \leq -1 .$$

Transformations with $\Lambda^0_0 \geq +1$ are called *orthochronous Lorentz transformations*.

We can thus divide homogeneous Lorentz transformations into 4 classes:

	$\det \Lambda = +1$	$\det \Lambda = -1$
$\Lambda^0_0 \geq +1$	I	II
$\Lambda^0_0 \leq -1$	III	IV

Transformations in class I form a subgroup known as the *proper orthochronous Lorentz group*. The transformations in sets II, III and IV can all be obtained by combining a proper orthochronous transformation with one of the following:

Spatial Inversion: The parity operation $x^0 \rightarrow x^0; \underline{x} \rightarrow -\underline{x}$ for which

$$\Lambda^{\mu}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$

Time Reversal: $x^0 \rightarrow -x^0; \underline{x} \rightarrow \underline{x}$ for which

$$\Lambda^{\mu}_{\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

Space-Time Inversion: $x^0 \rightarrow -x^0$; $\underline{x} \rightarrow -\underline{x}$ for which

$$\Lambda^\mu{}_\nu = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The interval $s^2 = (x-y)^\mu(x-y)_\mu$ between two events with space-time co-ordinates x^μ and y^μ is *invariant* under Lorentz transformations.

Classification of four-vectors a^μ

$$\begin{aligned} a^2 > 0 && \text{time-like} \\ a^2 = 0 && \text{light-like} \\ a^2 < 0 && \text{space-like} \end{aligned}$$

Differential operators a^μ

$$\begin{aligned} \partial_\mu &\equiv \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c} \frac{\partial}{\partial t}, \nabla \right) \\ \partial^\mu &\equiv \frac{\partial}{\partial x_\mu} = \left(\frac{1}{c} \frac{\partial}{\partial t}, -\nabla \right) \\ \text{d'Alembertian: } \partial_\mu \partial^\mu &= \partial^\mu \partial_\mu = \partial^2 = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \quad (= \square^2) \end{aligned}$$

(N.B., sometimes \square^2 is called \square !!)

Momentum and energy

The conserved 4-momentum is denoted by:

$$\begin{aligned} p^\mu &\equiv \left(\frac{E}{c}, \underline{p} \right) \\ p^2 &= \frac{E^2}{c^2} - \underline{p} \cdot \underline{p} = m^2 c^2 \quad \text{for a free particle} \\ \text{or } E^2 &= |\underline{p}|^2 c^2 + m^2 c^4, \quad \text{the mass-shell condition.} \end{aligned}$$

where m is the (rest) mass. Particles for which $p^\mu p_\mu = m^2 c^2$ are said to be *on their mass shell*.

Electromagnetic Theory

Maxwell's Equations

Maxwell's first equation in free space in SI units is

$$\nabla \cdot \underline{E} = \frac{\rho}{\epsilon_0},$$

where \underline{E} is the electric field, ρ is the charge density, and ϵ_0 is the permittivity of free space. For convenience, particle physicists generally use a system of units where $\epsilon_0 = 1$ (see Appendix C of *Gauge Theories in Particle Physics*, I. J. R. Aitchison & A. J. G. Hey (Adam Hilger) for details). This system is called the *rationalised cgs* or *Heaviside-Lorentz* system. It is used in all(?) books on relativistic QM, so we shall adopt it here. Maxwell's equations (in a different order!) can be written:

$$\begin{aligned} \text{ME1} \quad \underline{B} &= \nabla \times \underline{A} \\ \text{ME2} \quad \underline{E} &= -\frac{1}{c} \frac{\partial \underline{A}}{\partial t} - \nabla \Phi \\ \text{ME3} \quad \nabla \times \underline{B} &= \underline{j} + \frac{1}{c} \frac{\partial \underline{E}}{\partial t} \\ \text{ME4} \quad \nabla \cdot \underline{E} &= \rho \end{aligned}$$

Note that we have written the first two equations in terms of the *magnetic vector potential* \underline{A} and the potential Φ ; the usual forms are obtained by taking $\nabla \cdot$ (ME1) and $\nabla \times$ (ME2).

Equations ME3 & ME4 lead to the continuity equation

$$\nabla \cdot \underline{j} + \frac{1}{c} \frac{\partial \rho}{\partial t} = 0.$$

If we define the electromagnetic current four-vector $j_{\text{em}}^\mu \equiv (\rho, \underline{j})$ then

$$\partial_\mu j_{\text{em}}^\mu = 0 \quad \text{expresses charge conservation.}$$

The Four-Vector Potential

The electromagnetic four-vector potential is defined by

$$A^\mu \equiv (\Phi, \underline{A})$$

Substituting ME1 & ME2 into ME3 yields:

$$\begin{aligned} \nabla \times (\nabla \times \underline{A}) &\equiv \nabla (\nabla \cdot \underline{A}) - \nabla^2 \underline{A} = \underline{j} - \frac{1}{c^2} \frac{\partial^2 \underline{A}}{\partial t^2} - \frac{1}{c} \nabla \left(\frac{\partial \Phi}{\partial t} \right) \\ \text{or} \quad \square^2 \underline{A} + \nabla \left[(\nabla \cdot \underline{A}) + \frac{1}{c} \frac{\partial \Phi}{\partial t} \right] &= \underline{j}. \end{aligned}$$

Similarly ME4 yields:

$$-\frac{1}{c} \frac{\partial}{\partial t} (\nabla \cdot \underline{A}) - \nabla^2 \Phi = \rho.$$

We can combine both equations into a covariant form

$$\boxed{\square^2 A^\mu - \partial^\mu \partial_\nu A^\nu = j_{\text{em}}^\mu}. \quad (2)$$

Gauge Transformations

We are free to define new potentials \tilde{A} and $\tilde{\Phi}$ via gauge transformations

$$\tilde{A} = \underline{A} - \nabla\chi \quad \text{and} \quad \tilde{\Phi} = \Phi + \frac{1}{c} \frac{\partial\chi}{\partial t}$$

or, in manifestly covariant notation,

$$\tilde{A}^\mu = A^\mu + \partial^\mu\chi$$

without changing the physical fields \underline{E} and \underline{B} . In particular, therefore, we can arrange that the potentials satisfy the *Lorentz condition*

$$\begin{aligned} \underline{\nabla} \cdot \underline{A} + \frac{1}{c} \frac{\partial\phi}{\partial t} &= 0 \\ \text{i.e.,} \quad \partial_\nu A^\nu &= 0, \end{aligned}$$

in which case the PDEs for \underline{A} and Φ decouple to give

$$\begin{aligned} \square^2 \underline{A} &= \underline{j} \\ \square^2 \Phi &= \rho \\ \text{i.e.,} \quad \square^2 A^\mu &= j_{\text{em}}^\mu \quad \text{or} \quad \partial^2 A^\mu = j_{\text{em}}^\mu. \end{aligned}$$

Technical aside: Imposing the Lorentz condition doesn't fix the gauge uniquely since

$$\underline{\nabla} \cdot \underline{\tilde{A}} + \frac{1}{c} \frac{\partial\tilde{\Phi}}{\partial t} = \underline{\nabla} \cdot \underline{A} + \frac{1}{c} \frac{\partial\Phi}{\partial t} - \nabla^2\chi + \frac{1}{c^2} \frac{\partial^2\chi}{\partial t^2},$$

so that if the original potential A^μ satisfies $\partial_\mu A^\mu = 0$, we can arrange for $\partial_\mu \tilde{A}^\mu = 0$ provided that χ is a solution of $\square^2\chi = 0$.

Covariant Form of Maxwell's Equations

The *Maxwell tensor*, or *electromagnetic field strength tensor*, is defined to be

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu;$$

i.e., the four-dimensional ‘curl’ of the 4-vector potential, and is manifestly gauge-invariant. It is straightforward to relate the components of F to those of the electric and magnetic fields:

$$F^{0i} = \partial^0 A^i - \partial^i A^0 = \frac{1}{c} \frac{\partial}{\partial t} A^i + \nabla^i \Phi \equiv -E^i$$

where the index i runs over the values 1, 2 and 3 only. Similarly,

$$F^{ij} = \partial^i A^j - \partial^j A^i = -\nabla^i A^j + \nabla^j A^i = -\varepsilon^{ijk} B^k.$$

It is clear from the antisymmetry of $F^{\mu\nu}$ that all other components are now determined. We can write F as a 4×4 array with rows and columns labelled by the 4 values 0, 1, 2, 3 of the indices

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}.$$

Note: This differs by an overall sign from Mandl & Shaw, who define

$$F^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\nu.$$

It is easy to see that eqn. 2 can be written as

$$\boxed{\partial_\mu F^{\mu\nu} = j_{\text{em}}^\nu}.$$

In this form, it is very clear that the electromagnetic current 4-vector is conserved

$$\partial_\nu j_{\text{em}}^\nu = \partial_\nu \partial_\mu F^{\mu\nu} \equiv 0$$

because of the antisymmetry of $F^{\mu\nu}$.

Interaction of matter with EM fields

Classically, the Hamiltonian for a free non-relativistic particle is

$$H = \frac{|\underline{p}|^2}{2m}.$$

The Hamiltonian for a particle of charge q interacting with an electromagnetic field is

$$H_{\text{em}} = \frac{|\underline{p} - q\underline{A}/c|^2}{2m} + q\Phi$$

in Heaviside-Lorentz units. (As we saw earlier, the equation in SI units is the same, but has no factor of c).

This Hamiltonian is chosen so that the force on the particle due to the EM-field is

$$\underline{F} = q \left(\underline{E} + \frac{1}{c} \underline{v} \times \underline{B} \right),$$

i.e., the familiar Lorentz force. (The factor of c is absent in SI units).

Note: The interacting Hamiltonian can be obtained from the free one by making the replacements

$$\begin{aligned} H &\rightarrow H - q\Phi \\ \underline{p} &\rightarrow \underline{p} - q\underline{A}/c \quad (\text{in Heaviside-Lorentz units}); \end{aligned}$$

this is known as the *minimal coupling* prescription. We shall use this prescription to construct wave equations for particles interacting with EM fields.

Relativistic Quantum Mechanics

The Klein-Gordon equation

Recall that the Schrödinger equation for a free particle

$$\left(-\frac{\hbar^2}{2m} \nabla^2 \right) \Psi(\underline{r}, t) = i\hbar \frac{\partial}{\partial t} \Psi(\underline{r}, t)$$

can be obtained from the (non-relativistic) classical total energy

$$E = \frac{|\underline{p}|^2}{2m} = H$$

by means of the operator substitution prescriptions

$$E \rightarrow i\hbar \frac{\partial}{\partial t} \quad \text{and} \quad \underline{p} \rightarrow -i\hbar \underline{\nabla}.$$

The relativistic expression for the total energy of a free particle is

$$E^2 = |\underline{p}|^2 c^2 + m^2 c^4;$$

Schrödinger (& Klein, Gordon, & Fock) suggested this as a starting point, thus obtaining

$$-\hbar^2 \frac{\partial^2}{\partial t^2} \phi(\underline{r}, t) = -\hbar^2 c^2 \nabla^2 \phi(\underline{r}, t) + m^2 c^4 \phi(\underline{r}, t) \quad (3)$$

which is Schrödinger's relativistic wave equation for a free particle, usually known as the Klein–Gordon (KG) equation (Schrödinger discarded it in favour of the non-relativistic equation that bears his name. Why? See later). We can write the KG equation in a manifestly covariant form as

$$\left(\square^2 + \frac{m^2 c^2}{\hbar^2} \right) \phi(x) = 0 \quad \text{or} \quad \left(\partial^2 + \frac{m^2 c^2}{\hbar^2} \right) \phi(x) = 0,$$

where x is the usual four-vector (ct, x^1, x^2, x^3) . Thus, in covariant form, the operator prescription is

$$\hat{p}^\mu \rightarrow i\hbar \frac{\partial}{\partial x_\mu} = i\hbar \left(\frac{1}{c} \frac{\partial}{\partial t}, -\underline{\nabla} \right).$$

Note: for a *massless* particle $m = 0$, the KG equation reduces to the classical wave equation.

Free particle solutions

By substitution into the KG equation (3) we see it has plane-wave solutions

$$\phi(\underline{r}, t) = \exp\{i\underline{k} \cdot \underline{r} - i\omega t\}$$

provided that ω , \underline{k} & m are related by

$$\hbar^2 \omega^2 = \hbar^2 c^2 |\underline{k}|^2 + m^2 c^4.$$

Taking the square root, we obtain $\hbar\omega = \pm \{\hbar^2 c^2 |\underline{k}|^2 + m^2 c^4\}^{1/2}$.

Such solutions are readily seen to be eigenfunctions of the momentum operator and of the energy operator, with eigenvalues $\underline{p} \equiv \hbar \underline{k}$ and $E \equiv \hbar\omega$ respectively.

Thus, if we interpret $\hbar\omega$ as an allowed energy of the free particle solution, there is an ambiguity in the *sign* of the total energy: \exists both +ve and –ve energy solutions. We shall see later that both types of solution have a rôle to play in quantum field theory, where the negative energy solutions are associated with the existence of anti-particles.

If we define the four-vector $k^\mu \equiv \left(\frac{\omega}{c}, \underline{k}\right)$ then we can write the solution in covariant form

$$\phi(x) \equiv \exp(-ik \cdot x) \equiv \exp(-ik^\mu x_\mu) \equiv \exp(-ip^\mu x_\mu/\hbar)$$

and thus interpret the four-momentum as $p^\mu = \hbar k^\mu$.

Note that, if $\phi(x)$ is a solution of the KG equation, then so is $\phi^*(x)$. The general solution can be written as a superposition of plane wave solutions

$$\phi(x) = \int \frac{d^3 p}{2E_p(2\pi)^3} [f(p) \exp(-ip \cdot x/\hbar) + g(p) \exp(ip \cdot x/\hbar)]$$

where the peculiar-looking factors in the denominator are included for future convenience, and $E_p \equiv +\sqrt{|\underline{p}|^2 c^2 + m^2 c^4}$.

Continuity equation and probability interpretation

Denote the Schrödinger equation by (SE) and its complex-conjugate by (SE)^{*}, then defining

$$\Psi^*(SE) - \Psi(SE)^*,$$

we get a continuity equation

$$\frac{\partial}{\partial t} \rho + \underline{\nabla} \cdot \underline{j} = 0,$$

where $\rho = \Psi^* \Psi$ and $\underline{j} = -\frac{i\hbar}{2m} (\Psi^* \underline{\nabla} \Psi - \Psi \underline{\nabla} \Psi^*)$

are the probability density and probability current respectively. (As before, integrate over any volume, and use the divergence theorem to see why). We can repeat this for the Klein Gordon equation, and obtain the quantities

$$\begin{aligned} \rho &= \frac{i\hbar}{2mc^2} \left(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t} \right) \\ \underline{j} &= -\frac{i\hbar}{2m} (\phi^* \underline{\nabla} \phi - \phi \underline{\nabla} \phi^*). \end{aligned}$$

1. \underline{j} is *identical* in form to the non-relativistic Schrödinger current (with the appropriate normalisation).
2. ρ can be shown to reduce to $\phi^* \phi$ in the non-relativistic limit.
3. The candidate for the probability density, $\rho(x)$, is no longer positive definite (negative energy solutions have $\rho(x) < 0$ (exercise)). Therefore there is no *simple* probability density interpretation. (But see later).

Klein-Gordon equation in an external electromagnetic field

We saw that we can introduce the interaction with an external electromagnetic field by replacing

$$E \rightarrow E - q\Phi \quad \text{and} \quad \underline{p} \rightarrow \underline{p} - q\underline{A}/c$$

in the classical non-relativistic Hamiltonian: the same is true in the relativistic case. In terms of four-vectors

$$p^\mu \rightarrow p^\mu - \frac{q}{c} A^\mu \quad \text{where} \quad A^\mu = (\Phi, \underline{A})$$

the energy momentum relation becomes

$$(E - q\Phi)^2 = c^2 \left(\underline{p} - \frac{q}{c} \underline{A} \right)^2 + m^2 c^4,$$

so that operator substitution gives

$$\left(i\hbar \frac{\partial}{\partial t} - q\Phi(x) \right)^2 \phi(x) = c^2 \left(-i\hbar \nabla - \frac{q}{c} \underline{A} \right)^2 \phi(x) + m^2 c^4 \phi(x).$$

Expanding this expression carefully, we get

$$\begin{aligned} & \left(-\hbar^2 \frac{\partial^2}{\partial t^2} - 2iq\hbar\Phi \frac{\partial}{\partial t} - iq\hbar \frac{\partial\Phi}{\partial t} + q^2\Phi^2 \right) \phi(x) \\ &= (-\hbar^2 c^2 \nabla^2 + 2iq\hbar c \underline{A} \cdot \nabla + iq\hbar c \nabla \cdot \underline{A} + q^2 |\underline{A}|^2 + m^2 c^4) \phi(x). \end{aligned}$$

This is the most general form of the equation. If the potentials are time-independent, then separable solutions of the form

$$\phi(\underline{r}, t) = u(\underline{r}) \exp(-iEt/\hbar)$$

are possible, giving

$$(E - q\Phi)^2 u(\underline{r}) = (-\hbar^2 c^2 \nabla^2 + 2iq\hbar c \underline{A} \cdot \nabla + iq\hbar c \nabla \cdot \underline{A} + q^2 |\underline{A}|^2 + m^2 c^4) u(\underline{r}).$$

Electrostatic potentials

For the special case of a static spherically symmetric electrostatic potential Φ with $\underline{A} = 0$, we can write

$$(-\hbar^2 c^2 \nabla^2 + m^2 c^4) u(\underline{r}) = (E - q\Phi(r))^2 u(\underline{r}).$$

We can separate variables in spherical polar coordinates in the standard way

$$u(r, \theta, \phi) = R_l(r) Y_{lm}(\theta, \phi),$$

where Y_{lm} are the usual spherical harmonics which satisfy $\hat{L}^2 Y_{lm} = \hbar^2 l(l+1) Y_{lm}$ and $\hat{L}_z Y_{lm} = \hbar m Y_{lm}$, and $R_l(r)$ satisfies the radial equation

$$-\left[\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) - \frac{l(l+1)}{r^2} \right] R_l(r) = \frac{[E - q\Phi]^2 - m^2 c^4}{\hbar^2 c^2} R_l(r)$$

which reduces to the usual radial equation in the non-relativistic limit. To see this, set

$$E' = E - mc^2 \quad \text{then}$$

$$\begin{aligned} \frac{[E' + mc^2 - q\Phi]^2 - m^2 c^4}{\hbar^2 c^2} &= \frac{m^2 c^4 + 2mc^2 [E' - q\Phi(r)] + [E' - q\Phi(r)]^2 - m^2 c^4}{\hbar^2 c^2} \\ &= \frac{2m}{\hbar^2} (E' - q\Phi) + O\left(\frac{1}{c^2}\right). \end{aligned}$$

See *Quantum Theory* lecture notes, or advanced QM textbooks, for the *solution of the Coulomb problem*.

The Dirac Equation

Dirac tried to avoid the twin difficulties of negative probability and negative energies by proposing a relativistic equation that is *linear* in $\partial/\partial t$, hoping to prevent the appearance of time derivatives in the ‘probability density’. He argued that the equation should also be linear in the spatial derivatives because relativity treats space and time on an equal footing. In this way he also hoped to avoid the sign ambiguity in taking the square-root of E^2 .

Therefore, we start with a Hamiltonian form of the equation

$$i\hbar \frac{\partial}{\partial t} \psi(\underline{r}, t) = \hat{H}\psi(\underline{r}, t)$$

and write

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \psi(\underline{r}, t) &= -i\hbar c \left(\alpha^1 \frac{\partial}{\partial x^1} + \alpha^2 \frac{\partial}{\partial x^2} + \alpha^3 \frac{\partial}{\partial x^3} \right) \psi(\underline{r}, t) + \beta mc^2 \psi(\underline{r}, t) \\ &= (c\underline{\alpha} \cdot \hat{\underline{p}} + \beta mc^2) \psi(\underline{r}, t) = \hat{H}\psi(\underline{r}, t). \end{aligned} \quad (4)$$

To start with, we wish to construct a free particle equation, so no terms in the Hamiltonian \hat{H} should depend on \underline{r} or t as these would describe forces. By assumption the α^i and β are independent of derivatives, therefore $\underline{\alpha}$ and β commute with \underline{r} , t , $\hat{\underline{p}}$ and E but not necessarily with each other.

Now we require relativistic invariance to be maintained, i.e., $E^2 = |\underline{p}|^2 c^2 + m^2 c^4$. Dirac therefore demanded that

$$\hat{H}^2 \psi(\underline{r}, t) = (c^2 |\hat{\underline{p}}|^2 + m^2 c^4) \psi(\underline{r}, t). \quad (5)$$

From equation (4) we have

$$\hat{H}^2 \psi(\underline{r}, t) = (c\underline{\alpha} \cdot \hat{\underline{p}} + \beta mc^2) (c\underline{\alpha} \cdot \hat{\underline{p}} + \beta mc^2) \psi(\underline{r}, t).$$

Expand the RHS of this equation, being very careful about the ordering of the as yet undetermined quantities α^i and β

$$\begin{aligned} &\hat{H}^2 \Psi(\underline{r}, t) \\ &= \left\{ c^2 [(\alpha^1)^2 (\hat{p}^1)^2 + (\alpha^2)^2 (\hat{p}^2)^2 + (\alpha^3)^2 (\hat{p}^3)^2] + m^2 c^4 \beta^2 \right\} \psi(\underline{r}, t) \\ &\quad + \left\{ (\alpha^1 \alpha^2 + \alpha^2 \alpha^1) \hat{p}^1 \hat{p}^2 + (\alpha^2 \alpha^3 + \alpha^3 \alpha^2) \hat{p}^2 \hat{p}^3 + (\alpha^3 \alpha^1 + \alpha^1 \alpha^3) \hat{p}^1 \hat{p}^3 \right\} \psi(\underline{r}, t) \\ &\quad + mc^3 \left\{ (\alpha^1 \beta + \beta \alpha^1) \hat{p}^1 + (\alpha^2 \beta + \beta \alpha^2) \hat{p}^2 + (\alpha^3 \beta + \beta \alpha^3) \hat{p}^3 \right\} \psi(\underline{r}, t). \end{aligned}$$

Condition (5) is satisfied if

$$\begin{aligned} (\alpha^1)^2 &= (\alpha^2)^2 = (\alpha^3)^2 = \beta^2 = 1 \\ \alpha^i \alpha^j + \alpha^j \alpha^i &= 0 \quad (i \neq j) \\ \alpha^i \beta + \beta \alpha^i &= 0 \end{aligned}$$

or, more compactly, as

$$\begin{aligned} \{\alpha^i, \alpha^j\} &= 2\delta^{ij} \quad \text{the anticommutator of } \alpha^i \text{ and } \alpha^j, \\ \{\alpha^i, \beta\} &= 0, \quad \beta^2 = 1. \end{aligned}$$

1. From these relations it is clear that the α^i and β cannot be just numbers. Let us therefore assume they are *matrices*. Since \hat{H} is hermitian, they must be *square* ($n \times n$) matrices.
2. The square of each matrix α^i , β is the unit matrix. Since they are hermitian, their eigenvalues are real, therefore all eigenvalues must be ± 1 .
3. $\text{tr } \alpha^i = \text{tr } \beta = 0$.

$$\begin{aligned}\text{Proof: } \text{tr } \alpha^i &= \text{tr}(\beta^2 \alpha^i) = \text{tr}(\beta \alpha^i \beta) \quad (\text{using } \text{tr}(AB) = \text{tr}(BA)) \\ &= -\text{tr}(\alpha^i \beta^2) \quad (\text{using } \alpha^i \beta = -\beta \alpha^i) \\ &= -\text{tr } \alpha^i = 0.\end{aligned}$$

Therefore, since the eigenvalues are ± 1 , n must be *even*. It is not possible to find a set of hermitian traceless 2×2 matrices which satisfy the anti-commutation relations — the 3 Pauli matrices σ^i are good start, but there is no 4th matrix. The simplest representation is 4×4 . The *standard representation* has β diagonal:

$$\begin{aligned}\beta &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \\ \alpha^1 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \alpha^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \alpha^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.\end{aligned}$$

We usually write these in block 2×2 form as:

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \underline{\alpha} = \begin{pmatrix} 0 & \sigma \\ \underline{\sigma} & 0 \end{pmatrix}$$

with each element a 2×2 submatrix, and where σ^i are the usual Pauli matrices.

Exercise: Check that these matrices satisfy the correct anti-commutation relations.

Since the Hamiltonian is a 4×4 matrix, the wave-function $\psi(\underline{r}, t)$ that it acts on must be a *4-component column vector*:

$$\psi(\underline{r}, t) = \begin{pmatrix} \psi_1(\underline{r}, t) \\ \psi_2(\underline{r}, t) \\ \psi_3(\underline{r}, t) \\ \psi_4(\underline{r}, t) \end{pmatrix}$$

Probability Density

The Dirac equation for a free particle is

$$i\hbar \frac{\partial}{\partial t} \psi(\underline{r}, t) = (-i\hbar c \underline{\alpha} \cdot \underline{\nabla} + \beta mc^2) \psi(\underline{r}, t). \quad (6)$$

Construction of the probability density is straightforward. Take the Hermitian conjugate of equation (6) (i.e., complex conjugate and transpose)

$$-i\hbar \frac{\partial}{\partial t} \psi^\dagger(\underline{r}, t) = \psi^\dagger(\underline{r}, t) \left(i\hbar c \underline{\alpha} \cdot \underline{\nabla} + \beta mc^2 \right) \quad (7)$$

Note that ψ^\dagger is a row vector. Multiply (6) on the left by ψ^\dagger and (7) on the right by ψ and subtract to obtain

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} (\psi^\dagger \psi) &= -i\hbar c \left(\psi^\dagger \underline{\alpha} \cdot \underline{\nabla} \psi + \psi^\dagger \underline{\alpha} \cdot \underline{\nabla} \psi \right) \\ &= -i\hbar c \left(\psi^\dagger \underline{\alpha} \cdot \underline{\nabla} \psi + (\underline{\nabla} \psi^\dagger) \cdot \underline{\alpha} \psi \right) \\ &= -i\hbar c \underline{\nabla} \cdot (\psi^\dagger \underline{\alpha} \psi), \end{aligned}$$

which can be written as a continuity equation

$$\frac{1}{c} \frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \underline{j} = 0$$

where

$$\rho = \psi^\dagger \psi \quad \text{and} \quad \underline{j} = \psi^\dagger \underline{\alpha} \psi.$$

We can write the continuity equation in covariant form as

$$\partial_\mu j^\mu = 0 \quad \text{with} \quad j^\mu = (\rho, \underline{j}).$$

This implies that $\psi^\dagger \psi$ transforms like the zero-component of a 4-vector, with $\psi^\dagger \underline{\alpha} \psi$ the corresponding space part.

Note that $\rho = |\psi|^2$ is a positive definite quantity as is required of a probability density.

Free Particle Solutions

Let us look for plane-wave solutions of the form

$$\psi(\underline{r}, t) = \exp(-ik_\mu x^\mu) u(\underline{p}) \equiv \exp \left\{ -\frac{i}{\hbar} (cp_0 t - \underline{p} \cdot \underline{r}) \right\} u(\underline{p})$$

with $u(\underline{p})$ a 4-component column vector.

Substituting this into the Dirac equation (6), yields

$$p^0 u = (\underline{\alpha} \cdot \underline{p} + \beta mc) u.$$

The trial solution presumably represents a particle of energy cp^0 and momentum \underline{p} . Writing out the previous equation by substituting the matrices β and α^i gives a set of four simultaneous linear equations, which we write in matrix form as

$$\begin{pmatrix} (-p^0 + mc) & 0 & p^3 & (p^1 - ip^2) \\ 0 & (-p^0 + mc) & (p^1 + ip^2) & -p^3 \\ p^3 & (p^1 - ip^2) & -(p^0 + mc) & 0 \\ (p^1 + ip^2) & -p^3 & 0 & -(p^0 + mc) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = 0. \quad (8)$$

The condition for non-trivial solutions for u_i is that the determinant of the matrix vanishes. On multiplying out the determinant, we find (exercise!)

$$\{m^2c^2 + |\underline{p}|^2 - (p^0)^2\}^2 = 0$$

which is exactly the required energy-momentum relation. Of course, this had to happen because we constructed the matrices α^i and β so that it would!

Taking the square-root of the energy-momentum relation, we have

$$p^0 = \pm \{m^2c^2 + |\underline{p}|^2\}^{1/2}$$

so the negative energy solutions are still with us!

Positive energy solutions: Let us first choose $p^0 > 0$, i.e.,

$$p^0 = + (m^2c^2 + |\underline{p}|^2)^{1/2} \equiv p_+^0 = \frac{E}{c} \quad (E > 0);$$

substituting into the first two rows of equations (8), we obtain

$$\begin{aligned} (-p_+^0 + mc)u_1 + p^3 u_3 + (p^1 - ip^2)u_4 &= 0 \\ (-p_+^0 + mc)u_2 + (p^1 + ip^2)u_3 - p^3 u_4 &= 0. \end{aligned}$$

Since only two of the equations are independent, we are free to *choose* two of the u_α arbitrarily. The usual *convention* is

$$\begin{aligned} \{u_1 = 1 \text{ and } u_2 = 0\} \\ \text{or } \{u_1 = 0 \text{ and } u_2 = 1\}. \end{aligned}$$

(Any other choice is a linear combination of these two). The first choice gives

$$u_3 = \frac{p^3}{p_+^0 + mc}, \quad u_4 = \frac{p^1 + ip^2}{p_+^0 + mc};$$

whilst the second one gives

$$u_3 = \frac{p^1 - ip^2}{p_+^0 + mc}, \quad u_4 = \frac{-p^3}{p_+^0 + mc}.$$

Negative energy solutions: Now choose $p^0 < 0$, ie

$$p^0 = - (m^2c^2 + |\underline{p}|^2)^{1/2} \equiv p_-^0 = - \frac{E}{c} \quad (E > 0)$$

For negative energy solutions, it is *conventional* to make the arbitrary *choice*

$$\begin{aligned} \{u_3 = 1 \text{ and } u_4 = 0\} \\ \text{or } \{u_3 = 0 \text{ and } u_4 = 1\}. \end{aligned}$$

The first choice gives

$$u_1 = \frac{p^3}{p_-^0 - mc}, \quad u_2 = \frac{p^1 + ip^2}{p_-^0 - mc};$$

whilst the second one gives

$$u_1 = \frac{p^1 - ip^2}{p_-^0 - mc}, \quad u_2 = \frac{-p^3}{p_-^0 - mc}.$$

Summary: The positive energy solutions with four momentum $p_+^\mu = (p_+^0, \underline{p})$ are

$$\omega^1(\underline{p}) = \begin{pmatrix} 1 \\ 0 \\ \frac{p^3}{p_+^0 + mc} \\ \frac{p^1 + ip^2}{p_+^0 + mc} \end{pmatrix} \quad \text{and} \quad \omega^2(\underline{p}) = \begin{pmatrix} 0 \\ 1 \\ \frac{p^1 - ip^2}{p_+^0 + mc} \\ \frac{-p^3}{p_+^0 + mc} \end{pmatrix}.$$

The negative energy solutions with spatial momentum $-\underline{p}$, i.e., with $p_-^\mu = (p_-^0, -\underline{p})$, are

$$\omega^3(\underline{p}) = \begin{pmatrix} -\frac{p^1 - ip^2}{p_-^0 - mc} \\ \frac{p^3}{p_-^0 - mc} \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \omega^4(\underline{p}) = \begin{pmatrix} -\frac{p^3}{p_-^0 - mc} \\ -\frac{p^1 + ip^2}{p_-^0 - mc} \\ 1 \\ 0 \end{pmatrix}.$$

N.B., conventions in labelling the ω^α differ.

Rest-frame solutions When $\underline{p} = 0$ we have

$$\omega^1(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \omega^2(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \omega^3(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \omega^4(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

so that

$$\psi^{(1)} = \exp(-imc^2t/\hbar) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \psi^{(2)} = \exp(-imc^2t/\hbar) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

are degenerate in energy. Therefore, there should be another operator which commutes with the Hamiltonian and whose eigenvalues label the states; such an operator is

$$\Sigma^3 \equiv \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}.$$

The rest-frame *four-component spinors* $\omega^\alpha(0)$ are eigenvectors of Σ^3 with eigenvalues ± 1 ; this suggests that we interpret the Dirac equation as describing a spin 1/2 particle. If we introduce the three matrices

$$\underline{\Sigma} \equiv \begin{pmatrix} \sigma & 0 \\ 0 & \underline{\sigma} \end{pmatrix}$$

$$\left. \begin{array}{l} \text{then } (\frac{1}{2}\hbar\underline{\Sigma}) \cdot (\frac{1}{2}\hbar\underline{\Sigma}) = \frac{3}{4}\hbar^2\hat{1} \\ \text{and } \frac{1}{2}\hbar\Sigma^3 \text{ has eigenvalues } \pm\frac{1}{2}\hbar \end{array} \right\} \implies \hat{s} = \frac{1}{2}\hbar\underline{\Sigma},$$

i.e., $\frac{1}{2}\hbar\underline{\Sigma}$ is the spin operator.

Note, however, that $\underline{\Sigma}$ does not commute with the hamiltonian operator $c\underline{\alpha} \cdot \underline{p} + \beta mc^2$ in any frame other than the rest frame, $\underline{p} = 0$. Note also that the operator $\underline{\hat{L}} = \underline{\hat{r}} \times \underline{\hat{p}}$ does not commute with the Hamiltonian. However, the operator

$$v\hat{J} = \underline{\hat{L}} + \frac{1}{2}\hbar\underline{\Sigma}$$

does commute with the Hamiltonian, suggesting that it may be interpreted as a total angular momentum operator (tutorial).

When $\underline{p} \neq 0$ there are two independent states for any given four-momentum. Therefore there exists some operator that commutes with $c\underline{\alpha} \cdot \underline{p} + \beta mc^2$ and labels the states. The *helicity operator*

$$\hat{h}(\underline{p}) = \begin{pmatrix} \frac{\sigma \cdot \underline{p}}{|\underline{p}|} & 0 \\ 0 & \frac{\sigma \cdot \underline{p}}{|\underline{p}|} \end{pmatrix}$$

commutes with $c\underline{\alpha} \cdot \underline{p} + \beta mc^2$ and has eigenvalues ± 1 . Therefore, general plane wave states with $\underline{p} \neq 0$ can be chosen to be helicity states. (It still remains to *interpret* the negative energy states).

Two-by-two block form of the Dirac equation

So far, we have constructed plane wave solutions of the Dirac equation by brute force. A more elegant formalism, which is also useful for considering the non-relativistic limit, is obtained by writing the free-particle spinor as

$$u = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$$

where ϕ and χ are two-component spinors. The Dirac equation then becomes

$$p^0 \begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} mc & \underline{\sigma} \cdot \underline{p} \\ \underline{\sigma} \cdot \underline{p} & -mc \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix}.$$

Note that the elements of the 2×2 *block matrix* are themselves 2×2 matrices. More explicitly,

$$\begin{aligned} p^0\phi &= mc\phi + \underline{\sigma} \cdot \underline{p}\chi \\ p^0\chi &= \underline{\sigma} \cdot \underline{p}\phi - mc\chi. \end{aligned}$$

The second of these equations gives

$$\chi = \frac{\underline{\sigma} \cdot \underline{p}}{p^0 + mc} \phi$$

which we can substitute back into the first equation to obtain

$$p^0 \phi = \left\{ mc + \frac{(\underline{\sigma} \cdot \underline{p})^2}{p^0 + mc} \right\} \phi,$$

but $(\underline{\sigma} \cdot \underline{p})^2 = |\underline{p}|^2$ (exercise), therefore

$$(p^0)^2 + p^0 mc = p^0 mc + (mc)^2 + |\underline{p}|^2$$

which reduces to the usual relation between energy & momentum.

General positive-energy plane-wave solutions can be written as

$$\omega^{1,2}(\underline{p}) = \begin{pmatrix} \phi^{1,2} \\ \frac{c\underline{\sigma} \cdot \underline{p}}{E + mc^2} \phi^{1,2} \end{pmatrix}.$$

The choices

$$\phi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \phi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

reproduce the four-component solutions obtained above.

Similarly, the negative energy solutions can be written (for $p_-^\mu = (p_-^0, -\underline{p})$)

$$\omega^{3,4}(\underline{p}) = \begin{pmatrix} \frac{c\underline{\sigma} \cdot \underline{p}}{E + mc^2} \chi^{1,2} \\ \chi^{1,2} \end{pmatrix}$$

if we make the choice of two-component spinors

$$\chi^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \chi^2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The reason for this (seemingly perverse!) choice will become clear when we interpret the negative energy states.

More on the Dirac Equation

The Dirac Equation in an Electromagnetic Field

We now investigate the Dirac equation for a particle in an EM field. The Dirac equation for a particle of charge q interacting with an EM field A^μ is obtained from the minimal coupling prescription

$$p^\mu \rightarrow p^\mu - \frac{q}{c} A^\mu \quad \text{where} \quad A^\mu = (A^0, \underline{A})$$

Therefore

$$i\hbar \frac{\partial}{\partial t} \psi(\underline{r}, t) = \left\{ c\underline{\alpha} \cdot (\hat{\underline{p}} - \frac{q}{c} \underline{A}) + \beta mc^2 + qA^0 \right\} \psi(\underline{r}, t) = \hat{H}_D \psi(\underline{r}, t).$$

Reduction to the Pauli equation

Firstly, we investigate the non-relativistic (NR) limit of the Dirac equation. We know that an adequate NR description of spin needs only two-component spinors. For the free particle positive-energy spinors $\omega^{1,2}$, we wrote

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \quad \text{where} \quad \chi = \frac{c\sigma \cdot \underline{p}}{E + mc^2} \phi.$$

Thus $\chi \simeq \frac{v}{c}\phi$ which suggests that we write

$$\psi = \begin{pmatrix} \Psi \\ \Phi \end{pmatrix}$$

and can therefore regard the lower components being ‘small’ in the non-relativistic limit. **N.B.**, Ψ and Φ are **not** the two-component free-particle solutions of the previous lecture!

We write the Hamiltonian as $\hat{H}_1 + mc^2$ in anticipation of the NR limit. Thus

$$\hat{H}_1 = c\underline{\alpha} \cdot \left(-i\hbar\underline{\nabla} - \frac{q}{c}\underline{A} \right) + \beta mc^2 + qA^0 - mc^2.$$

In two-component notation:

$$\begin{aligned} \hat{H}_1 \begin{pmatrix} \Psi \\ \Phi \end{pmatrix} &= \\ \begin{pmatrix} 0 & c\underline{\sigma} \cdot \left(-i\hbar\underline{\nabla} - \frac{q}{c}\underline{A} \right) \\ c\underline{\sigma} \cdot \left(-i\hbar\underline{\nabla} - \frac{q}{c}\underline{A} \right) & 0 \end{pmatrix} \begin{pmatrix} \Psi \\ \Phi \end{pmatrix} &- 2mc^2 \begin{pmatrix} 0 \\ \Phi \end{pmatrix} + qA^0 \begin{pmatrix} \Psi \\ \Phi \end{pmatrix} \end{aligned}$$

which, on multiplying out, becomes

$$\hat{H}_1\Psi = c\underline{\sigma} \cdot \left(-i\hbar\underline{\nabla} - \frac{q}{c}\underline{A} \right) \Phi + qA^0\Psi \quad (9)$$

$$\hat{H}_1\Phi = c\underline{\sigma} \cdot \left(-i\hbar\underline{\nabla} - \frac{q}{c}\underline{A} \right) \Psi + qA^0\Phi - 2mc^2\Phi. \quad (10)$$

We can rewrite equation (10)

$$\left(\hat{H}_1 - qA^0 + 2mc^2 \right) \Phi = c\underline{\sigma} \cdot \left(-i\hbar\underline{\nabla} - \frac{q}{c}\underline{A} \right) \Psi.$$

Thus provided all matrix elements of $\hat{H}_1 \ll mc^2$ and $|qA^0| \ll mc^2$ we can drop them on the LHS as a first approximation, so

$$\Phi \simeq \frac{c\underline{\sigma} \cdot \left(-i\hbar\underline{\nabla} - \frac{q}{c}\underline{A} \right)}{2mc^2} \Psi$$

and we can rewrite equation (9) as

$$\hat{H}_1\Psi = \frac{1}{2m} \left\{ \underline{\sigma} \cdot \left(-i\hbar\underline{\nabla} - \frac{q}{c}\underline{A} \right) \right\}^2 \Psi + qA^0\Psi. \quad (11)$$

The operator on the RHS is the Pauli Hamiltonian. Multiplying out the first term on the RHS we obtain:

$$\frac{1}{2m} \left\{ \underline{\sigma} \cdot \left(-i\hbar \nabla - \frac{q}{c} \underline{A} \right) \right\}^2 \Psi = \\ \left\{ \frac{1}{2m} (\underline{\sigma} \cdot \hat{p})^2 - \frac{q}{2mc} (\underline{\sigma} \cdot \hat{p})(\underline{\sigma} \cdot \underline{A}) - \frac{q}{2mc} (\underline{\sigma} \cdot \underline{A})(\underline{\sigma} \cdot \hat{p}) + \frac{q^2}{2mc^2} (\underline{\sigma} \cdot \underline{A})^2 \right\} \Psi.$$

Using the commutation and anticommutation relations for the Pauli matrices, $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$ and $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$, we can derive the following identity for any two vectors \underline{A} and \underline{B}

$$(\underline{\sigma} \cdot \underline{A})(\underline{\sigma} \cdot \underline{B}) = \underline{A} \cdot \underline{B} + i\underline{\sigma} \cdot \underline{A} \times \underline{B}$$

Remembering that \underline{A} and \hat{p} do not in general commute we can use an arbitrary function $f(\underline{r})$ to evaluate

$$\begin{aligned} (\underline{\sigma} \cdot \hat{p})(\underline{\sigma} \cdot \underline{A})f(\underline{r}) &= \hat{p} \cdot (\underline{A}f(\underline{r})) + i\underline{\sigma} \cdot \hat{p} \times (\underline{A}f(\underline{r})) \\ &= \underline{A} \cdot \hat{p}f(\underline{r}) - i\hbar(\nabla \cdot \underline{A})f(\underline{r}) + i\underline{\sigma} \cdot (-i\hbar \nabla \times (\underline{A}f(\underline{r}))) \\ &= \underline{A} \cdot \hat{p}f(\underline{r}) - i\hbar(\nabla \cdot \underline{A})f(\underline{r}) + \hbar\underline{\sigma} \cdot \underline{B}f(\underline{r}) - i\underline{\sigma} \cdot (\underline{A} \times \hat{p})f(\underline{r}); \end{aligned}$$

similarly

$$(\underline{\sigma} \cdot \underline{A})(\underline{\sigma} \cdot \hat{p})f(\underline{r}) = \underline{A} \cdot \hat{p}f(\underline{r}) + i\underline{\sigma} \cdot (\underline{A} \times \hat{p})f(\underline{r}).$$

Substituting back into the Pauli Hamiltonian, we obtain

$$\hat{H}_1 \Psi = \left\{ \frac{\hat{p}^2}{2m} - \frac{q}{mc} (\underline{A} \cdot \hat{p}) + \frac{iq\hbar}{2mc} (\nabla \cdot \underline{A}) - \frac{q\hbar}{2mc} (\underline{\sigma} \cdot \underline{B}) + \frac{q^2 A^2}{2mc^2} + qA^0 \right\} \Psi$$

where $\underline{B} = \nabla \times \underline{A}$ is the magnetic field. The ‘spin term’ is

$$-\frac{q}{mc} (\hat{s} \cdot \underline{B}) = +g_s \frac{e}{2mc} (\hat{s} \cdot \underline{B})$$

for an electron of charge $q = -e$. Thus the Dirac equation predicts that $g_s = 2$ in good agreement with experiment. Recall that $g = 1$ in classical physics and therefore $g_s = 2$ has to be put in ‘by hand’ when doing NR atomic physics.

It is often claimed that to get $g_s = 2$ requires relativity. This is of course true in the sense that it falls out *naturally* from the Dirac equation. However, Feynman pointed out that if we start out (somewhat artificially) with the two-component wave equation

$$\frac{(\underline{\sigma} \cdot \hat{p})^2}{2m} \Psi = i\hbar \frac{\partial}{\partial t} \Psi$$

which is equivalent to Schrödinger’s equation for a free particle (check it!), then the minimal coupling prescription yields the Pauli Hamiltonian, equation (11), and hence $g_s = 2$.

Finally, note that we can write the Pauli equation in the compact form

$$i\hbar \frac{\partial}{\partial t} \Psi = \left\{ \frac{(\hat{p} - q\underline{A}/c)^2}{2m} - \frac{q\hbar}{2mc} (\underline{\sigma} \cdot \underline{B}) + qA^0 \right\} \Psi.$$

Covariant form of the Dirac equation

In most relativistic applications of the Dirac equation, a covariant notation is used. Defining the ‘natural’ system of units, $\hbar = c = 1$, the Dirac equation for a free particle is

$$i\frac{\partial}{\partial t}\psi(\underline{r},t) = (-i\underline{\alpha} \cdot \underline{\nabla} + \beta m)\psi(\underline{r},t). \quad (12)$$

If we multiply by β

$$i\beta\frac{\partial}{\partial t}\psi(\underline{r},t) = (-i\beta\underline{\alpha} \cdot \underline{\nabla} + m)\psi(\underline{r},t)$$

and introduce the matrices

$$\begin{aligned}\gamma^0 &\equiv \beta \\ \gamma^i &\equiv \beta\alpha^i,\end{aligned}$$

then we may write equation (12) as

$$\left\{ i\left(\gamma^0\frac{\partial}{\partial x^0} + \gamma^i\frac{\partial}{\partial x^i}\right) - m \right\} \psi(x) = 0$$

where $x = x^\mu$ ($\mu = 0, \dots, 3$), or more compactly as

$$\begin{aligned}(i\gamma^\mu\partial_\mu - m)\psi(x) &= 0, \\ \text{or} \quad (i\cancel{d} - m)\psi(x) &= 0\end{aligned}$$

where we have introduced the Feynman ‘slash’ notation

$$\cancel{a} \equiv \gamma^\mu a_\mu = \gamma_\mu a^\mu$$

pronounced ‘ a -slash’; similarly, \cancel{d} is ‘ d -slash’.

Positive energy plane-wave solutions of the type $\psi = \exp(-ip \cdot x)u$ thus satisfy

$$(\gamma^\mu p_\mu - m)u \equiv (\cancel{p} - m)u = 0,$$

whilst negative energy (negative four-momentum) solutions $\psi = \exp(+ip \cdot x)v$ satisfy

$$(\gamma^\mu p_\mu + m)v \equiv (\cancel{p} + m)v = 0.$$

It is easy to verify that the gamma matrices satisfy the anticommutation relations known as the *Clifford algebra*

$$\boxed{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \equiv \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}}.$$

In the standard representation of $\underline{\alpha}$ and β

$$\underline{\gamma} = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix} \quad \text{and} \quad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

so that γ^0 is hermitian $\gamma^{0\dagger} = \gamma^0$, and the γ^i are anti-hermitian $\gamma^{i\dagger} = -\gamma^i$, and $(\gamma^0)^2 = 1$ and $(\gamma^i)^2 = -1$.

It is also convenient to work with

$$\overline{\psi} \equiv \psi^\dagger\gamma^0$$

and similarly

$$\bar{u} \equiv u^\dagger \gamma^0,$$

where $\bar{\psi}$ is pronounced ‘ ψ -bar’.

The new notation treats space and time on an (even more) equal basis, and is known as the *covariant formulation*. We shall derive the properties of the Dirac wave-function $\psi(x)$ under Lorentz boosts and use them to *prove* that the conserved current

$$\begin{aligned} j^\mu &\equiv (\psi^\dagger \psi, \psi^\dagger \underline{\alpha} \psi) \\ &= \bar{\psi} \gamma^\mu \psi \end{aligned}$$

does indeed transform as a 4-vector under Lorentz transformations. Similarly, we shall show that the quantity $\bar{\psi} \psi$ is unchanged, a Lorentz scalar.

Relativistic Quantum Field Theory

Introduction

Motivation

Last year you learned something about Relativistic Quantum Mechanics (RQM), in particular, the Klein Gordon equation (KGE), and the Dirac equation (DE). These equations are the correct starting points for fully-relativistic quantum theories of spin-0 and spin- $\frac{1}{2}$ particles respectively, but there are problems with the interpretation of their solutions within the framework of RQM, both equations having “negative energy” solutions, for example. Furthermore, the naïve interpretation of the KG continuity equation leads to negative probabilities for negative energy particles — whatever that may mean. These problems can be solved (or at least circumvented) for free particles in RQM, but the addition of interactions leads to more severe complications.

As an example, consider a collision between an electron (e^-) and a positron (e^+): we observe experimentally that the particles may annihilate each other, and if they are energetic enough several e^+e^- pairs may be produced. We conclude that we need a relativistic quantum theory that can deal with an *arbitrary* number of particles.

One may reach the same conclusion using a naïve uncertainty principle argument. Start with the “energy-time uncertainty principle”: $\Delta E \Delta t > \hbar$. For sufficiently small Δt , we have $\Delta E > mc^2$, and once again we conclude that we need a formalism that’s capable of dealing with an arbitrary number of particles. Such a formalism is Quantum Field Theory (QFT).

The purpose of this course is to introduce you to the formalism and the physics of Quantum Field Theory. QFT provides us with the theoretical framework for understanding all of the fundamental interactions, with the possible exception of gravity. It has been staggeringly successful in its application to Quantum Electrodynamics (QED). Consider, for example, the calculation of the anomalous magnetic moment of the electron, characterised by $a \equiv (g-2)/2$ for the electron:

$$\begin{aligned} a_{\text{theor}} &= 1159651941(128) \times 10^{-12} \\ a_{\text{exptl}} &= 1159652188.4(4.3) \times 10^{-12} \end{aligned}$$

By the end of the course you should be able to perform lowest-order perturbation theory calculations in QED using Feynman diagram methods.

Relativistic Wave Equations

We begin by recalling the single-particle relativistic wave equations for free particles: massive spin-0, massless spin-1, and massive spin- $\frac{1}{2}$ free particles respectively. For detailed derivations, see the Revision Notes.

We shall work in ‘natural’ units, where $\hbar = c = 1$.

The Klein-Gordon Equation

$$(\partial^2 + m^2)\phi(x) = 0 \quad \text{or} \quad (\square^2 + m^2)\phi(x) = 0$$

has free-particle solutions of the form

$$\phi(x) = \exp(-ip \cdot x) = \exp(-ip^\mu x_\mu)$$

provided that $E^2 \equiv p_0^2 = |\underline{p}|^2 + m^2$. (Remember that $c = 1$ in natural units).

Note that if $\phi(x)$ is a solution, then so is $\phi^*(x) \equiv \exp(+ip \cdot x)$.

Electromagnetic Vector Potential Equation

$$\partial^2 A^\mu - \partial^\mu \partial_\nu A^\nu = 0 \tag{1}$$

You probably won’t recognise this as the equation for massless spin-1 particles. If you don’t be patient, all will be revealed later.

Define the **electromagnetic field strength tensor**

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$$

the four-dimensional ‘curl’ of the 4-vector potential. It is also the commutator of the covariant derivative $D^\mu = \partial^\mu - ieA^\mu$,

$$[D^\mu, D^\nu]f = -ie(\partial^\mu A^\nu + A^\mu \partial^\nu - \partial^\nu A^\mu - A^\nu \partial^\mu)f = -ie(\partial^\mu A^\nu - \partial^\nu A^\mu)f = -ieF^{\mu\nu}f.$$

Why isn’t this commutator a differential operator acting on the arbitrary function f ?

$F^{\mu\nu}$ is a second-rank tensor whose components are the electric field \underline{E} , and the magnetic field \underline{B} . It can be represented by an array whose rows and columns are labelled by the 4 values of the indices $\{0, 1, 2, 3\}$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}.$$

It is easy to see that equation (1) can be written as

$$\partial_\mu F^{\mu\nu} = 0.$$

The Dirac Equation

$$i \frac{\partial}{\partial t} \psi(\underline{r}, t) = \{-i\underline{\alpha} \cdot \nabla + \beta m\} \psi(\underline{r}, t)$$

The hermitian matrices α^i and β satisfy the Dirac algebra:

$$\begin{aligned} \{\alpha^i, \alpha^j\} &= 2\delta^{ij} \\ \{\alpha^i, \beta\} &= 0, \quad \beta^2 = 1. \end{aligned} \quad (2)$$

We shall usually work with the covariant form of the Dirac equation:

$$(i\gamma^\mu \partial_\mu - m) \psi(x) = 0 \quad \text{or} \quad (i\cancel{D} - m) \psi(x) = 0$$

where $\gamma^0 \equiv \beta$ and $\gamma^i \equiv \beta\alpha^i$, and the γ matrices satisfy the *Clifford algebra*

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}.$$

It is useful to introduce a fifth Dirac γ matrix, defined as follows:

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \equiv \gamma_5.$$

This extra matrix has two readily-established general properties

$$(\gamma^5)^2 = 1 \quad \text{and} \quad \{\gamma^5, \gamma^\mu\} = 0$$

which may be derived from the Clifford algebra property of the four matrices γ^μ .

Positive energy free-particle solutions of the Dirac equation are of the form

$$\psi(x) = \exp(-ip \cdot x) u(p)$$

where the 4-component, free-particle spinor, $u(p)$, satisfies

$$(\gamma^\mu p_\mu - m) u \equiv (\cancel{p} - m) u = 0,$$

whilst negative energy (negative 4-momentum) solutions are of the form

$$\psi(x) = \exp(+ip \cdot x) v(p)$$

with

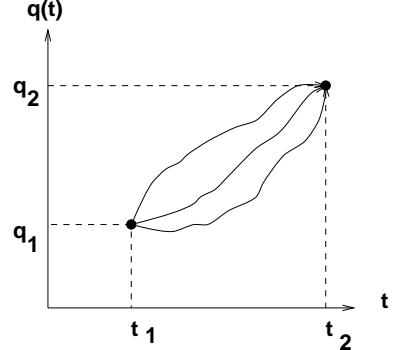
$$(\gamma^\mu p_\mu + m) v \equiv (\cancel{p} + m) v = 0.$$

Lagrangian Field Theory

Classical Mechanics of Point Particles

Consider the motion in one space dimension of a particle of mass m in a conservative force field. Let q be the generalized coordinate of the particle, $\dot{q} \equiv dq/dt$ its velocity, and $L(q, \dot{q}) \equiv T(\dot{q}) - V(q)$, the Lagrangian. Then *Hamilton's Principle*, also known as the *Principle of Least Action*, states that the particle motion is determined by

$$\delta S \equiv \delta \int_{t_1}^{t_2} dt L(q, \dot{q}) = 0.$$



This principle simply tells us that the classical path, $q(t)$ of the particle traveling from $q(t_1) = q_1$ to $q(t_2) = q_2$ is such that the *action* S is stationary. In other words, small variations from this path: $q(t) \rightarrow q(t) + \delta q(t)$ leave S unchanged to first order in δq .

This is a simple problem in calculus of variations. We consider a small variation $\delta q(t)$ in the function $q(t)$, subject to the condition that the values of q at the end-points are unchanged

$$\delta q(t_1) = \delta q(t_2) = 0.$$

To first order, the variation in $L(q, \dot{q})$ is

$$\delta L = \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \quad \text{where} \quad \delta \dot{q} \equiv \frac{d}{dt} \delta q;$$

thus the variation in the action is

$$\delta S = \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \frac{d}{dt} \delta q \right).$$

Integrating the second term by parts and noting that the integrated term

$$\left[\frac{\partial L}{\partial \dot{q}} \delta q \right]_{t_1}^{t_2}$$

vanishes at the limits, we obtain

$$\delta S = \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right] \delta q(t).$$

For S to be stationary, this variation must vanish for an *arbitrary* small variation $\delta q(t)$, which is only possible if the integrand vanishes identically. Thus we obtain the so-called *Euler–Lagrange Equation*

$$\boxed{\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) = 0}.$$

The generalization to 3 dimensions is straightforward

$$\boxed{\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0 \quad \text{for } i = 1, 2, 3}.$$

We can define *generalized momenta*, also known as *canonical momenta*,

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$$

and define the Hamiltonian function

$$\boxed{H(p, q) \equiv \sum_i p_i \dot{q}_i - L}.$$

From the LHS

$$dH = \sum_i \left(\frac{\partial H}{\partial q_i} \delta q_i + \frac{\partial H}{\partial p_i} \delta p_i \right)$$

and from the RHS

$$dH = \sum_i \left(\dot{q}_i \delta p_i + p_i \delta \dot{q}_i - \frac{\partial L}{\partial q_i} \delta q_i - \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) = \sum_i (\dot{q}_i \delta p_i - \dot{p}_i \delta q_i),$$

where from the Euler–Lagrange equations $\dot{p}_i = \partial L / \partial \dot{q}_i$. Comparing the LHS and RHS we obtain *Hamilton's equations*

$$\boxed{\frac{\partial H}{\partial p_i} = \dot{q}_i \quad \text{and} \quad \frac{\partial H}{\partial q_i} = -\dot{p}_i}.$$

For functions $A(p, q)$ and $B(p, q)$ we define the *Poisson Bracket* as

$$\{A, B\} \equiv \sum_i \left(\frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} - \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} \right);$$

Hamilton's equations then can be expressed as

$$\boxed{\dot{q}_i = \{H, q_i\} \quad \text{and} \quad \dot{p}_i = \{H, p_i\}}.$$

Notice the similarity in form to the Heisenberg Operator equations of motion in quantum mechanics.

The generalization to a system of N particles is also straightforward. We now have a set of generalized coordinates, q_i^α , where α labels the different particles with masses m^α , say. We can simplify the notation by simply extending the range of the index i from 1, 2, 3 to 1, 2, ..., 3N with the conventions that $m_1 = m_2 = m_3 \equiv m^1$, etc.

Exercise: Establish the equivalence to the more conventional approach by showing that the Euler–Lagrange equation for a single point particle is just Newton's equation of motion.

Continuous Systems

The analysis of the previous section is fine for systems with a finite number of degrees of freedom, such as a set of N massive particles connected by a massless string, executing transverse vibrations, as shown in Figure 1(a), but how do we deal with a continuous system, such as a massive string vibrating transversely as in Figure 1(b)? In one space dimension we have an amplitude ϕ which depends on a continuous ‘label’ x . Here, ϕ plays the role of q , while x is analogous to α (or i). As you might guess, the correct procedure is to replace

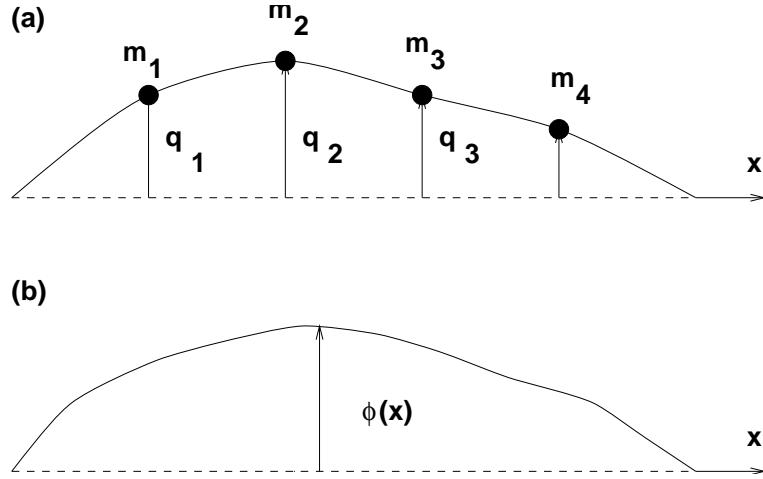


Figure 1: *Transverse oscillations of (a) a set of point masses connected by a massless string and (b) a massive string.*

summation over i by integration over x , or more generally, a three-dimensional integral over the spatial coordinates. The Lagrangian becomes not a sum but an integral over a *Lagrangian density* \mathcal{L} , which depends, in general, on the field, ϕ , and its derivatives with respect to space and time, which we will denote in the general case by $\partial_\mu \phi$. L is said to be a *functional* of ϕ and $\partial_\mu \phi$. Thus, for our string

$$L = \int_0^\ell dx \mathcal{L}(\phi, \dot{\phi}, \phi'),$$

where $\dot{\phi} = \partial\phi/\partial t$ and $\phi' = \partial\phi/\partial x$.

The action integral is

$$S = \int_{t_1}^{t_2} dt \int_0^\ell dx \mathcal{L}(\phi, \dot{\phi}, \phi').$$

If we consider small variations, $\delta\phi(x, t)$, which vanish at t_1 and t_2 , and also at $x = 0$ and $x = \ell$, the fixed end points of the string, we find that

$$\delta S = \int_{t_1}^{t_2} dt \int_0^\ell dx \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \frac{\partial}{\partial t} \delta\phi + \frac{\partial \mathcal{L}}{\partial \phi'} \frac{\partial}{\partial x} \delta\phi \right).$$

Integrating the second term by parts with respect to t and the third term with respect to x yields

$$\delta S = \int_{t_1}^{t_2} dt \int_0^\ell dx \left[\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) - \frac{\partial}{\partial x} \left(\frac{\partial \mathcal{L}}{\partial \phi'} \right) \right] \delta\phi(x, t).$$

Again, we can only satisfy Hamilton's Principle for arbitrary small variations of ϕ if the integrand vanishes identically, yielding the Euler–Lagrange equations

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) - \frac{\partial}{\partial x} \left(\frac{\partial \mathcal{L}}{\partial \phi'} \right) = 0}.$$

Classical Relativistic Fields

Let us now consider the more general case of a classical scalar field, $\phi(x)$. We restrict ourselves to considering theories which can be derived using Hamilton's Principle from an action integral involving a Lorentz-invariant functional \mathcal{L} of the field ϕ and its first derivatives $\partial_\mu \phi$,

$$\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi).$$

We define the action integral S_Ω for an arbitrary region Ω of space-time by

$$\boxed{S_\Omega = \int_{\Omega} d^4x \mathcal{L}(\phi, \partial_\mu \phi)}.$$

Proceeding as before, we consider variations $\delta\phi(x)$ which vanish on the surface $\partial\Omega$ bounding Ω . The corresponding change in the action must vanish

$$\delta S_\Omega = \int_{\Omega} d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta\partial_\mu \phi \right] = 0$$

which, on noting that

$$\delta\partial_\mu \phi = \partial_\mu \delta\phi,$$

can be integrated by parts to give

$$\delta S_\Omega = \int_{\Omega} d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) \right] \delta\phi = 0.$$

Requiring this to hold for arbitrary variations $\delta\phi$ and regions Ω leads to the Euler–Lagrange equations

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) = 0}.$$

This is the equation of motion of the field ϕ .

The canonical ‘momentum’ corresponding to ϕ is defined by

$$\pi(x) \equiv \frac{\partial \mathcal{L}(\phi, \partial_\mu \phi)}{\partial \dot{\phi}(x)},$$

and the Hamiltonian is written as the volume integral over a Hamiltonian density

$$H = \int d^3x \mathcal{H}(\pi(x), \phi(x)), \quad \mathcal{H} = \pi\dot{\phi} - \mathcal{L}.$$

More generally, for a system of n independent fields $\phi_r(x)$, $r = 1, 2, \dots, n$ we obtain n field equations

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi_r} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_r)} \right) = 0, \quad r = 1, 2, \dots, n}.$$

The Klein-Gordon Field

As our first example, we show that the Klein–Gordon equation can be obtained as the Euler–Lagrange equation of motion of a classical field $\phi(x)$ described by the Lagrangian density

$$\boxed{\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2}.$$

Proof.

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi, \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi;$$

giving the Euler–Lagrange equation

$$\partial_\mu\partial^\mu\phi + m^2\phi = 0,$$

which we can write as

$$(\square^2 + m^2)\phi = 0.$$

The first term in \mathcal{L} is usually referred to as the *kinetic term*, while the second term is known as the *mass term*.

The field or ‘momentum density’ conjugate to $\phi(x)$ is given by

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}(x),$$

yielding a Hamiltonian density

$$\mathcal{H} = \pi(x)\dot{\phi}(x) - \mathcal{L}(\phi, \partial_\mu\phi) = \frac{1}{2}[\pi^2(x) + (\nabla\phi)^2 + m^2\phi^2].$$

The Free Electromagnetic Field

Consider the Lagrangian density

$$\boxed{\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}.$$

This can be written

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\mu A_\nu)(\partial^\nu A^\mu).$$

Treating the components of the 4-vector potential A_ν as independent fields, we have

$$\frac{\partial\mathcal{L}}{\partial A_\nu} = 0 \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\partial^\mu A^\nu + \partial^\nu A^\mu;$$

giving as the Euler–Lagrange equations

$$\partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) = -\partial_\mu \partial^\mu A^\nu + \partial^\nu (\partial_\mu A^\mu) = 0,$$

which we can rewrite as

$$\square^2 A^\nu - \partial^\nu (\partial_\mu A^\mu) = 0,$$

recognizable as equivalent to Maxwell's equations in the absence of sources.

The canonically conjugate field is

$$\pi^\mu(x) = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = -F^{0\mu}(x).$$

Note. The antisymmetry of the Maxwell tensor implies that $\pi^0(x) \equiv 0$, which will lead to problems later when we try to quantize the electromagnetic field by imposing canonical commutation relations!

In the presence of sources, corresponding to the electromagnetic current 4-vector $j_{\text{em}}^\mu(x)$, we can derive the field equations from the Lagrangian density

$$\boxed{\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_{\text{em}}^\mu(x)A_\mu(x)}.$$

It is left as an exercise to show that the corresponding Euler–Lagrange equations are

$$\square^2 A^\nu - \partial^\nu(\partial_\mu A^\mu) = j_{\text{em}}^\nu.$$

The Dirac Field

The Lagrangian density can be taken to be

$$\boxed{\mathcal{L} = \bar{\psi}(x) \left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m \right) \psi(x)}$$

if we treat $\psi(x)$ and $\bar{\psi}(x)$ as independent fields. Varying with respect to $\bar{\psi}$ gives

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = \left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m \right) \psi(x) \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial (\partial_\mu \bar{\psi})} = 0$$

so that the Euler–Lagrange equation is just the Dirac equation:

$$\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m \right) \psi(x) = 0.$$

If instead we vary with respect to the field $\psi(x)$ we find that

$$\frac{\partial \mathcal{L}}{\partial \psi} = -m\bar{\psi}(x) \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} = i\bar{\psi}(x)\gamma^\mu,$$

giving an Euler–Lagrange equation for the adjoint spinor

$$\boxed{i\frac{\partial \bar{\psi}(x)}{\partial x^\mu}\gamma^\mu + m\bar{\psi}(x) = 0}.$$

The conjugate fields, or ‘momentum densities’, corresponding to ψ and $\bar{\psi}$ are then

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\psi^\dagger \quad \text{and} \quad \bar{\pi}(x) = \frac{\partial \mathcal{L}}{\partial \dot{\bar{\psi}}} = 0.$$

The Hamiltonian density is thus

$$\mathcal{H} = \pi(x)\dot{\psi}(x) - \mathcal{L} = i\psi^\dagger(x)\dot{\psi}(x) - \mathcal{L} = i\bar{\psi}(x)\gamma^0 \frac{\partial \psi}{\partial x^0} - \mathcal{L} = \bar{\psi}(x) \left(-i\gamma^i \frac{\partial}{\partial x^i} + m \right) \psi(x).$$

Symmetries and Noether's Theorem

If the action is invariant under a group of symmetry transformations then we can construct corresponding conserved quantities. For example, we can show that invariance of the action, or equivalently of the equations of motion, under translations in time leads to energy conservation while spatial translational invariance gives rise to momentum conservation.

Consider the simplest case where \mathcal{L} itself is invariant under the field variation

$$\phi(x) \rightarrow \phi'(x) = \phi(x) + \delta\phi(x).$$

Now the variation in \mathcal{L} arising from variation of the fields is

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right),$$

where we have used the Euler–Lagrange equation in the second step.

Invariance implies that $\delta\mathcal{L} = 0$ and hence the continuity equation

$$\partial_\mu j^\mu(x) = 0$$

with

$$j^\mu(x) = \boxed{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi}.$$

The corresponding “charges”

$$Q = \int d^3x j^0(x)$$

are constant in time: from the continuity equation

$$\dot{Q} = \int d^3x \frac{\partial}{\partial x^0} j^0(x) = - \int d^3x \frac{\partial}{\partial x^i} j^i(x).$$

Using the divergence theorem, we can rewrite the RHS as a surface integral and assuming that the fields, and hence the currents, vanish sufficiently fast at infinity as usual, we obtain

$$\dot{Q} = 0.$$

More generally, we can allow \mathcal{L} to be invariant up to a 4-divergence

$$\mathcal{L} \rightarrow \mathcal{L} + \partial_\mu\Lambda^\mu \tag{3}$$

for some Λ^μ . It is straightforward to show that the Euler–Lagrange equations of motion are invariant under such a change. Setting

$$\delta\mathcal{L} = \partial_\mu\Lambda^\mu$$

the conserved current is

$$j^\mu = \boxed{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi - \Lambda^\mu}.$$

This result is an example of *Noether's Theorem*: for each continuous symmetry of \mathcal{L} , there is a conserved current.

We can also apply these considerations to space-time transformations, such as infinitesimal translations

$$x^\mu \rightarrow x'^\mu = x^\mu + a^\mu$$

which induce a transformation of the scalar fields

$$\phi(x) \rightarrow \phi(x + a) = \phi(x) + a^\mu \partial_\mu \phi(x).$$

The Lagrangian density is also a scalar and so must transform in the same way

$$\mathcal{L} \rightarrow \mathcal{L} + a^\mu \partial_\mu \mathcal{L} = \mathcal{L} + a^\nu \partial_\mu (\delta^\mu_\nu \mathcal{L}).$$

Comparing with equation 3 we see that

$$\partial_\mu \Lambda^\mu = a^\nu \partial_\mu (\delta^\mu_\nu \mathcal{L}).$$

We now have four separately conserved currents. For example, if we consider time translations so that $\nu = 0$ only, we have $\delta\phi = a^0 \partial_0 \phi$ and the conserved current is

$$j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} a^0 \partial_0 \phi - a^0 \delta^\mu_0 \mathcal{L}.$$

We obtain similar conserved currents for the three spatial translations so that, scaling out the irrelevant parameters, we can write the four conserved currents as

$$T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L} \delta^\mu_\nu,$$

one for each value of the index ν . We can also write this as

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - \mathcal{L} g^{\mu\nu}.$$

This is the so-called *stress-energy tensor* or *energy-momentum tensor* of the field ϕ . The conserved “charge” associated with the $\nu = 0$ component is just our old friend, the Hamiltonian

$$\int d^3x T^{00} = \int d^3x \left(\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial^0 \phi - \mathcal{L} \right) = \int d^3x \mathcal{H} = H$$

and is associated with time translations.

The conserved charges associated with spatial translations are

$$P^i = \int d^3x T^{0i} = \int d^3x \pi \partial^i \phi = - \int d^3x \pi \partial_i \phi,$$

and we interpret them as the components of the momentum carried by the field.

Relativistic Quantum Field Theory

Covariance of Relativistic Wave Equations

Covariance of a relativistic equation means the *form* of the equation is the same in all inertial frames

Let us consider two inertial coordinate systems x^μ and x'^μ , which are connected by the Lorentz transformation Λ_ν^μ

$$x'^\mu = \Lambda^\mu_\nu x^\nu. \quad (1)$$

(Convince yourself by doing some explicit examples that $x_\nu = \Lambda^\mu_\nu x'_\mu$).

Klein-Gordon Equation

Let us denote by $\phi(x)$ the scalar field as observed in the x^μ coordinate system and by $\phi'(x')$ the same field as observed at the same physical point, but in the x'^μ coordinate system. Then for a scalar field these two values are equal

$$\phi'(x') = \phi(x). \quad (2)$$

This relation defines the function $\phi'(x')$ in terms of the function $\phi(x)$. In particular, by using this and equation (1) we get $\phi'(x') = \phi(\Lambda^{-1}x')$.

Recall the Klein–Gordon equation from Lecture 1

$$\left(\frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x_\mu} + m^2 \right) \phi(x) = 0. \quad (3)$$

We note the chain rule for partial derivatives

$$\frac{\partial}{\partial x^\mu} = \frac{\partial x'^\nu}{\partial x^\mu} \frac{\partial}{\partial x'^\nu} = \Lambda^\nu_\mu \frac{\partial}{\partial x'^\nu}; \quad (4)$$

thus writing equation (3) in the prime coordinates we obtain

$$\left(\Lambda^\nu_\mu \Lambda^\mu_{\bar{\nu}} \frac{\partial}{\partial x'^\nu} \frac{\partial}{\partial x'^{\bar{\nu}}} + m^2 \right) \phi'(x') = 0, \quad (5)$$

where we used equation (2). Recalling that $\Lambda^\nu_\mu \Lambda^\mu_{\bar{\nu}} = g^\nu_{\bar{\nu}}$ we see that equations (3) and (5) are of the same form.

Electromagnetic Vector Potential Equation

In this case

$$A'^\mu(x') = \Lambda^\mu_\nu A^\nu(x),$$

so that

$$F'^{\mu\nu}(x') = \Lambda^\mu_{\bar{\mu}} \Lambda^\nu_{\bar{\nu}} F^{\bar{\mu}\bar{\nu}}(x).$$

Recall the equation of motion for the field tensor

$$\frac{\partial}{\partial x^\mu} F^{\mu\nu}(x) = 0;$$

expressing this equation in terms of the prime coordinates, it becomes

$$\Lambda^{\bar{\mu}}{}_\mu \frac{\partial}{\partial x'^{\bar{\mu}}} \Lambda_{\bar{\mu}}{}^\mu \Lambda_{\bar{\nu}}{}^\nu F'^{\bar{\mu}\bar{\nu}}(x') = 0. \quad (6)$$

Noting that $\Lambda^{\bar{\mu}}{}_\mu \Lambda_{\bar{\mu}}{}^\mu = g^{\bar{\mu}\bar{\mu}}$, equation (6) becomes

$$\Lambda_{\bar{\nu}}{}^\nu \frac{\partial}{\partial x'^{\bar{\mu}}} F'^{\bar{\mu}\bar{\nu}}(x') = 0. \quad (7)$$

Multiplying both sides by $\Lambda^{\bar{\nu}}{}_\nu$, we then obtain the desired form

$$\frac{\partial}{\partial x'^{\bar{\mu}}} F'^{\bar{\mu}\bar{\nu}}(x') = 0.$$

Dirac Equation

Recall from Lecture 1 the covariant form of the Dirac equation

$$\boxed{(i\gamma^\mu \partial_\mu - m) \psi(x) = 0} \quad \text{or} \quad \boxed{(i\cancel{\partial} - m) \psi(x) = 0}. \quad (8)$$

If the Dirac equation is truly covariant, we expect it to have the same form in any inertial frame. In particular, if we take the wavefunction $\psi(x)$ in equation (8) to describe a particular physical state of an electron, as determined by an observer in an inertial frame F , we expect that a second observer in an inertial frame F' will describe the same state by a wavefunction $\psi'(x')$ satisfying a similar equation

$$\left(i\tilde{\gamma}^\mu \frac{\partial}{\partial x'^\mu} - m \right) \psi'(x') = 0, \quad (9)$$

where the matrix coefficients $\tilde{\gamma}^\mu$ satisfy the same Clifford algebra

$$\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} \equiv \tilde{\gamma}^\mu \tilde{\gamma}^\nu + \tilde{\gamma}^\nu \tilde{\gamma}^\mu = 2g^{\mu\nu}$$

and hence can be taken without loss of generality to be the same representation as in F .

The problem is then to find the relation between $\psi(x)$ and $\psi'(x')$. Both are 4-component column matrices, so that we expect there to be some 4×4 matrix $S(\Lambda)$ which relates them and depends in some way on the Lorentz transformation Λ connecting the two inertial frames,

$$\psi'(x') = \psi'(\Lambda x) = S(\Lambda)\psi(x) = S(\Lambda)\psi(\Lambda^{-1}x'). \quad (10)$$

Now the equivalence of the frames F and F' ensures that S must be non-singular and thus

$$\psi(x) = S(\Lambda)^{-1}\psi'(x') = S(\Lambda)^{-1}\psi'(\Lambda x);$$

but we could equally well write, using equation (10),

$$\psi(x) = S(\Lambda^{-1})\psi'(\Lambda x)$$

and so we conclude that

$$S(\Lambda^{-1}) = S(\Lambda)^{-1}.$$

Having established the properties of S , we now have to find it!

Using these last two results, we can rewrite the Dirac equation in F , equation (8), as

$$\left(i\gamma^\mu S(\Lambda)^{-1} \frac{\partial}{\partial x^\mu} - m S(\Lambda)^{-1} \right) \psi'(x') = 0.$$

If we left multiply this by $S(\Lambda)$ we obtain

$$\left(iS(\Lambda) \gamma^\mu S(\Lambda)^{-1} \frac{\partial}{\partial x^\mu} - m \right) \psi'(x') = 0;$$

using the chain rule for partial derivatives (4) we can write

$$\left(iS(\Lambda) \Lambda^\nu_\mu \gamma^\mu S(\Lambda)^{-1} \frac{\partial}{\partial x^\nu} - m \right) \psi'(x') = 0.$$

Comparing this with equation (9), we see that it is identical provided that

$$S(\Lambda) \Lambda^\nu_\mu \gamma^\mu S(\Lambda)^{-1} = \gamma^\nu$$

which can equally well be written

$$\boxed{\Lambda^\mu_\nu \gamma^\nu = S(\Lambda)^{-1} \gamma^\mu S(\Lambda)}. \quad (11)$$

Covariance under boosts

As a special case of a homogeneous Lorentz transformation, consider a boost in standard orientation in the *negative* x^1 direction, for which Λ has the form

$$\Lambda = \begin{pmatrix} \cosh \omega & \sinh \omega & 0 & 0 \\ \sinh \omega & \cosh \omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

then the claim is that the corresponding transformation matrix, which we denote by S_B , is given by

$$\boxed{S_B = \exp\left(\alpha^1 \frac{\omega}{2}\right) = \exp\left(\gamma^0 \gamma^1 \frac{\omega}{2}\right)}.$$

Partial Proof. We first note that the exponential can be rewritten as follows

$$\begin{aligned} \exp\left(\alpha^1 \frac{\omega}{2}\right) &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\alpha^1 \omega}{2}\right)^n = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \left(\frac{\omega}{2}\right)^{2n} + \sum_{n=0}^{\infty} \frac{\alpha^1}{(2n+1)!} \left(\frac{\omega}{2}\right)^{2n+1} \\ &= \cosh \frac{\omega}{2} + \alpha^1 \sinh \frac{\omega}{2} = \begin{pmatrix} \cosh \frac{\omega}{2} & \sigma^1 \sinh \frac{\omega}{2} \\ \sigma^1 \sinh \frac{\omega}{2} & \cosh \frac{\omega}{2} \end{pmatrix}, \end{aligned}$$

where we have used the standard representation and the fact that $(\alpha^1)^2 = 1$.

Now using the result that $S(\Lambda^{-1}) = S(\Lambda)^{-1}$ we have for the case $\mu = 0$

$$\begin{aligned} S_B^{-1}\gamma^0 S_B &= \begin{pmatrix} \cosh \frac{\omega}{2} & -\sigma^1 \sinh \frac{\omega}{2} \\ -\sigma^1 \sinh \frac{\omega}{2} & \cosh \frac{\omega}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cosh \frac{\omega}{2} & \sigma^1 \sinh \frac{\omega}{2} \\ \sigma^1 \sinh \frac{\omega}{2} & \cosh \frac{\omega}{2} \end{pmatrix} \\ &= \begin{pmatrix} \cosh \frac{\omega}{2} & -\sigma^1 \sinh \frac{\omega}{2} \\ -\sigma^1 \sinh \frac{\omega}{2} & \cosh \frac{\omega}{2} \end{pmatrix} \begin{pmatrix} \cosh \frac{\omega}{2} & \sigma^1 \sinh \frac{\omega}{2} \\ -\sigma^1 \sinh \frac{\omega}{2} & -\cosh \frac{\omega}{2} \end{pmatrix} \\ &= \begin{pmatrix} \cosh \omega & \sigma^1 \sinh \omega \\ -\sigma^1 \sinh \omega & -\cosh \omega \end{pmatrix} = \gamma^0 \cosh \omega + \gamma^1 \sinh \omega, \end{aligned}$$

while

$$\Lambda^0{}_\nu \gamma^\nu = \gamma^0 \cosh \omega + \gamma^1 \sinh \omega;$$

thus

$$S_B^{-1}\gamma^0 S_B = \Lambda^0{}_\nu \gamma^\nu.$$

Similar arguments can be used to establish the result for the $\mu = 1, 2$ and 3 components.

It is easy to generalize this result to the case of a boost from the frame F to a frame F' moving with velocity $-\underline{v}$ relative to F

$$S_B = \exp \left(\frac{\underline{\alpha} \cdot \underline{v}}{|v|} \frac{\omega}{2} \right).$$

Covariance under rotations

As a second special case of a homogeneous Lorentz transformation, consider a rotation through an angle θ about the z axis, for which Λ has the form

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & \sin \theta & 0 \\ 0 & -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

the corresponding transformation matrix S_R is given by

$$S_R = \exp \left(i \Sigma^3 \frac{\theta}{2} \right).$$

The proof uses similar techniques to the case of boosts, noting that since $(\Sigma^3)^2 = 1$ we can write

$$S_R = \cos \frac{\theta}{2} + i \Sigma^3 \sin \frac{\theta}{2},$$

and so on. A noteworthy feature is the appearance of the half-angle, which means that a Dirac spinor changes sign under a rotation through 2π , and only returns to itself after a rotation through 4π .

The generalization to a rotation about an axis in the direction of a unit vector \underline{n} is also straightforward

$$S_R = \exp\left(i\Sigma \cdot \underline{n} \frac{\theta}{2}\right).$$

Note: The operator S_R is unitary, $S_R^\dagger = S_R^{-1}$, whereas S_B is not, $S_B^\dagger = S_B \neq S_B^{-1}$. However, both satisfy the relation

$$S^{-1} = \gamma^0 S^\dagger \gamma^0$$

which is therefore true in general, since any homogeneous Lorentz transformation can be built from a combination of boosts and rotations.

We can use this property to establish the transformation properties of the conjugate spinors

$$\bar{\psi}'(x') \equiv \psi'^\dagger(x')\gamma^0 = \psi^\dagger(x)S^\dagger(\Lambda)\gamma^0 = \psi^\dagger(x)\gamma^0S^{-1}(\Lambda) = \bar{\psi}(x)S^{-1}(\Lambda).$$

Dirac Bilinears

We can now establish the Lorentz transformation properties of various so-called *Dirac bilinears*, which are constructs of the generic form $\bar{\psi}(x)\Gamma\psi(x)$, where Γ denotes some combination of Dirac γ matrices. It will turn out that there is only a finite number (16) of linearly independent possibilities for constructing such objects.

Scalar: It is easy to see that $\bar{\psi}\psi$ is scalar

$$\bar{\psi}'(x')\psi'(x') = \bar{\psi}(x)S^{-1}(\Lambda)S(\Lambda)\psi(x) = \bar{\psi}(x)\psi(x).$$

Vector: $j^\mu \equiv \bar{\psi}\gamma^\mu\psi$ transforms as a 4-vector under Lorentz transformations

$$j'^\mu(x') \equiv \bar{\psi}'(x')\gamma^\mu\psi'(x') = \bar{\psi}(x)S^{-1}(\Lambda)\gamma^\mu S(\Lambda)\psi(x) = \bar{\psi}(x)\Lambda^\mu{}_\nu\gamma^\nu\psi(x) = \Lambda^\mu{}_\nu j^\nu(x).$$

Tensor: It is useful to construct a set of 4×4 matrices from the Dirac γ -matrices as follows

$$\sigma^{\mu\nu} \equiv \frac{i}{2} [\gamma^\mu, \gamma^\nu].$$

By construction, $\sigma^{\nu\mu} = -\sigma^{\mu\nu}$, so that there are only six distinct matrices, which we could take to be $\sigma^{01}, \sigma^{02}, \sigma^{03}, \sigma^{12}, \sigma^{13}, \sigma^{23}$. It then follows that the bilinear $\bar{\psi}\sigma^{\mu\nu}\psi$ transforms as an antisymmetric rank 2 tensor under Lorentz transformations. This is left as an exercise.

Pseudoscalar: If we consider the bilinear $\bar{\psi}\gamma^5\psi$ under proper Lorentz transformations it appears to behave as a scalar. This can be seen by noting that

$$S_B^{-1}\gamma^5 S_B = \gamma^5 \quad \text{and} \quad S_R^{-1}\gamma^5 S_R = \gamma^5 \rightarrow S^{-1}\gamma^5 S = \gamma^5.$$

However, life gets more complicated if we admit improper transformations such as parity. Suppose we denote by P the matrix which relates the Dirac wavefunctions in the frames F and F' when they are related simply by spatial inversion

$$\psi'(x') = P\psi(x) \quad \text{where } x'^0 = x^0, x'^i = -x^i.$$

The condition on P is as before

$$P^{-1}\gamma^\mu P = \Lambda^\mu_\nu \gamma^\nu,$$

where now Λ is given by

$$\Lambda = \text{diag}(1, -1, -1, -1).$$

You can easily verify that this is satisfied by the choice

$$P = e^{i\phi} \gamma^0,$$

where the phase factor $e^{i\phi}$ is of no physical significance and is usually taken to be ± 1 or $\pm i$. Note that P is unitary and satisfies

$$P^{-1} = \gamma^0 P^\dagger \gamma^0,$$

so the transformation of the conjugate spinor $\bar{\psi}$ is

$$\bar{\psi}'(x') \equiv \psi'(x')^\dagger \gamma^0 = [P\psi(x)]^\dagger \gamma^0 = \psi(x)^\dagger P^\dagger \gamma^0 = \bar{\psi} \gamma^0 P^\dagger \gamma^0 = \bar{\psi}(x) P^{-1}.$$

Moreover

$$\begin{aligned} P^{-1} \gamma_5 P &= P^{-1} i \gamma^0 \gamma^1 \gamma^2 \gamma^3 P \\ &= i P^{-1} \gamma^0 (P P^{-1}) \gamma^1 (P P^{-1}) \gamma^2 (P P^{-1}) \gamma^3 P \\ &= i (P^{-1} \gamma^0 P) (P^{-1} \gamma^1 P) (P^{-1} \gamma^2 P) (P^{-1} \gamma^3 P) \\ &= i (\Lambda^0_0 \gamma^0) (\Lambda^1_1 \gamma^1) (\Lambda^2_2 \gamma^2) (\Lambda^3_3 \gamma^3) \\ &= (\det \Lambda) (i \gamma^0 \gamma^1 \gamma^2 \gamma^3) = -\gamma^5. \end{aligned}$$

Note: The rest-frame spinors are eigenstates of P and the positive and negative energy solutions have opposite eigenvalues or *intrinsic parities*.

Now consider the transformation properties of the bilinear $\bar{\psi} \gamma^5 \psi$ under parity

$$\bar{\psi}'(x') \gamma^5 \psi'(x') = \bar{\psi}(x) P^{-1} \gamma^5 P \psi(x) = -\bar{\psi}(x) \gamma^5 \psi(x),$$

Thus we see that this bilinear has the characteristics of a *pseudoscalar* quantity: it transforms as a scalar under proper transformations, but picks up a minus sign under improper transformations.

Pseudovector: We now consider the bilinear $\bar{\psi} \gamma^5 \gamma^\mu \psi$

$$\bar{\psi}'(x') \gamma^5 \gamma^\mu \psi'(x') = \bar{\psi}(x) S^{-1} \gamma^5 S S^{-1} \gamma^\mu S \psi(x) = \Lambda^\mu_\nu \bar{\psi}(x) S^{-1} \gamma^5 S \gamma^\nu \psi(x).$$

We already know that for proper transformations $S^{-1} \gamma^5 S = \gamma^5$, so in this case we have

$$\bar{\psi}'(x') \gamma^5 \gamma^\mu \psi'(x') = \Lambda^\mu_\nu \bar{\psi}(x) \gamma^5 \gamma^\nu \psi(x)$$

with the bilinear transforming as a 4-vector. However, for spatial inversion we find that

$$\bar{\psi}'(x') \gamma^5 \gamma^\mu \psi'(x') = \bar{\psi}(x) P^{-1} \gamma^5 P P^{-1} \gamma^\mu P \psi(x) = -\Lambda^\mu_\nu \bar{\psi}(x) \gamma^5 \gamma^\nu \psi(x),$$

where here $\Lambda = \text{diag}(1, -1, -1, -1)$. We thus find that

$$\begin{aligned} \bar{\psi}'(x') \gamma^5 \gamma^0 \psi'(x') &= -\bar{\psi}(x) \gamma^5 \gamma^0 \psi(x) \\ \bar{\psi}'(x') \gamma^5 \gamma^i \psi'(x') &= \bar{\psi}(x) \gamma^5 \gamma^i \psi(x). \end{aligned}$$

The fact that the 3-vector part does not change sign under inversion is characteristic of a *pseudovector* or *axial vector* in three dimensions, and so this Dirac bilinear is known as a pseudovector.

Summary: The set of 16 bilinears that we have discussed is in fact a complete set. If you try to form more complicated bilinears, you will find that by applying the Dirac algebra, they can always be reduced to linear combinations of the following

$\bar{\psi}\psi$	scalar	1
$\bar{\psi}\gamma^\mu\psi$	vector	4
$\bar{\psi}\sigma^{\mu\nu}\psi$	tensor	6
$\bar{\psi}\gamma^5\gamma^\mu\psi$	axial vector	4
$\bar{\psi}\gamma^5\psi$	pseudoscalar	1

Covariant description of spin

Recall from the revision notes that the spin operator for a positive-energy spin- $\frac{1}{2}$ Dirac particle in its rest frame is $\frac{\hbar}{2}\underline{\Sigma}$. Recall also that Dirac spinors are eigenstates of Σ^3 in the particle's rest-frame only. This reflects the fact that spin is only well-defined in the rest frame of a particle. How then are we to characterise the states of a free, spin- $\frac{1}{2}$ particle in a covariant fashion?

In the rest frame we can always construct spinors describing particles polarized along any desired direction \check{s} , where \check{s} is a unit vector, by a simple generalization of the procedures used in NRQM. Let us denote the 4-momentum in the rest frame by $\check{p} = (m, \underline{0})$. Then, by construction

$$\underline{\Sigma} \cdot \underline{s} u(\check{p}, \check{s}) = u(\check{p}, \check{s}). \quad (12)$$

Now introduce the Lorentz transformation which takes us from the rest frame to a frame in which the particle has 4-momentum p^μ

$$p^\mu = \Lambda^\mu{}_\nu \check{p}^\nu;$$

the *polarization* or *spin 4-vector* is then *defined* to be

$$s^\mu = \Lambda^\mu{}_\nu \check{s}^\nu$$

where \check{s}^ν has components $(0, \underline{\check{s}})$. By construction, s^μ has the following properties

$$s^\mu s_\mu = \check{s}^\mu \check{s}_\mu = -1; \quad p^\mu s_\mu = \check{p}^\mu \check{s}_\mu = 0.$$

We still need to express the rest-frame polarization condition in a manifestly covariant form by finding an operator which

1. reduces to $\underline{\Sigma} \cdot \underline{s}$ in the rest frame;
2. commutes with \not{p} , since the spinor $u(p, s)$ is an eigenstate of \not{p} by virtue of the Dirac equation: $\not{p}u(p, s) = m u(p, s)$.

A suitable operator is $\gamma^5 \not{p} = \gamma^5 \gamma^\mu s_\mu$. In the rest frame in the standard representation we have

$$\gamma^5 \not{p} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\underline{\sigma} \cdot \check{s} \\ \underline{\sigma} \cdot \check{s} & 0 \end{pmatrix} = \begin{pmatrix} \underline{\sigma} \cdot \check{s} & 0 \\ 0 & -\underline{\sigma} \cdot \check{s} \end{pmatrix}$$

so that, at least for positive energy spinors, the rest-frame condition equation (12) may be written as

$$\gamma^5 \not{p} u(\check{p}, \check{s}) = u(\check{p}, \check{s}).$$

Notes: It is easy to show that the operator $\gamma^5 \not{p}$ has the properties,

$$[\gamma^5 \not{p}, \not{p}] = 0 \quad \text{and} \quad (\gamma^5 \not{p})^2 = 1 ;$$

so the general positive energy spinor $u(p, s)$ may be specified covariantly by the conditions

$$(\not{p} - m)u(p, s) = 0 \quad \text{and} \quad \gamma^5 \not{p} u(p, s) = u(p, s) .$$

We handle the negative energy (negative 4-momentum) solutions by letting $v(p, s)$ denote a negative energy solution with polarization in the direction $-\underline{s}$ in the rest frame, that is

$$\underline{\Sigma} \cdot \underline{s} v(p, s) = -v(p, s)$$

so that the covariant conditions may be written

$$(\not{p} + m)v(p, s) = 0 \quad \text{and} \quad \gamma^5 \not{p} v(p, s) = v(p, s) .$$

Relation between relativistic equations and the Lorentz group *

A Group G is a set with a rule for assigning to every (ordered) pair of elements a third element, satisfying

- if $f, g \in G$ then $h = fg \in G$ (closure);
- for $f, g, h \in G$, $f(gh) = (fg)h$ (associativity);
- There is an identity element 1, such that $1f = f1 = f \forall f \in G$;
- every element $f \in G$ has an inverse f^{-1} such that $ff^{-1} = f^{-1}f = 1$.

The Lorentz transformations are 4×4 matrices Λ that relate how coordinates in two inertial reference frames are related $x'^\nu = \Lambda^\nu_\mu x^\mu$ (some examples of Lorentz transformations are given in the revision notes). Lorentz transformations form a continuous group called $\text{SO}(3, 1)$. It has an infinite number of elements, since the parameters of rotation and boost take on a continuum of values. A general Lorentz transformation has 6 real parameters $(\underline{\theta}, \underline{\omega})$ to which there correspond 6 generators

$$J_1 = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad J_2 = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad J_3 = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$K_1 = -i \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = -i \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_3 = -i \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The generators of rotation \underline{J} are Hermitian and those of boost \underline{K} are anti-Hermitian. Infinitesimal transformations are given by, for example for a boost in the z -direction

$$\Lambda(\delta\omega_3) = 1 + iK_3\delta\omega_3,$$

so that the corresponding finite boost is $\Lambda(\omega_3) = \exp(iK_3\omega_3)$. A general Lorentz transformation (group element) is

$$\Lambda = \exp(i\underline{J} \cdot \underline{\theta} + i\underline{K} \cdot \underline{\omega}). \quad (13)$$

The commutation relations of the generators can be calculated to give

$$\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk}J_k \\ [K_i, K_j] &= -i\epsilon_{ijk}J_k \\ [J_i, K_j] &= i\epsilon_{ijk}K_k. \end{aligned}$$

It is easier to understand the algebra by first defining

$$\begin{aligned} \underline{A} &\equiv \frac{1}{2}(\underline{J} + i\underline{K}) \\ \underline{B} &\equiv \frac{1}{2}(\underline{J} - i\underline{K}) \end{aligned}$$

then,

$$\begin{aligned}[A_i, A_j] &= i\epsilon_{ijk}A_k \\ [B_i, B_j] &= i\epsilon_{ijk}B_k \\ [A_i, B_j] &= 0.\end{aligned}$$

Observe that \underline{A} and \underline{B} both have the algebra of $SU(2)$ (compare to commutation relations for the Pauli matrices). Thus \underline{A} and \underline{B} each generate $SU(2)$, so the Lorentz algebra is $SU(2) \times SU(2)$. States transforming in a well-defined way will be labelled by two angular momenta (j, j') , the first corresponding to \underline{A} and second to \underline{B} .

Representations of $SU(2) \times SU(2)$

- $(0, 0)$ is the simplest and corresponds to the scalar field case. The field transforms trivially under a Lorentz transformation.
- $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ — called the *fundamental representation* — the algebra of $SU(2)$ for angular momentum $\frac{1}{2}$ is explicitly realized by the set of Pauli matrices $\frac{1}{2}\underline{\sigma}$. A general Lorentz transformation in terms of \underline{A} and \underline{B} can be written as

$$\Lambda = \exp(i\underline{A} \cdot \underline{\Theta}_1 + i\underline{B} \cdot \underline{\Theta}_2).$$

If this is compared to equation (13) it implies $\underline{\Theta}_1 = \underline{\theta} - i\underline{\omega}$ and $\underline{\Theta}_2 = \underline{\theta} + i\underline{\omega}$.

Examining now the two cases, under $(\frac{1}{2}, 0)$ one state transforms as spin $\frac{1}{2}$ and the other as a scalar; this requires $\underline{B} = 0$, which implies that in this representation $\underline{J} = \frac{1}{2}\underline{\sigma}$ and $\underline{K} = -\frac{i}{2}\underline{\sigma}$. Thus a spinor ϕ_1 transforms as

$$\phi_1 \rightarrow \exp\left(\frac{i}{2}\underline{\sigma} \cdot (\underline{\theta} - i\underline{\omega})\right) \phi_1 \equiv M\phi_1. \quad (14)$$

Similarly for $(0, \frac{1}{2})$ the spinor ϕ_2 associated with it transforms as

$$\phi_2 \rightarrow \exp\left(\frac{i}{2}\underline{\sigma} \cdot (\underline{\theta} + i\underline{\omega})\right) \phi_2 \equiv N\phi_2. \quad (15)$$

If we want to consider also improper Lorentz transformations, we can no longer consider these 2-spinors separately. For example under parity, velocity $\underline{v} \rightarrow -\underline{v}$, hence the generator \underline{K} changes sign $\underline{K} \rightarrow -\underline{K}$ like a vector but \underline{J} does not change sign $\underline{J} \rightarrow \underline{J}$ like an axial vector. Thus under parity the representations get interchanged $(j, 0) \leftrightarrow (\bar{0}, j)$, thus $\phi_1 \leftrightarrow \phi_2$. To have a spinor in which a parity transformation can be done, it requires a 4-spinor

$$\psi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

which under Lorentz transformation behaves as

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \rightarrow \begin{pmatrix} \exp\left(\frac{i}{2}\underline{\sigma} \cdot (\underline{\theta} - i\underline{\omega})\right) & 0 \\ 0 & \exp\left(\frac{i}{2}\underline{\sigma} \cdot (\underline{\theta} + i\underline{\omega})\right) \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

and under parity

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

We will relabel the 2-spinors at this stage to a notation that better reflects their relation by parity transformation, thus

$$\phi_1 \rightarrow \phi_R \quad \phi_2 \rightarrow \phi_L.$$

Consider a spinor of mass m at rest $\phi_R(\underline{p} = 0)$ and a transformed one $\phi_R(\underline{p})$ with momentum \underline{p} , they are related by

$$\begin{aligned} \phi_R(\underline{p}) &= \exp\left(\frac{1}{2}\underline{\sigma} \cdot \underline{\omega}\right) \phi_R(0) \\ &= \left(\sqrt{\frac{\gamma+1}{2}} + \underline{\sigma} \cdot \hat{\underline{p}} \sqrt{\frac{\gamma-1}{2}}\right) \phi_R(0) \\ &= \frac{E+m+\underline{\sigma} \cdot \underline{p}}{\sqrt{2m(E+m)}} \phi_R(0). \end{aligned} \quad (16)$$

where we used $\cosh \frac{\omega}{2} = \sqrt{\frac{1}{2}(\gamma+1)}$, $\sinh \frac{\omega}{2} = \sqrt{\frac{1}{2}(\gamma-1)}$, and $\gamma = E(\underline{p})/m$. Similarly

$$\phi_L(\underline{p}) = \frac{E+m-\underline{\sigma} \cdot \underline{p}}{\sqrt{2m(E+m)}} \phi_L(0). \quad (17)$$

At rest, one cannot define spin as either left (L) or right (R) handed, so $\phi_R(0) = \phi_L(0)$, thus from equations (16) and (17) it implies

$$\begin{pmatrix} -m & p_0 + \underline{\sigma} \cdot \underline{p} \\ p_0 - \underline{\sigma} \cdot \underline{p} & -m \end{pmatrix} \begin{pmatrix} \phi_R(\underline{p}) \\ \phi_L(\underline{p}) \end{pmatrix} = 0, \quad (18)$$

which can be written as

$$(\gamma^\mu p_\mu - m)\psi(p) = 0, \quad (19)$$

where

$$\begin{aligned} \psi(p) &= \begin{pmatrix} \phi_R(\underline{p}) \\ \phi_L(\underline{p}) \end{pmatrix}, \\ \underline{\gamma} &= \begin{pmatrix} 0 & -\underline{\sigma} \\ \underline{\sigma} & 0 \end{pmatrix} \quad \text{and} \quad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \quad (20)$$

Equation (19) is the Dirac equation for massive spin- $\frac{1}{2}$ particles in momentum space. In particular equation (18) is in the Weyl representation. Noting that the matrices in equation (20) satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, one has freedom to choose other representations such as the Dirac representation given in Lecture 1 and primarily used in this course.

- $(\frac{1}{2}, \frac{1}{2})$ — this contains the four vector. To see this construct a general two index object $\psi_{\alpha\alpha'}$, in which the respective indices transform as $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, thus

$$\psi'_{\beta\beta'} = M_{\beta\alpha} N_{\beta'\alpha'} \psi_{\alpha\alpha'},$$

where M and N are the $SU(2)$ transformations defined in equations (14) and (15) respectively. Thus $\psi_{\alpha\alpha'}$ is a four component quantity. Consider the expression

$$A^\mu \equiv \text{tr}(\psi \sigma^2 \sigma^\mu) = \psi_{\alpha\alpha'} (\sigma^2 \sigma^\mu)_{\alpha'\alpha}, \quad (21)$$

where $\sigma^0 = \mathbb{I}$ and σ^i , $i = 1, 2, 3$, are the standard Pauli matrices. The above equation is simply re-expressing these four independent components of $\psi_{\alpha\alpha'}$ in a different way. Let us show that as implied by the notation A^μ indeed transforms like a Lorentz four-vector. Consider for example the case of a rotation ($\underline{\omega} = 0$)

$$\begin{aligned} A'^\mu &= \psi'_{\alpha\alpha'} (\sigma^2 \sigma^\mu)_{\alpha'\alpha} = M_{\alpha\beta} N_{\alpha'\beta'} \psi_{\beta\beta'} (\sigma^2 \sigma^\mu)_{\alpha'\alpha} \\ &\approx \left(1 + \frac{i}{2} \underline{\sigma} \cdot (\underline{\theta} - i\underline{\omega})\right)_{\alpha\beta} \left(1 + \frac{i}{2} \underline{\sigma} \cdot (\underline{\theta} + i\underline{\omega})\right)_{\alpha'\beta'} \psi_{\beta\beta'} (\sigma^2 \sigma^\mu)_{\alpha'\alpha} \\ &= A^\mu + \frac{i}{2} \underline{\theta} \cdot \left(\underline{\sigma}_{\alpha\beta} (\sigma^2 \sigma^\mu)_{\beta'\alpha} + \underline{\sigma}_{\alpha'\beta'} (\sigma^2 \sigma^\mu)_{\alpha'\beta}\right) \psi_{\beta\beta'} \\ &= A^\mu + \frac{i}{2} \underline{\theta} \cdot (\sigma^2 \sigma^\mu \underline{\sigma} + \underline{\sigma}^T \sigma^2 \sigma^\mu)_{\beta'\beta} \psi_{\beta\beta'}. \end{aligned}$$

Now observe that $\underline{\sigma}^T \sigma^2 = -\sigma^2 \underline{\sigma}$, so

$$A'^\mu \approx A^\mu + \frac{i}{2} \underline{\theta} \cdot (\sigma^2 [\sigma^\mu \underline{\sigma} - \underline{\sigma} \sigma^\mu])_{\beta'\beta} \psi_{\beta\beta'} = A^\mu + \frac{i}{2} \underline{\theta} \cdot (\sigma^2 [\sigma^\mu, \underline{\sigma}])_{\beta'\beta} \psi_{\beta\beta'}.$$

For $\mu = 0$ the commutator in the last line is zero, thus $A'^0 = A^0$ as expected for a rotation. For the three spatial directions recall that for Pauli matrices

$$[\sigma^i, \sigma^k] = 2i\epsilon^{ijk}\sigma^j$$

so that

$$A'^i \approx A^i - \theta^k \epsilon^{ikj} [(\sigma^2 \sigma^j)_{\beta'\beta} \psi_{\beta\beta'}] = A^i - \theta^k \epsilon^{ikj} A^j = (1 + i[J_k]^i{}_j \theta^k) A^j,$$

from comparison with equation (21) and using the explicit form for J given at the beginning of this lecture. Similarly one can show for a Lorentz boost that equation (21) transforms like a Lorentz four-vector (tutorial problem). A product of rotations and boosts can then generate any arbitrary Lorentz transformation.

Relativistic Quantum Field Theory

Canonical Quantization

Quantum Mechanics of Point Particles

A standard approach to quantizing the dynamics of a point particle is to introduce the canonical momenta p_i conjugate to the generalized coordinates q_i

$$p_i = \frac{\partial L}{\partial \dot{q}_i},$$

and then to define the Hamiltonian as the Legendre transform

$$H = \sum_i p_i \dot{q}_i - L.$$

The quantities q_i , p_i , and H are then interpreted as Hermitian operators on the Hilbert space of physical states and the following algebraic relations, known as *Canonical Commutation Relations* (CCRs), are imposed

$$\boxed{[q_i, p_j] = i\delta_{ij}} \quad \text{and} \quad \boxed{[q_i, q_j] = [p_i, p_j] = 0}.$$

The time development of states is determined by the Hamiltonian H through the Schrödinger equation

$$i \frac{d}{dt} |\Psi, t\rangle = H |\Psi, t\rangle;$$

this scheme is referred to as the *Schrödinger Picture* (SP).

It will prove more useful in general to work in the *Heisenberg Picture* (HP) in which the time development of the system is realized through the time evolution of the operators, and the states are time-independent. The relationship between the two pictures is defined by picking an arbitrary time t_0 at which the pictures coincide, and making a time-dependent unitary transformation of states and observables. In the SP we can write a formal solution of the Schrödinger equation

$$|\Psi, t\rangle = e^{-iH(t-t_0)} |\Psi, t_0\rangle;$$

the HP observable $A(t)$ that coincides with the SP observable A at time t_0 is then

$$A(t) = e^{iH(t-t_0)} A e^{-iH(t-t_0)},$$

while the HP and SP states are related via

$$|\Psi\rangle^{\text{HP}} = e^{iH(t-t_0)} |\Psi, t\rangle^{\text{SP}} = |\Psi, t_0\rangle^{\text{SP}}.$$

It is trivial to show that this preserves expectation values

$$\langle A \rangle_t^{\text{HP}} = \langle A \rangle_t^{\text{SP}}.$$

For the canonical coordinates and momenta we have

$$q_i(t) = e^{iH(t-t_0)} q_i e^{-iH(t-t_0)} \quad \text{and} \quad p_i(t) = e^{iH(t-t_0)} p_i e^{-iH(t-t_0)}.$$

The Hamiltonian itself remains unaltered in the HP

$$H(t) = e^{iH(t-t_0)} H e^{-iH(t-t_0)} = H,$$

which means that we are free to construct the Hamiltonian from HP operators at an arbitrary time t .

Of course, the canonical commutation relations still hold for HP operators at an arbitrary time t

$$[q_i(t), p_j(t)] = e^{iH(t-t_0)} [q_i, p_j] e^{-iH(t-t_0)} = i \delta_{ij} \quad \text{and} \quad [q_i(t), q_j(t)] = [p_i(t), p_j(t)] = 0.$$

A general HP operator obeys the *Heisenberg Equation of Motion*

$$\left[\frac{d}{dt} A(t) = i [H, A(t)] \right].$$

Exercise: Show that, for a single point particle of mass m , moving in a conservative force field $V(q)$ in one dimension

$$\dot{q}(t) = \frac{p(t)}{m} \quad \text{and} \quad \dot{p}(t) = -\frac{\partial V}{\partial q}.$$

These equations are formally the same as the classical Hamiltonian equations of motion; this result holds for any sufficiently simple dynamical system. In particular, it will be true for the quantum field theories that we consider.

Quantum Fields

We interpret the field $\phi(t, \underline{x})$ and its conjugate $\pi(t, \underline{x})$ as Heisenberg operators satisfying the usual CCRs. The Kronecker δ on the right-hand side becomes a 3-dimensional Dirac δ -function for the case of continuous fields, as can be shown by approximating the continuous system by a discrete lattice of small cells and then taking the continuum limit in which the cell size shrinks to zero (see Mandl & Shaw for details), leading to the *equal time commutation relations* (ETCRs)

$$[\phi(t, \underline{x}), \pi(t, \underline{x}')] = i\delta(\underline{x} - \underline{x}') \quad \text{and} \quad [\phi(t, \underline{x}), \phi(t, \underline{x}')] = [\pi(t, \underline{x}), \pi(t, \underline{x}')] = 0.$$

For the Klein–Gordon field this reduces to

$$[\phi(t, \underline{x}), \dot{\phi}(t, \underline{x}')] = i\delta(\underline{x} - \underline{x}') \quad \text{and} \quad [\phi(t, \underline{x}), \phi(t, \underline{x}')] = [\dot{\phi}(t, \underline{x}), \dot{\phi}(t, \underline{x}')] = 0.$$

Equations of Motion

The Heisenberg equation of motion for the scalar field is

$$\dot{\phi}(t, \underline{x}) = i [H, \phi(t, \underline{x})] = i \int d^3x' [\mathcal{H}(t, \underline{x}'), \phi(t, \underline{x})].$$

Making use of the CCRs we have

$$\begin{aligned} [\mathcal{H}(t, \underline{x}'), \phi(t, \underline{x})] &= \left[\frac{1}{2}\pi(t, \underline{x}')^2, \phi(t, \underline{x}) \right] \\ &= \frac{1}{2} \left(\pi(t, \underline{x}') [\pi(t, \underline{x}'), \phi(t, \underline{x})] + [\pi(t, \underline{x}'), \phi(t, \underline{x})] \pi(t, \underline{x}') \right) \\ &= -i\pi(t, \underline{x}') \delta(\underline{x} - \underline{x}'). \end{aligned}$$

Substituting in the equations of motion gives

$$\dot{\phi}(t, \underline{x}) = \int d^3x' \pi(t, \underline{x}') \delta(\underline{x} - \underline{x}') = \pi(t, \underline{x}).$$

Similarly, we have

$$\dot{\pi}(t, \underline{x}) = i [H, \pi(t, \underline{x})] = i \int d^3x' [\mathcal{H}(t, \underline{x}'), \pi(t, \underline{x})];$$

using the CCRs gives

$$\begin{aligned} [\mathcal{H}(t, \underline{x}'), \pi(t, \underline{x})] &= \left[\frac{1}{2} \left(\nabla' \phi(t, \underline{x}') \right)^2 + \frac{1}{2} m^2 \phi(t, \underline{x}')^2, \pi(t, \underline{x}) \right] \\ &= \frac{1}{2} \left(\nabla' \phi(t, \underline{x}') \cdot \nabla' [\phi(t, \underline{x}'), \pi(t, \underline{x})] + \nabla' [\phi(t, \underline{x}'), \pi(t, \underline{x})] \cdot \nabla' \phi(t, \underline{x}') \right) \\ &\quad + m^2 \phi(t, \underline{x}') [\phi(t, \underline{x}'), \pi(t, \underline{x})] + m^2 [\phi(t, \underline{x}'), \pi(t, \underline{x})] \phi(t, \underline{x}') \\ &= i \left(\nabla' \phi(t, \underline{x}') \cdot \nabla' \delta(\underline{x} - \underline{x}') + m^2 \phi(t, \underline{x}') \delta(\underline{x} - \underline{x}') \right), \end{aligned}$$

where ∇' denotes differentiation with respect to the components of \underline{x}' . Substituting into the equation of motion gives

$$\dot{\pi}(t, \underline{x}) = - \int d^3x' \left(\nabla' \phi(t, \underline{x}') \cdot \nabla' \delta(\underline{x} - \underline{x}') + m^2 \phi(t, \underline{x}') \delta(\underline{x} - \underline{x}') \right);$$

and integration by parts on the derivative term yields

$$\dot{\pi}(t, \underline{x}) = \nabla^2 \phi(t, \underline{x}) - m^2 \phi(t, \underline{x}).$$

Using the equation of motion for ϕ then gives

$$\ddot{\phi}(t, \underline{x}) = \dot{\pi}(t, \underline{x}) = (\nabla^2 - m^2) \phi(t, \underline{x}),$$

which implies that the Heisenberg operator $\phi(x)$ satisfies the Klein–Gordon equation

$$(\square^2 + m^2) \phi(x) = 0.$$

Mode Operators

Now that we know that the scalar field operator $\phi(x)$ satisfies the Klein–Gordon equation, we can expand it in terms of the complete set of positive energy plane-wave solutions in a manifestly covariant way

$$\begin{aligned} \phi(t, \underline{x}) &= \int \frac{d^4p}{(2\pi)^4} 2\pi \delta(p^2 - m^2) \theta(p^0) (a(\underline{p}) e^{-ip \cdot x} + a(\underline{p})^\dagger e^{ip \cdot x}) \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega(\underline{p})} \int dp^0 \delta(p^0 - \omega(\underline{p})) (a(\underline{p}) e^{-ip \cdot x} + a(\underline{p})^\dagger e^{ip \cdot x}) \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega(\underline{p})} (a(\underline{p}) e^{-ip \cdot x} + a(\underline{p})^\dagger e^{ip \cdot x}), \end{aligned}$$

where $\omega(\underline{p}) \equiv +\sqrt{\underline{p}^2 + m^2}$, and in the last line we implicitly set $p^0 = \omega(\underline{p})$. The coefficients $a(\underline{p})$ and $a(\underline{p})^\dagger$ are (time-independent) operators chosen so as to ensure that ϕ is Hermitian.

Note: Our conventions for the expansion differ from those of Mandl & Shaw, and of Peskin & Schroeder.

The conjugate field is then

$$\pi(x) = \dot{\phi}(x) = -\frac{i}{2} \int \frac{d^3 p}{(2\pi)^3} (a(\underline{p}) e^{-ip \cdot x} - a^\dagger(\underline{p}) e^{ip \cdot x}).$$

We see that

$$\begin{aligned} \int d^3 x \phi(t, \underline{x}) e^{ip' \cdot x} &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega(\underline{p})} \int d^3 x (a(\underline{p}) e^{i(p' - p) \cdot x} + a(\underline{p})^\dagger e^{i(p' + p) \cdot x}) \\ &= \int \frac{d^3 p}{2\omega(\underline{p})} (a(\underline{p}) \delta(\underline{p}' - \underline{p}) e^{i(\omega' - \omega)t} + a(\underline{p})^\dagger \delta(\underline{p}' + \underline{p}) e^{i(\omega' + \omega)t}) \\ &= \frac{1}{2\omega(\underline{p}')} (a(\underline{p}') + a(-\underline{p}')^\dagger e^{2i\omega't}), \end{aligned}$$

where $(p')^0 = \omega(\underline{p}')$, we have abbreviated $\omega = \omega(\underline{p})$ and $\omega' = \omega(\underline{p}')$, and we have used the completeness relation

$$\int d^3 x \exp(-ip \cdot x) = (2\pi)^3 \delta(\underline{p}).$$

Likewise we find

$$\int d^3 x \pi(x) e^{ip' \cdot x} = -\frac{i}{2} (a(\underline{p}') - a(-\underline{p}')^\dagger e^{2i\omega't}),$$

so we obtain

$$a(\underline{p}) = \int d^3 x e^{ip \cdot x} (\omega(\underline{p}) \phi(t, \underline{x}) + i\pi(t, \underline{x}))$$

and its Hermitian conjugate

$$a(\underline{p})^\dagger = \int d^3 x e^{-ip \cdot x} (\omega(\underline{p}) \phi(t, \underline{x}) - i\pi(t, \underline{x})).$$

We may now calculate the commutators for $a(\underline{p})$ and $a(\underline{p})^\dagger$

$$\begin{aligned} [a(\underline{p}), a(\underline{p}')^\dagger] &= \int d^3 x \int d^3 x' e^{ip \cdot x - ip' \cdot x'} [\omega(\underline{p}) \phi(t, \underline{x}) + i\pi(t, \underline{x}), \omega(\underline{p}') \phi(t, \underline{x}') - i\pi(t, \underline{x}')] \\ &= \int d^3 x \int d^3 x' e^{ip \cdot x - ip' \cdot x'} (\omega(\underline{p}) + \omega(\underline{p}')) \delta(\underline{x} - \underline{x}'). \end{aligned}$$

We have thus established that

$$[a(\underline{p}), a^\dagger(\underline{p}')] = 2\omega(\underline{p}) (2\pi)^3 \delta(\underline{p} - \underline{p}');$$

similarly we find

$$[a(\underline{p}), a(\underline{p}')] = [a^\dagger(\underline{p}), a^\dagger(\underline{p}')] = 0.$$

Thus we see that the commutation relations between the mode operators look like those of the quantum harmonic oscillator annihilation and creation operators.

Quantized Real Scalar Field & Particle Interpretation

Let us now express the Hamiltonian in terms of Fourier mode operators. Let

$$H = \frac{1}{2} \int d^3x \left[\pi(x)^2 + \left(\nabla \phi(x) \right)^2 + m^2 \phi(x)^2 \right] \equiv H_1 + H_2 + H_3,$$

and consider the first term

$$H_1 = -\frac{1}{8} \int d^3x \int \frac{d^3p}{(2\pi)^3} (a(\underline{p})e^{-ip \cdot x} - a(\underline{p})^\dagger e^{ip \cdot x}) \int \frac{d^3p'}{(2\pi)^3} (a(\underline{p}')e^{-ip' \cdot x} - a(\underline{p}')^\dagger e^{ip' \cdot x}),$$

recalling that we have implicitly set $p_0 = \omega(\underline{p})$. The spatial integration gives rise to two types of term, time-dependent and time-independent,

$$\begin{aligned} \int d^3x e^{\pm i(p+p') \cdot x} &= (2\pi)^3 \delta(\underline{p} + \underline{p}') \exp(\pm 2i\omega(\underline{p})t) \\ \int d^3x e^{\pm i(p-p') \cdot x} &= (2\pi)^3 \delta(\underline{p} - \underline{p}'). \end{aligned}$$

Since we know that all time-dependent terms in H must cancel (they do — exercise!), we will evaluate only the time-independent terms

$$\begin{aligned} H_1 &\rightarrow -\frac{1}{8} \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} (2\pi)^3 \delta(\underline{p} - \underline{p}') (-a(\underline{p})a(\underline{p}')^\dagger - a(\underline{p})^\dagger a(\underline{p}')) \\ &= \frac{1}{8} \int \frac{d^3p}{(2\pi)^3} (a(\underline{p})a(\underline{p})^\dagger + a(\underline{p})^\dagger a(\underline{p})). \end{aligned}$$

The other two terms in H can be dealt with similarly and it is left as a tutorial exercise to show that the time-independent contributions are

$$\begin{aligned} H_2 &\rightarrow \frac{1}{8} \int \frac{d^3p}{(2\pi)^3} \frac{|\underline{p}|^2}{\omega(\underline{p})^2} (a(\underline{p})a(\underline{p})^\dagger + a(\underline{p})^\dagger a(\underline{p})) \\ H_3 &\rightarrow \frac{1}{8} \int \frac{d^3p}{(2\pi)^3} \frac{m^2}{\omega(\underline{p})^2} (a(\underline{p})a(\underline{p})^\dagger + a(\underline{p})^\dagger a(\underline{p})), \end{aligned}$$

so that on adding the three contributions we obtain

$$H = \frac{1}{4} \int \frac{d^3p}{(2\pi)^3} (a(\underline{p})a(\underline{p})^\dagger + a(\underline{p})^\dagger a(\underline{p})).$$

It is now easy to show that the Fourier mode operators do indeed have the properties that we associate with the raising and lowering operators of the quantum harmonic oscillator

$$\begin{aligned} [H, a(\underline{p})^\dagger] &= \frac{1}{4} \int \frac{d^3p'}{(2\pi)^3} \left\{ a(\underline{p}')a(\underline{p}')^\dagger a(\underline{p})^\dagger + a(\underline{p}')^\dagger a(\underline{p}')a(\underline{p})^\dagger \right. \\ &\quad \left. - a(\underline{p})^\dagger a(\underline{p}')a(\underline{p}')^\dagger - a(\underline{p})^\dagger a(\underline{p}')^\dagger a(\underline{p}') \right\} \\ &= \frac{1}{4} \int \frac{d^3p'}{(2\pi)^3} \left\{ a(\underline{p}')a(\underline{p})^\dagger a(\underline{p}')^\dagger + a(\underline{p}')^\dagger a(\underline{p}')a(\underline{p})^\dagger \right. \\ &\quad \left. - a(\underline{p})^\dagger a(\underline{p}')a(\underline{p}')^\dagger - a(\underline{p}')^\dagger a(\underline{p})^\dagger a(\underline{p}') \right\} \\ &= \frac{1}{4} \int \frac{d^3p'}{(2\pi)^3} \{ [a(\underline{p}'), a(\underline{p})^\dagger] a(\underline{p}')^\dagger + a(\underline{p}')^\dagger [a(\underline{p}'), a(\underline{p})^\dagger] \} \\ &= \frac{1}{4} \int d^3p' 2\omega(\underline{p}) \{ \delta(\underline{p} - \underline{p}')a(\underline{p}')^\dagger + a(\underline{p}')^\dagger \delta(\underline{p}' - \underline{p}) \} = \omega(\underline{p})a(\underline{p})^\dagger; \end{aligned}$$

similarly we can show that

$$[H, a(\underline{p})] = -\omega(\underline{p})a(\underline{p}).$$

In effect, therefore, we have a quantum harmonic oscillator for each of the infinitely many Fourier modes of the field.

Particle Interpretation

The particle interpretation of the quantum theory follows from the observation that the mode operator $a(\underline{p})^\dagger$ acting on an energy eigenstate, with eigenvalue E , increases the energy of that state by $\omega(\underline{p}) = \sqrt{\underline{p}^2 + m^2}$ (and the momentum by \underline{p}), that is, it *creates* a “particle” of 4-momentum p . Explicitly

$$Ha(\underline{p})^\dagger |E\rangle = (a(\underline{p})^\dagger H + \omega(\underline{p})a(\underline{p})^\dagger) |E\rangle = (E + \omega(\underline{p})) a(\underline{p})^\dagger |E\rangle.$$

Likewise, the mode operator $a(\underline{p})$ *annihilates* a “particle” of 4-momentum p . Since the Hamiltonian is non-negative, we may define a ground state — the state of lowest energy — which is annihilated by the operators $a(\underline{p})$,

$$a(\underline{p}) |0\rangle = 0 \quad \forall \underline{p}. \tag{1}$$

There is one immediate undesirable side effect. We can calculate the ground-state energy by rewriting the Hamiltonian as

$$H = \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} (a(\underline{p})^\dagger a(\underline{p}) + \frac{1}{2} [a(\underline{p}), a(\underline{p})^\dagger]) = \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} (N(\underline{p}) + \omega(\underline{p})(2\pi)^3 \delta(\underline{p} - \underline{p}))$$

where the so-called *number-density* operators $N(\underline{p})$ are defined by

$$N(\underline{p}) \equiv a(\underline{p})^\dagger a(\underline{p}).$$

When the first term in H is applied to the ground state it yields zero by virtue of equation (1). However, we see that the second term is an infinite constant: for each mode it is proportional to $\delta(0)$, and performing the integral over all possible modes makes matters worse.

This implies that the ground state, usually referred to as the *vacuum state*, has infinite energy. The source of this infinity is clear: it is the sum (i.e., integral) of the zero-point energies of an infinite number of Fourier modes, each of which extends over the whole of space. However, since experiments only measure energy *differences* from the vacuum state, we may argue that we are free to ignore this infinite constant in calculations.

The Spectrum of the Real Scalar Field

The energy eigenstates are built from the vacuum or zero-particle state by acting on $|0\rangle$ with creation operators. The state $a(\underline{p})^\dagger a(\underline{q})^\dagger \dots |0\rangle$ is an eigenstate of H with energy $\omega(\underline{p}) + \omega(\underline{q}) + \dots$ and such states exhaust the spectrum. It is also clear that the particles satisfy Bose-Einstein statistics since the creation operators all commute.

We saw previously that the physical momentum operator is given by

$$\underline{P} = - \int d^3 x \pi(x) \nabla \phi(x).$$

We can obtain an expression for \underline{p} in terms of the mode operators by substituting the mode expansions of the fields ϕ and π , as we did for the Hamiltonian. In our shorthand notation, the result is (tutorial)

$$\underline{P} = \frac{1}{4} \int \frac{d^3 p}{(2\pi)^3} \frac{\underline{p}}{\omega(\underline{p})} (a(\underline{p})a(\underline{p})^\dagger + a(\underline{p})^\dagger a(\underline{p})) .$$

Again this can be rewritten in terms of the number-density operators:

$$\underline{P} = \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \frac{\underline{p}}{\omega(\underline{p})} (N(\underline{p}) + \frac{1}{2} [a(\underline{p}), a(\underline{p})^\dagger]) .$$

This time the commutator term does not pose a problem since the contribution from modes with momentum \underline{p} and $-\underline{p}$ cancel in the sum (i.e., integral) over modes — as expected, the vacuum has zero total momentum.

The state $a(\underline{p})^\dagger a(\underline{q})^\dagger \cdots |0\rangle$ is an eigenstate of \underline{p} with momentum $\underline{p} + \underline{q} + \cdots$

N.B.: the ‘physical’ momentum operator \underline{P} should not be confused with the canonical ‘momentum density’ $\pi(x)$.

Summary: This calculation shows that the operator $a(\underline{p})^\dagger$ “creates” momentum \underline{p} and energy $\omega(\underline{p}) = \sqrt{(\underline{p})^2 + m^2}$, so it is natural to interpret these quantum excitations of the field as *particles* since the free-particle mass-shell condition is obeyed. From now on, therefore, we will refer to $\omega(\underline{p})$ as $E_{\underline{p}}$ or just E , since it is interpreted as a particle energy.

Occupation Numbers

The Hermitian number-density operators $N(\underline{p})$ have readily-established properties

1. The eigenvalues of $N(\underline{p})$ are non-negative;
2. The eigenvalues are proportional to $n(\underline{p})$, where $n(\underline{p}) = 0, 1, 2, 3, \dots$ and are known as *occupation numbers*;
3. Multi-particle states may be specified in terms of the occupation numbers of each mode or momentum state: if the allowed momenta are denoted $\underline{p}_1, \underline{p}_2, \underline{p}_3, \dots$ then the states are $|n_1, n_2, n_3, \dots\rangle$ and the total number of particles is $n = n_1 + n_2 + n_3 + \dots$

The occupation number representation is important in many-body theory treatments of condensed matter & nuclear physics. It is sometimes called *Fock space*.

Normalization of States

The natural choice for normalization of the vacuum state is

$$\langle 0 | 0 \rangle = 1.$$

The one-particle states $|\underline{p}\rangle \propto a(\underline{p})^\dagger |0\rangle$ will feature prominently and so it is useful to fix their normalization. We choose the *Lorentz-invariant normalization*

$$|\underline{p}\rangle = a(\underline{p})^\dagger |0\rangle$$

so that

$$\langle \underline{p} | \underline{q} \rangle = \langle 0 | a(\underline{p}) a(\underline{q})^\dagger | 0 \rangle = \langle 0 | [a(\underline{p}), a(\underline{q})^\dagger] | 0 \rangle = (2\pi)^3 2E_{\underline{p}} \delta(\underline{p} - \underline{q}).$$

Why is this a Lorentz-invariant normalization? Consider a boost in the negative 3-direction

$$\begin{aligned} p'_1 &= p_1 \\ p'_2 &= p_2 \\ p'_3 &= \gamma(p_3 + \beta E) \\ E' &= \gamma(E + \beta p_3) \end{aligned}$$

and use the delta function identity

$$\delta(f(x) - f(x_0)) = \frac{1}{|f'(x_0)|} \delta(x - x_0)$$

to find the effect of the Lorentz boost on the 3-dimensional delta function

$$\begin{aligned} \delta(\underline{p} - \underline{q}) &= \delta(\underline{p}' - \underline{q}') \cdot \frac{dp'_3}{dp_3} = \delta(\underline{p}' - \underline{q}') \gamma \left(1 + \beta \frac{dE}{dp_3} \right) \\ &= \delta(\underline{p}' - \underline{q}') \frac{\gamma}{E} (E + \beta p_3) = \delta(\underline{p}' - \underline{q}') \frac{E'}{E}, \end{aligned}$$

where we observed that $E^2 = |\underline{p}|^2 + m^2$ so $dE/dp_3 = p_3/E$. Thus the quantity $E\delta(\underline{p} - \underline{q})$ is Lorentz-invariant.

This normalization has some important consequences. For example, the completeness relation for 1-particle states becomes

$$\hat{1} |_{\text{1-particle}} = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_{\underline{p}}} |\underline{p}\rangle \langle \underline{p}|.$$

It is also possible to show that $\frac{d^3 p}{(2\pi)^3 2E_{\underline{p}}}$ is a Lorentz-invariant integration measure by rewriting

$$\int \frac{d^3 p}{(2\pi)^3} \frac{f(E_{\underline{p}}, \underline{p})}{2E_{\underline{p}}} = \int \frac{d^4 p}{(2\pi)^4} 2\pi \delta(p^2 - m^2) f(p) \quad p^0 > 0$$

(the proof is left as an exercise).

Normal Ordering

We saw in the last lecture that when we expressed the Hamiltonian operator in terms of the number operators, we encountered an infinite contribution to the energy, associated with the zero-point energy of the Fourier modes. Our prescription was simply to subtract off the troublesome term. A formal way of handling this is to introduce the idea of the *normal*

product of field operators. In a normal product of field operators, all annihilation operators stand to the right of all creation operators.

Example: We first split the field into so-called positive and negative frequency parts

$$\phi(x) = \phi^+(x) + \phi^-(x),$$

where

$$\phi^+(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega(\underline{p})} a(\underline{p}) e^{-ip \cdot x} \quad \text{and} \quad \phi^-(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega(\underline{p})} a(\underline{p})^\dagger e^{ip \cdot x}.$$

The reason for the names is that if the exponential factors were single-particle wavefunctions in quantum mechanics, they would correspond to states of positive and negative energy (frequency) respectively. Note, though, that p^0 is always positive and that the spectrum of the Hamiltonian contains only positive energies. A similar decomposition can be carried out on $\pi(x)$.

Consider a product of field operators

$$\begin{aligned} \phi(x)\phi(y) &= (\phi^+(x) + \phi^-(x))(\phi^+(y) + \phi^-(y)) \\ &= \phi^+(x)\phi^+(y) + \phi^+(x)\phi^-(y) + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^-(y). \end{aligned}$$

The normal ordered form has in each term the creation operators commuted to the left of any annihilation operators; thus the normal ordered form of this product is

$$:\phi(x)\phi(y): = \phi^+(x)\phi^+(y) + \phi^-(y)\phi^+(x) + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^-(y).$$

Note that

$$\boxed{\langle 0|:\phi(x)\phi(y):|0\rangle = 0},$$

and quite generally the vacuum expectation value of any normal ordered product will vanish.

We can now see that the subtraction of the zero-point energy term from the Hamiltonian is equivalent to normal ordering the Hamiltonian

$$H \rightarrow :H: = H - \langle 0|H|0\rangle.$$

From now on, we will always use the normal-ordered form for H and \underline{p} implicitly if not explicitly.

Multiplets of Scalar Fields

If we have several scalar fields $\phi_r(x)$ $r = 1, 2, \dots, N$, each of mass m , then quantization proceeds through a straightforward generalization:

$$\boxed{\mathcal{L} = \frac{1}{2} \left[(\partial_\mu \phi_r(x)) (\partial^\mu \phi_r(x)) - m^2 \phi_r(x)^2 \right]},$$

where there is an implied sum over the label r . The conjugate fields are

$$\pi_r(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_r(x)} = \dot{\phi}_r(x),$$

and the ETCRs are

$$\begin{aligned} [\phi_r(t, \underline{x}), \pi_s(t, \underline{x}')] &= i\delta_{rs} \delta(\underline{x} - \underline{x}') \\ \text{and} \quad [\phi_r(t, \underline{x}), \phi_s(t, \underline{x}')] &= [\pi_r(t, \underline{x}), \pi_s(t, \underline{x}')] = 0. \end{aligned}$$

The mode expansion is

$$\phi_r(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega(\underline{p})} (a_r(\underline{p}) e^{-ip \cdot x} + a_r(\underline{p})^\dagger e^{ip \cdot x})$$

with CCRs for the mode operators

$$\begin{aligned} [a_r(\underline{p}), a_s(\underline{p}')^\dagger] &= \delta_{rs} 2\omega(\underline{p}) (2\pi)^3 \delta(\underline{p} - \underline{p}') \\ [a_r(\underline{p}), a_s(\underline{p}')] &= [a_r(\underline{p})^\dagger, a_s(\underline{p}')^\dagger] = 0. \end{aligned}$$

The conserved 4-momentum operator is

$$P^\nu = \int d^3 x T^{0\nu}(x),$$

with

$$T^{\mu\nu}(x) = : \left[\left(\partial^\mu \phi_r(x) \right) \left(\partial^\nu \phi_r(x) \right) - g^{\mu\nu} \mathcal{L}(x) \right]: \quad (\text{summed over } r).$$

As usual

$$[H, a_r(\underline{p})^\dagger] = E a_r(\underline{p})^\dagger \quad \text{and} \quad [H, a_r(\underline{p})] = -E a_r(\underline{p}),$$

so that $a_r(\underline{p})^\dagger$ creates particles of energy E

$$a_r(\underline{p})^\dagger |0\rangle = |\underline{p}, r\rangle.$$

One subtlety: the number operator for r -particles is that

$$N_r(\underline{p}) = a_r(\underline{p})^\dagger a_r(\underline{p}) \quad \text{with no sum over } r.$$

Complex Scalar Fields

The complex or non-Hermitian scalar field $\phi(x)$ can be expressed in terms of two Hermitian fields ϕ_1 and ϕ_2 : conventionally

$$\phi(x) = \frac{\phi_1(x) + i\phi_2(x)}{\sqrt{2}}.$$

The Lagrangian density is then

$$\boxed{\mathcal{L} = \left(\partial_\mu \phi(x)^\dagger \right) \left(\partial^\mu \phi(x) \right) - m^2 \phi(x)^\dagger \phi(x)}.$$

The fields $\phi(x)$ and $\phi(x)^\dagger$ are treated as independent degrees of freedom so that the field conjugate to $\phi(x)$ is

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)} = \dot{\phi}(x)^\dagger,$$

and that conjugate to $\phi(x)^\dagger$ is

$$\pi(x)^\dagger = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)^\dagger} = \dot{\phi}(x).$$

The ETCRs are

$$[\phi(t, \underline{x}), \pi(t, \underline{x}')] = [\phi(t, \underline{x})^\dagger, \pi(t, \underline{x}')^\dagger] = i \delta(\underline{x} - \underline{x}')$$

with all other commutators zero.

The Hamiltonian is

$$\begin{aligned} H &= \int d^3x \left(\pi(x)^\dagger \dot{\phi}(x)^\dagger + \pi(x) \dot{\phi}(x) - \mathcal{L}(x) \right) \\ &= \int d^3x \left(\pi(x)^\dagger \pi(x) + \nabla \phi(x)^\dagger \cdot \nabla \phi(x) + m^2 \phi(x)^\dagger \phi(x) \right), \end{aligned}$$

and the Heisenberg equations of motion are

$$(\square^2 + m^2)\phi(x) = 0 \quad \text{and} \quad (\square^2 + m^2)\phi(x)^\dagger = 0.$$

The mode expansion is

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega(\underline{p})} (a(\underline{p}) e^{-i\underline{p} \cdot \underline{x}} + b(\underline{p})^\dagger e^{i\underline{p} \cdot \underline{x}}).$$

Note that we can no longer assume the Hermiticity relation between the operator coefficients of the positive and negative frequency terms because $\phi(x)$ is not Hermitian.

The CCRs of the mode operators which reproduce the ETCRs are

$$[a(\underline{p}), a(\underline{p}')^\dagger] = [b(\underline{p}), b(\underline{p}')^\dagger] = 2\omega(\underline{p})(2\pi)^3 \delta(\underline{p} - \underline{p}')$$

and

$$[a(\underline{p}), b(\underline{p}')] = [a(\underline{p}), b(\underline{p}')^\dagger] = [a(\underline{p}), a(\underline{p}')] = [b(\underline{p}), b(\underline{p}')] = 0$$

together with others derived from these by Hermitian conjugation.

The operators $a(\underline{p})^\dagger$ and $b(\underline{p})^\dagger$ act as creation operators for two types of particle, with $a(\underline{p})$ and $b(\underline{p})$ the corresponding annihilation operators, as may be deduced from the commutators

$$\begin{aligned} [H, a(\underline{p})^\dagger] &= E a(\underline{p})^\dagger, & [H, a(\underline{p})] &= -E a(\underline{p}) \\ [H, b(\underline{p})^\dagger] &= E b(\underline{p})^\dagger, & [H, b(\underline{p})] &= -E b(\underline{p}). \end{aligned}$$

The conserved 4-momentum operator is

$$P^\nu = \int d^3x T^{0\nu}(x)$$

with

$$T^{\mu\nu}(x) = : \partial^\mu \phi(x)^\dagger \partial^\nu \phi(x) + \partial^\nu \phi(x)^\dagger \partial^\mu \phi(x) - \mathcal{L}(x) :.$$

In terms of mode operators,

$$P^\nu = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega(\underline{p})} p^\nu (a(\underline{p})^\dagger a(\underline{p}) + b(\underline{p})^\dagger b(\underline{p})).$$

Charge Conservation

In addition to the conserved 4-momentum, there is also a conserved charge

$$Q = \int d^3x j^0(x)$$

where

$$j^\mu(x) = i : \left[\phi(x)^\dagger \left(\partial^\mu \phi(x) \right) - \left(\partial^\mu \phi(x)^\dagger \right) \phi(x) \right] : \quad \text{and} \quad \partial_\mu j^\mu = 0.$$

In terms of the mode operators

$$Q = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega(\underline{p})} \left(a(\underline{p})^\dagger a(\underline{p}) - b(\underline{p})^\dagger b(\underline{p}) \right).$$

It is easy to establish that

$$\begin{aligned} [Q, a(\underline{p})^\dagger] &= a(\underline{p})^\dagger, & [Q, a(\underline{p})] &= -a(\underline{p}) \\ [Q, b(\underline{p})^\dagger] &= -b(\underline{p})^\dagger, & [Q, b(\underline{p})] &= b(\underline{p}), \end{aligned}$$

so that $a(\underline{p})^\dagger$ creates particles with ‘charge’ +1 while $b(\underline{p})^\dagger$ creates particles of ‘charge’ −1. This leads us to interpret the non-Hermitian field ϕ as a charged field.

We can identify the origin of the conserved charge as being the invariance of the Lagrangian density under the phase transformations

$$\begin{aligned} \phi(x) &\rightarrow \phi'(x) = e^{i\alpha} \phi(x) \approx (1 + i\alpha) \phi(x) \\ \phi(x)^\dagger &\rightarrow \phi'(x)^\dagger = e^{-i\alpha} \phi(x)^\dagger \approx (1 - i\alpha) \phi(x)^\dagger \quad \text{for small } \alpha. \end{aligned}$$

Thus $\delta\phi(x) = i\alpha\phi(x)$ and $\delta\phi(x)^\dagger = -i\alpha\phi(x)^\dagger$. Such transformations, where the phase α is a constant independent of space-time are called *global phase transformations* or *rigid phase transformations* or *gauge transformations of the first kind*.

Since the Lagrangian density itself is invariant $\Lambda^\mu = 0$

$$\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta\phi = i\alpha (\partial^\mu \phi^\dagger) \phi \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^\dagger} \delta\phi^\dagger = -i\alpha (\partial^\mu \phi) \phi^\dagger.$$

Rescaling by a factor $-\alpha$ yields a conserved current

$$j^\mu = i(\partial^\mu \phi) \phi^\dagger - i(\partial^\mu \phi^\dagger) \phi$$

which takes the advertised form after normal ordering.

Covariant Commutation Relations

The various canonical commutation relations between fields and their conjugate momentum densities are *equal time commutation relations*, so it is not immediately obvious that they lead to a Lorentz covariant quantum field theory. For the case of the real scalar field we shall illustrate the covariance by computing the commutator $[\phi(x), \phi(y)]$ for arbitrary space-time points x and y .

We first split ϕ into its positive and negative frequency parts $\phi = \phi^+ + \phi^-$, and note that

$$[\phi^+(x), \phi^+(y)] = [\phi^-(x), \phi^-(y)] = 0$$

since ϕ^+ only contains annihilation operators and ϕ^- only contains creation operators; thus

$$[\phi(x), \phi(y)] = [\phi^+(x), \phi^-(y)] + [\phi^-(x), \phi^+(y)].$$

Focusing on the first term, we have

$$\begin{aligned} [\phi^+(x), \phi^-(y)] &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega(\underline{p})} \int \frac{d^3 p'}{(2\pi)^3} \frac{1}{2\omega(\underline{p}')} [a(\underline{p}), a(\underline{p}')^\dagger] e^{-ip \cdot x + ip' \cdot y} \\ &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega(\underline{p})} e^{-ip \cdot (x-y)}; \end{aligned}$$

we therefore define the invariant function

$$\boxed{\Delta^+(x) \equiv -i \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega(\underline{p})} e^{-ip \cdot x}} \quad (2)$$

such that

$$[\phi^+(x), \phi^-(y)] = i\Delta^+(x - y).$$

Similarly

$$[\phi^-(x), \phi^+(y)] = -i\Delta^+(y - x) \equiv i\Delta^-(x - y)$$

where

$$\boxed{\Delta^-(x) \equiv \Delta^+(x)^* = i \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega(\underline{p})} e^{ip \cdot x}} \quad (3)$$

With these definitions, we can write

$$[\phi(x), \phi(y)] = i\Delta(x - y)$$

with Δ defined by

$$\boxed{\Delta(x) \equiv \Delta^+(x) + \Delta^-(x) = -2 \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega(\underline{p})} \sin p \cdot x}.$$

We see that $\Delta(x)$ is a odd real function of x as required by the commutation relations.

Note that Δ satisfies the Klein-Gordon equation

$$(\partial_x^2 + m^2)\Delta(x - y) = 0,$$

as do Δ^\pm , and may be written in manifestly covariant form as (tutorial)

$$\Delta(x) = -i \int \frac{d^4 p}{(2\pi)^4} (2\pi) \delta(p^2 - m^2) \varepsilon(p_0) e^{-ip \cdot x}$$

where $d^4 p = dp_0 d^3 p$ with the p_0 integration over $-\infty < p_0 < \infty$, and $\varepsilon(p_0)$ is defined by

$$\varepsilon(p_0) = \frac{p_0}{|p_0|} = \begin{cases} +1 & p_0 > 0 \\ -1 & p_0 < 0. \end{cases}$$

The invariance of Δ under proper Lorentz transformations is manifest, since each factor in the integrand is Lorentz-invariant (including $\varepsilon(p_0)$ since proper Lorentz transformations do not interchange past and future).

The vanishing of the ETCR

$$[\phi(t, \underline{x}), \phi(t, \underline{y})] = 0 = i\Delta(0, \underline{x} - \underline{y})$$

is an example of the more general result that the commutator of two fields vanishes for all space-like $x^\mu - y^\mu$, that is for $(x - y)^2 < 0$,

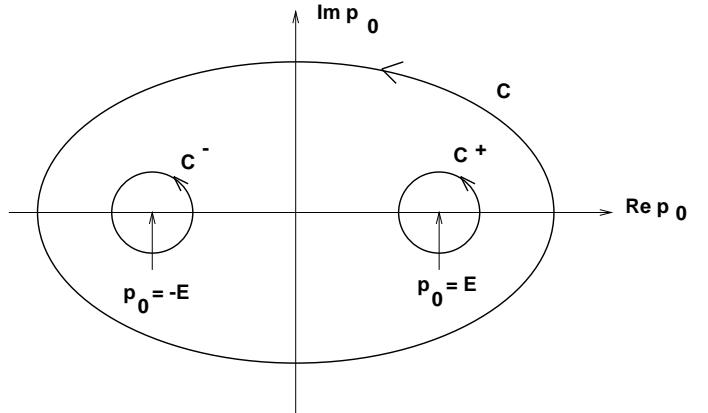
$$[\phi(x), \phi(y)] = 0, \quad (x - y)^2 < 0.$$

This result is known as the *microcausality condition*, since it implies that if the field is an observable then measurements at two points with space-like separation *cannot interfere with each other*: to do so, a signal would have to propagate at greater than the speed of light. For those interested the discussion of causality in Peskin & Schroeder is recommended.

It will prove useful to represent the various invariant Δ functions as contour integrals in the complex p_0 plane. We may write

$$\Delta^\pm(x) = -\frac{1}{(2\pi)^4} \int_{C_\pm} d^4 p \frac{e^{-ip \cdot x}}{p^2 - m^2}$$

with the contours C^\pm corresponding to Δ^\pm as shown in the adjacent figure.



To perform the p_0 integral, note that $p^2 - m^2 = p_0^2 - |\underline{p}|^2 - m^2 = p_0^2 - \omega(\underline{p})^2 = (p_0 - \omega(\underline{p})) (p_0 + \omega(\underline{p}))$.

The contour integration picks up the residue from the pole at $p_0 = \omega(\underline{p})$ or $p_0 = -\omega(\underline{p})$ and yields the previous results, equations (2) or (3), for Δ^\pm .

The function $\Delta(x)$ can be obtained from the same integral with the contour C which encloses both poles, as shown in the figure.

The Feynman Propagator

Since the commutators we have just considered are just numbers, not operators, we can equally well replace them by their vacuum expectation values and write

$$i\Delta^+(x - x') = \langle 0 | [\phi^+(x), \phi^-(x')] | 0 \rangle = \langle 0 | \phi^+(x) \phi^-(x') | 0 \rangle = \langle 0 | \phi(x) \phi(x') | 0 \rangle$$

since $\phi^+(x) | 0 \rangle = (\langle 0 | \phi^-(x))^\dagger = 0$; similarly

$$i\Delta^-(x - x') = \langle 0 | [\phi^-(x), \phi^+(x')] | 0 \rangle = -\langle 0 | \phi(x') \phi(x) | 0 \rangle.$$

We now define the *time-ordered product* of the fields to be

$$T(\phi(x)\phi(x')) = \begin{cases} \phi(x)\phi(x') & \text{if } t > t' \\ \phi(x')\phi(x) & \text{if } t' > t \end{cases};$$

using the unit step function (also known as the Heaviside step function) we can write this as

$$T(\phi(x)\phi(x')) = \theta(t - t')\phi(x)\phi(x') + \theta(t' - t)\phi(x')\phi(x).$$

The Feynman Δ function Δ_F is then defined to be the vacuum expectation value of this product

$$i\Delta_F(x - x') \equiv \langle 0 | T(\phi(x)\phi(x')) | 0 \rangle.$$

We see that Δ_F is related to the functions Δ^\pm by

$$\Delta_F(x) = \theta(t)\Delta^+(x) - \theta(-t)\Delta^-(x),$$

so that

$$\Delta_F(x) = \pm\Delta^\pm(x) \begin{cases} t > 0 \\ t < 0. \end{cases}$$

Physical Interpretation

For $t > t'$ the vacuum expectation value of the T -product reduces to $\langle 0 | \phi(x)\phi(x') | 0 \rangle$, which we interpret as the amplitude for a particle (which is created from the vacuum by the field at space-time position x') to propagate to space-time position x (where it is annihilated by the field).

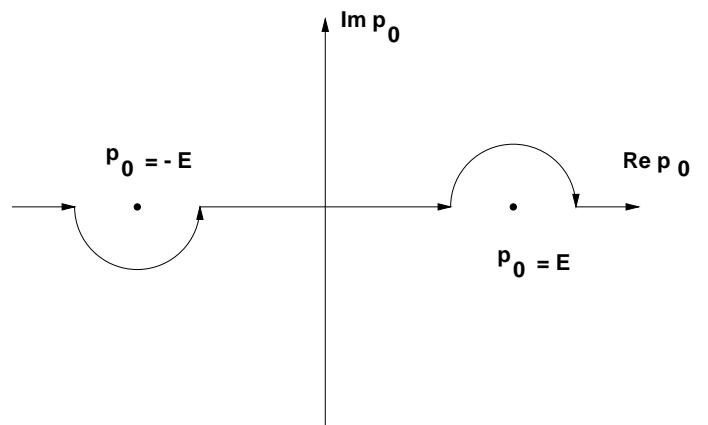
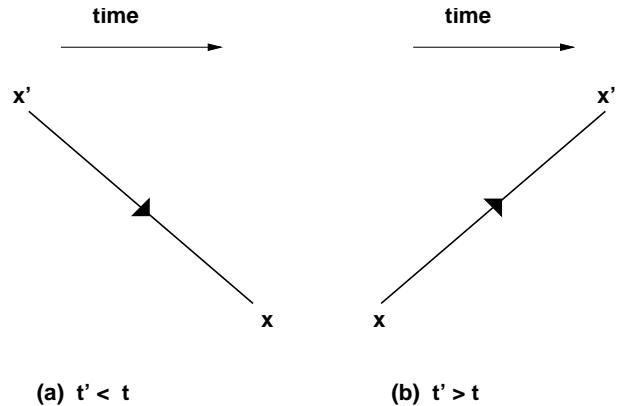
Similarly, when $t' > t$, $\langle 0 | \phi(x')\phi(x) | 0 \rangle$ represents the propagation of a particle from x to x' . We illustrate these ideas in the figure, where the lines represent the propagation of the particle from x to x' or vice versa.

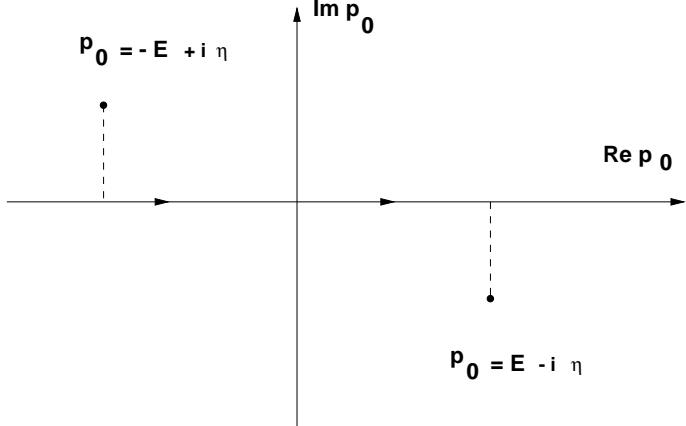
The function $i\Delta_F$ is referred to as the *Feynman propagator* for scalar or Klein-Gordon particles, it is sometimes also called the *scalar meson propagator*. It will form an important ingredient when we come to consider interactions between particles in perturbation theory and develop the Feynman diagram calculus.

The Feynman propagator also has a contour integral representation

$$\Delta_F(x) = \frac{1}{(2\pi)^4} \int_{C_F} d^4 p \frac{e^{-ip \cdot x}}{p^2 - m^2}$$

in the complex p_0 plane where the contour C_F is shown in the accompanying diagram. For $x_0 > 0$ we complete the contour with an infinite semicircle in the lower half-plane, enclosing the





right-hand pole, whereas for $x_0 < 0$ we close it in the upper half-plane, which picks up the left-hand pole.

Instead of deforming the contour round the poles at $p_0 = \pm\omega(\underline{p})$ we can instead move the singularities an infinitesimal distance $\mp i\eta$ off the real axis and integrate along the whole of the real axis: in other words we replace the previous integral by

$$\Delta_F(x) = \frac{1}{(2\pi)^4} \int d^4p \frac{e^{-ip \cdot x}}{p_0^2 - (\omega(\underline{p}) - i\eta)^2} = \frac{1}{(2\pi)^4} \int d^4p \frac{e^{-ip \cdot x}}{p^2 - m^2 + i\varepsilon}$$

where $\varepsilon \equiv 2\eta\omega(p)$ is a small positive number which we take to zero after performing the integration. This is known as *the $i\varepsilon$ prescription*. These arguments also apply to the complex scalar field, the Feynman propagator for charged scalar particles is

$$\langle 0 | T(\phi(x)\phi^\dagger(x')) | 0 \rangle = i\Delta_F(x - x').$$

Relativistic Quantum Field Theory

Quantization of the Dirac Field

The Number Representation for Fermions — General Discussion

Recall our earlier treatment of a set of real scalar fields, where the mode operators satisfy CCRs

$$\begin{aligned} [a_r(\underline{p}), a_s(\underline{p}')^\dagger] &= \delta_{rs} 2\omega(\underline{p})(2\pi)^3 \delta(\underline{p} - \underline{p}'), \\ [a_r(\underline{p}), a_s(\underline{p}')] &= [a_r(\underline{p})^\dagger, a_s(\underline{p}')^\dagger] = 0. \end{aligned}$$

The number-density operator $N_r(\underline{p}) = a_r(\underline{p})^\dagger a_r(\underline{p})$ satisfies the commutation relations

$$\begin{aligned} [N_r(\underline{p}), a_s(\underline{p}')^\dagger] &= \delta_{rs} 2\omega(\underline{p})(2\pi)^3 a_s(\underline{p})^\dagger \delta(\underline{p} - \underline{p}'), \\ [N_r(\underline{p}), a_s(\underline{p}')] &= -\delta_{rs} 2\omega(\underline{p})(2\pi)^3 a_s(\underline{p}) \delta(\underline{p} - \underline{p}'). \end{aligned} \quad (1)$$

We defined the vacuum state through $a_r(\underline{p})|0\rangle = 0$ and built up other states from linear superpositions of multiparticle energy-momentum eigenstates of the form

$$\left(a_{r_1}(\underline{p}_1)^\dagger \right)^{n_1} \left(a_{r_2}(\underline{p}_2)^\dagger \right)^{n_2} \cdots |0\rangle;$$

since all creation operators commute there is no restriction on the number of particles in each mode.

However, if we now turn to the case of spin- $\frac{1}{2}$ particles described by the Dirac equation experiment tells us that they must satisfy Fermi–Dirac statistics and must obey the restriction that no two identical fermions may occupy the same quantum state. How do we overcome the problem that the above mode CCRs imply Bose-Einstein statistics wherein the occupation number of each mode can be any integer?

The solution is to require mode operators for fermions to satisfy *anticommutation relations* instead of commutation relations,

$$\begin{aligned} \{a_r(\underline{p}), a_s(\underline{p}')^\dagger\} &= \delta_{rs} 2\omega(\underline{p})(2\pi)^3 \delta(\underline{p} - \underline{p}') \\ \{a_r(\underline{p}), a_s(\underline{p}')\} &= \{a_r(\underline{p})^\dagger, a_s(\underline{p}')^\dagger\} = 0 \end{aligned}$$

where $\{A, B\} \equiv AB + BA$ is the *anticommutator*; in particular notice that

$$a_r(\underline{p}) a_r(\underline{p}) = a_r(\underline{p})^\dagger a_r(\underline{p})^\dagger = 0.$$

Nevertheless, the commutation relations in equation (1) still hold, as you may verify, suggesting that we can still interpret $a_r(\underline{p})$, $a_r(\underline{p})^\dagger$, and $N_r(\underline{p})$ as annihilation, creation and number-density operators respectively; but now

$$a_r(\underline{p})^\dagger a_r(\underline{p})^\dagger |0\rangle = 0 \quad \text{although} \quad a_r(\underline{p})^\dagger |0\rangle = |\underline{p}, r\rangle$$

so that the occupation number of the state $|\underline{p}, r\rangle$ can only be 0 or 1, as we expect for identical fermions.

The Mode Expansion

For a single Dirac field, the Lagrangian and hence the Dirac equation are

$$\mathcal{L} = \bar{\psi}(x) (i\cancel{D} - m) \psi(x) \rightarrow (i\cancel{D} - m) \psi(x) = 0.$$

The latter has plane-wave solutions of the form

$$u(p, s) e^{-ip \cdot x} \quad \text{and} \quad v(p, s) e^{ip \cdot x}$$

where the (free-particle) spinors $u(p, s)$ and $v(p, s)$ satisfy the equations

$$(\cancel{p} - m) u(p, s) = 0 \quad \text{and} \quad (\cancel{p} + m) v(p, s) = 0.$$

We expand the (complex) Dirac field $\psi(x)$ in terms of the free-particle solutions

$$\psi(x) = \sum_s \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega(\underline{p})} \left[a_s(\underline{p}) u(p, s) e^{-ip \cdot x} + b_s^\dagger(\underline{p}) v(p, s) e^{ip \cdot x} \right] \equiv \psi^+(x) + \psi^-(x);$$

there is a corresponding expansion for the conjugate field $\bar{\psi}$

$$\bar{\psi}(x) = \sum_s \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega(\underline{p})} \left[b_s(\underline{p}) \bar{v}(p, s) e^{-ip \cdot x} + a_s(\underline{p})^\dagger \bar{u}(p, s) e^{ip \cdot x} \right] \equiv \bar{\psi}^+(x) + \bar{\psi}^-(x).$$

Remember that ψ , $\bar{\psi}$, and the free-particle solutions u, v, \bar{u} , and \bar{v} are all 4-component spinors.

We impose anticommutation relations on the mode operators

$$\begin{aligned} \{a_s(\underline{p}), a_{s'}^\dagger(\underline{p}')\} &= \{b_s(\underline{p}), b_{s'}^\dagger(\underline{p}')\} = \delta_{ss'} 2\omega(\underline{p}) (2\pi)^3 \delta(\underline{p} - \underline{p}') \\ \{a_s(\underline{p}), a_{s'}(\underline{p}')\} &= \{a_s(\underline{p}), b_{s'}(\underline{p})\} = 0 \\ \{a_s(\underline{p}), b_{s'}^\dagger(\underline{p}')\} &= \{b_s(\underline{p}), b_{s'}(\underline{p}')\} = 0 \end{aligned}$$

together with others obtained from these by Hermitian conjugation.

Normalization of Dirac Spinors

Before we obtain the corresponding canonical equal-time anticommutation relations for the field $\psi_\alpha(t, \underline{x})$ and its conjugate field $\pi = i\psi^\dagger$ we need to choose the normalization of the free-particle spinors $u(p, s)$ and $v(p, s)$. It is convenient to redefine them to incorporate a factor of $\sqrt{E + m}$; thus

$$u(p, s) = \sqrt{E + m} \begin{pmatrix} \phi^s \\ \frac{\sigma \cdot \underline{p}}{E + m} \phi^s \end{pmatrix} \quad \text{and} \quad v(p, s) = \sqrt{E + m} \begin{pmatrix} \frac{\sigma \cdot \underline{p}}{E + m} \chi^s \\ \chi^s \end{pmatrix}$$

where ϕ^s and χ^s are unit-normalized 2-component spinors. It is then simple to verify that $u^\dagger u = v^\dagger v = 2E$ and that $\bar{u}u = 2m = -\bar{v}v$.

N.B., this convention differs from that of many texts, including Mandl & Shaw and Bjorken & Drell (and from the conventions obtained by Lorentz-transforming the rest-frame spinors of the introduction).

It is now possible to derive the following necessary results (tutorial):

$$\sum_s u(p, s) \bar{u}(p, s) = \not{p} + m \equiv \Lambda_+$$

and

$$\sum_s v(p, s) \bar{v}(p, s) = \not{p} - m \equiv -\Lambda_-.$$

The operators Λ_{\pm} are *projection operators* for positive and negative 4-momentum solutions. Note that Λ_{\pm} are 4×4 matrices; in terms of Dirac spinor components the $\alpha\beta$ elements are

$$\Lambda_{+\alpha\beta} = \sum_s u_\alpha(p, s) \bar{u}_\beta(p, s)$$

and

$$\Lambda_{-\alpha\beta} = - \sum_s v_\alpha(p, s) \bar{v}_\beta(p, s).$$

Equal-time anticommutators for field operators

A straightforward computation yields (tutorial)

$$\left\{ \psi_\alpha(t, \underline{x}), i\psi_\beta^\dagger(t, \underline{x}') \right\} = i \delta_{\alpha\beta} \delta(\underline{x} - \underline{x}');$$

the equivalent version in terms of $\bar{\psi}$, rather than ψ^\dagger , is

$$\left\{ \psi_\alpha(t, \underline{x}), \bar{\psi}_\beta(t, \underline{x}') \right\} = \gamma_{\alpha\beta}^0 \delta(\underline{x} - \underline{x}').$$

We saw earlier that $\bar{\psi}$ does not have a conjugate field. The remaining equal-time anticommutation relations are thus

$$\left\{ \psi_\alpha(t, \underline{x}), \psi_\beta(t, \underline{x}') \right\} = \left\{ \bar{\psi}_\alpha(t, \underline{x}), \bar{\psi}_\beta(t, \underline{x}') \right\} = 0.$$

The Hamiltonian for the Dirac field is

$$\begin{aligned} H &= \int d^3x \left(i\psi^\dagger(x) \dot{\psi}(x) - \mathcal{L}(x) \right) \\ &= \int d^3x \left(i\bar{\psi}(x) \gamma^0 \partial_0 \psi(x) - \mathcal{L}(x) \right) \\ &= \int d^3x \left(-i\bar{\psi}(x) \underline{\gamma} \cdot \underline{\nabla} \psi(x) + m\bar{\psi}(x) \psi(x) \right). \end{aligned}$$

If we substitute in the mode expansions we find that (tutorial)

$$\begin{aligned} H &= \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{\omega(\underline{p})}{2\omega(\underline{p})} \left(a_s(\underline{p})^\dagger a_s(\underline{p}) - b_s(\underline{p}) b_s^\dagger(\underline{p}) \right) \\ &= \frac{1}{2} \sum_s \int \frac{d^3p}{(2\pi)^3} \left(a_s(\underline{p})^\dagger a_s(\underline{p}) + b_s^\dagger(\underline{p}) b_s(\underline{p}) - \delta_{ss} 2\omega(\underline{p}) (2\pi)^3 \delta(0) \right), \end{aligned}$$

so there is again a divergent ground-state energy, which we can remove by normal ordering. For obvious reasons, for fermion operators the normal-ordering prescription is to include a minus sign when two operators are interchanged

$$:\psi_\alpha\psi_\beta: = :(\psi_\alpha^+ + \psi_\alpha^-)(\psi_\beta^+ + \psi_\beta^-): = \psi_\alpha^+\psi_\beta^+ - \psi_\beta^-\psi_\alpha^+ + \psi_\alpha^-\psi_\beta^+ + \psi_\alpha^-\psi_\beta^-.$$

Note that *anticommutators* are *necessary* if we want a positive-energy spectrum of particle states. If we quantized the Dirac field with commutators the minus sign in the $b_s(\underline{p})b_s^\dagger(\underline{p})$ term in H would lead to a decrease in energy as more particles of type b are created. Note also that the anticommutation relations lead to a *negative* ground state energy.

It is left as an exercise to show that the Hamiltonian can be expressed as

$$H = \int d^3x T^{00}(x) \quad \text{with} \quad T^{\mu\nu}(x) = i:\bar{\psi}(x)\gamma^\mu\partial^\nu\psi(x):,$$

and that the conserved 3-momentum can be shown to be (tutorial)

$$\underline{P} = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{\underline{p}}{2\omega(\underline{p})} \left(a_s(\underline{p})^\dagger a_s(\underline{p}) + b_s^\dagger(\underline{p})b_s(\underline{p}) \right).$$

By calculating the commutators of the mode operators with H it is also easy to show that

$$[H, a_s(\underline{p})^\dagger] = \omega(\underline{p})a_s(\underline{p})^\dagger, \quad [H, a_s(\underline{p})] = -\omega(\underline{p})a_s(\underline{p})$$

and

$$[H, b_s^\dagger(\underline{p})] = \omega(\underline{p})b_s^\dagger(\underline{p}) \quad [H, b_s(\underline{p})] = -\omega(\underline{p})b_s(\underline{p})$$

so that $a_s(\underline{p})^\dagger$ and $b_s^\dagger(\underline{p})$ both raise the energy by E , while $a_s(\underline{p})$ and $b_s(\underline{p})$ lower the energy by E .

The eigenstates of the Hamiltonian are thus obtained by applying the creation operators to the ground state in the same way as for bosonic particles. The ground state is defined by

$$a_s(\underline{p})|0\rangle = b_s(\underline{p})|0\rangle = 0 \quad \forall \underline{p}, s.$$

The two-particle state

$$|p_1, s_1; p_2, s_2\rangle \equiv a_{s_1}^\dagger(\underline{p}_1)a_{s_2}^\dagger(\underline{p}_2)|0\rangle = -a_{s_2}^\dagger(\underline{p}_2)a_{s_1}^\dagger(\underline{p}_1)|0\rangle = -|p_2, s_2; p_1, s_1\rangle$$

is thus antisymmetric under interchange of particle labels, as we expect for identical fermions.

Charge Conservation

Invariance of the Lagrangian under $\psi \rightarrow e^{i\alpha}\psi$ leads to a conserved Noether current

$$j^\mu(x) = :\bar{\psi}(x)\gamma^\mu\psi(x):$$

with a corresponding conserved charge $Q = \int d^3x j^0(x)$. In terms of mode operators this is given by

$$Q = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega(\underline{p})} [a_s(\underline{p})^\dagger a_s(\underline{p}) - b_s^\dagger(\underline{p})b_s(\underline{p})],$$

and we interpret the particles associated with the two types of creation and annihilation operator as particles of opposite charge: electrons and positrons. Usually we rescale Q by a factor q , the fundamental quantum of electric charge so that Q becomes the electric charge operator

$$Q \rightarrow q \sum_s \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega(\underline{p})} [a_s(\underline{p})^\dagger a_s(\underline{p}) - b_s^\dagger(\underline{p}) b_s(\underline{p})].$$

Covariant Anticommutators for Fermion Fields

Just as for the scalar field, we can derive covariant anticommutation relations for the Dirac field. We leave it as a tutorial exercise to show that

$$\begin{aligned} \{\psi_\alpha^\pm(x), \bar{\psi}_\beta^\mp(y)\} &= i(i\cancel{p} + m)_{\alpha\beta} \Delta^\pm(x - y) \\ \{\psi_\alpha(x), \psi_\beta(y)\} &= \{\bar{\psi}_\alpha(x), \bar{\psi}_\beta(x)\} = 0, \end{aligned}$$

where Δ^\pm are the same invariant Δ functions that we introduced for the Klein–Gordon field. The non-vanishing anticommutator is usually written without explicit matrix indices as

$$\boxed{\{\psi^\pm(x), \bar{\psi}^\mp(y)\} = iS^\pm(x - y)}, \quad (2)$$

where we define

$$S^\pm(x) \equiv (i\cancel{p} + m) \Delta^\pm(x). \quad (3)$$

It then follows that

$$\{\psi(x), \bar{\psi}(y)\} = iS(x - y),$$

where

$$S(x) = S^+(x) + S^-(x) = (i\cancel{p} + m) \Delta(x)$$

where as before $\Delta(x) = \Delta^+(x) + \Delta^-(x)$.

We can use the various contour integral representations that we wrote down for the Δ functions to obtain similar representations of the corresponding S functions

$$S^\pm(x) = - \int_{C_\pm} \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot x} \frac{\cancel{p} + m}{p^2 - m^2},$$

where as before the contours C_\pm enclose the poles at $p_0 = \pm\omega(\underline{p})$ respectively. It is useful to rewrite this using the identity

$$(\cancel{p} \pm m)(\cancel{p} \mp m) = p^2 - m^2$$

as

$$\boxed{S^\pm(x) = - \int_{C_\pm} \frac{d^4 p}{(2\pi)^4} \frac{e^{-ip \cdot x}}{\cancel{p} - m}}.$$

The Fermion Propagator

Suppressing Dirac matrix indices, we define the *fermion propagator* as

$$\langle 0 | T(\psi(x)\bar{\psi}(x')) | 0 \rangle,$$

where for fermion fields the time-ordered product is defined by

$$T(\psi(x)\bar{\psi}(x')) = \theta(t - t')\psi(x)\bar{\psi}(x') - \theta(t' - t)\bar{\psi}(x')\psi(x) = \begin{cases} \psi(x)\bar{\psi}(x') & t > t' \\ -\bar{\psi}(x')\psi(x) & t' > t; \end{cases}$$

note the minus sign associated with the anticommuting nature of fermion fields.

We now calculate the fermion propagator in terms of S functions. Using equation (2), we have

$$\langle 0 | \psi(x)\bar{\psi}(x') | 0 \rangle = \langle 0 | \psi^+(x)\bar{\psi}^-(x') | 0 \rangle = \langle 0 | \{\psi^+(x), \bar{\psi}^-(x')\} | 0 \rangle = iS^+(x - x')$$

where we used $a_s(p)|0\rangle = b_s(p)|0\rangle = 0 \quad \forall p, s$; a similar calculation gives

$$\langle 0 | \bar{\psi}(x')\psi(x) | 0 \rangle = iS^-(x - x').$$

Combining these two results we have for the fermion propagator

$$\boxed{\langle 0 | T(\psi(x)\bar{\psi}(x')) | 0 \rangle = iS_F(x - x')} \quad (4)$$

where the Feynman S function is

$$S_F(x) \equiv \theta(t)S^+(x) - \theta(-t)S^-(x) = (i\partial + m)\Delta_F(x).$$

Again we can write down a representation in the complex p_0 plane where the p_0 integration is along the entire real axis provided we adopt the $i\varepsilon$ prescription to move the poles of the integrand off the real axis

$$\boxed{S_F(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{\not{p} + m}{p^2 - m^2 + i\varepsilon}}.$$

The interpretation of the Feynman propagator $S_F(x - x')$ is similar to that for scalars: for $t' < t$ it represents the creation of an electron at x' , its propagation from x' to x , and its annihilation at x ; whereas for $t' > t$ it represents creation of a positron at x , its propagation from x to x' , and annihilation at x' . (**Exercise:** why does the latter represent positron propagation rather than electron propagation?)

The Electromagnetic Field

Recall that the free-field Maxwell equations follow from the Lagrangian density

$$\mathcal{L}(x) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\mu A_\nu)(\partial^\nu A^\mu), \quad (5)$$

but we saw earlier that there is a problem in that the canonically conjugate field

$$\pi^\mu(x) = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = -F^{0\mu}(x)$$

vanishes for $\mu = 0$, so we cannot impose the usual ETCRs between $A^0(t, \underline{x})$ and $\pi^0(t, \underline{x}')$.

A simple solution to this difficulty is due to Fermi, who suggested simply omitting the second term from the Lagrangian of equation (5). This ‘fixes the gauge’ and the Lagrangian becomes

$$\mathcal{L}(x) = -\frac{1}{2}\partial_\mu A_\nu(x)\partial^\mu A^\nu(x) \quad (6)$$

giving for the conjugate fields

$$\pi^\mu(x) = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = -\dot{A}^\mu(x)$$

which are now all non-zero in general. It is straightforward to show that the corresponding field equations are

$$\partial^2 A^\nu(x) = 0 \quad (7)$$

but these are only equivalent to Maxwell’s equations $\partial^2 A^\nu(x) - \partial^\nu \partial_\mu A^\mu(x) = 0$ if the field satisfies the Lorentz condition

$$\partial_\mu A^\mu(x) = 0.$$

This suggests that we should first quantise the theory corresponding to the Fermi Lagrangian (6), and then impose the Lorentz condition as a constraint.

Aside: this quantization procedure is slightly different from the one adopted in most modern field theory texts wherein a ‘gauge-fixing’ term is added to the original Maxwell Lagrangian (5) in order to cancel the second term’s contribution to the equations of motion. The two methods are equivalent in the sense that they make the same physical predictions.

The Mode Expansion

We first note that equation (7) is a massless Klein–Gordon equation for each component of the vector potential, so we can take over many of the results we derived earlier for the real scalar field. The classical EM field is real, so the quantum EM field operator is Hermitian.

We expand the free electromagnetic field in terms of the complete set of plane-wave solutions

$$A^\mu(x) = \sum_r \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega(\underline{k})} \left(\epsilon_r^\mu(\underline{k}) a_r(\underline{k}) e^{-ik \cdot x} + \epsilon_r^{\mu*}(\underline{k}) a_r(\underline{k})^\dagger e^{ik \cdot x} \right)$$

with $k^2 = k_0^2 - |\underline{k}|^2 = 0$ and $k_0 = \omega(\underline{k})$. The quantities $\epsilon_r^\mu(\underline{k})$, which carry the Lorentz index μ , are known as *polarization vectors*; they can be chosen to be real. The label r runs over the values 0, 1, 2, 3, corresponding to the fact that for each \underline{k} there are four linearly independent polarization states. A particularly useful choice is

$$\epsilon_0^\mu(\underline{k}) = n^\mu = (1, 0, 0, 0) \quad \text{and} \quad \epsilon_r^\mu(\underline{k}) = \left(0, \underline{\epsilon}_r(\underline{k}) \right) \quad r = 1, 2, 3$$

where the three-vectors $\underline{\epsilon}_1(\underline{k})$ and $\underline{\epsilon}_2(\underline{k})$ are taken to be mutually orthogonal unit vectors which are also orthogonal to

$$\underline{\epsilon}_3(\underline{k}) \equiv \frac{\underline{k}}{|\underline{k}|}.$$

ϵ_1^μ and ϵ_2^μ are called *transverse polarization vectors*, ϵ_3^μ is the *longitudinal polarization vector* and ϵ_0^μ is called the ‘*scalar*’ or *timelike polarization vector*.

The polarization vectors satisfy the orthonormality and completeness relations (tutorial)

$$\epsilon_r^\mu(\underline{k})\epsilon_{s\mu}(\underline{k}) = g_{rs} \quad r, s = 0, 1, 2, 3$$

and

$$\sum_r g^{rr} \epsilon_r^\mu(\underline{k}) \epsilon_r^\nu(\underline{k}) = g^{\mu\nu}.$$

Equal Time Commutation Relations

Using the Fermi Lagrangian and the corresponding conjugate fields $\pi^\mu(x)$, we impose the ETCRs

$$\begin{aligned} [A^\mu(t, \underline{x}), \dot{A}^\nu(t, \underline{x}')] &= -ig^{\mu\nu}\delta(\underline{x} - \underline{x}') \\ [A^\mu(t, \underline{x}), A^\nu(t, \underline{x}')] &= [\dot{A}^\mu(t, \underline{x}), \dot{A}^\nu(t, \underline{x}')] = 0. \end{aligned}$$

Covariant Commutation Relations and the Photon Propagator

We can immediately write down the covariant commutation relations from those for the Klein–Gordon field, taking the zero-mass limit

$$[A^\mu(x), A^\nu(x')] = iD^{\mu\nu}(x - x')$$

where

$$D^{\mu\nu}(x) = \lim_{m \rightarrow 0} -g^{\mu\nu}\Delta(x).$$

Similarly, the *photon propagator* is

$$\langle 0 | T(A^\mu(x)A^\nu(x')) | 0 \rangle = iD_F^{\mu\nu}(x - x') \quad (8)$$

with

$$D_F^{\mu\nu} = \lim_{m \rightarrow 0} -g^{\mu\nu}\Delta_F(x) = -g^{\mu\nu} \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik \cdot x}}{k^2 + i\varepsilon}. \quad (9)$$

The photon interpretation follows from the commutation relations among the mode operators, which are easily shown to be

$$\begin{aligned} [a_r(\underline{k}), a_s(\underline{k}')^\dagger] &= -g_{rs} 2\omega(\underline{k})(2\pi)^3 \delta(\underline{k} - \underline{k}') \\ [a_r(\underline{k}), a_s(\underline{k}')] &= [a_r(\underline{k})^\dagger, a_s(\underline{k}')^\dagger] = 0 \end{aligned}$$

where $g_{rr} = -1$ for $r = 1, 2, 3$, so that for these values of r we have the usual boson commutation relations. However $g_{00} = 1$, so that it looks as if there is a problem for so-called ‘*scalar*’ or timelike photons: the roles of annihilation and creation operators seem to be interchanged.

The (normal-ordered) Hamiltonian is given by

$$H = \int d^3x : (\pi^\mu(x) \dot{A}_\mu(x) - \mathcal{L}(x)) :;$$

Substituting the mode expansions yields the expression (tutorial)

$$H = -\frac{1}{2} \sum_r \int \frac{d^3k}{(2\pi)^3} g^{rr} a_r(\underline{k})^\dagger a_r(\underline{k}),$$

so what are we to make of the minus sign when $r = 0$?

The Gupta–Bleuler Formalism

Despite the strange minus sign let us interpret the operators $a_r(\underline{k})$ and $a_r(\underline{k})^\dagger$ as annihilation and creation operators for transverse photons ($r = 1, 2$), longitudinal photons ($r = 3$), and scalar photons ($r = 0$). The vacuum state is the state containing no photons, therefore it satisfies

$$a_r(\underline{k}) |0\rangle = 0 \quad r = 0, 1, 2, 3;$$

a one photon state is obtained by applying a creation operator to the vacuum

$$a_r(\underline{k})^\dagger |0\rangle = |\underline{k}, r\rangle \quad r = 0, 1, 2, 3.$$

Despite the factor g^{rr} which appears in the expression for H in terms of mode operators, the energy is positive definite (because g^{rr} appears only as $(g^{rr})^2$). For example

$$H |\underline{k}, r\rangle = -\frac{1}{2} \sum_s \int \frac{d^3k'}{(2\pi)^3} g^{ss} a_s(\underline{k}')^\dagger a_s(\underline{k}') a_r(\underline{k})^\dagger |0\rangle = (g^{rr})^2 \omega(\underline{k}) a_r(\underline{k})^\dagger |0\rangle = \omega(\underline{k}) |\underline{k}, r\rangle$$

as you can easily verify from the mode operator commutation relations. Thus we may interpret the operator

$$N_r(\underline{k}) \equiv g^{rr} a_r(\underline{k})^\dagger a_r(\underline{k})$$

as the number-density operator for photons.

However, there are still some apparent difficulties. If we calculate the norm of the one photon state it is *negative* for scalar photons

$$\langle \underline{k}, r | \underline{k}, r \rangle = \langle 0 | a_r(\underline{k}) a_r^\dagger(\underline{k}) | 0 \rangle = \langle 0 | [a_r(\underline{k}), a_r^\dagger(\underline{k})] | 0 \rangle = -g_{rr} 2\omega(\underline{k}) (2\pi)^3 \delta(0)$$

if we choose to normalise the vacuum state so that $\langle 0 | 0 \rangle = 1$.

So far we have ignored the requirement that we must impose the Lorentz condition for the Fermi Lagrangian to yield the correct equations of motion. This reveals a new difficulty: if we take the Lorentz condition $\partial_\mu A^\mu = 0$ as an operator identity it is incompatible with the canonical commutation relations since

$$[\partial_\mu A^\mu(x), A^\nu(x')] = i\partial_\mu D^{\mu\nu}(x - x'),$$

and this is not identically zero. Gupta and Bleuler proposed that the Lorentz condition should be replaced as an operator identity by the weaker requirement that

$$\partial_\mu A^{\mu+} |\Psi\rangle = 0 \tag{10}$$

where $|\Psi\rangle$ is any physical state. This is a restriction on the allowed physical states in the theory, and it ensures that the Lorentz condition holds for expectation values. To see this first note the adjoint equation

$$\langle \Psi | \partial_\mu A^{\mu-}(x) = 0,$$

then we have

$$\langle \Psi | \partial_\mu A^\mu(x) |\Psi\rangle = \langle \Psi | (\partial_\mu A^{\mu+} + \partial_\mu A^{\mu-}) |\Psi\rangle = 0.$$

Substituting the mode expansion for $A^{\mu+}(x)$ into equation (10) and using the explicit representation of the polarization vectors that we defined in the last lecture gives (exercise)

$$\left(a_3(\underline{k}) - a_0(\underline{k}) \right) |\Psi\rangle = 0 \quad \forall \underline{k} \quad (11)$$

where we noted that $k_\mu \epsilon_r^\mu(\underline{k}) = 0$ for $r = 1, 2$; the corresponding adjoint equation is

$$\langle \Psi | \left(a_3^\dagger(\underline{k}) - a_0^\dagger(\underline{k}) \right) = 0 \quad \forall \underline{k}. \quad (12)$$

This means that when we compute the expectation value of H in a physical state $|\Psi\rangle$ we find that the contribution of scalar and longitudinal modes *cancel* because

$$\langle \Psi | (a_3(\underline{k})^\dagger a_3(\underline{k}) - a_0(\underline{k})^\dagger a_0(\underline{k})) |\Psi\rangle = \langle \Psi | a_3(\underline{k})^\dagger (a_3(\underline{k}) - a_0(\underline{k})) |\Psi\rangle = 0,$$

where we used equations (12) and (11) respectively, and hence

$$\langle \Psi | H |\Psi\rangle = \frac{1}{2} \sum_{r=1}^2 \int \frac{d^3 k}{(2\pi)^3} \langle \Psi | a_r(\underline{k})^\dagger a_r(\underline{k}) |\Psi\rangle.$$

In other words, only *transverse photons* contribute to the expectation value of the energy. For free EM fields the same is true for all other observables (this is not obvious). This is in accord with the usual assertion that for physical photons in free space there are only two linearly independent polarization states and these correspond to the transverse polarization vectors $r = 1, 2$. Note that equation (11) is a constraint on the linear combinations of longitudinal and scalar photons for each \underline{k} that may be present in a state $|\Psi\rangle$. It does not forbid longitudinal and scalar modes from the state but, as we have seen, these do not contribute to physical observables such as the energy.

In the presence of charges, the situation is more complicated and the longitudinal and scalar photons play an important role as virtual particles in intermediate states; they provide a covariant description of the ‘instantaneous’ Coulomb interaction between charges.

Interpretation of the Photon Propagator

It is convenient to introduce the 4-dimensional Fourier transform of the photon propagator

$$D_F^{\mu\nu}(x) = \int \frac{d^4 k}{(2\pi)^4} \tilde{D}_F^{\mu\nu}(k) e^{-ik \cdot x};$$

then we can write

$$\tilde{D}_F^{\mu\nu}(k) = -\frac{g^{\mu\nu}}{k^2 + i\varepsilon} = -\frac{\sum_r g^{rr} \epsilon_r^\mu(\underline{k}) \epsilon_r^\nu(\underline{k})}{k^2 + i\varepsilon}.$$

If we use the special choice of polarization vectors where $\epsilon_0^\mu = n^\mu = (1, 0, 0, 0)$ and

$$\epsilon_3^\mu = \left(0, \frac{\underline{k}}{|\underline{k}|} \right) = \frac{k^\mu - (k \cdot n)n^\mu}{\sqrt{(k \cdot n)^2 - k^2}},$$

we find that

$$\tilde{D}_F^{\mu\nu}(k) = \frac{1}{k^2 + i\varepsilon} \left(\sum_{r=1}^2 \epsilon_r^\mu(\underline{k}) \epsilon_r^\nu(\underline{k}) + \frac{(k^\mu - (k \cdot n)n^\mu)(k^\nu - (k \cdot n)n^\nu)}{(k \cdot n)^2 - k^2} - n^\mu n^\nu \right),$$

which we can regard as showing the contributions of transverse, longitudinal, and scalar photons to the photon propagator respectively.

We interpret the first term \tilde{D}_F^T as representing the propagation of transverse photons. The interpretation of the other two terms is most straightforward if we combine them into a term \tilde{D}_F^C proportional to $n^\mu n^\nu$ and a remainder \tilde{D}_F^R (exercise)

$$\begin{aligned} (\tilde{D}_F^C)^{\mu\nu}(k) &= \frac{n^\mu n^\nu}{(k \cdot n)^2 - k^2} = \frac{n^\mu n^\nu}{|\underline{k}|^2} \\ (\tilde{D}_F^R)^{\mu\nu}(k) &= \frac{1}{k^2 + i\varepsilon} \left(\frac{k^\mu k^\nu - (k \cdot n)(k^\mu n^\nu + k^\nu n^\mu)}{(k \cdot n)^2 - k^2} \right). \end{aligned}$$

Consider the first of these in configuration space

$$(D_F^C)^{\mu\nu}(x) = g^{\mu 0} g^{\nu 0} \int \frac{d^3 k}{(2\pi)^3} \frac{\exp(i\underline{k} \cdot \underline{x})}{|\underline{k}|^2} \int \frac{dk^0}{2\pi} e^{-ik^0 x^0} = \frac{g^{\mu 0} g^{\nu 0}}{4\pi |\underline{x}|} \delta(x^0).$$

We call this the *instantaneous Coulomb term*, since the spatial dependence is characteristic of the Coulomb potential and the time dependence is given by $\delta(x^0)$. When we use the photon propagator to describe the interaction between charged particles, we may regard the ‘instantaneous’ Coulomb interaction as arising from this particular combination of longitudinal and scalar photons.

We shall see later that the contribution of the remainder term to physical quantities actually vanishes.

Electromagnetic Interactions of Charged Particles

Recall that the prescription in Relativistic QM for introducing electromagnetic interactions into the Dirac equation is to use *minimal coupling*

$$p^\mu \rightarrow p^\mu - qA^\mu$$

or in terms of operators

$$\boxed{\partial_\mu \rightarrow \partial_\mu + iqA_\mu(x) \equiv D_\mu},$$

where the D_μ is known as the *covariant derivative*. For electrons $q = -e$ with $e > 0$. With this replacement the corresponding Dirac equation becomes

$$\boxed{(i\cancel{\partial} - m)\psi(x) = q\cancel{A}(x)\psi(x) = -e\cancel{A}(x)\psi(x)}.$$

It is easy to see that the Lagrangian density for which this is the Euler–Lagrange equation of motion is

$$\mathcal{L}(x) = \bar{\psi}(x)(i\cancel{D} - m)\psi(x). \quad (13)$$

We may write this as

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$$

where the first term is just the Lagrangian density for the free Dirac field

$$\mathcal{L}_0 = \bar{\psi}(x)(i\cancel{D} - m)\psi(x),$$

and the second term, known as the *interaction Lagrangian density*, is given by

$$\mathcal{L}_I = -q\bar{\psi}(x)\gamma_\mu\psi(x)A_\mu(x)$$

and represents the coupling between the Dirac and electromagnetic fields. In particular, \mathcal{L}_I can be written

$$\mathcal{L}_I = -j_{\text{em}}^\mu(x)A_\mu(x)$$

where $j_{\text{em}}^\mu(x) = q\bar{\psi}(x)\gamma^\mu\psi(x)$ is the conserved electromagnetic current density identified earlier.

Gauge Invariance

We have argued that only the electromagnetic field strengths $F_{\mu\nu}$ (*i.e.*, the \underline{E} & \underline{B} fields) have physical significance and that we can redefine the potential field $A_\mu(x)$ through a gauge transformation

$$A_\mu(x) \rightarrow \tilde{A}_\mu(x) = A_\mu(x) + \partial_\mu\chi(x).$$

However, it is easy to see that the Lagrangian density for the coupled theory is not invariant

$$\mathcal{L} \rightarrow \tilde{\mathcal{L}} = \mathcal{L} - q\bar{\psi}(x)\gamma^\mu\psi(x)\partial_\mu\chi(x).$$

Invariance may be restored if we demand that when we perform a gauge transformation of the field A_μ the Dirac fields transform as

$$\begin{aligned} \psi(x) &\rightarrow \tilde{\psi}(x) = e^{-iq\chi(x)}\psi(x) \\ \bar{\psi}(x) &\rightarrow \tilde{\bar{\psi}}(x) = \bar{\psi}(x)e^{iq\chi(x)}, \end{aligned}$$

a *local phase transformation* or *local gauge transformation*. Under these combined transformations of the vector potential and the fermion fields, the Lagrangian density remains invariant

$$\begin{aligned} \mathcal{L}_0 \rightarrow \tilde{\mathcal{L}}_0 &= \mathcal{L}_0 + q\bar{\psi}(x)\gamma^\mu\psi(x)\partial_\mu\chi(x) \\ \mathcal{L}_I \rightarrow \tilde{\mathcal{L}}_I &= \mathcal{L}_I - q\bar{\psi}(x)\gamma^\mu\psi(x)\partial_\mu\chi(x). \end{aligned}$$

The Gauge Principle

We can invert the reasoning of the last section and *demand* that the Dirac Lagrangian density be invariant under local phase transformations. Of course the derivative term in the free Lagrangian \mathcal{L}_0 is not invariant under such transformations, so we are forced to introduce a covariant derivative which transforms in the same way as the fields themselves under local phase transformations

$$D_\mu \psi(x) \rightarrow D_\mu \tilde{\psi}(x) = e^{-iq\chi(x)} D_\mu \psi(x),$$

which we can arrange by having

$$D_\mu \psi(x) = \partial_\mu \psi(x) + iqA_\mu(x)\psi(x)$$

with

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \chi(x).$$

This of course generates an extra term in the Lagrangian density which is precisely the interaction term \mathcal{L}_I .

It is also interesting to note that the commutator of two covariant derivatives

$$[D_\mu, D_\nu] = [\partial_\mu + iqA_\mu, \partial_\nu + iqA_\nu] = iq(\partial_\mu A_\nu - \partial_\nu A_\mu) = iqF_{\mu\nu}$$

is not a differential operator by just the familiar field strength tensor $F_{\mu\nu}$.

Relativistic Quantum Field Theory

Interactions and Feynman Diagrams

In the last lecture we saw that the Lagrangian density for the Dirac field interacting with the electromagnetic field is

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$$

where the first term is just the Lagrangian density for the free Dirac field

$$\mathcal{L}_0 = \bar{\psi}(x)(i\partial - m)\psi(x),$$

and

$$\mathcal{L}_I = -q\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x).$$

To obtain the full Lagrangian density for Quantum Electrodynamics (QED) we must add the Lagrangian density for the free electromagnetic field. Thus

$$\begin{aligned}\mathcal{L}_{\text{QED}} &= \mathcal{L}_{\text{Dirac}} + \mathcal{L}_I + \mathcal{L}_{\text{em}} \\ &= : \bar{\psi}(x)(i\partial - m)\psi(x) : + e : \bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x) : - \frac{1}{4} : F_{\mu\nu}F^{\mu\nu} :.\end{aligned}$$

We are unable to solve the corresponding set of coupled non-linear field equations exactly and must resort to perturbation theory, treating the interaction term \mathcal{L}_I as a perturbation.

The Interaction Picture

It will prove simplest to work in the *interaction picture* (also known as the Dirac picture) and study the equations of motion as a perturbation series known as the *S-matrix expansion*. For QED the dimensionless parameter which controls the perturbative expansion is the *fine structure constant* $\alpha \equiv e^2/4\pi \approx 1/137$. It is the smallness of this number which is chiefly responsible for the outstanding success of QED.

We now describe briefly the salient features of the interaction picture. We write the Hamiltonian as a free-field part H_0 and an interaction part H_I

$$H = H_0 + H_I.$$

The interaction picture is related to the Schrödinger picture by a time-dependent unitary transformation on states and operators

$$|\Psi, t\rangle_I = e^{iH_0(t-t_0)} |\Psi, t\rangle_S \quad \text{and} \quad \mathcal{O}^I(t) = e^{iH_0(t-t_0)} \mathcal{O}^S e^{-iH_0(t-t_0)},$$

where $H_0 = H_0^I = H_0^S$ is the non-interacting part of the Hamiltonian.

In the interaction picture operators satisfy Heisenberg-like equations of motion, but the time development is controlled by the free Hamiltonian H_0 . The state vectors $|\Psi, t\rangle_I$ are also time-dependent, but the time development is determined by the interaction Hamiltonian $H_I(t)$ (in the interaction picture)

$$\begin{aligned}H_I(t) &= e^{iH_0(t-t_0)} H_I^S e^{-iH_0(t-t_0)} \\ i\frac{\partial}{\partial t} |\Psi, t\rangle_I &= H_I(t) |\Psi, t\rangle_I\end{aligned}\tag{1}$$

where H_I^S is the interaction Hamiltonian in the Schrödinger picture. Thus in the limit of no interactions the interaction picture becomes the Heisenberg picture with time-independent state vectors. It is the interaction alone which leads to the state vector changing with time. For brevity, from now on, we shall drop the subscript I on the interaction picture state vector $|\Psi, t\rangle_I$.

The S -matrix

Equation (1) represents a time-dependent unitary transformation on states, so that the norm is preserved $\langle \Psi, t | \Psi, t \rangle = 1$ with an appropriate choice of normalization. In a scattering process we are interested in the transition between some initial state long before scattering takes place, in which there are given numbers of particles with specified momenta and spins widely separated so that they may be treated as non-interacting, to some final state when the particles are again far apart and may again be treated as free. We denote these states by $|\Psi, -\infty\rangle$ and $|\Psi, \infty\rangle$ respectively. The S -matrix, which should perhaps be called the scattering operator, relates these two states and may be defined by

$$|\Psi, \infty\rangle = S |\Psi, -\infty\rangle.$$

Of course, the final states are many and various, and all possibilities are contained in $|\Psi, \infty\rangle$. The probability corresponding to some particular final state $|f\rangle$ is given by

$$\left| \langle f | \Psi, \infty \rangle \right|^2$$

with suitable normalizations of the states. If we abbreviate the initial state to $|\Psi, -\infty\rangle = |i\rangle$ the required probability amplitude is the S -matrix element

$$\langle f | S | \Psi, -\infty \rangle = \langle f | S | i \rangle \equiv S_{fi}.$$

If we expand the state $|\Psi, \infty\rangle$ in terms of the complete set of orthonormal final states $|f\rangle$ we can write

$$|\Psi, \infty\rangle = \sum_f |f\rangle \langle f | \Psi, \infty \rangle = \sum_f |f\rangle S_{fi},$$

hence the normalization of $|\Psi, \infty\rangle$ requires

$$\langle \Psi, \infty | \Psi, \infty \rangle = \sum_{f,f'} S_{f'i}^* \langle f' | f \rangle S_{fi} = \sum_{f,f'} S_{f'i}^* \delta_{f'f} S_{fi} = \sum_f |S_{fi}|^2 = 1,$$

which expresses the unitarity of the S -matrix and the conservation of probability. It is important to recognise however that this is a more general result than in NRQM where particle number is conserved, because now particles can be destroyed and created.

We can integrate equation (1) to yield an integral equation for $|\Psi, t\rangle$

$$|\Psi, t\rangle = |\Psi, -\infty\rangle + \int_{-\infty}^t dt_1 \frac{\partial}{\partial t_1} |\Psi, t_1\rangle = |\Psi, -\infty\rangle - i \int_{-\infty}^t dt_1 H_I(t_1) |\Psi, t_1\rangle,$$

which we then solve by iteration

$$\begin{aligned} |\Psi, t\rangle &= |i\rangle + (-i) \int_{-\infty}^t dt_1 H_I(t_1) |\Psi, t_1\rangle \\ &\quad + (-i)^2 \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} dt_2 H_I(t_1) H_I(t_2) |\Psi, t_2\rangle. \end{aligned}$$

Repeated iteration yields an infinite series for the S -matrix

$$\begin{aligned} S &= \sum_{n=0}^{\infty} (-i)^n \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n H_I(t_1) H_I(t_2) \cdots H_I(t_n) \\ &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \cdots \int_{-\infty}^{\infty} dt_n T(H_I(t_1) H_I(t_2) \cdots H_I(t_n)), \end{aligned}$$

assuming that H_I contains an even number of fermion fields whereupon time ordering produces an even number of minus signs. (See Peskin & Schroeder for the origin of the $n!$).

Finally, we rewrite the last expression in terms of the Hamiltonian density

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \int d^4x_2 \cdots \int d^4x_n T(\mathcal{H}_I(x_1) \mathcal{H}_I(x_2) \cdots \mathcal{H}_I(x_n));$$

this is the Dyson expansion of the S -matrix.

Wick's Theorem

Only certain terms in the S -matrix will contribute to a given transition from a particular initial state $|i\rangle$ to a particular final state $|f\rangle$. The required terms must contain exactly the right annihilation operators to destroy the particles present in the initial state and the right creation operators to create the desired final state particles. Of course they may contain in addition operators which create and destroy other particles, which are thus only present in intermediate states and are called *virtual particles*.

The equations of motion for *interacting* fields in the interaction picture are governed by the *free* Hamiltonian H_0 , thus the equations of motion are the same as for free fields, and the solutions have the same form in terms of operators a^\dagger , a , etc.: this is why we use the interaction picture.

We can explicitly extract the virtual particles by re-expressing the S -matrix expansion as a sum of *normal products*. Remember that in a normal product of field operators all creation operators stand to the left of all annihilation operators, so such an operator product will first annihilate an existing set of particles and then create a new set, but not create and re-annihilate. This can be a bit confusing, so let's proceed by example.

Example: Consider Compton scattering: $e^- + \gamma \rightarrow e^- + \gamma$. The interaction Hamiltonian density is

$$\mathcal{H}_I(x) = -\mathcal{L}_I(x) = -e : \bar{\psi}(x) \gamma^\mu A_\mu(x) \psi(x) :$$

The normal product which contributes to Compton scattering must contain the operator product

$$\bar{\psi}^- A^- \psi^+ A^+;$$

ψ^+ and A^+ annihilate the initial electron & photon, whilst $\bar{\psi}^-$ and A^- create new ones.

The general definition of a normal-ordered product of field operators of the type A^\pm , ψ^\pm is

$$:A B \cdots W: = (-1)^p A' B' \cdots W'$$

where A', B', \dots, W' are the operators A, B, \dots, W re-ordered so that all creation operators (negative frequency parts) stand to the left of all annihilation operators (positive frequency

parts), and the index p is the number of interchanges of fermionic operators required to achieve this re-ordering.

From the definition of the normal product we have that for two field operators $A(x)$ and $B(x')$

$$A(x)B(x') - :A(x)B(x'): = \begin{cases} \{A^+(x), B^-(x')\} & \text{for two fermion fields} \\ [A^+(x), B^-(x')] & \text{otherwise.} \end{cases}$$

The right-hand side in either case is a c -number as opposed to an operator, and so we can replace it by its vacuum expectation value to obtain

$$A(x)B(x') = :A(x)B(x'): + \langle 0 | A(x)B(x') | 0 \rangle. \quad (2)$$

Now

$$:A(x)B(x'): = \pm :B(x')A(x): \quad (\text{exercise})$$

with the minus sign applying only to the case where A and B are both fermion fields. Provided that $x_0 \neq x'_0$ the two possible time orderings of equation (2) give Wick's Theorem for a product of two fields

$$\boxed{T(A(x)B(x')) = :A(x)B(x'): + \langle 0 | T(A(x)B(x')) | 0 \rangle}. \quad (3)$$

The vacuum expectation value on the right-hand side is denoted

$$\overline{A(x)B(x')} \equiv \langle 0 | T(A(x)B(x')) | 0 \rangle$$

and is called the *contraction* of $A(x)$ and $B(x')$. It will of course vanish unless one of the operators creates particles which are annihilated by the other operator. The non-vanishing contractions are of course just the particle propagators that we introduced earlier

$$\begin{aligned} \overline{\phi(x)\phi(x')} &= i\Delta_F(x-x') \\ \overline{\phi(x)\phi^\dagger(x')} &= \overline{\phi^\dagger(x')\phi(x)} = i\Delta_F(x-x') \\ \overline{\psi_\alpha(x)\bar{\psi}_\beta(x')} &= -\overline{\bar{\psi}_\beta(x')\psi_\alpha(x)} = iS_{F_{\alpha\beta}}(x-x') \\ \overline{A^\mu(x)A^\nu(x')} &= iD_F^{\mu\nu}(x-x'). \end{aligned}$$

We need to generalize equation (3) to the deal with the case of T products of more than two operators. We first define the generalized normal product

$$\overline{:ABCDEF \cdots JKLM \cdots:} = (-1)^p \overline{AKBCEL \cdots DF \cdots JM \cdots:}$$

where p is the number of interchanges of neighbouring fermion operators needed to change the order $ABC \cdots$ to $AKB \cdots$. For example,

$$\overline{: \psi_\alpha(x_1) \bar{\psi}_\beta(x_2) A^\mu(x_3) \bar{\psi}_\gamma(x_4) \bar{\psi}_\delta(x_5):} = (-1) \overline{\bar{\psi}_\beta(x_2) \bar{\psi}_\delta(x_5)} \overline{: \psi_\alpha(x_1) A^\mu(x_3) \bar{\psi}_\gamma(x_4):}$$

Wick's Theorem: For unequal times (i.e., *all* times are unequal)

$$\begin{aligned} T(ABCD \cdots WXYZ) &= :ABCD \cdots WXYZ: \\ &+ :\overline{ABC} \cdots YZ: + :\overline{ABC} \cdots YZ + \cdots + :ABC \cdots \overline{YZ}: \\ &+ :\overline{ABCD} \cdots WXYZ: + \cdots + :ABC \cdots \overline{WXYZ}: \\ &+ \cdots \end{aligned}$$

The right-hand side is the sum of all possible generalized normal products that can be formed from the given product of operators $ABCD \cdots WXYZ$; the first, second, and third lines representing all terms with zero, one, and two contractions respectively, and so on. Each such term contains all the operators in the same order in which they appear in the T product on the left. The proof is by induction and not too illuminating (see Peskin & Schroeder p. 90).

The S -matrix expansion contains so-called *mixed T products*

$$T(\mathcal{H}_I(x_1) \cdots \mathcal{H}_I(x_n)) = T(:A(x_1)B(x_1) \cdots : \cdots :A(x_n)B(x_n) \cdots:).$$

Wick extended his theorem to include such products

$$T(:A(x_1)B(x_1) \cdots: \cdots :(A(x_n)B(x_n) \cdots:) = T(A(x_1)B(x_1) \cdots A(x_n)B(x_n) \cdots)_{\text{no ETC}}$$

where ‘no ETC’ means exclude all equal-time contractions. (See Mandl & Shaw, pp. 105–6).

Feynman Diagrams in Configuration Space

In the last lecture we obtained the Dyson expansion of the S -matrix

$$S = \sum_{n=0}^{\infty} S^{(n)} \equiv \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \int d^4x_2 \cdots \int d^4x_n T(\mathcal{H}_I(x_1)\mathcal{H}_I(x_2) \cdots \mathcal{H}_I(x_n)).$$

In QED

$$\mathcal{H}_I(x) = -e : \bar{\psi}(x)\gamma^\mu A_\mu(x)\psi(x): = -e : (\bar{\psi}^+ + \bar{\psi}^-)(\mathcal{A}^+ + \mathcal{A}^-)(\psi^+ + \psi^-): \quad (4)$$

The terms that are obtained on multiplying out and normal-ordering correspond to eight basic processes involving electron scattering, positron scattering, electron-positron annihilation and electron-positron creation, and may be represented by *Feynman diagrams*. The top row corresponds to all processes in which a photon is annihilated. The first diagram corresponds to the normal-ordered term $:\bar{\psi}^-\mathcal{A}^+\psi^+$: in equation (4), the second to $:\bar{\psi}^+\mathcal{A}^+\psi^-$:, the third to $:\bar{\psi}^+\mathcal{A}^+\psi^+$:, and the fourth to $:\bar{\psi}^-\mathcal{A}^+\psi^-$:. The bottom row is the same but all $\mathcal{A}^+ \rightarrow \mathcal{A}^-$. All creation/annihilation processes occur at the space-time point x .

However, the processes shown cannot be observable physical processes for real particles: it is not possible to satisfy conservation of 4-momentum and at the same time, have the fermions and the photon satisfy the mass-shell conditions $p_1^2 = m^2$, $p_2^2 = m^2$, and $k^2 = 0$ respectively (except for the trivial case of scattering off a zero energy photon), thus

$$\langle f | S^{(1)} | i \rangle = 0.$$

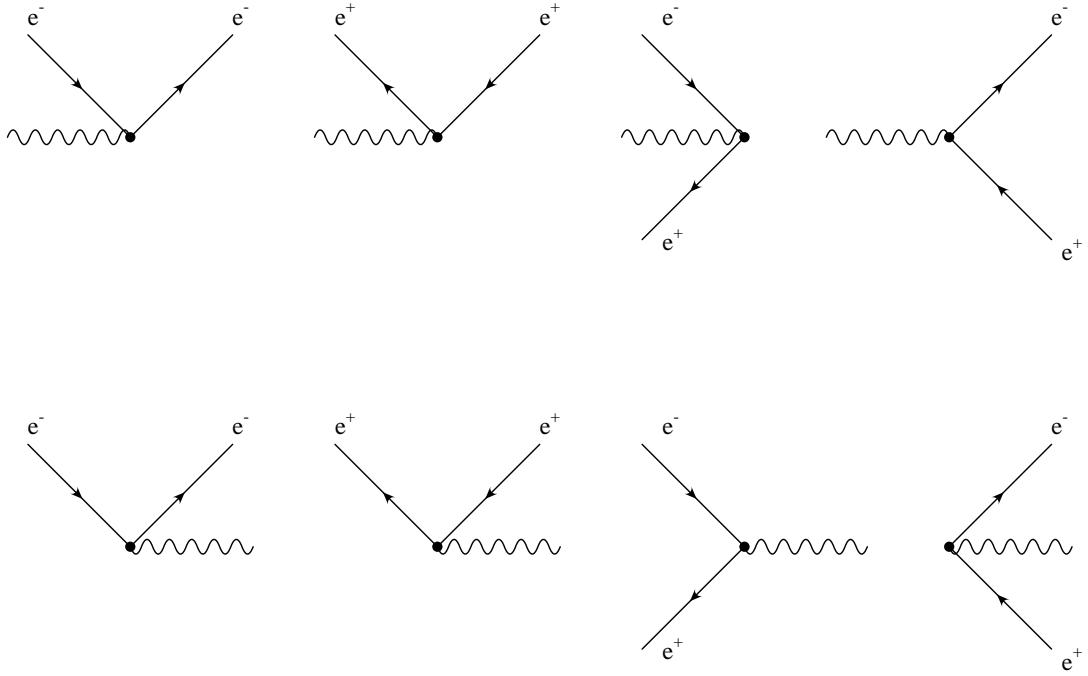


Figure 1: Feynman diagrams which represent the eight basic QED processes in lowest order of the Dyson expansion. By convention arrows on (external) positron lines point backwards.

More generally

$$\langle f | S^{(n)} | i \rangle = 0$$

for any process which violates a conservation law of the theory.

The first contribution to real observable physical processes arises at second order in the Dyson expansion of S

$$S^{(2)} = -\frac{e^2}{2!} \int d^4x_1 \int d^4x_2 T\left(:(\bar{\psi} A \psi)_{x_1} : (\bar{\psi} A \psi)_{x_2} :\right).$$

If we use Wick's Theorem we can write this as a sum of six contributions involving normal products and contractions

$$S^{(2)} = S_A^{(2)} + S_B^{(2)} + S_C^{(2)} + S_D^{(2)} + S_E^{(2)} + S_F^{(2)}$$

where

$$\begin{aligned} S_A^{(2)} &= -\frac{e^2}{2!} \int d^4x_1 \int d^4x_2 :(\bar{\psi} A \psi)_{x_1} (\bar{\psi} A \psi)_{x_2}: \\ S_B^{(2)} &= -\frac{e^2}{2!} \int d^4x_1 \int d^4x_2 \left(:(\bar{\psi} A \psi)_{x_1} \overline{(\bar{\psi} A \psi)_{x_2}}: + :(\overline{\bar{\psi} A \psi}_{x_1} \bar{\psi} A \psi)_{x_2}: \right) \\ S_C^{(2)} &= -\frac{e^2}{2!} \int d^4x_1 \int d^4x_2 :(\bar{\psi} \gamma^\mu \overline{A_\mu} \psi)_{x_1} (\bar{\psi} \gamma^\nu \overline{A_\nu} \psi)_{x_2}: \\ S_D^{(2)} &= -\frac{e^2}{2!} \int d^4x_1 \int d^4x_2 \left(:(\bar{\psi} \gamma^\mu \overline{A_\mu} \psi)_{x_1} (\bar{\psi} \gamma^\nu \overline{A_\nu} \psi)_{x_2}: + :(\overline{\bar{\psi} \gamma^\mu \overline{A_\mu} \psi}_{x_1} \bar{\psi} \gamma^\nu \overline{A_\nu} \psi)_{x_2}: \right) \end{aligned}$$

$$\begin{aligned} S_E^{(2)} &= -\frac{e^2}{2!} \int d^4x_1 \int d^4x_2 :(\bar{\psi} \not{A} \psi)_{x_1} (\bar{\psi} \not{A} \psi)_{x_2}: \\ S_F^{(2)} &= -\frac{e^2}{2!} \int d^4x_1 \int d^4x_2 :(\bar{\psi} \gamma^\mu A_\mu \psi)_{x_1} (\bar{\psi} \gamma^\nu A_\nu \psi)_{x_2}: \end{aligned}$$

The first term just corresponds to various combinations of two independent unphysical processes (at the points x_1 and x_2) of the kind illustrated in Figure 1.

The two terms in $S_B^{(2)}$ are actually identically equal. It is left as an exercise to show that

$$:(\bar{\psi} \not{A} \psi)_{x_1} (\bar{\psi} \not{A} \psi)_{x_2}: = :(\bar{\psi} \not{A} \psi)_{x_2} (\bar{\psi} \not{A} \psi)_{x_1}:,$$

so that by relabelling the integration variables in the second term we can write

$$S_B^{(2)} = -e^2 \int d^4x_1 \int d^4x_2 :(\bar{\psi} \not{A} \psi)_{x_1} (\bar{\psi} \not{A} \psi)_{x_2}: \quad (5)$$

which contains one fermion contraction, or fermion propagator, which we interpret as a virtual electron propagating from x_2 to x_1 when $t_2 < t_1$, or as a virtual positron propagating from x_1 to x_2 when $t_1 < t_2$. In the Feynman propagator these two cases are combined and we simply say that a virtual fermion propagates from x_2 to x_1 .

The remaining uncontracted operators in equation (5) serve to annihilate the initial-state particles and create the final-state particles, which are referred to generically as *external particles*.

Compton Scattering

Let us consider the Compton scattering process

$$e^- + \gamma \rightarrow e^- + \gamma$$

which corresponds to terms in $S_B^{(2)}$ which contain $\psi^+(x_2)$ to annihilate the initial electron and $\bar{\psi}^-(x_1)$ to create the final electron. Either $\not{A}^+(x_2)$ or $\not{A}^+(x_1)$ can annihilate the initial photon, and correspondingly either $\not{A}^-(x_1)$ or $\not{A}^-(x_2)$ creates the final photon. There are thus two terms in equation (5) which contribute to Compton scattering

$$S_{\text{Compton}}^{(2)} = S_a + S_b$$

where

$$\begin{aligned} S_a &= -e^2 \int d^4x_1 \int d^4x_2 \bar{\psi}^-(x_1) \gamma^\mu i S_F(x_1 - x_2) \gamma^\nu A_\nu^-(x_1) A_\mu^+(x_2) \psi^+(x_2) \\ S_b &= -e^2 \int d^4x_1 \int d^4x_2 \bar{\psi}^-(x_1) \gamma^\mu i S_F(x_1 - x_2) \gamma^\nu A_\nu^-(x_2) A_\mu^+(x_1) \psi^+(x_2). \end{aligned}$$

We may represent these two contributions by the Feynman diagrams shown in Figure (2). The electron propagates in the direction of the arrow on the electron line. The incident photon is absorbed at x_2 in the left-hand diagram, a virtual electron propagates from x_2 to x_1 , where the final photon is emitted, whereas in the right-hand diagram the final photon is

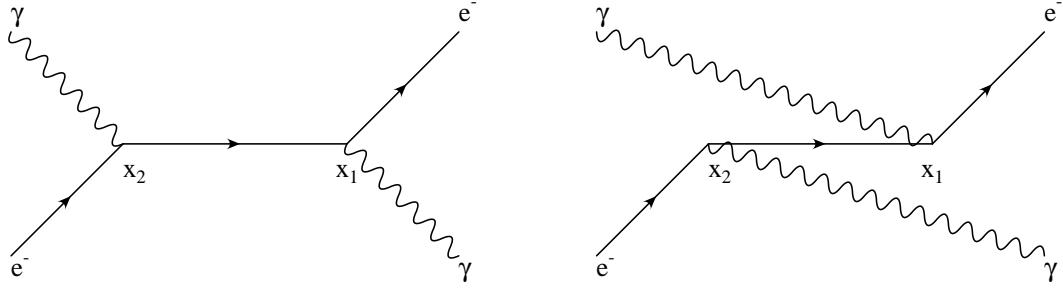


Figure 2: Feynman diagrams for Compton scattering, corresponding to S_a and S_b .

emitted at x_2 and the incident photon is absorbed at x_1 . Note that because the Feynman propagator describes both time-orderings for the virtual particle there is no time-ordering in Feynman diagrams. The fermion propagator, and hence the internal fermion line in each diagram, includes the contribution from a virtual positron propagating from x_1 to x_2 . Of course, equation (5) also describes all the other possible processes which involve the propagation of a virtual electron or positron. These are

- positron Compton scattering: $e^+ + \gamma \rightarrow e^+ + \gamma$;
- electron-positron annihilation into 2 photons: $e^- + e^+ \rightarrow \gamma + \gamma$;
- electron-positron pair production by two photons: $\gamma + \gamma \rightarrow e^- + e^+$.

The corresponding Feynman diagrams are shown in Figure 3. For example, the pair production process corresponds to the term:

$$S_{\text{pair}} = -e^2 \int d^4x_1 \int d^4x_2 \bar{\psi}^-(x_1) \gamma^\mu i S_F(x_1 - x_2) \gamma^\nu \psi^-(x_2) A_\mu^+(x_1) A_\nu^+(x_2).$$

Electron-Electron Scattering

The term $S_C^{(2)}$ contains a single contraction of A fields, with four fermion fields uncontracted and so contains the terms needed to describe electron-electron scattering, also known as Møller scattering

$$e^- + e^- \rightarrow e^- + e^-.$$

The relevant term is clearly

$$S_{\text{Møller}}^{(2)} = -\frac{e^2}{2!} \int d^4x_1 \int d^4x_2 :(\bar{\psi}^- \gamma^\mu \psi^+)_{x_1} (\bar{\psi}^- \gamma^\nu \psi^+)_{x_2} : i D_{F\mu\nu}(x_1 - x_2)$$

where we have substituted the photon propagator for the contraction of A fields.

Now this actually gives four contributions to electron-electron scattering, because either initial electron can be annihilated by either ψ^+ operator and likewise either final electron can be created by either $\bar{\psi}^-$ operator. Two pairs of terms differ only by relabelling of the dummy integration variables $x_1 \leftrightarrow x_2$, and so we need only consider one pair of terms, provided that we multiply the result by 2, thus getting rid of the $2!$ in the denominator. Quite generally, *we can omit the factor of $n!$ in the S-matrix expansion if we only consider topologically distinct Feynman diagrams.*

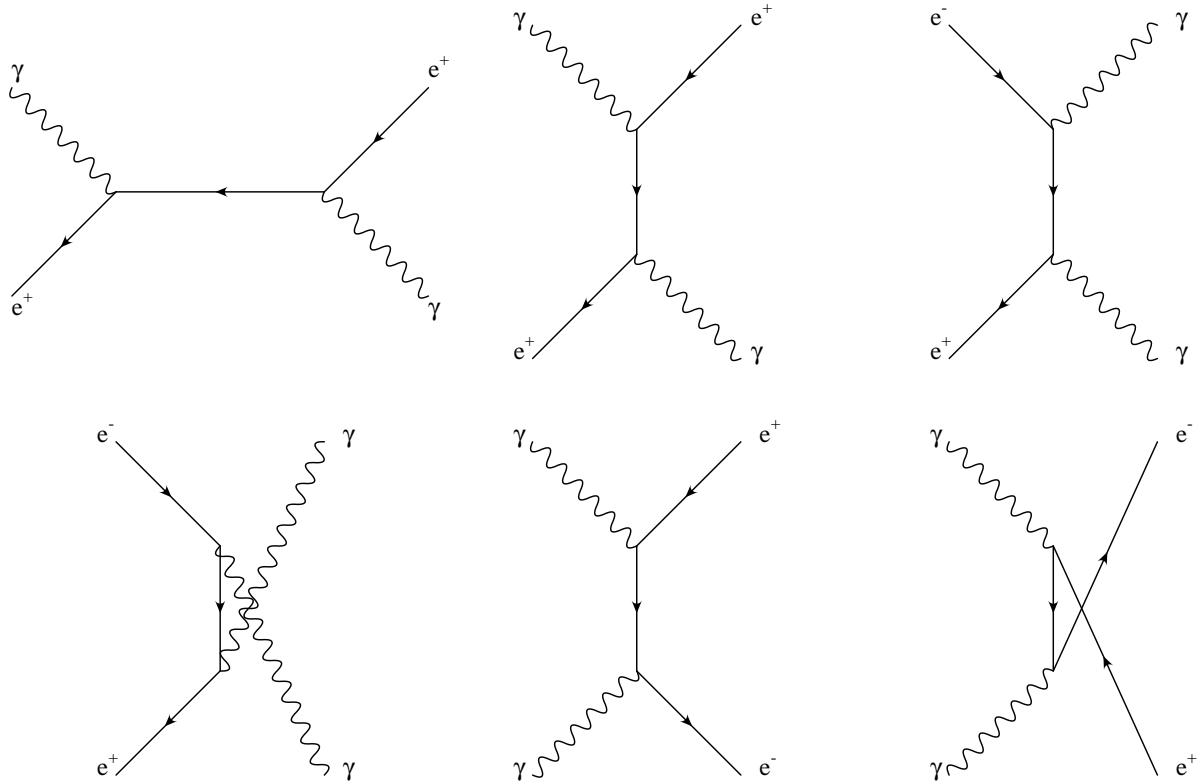


Figure 3: Feynman diagrams for Compton scattering of positrons, electron-positron annihilation into photons, and pair production.

Heuristic Analysis. Let us label the initial and final electrons 1, 2 and $1'$, $2'$ respectively. Let us also put the same labels on the specific fields which destroy (create) the initial (final) state particles — this is a bit artificial but will be fixed later.

We may then write the Møller scattering contribution to the S -matrix expansion as a sum of two terms depending on whether $\bar{\psi}^-(x_1)$ creates electron $1'$ or electron $2'$ respectively

$$\begin{aligned} S_{\text{Møller}}^{(2)} &= S_a + S_b \quad \text{where} \\ S_a &= -e^2 \int d^4 x_1 \int d^4 x_2 :(\bar{\psi}_{1'}^-\gamma^\mu \psi_1^+)_x_1 (\bar{\psi}_{2'}^-\gamma^\nu \psi_2^+)_x_2: iD_F{}_{\mu\nu}(x_1 - x_2) \\ S_b &= -e^2 \int d^4 x_1 \int d^4 x_2 :(\bar{\psi}_2^-\gamma^\mu \psi_1^+)_x_1 (\bar{\psi}_{1'}^-\gamma^\nu \psi_2^+)_x_2: iD_F{}_{\mu\nu}(x_1 - x_2). \end{aligned}$$

The point we wish to make here is that a relative minus sign between the two terms is generated when we perform the normal ordering of the fermion operators in order to give the sequence of creation and annihilation operators required to create/destroy the appropriate particles

$$-\bar{\psi}_{1'}^-(x_1)\bar{\psi}_{2'}^-(x_2)\psi_1^+(x_1)\psi_2^+(x_2) \quad \text{and} \quad +\bar{\psi}_{1'}^-(x_2)\bar{\psi}_{2'}^-(x_1)\psi_1^+(x_1)\psi_2^+(x_2).$$

The Møller scattering contribution to $S_B^{(2)}$ may be represented by the two topologically distinct Feynman diagrams shown in Figure 4. Note that permuting the vertices labelled x_1 and x_2 does not interchange the diagrams which therefore represent distinct contributions to the

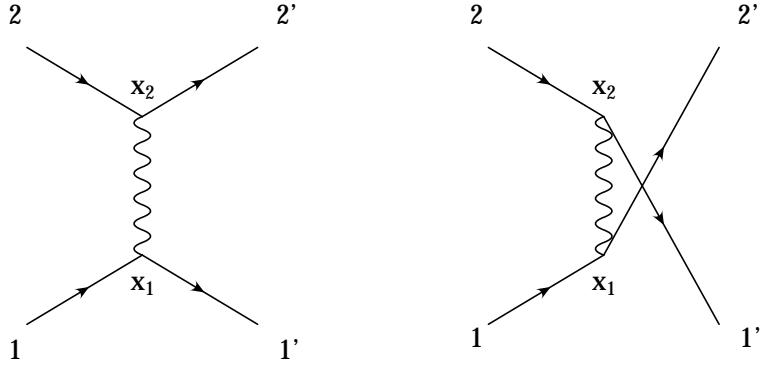


Figure 4: Feynman diagrams for electron-electron scattering.

scattering amplitude. When we take the matrix element of $S_{\text{M}\ddot{\text{o}}\text{l}\text{e}r}^{(2)}$ between initial and final momentum eigenstates we will indeed obtain expressions corresponding to both terms. The scattering amplitude is therefore also the sum of two terms, and is *antisymmetric* under interchange of the labels $1'$ and $2'$ of the two electrons in the final state. This argument is a bit of a hand-wave — the proper derivation is left as a tutorial problem.

We can generalize these arguments to any process in which the initial or final states contain several identical fermions: the amplitude is antisymmetric in the labels of the identical fermions. Furthermore, this extends to cases where there are initial-state electrons and final-state positrons (and vice versa) because the field $\psi(x)$ can annihilate an electron or *create a positron*. An example of this is provided by Bhabha scattering

$$e^- + e^+ \rightarrow e^- + e^+$$

for which there are again two contributing terms corresponding to the Feynman diagrams shown in Figure 5. The second diagram is obtained from the first by interchanging the initial electron line at x_1 and the final positron line at x_2 . It is left as an exercise to obtain the

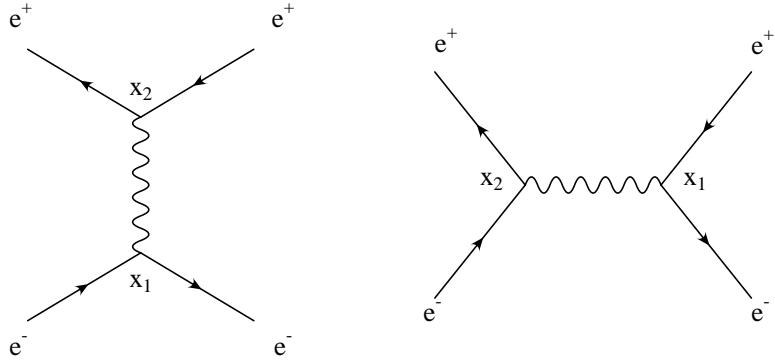


Figure 5: Feynman diagrams for electron-positron scattering.

corresponding contributions to the S -matrix.

Electron Self-Energy

The term $S_D^{(2)}$ contains only two uncontracted fermion fields, and includes both a fermion and a photon propagator. It describes two processes, with either an electron or a positron in both the initial and final state and is known as the *self-energy* of the electron or positron. Taking the electron self-energy as an example the expression reduces to

$$S_{\text{self-energy}}^{(2)} = -e^2 \int d^4x_1 \int d^4x_2 \bar{\psi}^-(x_1) \gamma^\mu iS_F(x_1 - x_2) \gamma^\nu \psi^+(x_2) iD_F{}_{\mu\nu}(x_1 - x_2),$$

which corresponds to the left-hand Feynman diagram in Figure 6. What does this represent

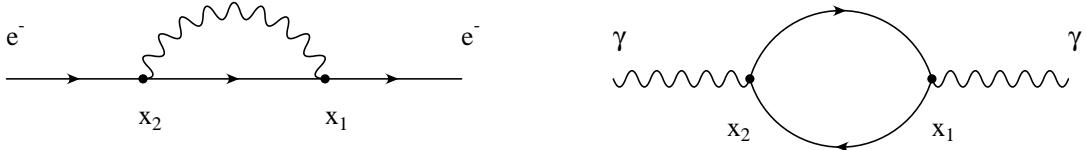


Figure 6: Feynman diagrams for electron and photon self-energies.

physically? It corresponds to the emission and re-absorption of a virtual photon by a so-called *bare* electron. The bare electron is constantly emitting and re-absorbing a ‘cloud’ of virtual photons and this process leads to the physical electron that we observe. The diagram above just represents the lowest-order contribution. The interaction changes the energy of the electron, that is it changes the mass from that of a bare electron. Evaluating the Feynman diagram leads to an integral which diverges due to infinite-momentum virtual particles in the loop. The cure for this disease is a procedure known as *renormalization*, which absorbs the divergence by re-expressing the results of the theory in terms of the observable physical mass of the electron, rather than the unobservable bare mass which appears in the Lagrangian. The fermion kinetic energy is also renormalized.

Photon Self-Energy

The term $S_E^{(2)}$ involves two external photons and no uncontracted fermion fields

$$S_E^{(2)} = -\frac{e^2}{2!} \int d^4x_1 \int d^4x_2 :(\bar{\psi} \not{A} \psi)_{x_1} (\bar{\psi} \not{A} \psi)_{x_2}:$$

It corresponds to the right-hand Feynman diagram of Figure 6, where a photon creates a virtual electron-positron pair which subsequently annihilate. Such diagrams are referred to as *vacuum polarization diagrams*. Infinite-momentum virtual particles propagating in the loop again give rise to an infinity and the photon field must be renormalized to absorb the infinity. The photon mass is not renormalized — it remains zero to all orders in perturbation theory.

The relevant terms in the normal product are

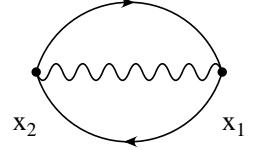
$$\begin{aligned} :(\bar{\psi}_\alpha (\not{A}^-)_{\alpha\beta} \psi_\beta)_{x_1} (\bar{\psi}_\gamma (\not{A}^+)_{\gamma\delta} \psi_\delta)_{x_2}: &= -\bar{\psi}_\delta(x_2) \bar{\psi}_\alpha(x_1) (\not{A}^-)_{\alpha\beta}(x_1) \bar{\psi}_\beta(x_1) \bar{\psi}_\gamma(x_2) (\not{A}^+)_{\gamma\delta}(x_2) \\ &= -\text{tr} [iS_F(x_2 - x_1) \not{A}^-(x_1) iS_F(x_1 - x_2) \not{A}^+(x_2)], \end{aligned}$$

where we have made the Dirac matrix indices explicit to reveal that a trace over these indices is involved.

Note: The minus sign in the last expression is characteristic of *a closed fermion loop*, as is the trace, which arises from an implicit sum over the spin states of the virtual electron-positron pair.

Vacuum Bubbles

The final term, $S_F^{(2)}$, corresponds to a diagram with *no external lines* at all and consequently causes no transitions. Such diagrams, known as *vacuum bubbles or vacuum diagrams*, can therefore be ignored when calculating scattering processes.



S-Matrix Elements

We are usually more interested in the matrix elements of S between given initial and final states rather than the terms in S itself. The initial and final states are specified in terms of the momenta and spin or polarization states of the incident and scattered particles. Calculation of the requisite matrix elements is thus more conveniently performed by Fourier transforming the fields in order to pick out the relevant annihilation and creation operators corresponding to the specified momenta and spins.

The propagators in momentum space are defined through

$$\overline{\psi(x_1)\bar{\psi}(x_2)} = iS_F(x_1 - x_2) = \frac{i}{(2\pi)^4} \int d^4p \tilde{S}_F(p) e^{-ip \cdot (x_1 - x_2)}$$

$$\overline{A^\mu(x_1)A^\nu(x_2)} = iD_F^{\mu\nu}(x_1 - x_2) = \frac{i}{(2\pi)^4} \int d^4k \tilde{D}_F^{\mu\nu}(k) e^{-ik \cdot (x_1 - x_2)}$$

so that

$$\boxed{\tilde{S}_F(p) = \frac{\not{p} + m}{p^2 - m^2 + i\varepsilon} = \frac{1}{\not{p} - m + i\varepsilon}} \quad \boxed{\tilde{D}_F^{\mu\nu}(p) = \frac{-g^{\mu\nu}}{k^2 + i\varepsilon}}.$$

The Fourier mode expansions of the fields are

$$\begin{aligned} \psi(x) &= \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega(\underline{p})} \left(a_s(\underline{p}) u(p, s) e^{-ip \cdot x} + b_s(\underline{p})^\dagger v(p, s) e^{ip \cdot x} \right) \\ \bar{\psi}(x) &= \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega(\underline{p})} \left(b_s(\underline{p}) \bar{v}(p, s) e^{-ip \cdot x} + a_s(\underline{p})^\dagger \bar{u}(p, s) e^{ip \cdot x} \right) \\ A^\mu(x) &= \sum_r \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega(\underline{k})} \left(\epsilon_r^\mu(\underline{k}) a_r(\underline{k}) e^{-ik \cdot x} + \epsilon_r^{\mu*}(\underline{k}) a_r(\underline{k})^\dagger e^{ik \cdot x} \right). \end{aligned}$$

Using the (anti-)commutators for the creation and annihilation operators and the fact that the vacuum is annihilated by the relevant annihilation operators we may easily show that

the effect of the uncontracted positive-frequency operators which occur in a given term in the S -matrix on single-particle states is

$$\begin{aligned}\psi^+(x) |e^-, \underline{p}, s\rangle &= u(p, s) e^{-ip \cdot x} |0\rangle \\ \bar{\psi}^+(x) |e^+, \underline{p}, s\rangle &= \bar{v}(p, s) e^{-ip \cdot x} |0\rangle \\ A^\mu + (x) |\gamma, \underline{k}, r\rangle &= \epsilon_r^\mu(\underline{k}) e^{-ik \cdot x} |0\rangle.\end{aligned}$$

Correspondingly, the effect of the uncontracted negative-frequency operators acting on the vacuum state to produce single-particle states is

$$\begin{aligned}\bar{\psi}^-(x) |0\rangle &= \sum_s \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega(\underline{p})} \bar{u}(p, s) e^{ip \cdot x} |e^-, \underline{p}, s\rangle \\ \psi^-(x) |0\rangle &= \sum_s \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega(\underline{p})} v(p, s) e^{ip \cdot x} |e^+, \underline{p}, s\rangle \\ A^\mu - (x) |0\rangle &= \sum_r \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega(\underline{k})} \epsilon_r^{\mu*}(\underline{k}) e^{ik \cdot x} |\gamma, \underline{k}, r\rangle.\end{aligned}$$

Feynman Diagrams in Momentum Space

First-Order Terms

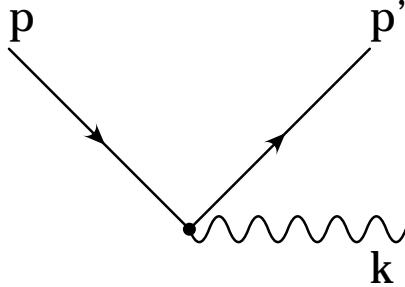


Figure 7: The basic vertex $e^- \rightarrow e^- + \gamma$.

Consider the basic vertex shown in Figure 7, which is called a *momentum space Feynman diagram*. The initial state consists of a single electron of momentum \underline{p} and spin s , while the final state consists of an electron of momentum \underline{p}' and spin s' together with a photon of momentum \underline{k} and polarization r . Thus

$$|i\rangle = |e^-, \underline{p}, s\rangle, \quad |f\rangle = |e^-, \underline{p}', s'; \gamma, \underline{k}, r\rangle.$$

The desired S -matrix element is

$$\begin{aligned}\langle f | S^{(1)} | i \rangle &= \langle e^-, \underline{p}', s'; \gamma, \underline{k}, r | ie \int d^4 x \bar{\psi}^-(x) \gamma^\mu A_\mu^-(x) \psi^+(x) | e^-, \underline{p}, s \rangle \\ &= ie \int d^4 x \left(\bar{u}(\underline{p}', s') e^{ip' \cdot x} \right) (\not{\epsilon}_r(\underline{k}) e^{ik \cdot x}) \left(u(\underline{p}, s) e^{-ip \cdot x} \right),\end{aligned}$$

where we have assumed without loss of generality that the photon polarization vector is real.

We can now perform the x -integration

$$\int d^4x e^{i(p'+k-p)\cdot x} = (2\pi)^4 \delta^{(4)}(p' + k - p)$$

which expresses conservation of 4-momentum at the vertex $p = p' + k$.

We write

$$\langle f | S^{(1)} | i \rangle = (2\pi)^4 \delta^{(4)}(p' + k - p) \mathcal{M},$$

where \mathcal{M} is called *the Feynman amplitude* and is given by

$$\mathcal{M} = ie\bar{u}(\underline{p}', s')\gamma^\mu\epsilon_{r\mu}(\underline{k})u(\underline{p}, s).$$

As we remarked earlier it is not possible to satisfy the 4-momentum conservation condition when all three particles are on-shell.

Second-Order Terms

Compton Scattering

As an example the momentum space Feynman diagrams for Compton scattering are shown in Figure 8. They look very similar to the x -space diagrams that we drew before, except that now the external lines are labeled by the momenta (and spins) of the incoming and outgoing particles and the vertices are no longer labeled by a space-time position.

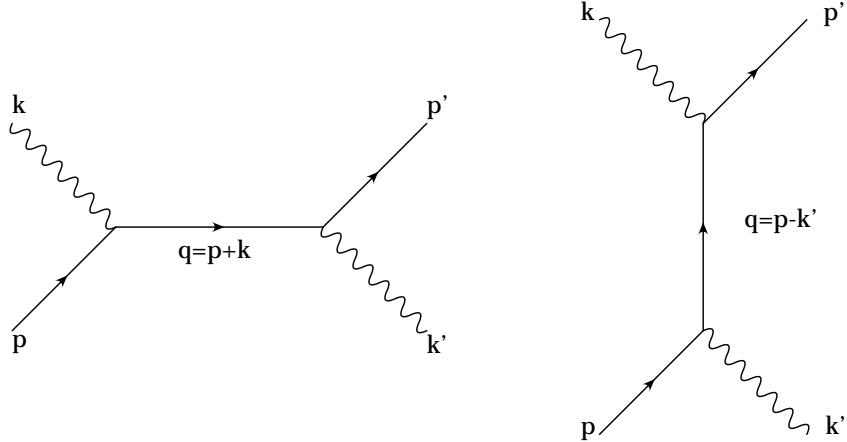


Figure 8: Momentum space Feynman diagrams for Compton scattering.

The first diagram corresponds to the contribution S_a

$$\begin{aligned} \langle f | S_a | i \rangle &= -e^2 \int d^4x_1 \int d^4x_2 \left(\bar{u}(\underline{p}', s') e^{ip'\cdot x_1} \right) \left(\epsilon_{r'}(\underline{k}') e^{ik'\cdot x_1} \right) \\ &\quad \times \int \frac{d^4q}{(2\pi)^4} i\tilde{S}_F(q) e^{-iq\cdot(x_1 - x_2)} \\ &\quad \times (\epsilon_r(\underline{k}) e^{-ik\cdot x_2}) (u(\underline{p}, s) e^{-ip\cdot x_2}). \end{aligned}$$

Again we can perform the space-time integrations which give

$$\int d^4x_1 e^{i(p'+k'-q)\cdot x_1} \int d^4x_2 e^{i(q-k-p)\cdot x_2} = (2\pi)^4 \delta^{(4)}(p' + k' - q)(2\pi)^4 \delta^{(4)}(q - p - k);$$

we see that 4-momentum is conserved at each vertex and in particular the δ functions imply the constraint

$$q = p + k = p' + k'$$

so that the 4-momentum of the virtual electron is fixed and equal to the total initial (and final) 4-momentum.

We can use one of the δ functions to perform the integral over q , giving

$$\langle f | S_a | i \rangle = (2\pi)^4 \delta^{(4)}(p' + k' - p - k) \mathcal{M}_a,$$

with the Feynman amplitude

$$\mathcal{M}_a = -e^2 \bar{u}(\underline{p}', s') \not{\epsilon}_{r'}(\underline{k}') i \tilde{S}_F(q = p + k) \not{\epsilon}_r(\underline{k}) u(\underline{p}, s).$$

We leave it as an exercise to compute the contribution from S_b corresponding to the second of the diagrams in Figure 8 and show that it is given by

$$\mathcal{M}_b = -e^2 \bar{u}(\underline{p}', s') \not{\epsilon}_r(\underline{k}) i \tilde{S}_F(q = p - k') \not{\epsilon}_{r'}(\underline{k}') u(\underline{p}, s).$$

Note: The factors u , \bar{u} , \tilde{S}_F , $\not{\epsilon}$ etc. are vectors and matrices in Dirac space and therefore the order in which they occur must be strictly adhered to. The rule is *following a fermion line in the sense of the arrows corresponds to writing the Dirac matrix-valued factors from right to left*.

Møller Scattering

The two momentum space diagrams are as shown in Figure 9.

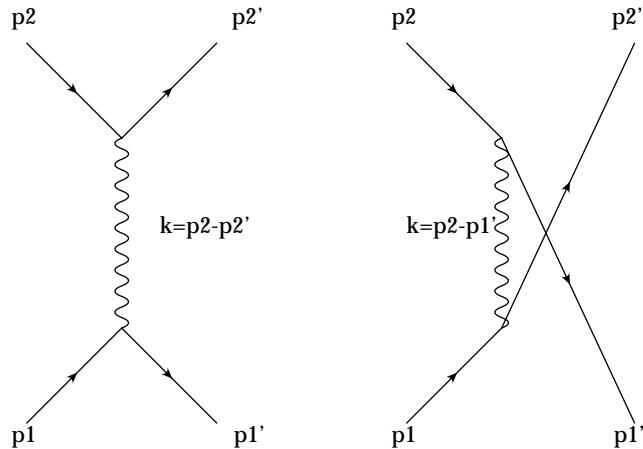


Figure 9: Momentum space Feynman diagrams for Møller scattering.

The relevant S -matrix element is

$$\langle f | S_{\text{Möller}}^{(2)} | i \rangle = (2\pi)^4 \delta^{(4)}(p'_1 + p'_2 - p_1 - p_2)(\mathcal{M}_a + \mathcal{M}_b),$$

where the Feynman amplitudes are given by

$$\begin{aligned} \mathcal{M}_a &= -e^2 \bar{u}(\underline{p}'_1, s'_1) \gamma^\mu u(\underline{p}_1, s_1) i \tilde{D}_{F_{\mu\nu}}(k = p_2 - p'_2) \bar{u}(\underline{p}'_2, s'_2) \gamma^\nu u(\underline{p}_2, s_2) \\ \mathcal{M}_b &= +e^2 \bar{u}(\underline{p}'_2, s'_2) \gamma^\mu u(\underline{p}_1, s_1) i \tilde{D}_{F_{\mu\nu}}(k = p_2 - p'_1) \bar{u}(\underline{p}'_1, s'_1) \gamma^\nu u(\underline{p}_2, s_2). \end{aligned}$$

Note: The relative minus sign between the two contributions is now explicit so that the antisymmetry of the amplitude under interchange of the final-state electron labels is manifest. Because the photon propagator is actually a function of k^2 rather than k , the sense of k is arbitrary and so there is no arrow on the internal photon line. However, it is necessary to *choose* a sense for k in order to define the 4-momentum conservation δ functions associated with each vertex.

Closed Loops

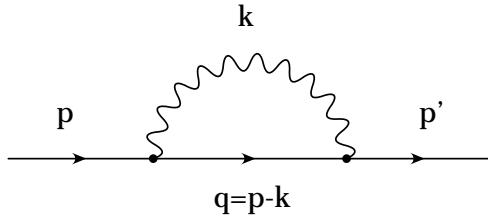


Figure 10: momentum space Feynman diagram for the electron self-energy.

For a loop diagram such as the electron self-energy there is a new feature: conservation of 4-momentum at the vertices does not determine the 4-momenta q and k of the virtual electron and photon $p = q + k = p'$, so it is plausible that the amplitude will involve a sum over the possible loop momenta. A detailed calculation gives

$$\langle f | S_{\text{self}}^{(2)} | i \rangle = (2\pi)^4 \delta^{(4)}(p - p') \mathcal{M},$$

with the Feynman amplitude

$$\mathcal{M} = -e^2 \int \frac{d^4 k}{(2\pi)^4} i \tilde{D}_{F_{\mu\nu}}(k) \bar{u}(p, s) \gamma^\mu i \tilde{S}_F(p - k) \gamma^\nu u(p, s).$$

Relativistic Quantum Field Theory

Transition Rates & Cross-Sections

The Feynman amplitude \mathcal{M} may be written as a sum of contributions

$$\mathcal{M} = \sum_{n=0}^{\infty} \mathcal{M}^{(n)}$$

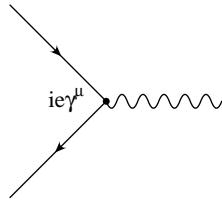
with the term $\mathcal{M}^{(n)}$ coming from the n th order term in the expansion of the S -matrix.

We have already seen several examples of how the n th order amplitude may be associated with a set of topologically distinct Feynman diagrams in momentum space containing n basic vertices. The *Feynman rules* establish a calculus that involves associating each feature of a Feynman diagram with a unique expression from which the amplitude is built.

Feynman rules for QED:

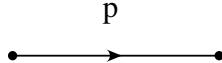
1. For each fermion-fermion-photon vertex, a factor

$$-iq\gamma^\mu \equiv ie\gamma^\mu$$



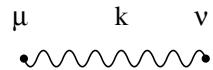
2. For each *internal* fermion line labelled by the 4-momentum p a fermion propagator

$$i\tilde{S}_F(p) = \frac{i}{p - m + i\varepsilon}$$



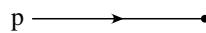
3. For each *internal* photon line labelled by 4-momentum k a photon propagator

$$i\tilde{D}_{F\mu\nu}(k) = \frac{-ig_{\mu\nu}}{k^2 + i\varepsilon}$$

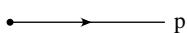


4. For each *external* line one of the following factors

- (a) For each initial electron $u(p, s)$



- (b) For each final electron $\bar{u}(p, s)$



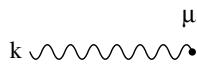
- (c) For each initial positron $\bar{v}(p, s)$



- (d) For each final positron $v(p, s)$



- (e) For each initial photon $\epsilon_\mu^r(k)$



- (f) For each final photon ϵ_μ^{*r}



5. The Dirac-matrix-valued factors are to be ordered so that reading from right to left corresponds to the sequence obtained by following the fermion line in the direction of its arrows.
6. For each closed fermion loop take the trace and include a factor of -1 .
7. The 4-momenta corresponding to the three lines which meet at a vertex are conserved. For each 4-momentum q not so fixed an integral
$$\frac{1}{(2\pi)^4} \int d^4 q;$$
one such integration occurs for each closed loop in the diagram.
8. Multiply the amplitude by a phase factor $(-1)^P$, where P is the number of interchanges of neighboring fermion operators required to achieve normal ordering.

Transition Rates and Cross-Sections

Having found out how to calculate the S -matrix elements S_{fi} we now turn to the problem of calculating scattering cross-sections. Given our definition of the Feynman amplitude \mathcal{M} for the transition from an initial state $|i\rangle$ to a final state $|f\rangle$ we note that

$$S_{fi} = \delta_{fi} + (2\pi)^4 \delta^{(4)}(P_f - P_i) \mathcal{M}. \quad (1)$$

Here, P_f is the total 4-momentum of the final-state particles and P_i the total 4-momentum of the initial-state particles.

The first term in equation (1) corresponds to no scattering and may be ignored in what follows.

Note: There are conventions for the Feynman amplitude which differ by a factor of i from the one which we are using. You may also see the notation

$$S_{fi} = \delta_{fi} + i(2\pi)^4 \delta^{(4)}(P_f - P_i) T_{fi},$$

where T_{fi} is referred to as the T matrix element.

Transition Rate

The probability for transition to a range of final states is

$$\mathcal{P} = \sum_f \left| \langle f | S | i \rangle \right|^2 = (2\pi)^4 \delta^{(4)}(0) \sum_f (2\pi)^4 \delta^{(4)}(P_f - P_i) |\mathcal{M}|^2,$$

where we have used the properties of the δ function to write

$$\left(\delta^{(4)}(P_f - P_i) \right)^2 = \delta^{(4)}(0) \delta^{(4)}(P_f - P_i).$$

How are we to interpret the factor of $\delta^{(4)}(0)$? Recall the completeness relation

$$(2\pi)^4 \delta^{(4)}(P) = \int d^4 x e^{-iP \cdot x}$$

which gives

$$(2\pi)^4 \delta^{(4)}(0) = \int d^4x = VT,$$

where VT denotes the (infinite) volume of space-time. One way of proceeding a little more rigorously would be to treat the entire system as enclosed by a finite spatial box of volume V , let the total elapsed time for the scattering be T , and take the limits $V \rightarrow \infty$ and $T \rightarrow \infty$ at the end of the calculation. Actually, what we want to compute is the *transition rate per unit volume* which is given by

$$w_{fi} = \frac{\mathcal{P}}{VT} = \sum_f (2\pi)^4 \delta^{(4)}(\underline{P}_f - \underline{P}_i) |\mathcal{M}|^2.$$

The sum over final states in practice means the transition rate to a group of states with momenta in an infinitesimal interval $(\underline{p}_f, \underline{p}_f + d\underline{p}_f)$. With our normalization of states, the completeness relation for one-particle states is

$$1 = \int \frac{d^4p}{(2\pi)^4} (2\pi) \delta(p^2 - m^2) \theta(p^0) |\underline{p}\rangle \langle \underline{p}| = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega(\underline{p})} |\underline{p}\rangle \langle \underline{p}|,$$

so that the required *density of final states* is

$$\frac{d^3p}{(2\pi)^3 2\omega(\underline{p})}.$$

There will be one such factor for each final-state particle.

Incident Flux

Our normalization of 1-particle states implies that

$$\langle \underline{p} | \underline{p} \rangle = (2\pi)^3 2\omega(\underline{p}) \delta^{(3)}(0),$$

and if we again consider a finite volume V we see that

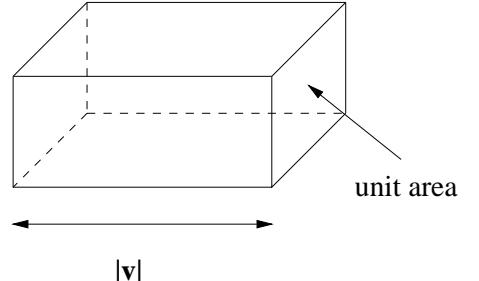
$$\frac{\langle \underline{p} | \underline{p} \rangle}{V} = 2\omega(\underline{p})$$

corresponding to a density of $2\omega(\underline{p})$ particles per unit volume.

It will be convenient to work in the LAB frame for two-body scattering where the target particle is at rest $\underline{p}_2 = (m, \underline{0})$. We denote the 4-momentum of the particles in the incident beam by $\underline{p}_1 = (E_1 \equiv \omega(\underline{p}_1), \underline{p}_1)$.

The *incident flux* I is defined to be the number of beam particles crossing unit area perpendicular to the beam direction per unit time, which is simply the density \times the magnitude of the velocity of the particles in the beam

$$I = 2E_1 \times |\underline{v}| = 2E_1 \frac{|\underline{p}_1|}{E_1} = 2|\underline{p}_1|.$$



With our normalization of states a unit volume contains $2E_2$ scattering centres (i.e., target particles) and so the transition rate per scattering centre per unit incident flux into the

element of phase space corresponding to the density of final states given above, which is called the *differential cross section*, is given by

$$d\sigma = (2\pi)^4 \delta^{(4)}(P_f - P_i) \frac{1}{2E_1 2E_2 |\underline{v}|} \left(\prod_f \frac{d^3 p_f}{(2\pi)^3 2E_{p_f}} \right) |\mathcal{M}|^2. \quad (2)$$

Note: The product $E_1 E_2 |\underline{v}|$ may be written in manifestly covariant form

$$E_1 E_2 |\underline{v}| = \sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2},$$

so that equation (2) holds in any Lorentz frame in which the two colliding particles are collinear such as for example the Centre of Momentum (CM) frame, as well as the LAB frame, provided we take $|\underline{v}|$ to be the magnitude of the *relative velocity* of the colliding particles.

The Two-Body Cross-Section

Consider the general elastic 2-particle to 2-particle cross-section

$$p_1 + p_2 \rightarrow p_3 + p_4$$

where we take the mass of particle 1 (and particle 3) to be m_1 and that of particle 2 (and particle 4) to be m_2 .

We can write equation (2) as

$$d\sigma = \frac{|\mathcal{M}|^2}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} d_{\text{LIPS}} \quad (3)$$

where

$$d_{\text{LIPS}} \equiv (2\pi)^4 \delta^{(4)}(p_3 + p_4 - p_1 - p_2) \times \frac{d^3 p_3}{(2\pi)^3 2E_3} \frac{d^3 p_4}{(2\pi)^3 2E_4}$$

is known as the *two-particle Lorentz-invariant phase space*.

We are interested in calculating the differential cross-section with respect to the polar and azimuthal angles of the outgoing scattered particle of 4-momentum p_3 , say. This is usually denoted by

$$\frac{d\sigma}{d\Omega}$$

and involves integrating over the remaining final state kinematic variables.

We can perform the integral over $d^3 p_4$ by using three of the four δ functions

$$\int d^3 p_4 \delta^{(4)}(p_3 + p_4 - p_1 - p_2) = \delta(E_3 + E_4 - E_1 - E_2)$$

where it is understood that now $\underline{p}_4 = \underline{p}_1 + \underline{p}_2 - \underline{p}_3$ and of course $E_4^2 = |\underline{p}_4|^2 + m_2^2$.

We will express the momentum space volume element for \underline{p}_3 in terms of polar variables:

$$d^3 p_3 = p_3^2 dp_3 d\Omega$$

where $p_3 \equiv |\underline{p}_3|$, and integrate over p_3 using the remaining δ function to yield the cross-section differential with respect to the element of solid angle $d\Omega$.

We now express the momentum space volume element for \underline{p}_3 in terms of polar variables

$$d^3 p_3 = p_3^2 d\underline{p}_3 d\Omega,$$

where $p_3 \equiv |\underline{p}_3|$, and note that

$$E_3^2 = p_3^2 + m_1^2 \Rightarrow E_3 dE_3 = p_3 dp_3;$$

thus the partially integrated phase space is

$$\int d_{\text{LIPS}} = \frac{1}{(4\pi)^2} d\Omega \frac{p_3 dE_3}{E_4} \delta(E_3 + E_4 - E_1 - E_2). \quad (4)$$

It only remains to carry out the integral over E_3 to obtain the required cross-section which is differential with respect to the solid angle of the scattered particle. So far the expression for the phase space is *valid in any Lorentz frame*.

Centre of Momentum (CM) Cross-Section

In the CM frame, we take the 4-momenta to be

$$p_1 = (E_1, \underline{p}); \quad p_2 = (E_2, -\underline{p}); \quad p_3 = (E_3, \underline{p}'); \quad p_4 = (E_4, -\underline{p}')$$

where $|\underline{p}| = |\underline{p}'|$ (because we are in the CM frame, and also $m_3 = m_1$, $m_4 = m_2$), and the total CM energy is given by

$$W \equiv E_1 + E_2 = \sqrt{p^2 + m_1^2} + \sqrt{p'^2 + m_2^2}.$$

where $p \equiv |\underline{p}|$. In this frame (again defining $p' \equiv |\underline{p}'|$)

$$E_3^2 = p'^2 + m_1^2 \quad \text{and} \quad E_4^2 = p'^2 + m_2^2 \quad \text{so that} \quad E_3 dE_3 = E_4 dE_4 = p' dp'.$$

Define W' to be the total final state energy $W' = E_3 + E_4$ and change integration variable from E_3 to W' in equation (4)

$$dW' = dE_3 + dE_4 = \frac{p' dp'}{E_3} + \frac{p' dp'}{E_4} = \frac{W'}{E_3 E_4} p' dp' = \frac{W'}{E_4} dE_3.$$

Thus the factor

$$p_3 \frac{dE_3}{E_4} \delta(E_3 + E_4 - E_1 - E_2)$$

in equation (4) may be written

$$\frac{p'}{W'} dW' \delta(W - W'),$$

which on integration enforces energy conservation $W' = W$, and yields for the integrated phase space

$$\boxed{\int d_{\text{LIPS}} = \frac{1}{(4\pi)^2} \frac{p}{W} d\Omega}.$$

The flux factor $4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}$ must now be evaluated in the CM frame

$$\begin{aligned} p_1 \cdot p_2 = E_1 E_2 + p^2 \rightarrow (p_1 \cdot p_2)^2 &= E_1^2 E_2^2 + p^4 + 2p^2 E_1 E_2 \\ &= (E_1^2 - p^2)(E_2^2 - p^2) + p^2(E_1 + E_2)^2 \\ &= m_1^2 m_2^2 + p^2 W^2. \end{aligned}$$

Thus

$$4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} = 4pW.$$

Finally, then, we have for the CM differential cross-section

$$\left(\frac{d\sigma}{d\Omega} \right)_{\text{CM}} = \frac{1}{64\pi^2 W^2} |\mathcal{M}|^2.$$

Trace Techniques for Fermion Spin Sums

So far we have assumed that the initial and final state particles all have specified spin or polarisation states and the Feynman amplitudes that we have written down correspond to this. For example, in electron-electron scattering, we considered scattering of an electron of 4-momentum p_1 and spin s_1 with an electron of 4-momentum p_2 and spin s_2 to give a final state containing electrons of 4-momentum p'_1 and spin s'_1 and 4-momentum p'_2 and spin s'_2 .

Often what is measured experimentally is the *unpolarised cross-section*, where the final state spins are not measured, and the initial spin states are all equally likely. In this situation we must therefore *sum* the computed cross-section over all final spin states and *average* over the initial spin states. In our example of Møller scattering

$$\left(\frac{d\sigma}{d\Omega} \right)_{\text{unpolarised}} = \frac{1}{4} \sum_{s_1} \sum_{s_2} \sum_{s'_1} \sum_{s'_2} \left(\frac{d\sigma}{d\Omega} \right)_{\text{Møller}}.$$

Consider a Feynman amplitude of the form

$$\mathcal{M} = \bar{u}(p', s') \Gamma u(p, s)$$

where Γ is a 4×4 matrix constructed from Dirac γ matrices. We saw that the Compton scattering amplitude for example had such a structure.

In order to compute the unpolarised cross-section we need therefore to consider

$$X = \frac{1}{2} \sum_s \sum_{s'} |\mathcal{M}|^2,$$

where $\frac{1}{2} \sum_s$ corresponds to the average over the 2 initial spin states and $\sum_{s'}$ the sum over final spin states.

Now $|\mathcal{M}|^2 = \mathcal{M} \mathcal{M}^*$ and since \mathcal{M} is a complex number — not a matrix — we can just as well write $|\mathcal{M}|^2 = \mathcal{M} \mathcal{M}^\dagger$ to give

$$X = \frac{1}{2} \sum_s \sum_{s'} \left(\bar{u}(p', s') \Gamma u(p, s) u^\dagger(p, s) \Gamma^\dagger \gamma^0 u(p', s') \right),$$

which we can rewrite as

$$X = \frac{1}{2} \sum_s \sum_{s'} \left(\bar{u}(p', s') \Gamma u(p, s) \bar{u}(p, s) \tilde{\Gamma} u(p', s') \right),$$

where we have defined

$$\tilde{\Gamma} \equiv \gamma^0 \Gamma^\dagger \gamma^0.$$

You should be able to show that if Γ is a product of γ matrices then $\tilde{\Gamma}$ is the product in the reverse order.

Making the Dirac indices explicit and using a summation convention enables us to write

$$X = \frac{1}{2} \sum_s \sum_{s'} \left(\bar{u}_\alpha(p', s') \Gamma_{\alpha\beta} u_\beta(p, s) \bar{u}_\gamma(p, s) \tilde{\Gamma}_{\gamma\delta} u_\delta(p', s') \right),$$

which we can re-order as

$$X = \frac{1}{2} \left(\sum_{s'} u_\delta(p', s') \bar{u}_\alpha(p', s') \right) \Gamma_{\alpha\beta} \left(\sum_s u_\beta(p, s) \bar{u}_\gamma(p, s) \right) \tilde{\Gamma}_{\gamma\delta}.$$

Now recall from Lecture 10 that

$$\boxed{\sum_s u(p, s) \bar{u}(p, s) = (\not{p} + m) \equiv \Lambda^+(p)},$$

so that we can re-write this expression as

$$X = \frac{1}{2} \left(\Lambda_{\delta\alpha}^+(p') \Gamma_{\alpha\beta} \Lambda_{\beta\gamma}^+(p) \tilde{\Gamma}_{\gamma\delta} \right) = \frac{1}{2} \text{tr} \left[\Lambda^+(p') \Gamma \Lambda^+(p) \tilde{\Gamma} \right] = \frac{1}{2} \text{tr} \left[(\not{p}' + m) \Gamma (\not{p} + m) \tilde{\Gamma} \right].$$

The traces which occur can be evaluated by means of various identities which you were invited to prove on the very first problem sheet, namely

1. $\gamma^\mu \gamma_\mu = 4$
2. $\gamma^\mu \not{a} \gamma_\mu = -2 \not{a}$
3. $\gamma^\mu \not{a} \not{b} \gamma_\mu = 4 \not{a} \cdot \not{b}$
4. $\gamma^\mu \not{a} \not{b} \not{c} \gamma_\mu = -2 \not{a} \not{b} \not{c}$
5. $\gamma^\mu \not{a} \not{b} \not{c} \not{d} \gamma_\mu = 2[\not{a} \not{b} \not{c} \not{d} + \not{c} \not{b} \not{a} \not{d}]$
6. $\text{tr}(\not{a} \not{b}) = 4 \not{a} \cdot \not{b}$
7. $\text{tr}(\not{a}_1 \not{a}_2 \cdots \not{a}_n) = 0$ if n is odd
8. $\text{tr}(\not{a} \not{b} \not{c} \not{d}) = 4[(a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c)]$
9. $\text{tr} \gamma^5 = 0$
10. $\text{tr} \gamma^5 \not{a} \not{b} = 0$

11. $\text{tr } \gamma^5 \not{d} \not{\psi} \not{\psi} \not{d} = 4i\varepsilon_{\alpha\beta\gamma\delta} a^\alpha b^\beta c^\gamma d^\delta$ where

$$\varepsilon_{\alpha\beta\gamma\delta} = \begin{cases} +1 & \text{if } \alpha, \beta, \gamma, \delta \text{ is an even permutation of } 0, 1, 2, 3; \\ -1 & \text{for odd permutations, and} \\ 0 & \text{if any two indices are equal.} \end{cases}$$

Example: Consider the case where $\Gamma = \not{d} = \gamma^\mu a_\mu$

$$\tilde{\Gamma} = \gamma^0 (\gamma^\mu)^\dagger a_\mu \gamma^0 = \gamma^0 \gamma^0 \gamma^0 a_0 - \gamma^0 \gamma^i \gamma^0 a_i = \gamma^0 \gamma^0 \gamma^\mu a_\mu = \gamma^\mu a_\mu = \Gamma,$$

where we have used the facts that in the standard representation $(\gamma^0)^\dagger = \gamma^0$ and $(\gamma^i)^\dagger = -\gamma^i$, while quite generally $(\gamma^0)^2 = 1$ and $\{\gamma^0, \gamma^i\} = 0$. Thus

$$X = \frac{1}{2} \text{tr} \left[(\not{p}' + m) \Gamma (\not{p} + m) \tilde{\Gamma} \right] = \frac{1}{2} \text{tr} \left[(\not{p}' + m) \not{d} (\not{p} + m) \not{d} \right].$$

Noting identity 7 which says that the trace of an odd number of γ matrices vanishes, we can simplify this to

$$X = \frac{1}{2} (\text{tr}(\not{p}' \not{d} \not{p} \not{d}) + m^2 \text{tr}(\not{d} \not{d})).$$

We use identity 8 on the first term and identity 6 on the second term

$$\begin{aligned} X &= \frac{1}{2} [4(p' \cdot a)(p \cdot a) - 4(p' \cdot p)(a \cdot a) + 4(p' \cdot a)(p \cdot a) + 4m^2(a \cdot a)] \\ &= 4(p' \cdot a)(p \cdot a) + 2a^2(m^2 - p' \cdot p). \end{aligned}$$

We can use similar methods in cases where the amplitude \mathcal{M} involves positron rather than electron spinors; we just need the second identity from Section 10

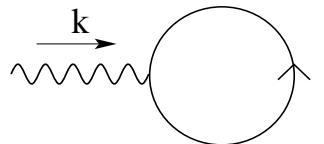
$$\boxed{\sum_s v(p, s) \bar{v}(p, s) = \not{p} - m \equiv -\Lambda_-(p)}.$$

Normal Ordering and the Vanishing QED Tadpole

Thus far, we have normal-ordered our Hamiltonian and used the “extended” form of Wick’s theorem to exclude all equal-time contractions. What happens if we don’t do either of these things? From Wick’s Theorem, the expression for $S^{(2)}$ will now contain contributions such as

$$S_{\text{tad}}^{(2)} = -\frac{e^2}{2!} \int d^4x_1 \int d^4x_2 \left(:(\bar{\psi} \gamma^\mu A_\mu \psi)_{x_1} (\bar{\psi} \gamma^\nu A_\nu \psi)_{x_2}: + :(\bar{\psi} \gamma^\mu A_\mu \psi)_{x_1} (\bar{\psi} \gamma^\nu A_\nu \psi)_{x_2}: \right)$$

The equal-time contractions at x_1 or x_2 result in (momentum-space) Feynman amplitudes which contain the following “tadpole” subdiagram



$$= ie \int \frac{d^4 p}{(2\pi)^4} \text{tr} \left(\gamma^\mu \frac{i(\gamma^\nu p_\nu + m)}{p^2 - m^2 + i\varepsilon} \right) = 0.$$

The first term in the loop-momentum integral vanishes because it is odd in p_ν , and the second one vanishes because the trace of an odd number of gamma matrices is zero.

There are no other equal-time contractions in QED, therefore *all* amplitudes involving equal-time contractions vanish identically. Normal-ordering is a bit of a red-herring — it may be viewed as a trick, allowing to omit all tadpole diagrams because they are identically zero.

Most field theories in 4D have tadpole diagrams, but they don't vanish in general. It turns out that they simply renormalise the parameters of the theory.

Calculation of some simple cross sections and decay rates is left to the tutorial.

Gauge Invariance

Measurable quantities such as cross-sections should be gauge invariant if the theory that we use to compute them is gauge-invariant. This implies that the Feynman amplitude should be gauge invariant. In particular the matrix element at a given order in perturbation theory, which corresponds to a sum of Feynman diagrams, must be gauge-invariant although the contributions from individual diagrams need not be.

Recall that under a gauge transformation

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \chi(x);$$

now for free photons, for the mode with 4-momentum k_μ ,

$$A_\mu^\pm(x) \propto \epsilon_\mu^r e^{\mp ik \cdot x},$$

so that adding an arbitrary multiple of k_μ to the polarization vector

$$\boxed{\epsilon_\mu^r \rightarrow \epsilon_\mu^r + Ck_\mu} \quad (5)$$

just amounts to a gauge transformation of A_μ with a particular choice of gauge function

$$\chi^\pm(x) \propto \pm iC e^{\mp ik \cdot x} \Rightarrow \partial_\mu \chi(x) \propto Ck_\mu e^{\mp ik \cdot x}.$$

We can see from the Feynman rules that for any process involving an external photon of 4-momentum k_μ and polarization ϵ_μ^r the amplitude is *linear in the photon polarization vector*. Thus we can write the Feynman amplitude in the form

$$\mathcal{M} = \epsilon_\mu^r(k) \mathcal{M}^\mu.$$

For the amplitude \mathcal{M} to be gauge-invariant therefore, we must have

$$\boxed{k_\mu \mathcal{M}^\mu = 0}. \quad (6)$$

More generally, *when any external photon polarization vector is replaced by the corresponding 4-momentum, the amplitude must vanish*.

Note that in a Lorentz gauge the Lorentz condition implies that the polarization vector satisfies the orthogonality condition

$$k^\mu \epsilon_\mu^r = 0,$$

and this is clearly preserved by the gauge transformation of equation (5) by virtue of the photon mass-shell condition $k^2 = 0$

$$k^\mu \epsilon_\mu^r \rightarrow k^\mu \epsilon_\mu^r + Ck^\mu k_\mu = k^\mu \epsilon_\mu^r.$$

An Example: Compton Scattering

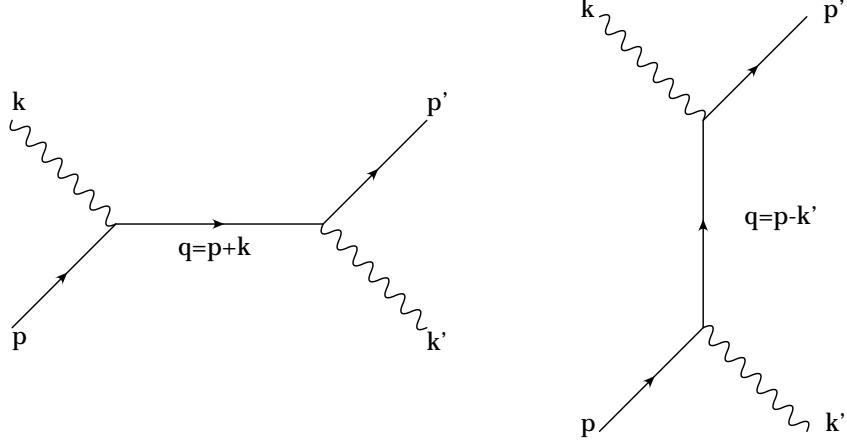


Figure 1: Momentum-space Feynman diagrams for Compton scattering.

The amplitude corresponding to the Feynman diagrams of Figure 1 is

$$\mathcal{M} = \mathcal{M}_a + \mathcal{M}_b,$$

where

$$\begin{aligned}\mathcal{M}_a &= -ie^2 \bar{u}(p', s') \not{\epsilon}' \frac{(\not{p} + \not{k} + m)}{(p + k)^2 - m^2} \not{\epsilon} u(p, s) \\ \mathcal{M}_b &= -ie^2 \bar{u}(p', s') \not{\epsilon} \frac{(\not{p}' - \not{k} + m)}{(p' - k)^2 - m^2} \not{\epsilon}' u(p, s),\end{aligned}$$

and we have used the abbreviated notation $\not{\epsilon} \equiv \not{\epsilon}^r(k)$ and $\not{\epsilon}' \equiv \not{\epsilon}^{r'}(k')$.

To check that the amplitude is gauge-invariant, we replace $\not{\epsilon}$ by \not{k} , and simplify the denominators by using the mass-shell conditions for the fermions and photons

$$\begin{aligned}(p + k)^2 - m^2 &= p^2 + 2p \cdot k + k^2 - m^2 = 2p \cdot k \\ (p' - k)^2 - m^2 &= p'^2 - 2p' \cdot k + k^2 - m^2 = -2p' \cdot k.\end{aligned}$$

The first term gives

$$\begin{aligned}-ie^2 \bar{u}(p', s') \not{\epsilon}' \frac{(\not{p} + \not{k} + m)}{2p \cdot k} \not{k} u(p, s) &= -ie^2 \bar{u}(p', s') \not{\epsilon}' \frac{\not{k}(-\not{p} + m) + 2p \cdot k}{2p \cdot k} u(p, s) \\ &= -ie^2 \bar{u}(p', s') \not{\epsilon}' u(p, s) \quad \text{using the Dirac equation.}\end{aligned}$$

The second term gives

$$\begin{aligned}-ie^2 \bar{u}(p', s') \not{k} \frac{(\not{p}' - \not{k} + m)}{-2p' \cdot k} \not{\epsilon}' u(p, s) &= ie^2 \bar{u}(p', s') \frac{(-\not{p}' + m)\not{k} + 2p' \cdot k}{2p' \cdot k} \not{\epsilon}' u(p, s) \\ &= ie^2 \bar{u}(p', s') \not{\epsilon}' u(p, s) \quad \text{using the Dirac equation.}\end{aligned}$$

Adding the two contributions gives zero so that the amplitude \mathcal{M} is gauge-invariant, although the contributions of the individual diagrams \mathcal{M}_a and \mathcal{M}_b are not. It is left as an exercise to show that a similar result is obtained if $\not{\epsilon}'$ is replaced by \not{k}' .

Polarization Sums

Given that

$$\mathcal{M} = \epsilon_\mu^r(\underline{k}) \mathcal{M}^\mu$$

for a process involving an external photon of 4-momentum k^μ and polarization ϵ_μ^r , the unpolarized cross-section will be proportional to

$$X = \sum_{r=1}^2 |\mathcal{M}|^2 = \mathcal{M}^\mu \mathcal{M}^{\nu*} \sum_{r=1}^2 \epsilon_\mu^r \epsilon_\nu^r. \quad (7)$$

Recall that the polarization vectors satisfy the completeness relation

$$\sum_{r=0}^3 g_{rr} \epsilon_\mu^r \epsilon_\nu^r = g_{\mu\nu},$$

so that the sum over the physical transverse polarizations is

$$\sum_{r=1}^2 \epsilon_\mu^r \epsilon_\nu^r = -g_{\mu\nu} + \epsilon_\mu^0 \epsilon_\nu^0 - \epsilon_\mu^3 \epsilon_\nu^3. \quad (8)$$

Recall also that $\epsilon_\mu^0 = n_\mu$ where n_μ is a time-like unit vector which we took to be $(1, 0, 0, 0)$. The space-like or longitudinal polarization vector ϵ_μ^3 lies in the direction of propagation of the photon \underline{k}

$$\epsilon_\mu^3 = \left(0, \frac{\underline{k}}{|\underline{k}|} \right).$$

This may be written covariantly for an on-shell photon as

$$\epsilon_\mu^3 = \frac{k_\mu - (k \cdot n)n_\mu}{(k \cdot n)}.$$

Substituting into equation (8) we find that the $n_\mu n_\nu$ terms cancel and thus

$$\sum_{r=1}^2 \epsilon_\mu^r \epsilon_\nu^r = -g_{\mu\nu} - \frac{1}{(k \cdot n)^2} [k_\mu k_\nu - (k \cdot n)(k_\mu n_\nu + k_\nu n_\mu)].$$

We can substitute this result into equation (7) and noting the gauge invariance condition (6) we find that

$$\boxed{\sum_{r=1}^2 |\mathcal{M}|^2 = -\mathcal{M}^\mu \mathcal{M}_\mu^*}.$$

This result is easily extended to amplitudes for processes involving more than one external photon.

In general it is advantageous to make some special choice of gauge in order to simplify calculations, but of course explicit gauge invariance is lost once a particular gauge is fixed.

Example: In Compton scattering, it is usual to pick a Lorentz gauge in which the external photons have polarization vectors of the form

$$\epsilon_\mu = (0, \underline{\epsilon}), \quad \epsilon'_\mu = (0, \underline{\epsilon}'),$$

so that

$$\epsilon \cdot k = -\underline{\epsilon} \cdot \underline{k} = 0 \quad \text{and} \quad \epsilon' \cdot k' = -\underline{\epsilon}' \cdot \underline{k}' = 0.$$

In the LAB frame the analysis is further simplified since the 4-momentum of the target electron is $p^\mu = (m, \underline{0})$ which means that

$$p \cdot \epsilon = p \cdot \epsilon' = 0.$$

For full details of the complete calculation of the Compton cross-section you are referred to the texts by Mandl & Shaw or by Aitchison & Hey.