Introduction into higher-spin gauge theory

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Abstract

These are lecture notes for a three-month course given at Utrecht University in Spring 2014. The notes are available at `http://www.staff.science.uu.nl/~3021017/higherspin.htm` .

Symmetries play a fundamental role in physics. These notes give an introduction to field theories exhibiting various types of symmetries, including Yang-Mills theory, supersymmetric field theories, gravity, supergravity and higher-spin gauge theories. The latter are highly symmetric relativistic theories conjectured to underlay quantum gravity and String Theory.

Topics covered include the Wigner classification of elementary particles, differential forms and Cartan-MacDowell-Mansouri gravity, and star-product algebras. Implications of higher symmetries for the fundamental concepts of space-time geometry as well as relevant aspects of the AdS/CFT correspondence are discussed.

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1 Introduction

To begin with let me briefly recall the present-day state of the art in the modern theory of fundamental interactions. Today, we fairly well understand the theory of electromagnetism which we face in our everyday life, as well as the theory of weak and strong interactions which govern nuclear and particle physics. All these interactions are described by the so-called Standard Model (SM) based on the group $U(1) \times SU(2) \times SU(3)$. The greatest achievement of the last years was the discovery of the Higgs boson predicted long ago by Englert, Higgs, Kibble (and, maybe, some others) on purely theoretical grounds resulting from the symmetry pattern of the theory associated with the part $U(1) \times SU(2)$ of the SM group.

This theory is quantum and relativistic, being quantum field theory (QFT). The creation of QFT was a tremendously important achievement reached after decades of efforts of the best physicists. Quantum mechanics is believed to underlay the physics of the microscopic world. Let me stress that this does not necessarily mean that it can only effect specific phenomena like those studied by accelerators or fine condensed-matter experiments. Indeed, it is everywhere around us and even inside us since the atoms that we consist of are stabilized by quantum mechanics. Also, lasers and superconductivity are of course pure quantum phenomena. Quantum mechanics is one of the cornerstones of the modern theoretical physics. By now, it has been tested fantastically well. Still I do not think that we understand the deep reason of why the rules of quantum mechanics work so well. This means that we may not quite understand the origin of quantum mechanics.

Apart from the SM, we understand the classical theory of gravity that perfectly describes the macroscopic gravitational phenomena from the scale of everyday life to that of Universe. Einstein showed that gravity is a theory of space-time geometry, *i.e.*, of space-time. What is still missing, however, is the theory of Quantum Gravity. There are very different opinions on this point. Some will not agree saying that String Theory or Loop Gravity provide models of Quantum Gravity. Some other would say that even though Quantum Gravity has not been constructed, this does not actually matter since the effects of quantum gravity are negligible at the energy scales available in any Earth experiment, requiring extremely short distances of the order of the Planck length 10^{-33} cm (or, equivalently, extremely high energies of 10^{19} GeV, to be compared with the Higgs mass 175 GeV). So this group of people is inclined to say that Quantum Gravity is a too little cloud to care about.

As I hope you have heard, in 1900 the famous German mathematician Hilbert tried to put physics into a rigorous mathematical framework (this is the sixth Hilbert problem). Those days it looked like this was possible except that, according to the expression of Lord Kelvin, there were two little clouds at the horizon: the Michelson-Morley experiment and, the ultraviolet catastrophe of black-body radiation computed by means of classical physics. As we know now, one of these clouds evolved into Special Relativity while the other one evolved into Quantum Mechanics, which are the cornerstones of modern physics. On the other hand, since General Relativity is a logical development of Special Relativity, the present absence of Quantum Gravity implies that the two little clouds at the horizon of physics of the 19th century did not disappear, but rather joined to form a single mysterious cloud at the horizon of physics of the 21st century.

Myself, I believe that a satisfactory theory of Quantum Gravity has not yet been constructed, but the understanding of Quantum Gravity may lead us to new fruitful insights into both the structure of space-time and Quantum Mechanics. But how can we hope to make progress in this direction having no direct experimental support?

Probably the only chance is to study possible symmetries of a yet hypothetical theory. An important property of field-theoretical models is that sometimes they are based on symmetries that are not visible in everyday life, but may become manifest at a high-enough energy scale (equivalently, small-enough distance scale). For example the $U(1) \times SU(2)$ symmetry of electroweak theory becomes manifest at energies above the mass of the Higgs boson. Analogously, one can speculate that Quantum Gravity, whatever it is, may exhibit new symmetries that are not visible (broken) at energies below the Planck scale.

Indeed, the achievements of the known theories of fundamental interactions are based on various symmetry principles. For example, one can say that General Relativity is based on the Einstein equivalence principle (coordinate independence). A remarkable extension of this theory, supergravity, results from including supersymmetry as we will learn in these lectures. String Theory, which is the most sophisticated theory considered these days, is based on a certain conformal symmetry principle.

To go beyond the usual framework, the key question is what kind of higher symmetries one can expect at trans-Planckian energies. The main aim of these lectures is precisely to address this question. We shall see that higher symmetries that go beyond usual low-energy symmetries deserve to be called *higher-spin symmetries*, while the corresponding theories exhibiting these symmetries are *higher-spin gauge theories*. Among other things, this fact indicates that higher-spin theory should be closely related to String Theory because the latter also contains higher-spin states. One difference between String Theory and higher-spin gauge theory is that the former contains massive higher-spin states while the latter only describes massless higher-spin states. These properties fit the expectation that higher-spin gauge theory describes ultra-high energies that are large compared to all possible mass scales, including the Planck scale. On the other hand, String Theory contains higher-spin states of large (presumably Planck) mass scales, which indicates that String Theory should admit an interpretation as a higher-spin theory with spontaneously broken higher-spin symmetries. Hence, by studying higher-spin gauge theory we hope to get a better understanding of String Theory.

Fortunately, higher-spin symmetries are very restrictive. The class of higher-spin theories that are compatible with higher-spin symmetry is very limited. (In fact it is so limited that for many years people believed that no consistent higher-spin gauge theory exists at all.) This gives a good chance to understand physics at trans-Planckian energies, assuming that these theories are organized by some deep symmetries.

It seems reasonable to expect that progress toward Quantum Gravity can significantly affect our understanding of fundamental properties of both space-time and Quantum Mechanics. In these lectures we will see that the standard concepts of Riemannian geometry are incompatible with higher-spin symmetries. Moreover, the resulting higher-spin theory suggests that such fundamental concepts as a point of space-time (local event) and the dimension of space-time are dynamical, depending on the physical phenomena in question.

Definitely, these days we experience a dramatic change of the physical picture of Nature, and those who start working in fundamental physics do it just in time. A celebrated example of such a change is the AdS/CFT or holographic duality that relates theories in different space-time dimensions, allowing to compute observables in a d-dimensional theory via computation in a $(d+1)$-dimensional theory and vice versa. During the last few years this duality was actively and successfully tested in the framework of higher-spin gauge theory. Moreover, many people believe that these studies of higher-spin theories will allow us to understand the very origin of holographic duality. In fact, I believe that to a large extent this already has happened.

These lecture notes contains three types of problems. *Exercises* ask you to check a statement that is made in the lecture notes, and should be quite straightforward. *Problems* are similar to exercises, but require some more thought and effort. Problems that can be skipped upon a first reading are marked as 'optional'. *Starred problems* are likely to be elaborated upon in future versions of these notes. These usually go into some details that are important but were omitted from the lectures in view of time.

2 Symmetries in relativistic field theory

2.1 Groups and Lie algebras

Any time when some object or setup does not change under certain transformations this is tantamount to saying that it possesses a symmetry. Usually symmetries form a *group* G, which means that if $g_1, g_2, g_3 \in G$ are some symmetry transformations then

- The composition $g_1 g_2$ (*i.e.*, successive application of first g_2 and then g_1) belongs to G.
- G contains a unit element e such that $eg = g = ge$ for all $g \in G$. In terms of symmetry transformations e is associated with the trivial transformation that does not change anything.
- The group product is associative: $(g_1 g_2) g_3 = g_1 (g_2 g_3)$.
- Existence of the inverse element: $\forall g \in G$ there exists a unique $g^{-1} \in G$ such that $g g^{-1} = g^{-1} g = e$.

From the perspective of symmetry transformations the first three group axioms are almost automatic. The fourth axiom, restricting us to the consideration of only invertible transformations, is an additional condition.

For *Lie groups*, which describe transformations (smoothly) depending on continuous parameters, it is often sufficient to consider infinitesimal group transformations that can be represented in the form

$$g = e + i\epsilon^a T_a + O(\epsilon^2)\,, \tag{2.1}$$

where ϵ^a $(a = 1, 2, \ldots N)$ are infinitesimal parameters that describe group elements close to e, while T_a are the so-called *generators*, which provide a basis of the space of infinitesimal transformations. This means that any group element infinitesimally close to the unit element can be represented in the form (2.1)) for some infinitesimal parameters ϵ^a. We will often abbreviate (2.1)) as $g \cong e + i\epsilon^a T_a$. The factor of i is usually introduced in the physics literature in order to work with Hermitian operators T_a as is appropriate in the context of Quantum Mechanics.

Let $g_1 \cong e + i\epsilon_1^a T_a$ and $g_2 \cong e + i\epsilon_2^a T_a$ be some group elements close to unity. Consider the element $g_1 g_2 g_1^{-1} g_2^{-1}$. Using that $g^{-1} \cong e - i\epsilon^a T_a$ it is not difficult to see that

$$g_1 g_2 g_1^{-1} g_2^{-1} = e + \epsilon_2^a \epsilon_1^b [T_a\,, T_b] + O(\epsilon^3)\,, \tag{2.2}$$

where the commutator is defined as

$$[T_a\,, T_b] := T_a T_b - T_b T_a\,. \tag{2.3}$$

To check (2.2)) it is useful to use that $g_1 g_2 g_1^{-1} g_2^{-1} = e$ for $g_1 = e$ or $g_2 = e$ which implies that all terms that depend only on ϵ_1 or ϵ_2 on the right-hand side of (2.2)) must cancel, *i.e.*, the only nontrivial terms are those proportional to $\epsilon_1\epsilon_2$.

Since $g_1 g_2 g_1^{-1} g_2^{-1}$ itself is infinitesimally close to the unit element e, and any such element should admit a representation (2.1), this requires

$$[T_a\,, T_b] = i f_{ab}^c T_c \tag{2.4}$$

for some constants f_{ab}^c known as the *structure constants* of G. Thus

$$g_1 g_2 g_1^{-1} g_2^{-1} \cong e + i\epsilon_{2,1}^a T_a\,, \tag{2.5}$$

with

$$\epsilon_{2,1}^a := f_{bc}^a \epsilon_2^b \epsilon_1^c\,. \tag{2.6}$$

One can see that the commutator (2.3) obeys the *Jacobi identity*)

$$[R, [S, T]] + [S, [T, R]] + [T, [R, S]] \equiv 0\,. \tag{2.7}$$

This implies that the structure constants f_{ab}^c have to obey the following Jacobi identity

$$f_{ab}^d f_{dc}^e + f_{bc}^d f_{da}^e + f_{ca}^d f_{db}^e = 0\,. \tag{2.8}$$

Also, by (2.4)), the structure constants f^c_{ab} are antisymmetric

$$f^c_{ab} = -f^c_{ba}\,. \tag{2.9}$$

These two properties provide a definition of a fundamental mathematical object that plays central role in many problems in physics: any set of structure constants obeying (2.8)) and (2.9)) defines some *Lie algebra* g. The number N of generators T_a is the *dimension* of g.

Any set of matrices $T_a{}^\alpha{}_\beta = (T_a)^\alpha{}_\beta$ obeying (2.4)) provides a *representation* or *module* of g. Note that a Lie algebra g may have (and usually has) different representations where, for example, the indices α, β take different numbers of values. Moreover, any Lie algebra admits a specific representation, called the *adjoint representation*, with $N \times N$ matrices T_a realized by the structure constants themselves:

$$(T_a)^b{}_c := i f^b_{ac}\,.$$

Problem 2.1. Check that this does indeed give a representation of g.

Thus, any Lie group gives rise to some Lie algebra. Properties of the former are to a large extent determined by those of the latter, which is usually easier to work with. One of the most important examples of a Lie group and corresponding Lie algebra is provided by special relativistic symmetries.

2.2 Poincaré group

A model is called *relativistic* if it is invariant under the following Poincaré symmetry transformations of space-time coordinates x^n ($n = 0, 1, \dots, d-1;\ x^0 = t$) of Minkowski space[1]

$$x^n \to x'^n = \Lambda^n{}_m x^m + a^n\,, \tag{2.10}$$

$$\Lambda^n{}_m \Lambda^{n'}{}_{m'} \eta^{mm'} = \eta^{nn'}\,, \tag{2.11}$$

where η^{mn} is the inverse of the Minkowski metric, which we take to be of the mostly-minus form

$$\eta_{nm} = \begin{pmatrix} +1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix}.$$

The parameters $\Lambda^n{}_m$ in (2.10)), obeying (2.11)), form the pseudo-orthogonal *Lorentz group* $O(d-1, 1)$, which is $O(3, 1)$ at $d = 4$. The parameters a^n form the group of translations of Minkowski space. Altogether, translations and Lorentz transformations form the *Poincaré group*, which is often called the *inhomogeneous* Lorentz

[1] In the literature, indices of Lorentz tensors are often denoted by Greek letters $\mu, \nu \dots$. For the sake of future convenience we prefer to use Latin indices $m, n \dots$, which is more standard in the context of Cartan's formulation of gravity introduced in Section 8.

group and is denoted by $IO(d-1,1)$. Dimension of $IO(d-1,1)$ is $\frac{d(d+1)}{2}$ since it describes $\frac{d(d-1)}{2}$ Lorentz rotations and d translations.

The condition (2.11)) implies that

$$\det{}^2\left|\Lambda^n{}_m\right| = 1\,. \tag{2.12}$$

One can impose the stronger condition that

$$\det\left|\Lambda^n{}_m\right| = 1\,. \tag{2.13}$$

Such matrices form the *special* Lorentz group denoted by $SO(d-1,1)$. The associated subgroup of the Poincaré group is denoted by $ISO(d-1,1)$.

The statement that elements of special groups must have unit determinant translates to the statement that elements of the corresponding Lie algebra must be traceless.

2.3 Examples of relativistic fields

2.3.1 Scalar field

Scalar fields are described by the Klein-Gordon equation

$$(\Box + m^2)C = 0\,, \qquad \text{with} \quad \Box := \eta^{nm}\frac{\partial^2}{\partial x^n \partial x^m}\partial_n\partial_m\,, \qquad \partial_n := \frac{\partial}{\partial x^n}\,. \tag{2.14}$$

Poincaré transformations act on the scalar field as

$$C(x^n) \;\to C'(x^n) := (gC)(x^n) = C(\Lambda^n{}_m x^m + a^n)\,. \tag{2.15}$$

Exercise 2.2. Check the relativistic invariance of the Klein-Gordon equation.

The Poincaré (Lie) algebra $io(d-1,1)$ represents infinitesimal elements of the Poincaré group

$$g \cong e + i\epsilon^n P_n + \frac{i}{2}\epsilon^{nm}L_{nm}\,, \tag{2.16}$$

where ϵ^n and $\epsilon^{nm} = -\epsilon^{mn}$ are infinitesimal parameters of translations and Lorentz transformations, respectively. To see that ϵ^{nm} must indeed be antisymmetric we subsitute

$$\Lambda^n{}_m \cong \delta^n_m + \epsilon^n{}_m \tag{2.17}$$

in (2.11) to obtain $\epsilon^{nm} + \epsilon^{mn} = 0$ (indices are raised by the Minkowski metric).

(2.17) can be written in the form (2.1) as

$$\Lambda^n{}_m \cong \delta^n_m + \frac{i}{2}\epsilon^{kl}T_{kl}{}^n{}_m\,, \tag{2.18}$$

where $T_{kl}{}^{nm} = i(\delta^n_l\delta^m_k - \delta^n_k\delta^m_l)$ are the generators of the Lorentz algebra $o(d-1,1)$ in its *vector representation*, *i.e.*, the infinitesimal transformation law of a (Lorentz) vector A^n is

$$\delta A^n = \frac{i}{2}\epsilon^{kl}T_{kl}{}^n{}_m A^m = \epsilon^n{}_m A^m\,. \tag{2.19}$$

In the scalar-field case computation of $\delta C(x) := C'(x) - C(x) + O(\epsilon^2)$ from (2.15) and (2.16) gives that infinitesimal Poincaré transformations act by

$$P_n = -i\partial_n \,, \qquad L_{nm} = i\left(x_n\partial_m - x_n\partial_m\right) \,. \tag{2.20}$$

From here it follows that

$$[P_n \,, P_m] = 0 \,, \tag{2.21}$$

$$[L_{nm} \,, P_k] = i(\eta_{mk}P_n - \eta_{nk}P_m) \,, \tag{2.22}$$

$$[L_{nm} \,, L_{kl}] = i(\eta_{mk}L_{nl} - \eta_{nk}L_{ml} - \eta_{ml}L_{nk} + \eta_{nl}L_{mk}) \,. \tag{2.23}$$

By construction, these relations form a Lie algebra. called *Poincaré algebra* $io(d-1,1)$. Their form is independent of the particular relativistic system in question. As we will see in Section 3 these relations determine to a large extent the properties of various relativistic models.

In these lectures we will discuss possible types of deformations and/or extensions of these relations. Note that the Poincaré algebra is itself a deformation of the non-relativistic Galilean algebra with the speed of light c as a deformation parameter, *i.e.*, the Galilean algebra can be derived from the Poincaré algebra in the limit $c \to \infty$.

2.3.2 Spin-1/2 fermions

Matter in our world is mostly built from spin-$\frac{1}{2}$ fermions. In field theory they are described by the Dirac equation[2]

$$\left(i\gamma^n{}_{\hat\alpha}{}^{\hat\beta}\,\partial_{\mathsf{n}} + m\,\delta^{\hat\beta}_{\hat\alpha}\right)\psi_{\hat\beta}(x) = 0 \,, \tag{2.24}$$ *

where $\gamma^{n\hat\beta}_{\ \hat\alpha}$ are γ-matrices obeying the Clifford anticommutation relations

$$\{\gamma^n \,, \gamma^m\} = 2\eta^{nm} I \,, \tag{2.25}$$

where the anticommutator is defined as usual

$$\{A \,, B\} := AB + BA \,, \tag{2.26}$$

spinor indices $\hat\alpha$, $\hat\beta$ are implicit, and I is the unit matrix in spinor space, *i.e.*, $I^{\hat\beta}_{\hat\alpha} = \delta^{\hat\beta}_{\hat\alpha}$.

(In this review we use the mostly minus convention (2.12) for the metric tensor η_{nm}. One could use the mostly plus convention as well. In terms of η_{nm}, the transition from one convention to another results from the sign change of η. In terms of γ-matrices this is more subtle because γ-matrices in the two bases differ by a factor of i. This has the effect that some important equations, including the Dirac equation, have different forms in the mostly plus and mostly minus setups.)

[2]Throughout these notes we put hats on the indices of *Dirac* spinors in four dimensions, so $\hat\alpha = 1, \ldots, 4$. We reserve ordinary Greek indices for two-component spinors, see Section 10.4.

To check that the Dirac equation is Poincaré invariant one observes that generators of the Lorentz algebra acting on spinor indices are realized in terms of γ-matrices as follows:

$$M_{nm\hat{\alpha}}{}^{\hat{\beta}} := \frac{i}{4}[\gamma_n\,,\gamma_m]_{\hat{\alpha}}{}^{\hat{\beta}}\,. \tag{2.27}$$

To see that these M_{nm} obey the commutation relations (2.23) it is convenient first to check that

$$[M_{nm}\,,\gamma_k] = i(\eta_{mk}\gamma_n - \eta_{nk}\gamma_m)\,. \tag{2.28}$$

Note that the space of spinors $\psi_{\hat{\alpha}}$ of any pseudo-orthogonal algebra $o(p,q)$ can be understood as a (left) module of the Clifford algebra (2.25) with η_{nm} being the $o(p,q)$-invariant metric of signature (p,q). Thus, such spinors also form $o(p,q)$-modules with $o(p,q)$ generators M_{nm} (2.27). We will discuss spinors in some more detail in Section 10.

Now it is not difficult to see that Dirac equation is invariant under infinitesimal Poincaré transformations acting by

$$\delta\psi(x) = i\left(\epsilon^n P_n + \frac{1}{2}\epsilon^{nm}(L_{nm} + M_{nm})\right)\psi(x) \tag{2.29}$$

with P_n and L_{nm} as in (2.20).

Note that there is an important difference between integer and half-integer spins. Mathematically, half-integer spins do not transform under the Poincaré group (*cf.* the extra term with M_{nm} in (2.29)) but rather under its double covering group, so that the Lorentz subgroup $SO(3,1)$ of the Poincaré group is replaced by its double cover called $Spin(3,1)$. This is because fields of half-integer spins have the peculiar property that, instead of leaving fermions invariant, a spatial rotation over 2π changes the sign of fermionic fields. Hence, in the fermion sector, the unit transformation is represented by a rotation over 4π, which doubles the usual rotation leaving invariant space-time. The reason why this effect is not visible in the world around us is that fermions always enter in bilinear combinations, which *are* invariant under usual rotations. Note also that $Spin(3,1)$ and $SO(3,1)$ provide examples of different Lie groups which have the same Lie algebra (very much like $SU(2)$ and $SO(3)$ do).

2.3.3 Spin one

Here dynamical variables are given by a four-vector field $A_n(x)$. Its field strength is

$$F_{nm}(x) := \partial_n A_m(x) - \partial_m A_n(x)\,. \tag{2.30}$$

By its definition, F_{mn} obeys the *Bianchi identities*

$$\partial_n F_{kl} + \partial_k F_{ln} + \partial_l F_{nk} = 0 \tag{2.31}$$

and remains invariant under gradient transformations

$$\delta A_n(x) = \partial_n \varepsilon(x)\,, \tag{2.32}$$

where $\varepsilon(x)$ is an arbitrary gauge parameter.[3]

The massive spin-one field equation, called the *Proca equation*, is

$$\partial_n F^n{}_m + m^2 A_m = 0\,. \tag{2.33}$$

Notice that the mass term in (2.33) breaks invariance under (2.32).

As a consequence of the antisymmetry of the field strength (2.33) gives

$$m^2 \partial_n A^n = 0\,. \tag{2.34}$$

Hence, at $m \neq 0$ Eq. (2.33) is equivalent to

$$\Box A_n + m^2 A_n = 0\,, \qquad \partial_n A^n = 0\,. \tag{2.35}$$

The equation $\partial_n A^n = 0$ is called the *Lorentz condition*.

The spin-one equations are invariant under the following action of the Poincaré group:

$$A^m(x^n) \to A'^m(x^n) := (gA)^m(x^n) = \Lambda^m{}_k A^k(\Lambda^n{}_l x^l + a^n)\,. \tag{2.36}$$

Here the Lorentz transformation acts not only on the argument but also on the vector index of $A_m(x)$. Indeed, recall that the infinitesimal transformation law for the *vector representation* of any pseudo-orthogonal Lie algebra $o(p,q)$ that leaves invariant the metric η^{nm} is

$$\delta A^n = \epsilon^n{}_m A^m\,, \tag{2.37}$$

where $\epsilon_{nm} = -\epsilon_{mn}$ is an infinitesimal rotation parameter., *cf.* Eq. (2.19).

Note that P_n acting by zero in the vector representation is only consistent because the translation operators of the Poincaré algebra have the property that their commutator with any other element of the Poincaré algebra is again a translation operator.[4]

An example of a massless ($m = 0$) spin-one field is the usual electromagnetic field. In addition to the Bianchi identities (2.31), the Maxwell equations contain

$$\partial_n F^n{}_m = 4\pi\, J_m\,, \tag{2.38}$$

where J_m is the electric current built from some other fields as discussed in some more detail in Section 6. Note that the Bianchi identity (2.31) requires the current J^n to obey the conservation condition $\partial_n J^n = 0$.

Free massless spin-one particles are described by the Maxwell equations (2.38) at $J_n = 0$. In this case, the Lorentz condition does *not* follow from the Maxwell equations. However, since the vector potential enters the Maxwell equations only via the field strength, (2.38) is *invariant* under the gauge transformation (2.32). Fields that differ by a gauge transformation are physically equivalent. This allows one

[3]Throughout this review we use $\varepsilon(x)$ for local parameters that are arbitrary functions of space-time coordinates and $\epsilon(x)$ to denote global parameters that depend on space-time coordinates in a specific way. See also Section 6.

[4]Mathematically this means that translations form an ideal of the Poincaré algebra. At the group level this means that the translation subgroup of the Poincaré group is normal.

to impose the Lorentz condition as a *gauge condition*. In this gauge, the Maxwell equations again take the form (2.35) at $m^2 = 0$.

Historically, the principles of relativistic invariance were deduced from the analysis of the Maxwell equations. These days we can say that special relativity is an elementary consequence of the Maxwell equations. Saying this, one has to remember however that the actual interval between the formulation of the Maxwell equations by Maxwell and special relativity by Einstein (and others, including in the first place Lorentz) was about fifty years. This is an impressive and thoughtful example of how long it can take to understand and appreciate a theory and its ideas.

3 Elementary particles as Poincaré-modules

The statement that the Poincaré group leaves some relativistic equation invariant means that the Poincaré transformations map solutions to solutions. This is tantamount to saying that the space of solutions of a particular relativistic equation forms a Poincaré-module.

For example, in the case of the module of the Poincaré algebra associated with the Klein-Gordon equation, the operators of translations are represented by P_n (**??**) while the module space V is the linear space of all solutions of the Klein-Gordon equation of a certain class (*e.g.*, having a finite norm and/or obeying certain boundary conditions). The most important class is associated with unitary modules.

A module is called *unitary* if it is compatible with the principles of Quantum Mechanics when V is identified with the space of states of a quantum theory. Namely, the space V should be endowed with a positive-definite semi-linear form $\langle\psi|\psi'\rangle$ for $\psi, \psi' \in V$ (recall that *positive-definite* means that $\langle\psi|\psi\rangle > 0$ for all $\psi \neq 0$) that is invariant under a symmetry transformation represented by an operator U, *i.e.*,

$$\langle U\psi|U\psi'\rangle = \langle\psi|\psi'\rangle \,. \tag{3.1}$$

The aim of this section is first to clarify the structure of the Poincaré-modules associated with the usual relativistic field equations and then to classify all possible types of elementary particles as unitary Poincaré-modules.

In this classification, different elementary particles are identified with different irreducible unitary Poincaré-modules. Recall that a g-module V is called *irreducible* if V does not contain a subspace V' different from V and $\emptyset$, which is itself a g-module$V' \subseteq V$ which is itself a g-module, except for $\emptyset$ and V itself. In other words: irreducible modules do not contain nontrivial submodules. Irreducibility is a mathematical equivalent of the property that a particle is elementary, hence containing no more elementary constituents. The property that elementary particles are associated to linear spaces where Poincaré symmetry acts expresses the fact that an elementary particle is described by linear relativistic equations: it describes * asymptotic state, which is separated from any other particles with which it could otherwise interact via nonlinear corrections to the field equations.

3.1 Lower-spin examples

3.1.1 Spin 0

Consider again the Klein-Gordon (KG) equation (2.14). As any other linear differential equation with constant coefficients it is most conveniently analyzed by the Fourier transform, setting

$$C(x) = \int d^4p \;\tilde{C}(p) \exp ip_n x^n \,. \tag{3.2}$$

Substitution into the KG equation gives the condition

$$(p^2 - m^2)\tilde{C}(p) = 0\,, \qquad p^2 := p_n p_m \eta^{nm} \tag{3.3}$$

which implies that

$$\tilde{C}(p) \propto \delta(p^2 - m^2)\,. \tag{3.4}$$

Since the equation

$$(p^0)^2 = \vec{p}^{\,2} + m^2\,, \qquad \vec{p} = (p_1, p_2, \dots, p_{d-1}) \tag{3.5}$$

admits two solutions

$$p^0 = \pm\sqrt{\vec{p}^{\,2} + m^2} \tag{3.6}$$

we obtain

$$\tilde{C}(p) = \delta(p^2 - m^2)\left(\theta(p^0)\tilde{C}^+(\vec{p}) + \theta(-p^0)\tilde{C}^-(-\vec{p})\right)\,, \tag{3.7}$$

where $\theta(p^0)$ is the Heaviside step-function defined by

$$\theta(u) = \begin{cases} 1 & \text{at} \quad u \geq 0\,, \\ 0 & \text{at} \quad u < 0\,. \end{cases} \tag{3.8}$$

The two functions $\tilde{C}^{\pm}(\vec{p})$ of $(d-1)$ variables $\vec{p}$ determine the general solution of the KG equation. This is in accordance with the standard Cauchy problem where a general solution of the KG equation is determined in terms of two functions of $(d-1)$ variables, $C(0, \vec{x})$ and $\frac{\partial}{\partial x^0} C(x^0, \vec{x})|_{x^0=0}$.

The energy operator E is identified with P^0 (2.20):

$$E = -i\frac{\partial}{\partial x_0}\,. \tag{3.9}$$

Its eigenvalues on the solutions $\tilde{C}^{\pm}(\vec{p})$ are equal to $\pm\sqrt{\vec{p}^2 + m^2}$. Therefore, $\tilde{C}^+(\vec{p})$ and $\tilde{C}^-(\vec{p})$ are called *positive-* and *negative-frequency* solutions, respectively.

Since infinitesimal transformations cannot affect the sign of the time arrow, the positive and negative branches of the solutions form independent modules $V^+(m, 0)$ and $V^-(m, 0)$ of the Poincaré algebra. These modules are the linear spaces of functions of spatial momenta, $\tilde{C}^+(\vec{p})$ and $\tilde{C}^-(\vec{p})$, respectively. In fact, $V^+(m, 0)$ and $V^-(m, 0)$ are conjugate if the original field

$$C(x) = C^+(x) + C^-(x) \tag{3.10}$$

was real. It is enough to consider the module $V^+(m, 0)$ associated with positive-frequency solutions.

Problem 3.1. Find how infinitesimal Poincaré transformations act on $\tilde{C}^+(\vec{p})$, paying attention to the action of the Lorentz boosts.

This module is unitary since it admits a positive-definite invariant norm, which has the form

$$\|C^+\|^2 = \langle C^-|C^+\rangle = \int_{t=\mathrm{const}} \frac{d^{d-1}\vec{p}}{\sqrt{\vec{p}^2+m^2}} \tilde{C}^-(\vec{p})\tilde{C}^+(\vec{p})\,. \tag{3.11}$$

This norm is obviously positive definite. That it is Poincaré invariant follows from the fact that it can be written in the form of an "electric charge" (see also Section 6.2) for the complex fields $C^+(x)$ and $C^-(x)$:

$$\langle C^-|C^+\rangle = -i\int_{t=\mathrm{const}} d^{d-1}\,\vec{x}\ \left(C^-(t,\vec{x})\partial^0 C^+(t,\vec{x}) - \partial^0 C^-(t,\vec{x})C^+(t,\vec{x})\right)\,. \tag{3.12}$$

Indeed, the so-defined norm is conserved, *i.e.*, t-independent. Moreover, the conservation condition implies that it is insensitive to infinitesimal Lorentz rotations of the integration surface. As a result, Eq. (3.12) defines a Poincaré-invariant norm on the space of positive-definite solutions.

Exercise 3.2. Use (3.2) to show that the right-hand sides of (3.11) and (3.12) are equal.

3.1.2 Spin 1/2

From the Dirac equation (2.24), *i.e.*,

$$(i\partial\!\!\!/ + m)\psi(x) = 0\,, \qquad \partial\!\!\!/ := \gamma^n \frac{\partial}{\partial x^n} \tag{3.13}$$

it follows that the components $\psi_{\hat{\alpha}}(x)$ each obey the Klein-Gordon equation

$$(\Box + m^2)\psi_{\hat{\alpha}}(x) = 0\,. \tag{3.14}$$

Indeed, multiplying (3.13) from the left by $i\partial\!\!\!/ - m$ we obtain $(\partial\!\!\!/^2 + m^2)\psi_{\hat{\alpha}} = 0$, while the basic relations (2.25) for γ-matrices imply that $\partial\!\!\!/^2 = \frac{1}{2}\partial^n\partial^m\{\gamma_n\,,\gamma_m\} = \Box$.

Thus, analogously to the spin-zero case, we obtain that the Fourier modes for spin-1/2 fields

$$\psi_{\hat{\alpha}}(x) = \int d^d p\, \tilde{\psi}_{\hat{\alpha}}(p) \exp i p_n x^n \tag{3.15}$$

admit a decomposition in positive and negative frequencies

$$\tilde{\psi}_{\hat{\alpha}}(p) = \delta(p^2 - m^2)\Big(\theta(p^0)\tilde{\psi}^+_{\hat{\alpha}}(\vec{p}) + \theta(-p^0)\tilde{\psi}^-_{\hat{\alpha}}(-\vec{p})\Big)\,. \tag{3.16}$$

Let us introduce the following operators

$$\Pi^\pm := \frac{1}{2}(I \pm m^{-1}p^n\gamma_n)\,. \tag{3.17}$$

On the mass shell, *i.e.*, at $p^2 = m^2$, these operators have the properties of a complete set of projectors:

$$\Pi^+ + \Pi^- = I\,, \qquad \Pi^\pm \Pi^\pm = \Pi^\pm\,, \qquad \Pi^\pm \Pi^\mp = 0\,. \tag{3.18}$$

Moreover, it can be easily seen that Π^+ and Π^- have equal rank because $tr(\gamma_n) = 0$. The Dirac equation then implies that

$$\Pi^-{}_{\hat\alpha}{}^{\hat\beta}\tilde\psi^+_{\hat\beta} = 0\,, \qquad \Pi^+{}_{\hat\alpha}{}^{\hat\beta}\tilde\psi^-_{\hat\beta} = 0\,, \tag{3.19}$$

i.e.,

$$\tilde\psi^+_{\hat\beta} = (\Pi^+\tilde\psi)_{\hat\beta}\,, \qquad \tilde\psi^-_{\hat\beta} = (\Pi^-\tilde\psi)_{\hat\beta}\,. \tag{3.20}$$

This means that the Dirac equation does not only restrict the dependence on space-time coordinates via the Klein-Gordon equation, but also sets half of the components of $\psi_{\hat\alpha}$ to zero in a way that depends on the space-time momenta. In particular, in the rest frame, with $p^0 = m$ and $\vec p = 0$, the projectors $\Pi^\pm$ have the form

$$\Pi^\pm{}_{\hat\alpha}{}^{\hat\beta} = \frac{1}{2}(\delta^{\hat\beta}_{\hat\alpha} \pm \gamma^0{}_{\hat\alpha}{}^{\hat\beta})\,. \tag{3.21}$$

From this formula it follows that the operators $\Pi^\pm{}_{\hat\alpha}{}^{\hat\beta}$ are invariant under spatial rotations that leave the momentum p^n invariant. Hence, the projections of $o(d-1,1)$-spinors by the projectors $\Pi^\pm{}_{\hat\alpha}{}^{\hat\beta}$ give $o(d-1)$-spinors. We conclude that the degrees of freedom of the Dirac field are parametrized by a spinor of $o(d-1)$ at a given momentum. Since the Dirac equation is Poincaré invariant, its solutions form a Poincaré-module. As before, the spaces of positive- and negative-frequency solutions $\psi^+_{\hat\alpha}$ and $\psi^-_{\hat\alpha}$ form two irreducible (and conjugate) Poincaré-modules $V^\pm(m, \frac{1}{2})$. As linear spaces they are spaces of functions of momenta $\vec p$ carrying spinor indices of $o(d-1)$.

Again this Poincaré-module is unitary with the positive-definite norm given by the conserved "electric charge" defined with respect to the positive and negative frequencies, *i.e.*,

$$\|\psi^+\|^2 = \langle\psi^-|\psi^+\rangle = \int_{t=\mathrm{const}} d^{d-1}\,\vec x\;\; \psi^{-\hat\alpha}(t,\vec x)\psi^+_{\hat\alpha}(t,\vec x)\,. \tag{3.22}$$

We will come back to this issue in Section 6.2.

3.1.3 Spin one

Massive fields of spin one, described by the Proca equation, can be analyzed in a way analogous to the examples of spin 0 and 1/2. Here one should only take into account that the Lorentz condition implies that the time component of the vector field $\tilde A^\pm_0(p)$ is zero in the rest frame. The positive-frequency module therefore is characterized by $d-1$ functions of the spatial momenta $\vec p$. These can be thought of as forming a vector representation of $o(d-1)$.

Problem 3.3. Work out the details of the interpretation of the space $V^{\pm}(m,1)$ of solutions of the Proca equation as a unitary Poincaré module.

Suggestion. Use that the norm for the massive case has the same form as in Eq. (3.33), taking into account the Lorentz condition. Compare the result with the spin-zero norm (3.12).

The massless case is less straightforward. The equations for a spin-one massless field are the Maxwell equations (2.38) at $J = 0$

$$\Box A_n - \partial_n \partial^m A_m = 0\,. \tag{3.23}$$

We observe that this equation does not imply the Klein-Gordon equation. This is because Maxwell theory is invariant under gradient transformations (2.32)

$$A_n \to A'_n(x) = A_n(x) + \partial_n \varepsilon(x). \tag{3.24}$$

Using this symmetry one can fix the Lorentz gauge setting

$$\partial^n A_n = 0\,. \tag{3.25}$$

In this gauge, the Maxwell equations amount to the massless Klein-Gordon equation for every component of A_n

$$\Box A_n = 0\,. \tag{3.26}$$

However, there is a subtlety: the Lorentz gauge (3.25) is not complete. Indeed, as follows from (**??**), the Lorentz gauge is preserved by residual gauge transformations (3.24) with any gauge parameter $\varepsilon(x)$ that itself obeys the massless Klein-Gordon equation

$$\Box \varepsilon(x) = 0\,. \tag{3.27}$$

Performing a Fourier transform we obtain

$$\tilde{A}_n(p) = \delta(p^2)\Big(\theta(p^0)\tilde{A}^+_n(\vec{p}) + \theta(-p^0)\tilde{A}^-_n(-\vec{p})\Big)\,, \tag{3.28}$$

$$\tilde{\varepsilon}(p) = \delta(p^2)\Big(\theta(p^0)\tilde{\varepsilon}^+(\vec{p}) + \theta(-p^0)\tilde{\varepsilon}^-(-\vec{p})\Big)\,. \tag{3.29}$$

Recall that a vector p_n obeying $p^2 = 0$ is called light-like. If $p_0 > 0$, by an appropriate spatial rotation it can always be put into the form

$$p_0 = p_{d-1} = \omega\,, \quad p_{\mathbf{i}} = 0\,, \qquad \mathbf{i} = 1, \ldots, d-2 \tag{3.30}$$

with some positive ω. Such momentum will be called *standard*. For a standard momentum the Lorentz condition (3.25) gives

$$\tilde{A}_0 - \tilde{A}_{d-1} = 0\,. \tag{3.31}$$

On the other hand, the gauge transformation (3.24) gives

$$\delta\tilde{A}_0 = \omega\tilde{\varepsilon}\,, \qquad \delta\tilde{A}_{d-1} = \omega\tilde{\varepsilon}\,. \tag{3.32}$$

This allows us to gauge-fix $\tilde{A}_0$ to zero. By virtue of (3.31) this also implies $\tilde{A}_{d-1} = 0$. As a result, in this gauge, a free massless vector field is described by $d-2$ components $\tilde{A}_{\mathbf{i}}(\vec{p})$ with $\mathbf{i} = 1, \ldots, d-2$ constituting a vector of $o(d-2)$. In particular, at $d = 4$ a massless spin-one field has two independent components. This fits the well-known fact that a photon carries two polarizations.

The invariant norm on $V^+(0,1)$ can again be defined as an "electric charge" for positive and negative frequencies

$$\|A^+\|^2 = \langle A^-|A^+\rangle = -i\sum_{k=1}^{d-1}\int_{t=\mathrm{const}} d^{d-1}\vec{x}\ \left(A_k^-(t,\vec{x})F_{0k}^+(t,\vec{x}) - F_{0k}^-(t,\vec{x})A_k^+(t,\vec{x})\right)\,. \tag{3.33}$$

There is however another subtlety: this norm is only positive-*semi*definite. Namely, it vanishes on the pure gauge components $A_n = \partial_n\varepsilon$. In fact, pure gauge modes are orthogonal to any state, including themselves, thus being *null states*. Indeed, if $A'_n(x) = A_n(x) = \partial_n\varepsilon$ then the ε-dependent term contributes only through

$$-i\sum_{k=1}^{d-1}\int_{t=\mathrm{const}} d^{d-1}\vec{x}\ \left(\partial_k\varepsilon^-(t,\vec{x})F_{0k}^+(t,\vec{x}) - F_{0k}^-(t,\vec{x})\partial_k\varepsilon^+(t,\vec{x})\right)\,, \tag{3.34}$$

which is a total derivative by virtue of the Gauss law $\sum_k \partial_k F_{0k} = 0$. This property is consistent with the expectation that pure gauge degrees of freedom should not contribute to any physical phenomena. At the same time, the norm (3.33) turns out to be positive-definite on the space of physical states V^{phys} as one can see e.g. in the particular gauge $\tilde{A}_0 = \tilde{A}_{d-1} = 0$.

Mathematically, the space of physical states V^{phys} is the quotient of the space of all states $V^{\pm}(0,1)$ over the space of pure gauge states V^{gauge},

$$V^{\mathrm{phys}} = V^{\pm}(0,1)/V^{\mathrm{gauge}}\,. \tag{3.35}$$

This is tantamount to saying that states that differ by a gauge transformation are physically equivalent.

Note that the appearance of null states usually signals gauge symmetries.

3.1.4 Relation to QFT

In the above examples we have seen that unitary Poincaré-modules are realized by the spaces of positive- and negative-frequency solutions of relativistic field equations. The invariant positive-definite norm was defined in a quantum-mechanical fashion, paring complex conjugate positive- and negative-frequency solutions. So, it may look a bit confusing that we have arrived at the quantum picture without actual quantization via the introduction of (noncommutative) field operators.

In fact, the decomposition of the space of solutions of relativistic field equations into positive- and negative-frequency solutions is the main element of the quantization prescription. In particular, one should keep in mind that in the usual local field theory it is not possible to write field equations that would only describe positive or negative frequencies. In other words, the decomposition into positive or negative frequencies is a nonlocal operation. Let me briefly recall the relevant part of QFT.

In QFT we quantize fields replacing, say, a scalar field $C(x)$ by an operator $\hat{c}(x)$, denoted by a lower-case c for future convenience. Then we expand $\hat{c}(x)$ in positive and negative frequencies

$$\hat{c}(x) = \hat{c}^+(x) + \hat{c}^-(x)\,, \tag{3.36}$$

identifying $\hat{c}^+(x)$ and $\hat{c}^-(x) = \hat{c}^+(x)^\dagger$ with creation and annihilation operators. Elementary (*i.e.*, basis) creation and annihilation operators are identified with appropriately normalized Fourier modes $\hat{\tilde{c}}^-(-\vec{p})$ and $\hat{\tilde{c}}^+(\vec{p})$ of

$$\hat{\tilde{c}}(p) = \delta(p^2 - m^2)\left(\theta(p^0)\hat{\tilde{c}}^+(\vec{p}) + \theta(-p^0)\hat{\tilde{c}}^-(-\vec{p})\right)\,. \tag{3.37}$$

The space of single-particle states is spanned by eigenstates with various definite momenta $\vec{p}$

$$\hat{\tilde{c}}^+(\vec{p})|0\rangle\,. \tag{3.38}$$

Indeed, any single-particle state has the form

$$|C\rangle = \int d^{d-1}\vec{p}\,\tilde{C}^-(\vec{p})\hat{\tilde{c}}^+(\vec{p})|0\rangle\,, \tag{3.39}$$

where $\tilde{C}^-(\vec{p})$ is an arbitrary (well-behaved) complex function of the spatial momenta. The function $\tilde{C}^-(\vec{p})$ can be identified with the "classical" negative-frequency solution of the KG equation. Moreover, the state $|C\rangle$ can again be written Lorentz covariantly using the current formula (3.12) pairing the operator solution of the KG equation $\hat{\tilde{c}}^+(-\vec{p})$ with the classical solution $\tilde{C}^-(\vec{p})$. If we now compute the norm $\langle C|C\rangle$ it will take the form of some bilinear functional of the "classical" solution $C^-(x)$ (equivalently, of its complex conjugate $C^+(x)$). By construction, this norm is Lorentz invariant. In a consistent unitary theory like KG theory it should be positive definite. This is the norm which we considered in the examples above.

Thus, the idea is that we can actually skip some standard QFT steps to analyze the unitarity of Poincaré-modules. Indeed, the decomposition into positive and negative frequencies is in fact the main ingredient of the quantization procedure. Other elements of the procedure are convenient, but in many cases not necessary.

Now we are in a position to see how this analysis of lower-spin fields can be generalized to elementary particles of any other type.

3.2 Wigner classification of relativistic fields

In his important paper, Wigner suggested that elementary particles should be associated with various irreducible unitary Poincaré-modules and solved the problem of classifying the unitary Poincaré modules in four-dimensional Minkowski space, hence describing all possible types of elementary particles in four dimensions. Based on the examples of lower-spin fields considered above, in this section I explain the main idea of the Wigner analysis.

Let a vector space V form a unitary Poincaré-module. In the scalar-field case, $V = V^+(m, 0)$ was the space of positive-frequency solutions of the Klein-Gordon equation.

Generally, since the operators P_n from (2.20) are Hermitean and commute with each other, they can be simultaneously diagonalized. Let V_p be the subspace of V such that

$$P_n V_p = p_n V_p \tag{3.40}$$

where p_n are fixed eigenvalues of P_n. Note that in the example of a scalar field the diagonalization of P_n was achieved by the Fourier transform, while V_p consisted of positive-frequency modes $C^+(p)$ at given p_n. As such, in the example of scalar fields the space V_p was one dimensional for any allowed p_n.

The operator

$$C_2 := P_n P_m \eta^{nm} \tag{3.41}$$

commutes with all generators of the Poincaré algebra.[5] Indeed, it commutes with P_k since the operators of momenta commute with each other, and with the Lorentz generators M_{nm} since C_2 is manifestly Lorentz covariant. As a result, for an irreducible module, C_2 has to take some fixed value that characterizes the chosen module. Mathematically, this is a consequence of the so-called Schur Lemma. In the case of interest this is easy to see by observing that C_2 is diagonal on V_p (*i.e.*, with respect to different Fourier modes $\exp i p_n x^n$), and since Poincaré transformations commute with C_2 they can only relate those V_p that have the same value for C_2. Thus, for an irreducible module

$$C_2 = m^2 I \,, \tag{3.42}$$

where m^2 is some number. We recognize that this condition is nothing else than the mass-shell condition due to the KG equation familiar from the lower-spin examples considered above. Thus, from the perspective of representation theory, the KG equation is a consequence of the irreducibility condition. So every irreducible (elementary) relativistic field should obey the KG equation for some m^2.

A priori, there are three cases:

- massive particles, with $m^2 > 0$,
- massless particles, with $m^2 = 0$,
- tachyons, with $m^2 < 0$.

The third case is pathological from the physical point of view. In particular, from the Hamiltonian viewpoint tachyons have an up-side-down potential which renders them unstable. A related property is that in the tachyonic case it is not possible to distinguish between positive and negative frequencies in a Lorentz-covariant way.

[5] Operators that commute with all elements of a Lie algebra h are called *Casimir operators* of h. Thus C_2 is a Casimir operator of the Poincaré algebra. More precisely, it is the quadratic Casimir operator. The Poincaré algebra has also other (higher) Casimir operators, but those will not be used in the discussion below.

This essentially means that such a system cannot be quantized in a consistent way. Therefore the tachyonic case will not be considered here.

The interesting cases are those of massive and massless particles. It should be noted that in the case of $m^2 = 0$ there is also a degenerate subcase where $p_n = 0$, which corresponds to constant tensors that do not depend on space-time coordinates. In the context of field theory such "constant fields" are of little interest. However, they will play a role in the analysis of Section 3.2.3.

3.2.1 Massive case

Let us first consider the massive case with $m^2 > 0$. As in the example of a scalar, there are two branches of positive- and negative-frequency solutions, namely with $p_0 > 0$ and $p_0 < 0$. It is enough to consider the positive-frequency branch with $p_0 > 0$.

We observe that all p_n such that $p_n p_m \eta^{nm} = m^2$ and $p_0 > 0$ are related by a Lorentz transformation. This means that any p_n of this type can be obtained by a Lorentz transformation from any other.

The problem is to find the possible structure of the space V_p at a given p. (In the sequel we skip the label '+' indicating that V_p^+ is the space of positive-frequency states.)

Since all V_p with different p on the mass shell $p^2 = m^2$ are related by Lorentz transformations it is enough to analyze the problem at any particular p_n. The most convenient choice is

$$p_n = (m, 0, \dots, 0) \,. \tag{3.43}$$

This choice is usually referred to as the *rest frame* because it corresponds to the case where the velocity associated with $\vec{p}$ is zero.

The structure of V_p at given p_n is determined by those Lorentz transformations that leave (3.43) invariant. The group of such transformations is called the *Wigner little group* while its Lie algebra is called the *Wigner little algebra*. Clearly, the little algebra in the massive case is $o(d-1)$. Indeed, this is the part of the Lorentz algebra that does not act on the time component of vectors. As such, it describes infinitesimal spatial rotations of $(d-1)$-dimensional space. Thus we arrive at the important conclusion that, in the massive case, any V_p should form an $o(d-1)$-module. The problem therefore reduces to the question which modules of $o(d-1)$ can correspond to massive particles. The answer is that any V_p that forms a *unitary* $o(d-1)$-module leads to a unitary Poincaré-module. I will not focus on the formal mathematical proof of this result because it will follow from the explicit construction of the field equations in Section 3.3.1, which give rise to the complete list of modules realized by their positive-frequency solutions.

So the problem of classification of possible elementary particles as Poincaré-modules is reduced to the problem of the classification of unitary $o(d-1)$-modules. The list of unitary $o(d-1)$-modules is well known.

The simplest example is the *trivial* module on which $o(d-1)$ does not act at all. In the notation of Young diagrams, which we will explain below, it is denoted by $\bullet$.

The relativistic field that does not transform under space rotations is of course the scalar field considered above. In this case the space V_p is one dimensional.

A less trivial example is that of the *vector* $o(d-1)$-module. In this case, elements of V_p are $o(d-1)$-vectors. Let us call them $\widetilde{A}_i(p)$ with $i = 1, \ldots, d-1$. This type of modules is denoted by the Young diagram $\square$. Analogously to our analysis of a scalar, $\widetilde{A}_i(p)$ can be identified with the Fourier components of a relativistic vector field $A_n(x)$ that obeys two conditions. One is the Klein-Gordon equation

$$(\square + m^2)A_n(x) = 0\,, \tag{3.44}$$

which restricts momenta to the condition (3.42). The other one is the Lorentz (divergencelessness) condition

$$\partial_n A^n(x) = 0\,. \tag{3.45}$$

In the rest frame, where only p_0 is nonzero, the latter implies that $\widetilde{A}_0(p) = 0$ and leaves the spatial components $\widetilde{A}_i(p)$ unrestricted. Upon Fourier transforming, $\widetilde{A}_i(p)$ represent elements of the space V_p forming the vector $o(d-1)$-module.

3.2.2 Irreducible $o(N)$-modules and Young diagrams

One proceeds analogously for any other tensor $o(N)$-module. For example, instead of $\widetilde{A}_i(p)$ from the vector-field case, consider a second-rank tensor $\phi_{i,j}$. It contains three $o(N)$-irreducible parts:

$$\phi_{i,j} = S^{\perp}_{i,j} + A_{i,j} + \frac{1}{N}\delta_{ij}\phi^{\mathrm{tr}}\,, \tag{3.46}$$

where

$$\phi^{\mathrm{tr}} := \phi_{kl}\delta^{kl} \tag{3.47}$$

is the trace part of $\phi_{i,j}$,

$$S^{\perp}_{i,j} := \frac{1}{2}(\phi_{i,j} + \phi_{j,i}) - \frac{1}{N}\delta_{ij}\phi^{\mathrm{tr}} \tag{3.48}$$

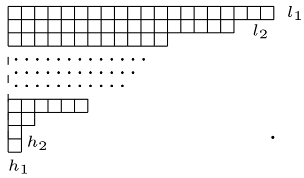
is the symmetric traceless part of $\phi_{i,j}$ and

$$A_{i,j} := \frac{1}{2}(\phi_{i,j} - \phi_{j,i}) \tag{3.49}$$

is the antisymmetric part of $\phi_{i,j}$. In the notation of Young diagrams, ϕ^{tr}, $S^{\perp}$ and A are represented by $\bullet$, $\square\!\square$ and $\begin{smallmatrix}\square\\\square\end{smallmatrix}$, respectively. Irreducibility of these modules expresses the fact that it is not possible to impose further conditions on the traceless tensors of definite symmetry to single out a submodule.

Now consider a rank-r tensor $\phi_{i_1,i_2,\ldots}(p)$ that carries r spatial indices taking $N = d-1$ values. To make the tensor irreducible, first of all one has to require $\phi_{i_1,i_2,\ldots}$ to be traceless, which means that contraction of any pair of indices with the Kronecker delta, which is the spatial part of the Minkowski metric, gives zero. For example, $\phi_{i_1,i_2,\ldots}\delta^{i_1i_2} = 0$, and likewise for any other pair of indices. In addition, $\phi_{i_1,i_2,\ldots}$ should have some definite symmetry properties.

Different symmetry properties of rank-r tensors $\phi_{i_1,i_2,\ldots}$ can be represented by various *Young diagrams*. By definition, these consist of r cells $\square$ and are such that the length of any row l_n does not exceed the length of any row l_m above it, *i.e.*, $l_m \geq l_n$ at $m \leq n$:



$$. \tag{3.50}$$

If a tensor has the symmetry property of a Young diagram $Y(l_1, l_2, l_3, \ldots)$, this means that $\phi_{i_1,i_2,\ldots i_{l_1},j_1,j_2,\ldots j_{l_2},k_1,k_2,\ldots k_{l_3},\ldots}$ is symmetric with respect to the indices i, the indices j, the indices k, *etc*, and such that symmetrization of any of the indices j with *all* indices i gives zero, symmetrization of any of the indices k with *all* indices i or *all* indices j gives zero, *etc*.

In summary, to be $o(N)$-irreducible, a tensor has to have definite symmetry properties and also be traceless with respect to any pair of indices

$$\phi_{\ldots i \ldots j \ldots}\delta^{ij} = 0\,. \tag{3.51}$$

With a slight abuse of terminology, when referring to a $o(N)$-*Young diagram*, we will only mention the Young diagram encoding the symmetry properties, keeping the tracelessness condition implicit.

We leave it to the reader as an exercise to check that $\bullet$, $\square$, $\square\!\square$ and $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ indeed give rise to a scalar, vector, traceless symmetric tensor, and an antisymmetric tensor and that the single-column diagrams $\begin{smallmatrix}\square\\\vdots\\\square\end{smallmatrix}$ describe totally antisymmetric tensors.

Let us consider a couple of examples of mixed symmetry tensors that are neither totally symmetric $\square\cdots\square$ nor totally antisymmetric $\begin{smallmatrix}\square\\\vdots\\\square\end{smallmatrix}$.

By definition, a rank-three tensor $\phi^S_{nm,k}$ has the symmetry of the *hook* diagram $\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$ if it is symmetric in n, m and such that (total) symmetrization over n, m, k gives zero

$$\phi^S_{nm,k} + \phi^S_{mk,n} + \phi^S_{kn,m} = 0\,. \tag{3.52}$$

Alternatively, we can consider the tensor

$$\phi^A_{[nm],k} := \phi^S_{nk,m} - \phi^S_{mk,n} \tag{3.53}$$

which is antisymmetric in the indices n and m (to stress this, we put the indices n and m in square brackets). Clearly, $\phi^A_{[nm],k}$ obeys the condition

$$\phi^A_{[nm],k} + \phi^A_{[mk],n} + \phi^A_{[kn],m} = 0 \tag{3.54}$$

which implies that the total *antisymmetrization* over all three indices gives zero.

Reversely, $\phi^S_{nk,m}$ can be expressed in terms of $\phi^A_{[nm],k}$ as

$$\phi^S_{nk,m} = \frac{1}{3}(\phi^A_{[nm],k} + \phi^A_{[km],n})\,. \tag{3.55}$$

We leave it to the reader to check the coefficient using the property (3.52).

It is customary to say that $\phi^S_{nm,k}$ and $\phi^A_{[nm],k}$ represent the hook diagram in the *symmetric* and *antisymmetric basis*, respectively.

Analogously, a tensor having the symmetry of the *window* diagram can be represented either by $\phi^S_{nm,kl}$, which is symmetric both in nm and in kl and has the property that symmetrization over any three indices gives zero, or by $\phi^A_{[nm],[kl]}$, which is antisymmetric both in nm and in kl and has the property that antisymmetrization over any three indices gives zero.

Problem 3.4. Prove that $\phi^S_{nm,kl} = \phi^S_{kl,nm}$ and $\phi^A_{[nm],[kl]} = \phi^A_{[kl],[nm]}$.

Problem 3.5. Find the explicit relations between $\phi^S_{nm,kl}$ and $\phi^A_{[nm],[kl]}$.

Generally, tensors in the antisymmetric basis of Young diagrams have the following properties: $\phi^A_{[n_1\dots n_{h_1}],[m_1\dots m_{h_2}],[k_1\dots k_{h_3}],\dots}$ is totally antisymmetric with respect to any group of indices n_i, m_i, k_i, *etc*, and such that the antisymmetrization of any index m with *all* indices n gives zero as well as the antisymmetrization of any index k with *all* indices m or all n, etc. Here h_i is the height of the i^{th} column of the same Young diagram. Clearly, $h_i \geq h_j$ at $i \leq j$.

Problem* 3.6. Check that if a Young diagram has the wrong shape, *i.e.*, if $l_n < l_m$ or $h_n < h_m$ for some $n < m$, any tensor associated with it is zero.

Since in the antisymmetric basis the antisymmetry associated with each column is explicit, it is always possible to dualize any antisymmetric tensor with the help of $o(N)$-epsilon symbol, replacing, say, $\phi_{[m_1\dots m_h]}$ by

$$\tilde{\phi}_{[m_1\dots m_{N-h}]} = \frac{1}{h!}\varepsilon_{[m_1\dots m_{N-h}]}{}^{[k_1\dots k_h]}\phi_{[k_1\dots k_h]}\,. \tag{3.56}$$

This allows us to consider only $o(N)$-tensors associated with such Young diagrams that the heights of their columns do not exceed $\frac{N}{2}$:

$$h_i \leq \frac{N}{2} \qquad \text{for all } i\,. \tag{3.57}$$

If N is even and h_1 takes the maximal value $h_1 = N/2$ one can also impose the (anti)self-duality condition relating $\phi_{\dots}$ and $\tilde{\phi}_{\dots}$. This is actually the last step which is necessary to do to make an $o(N)$-module irreducible. We consider the (anti)self-dual conditions along with the related field equations in Section 3.4.

So far we only considered massive bosons. Massive fermions can be considered analogously. As mentioned in Section 2.3.2, they form modules of the double cover of the Lorentz group $Spin(d-1,1)$ and are classified by representations of the Wigner little group $Spin(d-1)$. In practice, various unitary irreducible representations of $Spin(d-1)$ are given by tensor-spinors $\psi_{i_1,i_2,\dots,i_{l_1},j_1,j_2,\dots j_{l_2},k_1,k_2,\dots k_{l_3},\dots;\hat{\alpha}}$ that carry one spinor index $\hat{\alpha}$ and a number of vector indices that have some definite symmetry type as before. In addition, irreducible tensor-spinors obey the γ-*transversality condition* that contraction of any tensor index with a γ-matrix acting on the spinor index gives zero

$$\gamma^i{}_{\hat{\alpha}}{}^{\hat{\beta}}\psi_{\dots i\dots;\hat{\beta}} = 0\,. \tag{3.58}$$

Note that, by virtue of (2.25) and (3.58), $\psi_{i_1,i_2,\dots i_{l_1},j_1,j_2,\dots j_{l_2},k_1,k_2,\dots k_{l_3},\dots;\hat{\alpha}}$ obeys a tracelessness condition analogous to (3.51).

Using that massive fields are characterized by $o(d-1)$-modules we can now analyze the possible types of massless fields in different dimensions. For example, massive fields in $4d$ Minkowski space are characterized by such Young diagrams that $h_i \leq \frac{3}{2}$. This means that possible massive fields in four dimensions are associated with various totally symmetric tensors or tensor-spinors. The same happens for massive fields in $3d$ Minkowski space where $h_i \leq 1$, which is the same condition since h_i is integer. Thus, massive fields in four and three dimensions are described by a single number l which is the length of a one-row Young diagram. It is identified with the spin s of the field in the bosonic case and with $s - 1/2$ for fermions.

For $d = 2$ or $d = 1$ we obtain the condition that $h_i = 0$. The only diagram that has all heights equal zero is the trivial diagram $\bullet$ associated with scalar and spinor field in the bosonic and fermionic case, respectively. Thus, the Wigner analysis shows that in the case of two and one space-time dimensions there are only two possible types of propagating particles, namely scalars and spinors.

On the other hand, for $d = 5$ or 6, our analysis shows that $h_i \leq 2$. As a result, in this case the possible types of massive fields are described by various two-row Young diagrams and, hence, by two numbers l_1 and $l_2 \leq l_1$ identified with the lengths of the two rows. These are "spins" of massive fields in five and six space-time dimensions.

3.2.3 Massless case

In the massless case the problem is to analyze the structure of V_p with light-like momenta p_n obeying $p_n p_m \eta^{nm} = 0$. It is enough to analyze the problem at any given light-like p_n with $p_0 > 0$. For example, it can be chosen in the form

$$p_n = (\omega, 0, \dots, 0, \omega) \tag{3.59}$$

with some $\omega > 0$ ($\omega = 0$ corresponds to the case of constant fields).

First of all we have to find the Wigner little algebra, which is the subalgebra of the Lorentz algebra that leaves p_n invariant. Obviously, it contains the algebra $o(d-2)$ that acts on the vanishing components of p_n. Its generators are

$$L_{\mathbf{ij}}\,, \qquad \mathbf{i},\mathbf{j} = 1, \dots, d-2\,. \tag{3.60}$$

However, this is not the whole story. Indeed, it is not difficult to see that the following combinations of the Lorentz generators also leave p_n invariant:

$$L_{-\mathbf{i}} := L_{0\mathbf{i}} - L_{d-1\,\mathbf{i}}\,. \tag{3.61}$$

The generators $L_{-\mathbf{i}}$ together with $L_{\mathbf{ij}}$ obey the following commutation relations

$$[L_{\mathbf{ij}}\,, L_{-\mathbf{k}}] = -i(\delta_{\mathbf{jk}}L_{-\mathbf{i}} - \delta_{\mathbf{ik}}L_{-\mathbf{j}})\,, \tag{3.62}$$

$$[L_{-\mathbf{i}}\,, L_{-\mathbf{j}}] = 0\,. \tag{3.63}$$

As a Lie algebra, the algebra formed by $L_{\mathbf{ij}}$ and $L_{-\mathbf{i}}$ is equivalent to the algebra $e_{d-2} := iso(d-2)$ of motions (*i.e.*, symmetries that leave the metric invariant) of $(d-2)$-dimensional Euclidean space, where $L_{-\mathbf{i}}$ are interpreted as translation generators.

Thus, in the massless case, V_p has to form an irreducible unitary module of $iso(d-2)$. There is however a mathematical theorem saying that unitary representations of non-compact algebras, to which $iso(d-2)$ belongs, cannot admit faithful (*i.e.*, with all generators of the algebra in question acting nontrivially) finite-dimensional representations. In the case of $iso(d-2)$ this is actually easy to see.

Indeed, unitary representations of $iso(d-2)$ can be constructed using the same method as we used for the Poincaré algebra. In particular, if $iso(d-2)$-translations associated with $L_{-\mathbf{i}}$ would act nontrivially, in the basis where $L_{-\mathbf{i}}$ are diagonalized, their action has to be represented by some quasi-momenta $p_{\mathbf{i}}$ and, hence, modules of $iso(d-2)$ are realized in the spaces of functions of $p_{\mathbf{i}}$. Such spaces are obviously infinite dimensional except for the particular case where $L_{-\mathbf{i}}$ do act trivially, *i.e.*, by zero. (Hence, the latter modules are not faithful.) This is precisely the exceptional case of zero momentum mentioned in the beginning of our analysis of representations of the Poincaré algebra $iso(d-1,1)$ as non interesting in the context of field theory. In the context of representations of the massless little algebra, this case is however most interesting since only in this case the spaces V_p can be finite dimensional, which effectively means that the corresponding field contains at most a finite number of components, carrying a finite spin.

We conclude that the possible massless particles described by fields with a finite number of components are characterized by various representations of the algebra $o(d-2)$. Abusing terminology, $o(d-2)$ is often called the *massless Wigner algebra* instead of $iso(d-2)$. The structure of modules of $o(d-2)$ is of course fully analogous to that of the massive little algebra $o(d-1)$. We therefore will not repeat it here. This conclusion is in agreement with the example of a massless spin-one field whose degrees of freedom have been shown in Section 3.1.3 to be described by a $o(d-2)$-vector.

As in the massive case, the Wigner classification allows one to analyze possible types of massless fields in various dimensions. These are analogous to the massive case with the shift of d to $d-1$. Namely, massless fields in four- and five-dimensional Minkowski space are described by totally symmetric representations of the little algebra. The respective fields are called *totally symmetric*. They are characterized by a spin, which is the rank of respective symmetric tensors for bosons and the rank + 1/2 for fermions. Since in the old days people did not pay much attention to space-times of dimensions other than four, symmetric fields received most attention.

Another consequence is that in dimensions $d = 3$ and lower, the only possible types of massless fields are the scalar and spinor. This is indeed true. One manifestation of this fact is that there are no propagating gravitational waves in $d \leq 3$ and another is that the three-dimensional electrodynamics is equivalent to the theory of a scalar field.

Wigner actually elaborated also the case of *infinite-* or *continuous-spin* representations where $iso(d-2)$ does act faithfully. It is still unclear whether or not such infinite spin representations can play an important role for applications. We will not consider them here, and suffice by mentioning that different types of infinite-spin fields are classified by various types of finite-dimensional $o(d-3)$-modules. To show this one can analyze unitary modules of $iso(d-2)$ along the lines of the analysis of $iso(1,d-1)$ in Section 3.2.1.

For more details about unitary Poincaré-modules and Young diagrams we refer the reader to [1].

3.3 Field equations

By analogy with the lower-spin examples considered above, it is not difficult to write down relativistic equations that give rise to various unitary irreducible Poincaré-modules as the spaces of their positive-frequency solutions. In particular, this completes the construction of the full Poincaré-modules starting from the Wigner little algebra. As in the lower-spin examples, the respective positive-definite (or positive-semidefinite, in the case of massless tensor fields) norms, making the modules unitary, are defined as *electric charges* with respect to positive and negative frequencies.

3.3.1 Massive fields

Let the $o(d-1)$-module V_p be the space of tensors $\phi_{i_1,i_2,\ldots}$ ($i_k = 1,\ldots,d-1$) that are traceless and possess some definite symmetry properties as discussed in Section 3.2.2. The problem is to find Poincaré-invariant field equations which reproduce V_p at a given momentum p. To this end it suffices to take a Lorentz tensor field $\Phi_{n_1,n_2,\ldots}(x)$ ($n_j = 0,\ldots,d-1$) that has the same symmetry properties in its Lorentz indices, is Lorentz-traceless, and obeys the Klein-Gordon equation

$$(\Box + m^2)\Phi_{n_1,n_2,\ldots}(x) = 0 \qquad (n_j = 0,\ldots,d-1) \tag{3.64}$$

along with the divergencelessness conditions

$$\partial_n \Phi_{\ldots}{}^{n}{}_{\ldots}(x) = 0 \tag{3.65}$$

to be satisfied for contractions of ∂_n with any index of $\Phi_{n_1,n_2,\ldots}(x)$.

In the rest frame with momentum (3.43) the divergencelessness conditions (3.65) imply that all those components of $\Phi_{n_1,n_2,\ldots}(x)$, which carry at least one index equal to zero, vanish. After that, the Lorentz-tracelessness condition reduces to the $o(d-1)$-tracelessness condition. The resulting irreducible $o(d-1)$-tensor at the rest-frame momentum just represents the appropriate $o(d-1)$-module $\phi_{i_1,i_2,\ldots}$.

The story for fermions is analogous. Let V_p be the space of tensor-spinors $\psi_{i_1,i_2,\ldots;\alpha}$ ($i_j = 1,\ldots,d-1$ and α is a spinor index of $o(d-1)$) that are γ-traceless and possess definite symmetry properties. To find Poincaré-invariant field equations which reproduce V_p at a given momentum p it suffices to take a Lorentz tensor-spinor field $\Psi_{n_1,n_2,\ldots;\hat{\alpha}}(x)$ that has the same symmetry properties in its Lorentz indices, is γ-traceless, and obeys the Dirac equation

$$(i\gamma^n\partial_n + m)\Psi_{n_1,n_2,\ldots;\hat{\alpha}}(x) = 0 \tag{3.66}$$

along with the divergencelessness conditions

$$\partial_n \Psi_{\dots}{}^{n}{}_{\dots}(x) = 0 \tag{3.67}$$

to be satisfied for contractions of ∂_n with any index of $\Psi_{n_1,n_2,\dots}(x)$. Here $n_j = 0,\dots,d-1$ while $\hat{\alpha}$ is a spinor index of $o(d-1,1)$.

As in the spin-1/2 example, the Dirac equation projects to zero half of the components of $\Psi_{n_1,n_2,\dots}(x)$ by a condition analogous to (3.19). In the rest frame these conditions reduce $\Psi_{n_1,n_2,\dots;\hat{\alpha}}(x)$ to the $o(d-1)$-module represented by $\psi_{i_1,i_2,\dots;\underline{\hat{\alpha}}}$ where the $o(d-1)$-spinor index $\underline{\hat{\alpha}}$ takes half the number of values compared to the $o(d-1,1)$-spinor index $\hat{\alpha}$.

A particularly interesting and the simplest class of massive fields is represented by fields $\Phi_{n_1,n_2,\dots n_s}(x)$ which are totally symmetric traceless tensors of rank s

$$\Phi_{\dots n_k \dots n_l \dots}(x) = \Phi_{\dots n_l \dots n_k \dots}(x)\,. \tag{3.68}$$

Such fields have the symmetry properties of Young diagrams with a single row of length s, and are called *spin-s symmetric fields*. Analogously, tensor-spinor fields $\Psi_{n_1,n_2,\dots,n_s;\hat{\alpha}}(x)$, which are symmetric in the indices $n_1,n_2,\dots,n_s$, are called *spin-$(s+1/2)$ symmetric fields*.

The fields that are neither totally symmetric nor totally antisymmetric are called *mixed-symmetry fields*. In particular, such fields are important because massive states of mixed symmetry type are present among the states of String Theory.

3.3.2 Massless fields

In Section 3.2.3 we argued that massless unitary modules are associated with various finite-dimensional modules of the massless little algebra $o(d-2)$. Let the $o(d-2)$-module be the linear space of $o(d-2)$-tensors $\phi_{\mathbf{i}_1,\mathbf{i}_2,\dots}$ ($\mathbf{i}_i = 1,\dots,d-2$) that are traceless and possess definite symmetry properties. To realize the massless Poincaré-module associated with this $o(d-2)$-module we consider the Lorentz tensor field $\Phi_{n_1,n_2,\dots}(x)$ ($n_j = 0,\dots,d-1$) that has the same symmetry properties in its Lorentz indices, is Lorentz-traceless, and obeys the Klein-Gordon equation

$$\Box \Phi_{n_1,n_2,\dots}(x) = 0\,, \qquad (n_j = 0,\dots,d-1). \tag{3.69}$$

Also we impose the divergencelessness gauge conditions

$$\partial_n \Phi_{\dots}{}^{n}{}_{\dots}(x) = 0 \tag{3.70}$$

for any index of $\Phi_{n_1,n_2,\dots}(x)$. However, in the massless case, the space of solutions of these equations is not irreducible because, like in the example of electrodynamics, there is a Poincaré invariant subclass of solutions that have the following *on-shell* gauge form

$$\delta \Phi_{n_1,n_2,n_3,\dots}(x) = \Pi(\partial_{n_1} \varepsilon_{n_2,n_3,\dots}(x))\,. \tag{3.71}$$

Here Π is the projector to the symmetry type of $\Phi_{n_1,n_2,n_3,\dots}(x)$ and $\varepsilon_{n_2,n_3,\dots}(x)$ are various possible gauge-symmetry parameters that can give a non-zero contribution to $\delta\Phi_{n_1,n_2,n_3,\dots}(x)$ and themselves possess a definite symmetry, are Lorentz-traceless, and obey the "on-shell" conditions

$$\Box\varepsilon_{n_2,n_3,\dots}(x) = 0 \tag{3.72}$$

as well as the divergencelessness conditions

$$\partial_n \varepsilon_{\dots}{}^{n}{}_{\dots}(x) = 0\,. \tag{3.73}$$

For example, for $\Phi_{nm,k}$ that has the symmetry of the hook diagram $\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ in the symmetric basis, both symmetric and antisymmetric gauge parameters $\varepsilon^S_{nm} = \varepsilon^S_{mn}$ and $\varepsilon^A_{[nm]} = -\varepsilon^A_{[mn]}$ contribute to the variation. The effect of the projector Π is to put the variation into a form that is consistent with the properties of $\Phi_{nm,k}$:

$$\delta\Phi_{nm,k} = \delta^S\Phi_{nm,k} + \delta^A\Phi_{nm,k}\,, \tag{3.74}$$

$$\delta^S\Phi_{nm,k} = \partial_n\varepsilon^S_{km} + \partial_m\varepsilon^S_{kn} - 2\partial_k\varepsilon^S_{nm}\,, \tag{3.75}$$

$$\delta^A\Phi_{nm,k} = \partial_n\varepsilon^A_{km} + \partial_m\varepsilon^A_{kn}\,. \tag{3.76}$$

These formulae respect both the symmetry properties and, taking into account (3.73), the tracelessness conditions of $\Phi_{nm,k}$.

Since the on-shell gauge solutions form a Poincaré-invariant space, *i.e.*, a submodule of the space of solutions to (3.69) and (3.70), it should be factored out. In physical terms this means that in order to have an irreducible module of a massless particle it is necessary to demand gauge invariance with respect to the transformation (3.71).

To see how this construction reproduces Wigner's results consider its consequences for the fixed momentum $\mathbf{p}_n$ (3.59). Using the notations $A_\pm = A_0 \pm A_{d-1}$, we observe that conditions (3.70) imply that all those components of $\Phi_{n_1,n_2,\dots}$ that carry at least one index $'-'$ are zero. On the other hand, the gauge transformations (3.71) imply that all those components of $\Phi_{n_1,n_2,\dots}$ that carry at least one index $'+'$ are pure gauge (unphysical). As a result, the physical components are the $\Phi_{\mathbf{i}_1,\mathbf{i}_2,\dots} = \phi_{\mathbf{i}_1,\mathbf{i}_2,\dots}$ that do not contain $'+'$ and/or $'-'$ indices. These precisely realize the $o(d-2)$-modules of the Wigner analysis from Section 3.2.3.

The analysis for fermions is analogous.

Analogously to the case of electrodynamics the on-shell gauge parameters $\varepsilon_{n_1,n_2,\dots}$ are residual gauge symmetries in the corresponding off-shell theory remaining after imposing the Lorentz-invariant gauge conditions (3.73). For mixed-symmetry massless fields the new phenomenon of gauge symmetries for gauge symmetries occurs. For instance, in the example of the hook massless fields consider the following gauge transformation of the gauge parameters ε^S_{nm} and ε^A_{nm}

$$\delta\varepsilon^S_{nm} = \partial_n\varepsilon_m + \partial_m\varepsilon_n\,, \tag{3.77}$$

$$\delta\varepsilon^A_{nm} = \partial_n\varepsilon_m - \partial_m\varepsilon_n \tag{3.78}$$

with a new parameter ε_n that itself obeys $\Box\varepsilon_n = 0$ and $\partial_n\varepsilon^n = 0$. The point is that such a transformation of ε^S_{nm} and ε^A_{nm} does not affect the gauge variation (3.74) of the hook field.

The gauge-for-gauge symmetries do not appear in the particularly interesting class of massless fields represented by fields $\Phi_{n_1,n_2,\ldots,n_s}$ which are totally symmetric traceless tensors of rank s

$$\Phi_{\ldots n_k \ldots n_l \ldots} = \Phi_{\ldots n_l \ldots n_k \ldots} . \tag{3.79}$$

As in the massive case, such fields are called spin-s symmetric fields. Analogously, tensor-spinor fields $\Psi_{n_1,n_2,\ldots;\hat{\alpha}}$, which are symmetric in the indices $n_1, n_2, \ldots$, are called spin-$(s+\frac{1}{2})$ symmetric fields. In this review I will mainly focus on totally symmetric massless fields because a consistent non-linear theory of higher-spin fields has been developed so far only for this class. The theory of mixed-symmetry fields is more complicated and less elaborated.

Note that the Wigner classification suggests that massless fields in d dimensions should somehow correspond to massive fields in $d-1$ dimensions. This is indeed true, and the correspondence can be made very precise using the mechanism of compactification on a circle S^1 in d dimensions as explained in Section 4.3.

3.4 Self-duality

Let an $o(N)$-module be realized as a linear space of traceless tensors $\phi_{[m_1,\ldots,m_{h_1}],[n_1,\ldots,n_{h_2}]\ldots}$ having definite symmetry properties in the antisymmetric basis. If $N = 4n$ and $h_1 = 2n$ the following (anti)self-duality condition can be imposed

$$\phi_{[m_1,\ldots,m_{2n}],\ldots} = \pm \frac{1}{(2n)!} \epsilon_{[m_1\ldots m_{2n}]}{}^{[k_1\ldots k_{2n}]} \phi_{[k_1,\ldots,k_{2n}],\ldots} \tag{3.80}$$

which finally makes the $o(4n)$-module irreducible. Let us note that such a condition can be imposed only on a set of antisymmetrized $2n$ indices. This implies that it can be written only for the indices of the first column or some other column of the same maximal height. However, tensors having definite symmetry properties, can be shown to be symmetric under the exchange of columns of equal heights, which means that any other possible selfduality condition is equivalent to (3.80).

Note that for $N = 4n+2$, the (anti)self-duality condition (3.80) is inconsistent having as a consequence of its double application $\phi_{[m_1,\ldots,m_{h_1}],\ldots} = -\phi_{[m_1,\ldots,m_{h_1}],\ldots}$ (recall that $h_1 = 2n+1$). This can be formally resolved by adding an additional factor of i to the right-hand side of (3.80)

$$\phi_{[m_1,\ldots,m_{2n+1}],\ldots} = \pm \frac{i}{(2n+1)!} \epsilon_{[m_1\ldots m_{2n+1}]}{}^{[k_1\ldots k_{2n+1}]} \phi_{[k_1,\ldots,k_{2n+1}],\ldots} . \tag{3.81}$$

In this case, due to the imaginary unit, the (anti)self-duality condition effectively implies that the field $\phi_{[m_1,\ldots,m_{h_1}],\ldots}$ is complex while the "(anti)self-duality" condition then relates its real part to the dual of the imaginary part and vise versa. So, we have the same number of degrees of freedom as for a real field, and from the point of view of $o(4n+2)$-modules one is still left with an unrestricted real field $\phi_{[m_1,\ldots,m_{h_1}],\ldots}$.

Nevertheless, since the space of positive-frequency solutions V_p^+ is genuinely complex, such a (anti)self-duality condition can play a role in field theory by further restricting the space of solutions of relativistic equations.

In the fermionic case, whenever possible, the $o(N)$-spinors can be required to be Majorana (*i.e.*, real or purely imaginary in an appropriate basis) and/or chiral, *i.e.*, obeying

$$\Gamma_{\hat{\alpha}}{}^{\hat{\beta}}\psi_{\dots;\hat{\beta}} = \pm\psi_{\dots;\hat{\alpha}} \tag{3.82}$$

with

$$\Gamma := i^{\frac{N(N-1)}{2}}\frac{1}{N!}\epsilon_{n_1 n_2 \dots n_N}\gamma^{n_1}\gamma^{n_2}\dots\gamma^{n_N} \tag{3.83}$$

normalized in such a way that

$$\Gamma^2 = I\,. \tag{3.84}$$

Note that in the fermionic case, the (anti)self-duality condition for γ-transversal tensor-spinors with $h_1 = N/2$ is a consequence of the chirality of the spinor. We leave it to the reader to check this fact.

Since (anti)self-dual fields will not play an important role in the rest of this review I will not go into details of their formulation, making just a few brief comments.

In the massive case with the little Wigner algebra $o(d-1)$, (anti)self-dual equations can be considered in odd dimensions. To reproduce the (anti)self-duality condition in the case of $h_1 = (d-1)/2$ with odd d one has to impose the equation

$$\partial_{[n_1}\Phi_{n_2,\dots n_{h_1+1}],\dots} = \pm\frac{i^{h_1+1}}{(h_1+1)!}\, m\epsilon_{n_1,\dots n_{h_1+1}}{}^{k_1,\dots k_{h_1}}\Phi_{k_1,\dots k_{h_1},\dots}\,, \tag{3.85}$$

where the brackets '[]' now denote the antisymmetrization normalized to be a projector, *i.e.*, $[[\dots]] = [\dots]$. This equation makes sense for real fields $\Phi_{k_1,\dots k_{h_1},\dots}$ if $d = 4n-1$ that, in particular, includes the case of $d = 3$, for which it applies to the totally symmetric massive fields. In this case, the equations (3.85) can be derived from a Lagrangian. For $d = 4n+1$ equation (3.85) manifestly involves the imaginary unit, not allowing a Lagrangian formulation in terms of real fields.

For spinors associated with Young diagrams of the maximal height $h_1 = (d-1)/2$ for odd d, the chirality condition with respect to $o(d-1,1)$ implies the (anti)self-duality condition.

For massless fields in Minkowski space-time of even dimension d that have symmetry properties of a Young diagram of maximal height, *i.e.*, $h_1 = d/2-1$, one can in addition impose the (anti)self-duality conditions in the form

$$\partial_{[n_1}\Phi_{n_2,\dots n_{h_1+1}],\dots} = \pm\frac{i^{h_1}}{(h_1+1)!}\epsilon_{n_1,\dots n_{h_1+1}}{}^{k_1,\dots k_{h_1+1}}\partial_{k_1}\Phi_{k_2,\dots k_{h_1+1},\dots}\,. \tag{3.86}$$

We leave it to the reader to check that this reproduces the (anti)self-duality property of the representation of the massless little algebra.

Note that the condition (3.86) explicitly contains the imaginary unit i for $d = 4, 8, 12$, *etc* while being real for $d = 2, 6, 10$, *etc*. This means that it can be imposed on real fields in any $d = 4n+2$ but not in any $d = 4n$, including in particular our four-dimensional Minkowski space-time.

4 Lagrangian formulation for symmetric fields

4.1 Motivation

Due to Wigner, we know all types of elementary particles that can exist in QFT. Moreover, we know the form of their field equations. For the four-dimensional case these field equations were written down by Dirac in the thirties of the last century. What is next? In fact, this is just the beginning of a long way.

First of all, our ultimate goal is of course a theory of interacting fields. A free theory itself is not too interesting since, being non-interacting with anything else, free particles can neither see anything nor be seen. What can make a system non-trivial and interesting are interactions between different particles. In terms of field equations, interactions are described by nonlinearities responsible for scattering processes as well as for annihilation and creation processes in QFT. When elementary particles are far from each other, however, the interactions become negligible and particles become free. The study of free fields is therefore very important for understanding the elementary building blocks of various relativistic models. In Section 5 we start the study of interacting theories. However, even at the free-field level, a lot of issues remain to be analyzed.

First of all, one should be aware that the equations presented above are not the only ones that can be used to describe the same particles (*i.e.*, unitary Poincaré-modules). For example, a massless spin-one field can be described either in terms of the vector potential A_n as we did, or in terms of the field strength F_{nm}. The physics of these two formulations, as well as the representation-theory pattern, is the same at least at the free-field level, though the equations have a different form. In such cases the two systems are called *dual* to each other. Usually there exist many (in fact, infinitely many) dual formulations for the same physical system. We will see more examples of such dualities in the rest of this review. However, in the physically important cases, there exists a distinguished *ground* system that underlies the Lagrangian formulation. In the example of spin one this is the formulation in terms of A_n.

There are many reasons why it is useful to have a Lagrangian formulation. One is that so far we have only considered free elementary particles described by linear field equations. To simplify the problem of the construction of a nonlinear theory it is very useful to derive a Lagrangian formulation for the fields in question. The simplest way to understand this is to observe that the systems of field equations considered so far are in most cases overdetermined: we have more equations than field variables. This has the dramatic consequence that as soon as we try to nonlinearly deform our system, most likely it will become mathematically inconsistent, allowing no solutions at all. In other words, a naive nonlinear deformation of the field equations in the form we have defined them so far most likely will not make any sense.

If, on the other hand, a system of partial differential equations is derived from an action principle, having the form

$$\frac{\delta S}{\delta \phi^i} = 0\,, \tag{4.1}$$

then the number of field equations matches the number of field variables, thus protecting one from immediately running into the consistency problem.It should be noted however that, resolving some of the problems, the Lagrangian formulation on its own does not guarantee that the system is free of difficulties of various kinds. Needless to say the Lagrangian formulation plays a key role for the quantization procedure via the Feynman path integral.

The program of reformulating relativistic equations in a Lagrangian form was originally formulated by Fierz and Pauli in the thirties of the last century. The tricky part of the problem is that to reach a Lagrangian formulation for some relativistic field ϕ it turns out to be necessary to introduce so-called *supplementary* fields χ which are not physical and turn out to be zero by virtue of the field equations and/or gauge symmetries.

4.2 Symmetric fields and Fronsdal formulation

The Lagrangian formulation of symmetric *massive* fields of any spin was solved by Singh and Hagen in 1974. The minimal set of fields necessary for a Lagrangian formulation in this case contains the following set of traceless symmetric tensors

$$\phi_{n_1\ldots n_s},\; \underbrace{\phi_{n_1\ldots n_{s-2}},\;\; \phi_{n_1\ldots n_{s-3}}\,,\;\ldots,\;\; \phi}_{\text{supplementary fields}}\,.$$

I will not consider massive fields in detail but instead focus on the theory of *massless* fields, which, in fact, was deduced from the massless limit of the Sing-Hagen theory by Fronsdal in 1978. Fronsdal theory is usually taken as a starting point in the analysis of higher-spin gauge theory. The construction of the massless Lagrangian is much simpler than that of the massive one. On the other hand, using dimensional reduction, it is easy to deduce the massive theory from the massless theory in one dimension higher (see Section 4.3).

According to Fronsdal, the Lagrangian for a massless symmetric field of any spin s is formulated in terms of just two symmetric traceless fields of ranks s and $s-2$

$$\phi_{n_1\ldots n_s}\,, \qquad \phi_{n_1\ldots n_{s-2}}\,.$$

These two traceless fields can be unified into a single field $\varphi_{n_1\ldots n_s}$

$$\varphi_{n_1\ldots n_s} := \phi_{n_1\ldots n_s} + \phi_{(n_1\ldots n_{s-2}}\eta_{n_{s-1}n_s)}\,, \tag{4.2}$$

where all indices n_i are symmetrized. The so-defined field $\varphi_{n_1\ldots n_s}$ obeys the *double-tracelessness* condition

$$\eta^{n_1n_2}\eta^{n_3n_4}\varphi_{n_1\ldots n_s} = 0\,. \tag{4.3}$$

Note that we use the convention that $((\ldots)) = (\ldots)$, which means that

$$A_{(n_1,n_2,\ldots,n_p)} = \frac{1}{p!}\Big(\sum A_{n_{i_1},n_{i_2},\ldots,n_{i_p}}\Big)\,, \tag{4.4}$$

where the summation on the right-hand side of (4.4) is over all the $p!$ permutations of the indices $n_1, n_2, \ldots, n_p$.

We will formulate Fronsdal theory in terms of $\varphi_{n_1\dots n_s}$. The key property of Fronsdal theory is that it possesses the gauge symmetry

$$\delta\varphi_{k_1\dots k_s}(x) := \partial_{(k_1}\varepsilon_{k_2\dots k_s)}(x)\,. \tag{4.5}$$

Here $\varepsilon_{k_1\dots k_{s-1}}(x)$ is an arbitrary symmetric tensor that is traceless

$$\eta^{n_1 n_2}\varepsilon_{n_1\dots n_{s-1}}(x) = 0\,. \tag{4.6}$$

As should be expected at the free field level, the gauge transformations (4.5) are Abelian. One of the main issues in the construction of higher-spin theory is to identify a non-Abelian higher-spin symmetry that underlies a nonlinear higher-spin theory, reproducing the symmetries (4.5) in the free limit.

For a spin-zero field, there is no associated gauge symmetry. For the case of spin one, the transformation (4.5) reproduces the transformation law of electrodynamics. In section 5 we first consider its non-Abelian deformation in Yang-Mills theory. Then in Section 7 we explain how the Cartan-Einstein theory of gravity results from the non-Abelian deformation of the spin-two gauge symmetry with vector parameter ε_n. Based on these examples we then will be able to consider supergravity and nonlinear higher-spin gauge theories.

Note that the property that $\varepsilon_{n_1\dots n_{s-1}}$ is traceless implies that the double-tracelessness condition (4.3) is gauge invariant. Indeed, in the gauge variation of the double trace $\eta^{n_1n_2}\eta^{n_3n_4}\varphi_{n_1\dots n_s}$ at least one Minkowski metric will hit a pair of indices of the gauge parameter, thus giving zero.

The field equations in Fronsdal theory can be put in the form

$$\mathcal{R}_{k_1\dots k_s}(\varphi(x)) = 0\,, \tag{4.7}$$

where the *Fronsdal tensor* $\mathcal{R}_{k_1\dots k_s}(\varphi(x))$ is defined as

$$\mathcal{R}_{k_1\dots k_s}(\varphi) := \Box\varphi_{k_1\dots k_s} - s\,\partial_{(k_1}\partial^n\varphi_{k_2\dots k_s)n} + \frac{s(s-1)}{2}\partial_{(k_1}\partial_{k_2}\varphi_{k_3\dots k_s)n}{}^{n}\,. \tag{4.8}$$

Its important property is that $\mathcal{R}_{k_1\dots k_s}(\varphi(x))$ is gauge invariant

$$\delta\mathcal{R}_{k_1\dots k_s}(\varphi(x)) = \mathcal{R}_{k_1\dots k_s}(\delta\varphi(x)) = 0\,. \tag{4.9}$$

sdelat' We leave it to the reader as a simple exercise to check this.

The number of equations in (4.7) is the same as the number of field variables since, as is not hard to see, the Fronsdal tensor obeys the double-tracelessness condition

$$\eta^{n_1n_2}\eta^{n_3n_4}\mathcal{R}_{n_1\dots n_s} = 0\,. \tag{4.10}$$

Before formulating the action we check that the Fronsdal equation (4.7) indeed describes massless fields of spin s. To this end, consider a gauge variation of the trace part of the Fronsdal field

$$\delta\varphi_n{}^{nm_1\dots m_{s-2}} \propto \partial_n\varepsilon^{nm_1\dots m_{s-2}}\,. \tag{4.11}$$

This allows us to choose the partial gauge

$$\varphi_n{}^{nm_3\ldots m_s} = 0\,.$$

This restricts the gauge-symmetry parameters by the condition $\partial_n \varepsilon^{nm_2\ldots m_{s-1}} = 0$. Contracting a couple of indices k_i in the Fronsdal equation (4.7) it now follows that

$$\partial_n \partial_m \varphi^{nm\ldots} = 0\,. \tag{4.12}$$

Taking into account that

$$\delta \partial_n \varphi^{nm_1\ldots m_{s-1}} = \frac{1}{s} \Box \varepsilon^{m_1\ldots m_{s-1}}$$

and using (4.12) we can choose the additional gauge

$$\partial_n \varphi^{nm_1\ldots m_{s-1}} = 0\,.$$

The residual gauge-symmetry parameter $\varepsilon^{m_1\ldots m_{s-1}}$ obeys

$$\Box \varepsilon^{m_1\ldots m_{s-1}} = 0\,, \qquad \partial_n \varepsilon^{n\,m_2\ldots m_{s-1}} = 0\,, \qquad \varepsilon^n{}_n{}^{m_3\ldots m_{s-1}} = 0\,.$$

The remaining field equations coincide with those that were associated with massless fields of spin s in Section 3.2.3:

$$\Box \varphi^{m_1\ldots m_s} = 0\,, \qquad \varphi^n{}_{nm_3\ldots m_s} = 0\,, \qquad \partial_n \varphi^{nm_2\ldots m_s} = 0\,.$$

We see that the components $\varphi^n{}_{nm_3\ldots m_s}$ are supplementary. They do not describe nontrivial degrees of freedom, being pure gauge.

The action that leads to the Fronsdal equations has the form

$$S[\varphi] := \frac{(-1)^{s+1}}{2} \int d^d x \left(\varphi^{m_1\ldots m_s} \mathcal{R}_{m_1\ldots m_s}(\varphi) - \frac{s(s-1)}{4} \varphi_n{}^{n\,m_3\ldots m_s} \mathcal{R}^k{}_{k\,m_3\ldots m_s}(\varphi) \right) \tag{4.13}$$

The overall sign here is introduced to have the standard normalization of the kinetic term leading to a positive-definite Hamiltonian in the mostly minus signature of the Minkowski metric.

Now we have to check two things. One is that the action is gauge invariant. The other is that it does indeed give rise to the Fronsdal equations (4.7).

Both of these can be most easily checked using the observation that the bilinear action is symmetric in the sense that

$$\begin{aligned} &\int d^d x \left(\delta\varphi^{m_1\ldots m_s} \mathcal{R}_{m_1\ldots m_s}(\varphi) - \frac{1}{4} s(s-1) \delta\varphi_n{}^{n\,m_3\ldots m_s} \mathcal{R}^k{}_{k\,m_3\ldots m_s}(\varphi) \right) \\ &= \int d^d x \left(\varphi^{m_1\ldots m_s} \mathcal{R}_{m_1\ldots m_s}(\delta\varphi) - \frac{1}{4} s(s-1) \varphi_n{}^{n\,m_3\ldots m_s} \mathcal{R}^k{}_{k\,m_3\ldots m_s}(\delta\varphi) \right) \end{aligned} \tag{4.14}$$

for generic $\delta\varphi^{m_1\ldots m_s}$. Indeed, the expression

$$\int d^d x \left(\varphi^{m_1\ldots m_s} \mathcal{R}_{m_1\ldots m_s}(\psi) - \frac{1}{4} \alpha\, s(s-1) \varphi_n{}^{n\,m_3\ldots m_s} \mathcal{R}^k{}_{k\,m_3\ldots m_s}(\psi) \right)$$

with an arbitrary constant α contains six types of terms. By integration by parts each of the terms

$$\int d^dx\ \varphi_{n_1\ldots n_s}\Box\psi^{n_1\ldots n_s}\,, \tag{4.15}$$

$$\int d^dx\ \varphi^k{}_{k\,n_3\ldots n_s}\Box\psi_l{}^{l\,n_3\ldots n_s}\,, \tag{4.16}$$

$$\int d^dx\ \varphi^k{}_{n_2\ldots n_s}\partial_k\partial_l\psi^{l\,n_2\ldots n_s} \tag{4.17}$$

$$\int d^dx\ \varphi^{km}{}_{m\,n_4\ldots n_s}\partial_k\partial_l\psi_p{}^{pl\,n_4\ldots n_s} \tag{4.18}$$

is obviously symmetric with respect to the exchange $\varphi \leftrightarrow \psi$. The two remaining terms are proportional to

$$\int d^dx \left(\varphi^{m_1\ldots m_s}\partial_{m_1}\partial_{m_2}\psi^k{}_{km_3\ldots m_s} + \alpha\,\varphi_n{}^{n\,m_3\ldots m_s}\partial_k\partial_l\psi^{kl}{}_{m_3\ldots m_s}\right). \tag{4.19}$$

For general α, this is not symmetric under the exchange $\varphi \leftrightarrow \psi$. However, at $\alpha = 1$ it is. This proves (4.14), at the same time explaining the relative coefficient in the action (4.13).

The gauge invariance of the action is now obvious by virtue of (4.14) because the gauge variation of the action $\delta S = 0$ is expressed via $\delta\mathcal{R}_{n_1\ldots n_s}(\varphi)$, which is zero. On the other hand, a general variation of the action gives field equations of the form

$$G_{m_1\ldots m_s}(\varphi) := \mathcal{R}_{m_1\ldots m_s}(\varphi) - \frac{1}{4}s(s-1)\eta_{(m_1m_2}\mathcal{R}_{m_3\ldots m_s)n}{}^n(\varphi) = 0\,. \tag{4.20}$$

By a nondegenerate linear transformation these equations are equivalent to (4.7). Note that the Fronsdal tensor (4.8) is a generalization of the linearized Ricci tensor for spin two while the tensor $G_{m_1\ldots m_s}(\varphi)$ (4.20) generalizes the linearized Einstein tensor for spin two.

It is not very difficult to show that, up to an overall constant, the action (4.13) is the only one that contains two derivatives of $\varphi_{n_1\ldots n_s}$ and is invariant under the gauge transformations (4.5). Indeed, the most general Poincaré-invariant action that is quadratic in the Fronsdal field $\varphi_{n_1\ldots n_s}$ and contains two derivatives is of the form

$$\int d^dx \Big(a_1\,\varphi_{n_1\ldots n_s}\Box\varphi^{n_1\ldots n_s} + a_2\,\varphi^k{}_{k\,n_3\ldots n_s}\Box\varphi_l{}^{l\,n_3\ldots n_s} + a_3\,\varphi^k{}_{n_2\ldots n_s}\partial_k\partial_l\varphi^{l\,n_2\ldots n_s}$$
$$+ a_4\,\varphi^{kl}{}_{n_3\ldots n_s}\partial_k\partial_l\varphi_m{}^{m\,n_3\ldots n_s} + a_5\,\varphi^{km}{}_{m\,n_4\ldots n_s}\partial_k\partial_l\varphi_p{}^{pl\,n_4\ldots n_s}\Big) \tag{4.21}$$

for some coefficients $a_1, \ldots, a_5$.

To prove that, up to an overall constant and total derivatives, the Fronsdal action (4.13) is the only action of the form (4.21) that is invariant under the gauge transformations (4.5) it suffices to observe that if there were an independent gauge-invariant action of the form (4.21) then there would exist another one with $a_1 = 0$.

Hovewer it is easy to show that there are no nonzero gauge-invariant actions with this property.

For lower spins (4.20) reproduces the Klein-Gordon and Maxwell equations in the cases of spin zero and one, respectively.

For spin two, the Fronsdal equations turn out to be equivalent to the linearized Einstein equations for the gravitational field. Given that lower-spin gauge fields play a fundamental role in physics, the key question is what kind of nontrivial theory may underly gauge fields of all higher spins. The goal is to find a nonlinear higher-spin theory such that in the free-field limit it amounts to the Fronsdal theory and such that Abelian higher-spin gauge symmetries related to the parameters $\varepsilon_{m_1\dots m_{s-1}}$ deform to a non-Abelian higher-spin gauge symmetry.

We skip the details of symmetric massless fields of half-integer spins, which is also available due to Fang and Fronsdal, extending the cases of spins 1/2 and 3/2 to all half-integer spins [2].

4.3 Massive fields from massless fields

In Section 3.2 we have seen that the Wigner little groups for the massless and massive cases differ by a shift of the space-time dimension by one. Indeed, the massive little group $O(d-1)$ in d dimensions coincides with the massless little group in $d+1$ dimensions. This fact has several important field-theoretical implications.

One observation is that, knowing a massless theory in $d+1$-dimensional Minkowski space, one can obtain a massive theory for a field of the same symmetry type in d-dimensional Minkowski space by the dimensional reduction of a $d+1$-dimensional model on a circle. Namely, starting from a real massless field $\Phi_{\mathcal{I}}(X)$ in $D+1$-dimensional space-time with coordinates X^M (where $M=0,\dots,d$ and $\mathcal{I}$ is a composite index labelling a set of tensor fields together with their tensorial indices), one takes the coordinates to be

$$X^M = (x^n\,,\varphi)\,,\qquad n=0,\dots,d-1\,, \tag{4.22}$$

where the periodic coordinate φ parametrizes a circle of a radius m^{-1}, *i.e.*, $\varphi\sim\varphi+2\pi m^{-1}$. Let us write $\partial_N :=\partial/\partial X^N$ and $\partial_n :=\partial/\partial x^n$.

Then, taking $\Phi_{\mathcal{I}}(X)$ of the form

$$\Phi_{\mathcal{I}}(X) = e^{im\varphi}\,\phi_{\mathcal{I}}(x) + e^{-im\varphi}\,\bar\phi_{\mathcal{I}}(x)\,, \tag{4.23}$$

by substituting it into the Lagrangian and field equations for the real massless field in $d+1$ dimensions, and integrating over the cyclic coordinate, one gets the Lagrangian and equations of motion for a complex field of mass m in d dimensions. This result is based on the fact that the massless D'Alambert operator in $d+1$ dimensions reduces to the massive Klein-Gordon operator in d dimensions when acting on (4.23):

$$\partial_M\partial^M\Phi_{\mathcal{I}}(X) = e^{im\varphi}(\partial_n\partial^n+m^2)\phi_{\mathcal{I}}(x) + e^{-im\varphi}(\partial_n\partial^n+m^2)\bar\phi_{\mathcal{I}}(x)\,. \tag{4.24}$$

It is important to note that if the original massless theory in $d+1$ dimensions possessed gauge symmetries with a set of gauge parameters $\Lambda_{\mathcal{J}}$, these get translated into gauge symmetries of the massive theory in d dimensions in a way similar to (4.23):

$$\Lambda_{\mathcal{J}}(X) = e^{im\varphi}\,\lambda_{\mathcal{J}}(x) + e^{-im\varphi}\,\lambda_{\mathcal{J}}(x)\,. \tag{4.25}$$

In the massive case, however, the gauge symmetries become *Stueckelberg* (shift) symmetries because differentiation with respect to φ is equivalent to multiplication by im.

To illustrate how this procedure works let us consider the example of a spin-one field. We start with the vector potential in $d+1$ dimensions $A_M(X)$ with the Maxwell Lagrangian integrated over the circle

$$L = -\frac{1}{4g^2}\int_0^{2\pi/m} d\varphi F_{MN}F^{MN}\,, \qquad F_{MN} = \partial_M A_N - \partial_N A_M\,, \tag{4.26}$$

which is invariant under the gauge transformations

$$\delta A_M = \partial_M \Lambda\,. \tag{4.27}$$

To perform reduction to d dimensions we set

$$A_N(X) = \begin{cases} e^{im\varphi}\,A_n(x) + e^{im\varphi}\,\bar{A}_n(x) & \text{at} \quad N = n\,, \\ i(e^{im\varphi}\,\phi(x) - e^{-im\varphi}\,\bar{\phi}(x)) & \text{at} \quad N = d\,. \end{cases} \tag{4.28}$$

The appearance of the factor of i in the definition of $A_d(x)$ is just a convenient choice of conventions (the imaginary unit i can be absorbed by redefining ϕ). This gives the d-dimensional field strengths

$$F_{MN} = \begin{cases} \partial_m A_n - \partial_n A_m & \text{at} \quad M = m, N = n\,, \\ i(\partial_n \phi - m A_n) & \text{at} \quad M = m, N = d\,, \end{cases} \tag{4.29}$$

and

$$\bar{F}_{MN} = \begin{cases} \partial_m \bar{A}_n - \partial_n \bar{A}_m & \text{at} \quad M = m, N = n\,, \\ -i(\partial_n \bar{\phi} - m \bar{A}_n) & \text{at} \quad M = m, N = d\,, \end{cases} \tag{4.30}$$

which are invariant under the gauge transformations

$$\delta A_n = \partial_n \lambda\,, \tag{4.31}$$

$$\delta\phi(x) = m\lambda(x)\,, \tag{4.32}$$

and their complex conjugates. With these definitions, the d-dimensional Lagrangian, which follows from Eq. (4.26), has the form

$$L = -\frac{1}{2mg^2}\Big(F_{nm}\bar{F}^{nm} + 2(\partial_n\phi - mA_n)(\bar{\partial}^n\phi - m\bar{A}^n)\Big)\,. \tag{4.33}$$

Using the Stueckelberg gauge symmetry (4.32) we can gauge fix the scalar field to zero,

$$\phi = 0\,, \tag{4.34}$$

and thus reproduce the Proca Lagrangian (2.33) for a massive complex spin-one field. By identifying A_m with $\bar{A}_m$ one gets the Lagrangian for the real massive vector field.

In a similar way one can apply the dimensional reduction procedure to a massless field of any symmetry type and get the Lagrangian formulation for the corresponding massive field in one dimension less, which is invariant under gauge symmetries of the Stueckelberg type.

We conclude that in the case of Minkowski space it is not even necessary to study massive free fields independently because it is easier to obtain them from the theory of massless fields by dimensional reduction. The appearance of the Stueckelberg symmetries by virtue of this mechanism is, in fact, a very useful feature because these gauge symmetries control degrees of freedom in the system and are particularly important for the analysis of interactions of massive higher-spin fields.

5 Yang-Mills theory

Yang-Mills theory is a theory of interacting massless fields of spin one. It plays a central role in the modern theory of fundamental interactions. In this section we explain the basic elements of Yang-Mills theory with some emphasis on those that are important for higher-spin gauge theories.

5.1 Local symmetries

Poincaré transformations provide an example of *global* (or equivalently, *rigid*) symmetries: the parameters of the Lorentz transformations $\Lambda^n{}_m$ and translations a^n are independent of the space-time coordinates x^n.

From the point of view of causality and the finiteness of the speed of light (*i.e.*, propagation of signals) it is somewhat strange that the symmetry-transformation parameters are constant in the entire space-time. Most likely this means that the metric η^{nm} exhibiting these symmetries was prepared during a very long process in the past (say, inflation in cosmology). From the perspective of relativistic physics it would be much more natural if symmetries are such that their parameters *do* depend on the space-time points, so that the transformations are independent at different points of space-time.

As we learned in Section 3.3.2, most massless fields are gauge fields with gauge parameters being arbitrary functions of the space-time coordinates. It is this property that makes them most interesting. After understanding the structure of free massless (*i.e.*, gauge) spin-s fields, the most important problem is to find a nonlinear deformation of one or another gauge theory that accounts for interactions of the corresponding particles. The main principle here is that the nonlinear theory should respect the gauge symmetries. As a result, gauge symmetries highlight a way towards a theory of fundamental interactions. We start with the simplest example, namely that of massless spin-one fields, which plays a key role in the present-day understanding of the physics of the fundamental interactions including our everyday life, because we see the world around us by virtue of the Maxwell equations that describe electromagnetic phenomena mediated by the spin-one massless electromagnetic field. In Section 8 we will see how gauge fields associated with space-time symmetries lead to the theory of gravity.

5.2 Maxwell field

The electromagnetic field strength, and hence Maxwell theory, is invariant under the gradient transformations (2.32)

$$A_n \to A'_n(x) = A_n(x) + \partial_n \varepsilon(x)\,,$$

where $\varepsilon(x)$ is an arbitrary function.

This type of symmetry is much more natural than global symmetries are from the perspective of causality: the values of $\varepsilon(x)$ at different space-time points are independent of each other. Symmetries whose parameters depend arbitrarily on space-time points are called *local* or *gauge* symmetries.

In the case of electrodynamics different symmetry transformations commute with each other

$$\Big(A_n(x) + \partial_n \varepsilon_1(x)\Big) + \partial_n \varepsilon_2(x) = \Big(A_n(x) + \partial_n \varepsilon_2(x)\Big) + \partial_n \varepsilon_1(x).$$

Such symmetries are called *Abelian.* Mathematically, in the case of electrodynamics the gauge group is $U(1)$. In fact, the gauge symmetry (4.5) of any massless Fronsdal field is Abelian. Symmetries that do not all commute with each other are called *non-Abelian.*

5.3 Yang-Mills fields

The non-Abelian generalization of electrodynamics leads to *Yang-Mills theory.* In the beginning of these lectures I have mentioned that the gauge group $U(1) \times SU(2) \times SU(3)$ underlies the modern theory of electroweak ($U(1) \times SU(2)$) and strong ($SU(3)$) interactions known as the Standard Model. This symmetry is supported by Yang-Mills fields.

In the Yang-Mills case, there are several vector fields A_n^a. The precise number depends on the choice of the gauge group. All these fields carry spin one and zero rest mass. Though they can be strongly interacting, as *e.g.* gluons in the $SU(3)$ theory of strong interactions, or they can acquire a non-zero mass by virtue of the Higgs phenomenon, as $W^\pm$ and Z bosons do in the theory of electroweak interactions, the starting point for the whole construction is to consider massless fields of spin one:

$$m = 0 \Rightarrow \text{ gauge invariance}\,.$$

Let us recall the general structure of Yang-Mills theory in some more detail. There are two types of fields in these theories, namely matter fields and Yang-Mills fields. The latter carry a space-time vector index n as well as some matrix indices α, β:

$$A_n{}^\alpha{}_\beta(x)\,.$$

They can be represented via the A_n^a as follows

$$A_n{}^\alpha{}_\beta(x) = A_n^a(x) t_a{}^\alpha{}_\beta\,,$$

where the $t_a{}^\alpha{}_\beta$ form a set of linearly independent matrices with respect to the indices α, β, and are labeled by the index a which takes some N values. This means that there are just N Yang-Mills vector fields $A_n{}^a$. The matrices $t_a{}^\alpha{}_\beta$ are required to form a Lie algebra with respect to the matrix commutator

$$[t_a\,, t_b] = f_{ab}^c t_c \tag{5.1}$$

or, making the matrix indices explicit,

$$t_a{}^\alpha{}_\gamma t_b{}^\gamma{}_\beta - t_b{}^\alpha{}_\gamma t_a{}^\gamma{}_\beta = f_{ab}^c t_c{}^\alpha{}_\beta\,.$$

Here the normalization of the generators t_a differs by a factor of i from that of the Hermitian generators T_a in Section 2.1. Hence, the generators t_a of the gauge algebra are assumed to be anti-Hermitian.

We can say that A_n^a is valued in some Lie algebra h. It should be stressed that the list of possible Lie algebras and, hence, the possible sets of Yang-Mills fields is quite restrictive (although infinite).

The Yang-Mills field strength is defined as follows:

$$F_{nm}(x) := \partial_n A_m(x) - \partial_m A_n(x) + g\,[A_n(x)\,, A_m(x)] \tag{5.2}$$

or, more explicitly,

$$F_{nm}^a(x) = \partial_n A_m^a(x) - \partial_m A_n^a(x) + g f_{bc}^a A_n^b(x) A_m^c(x)\,. \tag{5.3}$$

The parameter g is called the *coupling constant.*

Infinitesimal Yang-Mills gauge transformations have the form

$$\delta A_n(x) = D_n^{\mathrm{ad}} \varepsilon(x) \tag{5.4}$$

where the *covariant derivative in the adjoint representation*, D_n^{ad}, is defined by

$$D_n^{\mathrm{ad}} \varepsilon(x) = \partial_n \varepsilon(x) + g[A_n(x)\,, \varepsilon(x)]\,, \tag{5.5}$$

and the infinitesimal gauge parameter $\varepsilon^a(x)$ is valued in the Lie algebra h. Here we use shorthand notation skipping the index a instead of writing

$$(D_n^{\mathrm{ad}} \varepsilon(x))^a = \partial_n \varepsilon^a + g f_{bc}^a A_n^b \varepsilon^c\,. \tag{5.6}$$

In contrast to the Yang-Mills field, the field strength transforms homogeneously under the Yang-Mills gauge transformations (5.4)

$$\delta F_{nm}(x) = g[F_{nm}(x)\,, \varepsilon(x)]\,. \tag{5.7}$$

Note that non-Abelian field strengths are covariant under gauge transformations, but not invariant as they are in the Abelian case.

We leave it to the reader to check the transformation law (5.7) and that the commutator of two infinitesimal Yang-Mills transformations with parameters ε_1 and ε_2 gives again an infinitesimal gauge transformation with gauge parameter (*cf.* Eq. (2.6))

$$\varepsilon_{1,2}(x) = g[\varepsilon_1(x)\,,\varepsilon_2(x)]\,. \tag{5.8}$$

For non-Abelian Lie algebras $\varepsilon_{1,2}(x) \neq 0$. If it is zero for all ε_1 and ε_2, the algebra is *Abelian.*

A useful formula for the variation of the Yang-Mills field strength under a general variation of $A_n(x)$ is

$$\delta F_{nm}(x) = D_n^{\rm ad}\delta A_m(x) - D_m^{\rm ad}\delta A_n(x)\,. \tag{5.9}$$

Plugging the gauge transformation law into (5.9) and comparing with (5.7) we conclude that

$$g[F_{nm}\,,V] = (D_n^{\rm ad}D_m^{\rm ad} - D_m^{\rm ad}D_n^{\rm ad})V \tag{5.10}$$

where both F_{nm} and V are treated as matrices in the adjoint representation, *i.e.*, ${F_{nm}}^b{}_c{:=}F^a_{nm}f^b_{ac}$ and $V^b{}_c{:=}V^a f^b_{ac}$. The short-hand notation for this relation is

$$g\,F_{nm} = [D_n^{\rm ad}, D_m^{\rm ad}]\,. \tag{5.11}$$

5.4 Yang-Mills Lagrangian and field equations

The Lagrangian

$$L = -\frac{1}{4}tr(F_{nm}F_{kl})\eta^{nk}\eta^{ml} = -\frac{1}{4}tr(F_{nm}F^{nm}), \tag{5.12}$$

and hence the Yang-Mills action

$$S = \int L\,, \tag{5.13}$$

is gauge invariant since

$$tr([F_{nm}\,,\varepsilon]F_{kl}) + tr(F_{nm}[F_{kl}\,,\varepsilon]) = tr([F_{nm}F_{kl}\,,\varepsilon]) = 0$$

by virtue of the Jacobi identity and the known cyclic property of the trace operation

$$tr(RS) = tr(SR)\,.$$

Here the trace is taken over the representation of h in which $A_n = A^a_n {t_a}^\alpha{}_\beta$ and $F_{nm} = F^a_{nm}{t_a}^\alpha{}_\beta$ are valued. In the adjoint representation this gives

$$L = -\frac{1}{4}g_{ab}(F^a_{nm}F^b_{kl})\eta^{nk}\eta^{ml}\,, \tag{5.14}$$

where

$$g_{ab} := f^c_{ad} f^d_{bc} \tag{5.15}$$

is called the *Killing metric* of the Lie algebra h. We leave it as an exercise to check that (5.14) results from (5.12) in the adjoint representation of h.

The Yang-Mills field equations, which follow from the Yang-Mills Lagrangian (5.12) with the aid of (5.9), are

$$D^{\rm ad}_n F^{nl} = 0\,. \tag{5.16}$$

As a consequence of its definition the Yang-Mills strength (5.3) obeys the Bianchi identity

$$D^{\rm ad}_n F_{ml} + D^{\rm ad}_m F_{ln} + D^{\rm ad}_l F_{nm} = 0\,. \tag{5.17}$$

Checking this one should appreciate the role of the Jacobi identity of the underlying Lie algebra.

Equations (5.16) and (5.17) provide the generalization of the Maxwell equations of electrodynamics in the absence of currents to the non-Abelian case. Since any commutator above is accompanied by a factor of the coupling constant, in the limit $g \to 0$ the Yang-Mills equations amount to the Maxwell equations for A^a_n at every a. The other way around, one can say that the Yang-Mills equations provide a non-linear deformation of the free (linear) Maxwell equations for the set of fields A^a_n. In this interpretation, the coupling constant g serves as a deformation parameter. This phenomenon is very general: any nonlinear theory can be considered as a deformation of a free theory with the coupling constants being deformation parameters.

In the case of Yang-Mills theory we do not consider arbitrary deformations, but only those preserving the number of gauge symmetries. This does not mean that the form of the gauge transformations remains intact. Their form also gets modified by a g-dependent term in the transformation law (5.4). This phenomenon is also quite general. Consistent deformations of gauge theories should preserve the number of gauge symmetries. It should be stressed that a nonlinear deformation affects the form of the algebra of gauge transformations, deforming an Abelian theory to non-Abelian. This is again a very general phenomenon.

5.5 Interactions with matter

It is important to introduce interactions of Yang-Mills fields with matter. This is possible when matter fields belong to some module of the Yang-Mills symmetry algebra h. In fact, the appearance of any (linearly realized) symmetry means that the dynamical variables, *i.e.*, fields, form an h-module. In the case of so-called *internal* symmetries h underlying Yang-Mills theory, this is just the extension of Poincaré-modules to the larger symmetry algebra $iso(3,1) \oplus h$ with h commuting with the Poincaré generators. For example, the three colors of quarks just indicate that they form a three-dimensional $SU(3)$-module. Different choices of h-modules imply that the particles form different multiplets with respect to the innternal symmetry.

For example, let a set of scalar fields $C^\alpha(x)$ form an h-module. This simply means that there exists a set of matrices $t_a{}^\alpha{}_\beta$ that obey the commutation relations (5.1). This allows us to introduce the transformation law for $C^\alpha(x)$ under a YM gauge-symmetry parameter $\varepsilon^a(x)$ as follows

$$\delta C^\alpha(x) = -g\varepsilon^a(x) t_a{}^\alpha{}_\beta C^\beta(x)\,. \tag{5.18}$$

Introduce the *covariant derivative* in the representation determined by the $t_a{}^\alpha{}_\beta$

$$(D_n C)^\alpha(x) := \partial_n C^\alpha(x) + g A_n^a(x) t_a{}^\alpha{}_\beta C^\beta(x)\,. \tag{5.19}$$

Note that this gives (5.5) for the adjoint representation.

We observe that the covariant derivative has the following fundamental property

$$\delta(D_n C)^\alpha(x) = -g\varepsilon^a(x) t_a{}^\alpha{}_\beta (D_n C^\beta(x))\,, \tag{5.20}$$

i.e., the covariant derivative of a field C transforms the same way as C.

Here it is crucial that the usual derivative hits the x-dependence of the parameter $\varepsilon^a(x)$, producing additional (non-covariant) terms in the variation of $\partial_n C^\alpha(x)$. The role of the A_n^a-dependent term in the covariant derivative is just to cancel these additional terms against the inhomogeneous term in the transformation law (5.4) of A_n^a.

It should be stressed that the covariant derivative is nontrivial even in an Abelian Yang-Mills theory. An important example is provided by electrodynamics. Here charged fields form nontrivial modules of the Abelian group $U(1)$. These are realized by a complex field $\phi(x)$ along with its complex conjugate $\bar\phi(x)$ transforming as

$$\delta\phi(x) = ie\,\varepsilon(x)\phi(x)\,, \qquad \delta\bar\phi(x) = -ie\,\varepsilon(x)\bar\phi(x)\,, \tag{5.21}$$

where e is the electric charge of $\phi(x)$. Note that $\phi(x)$ may carry any Lorentz tensor and/or spinor indices representing a field of any spin and mass.

Covariantization of derivatives makes it possible to write covariant field equations and Lagrangians simply by replacing usual derivatives by covariant ones. In particular, the covariant version of the Klein-Gordon equation is

$$\eta^{nm}(D_n D_m C)^\alpha + m^2 C^\alpha = 0\,. \tag{5.22}$$

Note that these equations are nonlinear because the covariant derivative contains the Yang-Mills field $A_n^a(x)$.

One can proceed analogously at the action level by writing the covariantized Lagrangian for the scalar field as

$$L = \frac{1}{2} U_{\alpha\beta}(\eta^{nm} D_n C^\alpha D_m C^\beta - m^2 C^\alpha C^\beta) - V(C)\,, \tag{5.23}$$

where $U_{\alpha\beta}$ is some *h-invariant* symmetric bilinear form, which means that

$$t_a{}^\gamma{}_\alpha U_{\gamma\beta} + t_a{}^\gamma{}_\beta U_{\alpha\gamma} = 0\,, \tag{5.24}$$

and $V(C)$ is some *h-invariant potential*, which means that $\delta V(C) = 0$, *i.e.*,

$$t_a{}^\beta{}_\alpha \frac{\partial}{\partial C^\beta} V(C) = 0 \,. \tag{5.25}$$

The mass term $m^2 U_{\alpha\beta} C^\alpha C^\beta$ provides a particular case of such a potential.

As an exercise we leave to the reader to check that the metric of signature (p, q), e.g. the Euclidean or Minkowski metric, is $o(p, q)$-invariant in the sense (5.24) and that the Killing metric (5.15) is h-invariant. In the latter case, one can start by showing that the structure constants themselves are h-invariant.

Sometimes it is more convenient to use a different normalization of the Yang-Mills field, setting

$$A'^a_n := g\, A^a_n \,, \qquad F'^a_{nm} := g\, F^a_{nm} \,. \tag{5.26}$$

With this normalization the covariant derivatives and field strengths become g-independent

$$D'_n = D_n\big|_{g=1} \,, \qquad F'_{nm} = F_{nm}\big|_{g=1} \,. \tag{5.27}$$

In this normalization the dependence on the Yang-Mills coupling constant disappears everywhere except in the action of the Yang-Mills field itself, so that the total Lagrangian instead takes the form

$$L = -\frac{1}{4g^2} tr(F'_{nm} F'^{nm} + L^{\mathrm{mat}}(\phi^{\mathrm{mat}}, A') \,, \tag{5.28}$$

where $L^{\mathrm{mat}}(\phi^{\mathrm{mat}}, A')$ is the Lagrangian for matter fields ϕ^{mat}, covariantized with respect to Yang-Mills symmetries.

In this section, we have not considered examples of concrete Yang-Mills algebras. We suggest to the reader to work out the details of Yang-Mills theory for the case of the Lie algebra $h = su(2)$. To this end one can choose the basis elements e.g. of the form

$$\tau_\pm = \sigma_1 \pm i\sigma_2 \,, \qquad \tau_0 = \sigma_3 \,, \tag{5.29}$$

where σ_i are the Pauli matrices, and work out the details of the form of the Yang-Mills curvatures and the Lagrangian in terms of the respective fields $A^\pm_n$, A^0_n. **This problem has to be worked out in the text** Yang-Mills fields associated with the orthogonal algebra will be considered later on in Section 7 in the context of gravity.

6 Symmetries and currents

6.1 Global symmetries

In Section 5 we started from global symmetries and argued that local symmetries are more natural from a relativistic point of view. Reversely, global symmetries can be interpreted in the context of gauge theories as follows. Suppose that $A^a_n = 0$. The general gauge transformations (5.4) do not leave this condition invariant. However, there are residual symmetries with the parameters $\varepsilon^a(x)$ obeying the condition

$$\delta A^a_n\big|_{A=0} = 0 \quad \Rightarrow \quad \partial_n \varepsilon^a(x) = 0 \,, \tag{6.1}$$

i.e., $\varepsilon^a(x) = \epsilon^a_{\rm gl}$ with x–independent $\epsilon^a_{\rm gl}$, which are the parameters of a global symmetry. It should be stressed that while leaving the condition $A^a_n = 0$ invariant, the residual global symmetry does act nontrivially on the matter fields via the transformation law (5.18), giving

$$\delta_{\rm gl} C^\alpha(x) = -g\epsilon^a_{\rm gl} t_a{}^\alpha{}_\beta C^\beta(x)\,. \tag{6.2}$$

Indeed, the Lagrangian

$$L = \frac{1}{2} U_{\alpha\beta}(\eta^{nm}\partial_n C^\alpha \partial_m C^\beta - m^2 C^\alpha C^\beta) - V(C) \tag{6.3}$$

is invariant under the global symmetry transformations (6.2).

An important lesson from this simple example is that global symmetries can be interpreted as remnants of some local symmetries that leave invariant a particular "vacuum" solution for the gauge fields.

6.2 Conserved charges and currents

6.2.1 Noether theorem

It is well known that global symmetries induce conserved charges. Let some action

$$S[\phi] = \int d^d x\, L(\phi, \partial\phi, \partial^2\phi, \ldots) \tag{6.4}$$

with field variables ϕ^α possess a global symmetry under the infinitesimal transformation law

$$\delta\phi^\alpha = R^\alpha_a(\phi, \partial\phi, \ldots)\epsilon^a \tag{6.5}$$

with some field-dependent coefficients $R^\alpha_a(\phi, \partial\phi, \ldots)$. Then the *Noether theorem* states that there exist as many currents $J^n_a(\phi, \partial\phi, \ldots)$ as there are parameters ϵ^a, and these currents are conserved as a consequence of the field equations. This means that

$$\partial_n J^n_a(\phi, \partial\phi, \ldots) = \int d^d x\, V^\alpha_a(\phi(x), \partial\phi(x), \ldots)\frac{\delta S}{\delta\phi^\alpha(x)}\,, \tag{6.6}$$

where the coefficients $V^\alpha_a(\phi(x), \partial\phi(x), \ldots)$ are constructed from ϕ and its derivatives, while $\frac{\delta S}{\delta\phi^\alpha(x)}$ is the left-hand side of the Euler-Lagrange field equations

$$\frac{\delta S}{\delta\phi^\alpha(x)} = 0\,. \tag{6.7}$$

Recall that the functional derivative (*Euler derivative*) $\frac{\delta S}{\delta\phi^\alpha(x)}$ of a functional $S(\phi)$ is defined as

$$\delta S := S(\phi + \delta\phi) - S(\phi) = \int d^d x \frac{\delta S}{\delta\phi^\alpha(x)}\delta\phi^\alpha(x) + o(\delta\phi(x)) \tag{6.8}$$

for any infinitely differentiable infinitesimal $\delta\phi^\alpha(x)$ having compact support, *i.e.*, $\delta\phi^\alpha(x) \neq 0$ in a compact domain of $\mathbb{R}^d$. Such properties of $\delta\phi^\alpha(x)$ make it possible to freely integrate by parts, allowing to bring the variation to the form (6.8).

A simpler way to write (6.6) is

$$\partial_n J_a^n(\phi, \partial\phi, \ldots) \approx 0\,, \tag{6.9}$$

where $\approx$ denotes the equality modulo terms that vanish on-shell, *i.e.*, as a consequence of the field equations (6.7).

A simple proof of the Noether theorem is as follows. Consider the transformation (6.5) where the parameters of the transformations are allowed to be arbitrary functions of x that vanish at infinity, replacing $\epsilon^\alpha \to \varepsilon^\alpha(x)$. We observe that under the transformations

$$\delta_{\varepsilon(x)}\phi^\alpha = R_a^\alpha(\phi, \partial\phi, \ldots)\varepsilon^a(x) \tag{6.10}$$

the variation of the action has to be of the form

$$\delta_{\varepsilon(x)}S = -\int d^dx\, J_a^n(\phi, \ldots)\partial_n\varepsilon^a(x) \tag{6.11}$$

for some $J_a^n(\phi, \ldots)$. Indeed, since the action is invariant under the global symmetry with x-independent parameters ϵ^a, the variation should necessarily contain at least one derivative of $\varepsilon^a(x)$. If it contains more than one derivative, these can always be integrated by parts to reduce the variation to the form (6.11). Eq. (6.11) defines the current $J_a^n(\phi, \ldots)$. On the other hand, integrating by parts, (6.11) can be represented in the form

$$\delta_{\varepsilon(x)}S = \int d^dx\, \partial_n J_a^n(\phi, \ldots)\varepsilon^a(x)\,. \tag{6.12}$$

Being a local variation of the action, the left-hand side is zero by virtue of the field equations. Then the right-hand side implies that J_a^n is conserved on the field equations.

As useful exercises we leave it to the reader to check the following statements.

The Noether current associated with the global transformation (6.2) of the scalar field has the form

$$J_a^m = -gU_{\alpha\beta}\eta^{nm}C^\alpha(x)t_a{}^\beta{}_\gamma\partial_n C^\gamma(x)\,. \tag{6.13}$$

The electromagnetic current for a complex scalar field $C(x)$ with Lagrangian

$$L = \eta^{nm}\partial_n\bar{C}(x)\partial_m C(x) - m^2\bar{C}(x)C(x)\,, \tag{6.14}$$

which is associated with the global phase transformation

$$\delta C(x) = ie\epsilon\, C(x)\,, \qquad \delta\bar{C}(x) = -ie\epsilon\,\bar{C}(x)\,, \tag{6.15}$$

is

$$J^n = ie\Big(\bar{C}(x)\partial^n C(x) - \partial^n\bar{C}(x)C(x)\Big)\,. \tag{6.16}$$

J^n is conserved due to the field equations of $C(x)$.

Electromagnetic current for a spin-1/2 field

$$J^n = ie\bar{\psi}^{\hat{\alpha}}(x)\gamma^n{}_{\hat{\alpha}}{}^{\hat{\beta}}\psi_{\hat{\beta}}(x) \tag{6.17}$$

is conserved by virtue of the Dirac equation.

The conserved current associated with the translational invariance of the scalar-field Lagrangian L (6.3) under

$$\delta C^{\alpha}(x) = -\epsilon^{n}\partial_{n}C^{\alpha}(x) \tag{6.18}$$

has the form

$$J_{n}{}^{m} = U_{\alpha\beta}\partial_{n}C^{\alpha}(x)\partial^{m}C^{\beta}(x) - \delta_{n}^{m}L\,. \tag{6.19}$$

This tensor is called the *stress tensor* and, with lowered index m, is conventionally denoted by T_{nm}. Note that the so-defined T_{nm} is symmetric in its indices.

6.2.2 Conserved charges

Conserved currents have numerous applications in physics. In particular they generate conserved charges

$$Q_{a} = \int_{t=\mathrm{const}} d^{d-1}\vec{x}\, J_{a}^{0}(\phi(t,\vec{x}),\ldots)\,, \tag{6.20}$$

where $x^{n} = (t,\vec{x})$ is decomposed into the time component t and spatial components $\vec{x}$. The charges are conserved in the sense that they do not depend on t and, more generally, on local variations of the integration surface, including the Lorentz rotations.

As an exercise we leave it to the reader to prove that Q_a does not depend on t provided that the fields are localized in a finite region of space or fall down fast enough at the spatial infinity.

To be invariant under Lorentz transformations, the charge should not carry any tensor indices. In particular, this is the case for the electric charge which carries no tensor indices a. This is why this charge plays a special role in the construction of the invariant norm on the space of positive-frequency solutions, which is equivalent to the space of single-particle states in the quantum field theory. Let us stress that the current (6.16) is conserved for any two solutions C and $\bar{C}$ of the Klein-Gordon equation. In particular the current and the related charge are conserved if C and $\bar{C}$ are identified with C^{+} and C^{-}, respectively, as discussed in Section 3. It is this charge that defines the invariant positive-definite scalar product on the space of single-particle states of the scalar field as well as all other relativistic fields considered in Section 3.

The electric charge for the spin-1/2 field resulting from the conserved current (6.17) gives rise to the norm (3.22) (up to a factor), identifying $\psi_{\hat{\alpha}}$ with the positive-frequency part $\psi^{+}_{\hat{\alpha}}$. (To see this one should take into account that $\bar{\psi}^{\hat{\alpha}} = \psi^{\dagger\hat{\beta}}\gamma^{0}{}_{\hat{\beta}}{}^{\hat{\alpha}}$ where $\psi^{\dagger\hat{\alpha}}$ is complex conjugate to $\psi_{\hat{\alpha}}$.)

6.2.3 Noether-current interactions

Another important application is that conserved currents determine the lowest-order terms describing interactions of matter fields with gauge fields. Indeed, to construct a model with local symmetry parameters $\varepsilon^a(x)$ starting from a model with action S that possesses a global symmetry with the associated current $J^n_a(\phi,\ldots)$ it is enough to introduce the term

$$S \Rightarrow S + \Delta S\,, \qquad \Delta S = \int d^dx A^a_n J^n_a(\phi,\ldots)\,, \tag{6.21}$$

where A^a_n is a gauge field that has the transformation law

$$\delta A^a_n(x) = \partial_n \varepsilon^a(x) + \ldots \tag{6.22}$$

and the ellipsis denote some further field-dependent terms in the transformation law. This approach applies not only to Yang-Mills fields but to any other global symmetries and gauge fields. It is called the *Noether-current procedure.* In the Abelian case of electrodynamics, the interaction between the electromagnetic field and electrons has just the Noether-current form for the electric charge of spin-1/2 fields.

In the specific case of Yang-Mills theory we also know the full answer. Here, however, the Noether-current term captures only the cubic term. We leave it to the reader to check that the part of the Yang-Mills action cubic in the Yang-Mills fields A^a_n has the form (6.21).

6.2.4 Charge conservation versus gauge invariance

The construction of currents in gauge theories has an important subtlety however. The definition of currents via (6.11) is of course applicable in the case where ϕ are gauge fields (for example, some of the Fronsdal fields). However, it does not guarantee that the current defined via (6.11) is gauge invariant whenever the action S was gauge invariant. And in fact, in most cases it is not. Nevertheless, it turns out that the conserved charges (6.20) remain gauge invariant, which in fact means that the gauge variation of the conserved current is a total derivative

$$\delta_\varepsilon J^n_a(\phi,\ldots) = \partial_m J^{nm}(\phi,\ldots,\varepsilon)\,, \qquad J^{nm}(\phi,\ldots,\varepsilon) = -J^{mn}(\phi,\ldots,\varepsilon)\,. \tag{6.23}$$

As an exercise we suggest the reader to consider a free complex spin-one massless field. Derive its electric current and check its gauge invariance. Check the gauge invariance of the electric charge, which has a form analogous to (3.33). Make sure that the latter property implies the decoupling of the pure gauge modes as null states.

Similarly, the stress tensor for the massless spin-two field is not gauge invariant, while the corresponding charges are. This fact means, in particular, that the notion of the density of energy and momentum does not make sense in the presence of gravitational field. Nevertheless, the total energy and momenta of the system are well-defined.

In fact this property is very general: it is possible to prove that conserved charges resulting from a gauge-invariant action are always gauge invariant. For the proof it is enough to assume that the commutator of a global symmetry with a gauge transformation is a gauge transformation.

Possible gauge non-invariance of conserved currents built from gauge fields illustrates the difficulties of the construction of interactions between different gauge fields. Indeed, let ϕ_1 and ϕ_2 be two different gauge fields. For example, $\phi_1 = A_n$ can be a spin-one massless field while ϕ_2 can be one of the higher-spin Fronsdal fields. The naive Noether-current interaction (6.21) then preserves the gauge invariance with respect to the spin-one field A_n, but can break the gauge invariance with respect to ϕ_2. This is actually the origin of the difficulty of introducing of higher-spin interactions. The current J^n_a in ΔS would break the Fronsdal symmetry (4.5). We will see how this problem can be resolved in due course.

6.2.5 Summary

There are two types of symmetries: global (rigid) and local. Local symmetries require specific fields, called gauge fields, that make the setup covariant under transformations with gauge-symmetry parameters that are arbitrary functions of the space-time coordinates. The condition that the action is invariant under local gauge symmetries largely restricts the possible structure of nonlinearities. For example, in Yang-Mills theory gauge symmetry relates the coupling constants of the cubic and quartic interaction terms.

Global symmetries can be thought of as particular local symmetries. Gauge fields couple to conserved currents associated with the global symmetries. We see that such seemingly different entities as global symmetries, currents, local symmetries and gauge fields are all closely related to each other, highlighting different aspects of the same theory.

It should be stressed that the gauge-symmetry principle heavily restricts the structure of the theory. For instance, to formulate a gauge-invariant theory of spin-one particles one has to start with a specific set of fields, namely those associated with some Lie algebra. Even though there are many options, the possible sets of fields are heavily restricted by this condition. Secondly, the gauge symmetry relates the coupling constants of different types of interactions (*i.e.*, nonlinearities). In particular, in Yang-Mills theory, the coupling constants for the cubic term $gA^2\partial A$ and the quartic term g^2A^4 are expressed in terms of one and the same Yang-Mills coupling constant g. In fact, it is this property that improves the quantum behavior of the theory because it also restricts the structure of possible divergencies. Ultimately, one may hope to find such strong symmetries that will leave no room for divergencies at all.

Before proceeding with the analysis of other types of gauge theories I would like to recall a story about Yang-Mills theory. When Yang and Mills discovered their theory in the mid fifties of the last century, Yang gave a presentation at the Princeton Institute for Advanced Studies. Pauli, who was present in the audience, behaved in a very critical and even aggressive way. (Actually he was famous for that, but this time it was worse than usually.) He was repeatedly asking "what is the mass of the field in question?" Yang was trying to avoid answering as long as possible but eventually admitted that it is zero. This was what Pauli was awaiting for, since those days that sounded as nonsense: the only known massless particle was the photon, so it looked like there was no room for more massless particles, especially mutually interacting. The rumor was that actually Pauli himself knew the theory by that time, but since he saw no applications for it, he never published the work and behaved particularly aggressively when other people presented this theory. This story has an instructive moral: mathematically beautiful theories usually are very well protected by themselves. It was simply too early to judge Yang-Mills theory in the fifties because neither the Higgs phenomenon, allowing to give a nonzero mass to the Yang-Mills fields, nor the asymptotic-freedom phenomenon and, more generally, confinement were known at the time. The first phenomenon led to the construction of the modern theory of electroweak interactions, while the second one led to chromodynamics, *i.e.*, the theory of strong interactions that does not allow one to observe the respective Yang-Mills fields known as gluons as independent free fields at low energies.

A special property of gauge fields, which is responsible for local symmetries, is that they have zero rest mass like for the case of electromagnetic or Yang-Mills fields. Hence, the study of all possible symmetries and that of different types of massless fields is to a large extent the same problem. As we know, there is an infinite variety of different types of massless fields which are characterized by their spin and are associated with different symmetries. In particular, Yang-Mills fields have spin one. Other possible types of fields have higher spins.

Actually physics is full of unexpected turns like this that change the fate of the theory. Another prominent example is String Theory which was originally designed for the description of hadronic physics. Having a massless spin-two field in its consistent formulation it did not serve the original purpose very well, but instead became a very interesting theory of gravity. So, studying new models we should be open to their unusual properties and unexpected new reinterpretations.

There will be at least two more occasions to mention Pauli: once in the context of gravity and once in the context of supergravity. This great scientist was probably too critical not only to others but in the first place to himself.

7 Diffeomorphisms and differential forms

7.1 Coordinate transformations

The fundamental principle underlying Einstein gravity is the prominent *equivalence principle* which requires a theory to be invariant under any change of coordinates

$$x^{\underline{n}} = x^{\underline{n}}(x')$$

where $x^{\underline{n}}(x')$ is an arbitrary smooth enough invertible map from coordinates $x'^{\underline{n}}$ to the coordinates $x^{\underline{n}}$. We have changed the notation since later on we will need to distinguish between underlined *base* (or *world*) indices $\underline{m}, \underline{n}, \dots$ and ordinary *fiber* indices $m, n, \dots$. In this section we will only need base indices. The difference between the two types of indices will be discussed in Section 8. As we shall see, in the Cartesian coordinate system of flat Minkowski space the two types of indices can be identified, which allowed us to use indices $n, m, \ldots$ in the consideration so far.

Infinitesimally the transformation law can be written in the form

$$x^{\underline{n}} \cong x'^{\underline{n}} + \varepsilon^{\underline{n}}(x)\,, \tag{7.1}$$

where $\varepsilon^{\underline{n}}(x)$ is an arbitrary *vector field*, which can be interpreted as a gauge-symmetry parameter in gravity. It is worth noting that this gauge parameter is the same as that of the spin-two Fronsdal theory.

The equivalence principle is the condition that physical laws have to be insensitive to the particular choice of coordinate system. Since an arbitrary change of coordinates is equivalent to going from one non-inertial coordinate system to another, the equivalence principle implies that the transition from one coordinate system to another is equivalent to switching on a gravitational field. Geometrically, this principle is absolutely natural. For example, a plane is still a plane in any coordinates, say, Cartesian or polar. The coordinate choice is a matter of convenience rather than a physical phenomenon.

Let us first consider how a change of coordinates acts on different types of fields. Scalars are demanded to keep the same values at any point of space-time independently of the coordinate choice. This means that

$$C'(x'(x)) = C(x)\,. \tag{7.2}$$

Infinitesimally, consider the variation

$$\delta C(x) := C'(x) - C(x)\,. \tag{7.3}$$

By virtue of (7.1) this implies

$$\delta C(x) = C'(x' + \varepsilon(x)) - C(x) \cong \varepsilon^{\underline{n}}(x)\frac{\partial C(x)}{\partial x^{\underline{n}}}\,, \tag{7.4}$$

where terms of order ε^2 are neglected.

A *covariant vector* $A_{\underline{n}}(x)$ transforms as the derivative of a scalar $\frac{\partial C}{\partial x^{\underline{n}}}$, *i.e.*,

$$A'_{\underline{n}}(x') = \frac{\partial x^{\underline{m}}(x')}{\partial x'^{\underline{n}}} A_{\underline{m}}(x(x'))\,. \tag{7.5}$$

With the help of the text-book formula

$$\frac{\partial x^{\underline{m}}(x')}{\partial x'^{\underline{n}}}\frac{\partial x'^{\underline{n}}(x)}{\partial x^{\underline{k}}} = \delta^{\underline{m}}_{\underline{k}} \tag{7.6}$$

this is equivalent to

$$A_{\underline{n}}(x) = \frac{\partial x'^{\underline{m}}(x)}{\partial x^{\underline{n}}} A'_{\underline{m}}(x'(x))\,. \tag{7.7}$$

A *contravariant vector* $B^{\underline{n}}(x)$ transforms so that $B^{\underline{n}}(x)A_{\underline{n}}(x)$ is a scalar

$$B'^{\underline{n}}(x'(x)) = \frac{\partial x'^{\underline{n}}(x)}{\partial x^{\underline{m}}} B^{\underline{m}}(x)\,. \tag{7.8}$$

In particular, the differential $dx^{\underline{n}}$ transforms this way. As a result we obtain the very important (though geometrically obvious) result that the differential operator

$$\mathrm{d} := dx^{\underline{n}} \frac{\partial}{\partial x^{\underline{n}}} \tag{7.9}$$

applied to a scalar C gives an invariant object $\mathrm{d}C$.

Infinitesimally, the transformation laws for covariant and contravariant vectors are

$$\delta A_{\underline{n}}(x) = \partial_{\underline{n}}(\varepsilon^{\underline{m}}(x)) A_{\underline{m}}(x) + \varepsilon^{\underline{m}}(x) \partial_{\underline{m}} A_{\underline{n}}(x)\,, \tag{7.10}$$

$$\delta B^{\underline{n}}(x) = -\partial_{\underline{m}}(\varepsilon^{\underline{n}}(x)) B^{\underline{m}}(x) + \varepsilon^{\underline{m}}(x) \partial_{\underline{m}} B^{\underline{n}}(x)\,. \tag{7.11}$$

Note that $B^{\underline{n}} A_{\underline{n}}$ transforms as a scalar.

Tensors carrying covariant and contravariant indices transform analogously with the Jacobi matrices acting on all indices

$$A_{\underline{n},\underline{m}\dots}{}^{\underline{k},\underline{l}\dots}(x) = \frac{\partial x'^{\underline{n}'}(x)}{\partial x^{\underline{n}}} \frac{\partial x'^{\underline{m}'}(x)}{\partial x^{\underline{m}}} \dots \frac{\partial x^{\underline{k}}(x')}{\partial x'^{\underline{k}'}} \frac{\partial x^{l}(x')}{\partial x'^{\underline{l}'}} \tilde{A}_{\underline{n}',\underline{m}'\dots}{}^{\underline{k}',\underline{l}'\dots}(x'(x))\,. \tag{7.12}$$

Infinitesimally, the matrices $\partial_{\underline{m}} \varepsilon^{\underline{n}}(x)$ act with appropriate signs on all indices as well. For example, the transformation law of a rank-two symmetric tensor $g_{\underline{nm}}$ is

$$\delta g_{\underline{m}\,\underline{n}}(x) = \partial_{\underline{n}}(\varepsilon^{\underline{k}}(x)) g_{\underline{k}\,\underline{m}}(x) + \partial_{\underline{m}}(\varepsilon^{\underline{k}}(x)) g_{\underline{k}\,\underline{n}}(x) + \varepsilon^{\underline{k}}(x) \partial_{\underline{k}} g_{\underline{n}\,\underline{m}}(x)\,.$$

There is an important notion of the *Lie derivative* which is just the first-order differential operator $\mathcal{L}_\varepsilon$ that appears in the infinitesimal coordinate transformation law of different types of fields. Note that $\mathcal{L}_V$ depends on a vector field $V^{\underline{n}}(x)$ where $V^{\underline{n}}$ is not demanded to be infinitesimally small. By its definition, the form of $\mathcal{L}_V$ depends on the tensor type of the objects it acts on. For example, from (7.4), (7.10) and (7.11) we see that

$$\mathcal{L}_V C = V^{\underline{n}}(x) \frac{\partial C(x)}{\partial x^{\underline{n}}}\,, \tag{7.13}$$

$$\mathcal{L}_V A_{\underline{n}}(x) = \partial_{\underline{n}}(V^{\underline{m}}(x)) A_{\underline{m}}(x) + V^{\underline{m}}(x) \partial_{\underline{m}} A_{\underline{n}}(x)\,, \tag{7.14}$$

$$\mathcal{L}_V B^{\underline{n}}(x) = -\partial_{\underline{m}}(V^{\underline{n}}(x)) B^{\underline{m}}(x) + V^{\underline{m}}(x) \partial_{\underline{m}} B^{\underline{n}}(x)\,. \tag{7.15}$$

Invertible changes of coordinates obey (*cf.* (7.5))

$$\det \left| \frac{\partial x'^{\underline{m}}(x)}{\partial x^{\underline{n}}} \right| \neq 0\,. \tag{7.16}$$

Such maps form a group. Smooth changes of coordinates with infinitely differentiable functions $x^{\underline{n}}(x')$ are called *diffeomorphisms*. Infinitesimal diffeomorphisms form a Lie algebra. Its structure is characterized by the relation expressing the parameter $\varepsilon_{1,2}$ of the commutator of two infinitesimal diffeomorphisms represented by the Lie derivatives $\mathcal{L}_{\varepsilon_1}$ and $\mathcal{L}_{\varepsilon_2}$ with parameters $\varepsilon_1(x)$ and $\varepsilon_2(x)$.

Successive application of diffeomorphisms to a scalar gives

$$\mathcal{L}_{\varepsilon_2} \mathcal{L}_{\varepsilon_1} C(x) = \mathcal{L}_{\varepsilon_2}(\varepsilon_1^{\underline{n}} \partial_{\underline{n}} C(x)) = \varepsilon_2^{\underline{n}} \partial_{\underline{n}}(\varepsilon_1^{\underline{m}} \partial_{\underline{m}} C(x))\,. \tag{7.17}$$

Using that double derivatives are symmetric, $\partial_n \partial_m C = \partial_m \partial_n C$, this gives

$$\mathcal{L}_{\varepsilon_2} \mathcal{L}_{\varepsilon_1} - \mathcal{L}_{\varepsilon_1} \mathcal{L}_{\varepsilon_2} = \mathcal{L}_{\varepsilon_{1,2}}\,, \tag{7.18}$$

where

$$\varepsilon_{1,2}^{\underline{n}} = \varepsilon_2^{\underline{m}} \partial_{\underline{m}} \varepsilon_1^{\underline{n}} - \varepsilon_1^{\underline{m}} \partial_{\underline{m}} \varepsilon_2^{\underline{n}}\,. \tag{7.19}$$

Thus, as anticipated, the commutator of two diffeomorphisms is again a diffeomorphism. It should be stressed that formula (7.18) is true for diffeomorphisms acting on a tensor field of any type. This means that different types of tensor fields form different modules of the same Lie algebra of diffeomorphisms, just like different types of $o(N)$-tensors form different $o(N)$-modules.

Note that both the group and the Lie algebra of diffeomorphisms are non-Abelian.

Remark. If we instead think of diffeomorphisms as transformations of a predefined set fields only (in our case just containing the scalar field), we would obtain the opposite sign of $\varepsilon_{1,2}$ since in this case

$$\mathcal{L}_{\varepsilon_2} \mathcal{L}_{\varepsilon_1} C(x) = \mathcal{L}_{\varepsilon_2} (\varepsilon_1^{\underline{n}} \partial_{\underline{n}} C(x)) = \varepsilon_1^{\underline{n}} \partial_{\underline{n}} (\varepsilon_2^{\underline{m}} \partial_{\underline{m}} C(x))\,. \tag{7.20}$$

In the case of diffeomorphisms one can proceed in either way. The latter prescription is however preferable from the perspective of the study of new types of symmetries, like *e.g.* higher-spin symmetry, at a stage at which their geometric interpretation is not yet clear. For this reason we will actually use this latter convention. **Ostavit'?**In particular, this sign difference will matter in the analysis of supersymmetry in Section 11 effectively changing a sign of some of the commutation relations of the Poincaré algebra.

7.2 Differential forms

Contrary to the first derivative of a scalar, derivatives of tensors do not transform as tensors. Indeed,

$$\frac{\partial}{\partial x'^{\underline{n}}} A'_{\underline{m}}(x') = \frac{\partial x^{\underline{n}'}}{\partial x'^{\underline{n}}} \frac{\partial}{\partial x^{\underline{n}'}} \Big(\frac{\partial x^{\underline{m}'}}{\partial x'^{\underline{m}}} A_{\underline{m}'}(x) \Big) = \frac{\partial x^{\underline{n}'}}{\partial x'^{\underline{n}}} \frac{\partial x^{\underline{m}'}}{\partial x'^{\underline{m}}} \frac{\partial}{\partial x^{\underline{n}'}} A_{\underline{m}'}(x) + \frac{\partial^2 x^{\underline{m}'}}{\partial x'^{\underline{m}} \partial x'^{\underline{n}}} A_{\underline{m}'}(x)\,. \tag{7.21}$$

The first term on the right-hand side of this relation matches the tensor transformation law (7.12) while the second one does not.

In the framework of Riemannian geometry, to cancel the second term ordinary derivatives are replaced by covariant ones

$$\partial_{\underline{n}} \to D_{\underline{n}}\,, \qquad D_{\underline{n}} A_{\underline{m}} := \partial_{\underline{n}} A_{\underline{m}} - \Gamma^{\underline{k}}_{\underline{nm}} A_{\underline{k}}\,, \tag{7.22}$$

where the *Christoffel symbol* $\Gamma^{\underline{k}}_{\underline{nm}}$ is introduced in such a way that $D_{\underline{n}}$ becomes covariant under diffeomorphisms. I will not discuss details of the Riemannian geometry in this review turning instead to the so-called *Cartan formulation* of gravity based on the formalism of differential forms, which is more general and most convenient for our purpose.

7.2.1 Definition and properties

The key observation of the Cartan formalism of differential forms is that though the usual derivative of a vector $\partial_{\underline{n}} A_{\underline{m}}$ is not covariant, the antisymmetric combination

$$\partial_{\underline{n}} A_{\underline{m}} - \partial_{\underline{m}} A_{\underline{n}} \tag{7.23}$$

is. Indeed, being symmetric in $\underline{n}$ and $\underline{m}$, the non-tensorial term on the right-hand side of (7.21) drops out of the antisymmetric combination (7.23). A closely related property in Riemannian geometry is that the Christoffel symbol can be chosen to be symmetric, $\Gamma_{\underline{nm}}^{\underline{k}} = \Gamma_{\underline{mn}}^{\underline{k}}$, in which case

$$D_{\underline{n}} A_{\underline{m}}(x) - D_{\underline{m}} A_{\underline{n}}(x) = \partial_{\underline{n}} A_{\underline{m}}(x) - \partial_{m} A_{\underline{n}}(x).$$

The main idea of the Cartan formalism is to work solely with totally antisymmetric tensors with lowered indices (*i.e.*, antisymmetric covariant tensors) $A_{[\underline{n}_1,\dots,\underline{n}_p]}(x)$. Such tensors are called *differential forms*. The rank p of the tensor $A_{[\underline{n}_1,\dots,\underline{n}_p]}(x)$ is called the *degree* of the differential form.

To simplify notation hiding tensor indices, it is convenient to pack antisymmetric tensors into functions of auxiliary anticommuting variables[6] $\xi^{\underline{n}}$

$$\xi^{\underline{n}} \xi^{\underline{m}} = -\xi^{\underline{m}} \xi^{\underline{n}}. \tag{7.24}$$

Then a function of $\xi^{\underline{m}}$

$$A(\xi, x) = \sum_{p=0}^{\infty} \xi^{\underline{n}_1} \dots \xi^{\underline{n}_p} A_{[\underline{n}_1,\dots,\underline{n}_p]}(x) \tag{7.25}$$

contains a sum of differential forms of all degrees. In this notation, a differential form $A(\xi, x)$ has degree p provided that it is a degree-p homogeneous polynomial in ξ. Note that in d-dimensional space-time with $\underline{n} = 0, 1, \dots, d-1$, differential forms of degrees $p > d$ vanish. This is because any degree-p form with $p > d$ will contain the square of $\xi^{\underline{n}}$ at some $\underline{n}$, which is zero by virtue of (7.24). Hence, the sum in (7.25) is effectively from $p = 0$ to $p = d$.

Being simply some functions of the anticommuting variables $\xi^{\underline{n}}$, differential forms can be multiplied in a natural way.[7] For example, in the product of two differential forms $A_1^{p_1}(\xi, x)$ and $A_2^{p_2}(\xi, x)$ of degrees p_1 and p_2, the dependence on $\xi^{\underline{n}}$ automatically enforces the total antisymmetrization over all covariant tensor indices

$$A_1^{p_1}(\xi, x) A_2^{p_2}(\xi, x) = \xi^{\underline{n}_1} \dots \xi^{\underline{n}_{p_1+p_2}} A^{p_1}_{[\underline{n}_1 \cdots \underline{n}_{p_1}}(x) A^{p_2}_{\underline{n}_{p_1+1} \cdots \underline{n}_{p_1+p_2}]}(x). \tag{7.26}$$

The algebra of functions of anticommuting variables $\xi^{\underline{n}}$ is called the *Grassmann algebra*.

[6]Note that the variables $\xi^{\underline{n}}$ are customarily denoted by $dx^{\underline{n}}$. We avoid this notation firstly to shorten formulae and, secondly, to avoid possible confusion caused by the anticommutativity of the differentials originally identified with the variations of the commuting coordinates $x^{\underline{n}}$.

[7]In the literature such a product of antisymmetric tensors is usually denoted by the *wedge symbol* $\wedge$, writing $A_1^{p_1}(x) \wedge A_2^{p_2}(x)$. We will not use this notation either, assuming that the antisymmetrization is automatic due to the ξ-dependence.

In the Cartan formalism the indices of derivatives are antisymmetrized together with all other tensor indices, like in (7.23). In terms of variables $\xi^{\underline{n}}$ this means that the only allowed first-order differential operator is the so-called *exterior derivative* (7.9)

$$\mathrm{d} = \xi^{\underline{n}}\frac{\partial}{\partial x^{\underline{n}}}\,, \tag{7.27}$$

which acts as follows on the general differential form (7.25)

$$\mathrm{d}A(\xi,x) = \sum_p \xi^{\underline{n}_0}\dots\xi^{\underline{n}_p}\partial_{[\underline{n}_0}A_{\underline{n}_1,\dots,\underline{n}_p]}(x)\,. \tag{7.28}$$

Because the $\xi^{\underline{n}}$ are anticommuting while the derivatives $\frac{\partial}{\partial x^{\underline{n}}}$ are commuting and $\xi^{\underline{n}}\partial_{\underline{m}} = \partial_{\underline{m}}\xi^{\underline{n}}$., d has the fundamental property

$$\mathrm{d}^2 = 0\,. \tag{7.29}$$

Let me stress that we have already shown that, because of the antisymmetrization over all indices, the exterior differential is covariant, *i.e.*, it keeps the same form in any coordinate system.

Differential forms have many remarkable properties, playing an important role both in physics and in mathematics. Let me briefly list some of them before looking at examples of applications of differential forms. For more details we refer to [6].

Let $A_p(\xi,x)$ be a p-form in some p-dimensional space (manifold) M^p. Then it can be integrated over M^p

$$\int_{M^p} A_p = \frac{1}{p!}\int d^px\,\epsilon^{\underline{n}_1\cdots\underline{n}_p}A_{\underline{n}_1\cdots\underline{n}_p} \tag{7.30}$$

where $\epsilon^{\underline{n}_1\cdots\underline{n}_p}$ is the totally antisymmetric Levi-Civita symbol. As follows from the formula

$$J^{\underline{n}'_1}{}_{\underline{n}_1}\dots J^{\underline{n}'_p}{}_{\underline{n}_p}\,\epsilon^{\underline{n}_1\cdots\underline{n}_p} = \det|J|\,\epsilon^{\underline{n}'_1\cdots\underline{n}'_p} \tag{7.31}$$

and the transformation law (7.12) applied to differential forms, the integral (7.30) preserves its form in any coordinate system.

Let me introduce some terminology. A form $A(\xi,x)$ is *exact* if $A(\xi,x) = \mathrm{d}B(\xi,x)$ for some differential form $B(\xi,x)$. A form $A(\xi,x)$ is *closed* if $\mathrm{d}A(\xi,x) = 0$.

Obviously, any exact form is closed by virtue of (7.29). If there are closed forms on M that are not exact, they represent the *de Rham cohomology* of M.[8]

Let Σ^q be a q-dimensional hypersurface (submanifold) of M. The embedding of Σ^q into M can be given in terms of a map $x^{\underline{n}} = x^{\underline{n}}(y^{\underline{\alpha}})$, where $y^{\underline{\alpha}}$ are local coordinates of Σ^q ($\underline{\alpha} = 1,\dots q$). For example a closed contour Σ^1 in M can be given by a map $x^{\underline{n}}(t)$ with $t\in[0,2\pi]$ such that $x^{\underline{n}}(0) = x^{\underline{n}}(2\pi)$.

The *pullback* of an r-form A on M to Σ^q is the differential form

$$A_{\underline{\alpha}_1\cdots\underline{\alpha}_r}|_\Sigma(y) := \frac{\partial x^{\underline{n}_1}(y)}{\partial y^{\underline{\alpha}_1}}\dots\frac{\partial x^{\underline{n}_r}(y)}{\partial y^{\underline{\alpha}_r}}A_{\underline{n}_1\cdots\underline{n}_r}(x(y))\,. \tag{7.32}$$

[8]More precisely, the *cohomology space* (also often called *cohomology group*) $H^p(M) = C^p/E^p$ is the quotient space of the space C^p of closed p-forms over its subspace E^p of exact p-forms.

In particular, this allows one to integrate a q-form A over any q-dimensional hypersurface $\Sigma^q \subset M$ via integration of its pullback $A|_{\Sigma^q}$. In particular, one-forms can be integrated over any one-dimensional curve Σ^1.

It is important that, as the reader can easily see, the pullback of a closed (exact) form is closed (exact).

The *Stokes theorem* states that if A is a $(p-1)$-form on some p-dimensional manifold M, then

$$\int_M \mathrm{d}A(\xi, x) = \int_{\partial M} A|_{\partial M}(\xi, x)\,, \tag{7.33}$$

where ∂M denotes the boundary of M. If there is no boundary, like for instance in the cases of a sphere or torus, the right-hand side is zero.

The well-known formula

$$\int_a^b dt \frac{\partial f(t)}{\partial t} = f(b) - f(a) \tag{7.34}$$

is a particular case of the Stokes theorem.

To prove the Stokes theorem one can apply (7.34) to a polycube and then use the invariance of the integration of differential forms under coordinate transformations to deform the boundary of the polycube to an arbitrary deformed polycube.

Let a hypersurface $\Sigma \subset M$ have no boundary. Then, by virtue of the Stokes theorem, the integral $\int_\Sigma A$ is zero for exact A and is independent of local variations of Σ for closed A.

To check the last statement it suggices to consider a small variation of the integration surface and apply the Stokes theorem to the volume bounded by the original and deformed surfaces.

A few comments are in order. It should be stressed that the formalism of differential forms (also called the *exterior-algebra* formalism) does not assume the existence of a metric. The Stokes theorem (7.33) incorporates all known theorems on the integration of total derivatives including the familiar low-dimensional Green's and Stokes' theorems. In the formalism of differential forms total derivatives are represented by exact forms.

Now we consider some examples of application of the exterior-algebra formalism.

7.2.2 Conserved currents as closed forms

In terms of differential forms conserved currents $J^{\underline{n}}$ are dual to closed $(d-1)$-forms $\tilde{J}_{\underline{n}_1\cdots\underline{n}_{d-1}}$ which we will call *current forms.* The explicit relation is[9]

$$\tilde{J}_{\underline{n}_1\cdots\underline{n}_{d-1}} = \epsilon_{\underline{n}_1\cdots\underline{n}_d} J^{\underline{n}_d}\,. \tag{7.35}$$

It is not difficult to check that conserved currents are equivalent to closed current forms. Another important fact is that conserved currents that are related by (7.35)

[9]The reader familiar with Riemannian geometry should bear in mind that conserved currents are represented by vector densities (rather than by vectors) for which the conservation condition $\partial_{\underline{n}} J^{\underline{n}} = 0$ is covariant.

to exact forms can be written as $J^{\underline{n}} = \partial_{\underline{m}} F^{\underline{nm}}$ for some antisymmetric $F^{\underline{nm}} = -F^{\underline{mn}}$. Such currents are called *improvements*.

Let $\tilde{J}$ be a closed current form. The charge is defined as (*cf.* (6.20))

$$Q = \int_{\Sigma} \tilde{J}\,, \tag{7.36}$$

where Σ is a $(d-1)$-dimensional hypersurface. The charge conservation follows from the independence of the integral of local variations of Σ. Indeed, Σ can be identified with the usual space at given time t while its variation can be in particular represented by the time evolution.

Let us stress that the reformulation of the conservation law in terms of differential forms proves a much stronger result, namely that any local variation of the integration surface leaves the charge invariant. In particular, considering the variation of the spatial surface Σ under infinitesimal Poincaré transformations, which also cannot affect the charge, proves the Poincaré invariance of the norms on the space of positive-frequency solutions of relativistic equations.

Note that in application to relativistic fields, the condition that Σ has no boundary is equivalent to the condition that the fields fall off sufficiently fast at spatial infinity since this means that the contribution to the boundary integral at spatial infinity vanishes. In relativistic models the latter condition is absolutely natural because if some configuration of fields was localized in some finite region of space, the finiteness of the propagation speed of signals would imply that it will be localized at a finite region at any time in the future.

Since exact forms $\tilde{J}$ do not contribute to the conserved charges, the latter represent *current cohomology*, *i.e.*, the closed current forms modulo the exact ones.

The property that gauge symmetries change conserved currents by total derivatives now reads

$$\delta_{\rm gauge} \tilde{J} = \mathrm{d}H\,. \tag{7.37}$$

7.2.3 Yang-Mills fields as differential forms

Yang-Mills theory also admits a nice interpretation in terms of differential forms. Namely, the Yang-Mills field $A^a_{\underline{m}}$ can be interpreted as the one-form

$$A(\xi, x) := A^a(\xi, x) t_a = \xi^{\underline{n}} A^a_{\underline{n}}(x) t_a \tag{7.38}$$

valued in a Lie algebra h (*cf.* the index a). Analogously, the Yang-Mills field strength can be interpreted as a two-form

$$F(\xi, x) = F^a(\xi, x) t_a\,, \qquad F^a(\xi, x) = \frac{1}{2} \xi^{\underline{n}} \xi^{\underline{m}} F^a_{\underline{nm}}(x) \tag{7.39}$$

defined as

$$F^a(\xi, x) = \mathrm{d}A^a(\xi, x) + \frac{1}{2} g f^a_{bc} A^b(\xi, x) A^c(\xi, x)\,. \tag{7.40}$$

This relation can be compactly written as

$$F(\xi, x) = \mathrm{d}A(\xi, x) + \tfrac{1}{2}\, g\, [A(\xi, x), A(\xi, x)]\,. \tag{7.41}$$

napisat' Write down equations (5.9) and (5.10) from Section 5.3 in the compact notation used in (7.41).

An infinitesimal gauge transformation reads

$$\delta A^a(\xi,x) = \mathrm{d}\varepsilon^a(x) + g f^a_{bc} A^b(\xi,x)\varepsilon^c(x) = \mathrm{d}\varepsilon^a(x) + g[A(\xi,x)\,,\varepsilon(x)]^a\,, \tag{7.42}$$

$$\delta F^a(\xi,x) = g f^a_{bc} F^b(\xi,x)\varepsilon^c(x) = g[F(\xi,x)\,,\varepsilon(x)]^a\,, \tag{7.43}$$

where the infinitesimal gauge parameter $\varepsilon^a(x)$ is a zero-form, *i.e.*, an ordinary function of x. (Note that writing the second term on the right-hand side of (7.40) in the form of commutator may be confusing because now $A^b(\xi,x)$ anticommute as one-forms.)

A one-form $A^a(\xi,x)$ with the transformation law (7.42) is called a *connection* while the associated two-form $F^a(\xi,x)$ from (7.40) is called its *curvature*.

Using the notation of differential forms, the Bianchi identity (5.17) can now be written as

$$D^{\mathrm{ad}} F(\xi,x) = 0\,. \tag{7.44}$$

8 Cartan gravity

General relativity is a theory of the gravitational interaction. It can be interpreted as a theory of an interacting massless spin-two field, the *graviton*. In this section we explain basic elements of the Cartan formulation of gravity which is most appropriate for the analysis of its higher-spin generalizations.

In Cartan theory, the gravitational field is described by the Yang-Mills fields of the Poicaré algebra $iso(d-1,1)$. In a certain sense this means that gravity results from localization (*gauging*) of the space-like symmetries of Minkowski space. One should keep in mind however that, as explained below, though the dynamical fields of the Cartan gravity can be identified with the Yang-Mills fields of the Poicaré algebra, Cartan theory differs from the Yang-Mills theory of the Poincaré symmetry, having a different action principle and field equations.

8.1 Fields and symmetries

Since the generators of the Poincaré algebra are P_n and L_{nm}, its connection one-form also consists of two parts:

$$A(\xi,x) = -i\Big(e^n(\xi,x)P_n + \frac{1}{2}\omega^{nm}(\xi,x)L_{nm}\Big)\,. \tag{8.1}$$

Here $e^n(x)$ is the *vielbein one-form*, also called *(co)frame* and $\omega^{nm}(x)$ is called the *Lorentz connection* or *spin connection.* Since e^n and ω^{nm} are one-forms they can be expanded as

$$e^n(\xi,x) = \xi^{\underline{n}} e_{\underline{n}}{}^n(x)\,, \qquad \omega^{nm}(\xi,x) = \xi^{\underline{n}}\omega_{\underline{n}}{}^{nm}(x)\,. \tag{8.2}$$

Now we can appreciate the different roles of the underlined *base* indices, which are indices of differential forms, and ordinary *fiber* indices, which refer to tensors of the gauged Lorentz algebra. Indeed, through the coordinate-dependence of the one-forms $\xi^{\underline{n}}$ base indices depend on the choice of coordinates that we are working with. Fiber indices, on the other hand, label the Poincaré generators independently of the coordinates that we use.

In the case of gravity, it is customary to denote the curvatures by $\mathcal{R}$ to memorize Riemann. Like the connection (8.1) the Yang-Mills curvature for the Poincaré algebra

$$\mathcal{R}(\xi,x) = -i\Big(\mathcal{R}^n(\xi,x)P_n + \frac{1}{2}\mathcal{R}^{nm}(\xi,x)L_{nm}\Big) \tag{8.3}$$

has two different parts. The definition (7.40) along with the commutation relations of the Poincaré algebra (2.21), (2.22) and (2.23) show that the components of (8.3) and (8.1) are related by

$$\mathcal{R}^n(\xi,x) = \mathrm{d}e^n(\xi,x) + \omega^n{}_m(\xi,x)e^m(\xi,x)\,, \tag{8.4}$$

$$\mathcal{R}^{nm}(\xi,x) = \mathrm{d}\omega^{nm}(\xi,x) + \omega^n{}_k(\xi,x)\omega^{km}(\xi,x) \tag{8.5}$$

or, explicitly showing the indices of differential forms,

$$\mathcal{R}_{\underline{nm}}{}^n(x) = \partial_{\underline{n}}e_{\underline{m}}{}^n(x) + \omega_{\underline{n}}{}^n{}_m(x)e_{\underline{m}}{}^m(x) - (\underline{n}\longleftrightarrow\underline{m})\,, \tag{8.6}$$

$$\mathcal{R}_{\underline{nm}}{}^{nm}(x) = \partial_{\underline{n}}\omega_{\underline{m}}{}^{nm}(x) + \omega_{\underline{n}}{}^n{}_k(x)\omega_{\underline{m}}{}^{km}(x) - (\underline{n}\longleftrightarrow\underline{m})\,, \tag{8.7}$$

where the fiber indices are raised and lowered by the flat Minkowski metric η^{nm}.

The gauge transformations are

$$\delta e^n(\xi,x) = \mathrm{d}\varepsilon^n(x) + \omega^{nm}(\xi,x)\varepsilon_m(x) - \varepsilon^n{}_m(x)e^m(\xi,x)\,, \tag{8.8}$$

$$\delta\omega^{nm}(\xi,x) = \mathrm{d}\varepsilon^{nm}(x) + \omega^n{}_k(\xi,x)\varepsilon^{km}(x) + \omega^m{}_k(\xi,x)\varepsilon^{nk}(x)\,, \tag{8.9}$$

or, equivalently,

$$\delta e_{\underline{n}}{}^n(x) = \partial_{\underline{n}}\varepsilon^n(x) + \omega_{\underline{n}}{}^{nm}(x)\varepsilon_m(x) - \varepsilon^n{}_m(x)e_{\underline{n}}{}^m(x)\,, \tag{8.10}$$

$$\delta\omega_{\underline{n}}{}^{nm}(x) = \partial_{\underline{n}}\varepsilon^{nm}(x) + \omega_{\underline{n}}{}^n{}_k(x)\varepsilon^{km}(x) + \omega_{\underline{n}}{}^m{}_k(x)\varepsilon^{nk}(x)\,. \tag{8.11}$$

Here $\varepsilon^m(x)$ and $\varepsilon^{nm}(x)$ are infinitesimal gauge parameters of localized translations and Lorentz transformations, respectively.

It is convenient to introduce the following *Lorentz-covariant derivative*

$$D^{\mathrm{L}}A^n(x) := \mathrm{d}A^n(x) + \omega^n{}_m(\xi,x)A^m(x) \tag{8.12}$$

for any fiber vector A^n or, equivalently,

$$D^{\mathrm{L}}_{\underline{n}}A^n(x) = \partial_{\underline{n}}A^n(x) + \omega_{\underline{n}}{}^n{}_m(x)A^m(x)\,. \tag{8.13}$$

This is just the Yang-Mills covariant derivative (5.19) for the vector module of the Lorentz algebra. Recall that in a tensor module the Lorentz connection acts on all indices. For instance,

$$D^{\mathrm{L}}_{\underline{n}}A^{nm}(x) = \partial_{\underline{n}}A^{nm}(x) + \omega_{\underline{n}}{}^n{}_k(x)A^{km}(x) + \omega_{\underline{n}}{}^m{}_k(x)A^{nk}(x)\,. \tag{8.14}$$

In these terms one can see that

$$\mathcal{R}^n = D^{\mathrm{L}} e^n \tag{8.15}$$

and

$$D^{\mathrm{L}} D^{\mathrm{L}} A^n = \mathcal{R}^n{}_m A^m \,. \tag{8.16}$$

In the case of the Yang-Mills algebra $iso(d-1,1)$, the Bianchi identities (5.17) give

$$D^{\mathrm{L}} \mathcal{R}^n = e_m \mathcal{R}^{nm} \,, \tag{8.17}$$

$$D^{\mathrm{L}} \mathcal{R}^{nm} = 0 \,. \tag{8.18}$$

Writing explicitly the indices of the differential forms this is equivalent to

$$D^{\mathrm{L}}_{[\underline{n}} \mathcal{R}_{\underline{mk}]}{}^n = e_{[\underline{n}}{}^m \mathcal{R}_{\underline{mk}]}{}^n{}_m \,, \tag{8.19}$$

$$D^{\mathrm{L}}_{[\underline{n}} \mathcal{R}_{\underline{mk}]}{}^{nm} = 0 \,. \tag{8.20}$$

The Cartan formulation of gravity has two fundamental symmetry principles. The equivalence principle requires invariance under diffeomorphisms. This is the same as in the usual Einstein-Riemann approach. Diffeomorphisms act on the indices of differential forms, *i.e.*, on the base indices $\underline{n}, \underline{m}, etc$, as *e.g.* in (7.10). In the exterior-algebra formalism, the invariance under diffeomorphisms is manifest.

Another fundamental symmetry principle, which is specific for Cartan gravity, is *local Lorentz symmetry* with parameters ε_{nm}, which acts on the fiber indices $n, m, \ldots$. The reason why the local Lorentz symmetry does not play a role in the Riemannian approach is that the metric tensor is expressed via the frame one-form by the manifestly Lorentz-invariant relation

$$g_{\underline{nm}} = e_{\underline{n}}{}^n e_{\underline{m}}{}^m \eta_{nm} \,. \tag{8.21}$$

In this sense, local Lorentz symmetry acts trivially in the Riemannian approach.

In turn, the local Lorentz symmetry just guarantees that the nontrivial degrees of freedom are carried by the metric tensor. Indeed, the metric tensor has $d(d+1)/2$ components in d dimensions. This has to be compared with the vielbein which has d^2 components among which $d(d-1)/2$ are pure gauge due to the local Lorentz symmetry. From this perspective the role of local Lorentz symmetry is to compensate for the extra components of the vielbein $e_{\underline{n}}{}^n$ compared to those of the metric tensor g_{nm}.

From (8.21) it is clear that the condition that the metric g_{nm} is non-degenerate is equivalent to

$$\det e_{\underline{n}}{}^n \neq 0 \,. \tag{8.22}$$

Note that Eq. (8.21) implies that the frame one-form $e_{\underline{n}}{}^n$ can in some sense be interpreted as a square root of the metric tensor $g_{\underline{nm}}$.

8.2 Minkowski vacuum

Before explaining how to describe gravitational interactions within the Cartan formalism let us first show that flat Minkowski space can be understood as a solution to the equations

$$\mathcal{R}^{n\,m} = 0, \qquad \mathcal{R}^{n} = 0 \tag{8.23}$$

at the condition that the frame field is non-degenerate.

In Cartesian coordinates Minkowski space is described by

$$e_{\underline{n}}{}^{n} = \delta_{\underline{n}}{}^{n}\,. \tag{8.24}$$

Indeed, by (8.21) this gives $g_{\underline{nm}} = \eta_{\underline{nm}}$. Eq. (8.24) explains why it is possible to identify base and fiber indices in Minkowski space.

Since $\partial_{\underline{m}} e_{\underline{n}}{}^{n} = 0$ in Cartesian coordinates, from the equation $\mathcal{R}^{n} = 0$ it follows that

$$\omega_{\underline{n}}{}^{n\,m} = 0 \tag{8.25}$$

and hence $\mathcal{R}^{n\,m} = 0$ as well.

Because of covariance of the curvatures under diffeomorphisms due the formalism of differential forms, the vielbein $e_{\underline{n}}{}^{n}(x)$ and Lorentz connection $\omega_{\underline{n}}{}^{n\,m}(x)$ of the flat Minkowski space obey the conditions (8.23) in any coordinate system. The description of Minkowski space as a solution of the flatness conditions (8.23) without specification of particular coordinates is sufficient for most of applications.

More generally, one can see that a solution of the zero-curvature condition $\mathcal{R}^{a} = 0$ for any algebra h, if not restricted by some additional boundary conditions, possesses a global symmetry h. For the Poincaré algebra we obtain that any solution of $\mathcal{R}^{n} = 0$, $\mathcal{R}^{nm} = 0$ (in any coordinate system) possesses Poincaré symmetry.

Indeed, from Yang-Mills theory we know that the curvature transforms homogenously under the gauge transformations

$$\delta F^{a} = [F,\, \varepsilon]^{a}\,, \qquad \delta A^{a} = \mathrm{d}\varepsilon^{a} + [A,\, \varepsilon]^{a}. \tag{8.26}$$

This means that gauge transformations map solutions of the zero-curvature condition to other solutions of the zero-curvature condition.

Let A_0^{a} be some solution of the zero-curvature (or *flatness*) condition

$$F^{a}(A_0) = 0\,. \tag{8.27}$$

Which gauge transformations leave such a flat connection A_0^{a} invariant? Clearly, those with parameters obeying

$$\delta A^{a}_{0\underline{n}}(x) \equiv \partial_{\underline{n}} \varepsilon^{a}(x) + [A_{0\underline{n}}(x),\, \varepsilon(x)]^{a} = 0\,. \tag{8.28}$$

How many solutions does this system admit? The equations (8.28) determine all derivatives of the parameter $\varepsilon^{a}(x)$ in a point x via its values $\varepsilon^{a}(x)$ in x. If these equations are consistent in the sense that they respect the symmetry of second derivatives,

$$\partial_{\underline{n}} \partial_{\underline{m}} \varepsilon^{a}(x) = \partial_{\underline{m}} \partial_{\underline{n}} \varepsilon^{a}(x)\,, \tag{8.29}$$

such equations determine $\varepsilon^a(x)$ in a neighborhood of any point $x^{\underline{n}} = x_0^{\underline{n}}$ in terms of its values $\varepsilon^a(x_0)$ in $x_0^{\underline{n}}$. For example, knowing $\varepsilon^a(x_0)$ in $x_0^{\underline{n}}$, $\varepsilon^a(x)$ can be reconstructed by the Taylor expansion, where all derivatives at x_0 are reconstructed from $A^a_{0\underline{n}}$ and $\varepsilon^a(x_0)$ by the multiple differentiations of (8.28).

The compatibility condition (8.29) is in fact respected because the connection $A^a_{0\,\underline{n}}(x)$ obeys (8.27). Indeed, rewriting (8.28) as $D_{0\,\underline{n}}\varepsilon^a = 0$, the compatibility condition is satisfied as a consequence of the relation (5.10), *i.e.*,

$$(D_{0\,\underline{n}}D_{0\,\underline{m}} - D_{0\,\underline{m}}D_{0\,\underline{n}})\varepsilon^a = f^a_{bc}F^b_{\underline{n}\,\underline{m}}(A_0)\varepsilon^c = 0\,. \tag{8.30}$$

From this general analysis it follows in particular that the symmetry of a generic solution of the flatness conditions (8.23) is Poincaré symmetry.

Let us illustrate this idea by the derivation of the symmetries of Minkowski space in Cartesian coordinates (8.24), (8.25). In this case (8.28) gives

$$\partial_{\underline{n}}\varepsilon^n(x) - \varepsilon^n{}_m(x)e_{\underline{n}}{}^m = 0\,, \qquad \partial_{\underline{n}}\varepsilon^{n\,m}(x) = 0.$$

From here it follows that

$$\varepsilon^{k\,m}(x) = \epsilon^{k\,m} = const\,, \qquad \varepsilon^n(x) = \epsilon^n{}_m x^m + \epsilon^n\,,$$

where $\epsilon^{nm} = -\epsilon^{mn}$ and ϵ^n are x-independent. The parameter $\varepsilon^n(x)$ describes global infinitesimal transformations of the Poincaré group as particular gauge transformation of the Yang-Mills fields of the Poincaré group.

We argued that Yang-Mills gauge transformations of the Poincaré group map one coordinate system in Minkowski space to another. Why did Yang-Mills gauge transformations turn out to be related to diffeomorphisms?

There is an important relation between the two types of transformations. To understand this let us again consider a general Yang-Mills theory. Let $V^{\underline{n}}$ be a vector field of an infinitesimal diffeomorphism. Consider the Yang-Mills gauge transformation

$$\delta_{\varepsilon_V}A^a_{\underline{n}}(x) = \partial_{\underline{n}}\varepsilon^a_V(x) + [A_{\underline{n}}(x),\, \varepsilon_V(x)]^a \tag{8.31}$$

with a field-dependent gauge parameter

$$\varepsilon^a_V(x) = -V^{\underline{m}}(x)A^a_{\underline{m}}(x)\,. \tag{8.32}$$

For *covariantized diffeomorphisms*, representing a combination of diffeomorphism generated by $\mathcal{L}_V$ (7.14) and Yang-Mills transformations with gauge parameter (8.32), we obtain

$$(\mathcal{L}_V + \delta_{\varepsilon(V)})A^a_{\underline{n}} := \mathcal{L}_V A^a_{\underline{n}} - \partial_{\underline{n}}(V^{\underline{m}}A_{\underline{m}})^a - [A_{\underline{n}},\, V^{\underline{m}}A_{\underline{m}}]^a = V^{\underline{m}}F^a_{\underline{m}\,\underline{n}}\,. \tag{8.33}$$

As a result, if $F^a = 0$, then diffeomorphisms amount to gauge transformations. In the gravity case, from (8.32) it follows that

$$V^{\underline{m}}(x)e_{\underline{m}}{}^n = -\varepsilon^n(x)\,. \tag{8.34}$$

If the curvature tensor $\mathcal{R}$ is non zero, diffeomorphisms differ from the usual Yang-Mills gauge transformations by some $\mathcal{R}$-dependent terms.

The example of Minkowski space illustrates important general features of theories formulated in terms of differential forms. In particular, a theory that possesses a global symmetry h in its most symmetric vacuum state can usually be described in terms of the Yang-Mills fields of the algebra h. Then the most symmetric solution is that of the flatness equation $F^a = 0$. The latter provides a coordinate-independent way to describe such an h-invariant geometry. The explicit form of $A^a_{\underline{n}}$ (*e.g.*, $e_{\underline{n}}{}^n$ and $\omega_{\underline{n}}{}^{nm}(x)$) is not important in many respects, referring mainly to the coordinate choice. For most applications it suffices to use the fact that $F^a = 0$ with no reference to a particular coordinate system. We will see more examples of this phenomenon in the next sections.

8.3 Equations of motion

To introduce a nontrivial gravitational field we have to go beyond Minkowski geometry, relaxing the flatness conditions (8.23). Since we want to use frame one-form as the dynamical field, we have to impose constraints expressing the Lorentz connection via the frame one-form. The appropriate set of constraints is

$$\mathcal{R}^n = 0\,. \tag{8.35}$$

This is often called the *zero-torsion condition.* Making the base indices explicit, this is equivalent to

$$\mathcal{R}_{\underline{nm}}{}^n = 0\,. \tag{8.36}$$

We observe that this equation has as many components as the Lorentz connection $\omega_{\underline{n}}{}^{nm}$. From the explicit form of $\mathcal{R}_{\underline{nm}}{}^n$ (8.6) it follows that (8.36) is an algebraic equation on the Lorentz connection relating it to the frame field and its derivatives. At the condition (8.22) that the frame field is non-degenerate, Eq. (8.36) admits the following unique solution

$$\omega_{\underline{n},nm} = \mathcal{F}_{\underline{n}n,\,m} - \mathcal{F}_{\underline{n}m,\,n} + \mathcal{F}_{mn,\,\underline{n}}\,, \tag{8.37}$$

where

$$\mathcal{F}_{\underline{nm},}{}^n = \frac{1}{2}(\partial_{\underline{n}} e_{\underline{m}}{}^n - \partial_{\underline{m}} e_{\underline{n}}{}^n)\,. \tag{8.38}$$

Here the indices are translated from one type to another by the frame field, and the comma is used to keep track of the original types of indices. For instance,

$$\mathcal{F}_{\underline{n}n,}{}^m = e^{\underline{k}}{}_n \mathcal{F}_{\underline{n}\,\underline{k},}{}^m\,, \tag{8.39}$$

where $e^{\underline{k}}{}_n$ is the inverse frame obeying

$$e^{\underline{m}}{}_n e_{\underline{m}}{}^k = \delta^k_n\,, \qquad e^{\underline{m}}{}_n e_{\underline{n}}{}^n = \delta^{\underline{m}}_{\underline{n}}\,. \tag{8.40}$$

We leave it to the reader as an exercise to check that (8.37) solves (8.35).

To make contact with Riemannian geometry note that the equation (8.36) can be extended to the full *frame-field postulate* (*cf.* (7.22))

$$D_{\underline{n}} e_{\underline{m}}{}^{n} := \partial_{\underline{n}} e_{\underline{m}}{}^{n} - \Gamma_{\underline{nm}}^{\underline{k}} e_{\underline{k}}{}^{n} + \omega_{\underline{n}}{}^{n}{}_{m} e_{\underline{m}}{}^{m} = 0\,. \tag{8.41}$$

This equation contains two parts. Firstly, it implies $D_{\underline{n}} g_{\underline{mk}} = 0$, which is the *metric postulate* of Einstein-Riemann gravity. Secondly, it implies $D_{\underline{n}} e_{\underline{m}}{}^{n} - D_{\underline{m}} e_{\underline{n}}{}^{n} = 0$. The first expresses the symmetric part of the Christoffel symbol $\Gamma_{\underline{nm}}^{\underline{k}}$ in terms of $g_{\underline{nm}}$ while the latter amounts to (8.36) at the condition that the antisymmetric part of $\Gamma_{\underline{nm}}^{\underline{k}}$, called the *torsion tensor*, is zero. Clearly, (8.41) expresses the Christoffel symbol in terms of the Lorentz connection and the frame field.

The Riemann tensor can be recovered as follows. Using (8.15) and (8.16) from the zero-torsion condition (8.36) and the Bianchi identities (8.17) it follows that

$$e^{n} \mathcal{R}_{nm} = 0\,. \tag{8.42}$$

Making the base indices explicit, let us now introduce the tensor

$$\mathcal{R}_{\underline{nm},\underline{kl}} := e_{\underline{k}}{}^{k} e_{\underline{l}}{}^{l} \mathcal{R}_{\underline{nm},kl} \tag{8.43}$$

that carries only base indices. It is by construction antisymmetric in the first and second pairs of indices. The condition (8.42) then implies that the antisymmetrization over any three indices gives zero

$$\mathcal{R}_{[\underline{nm},\underline{k}]\underline{l}} = 0\,. \tag{8.44}$$

In fact, this means that the tensor $\mathcal{R}_{\underline{nm},\underline{kl}}$ has the symmetry of the window Young diagram $\boxplus$ in the antisymmetric basis.

Once the zero-torsion condition is imposed, $\mathcal{R}_{\underline{nm},\underline{kl}}$ is expressed in terms of the frame field $e_{\underline{n}}{}^{n}$ and its first and second derivatives. Carrying only base indices it is manifestly invariant under the local Lorentz transformations. In turn, from here it follows that $\mathcal{R}_{\underline{nm},\underline{kl}}$ can be expressed in terms of the metric tensor (8.21). For those familiar with Riemannian geometry this proves that $\mathcal{R}_{\underline{nm},\underline{kl}}$ is the Riemann tensor (maybe up to a factor). Thus, the Cartan formalism contains the Riemannian one.

The trace part of the Riemann tensor

$$\mathcal{R}_{\underline{nm}} := g^{\underline{kl}} \mathcal{R}_{\underline{nk},\underline{lm}} \tag{8.45}$$

is called the *Ricci tensor*. The *scalar curvature* $\mathcal{R}$ is the trace of the Ricci tensor

$$\mathcal{R} := g^{\underline{nm}} \mathcal{R}_{\underline{nm}} \equiv g^{\underline{nm}} g^{\underline{kl}} \mathcal{R}_{\underline{nk},\underline{lm}}\,. \tag{8.46}$$

As an exercise the reader can check that the so-defined Ricci tensor is symmetric as a consequence of (8.44),

$$\mathcal{R}_{\underline{nm}} = \mathcal{R}_{\underline{mn}}\,. \tag{8.47}$$

Note that in the literature different sign conventions for the Ricci tensor are commonly used. The definition (8.45) is chosen so that it matches the definition of the spin-two Fronsdal tensor (4.8) in the linearized approximation.

In the absence of matter fields the equations of motion for the gravitational field in Einstein theory require the Ricci tensor to vanish

$$\mathcal{R}_{\underline{nm}} = 0\,. \tag{8.48}$$

There is a useful way to rewrite the vacuum Einstein equations entirely in the language of differential forms. Instead of demanding that the Riemann tensor is traceless we can say that the curvatures of the Poincaré algebra obey the following equations

$$\mathcal{R}_n(x) = 0\,, \qquad \mathcal{R}_{nm}(x) = e^k(x)e^l(x)C_{nm,kl}(x)\,, \tag{8.49}$$

where the fiber tensor $C_{nm,kl}(x)$ has the properties of the traceless window diagram in the antisymmetric basis,

$$C_{nm,kl} = -C_{mn,kl} = -C_{nm,lk}\,, \qquad C_{[nm,k]l} = 0\,, \qquad \eta^{ml}C_{nm,kl} = 0\,. \tag{8.50}$$

In Eq. (8.49) we never use the inverse frame explicitly.

The traceless part of the Riemann tensor is called the *Weyl tensor*. Clearly, (8.49) implies both that the Ricci tensor is zero and that $C_{nm,kl}$ equals the Weyl tensor.

Let us now outline the connection of gravity with Fronsdal theory. Gravity can be thought of as the theory of a massless spin-two field called the graviton. It is just described by the spin-two Fronsdal field φ_{nm}. In the standard Riemannian approach it is associated with fluctuations of the metric tensor g_{nm} around the metric of flat Minkowski space

$$g_{nm}(x) = \eta_{nm} + \frac{1}{2}\varphi_{nm}(x)\,. \tag{8.51}$$

We observe that the parameter of diffeomorphisms $\varepsilon^n(x)$ has the same tensor type as the gauge parameter of the spin-two Fronsdal field. In the weak-field limit of gravity the transformation law of the frame field

$$e_{\underline{n}}{}^n = \delta_{\underline{n}}{}^n + e'_{\underline{n}}{}^n \tag{8.52}$$

is

$$\delta e'_{\underline{n}}{}^n = \partial_{\underline{n}}\varepsilon^n(x) + \varepsilon^n{}_m(x)e'_{\underline{n}}{}^m = o(e')\,. \tag{8.53}$$

At the linearized level the local Lorentz symmetry implies that only the symmetric part of the frame field,

$$\varphi_{nm} = \frac{1}{2}(e'_{n,m} + e'_{m,n})\,, \tag{8.54}$$

contributes. The gauge symmetry with parameter ε^n gives

$$\delta\varphi_{m\,n}(x) = \partial_{(n}\varepsilon_{m)}(x) + O(\varphi),$$

which is just the gauge transformation law of the free spin-two massless Fronsdal field.

The other way around, diffeomorphisms provide a non-Abelian deformation of the Abelian symmetries of the free equations of a massless spin-two field, while Einstein gravity provides a nonlinear deformation of the free massless spin-two system.

Its structure is heavily restricted by the gauge-symmetry principle. The gauge-invariant nonlinear equations of motion of a massless spin-two field, which contain at most two derivatives, turn out to be unique modulo nonlinear field redefinitions

$$\varphi_{nm} \to \varphi_{nm}(\varphi') = \varphi'_{nm} + \alpha\, \varphi'_{nk}\varphi'_m{}^k + \dots\,. \tag{8.55}$$

8.4 Action

The Einstein equations admit an action principle. Using the formalism of differential forms the *Einstein-Hilbert action* can be written in the Weyl form

$$S = -\frac{d(d-1)}{2\kappa}\int_{M^d} \epsilon_{n_1\dots n_d} \underbrace{e^{n_1}\dots e^{n_{d-2}}\mathcal{R}^{n_{d-1}n_d}}_{d\text{-form}}\,. \tag{8.56}$$

Here κ is the gravitational constant related to the Newton constant G as

$$\kappa = 8\pi G \qquad \text{at} \quad d=4\,. \tag{8.57}$$

To see the relation of (8.56) to the Einstein-Hilbert action first notice that it is equivalent to

$$S = -\frac{1}{4(d-2)!\kappa}\int d^dx\, \epsilon^{\underline{n}_1\dots\underline{n}_d}\epsilon_{n_1\dots n_d} e_{\underline{n}_1}{}^{n_1}\dots e_{\underline{n}_{d-2}}{}^{n_{d-2}}\mathcal{R}_{\underline{n}_{d-1}\underline{n}_d},{}^{n_{d-1}n_d} \tag{8.58}$$
$$= -\frac{1}{4(d-2)!\kappa}\int d^dx\, \epsilon^{\underline{n}_1\dots\underline{n}_d}\epsilon_{n_1\dots n_d} e_{\underline{n}_1}{}^{n_1}\dots e_{\underline{n}_{d-2}}{}^{n_{d-2}} e_{\underline{n}_{d-1}}{}^{p} e_{\underline{n}_d}{}^{q}\mathcal{R}_{pq},{}^{n_{d-1}n_d}\,.$$

Using formula (7.31) in the form

$$\epsilon^{\underline{n}_1\dots\underline{n}_d} e_{\underline{n}_1}{}^{n_1}\dots e_{\underline{n}_d}{}^{n_d} = \epsilon^{n_1\dots n_d}\ \det|e| \tag{8.59}$$

along with

$$\frac{1}{(d-2)!}\epsilon_{n_1\dots n_{d-2}ml}\epsilon^{n_1\dots n_{d-2}m'l'} = 2\,\delta^{m'}_{[m}\delta^{l'}_{l]} \tag{8.60}$$

we obtain

$$S = -\frac{1}{4(d-2)!\kappa}\int d^dx\ \det e\, \epsilon^{n_1\dots n_{d-2}pq}\epsilon_{n_1\dots n_d}\mathcal{R}_{pq},{}^{n_{d-1}n_d} = \frac{1}{2\kappa}\int d^dx\ \det e\, \mathcal{R}\,, \tag{8.61}$$

where $\mathcal{R}$ is the scalar curvature (8.46).

Once the Lorentz connection is expressed via the frame field, the scalar curvature $\mathcal{R}(e,\omega(e))$ is expressed entirely in terms of the metric tensor. Taking also into account that

$$\det\left|e_{\underline{n}}{}^n\right| = \det{}^{1/2}\left|(-1)^{d+1}g_{\underline{nm}}\right| \tag{8.62}$$

we find that the action (8.56) indeed coincides with the usual Einstein-Hilbert action written in terms of the metric tensor provided that the Lorentz connection is expressed via the frame field via the zero-torsion condition (8.35). Such approach is called the *second-order formalism*.

Alternatively, one can consider the so-called *first-order formalism* where both $e^a(x)$ and $\omega^{nm}(x)$ are treated as independent fields. Using the formula

$$\delta\mathcal{R}^{nm} = D^{\mathrm{L}}\delta\omega^{nm} \tag{8.63}$$

from Yang-Mills theory, *cf.* (5.9), we obtain

$$\delta S = -\frac{1}{2(d-2)!\kappa}\int d^dx\,\epsilon_{n_1\dots n_d}\Big(e^{n_1}\dots e^{n_{d-2}}D^{\mathrm{L}}\delta\omega^{n_{d-1}n_d}+(d-2)\delta e^{n_1}\dots e^{n_{d-2}}\mathcal{R}^{n_{d-1}n_d}\Big)\,.$$

Now we can use that the covariant derivatives can be integrated by parts by the Stokes theorem since

$$\int\big((D^{\mathrm{L}}A_a)B^a + A_a D^{\mathrm{L}}B^a\big) = \int \mathrm{d}(A_aB^a) = 0\,. \tag{8.64}$$

As a result,

$$\delta S = -\frac{1}{2(d-3)!\kappa}\int d^dx\,\epsilon_{n_1\dots n_d}\Big(e^{n_1}\dots D^{\mathrm{L}}(e^{n_{d-2}})\delta\omega^{n_{d-1}n_d} + \delta e^{n_1}\dots e^{n_{d-2}}\mathcal{R}^{n_{d-1}n_d}\Big)\,.$$

We observe that the part of the variation of the action with respect to the Lorentz connection ω^{nm} is proportional to the torsion tensor $D^{\mathrm{L}}e^n$, *i.e.*, the field equation for ω^{nm} just gives the zero-torsion condition (8.36).

On the other hand, the variation with respect to the frame field gives

$$\delta e_{\underline{n}}{}^{n}\left(\mathcal{R}^{\underline{n}}{}_{n} - \frac{1}{2}e^{\underline{n}}{}_{n}\mathcal{R}^{\underline{m}}{}_{\underline{m}}\right) = 0\,, \tag{8.65}$$

as is easy to see using that

$$\frac{1}{(d-3)!}\epsilon_{n_1\dots n_{d-3}mlk}\,\epsilon^{n_1\dots n_{d-3}m'l'k'} = 6\,\delta^{m'}_{[m}\delta^{l'}_{l}\,\delta^{k'}_{k]}\,. \tag{8.66}$$

The variation (8.65) gives the Einstein equations in the vacuum

$$G_{\underline{n}\,\underline{m}} := \mathcal{R}_{\underline{n}\,\underline{m}} - \frac{1}{2}g_{\underline{n}\,\underline{m}}\mathcal{R} = 0\,, \tag{8.67}$$

where $G_{\underline{nm}}$ is called the *Einstein tensor*.

One can check that $G_{\underline{n}\,\underline{m}}$ reproduces the spin-two tensor (4.20) in the linearized approximation.

The first-order and the second-order formalisms are dynamically equivalent as is most easily seen by virtue of the 1.5-order formalism which is the second-order formalism taking into account that $\omega(e)$ solves its field equations which makes it possible to neglect all the terms in the variation of the action that came from the variation of $\omega(e)$.

8.5 Gravitational interaction of matter fields

In the Cartan formalism it is easy to introduce gravitational interactions for any Poincaré-invariant matter Lagrangian $L(\partial\phi, \phi)$ by inserting the volume form

$$|e| := \frac{1}{d!}\epsilon_{n_1 \dots n_d} e^{n_1} \dots e^{n_d} \tag{8.68}$$

and replacing the usual derivatives ∂_n by the Lorentz covariant ones

$$\partial_n \Longrightarrow D^{\mathrm{L}}_n = e^{\underline{n}}{}_n D^{\mathrm{L}}_{\underline{n}} \tag{8.69}$$

so that the covariantized action becomes

$$S^{\mathrm{matt}} = \int d^d x \, |e| \, L(D^{\mathrm{L}}\phi, \phi) \,. \tag{8.70}$$

The so-defined action respects the fundamental symmetry principles of the Cartan formulation of gravity, namely diffeomorphism and local Lorentz symmetry.

Let us consider some examples of lower-spin matter fields.

For a scalar field the action is

$$S^{\mathrm{scalar}} = \frac{1}{2}\int d^d x \, |e| \Big(\partial_n C \partial_m C \eta^{nm} - m^2 C^2 - V(C)\Big) \,, \qquad \partial_n = e_n{}^{\underline{n}} \frac{\partial}{\partial x^{\underline{n}}}. \tag{8.71}$$

For a spin-1/2 spinor, the action is

$$S^{\mathrm{spinor}} = \int d^d x \, |e| \left(i\bar{\psi}^{\hat{\alpha}} \gamma^{\underline{n}}{}_{\hat{\alpha}}{}^{\hat{\beta}} (D^{\mathrm{L}}_{\underline{n}}\psi)_{\hat{\beta}} + m\bar{\psi}^{\hat{\alpha}}\psi_{\hat{\alpha}} \right)$$

where

$$\gamma^{\underline{n}}(x) = e^{\underline{n}}{}_n(x)\gamma^n \tag{8.72}$$

and

$$(D^{\mathrm{L}}_{\underline{n}}\psi)_{\hat{\alpha}} = \partial_{\underline{n}}\psi_{\hat{\alpha}} + \frac{1}{4}\omega_{\underline{n}}{}^{nm}[\gamma_n, \, \gamma_m]_{\hat{\alpha}}{}^{\hat{\beta}}\psi_{\hat{\beta}} \tag{8.73}$$

is the Lorentz-covariant derivative in the spinor module of the Lorentz algebra.

Here is an important comment. One of the significant advantages of the Cartan approach compared to the Riemannian one is that the former provides a covariant description of spinors, which cannot be done in the latter. The point is that spinor indices $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}, \dots$ only make sense as fiber indices, admitting no base counterparts. So the Cartan formalism is really necessary to include fermions into the consideration. In particular, it is this formalism which has to be used in supergravity.

Another way to see this is that while the Dirac matrices γ^n can be understood as a "square root" of the flat metric tensor,

$$\{\gamma^n \, , \gamma^m\} = 2\eta^{nm} I \,, \tag{8.74}$$

without introducing the frame field it is hard to introduce γ-matrices $\gamma^{\underline{n}}(x)$ as a square root of an arbitrary curved metric

$$\{\gamma^{\underline{n}}(x) \, , \gamma^{\underline{m}}(x)\} = 2g^{\underline{nm}}(x) \,. \tag{8.75}$$

On the other hand, in the presence of the frame field, this is easily achieved by virtue of (8.72).

The covariantized Yang-Mills action is

$$S^{\mathrm{YM}} = -\frac{1}{4g^2}\int d^dx\,|e|tr\,(F_{\underline{n}\,\underline{m}}F_{\underline{k}\,\underline{l}})\,g^{\underline{n}\,\underline{k}}g^{\underline{m}\,\underline{l}}. \tag{8.76}$$

Note that this action contains only usual derivatives because $A^a_{\underline{n}}$ is a one-form and $F^a_{\underline{nm}}$ is a two-form.

For the system which contains both gravity and matter fields the full action is the sum of the gravitational action and the covariantized matter action

$$S = S^{\mathrm{grav}} + S^{\mathrm{matt}}\,. \tag{8.77}$$

The field equations

$$\delta(S^{\mathrm{grav}} + S^{\mathrm{matt}}) = 0$$

now imply

$$G_{\underline{nm}} = \kappa T_{\underline{nm}}\,, \tag{8.78}$$

where

$$T^{\underline{n}}{}_n = \frac{\delta S^{\mathrm{matt}}}{\delta e_{\underline{n}}{}^n}\,. \tag{8.79}$$

Together with the zero-torsion condition these are the *Einstein equations.*

As an exercise we suggest to check that $T^{\underline{n}}{}_n$ from (8.79) coincides with the stress tensor associated with the invariance under Poincaré translations via the approach of Section 6.2.

8.6 The problem of the gravitational interaction of higher-spin fields

Once we know how to introduce the gravitational interaction of matter fields we can try to proceed the same way in the case of massless higher-spin fields. There is no problem with the covariantization of the Fronsdal action along the same lines as for the lower-spin fields. This will guarantee local Lorentz symmetry and invariance under diffeomorphisms. However, a difficulty arises due to the higher-spin gauge symmetries.

To respect diffeomorphisms and the local Lorentz symmetry we have to covariantize both the Fronsdal action (4.13)

$$S[\varphi,\partial\varphi] \Longrightarrow S^{\mathrm{cov}}[\varphi, D^{\mathrm{L}}\varphi] \tag{8.80}$$

and the higher-spin gauge transformation law (4.5)

$$\delta^{\mathrm{cov}}\varphi_{k_1\ldots k_s}(x) := D^{\mathrm{L}}_{(k_1}\varepsilon_{k_2\ldots k_s)}(x)\,. \tag{8.81}$$

Now we observe that when checking the gauge invariance of the Fronsdal action we commuted derivatives using that $\partial_{\underline{n}}\partial_{\underline{m}} = \partial_{\underline{m}}\partial_{\underline{n}}$. However, this property is no longer true for covariant derivatives because their commutator is proportional to the Lorentz curvature (*cf.* (8.16)). Computing a gauge variation of the covariantized action $\delta^{\rm cov}S^{\rm cov}$ one can see that this is no longer zero, but instead contains some terms proportional to the Lorentz curvature, *i.e.*, the Riemann tensor. Schematically,

$$\delta^{\rm cov}S^{\rm cov} \sim \int d^dx\, D^{\rm L}\mathcal{R}\,\varphi\,\varepsilon\,, \tag{8.82}$$

where the derivative $D^{\rm L}$ can act on any of the objects $\mathcal{R}$, φ or ε.

In practice, this expression is fairly complicated even for the case of spin three where all possible types of terms already contribute; hence I am not writing the explicit formula here. The appearance of curvature-dependent terms in the gauge variation of the covariantized Fronsdal action is the most direct manifestation of the problem of introducing the higher-spin interaction with gravity. At the same time, since the lowest-order interactions are represented by the Noether-current interactions, this is a manifestation of the property discussed in Section 6.2.4 implying that the stress tensor for higher-spin gauge fields is not gauge invariant.

Note that this phenomenon does not occur for the spin-one field because both the Yang-Mills field strength $F^a_{\underline{nm}}$ and the transformation $\delta A^a_{\underline{n}}$ contain usual derivatives rather than Lorentz-covariant ones. This is because $A^a_{\underline{n}}$ is a one-form, $F^a_{\underline{nm}}$ is a two-form and the gauge parameter ε^a is a zero-form, none of which carry fiber indices. As a result there is no conflict between the gravitational and Yang-Mills symmetries.

The nontrivial curvature-dependent contributions to the gauge variation of the action occur for spins $s \geq 3/2$. In the case of spin 3/2 it turns out to be possible to deform the gauge transformation law of the spin-two gravitational field under the spin-3/2 gauge transformation in such a way that the problematic terms cancel out. This opens a way towards the nonlinear theory of gauge fields of spin two and spin 3/2 known as *supergravity*.

For higher-spin fields with $s > 2$ a similar trick does not work. Namely, it looks like it is not possible to add some new terms to the action and/or modify the transformation laws in such a way that the higher-spin symmetry will be restored. This conclusion created the impression that it is impossible to introduce a consistent (*i.e.*, gauge invariant) interaction between gravity and higher-spin gauge fields. Because of the universal role of the gravitational interaction attributed to any matter, if true this conclusion would mean that a higher-spin gauge theory does not make much sense as a physical model.

In Section 14 I will actually explain how this problem can be avoided. The main trick is that one should start with higher-spin theory in the De Sitter, or better, anti-De Sitter space-time rather than in Minkowski one. So, as a next step I explain what these space-times are and how the corresponding version of the gravity theory is formulated.

9 (Anti-)De Sitter gravity

9.1 Cosmological constant

There is an important generalization of the Einstein action via the addition of a cosmological term

$$S = \frac{1}{2\kappa}\int |e|(\mathcal{R} - 2\Lambda)\,, \tag{9.1}$$

where the parameter Λ is called the *cosmological constant*.

In the absence of gravity, the addition of such a constant term to the Lagrangian does not affect the equations of motion. In the presence of gravity, however, this is no longer the case because of the volume form $|e|$. The vacuum (*i.e.*, in the absence of matter) Einstein equations with the cosmological term are

$$G_{\underline{nm}} + g_{\underline{nm}}\Lambda = 0\,. \tag{9.2}$$

9.2 (Anti-)De Sitter space

The maximally symmetric solutions of (9.2) obey the stronger conditions

$$\mathcal{R}_{\underline{n}\,\underline{m},\,\underline{p}\,\underline{k}} = -\Lambda'(g_{\underline{n}\,\underline{p}}g_{\underline{m}\,\underline{k}} - g_{\underline{n}\,\underline{k}}g_{\underline{m}\,\underline{p}}), \tag{9.3}$$

where

$$\Lambda = \frac{1}{2}(d-1)(d-2)\Lambda'\,. \tag{9.4}$$

We leave it to the reader to check that (9.2) holds as a consequence of (9.3).

Equations (9.3) are just the equations of De Sitter ($\Lambda > 0$) and anti-De Sitter ($\Lambda < 0$) space. The cosmological constant Λ can be treated as a deformation parameter from the degenerate case of $\Lambda = 0$ associated with flat Minkowski space.

Note that the De Sitter geometry describes the effect of dark energy in the present state of Universe, while anti-De Sitter space plays a fundamental role in the so-called AdS/CFT-correspondence, which is a remarkable duality between gravitational theories in $(d+1)$-dimensional anti-De Sitter space and d–dimensional field theories.

The relation of (anti-)De Sitter space to Minkowski space is analogous to that of the sphere to the plane. The sphere S^d is a d-dimensional Euclidean hypersurface in $(d+1)$-dimensional space, defined by the condition

$$X^A X^B \delta_{AB} = R^2\,, \tag{9.5}$$

where $A, B = 0, 1, ..., d$ are the indices of the ambient space and constant R is the radius of the sphere. From its definition it is obvious that the sphere is invariant under the group $O(d+1)$ which leaves the Kronecker delta δ_{AB} invariant.

Consider a point $X_0 \in S^d$. Clearly, $O(d+1)$ contains transformations that leave X_0^A invariant. These transformations form the *stability group* of X_0. An analysis fully analogous to that of the massive little group shows that the stability group of any point X_0 is $O(d)$.

In addition $O(d+1)$ contains elements that shift X_0^A. We should expect d independent $O(d+1)$-rotations of this type because S^d has d local coordinates. To check this one can compare the dimensions of the respective groups, taking into account that

$$\dim\, O(N) = \frac{N(N-1)}{2}\,. \tag{9.6}$$

As anticipated we find that the number of independent elements that shift X_0 is[10]

$$\frac{d(d+1)}{2} - \frac{d(d-1)}{2} = d = \dim\, S^d\,. \tag{9.7}$$

In the flat limit $R \to \infty$ the stability group of a point X_0^A remains $O(d)$ but the transformations that shift X_0 become the now commutative translations of a plane $\mathbb{R}^d$. Thus, in the flat limit $R \to \infty$ the group $O(d+1)$ contracts to the group of motions of the Euclidean plane, $e_d = iso(d)$.

Analogously to the sphere, De Sitter and anti-De Sitter spaces are quadrics in the $(d+1)$-dimensional ambient space. Let indices $A, B = 0, 1, ..., d$ be vector indices of either $o(d,1)$ for dS or $o(d-1,2)$ for AdS. We will also use the notation $A = (n, \bullet)$ where $n = 0, ..., d-1$ is a Lorentz index while $\bullet$ denotes the last value $A = d$. The $o(d,1)$ or $o(d-1,2)$-invariant metrics in the ambient space have different signatures:

$$AdS: \qquad \eta^{\bullet\bullet} = 1\,, \tag{9.8}$$

$$dS: \qquad \eta^{\bullet\bullet} = -1\,. \tag{9.9}$$

In either case it is assumed that η^{nm} is a d-dimensional Minkowski metric and $\eta^{\bullet n} = 0$.

Anti-De Sitter space is the hypersurface in the $(d+1)$-dimensional space described by the equation

$$X^A X^B \eta_{AB} = R^2\,, \tag{9.10}$$

where η_{AB} is the $o(d-1,2)$-invariant metric. According to this definition anti-De Sitter space is a one-sheet hyperboloid. **It would be nice to have a picture of AdS here** In this realization, the time coordinate turns out to be periodic, being associated with the phase in the plane $X^{0\bullet}$, *i.e.*,

$$X^0 = \sin(t)\sqrt{R^2 + \vec{X}^2}\,, \qquad X^\bullet = \cos(t)\sqrt{R^2 + \vec{X}^2}\,, \qquad \vec{X} = \left(X^1, \ldots, X^{d-1}\right). \tag{9.11}$$

This property is not very nice, destroying the causal structure of space-time, allowing to get back to the past after moving to the future in time (*i.e.*, admitting closed *time-like* geodesics). This problem is however easily resolved by the transition to the universal cover of the anti-De Sitter space, in which the S^1 associated with the

[10]Note that here we have an example of a general situation: every space S invariant under the action of a group G and such that G relates any two points of S (*i.e.*, acts *transitively*) is equivalent to the *space of conjugacy classes* G/H where $H \subset G$ is the stability subgroup of any point of S under the action of G.

time variables is replaced by $\mathbb{R}$ *i.e.*, the time-points t and $t+2\pi$ are not identified any more.

De Sitter space is the space described by the equation

$$X^A X^B \eta_{AB} = -R^2 \,, \tag{9.12}$$

where η_{AB} now is the $o(d,1)$-invariant metric. **It would be nice to have a picture of dS here**

Now we describe De Sitter and anti-De Sitter spaces in terms of the algebras $o(d,1)$ and $o(d-1,2)$, respectively.

The Yang-Mills fields of both $o(d,1)$ and $o(d-1,2)$ are one-forms $A^{AB} = -A^{BA}$. The curvature tensor is given by the standard expression (*cf.* the expression (8.5) for the Lorentz curvature)

$$R^{AB} = \mathrm{d}A^{AB} + A^A{}_C A^{CB} = -R^{BA} \,. \tag{9.13}$$

Here the indices are lowered by the invariant metric η_{AB}.

The gauge-transformation law has the form

$$\delta A^{AB} = \mathrm{d}\varepsilon^{AB} + A^A{}_C \varepsilon^{CB} + A^B{}_C \varepsilon^{AC} \,. \tag{9.14}$$

Let us decompose the gauge fields into irreducible components with respect to the Lorentz algebra $o(d-1,1)$ which belongs both to $o(d-1,2)$ and to $o(d,1)$. We set $A^{nm} = \omega^{nm}$ and $A^{n\bullet} = \lambda e^n$, where $\lambda \neq 0$ is some parameter of dimension $(\mathrm{cm})^{-1}$ introduced to make the frame-field e^n dimensionless since A^{AB} has dimension $(\mathrm{cm})^{-1}$.

This decomposition of the connection leads to the corresponding decomposition of the curvature into R^{nm} and $R^{n\bullet} = \lambda R^n$

$$R^n = \mathrm{d}e^n + \omega^n{}_m e^m \,, \tag{9.15}$$

$$R^{nm} = \mathrm{d}\omega^{nm} + \omega^n{}_k \omega^{km} - \lambda^2 \eta_{\bullet\bullet} e^n e^m \,. \tag{9.16}$$

We observe that these curvatures are analogous to those of the Poincaré algebra. The only difference is the last term on the right-hand side of (9.16). From the Lie-algebra point of view this term signals that the commutation relations of the momentum generators P_n have been modified to

$$[P_n \,, P_m] = -\eta_{\bullet\bullet} \lambda^2 L_{nm} \,, \tag{9.17}$$

i.e.,

$$AdS: \qquad [P_n \,, P_m] = -\lambda^2 L_{nm} \,, \tag{9.18}$$

$$dS: \qquad [P_n \,, P_m] = \lambda^2 L_{nm} \,. \tag{9.19}$$

Also one can see directly that these commutation relations follow from those of $o(d-1,2)$ and $o(d,1)$ via the identification

$$P_n = \lambda L_{n\bullet} \,. \tag{9.20}$$

Eq. (9.17) shows that the Lie algebras $o(d-1,2)$ and $o(d,1)$ both are *deformations* of the Poincaré Lie algebra $iso(d-1,1)$ with λ^2 being the deformation parameter. The other way around, the Poincaré Lie algebra $iso(d-1,1)$ is a limit (*contraction*) of both $o(d-1,2)$ and $o(d,1)$.

Now we observe that setting the $(A)dS$ curvatures to zero just reproduces the equations (9.3) with

$$AdS: \qquad \Lambda' = -\lambda^2\,, \tag{9.21}$$

$$dS: \qquad \Lambda' = \lambda^2\,. \tag{9.22}$$

From the general analysis of the zero-curvature conditions of Section 8.2 it follows that the equations (9.3) indeed describe space-times exhibiting $o(d-1,2)$ and $o(d,1)$ symmetries in the AdS and dS cases, respectively. It should be stressed that this conclusion is coordinate independent. At the same time one has to remember that it may be not true in the presence of additional boundary conditions.

The formalism of differential forms allows one to work with the relevant space-times without reference to a specific coordinate system. Nevertheless it is sometimes useful to choose some specific coordinates. In the context of the AdS/CFT-correspondence, a particularly important coordinate system is that due to Poincaré. The corresponding $o(d-1,2)$-connection can be chosen in the form

$$e^n = \frac{1}{\lambda z}\xi^n\,, \qquad \omega^{\mathbf{i}\circ} = \frac{1}{z}\xi^{\mathbf{i}}\,, \qquad \omega^{\mathbf{ij}} = 0\,, \tag{9.23}$$

where the indices $\mathbf{i}, \mathbf{j}, \ldots$ correspond to the coordinates of the $(d-1)$–dimensional slices of anti-De Sitter space while $\circ$ denotes the radial direction in anti-De Sitter space associated with the *Poincaré coordinate* z. Note that (9.23) implies in particular that $e_{\underline{n}}{}^n = \frac{1}{\lambda z}\delta^n_{\underline{n}}$. We leave it to the reader to check that the so-defined e^n and ω^{nm} form a flat $o(d-1,2)$-connection, *i.e.*, that the associated curvature tensors vanish.

The Poincaré coordinate z is chosen in such a way that the limit $z \to 0$ yields the spatial infinity of the AdS space. The coordinates $\vec{x}$ parametrize a flat $(d-1)$–dimensional space-time slice of AdS_d, while z parametrizes the slicing (foliation) of AdS_d by flat $(d-1)$–dimensional slices. This z can be interpreted as a radial coordinate going from infinity to inside of the AdS space. In the context of the AdS/CFT-correspondence, the infinity at $z \to 0$ is called *(conformal) infinity.* As already mentioned, the AdS/CFT-correspondence is a very unusual duality between theories of gravity formulated in AdS_{d+1} and d-dimensional field theories which in some sense live at the boundary $z = 0$ of AdS_{d+1} described by the coordinates $\vec{x}$.

Obviously, the metric corresponding to (9.23) is **napisat'**.

Note that the Poincaré coordinates do not cover the full AdS space because of the singularity at $z \to \infty$.

9.3 MacDowell-Mansouri action

The Weyl action (8.56) for gravity admits a beautiful generalization to AdS gravity first found by MacDowell and Mansouri in 1977 for the case of four dimensions [3].

Since this approach turns out to be very useful in application both to supergravity and to higher-spin gauge theories we explain it in some detail.

The *MacDowell-Mansouri action* is

$$S = \frac{3}{\kappa\lambda^2}\int_{M^4} \epsilon_{n\,m\,p\,q} R^{n\,m} R^{p\,q}\,, \tag{9.24}$$

where the integrand is a four-form. Substituting

$$R^{nm} = \mathcal{R}^{nm} - \lambda^2 e^n e^m \tag{9.25}$$

this action decomposes into three terms:

$$S = S_{-1} + S_0 + S_1 \tag{9.26}$$

$$S_{-1} = \frac{3}{\kappa\lambda^2}\int_{M^4} \epsilon_{n\,m\,p\,q}\mathcal{R}^{n\,m}\mathcal{R}^{p\,q}, \tag{9.27}$$

$$S_0 = -\frac{6}{\kappa}\int_{M^4} \epsilon_{n\,m\,p\,q} e^n e^m \mathcal{R}^{p\,q}, \tag{9.28}$$

$$S_1 = \frac{3\lambda^2}{\kappa}\int_{M^4} \epsilon_{n\,m\,p\,q} e^n e^m e^p e^q\,. \tag{9.29}$$

The action S_{-1} is a topological invariant. Indeed, by (5.9) its variation can be written as

$$\delta S_{-1} = \frac{6}{\kappa\lambda^2}\int_{M^4} \epsilon_{n\,m\,p\,q}\mathcal{R}^{n\,m} D^{\mathrm{L}}(\delta\omega^{p\,q})\,, \tag{9.30}$$

where D^{L} is the Lorentz-covariant derivative (8.12). Since $\epsilon_{n\,m\,p\,q}$ is a Lorentz-invariant tensor, the covariant derivative can be integrated by parts. The result is zero by virtue of the Bianchi identities $D^{\mathrm{L}}\mathcal{R}^{nm} = 0$ (8.18).

Thus, locally, S_{-1} is an integral of a total derivative which does not affect the field equations. Globally, this is a *topological* term called the *Gauss-Bonnet action*. It is topological since it is an integral of a closed form hence being independent on local variations of the manifold M^4. Mathematically it describes the Euler characteristic of the four-dimensional space-time manifold.

We leave it as an exercise to check that the four-form $\epsilon_{n\,m\,p\,q}\mathcal{R}^{n\,m}\mathcal{R}^{p\,q}$ is closed irrespective of the space-time dimension.

S_0 is the Einstein-Hilbert action in the Weyl form (8.56). S_1 is just the cosmological term by virtue of the determinant formula (8.68). Thus, the MacDowell-Mansouri action differs from the Hilbert-Einstein action with cosmological term by a topological term which does not contribute to the classical field equations. Note that the coefficient in front of the topological part of the MacDowell-Mansouri action is singular in the flat limit $\lambda \to 0$.

Let us discuss the symmetries of the MacDowell-Mansouri action. Since $\epsilon_{n\,m\,p\,q}$ is Lorentz invariant, the MacDowell-Mansouri action is manifestly invariant under local Lorentz symmetry. Due to the formalism of differential forms, the action is manifestly invariant under diffeomorphisms as well. However the original local $o(3,2)$-symmetry is explicitly broken since the tensor $\epsilon_{n\,m\,p\,q}$ is not $o(3,2)$-invariant.

It was noted by Stelle and West in 1980 that this can be interpreted as spontaneous symmetry breaking. To this end they introduced a Higgs-like field V^A that transforms as a vector of $o(3,2)$ and has a normalized $o(3,2)$-invariant norm

$$V^A(x)V^B(x)\eta_{AB} = 1\,. \tag{9.31}$$

The MacDowell-Mansouri action (9.24) can now be rewritten in the form

$$S = \frac{3}{\kappa\lambda^2}\int_{M^4} \epsilon_{ABCDE}R^{AB}R^{CD}V^E(x)\,. \tag{9.32}$$

This action is manifestly invariant under both diffeomorphisms and the local $o(3,2)$-symmetry. However, being nonzero as a consequence of (9.31), the field V^E breaks $o(3,2)$ down to the Lorentz algebra $o(3,1)$. This breaking is spontaneous.[11] Indeed, choosing a gauge with respect to the gauge algebra $o(3,2)$ it is possible to achieve the *standard gauge* $V^E(x) = \delta^E_\bullet$ in which V^E is a constant vector pointing in the direction additional to the Lorentz directions. In the standard gauge the Stelle-West action amounts to the MacDowell-Mansouri action. However in (9.32) now any other choice of $V^E(x)$ with $V^2(x) = 1$ is equally good. Note that the field $V^A(x)$, called the *compensator*, carries no new degrees of freedom compared to the original fields ω^{nm} and e^n: all degrees of freedom contained in V^A obeying (9.31) are eaten by the extra gauge symmetries in the gauge $o(3,2)$ compared to $o(3,1)$. Also, it should be noted that there is a deep parallel between the equation (9.31) on the field $V^A(x)$ and the quadratic equation (9.10) of the anti-De Sitter space in the coordinates X^A.

The introduction of the compensator field V^E makes it possible to give an invariant definition of the frame one-form:

$$\lambda E^A = DV^A = \mathrm{d}V^A + A^A{}_B V^B. \tag{9.33}$$

As a consequence of (9.31) it obeys

$$E^A V_A = 0\,. \tag{9.34}$$

In the standard gauge, (9.33) gives $E^\bullet = 0$ and $E^n = e^n$.

Now the condition that the frame-one form has maximal rank is invariant under all symmetries of the model, *i.e.*, under both diffeomorphisms and local $o(3,2)$ transformations. Without introduction of the compensator it was not clear how to introduce the non-degeneracy metric condition in a gauge-invariant way. Indeed, the condition that the naively defined frame one-form e^n is nondegenerate is not invariant under large (*i.e.*, not infinitesimal) $O(3,2)$ gauge transformations. Note that the local Lorentz group is simply the stability group of $V^A(x)$.

Finally, let me note that the MacDowell-Mansouri-Stelle-West action (9.32) admits a manifestly $o(d-1,2)$ -invariant generalization to any dimension $d \geq 4$

$$S \sim \int_{M^d} \epsilon_{A_1\dots A_{d+1}} V^{A_1}E^{A_2}\dots E^{A_{d-3}}R^{A_{d-2}\,A_{d-1}}R^{A_d\,A_{d+1}}. \tag{9.35}$$

[11]Recall, that spontaneous breakdown of a symmetry h means that the full theory is h symmetric but a particular vacuum solution is not. In the case in question, the breaking is spontaneous because any solution of (9.31) singles out a particular $V^A \neq 0$. As such, it breaks the $o(3,2)$-symmetry down to $o(3,1)$ which is the stability group of any fixed V^A obeying (9.31).

The difference is that for $d > 4$ the Gauss-Bonnet term is no longer topological and does contribute to the field equations. It can be shown, however, that it only affects nonlinear corrections and, despite containing higher derivatives, does not harm the phase structure of the theory.

Analogously, the MacDowell-Mansouri action admits a very simple generalization to supergravity. The latter is the gauge theory resulting from the localization of supersymmetry.

10 Clifford algebra and spinors

Since fermions are described by spinors in relativistic field theory, we consider some elements of the theory of spinors.

10.1 Clifford algebra

It is most convenient to introduce spinors as modules of the *Clifford algebra* $\mathcal{C}_n$ which is the associative algebra with unity I generated by the elements ϕ_n ($n = 0, \dots d-1$) that satisfy the relations

$$\{\phi_n\,,\phi_m\} = 2\eta_{nm}I\,, \qquad I\phi_n = \phi_n I = \phi_n\,, \qquad II = I \tag{10.1}$$

with $\{a, b\} = ab + ba$, and some symmetric tensor (*i.e.*, bilinear form) η_{nm} that in our analysis will be assumed to be non degenerate. In the case of real Clifford algebras one has to distinguish between different signatures of η_{nm}. To make things shorter, in the consideration below we consider the Clifford algebra $\mathcal{C}_d(\mathbb{C})$ over the complex numbers in which case the signature of η_{nm} does not matter because it can be changed by multiplying the generating elements ϕ_n by the appropriate factors of i.

Recall that an *algebra* is a vector space endowed with a product law $ab \in A$, for any $a \in A$ and $b \in A$, that obeys

$$(\lambda a + \mu b)c = \lambda ab + \mu bc\,, \qquad a(\lambda b + \mu c) = \lambda ab + \mu ac\,, \tag{10.2}$$

where λ and μ are any numbers (*i.e.*, elements of the field over which the algebra is defined; in this section we mostly focus on algebras over the field $\mathbb{C}$ of complex numbers). Note that a Lie algebra is of course an algebra. In most cases, it is, however, not associative.

Since any symmetrization of generating elements ϕ_n, ϕ_m in a product of elements of the Clifford algebra gives the unit element, the Clifford algebra is spanned by the elements

$$\phi_{n_1\dots n_p} := \phi_{[n_1}\dots\phi_{n_p]}\,, \tag{10.3}$$

which are totally antisymmetric in the indices $n_1, \dots, n_p$. Clearly, such tensors are nonzero only for $p \leq d$.

The element with $p = 0$ is identified with the unit element I.

Since the number N_p of independent components of an antisymmetric tensor $A_{[n_1 \dots n_p]}$ in d dimensions is

$$N_p = \frac{d!}{p!(d-p)!}\,, \tag{10.4}$$

the dimension of $\mathcal{C}_d$ is

$$\dim \mathcal{C}_d = \sum_{p=0}^{d} N_p = 2^d\,. \tag{10.5}$$

Clearly, the Dirac γ–matrices generate a representation of the Clifford algebra. The relevance of the Clifford algebra to relativistic physics relies in the first place on the fact that the elements

$$M_{nm} = \frac{i}{2}\phi_{nm} = \frac{i}{4}[\phi_n\,, \phi_m] \tag{10.6}$$

fulfill the $o(d)$-commutation relations with respect to commutator (*cf.* (2.27)). As a result, any $\mathcal{C}_d$-module gives an $o(d)$-module, where the generators of $o(d)$ are represented by M_{nm} (10.6). More precisely, the most relevant for applications in relativistic physics is the real Clifford algebra with metric η_{nm} of Minkowski signature, in which case M_{nm} are generators of Lorentz transformations.

Clifford algebras with odd and even d turn out to be essentially different. Indeed, consider the maximal element

$$\Gamma := i^{\frac{d(d-1)}{2}} \sqrt{det|\eta|}\phi_0\phi_1 \dots \phi_{d-1} \tag{10.7}$$

(*cf.* (3.83)). From (10.1) it follows that

$$\Gamma\phi_n = (-1)^{d-1}\phi_n\Gamma \tag{10.8}$$

and

$$\Gamma^2 = I\,. \tag{10.9}$$

The property (10.8) means that for odd d, the element Γ commutes with the ϕ_n and hence with any other element of the Clifford algebra. In other words, Γ is a *central element* of the Clifford algebra for odd d. The property (10.9) then allows us to introduce the projection operators

$$P^{\pm} := \frac{1}{2}(I \pm \Gamma)\ : \qquad P^{\pm}P^{\pm} = P^{\pm}\,, \qquad P^{\pm}P^{\mp} = 0\,, \qquad P^{+} + P^{-} = I\,, \tag{10.10}$$

which in turn allow us to decompose the Clifford algebra for odd d into a direct sum of two subalgebras

$$\mathcal{C}_d = \mathcal{C}_d^{+} \oplus \mathcal{C}_d^{-}\,, \qquad \mathcal{C}_d^{\pm} := P^{\pm}\mathcal{C}_d\,. \tag{10.11}$$

The orthogonality of the projectors implies that $\phi^+\phi^- = 0$ for any $\phi^+ \in \mathcal{C}_d^+$ and $\phi^- \in \mathcal{C}_d^-$. This means that $\mathcal{C}_d$ has a block-diagonal structure with two blocks associated with $\mathcal{C}_d^+$ and $\mathcal{C}_d^-$. **Narisovat'**

The question is what the algebras $\mathcal{C}_d^{\pm}$ are. The answer is that they are both equivalent to the Clifford algebra of one dimension lower, *i.e.*,

$$\mathcal{C}_d^{\pm} \cong \mathcal{C}_{d-1}\,, \qquad d \quad \text{odd}\,. \tag{10.12}$$

This result reduces the problem of the analysis of Clifford algebras in odd dimensions to the analysis of Clifford algebras in even dimensions.

We leave to the reader to check the following simple facts:
$\mathcal{C}_0(\mathbb{C})$ is the algebra of numbers, *i.e.*, $\mathbb{C}$;
$\mathcal{C}_1(\mathbb{C}) \cong \mathbb{C} \oplus \mathbb{C}$;
$\mathcal{C}_2(\mathbb{C})$ for $\eta_{nm} = \delta_{nm}$ is realized by the Pauli matrices $\phi_n = \sigma_n$, $n = 1, 2$;
$\mathcal{C}_3(\mathbb{C}) \cong \mathcal{C}_2(\mathbb{C}) \oplus \mathcal{C}_2(\mathbb{C})$;
as well as (10.12).

10.2 Spinors

In this section we analyze $\mathcal{C}_d$-modules. More precisely, for an associative algebra A one should distinguish between *left* and *right* modules in which elements of A act from the left

$$(ab)(v_{\mathrm{L}}) = a(b(v_{\mathrm{L}}))\,, \tag{10.13}$$

or the right

$$(ab)(v_{\mathrm{R}}) = b(a(v_{\mathrm{R}})) \tag{10.14}$$

on $v \in V$. An example of this is the action of the associative algebra of matrices $A_i{}^j$ on the vector spaces of columns $v_{\mathrm{L}\,i}$

$$A(v_{\mathrm{L}})_i = A_i{}^j v_{\mathrm{L}\,j} \tag{10.15}$$

and rows v_{R}^i

$$A(v_{\mathrm{R}})^j = v_{\mathrm{R}}^i A_i{}^j\,. \tag{10.16}$$

Since, due to the antisymmetry of the Lie bracket, for Lie algebras every left module is equivalent to some right module, and vice versa, both left and right $\mathcal{C}_d$-modules generate $o(d)$-modules.

To analyze $\mathcal{C}_d$-modules for even d we choose new generating elements ψ_a^- and ψ^{+a} with $a = 1, \ldots, d/2$ that obey the commutation relations

$$\{\psi_a^-\,, \psi^{+b}\} = \delta_a^b\,, \qquad \{\psi_a^-\,, \psi_b^-\} = 0\,, \qquad \{\psi^{+a}\,, \psi^{+b}\} = 0\,. \tag{10.17}$$

In this basis the defining relations of the Clifford algebra take the form of fermionic oscillator algebra familiar from quantum mechanics. This immediately allows us to construct a module of the Clifford algebra $\mathcal{C}_d$ in the form of a *Fock space*. Namely, introduce a Fock vacuum $|0\rangle$ annihilated by the elements ψ_a^-

$$\psi_a^- |0\rangle = 0\,. \tag{10.18}$$

Then a general element of the left module F_d of the Clifford algebra $\mathcal{C}_d$ for even d has the form

$$|v\rangle \in F_d : \qquad |v\rangle = \sum_{p=0}^{d/2} v_{a_1 \dots a_p} \psi^{+a_1} \dots \psi^{+a_p} |0\rangle \,, \tag{10.19}$$

where $v_{a_1 \dots a_p}$ are arbitrary totally antisymmetric coefficients. Indeed, the action of any element of the Clifford algebra on $v \in F_d$ gives some other element from F_d.

Elements of F_d are called *spinors*. Computation analogous to that for the Clifford algebra itself shows that the dimension of the space F_d is

$$\dim\, F_d = 2^{d/2} \,. \tag{10.20}$$

Another important fact is that, as a consequence of the identity $(1-1)^{d/2} = 0$, the subspaces F_d^E and F_d^O that consists of the elements (10.19) with odd and even p, respectively, have equal dimensions, *i.e.*,

$$\dim F_d^E = \dim F_d^O = 2^{d/2-1} \,. \tag{10.21}$$

F_d is an irreducible left $\mathcal{C}_d$–module. This is easy to prove by showing that, by acting with an appropriate element of the Clifford algebra, it is possible to map any element $|v\rangle \in F_d$ to any other $|v'\rangle \in F_d$.

Also it is not difficult to show that every irreducible left $\mathcal{C}_d$–module is isomorphic to F_d using that every left module contains a vacuum state annihilated by all ψ_a^-.

The right Fock module F_d^* is defined analogously

$$\langle v| \in F_d^* : \qquad \langle v| = \sum_{p=0}^{\frac{d}{2}} \langle 0| v^{a_1 \dots a_p} \psi^-_{a_1} \dots \psi^-_{a_p} \,, \qquad \langle 0|\psi^{+a} = 0 \,. \tag{10.22}$$

These results admit the following simple interpretation. The Clifford algebra $\mathcal{C}_d$ with even d is simply the algebra $Mat_{2^{d/2}}$ of $2^{d/2} \times 2^{d/2}$ matrices. Left and right modules of $\mathcal{C}_d$ are the spaces of columns and rows of this matrix algebra. The spinor index $\hat\alpha$ which we used before simply enumerates all elements $\chi_{\hat\alpha} \in F_d$ *i.e.*, $\hat\alpha = 1, \dots 2^{d/2}$. Elements of F_d^* are usually denoted as $\bar\chi^{\hat\alpha}$. General elements of $\mathcal{C}_d$ span all matrices $A_{\hat\alpha}{}^{\hat\beta}$. In particular ϕ_n are realized as γ–matrices $\gamma_{n\hat\alpha}{}^{\hat\beta}$. Note that, as anticipated, in four dimensions spinor indices take $2^2 = 4$ values. The Fock realization of spinor space gives rise to the concrete realization of the γ–matrices.

Forming a $\mathcal{C}_d$-module, F_d also forms an $o(d)$-module with $o(d)$-generators (10.6). This $o(d)$-module is however reducible. Indeed, though Γ anticommutes with ψ_n at even d, it commutes with any even function of ϕ_n and, in particular, with M_{nm}. Hence

$$F_d^\pm := P^\pm F_d \tag{10.23}$$

form invariant subspaces with respect to the action of $o(d)$.[12]

[12] Alternatively one can observe that the subspaces F_d^{E} and F_d^{O} of F_d that are spanned by the elements (10.19) with even and odd p, respectively, form invariant subspaces under the action of $o(d)$. It is not difficult to see that the decompositions $F_d = F_d^+ \oplus F_d^-$ and $F_d = F_d^O \oplus F_d^E$ are equivalent, *i.e.*, F_d^+ is isomorphic to either F_d^O or F_d^E while F_d^- to the other.

Elements of F_d^+ and F_d^- are called *chiral* (*left* and *right*) spinors. The dimension of $F_d^\pm$ is

$$\dim F_d^\pm = 2^{\frac{d}{2}-1} \,. \tag{10.24}$$

In particular, chiral spinors in four-dimensional space-time have two components.

Let us stress the following important difference between spinors and tensors. The dimension of any tensor-representation of the Lorentz algebra grows polynomially with the dimension of space-time d. In particular, the number of components of a general tensor $A_{n_1,n_2,\dots,n_k}$ in d dimensions is d^k.

On the other hand the dimension of the spinor representation grows exponentially with d. This makes it difficult to have higher-dimensional theories with balanced numbers of bosons and fermions as required by supersymmetry.

10.3 Charge-conjugation matrix

An important property of the Clifford algebra is that it admits a so-called *charge-conjugation matrix* $C_{\hat\alpha\hat\beta}$. The latter can be defined as providing a map between left and right modules of the Clifford algebra $\mathcal{C}_d$. This means that if $\tilde\chi^{\hat\alpha}$ forms a right module then

$$\chi'_{\hat\alpha} = C_{\hat\alpha\hat\beta}\tilde\chi^{\hat\beta} \tag{10.25}$$

forms a left module. The inverse relation is

$$\tilde\chi^{\hat\beta} = C^{\hat\beta\hat\alpha}\chi'_{\hat\alpha} \,, \tag{10.26}$$

where

$$C^{\hat\beta\hat\gamma}C_{\hat\gamma\hat\alpha} = \delta^{\hat\beta}_{\hat\alpha} \,. \tag{10.27}$$

Note that the relation (10.25) provides the map which is used in the usual charge conjugation in field theory that relates Dirac fermion fields with their conjugates.

The fact that the charge-conjugation matrix exists is a consequence of the definition of the Clifford algebra along with the fact that it admits the unique irreducible left and right modules F_d and F_d^*. An additional condition on the charge-conjugation matrix is that the γ-matrices have to obey the relations [13]

$$C_{\hat\beta\hat\gamma}\gamma_{n\,\hat\alpha}{}^{\hat\gamma} = -C_{\hat\gamma\hat\alpha}\gamma_{n\,\hat\beta}{}^{\hat\gamma} \,. \tag{10.28}$$

[13] Let me give some hints how one can see this. For an associative algebra A with the product law ab the *opposite algebra* A' has the product law $a \circ b = ba$. Every left (right) A-module is equivalent to a right (left) A'-module. If A admits an *involutive anti-automorphism* ρ, *i.e.*, a linear map of A to itself such that $\rho(ab) = \rho(b)\rho(a)$ and $\rho^2 = I$, then A is isomorphic to A'. This implies that every left A-module is equivalent to some right A-module and vice versa. If the irreducible left and right modules are unique, as is the case for the (complex) Clifford algebra, the existence of ρ proves their equivalence. In the Clifford-algebra case ρ can be defined by either of the relations $\rho_\pm(\phi_n) = \pm\phi_n$ which leave invariant the defining relations (10.1). These two definitions lead to two charge-conjugation matrices obeying the relations $C^\pm_{\hat\beta\hat\gamma}\gamma_{n\,\hat\alpha}{}^{\hat\gamma} = \pm C^\pm_{\hat\gamma\hat\alpha}\gamma_{n\,\hat\beta}{}^{\hat\gamma}$. In the context of $4d$ supersymmetry it is convenient to work with $C_{\hat\alpha\hat\beta} := C^-_{\hat\alpha\hat\beta}$, which convention is taken in the main text.

We see that $C_{\hat{\alpha}\hat{\beta}}$ serves as a kind of metric (symmetric or antisymmetric, depending on the dimension d) allowing to raise and lower spinor indices. Let us use $C_{\hat{\beta}\hat{\gamma}}$ to lower the upper index of any element $A_{\hat{\alpha}}{}^{\hat{\beta}}$ of the Clifford algebra by setting

$$A_{\hat{\alpha}\hat{\beta}} := C_{\hat{\beta}\hat{\gamma}} A_{\hat{\alpha}}{}^{\hat{\gamma}} \,. \tag{10.29}$$

Then (10.28) implies

$$\gamma_{n\,\hat{\alpha}\hat{\beta}} = -\eta \gamma_{n\,\hat{\beta}\hat{\alpha}} \tag{10.30}$$

with

$$C_{\hat{\alpha}\hat{\beta}} = \eta C_{\hat{\beta}\hat{\alpha}} \,, \qquad \eta^2 = 1 \,. \tag{10.31}$$

More generally, every matrix $\gamma_{[n_1,\dots n_p]\,\hat{\alpha}\hat{\beta}}$, including $C_{\hat{\alpha}\hat{\beta}}$ at $p = 0$, has definite symmetry

$$\gamma_{[n_1,\dots,n_p]\,\hat{\alpha}\hat{\beta}} = \eta(p,d)\,\gamma_{[n_1,\dots,n_p]\,\hat{\beta}\hat{\alpha}} \,, \qquad (\eta(p,d))^2 = 1 \,. \tag{10.32}$$

Problem* 10.1. Find the coefficients $\eta(p,d)$ including $\eta(0,d) \sim \eta$.

This problem has to be worked out

10.4 Spinors in three and four dimensions

Properties of spinors in various dimensions are determined by the properties of the charge conjugation matrix and chiral spinors. The latter depend on the signature of the metric tensor, *i.e.*, elaboration of these properties requires a full-fledged analysis of real Clifford algebras, which is far more complicated than that of the complex case going beyond the scope of this review. We just mention that these properties have *mod* 8 Bott periodicity implying that spinors have similar properties in d and $d+8$ dimensions, referring for more detail e.g. to [4]. In these lectures we will be mostly interested in spinors in four- and three-dimensional Minkowski space-times. Here we summarize the main properties of Minkowski spinors in these dimensions.

In both of these cases elementary spinors $\chi_{\hat{\alpha}}$ turn out to be real (Majorana) in the mostly plus signature. The invariant form of the *Majorana condition* is

$$\overline{\psi}^{\hat{\alpha}} = i\, C^{\hat{\alpha}\hat{\beta}} \psi_{\hat{\beta}} \,, \tag{10.33}$$

where the factor of i is introduced to keep $C^{\hat{\alpha}\hat{\beta}}$ real in the mostly minus signature. The charge-conjugation matrix $C^{\hat{\alpha}\hat{\beta}}$ is antisymmetric

$$C^{\hat{\alpha}\hat{\beta}} = -C^{\hat{\beta}\hat{\alpha}} \,. \tag{10.34}$$

The corresponding γ-matrices are pure imaginary in the mostly minus signature and symmetric in accordance with (10.30). The reader can check that matrices $\gamma_{[nm]\,\hat{\alpha}\hat{\beta}}$ are also symmetric

$$\gamma_{[nm]\,\hat{\alpha}\hat{\beta}} = \gamma_{[nm]\,\hat{\beta}\hat{\alpha}} \,. \tag{10.35}$$

In four dimensions $\gamma_{[nmk]}$ and $\gamma_{[nmkl]}$ are antisymmetric.

In the three-dimensional case spinor indices, which take two values, will be denoted by $\alpha, \beta, \ldots$ without hats. The charge conjugation matrix is $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$, $\epsilon_{12} = 1$. The matrices $\gamma_{n\alpha\beta}$ are given by symmetric 2×2 matrices,

$$\gamma_{n\alpha\beta} = \big(\delta_{\alpha\beta}\,, \sigma_{1\,\alpha\beta}\,, \sigma_{3\,\alpha\beta}\big)\,. \tag{10.36}$$

One can see that as a consequence of the projection to one of the irreducible components of the three-dimensional Clifford algebra, the matrices $\gamma_{[nm]\,\alpha\beta}$ are not independent, being related to $\gamma_{n\,\alpha\beta}$ by the epsilon symbol ϵ_{nmk}.

In the four-dimensional case one can introduce chiral or *Weyl* spinors $\chi^{\pm}_{\hat{\alpha}}$. However, they are not real because the matrix Γ (10.7) is pure imaginary in $d = 4$ with Lorentz signature. In other words, in $d = 4$ a spinor can be either Majorana or Weyl, but not Majorana-Weyl. (Note that Majorana-Weyl spinors do exist at $d = 2, 10$ [4] which is of primary importance in string theory.) Note that a Dirac spinor is equivalent to a pair of Majorana spinors. Alternatively it is equivalent to a pair of Weyl spinors that are not related by complex conjugation.

Since $4d$ Weyl spinors carry two complex components instead of the four real for Majorana spinors, it is convenient to use so-called *two-component* notations for $4d$ spinors, replacing the Majorana spinor $\chi_{\hat{\alpha}}$ by a pair of complex conjugate chiral spinors χ_α and $\bar{\chi}_{\dot{\alpha}}$ with $\alpha, \beta = 1, 2$, $\dot{\alpha}, \dot{\beta} = 1, 2$. In these notations, the γ-matrices are represented by a set of 2×2 Hermitian matrices $\sigma^n_{\alpha\dot{\alpha}}$

$$\sigma^n{}_{\alpha\dot{\alpha}} := \Big(\delta_{\alpha\dot{\alpha}}\,, \sigma^{\mathbf{i}}_{\alpha\dot{\alpha}}\Big)\,, \tag{10.37}$$

where $\mathbf{i} = 1, 2, 3$ and $\sigma^{\mathbf{i}}$ are the usual Pauli matrices. In these terms the charge conjugation matrix has a block-diagonal form and can be chosen in the form

$$C_{\alpha\beta} = \epsilon_{\alpha\beta}\,, \qquad C_{\dot{\alpha}\dot{\beta}} = \epsilon_{\dot{\alpha}\dot{\beta}}\,, \tag{10.38}$$

where

$$\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}\,, \qquad \epsilon_{\dot{\alpha}\dot{\beta}} = -\epsilon_{\dot{\beta}\dot{\alpha}}\,, \qquad \epsilon_{12} = 1\,. \tag{10.39}$$

We will use the following conventions, which slightly differ from those used in some text books:

$$A^\alpha = \epsilon^{\alpha\beta} A_\beta\,, \qquad A_\alpha = A^\beta \epsilon_{\beta\alpha}\,, \qquad \epsilon_{\gamma\beta}\epsilon^{\gamma\alpha} = \delta^\alpha_\beta\,. \tag{10.40}$$

Here, unlike for $C^{\hat{\alpha}\hat{\beta}}$, $\epsilon^{\alpha\beta}$ is defined not as the inverse of $\epsilon_{\alpha\beta}$ but as $\epsilon_{\alpha\beta}$ with indices raised by the rule (10.40), which effectively means that $\epsilon^{\alpha\beta} = -(\epsilon_{\alpha\beta})^{-1}$.

Because the spinor indices $\alpha, \beta, \ldots$ and $\dot{\alpha}, \dot{\beta}, \ldots$ take just two values their antisymmetrization is equivalent to the contraction

$$A_\alpha B_\beta - A_\beta B_\alpha = \epsilon_{\alpha\beta} A_\gamma B^\gamma\,, \qquad A_{\dot{\alpha}} B_{\dot{\beta}} - A_{\dot{\beta}} B_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}} A_{\dot{\gamma}} B^{\dot{\gamma}}\,. \tag{10.41}$$

The following simple identities which can be checked directly are often used

$$\sigma^n{}_{\alpha\dot{\alpha}} \sigma^{m\alpha\dot{\alpha}} = 2\eta^{nm}\,, \qquad \sigma^n{}_{\alpha\dot{\alpha}} \sigma_n{}^{\beta\dot{\beta}} = 2\delta^\beta_\alpha \delta^{\dot{\beta}}_{\dot{\alpha}}\,. \tag{10.42}$$

$$\sigma^n{}_{\alpha\dot{\alpha}} \sigma^{m\beta\dot{\alpha}} + (n \leftrightarrow m) = 2\delta^\beta_\alpha \eta^{nm}\,, \qquad \sigma^n{}_{\alpha\dot{\alpha}} \sigma^{m\alpha\dot{\beta}} + (n \leftrightarrow m) = 2\delta^{\dot{\beta}}_{\dot{\alpha}} \eta^{nm}\,. \tag{10.43}$$

The relation between σ-matrices and γ-matrices is

$$\gamma_{n\alpha}{}^{\beta} = 0\,, \qquad \gamma_{n\alpha}{}^{\dot\beta} = i\sigma_{n\alpha}{}^{\dot\beta}\,, \qquad \gamma_{n\dot\alpha}{}^{\dot\beta} = 0\,, \qquad \gamma_{n\dot\alpha}{}^{\beta} = i\sigma_{n}{}^{\beta}{}_{\dot\alpha}\,. \tag{10.44}$$

It is convenient to introduce the mutually conjugate matrices

$$\sigma^{nm}_{\alpha\beta} := \sigma^{[n}_{\alpha\dot\alpha}\sigma^{m]\dot\alpha}_{\beta}\,, \qquad \bar\sigma^{nm}_{\dot\alpha\dot\beta} := \sigma^{[n}_{\alpha\dot\alpha}\sigma^{m]\alpha}{}_{\dot\beta}\,, \tag{10.45}$$

which are antisymmetric in the indices n, m and symmetric in the respective pairs of indices α, β and $\dot\alpha, \dot\beta$.

In these terms (10.43) and (10.45) combine to

$$\sigma^{n}{}_{\alpha\dot\alpha}\sigma^{m\beta\dot\alpha} = \delta^{\beta}_{\alpha}\eta^{nm} + \sigma^{nm}{}_{\alpha}{}^{\beta}\,, \qquad \sigma^{n}{}_{\alpha\dot\alpha}\sigma^{m\alpha\dot\beta} = \delta^{\dot\beta}_{\dot\alpha}\eta^{nm} + \bar\sigma^{nm}{}_{\dot\alpha}{}^{\dot\beta}\,. \tag{10.46}$$

From (10.44) it follows that the Lorentz generators (10.6) take the form

$$M_{nm\alpha}{}^{\beta} = \frac{i}{2}\sigma_{nm\alpha}{}^{\beta}\,, \qquad M_{nm\alpha}{}^{\dot\beta} = 0\,, \qquad M_{nm\dot\alpha}{}^{\dot\beta} = \frac{i}{2}\sigma_{nm\dot\alpha}{}^{\dot\beta}\,, \qquad M_{nm\dot\alpha}{}^{\beta} = 0\,. \tag{10.47}$$

The formalism of two-component spinors is most convenient for practical computations. It will be used for the analysis of both supersymmetric models and higher-spin theories in four-dimensions.

11 Supersymmetry

The path towards more symmetric relativistic theories lies in the extension of the Poincaré algebra or its $(A)dS$ deformation to a larger algebra. A particularly important extension first suggested in 1971 by Golfand and Lichtman from Lebedev Institute in Moscow is called *supersymmetry*. *A priori*, there exist infinitely many extensions of the Poincaré algebra or its $(A)dS$ deformation. Most of them however do not lead to a consistent nontrivial quantum field theory. By 'consistent' I mean that it is unitary while by 'nontrivial' that it is nonlinear (interacting).

After considering concrete examples of supersymmetric models we will analyze the general pattern of supermultiplets.

11.1 Super-Poincaré algebra

The *N-extended supersymmetry* algebra or *N-extended super-Poincaré* algebra, which for brevity we denote by $iso_N(d-1,1)$, is the extension of the Poincaré algebra by *supergenerators* $Q^i_{\hat\alpha}$ where $\hat\alpha$ is a spinor index while i is an internal index which takes some N values and is not affected by the Poincaré transformations. We do not specify a type of a spinor index carried by the charge $Q^i_{\hat\alpha}$ (*i.e.*, whether it is Majorana and/or Weyl) since this depends on the space-time dimension. For a given dimension it is usually restricted as much as possible.

The algebra $iso_N(d-1,1)$ is spanned by the generators P_n, L_{nm} and $Q^i_{\hat\alpha}$. It contains the Poincaré algebra as a subalgebra, which means that P_n and L_{nm} still obey the commutation relations (2.21)–(2.23) but with an extra overall sign on the right-hand side, *cf.* the remark at the end of Section 7. Moreover, the Poincaré algebra can be interpreted as $iso_0(d-1,1)$. The relations that involve supergenerators are

$$[P_n\,, Q^i_{\hat\alpha}] = 0\,, \tag{11.1}$$

$$[L_{nm}\,, Q^i_{\hat\alpha}] = -M_{nm\hat\alpha}{}^{\hat\beta} Q^i_{\hat\beta}\,, \qquad M_{nm} := \frac{i}{4}[\gamma_n\,, \gamma_m]\,, \tag{11.2}$$

$$\{Q^i_{\hat\alpha}\,, \overline{Q}^{j\hat\beta}\} = 2\delta^{ij} P_n \gamma^n{}_{\hat\alpha}{}^{\hat\beta}\,. \tag{11.3}$$

The novelty compared to the case of Lie algebras is that the $Q^i_{\hat\alpha}$ obey anticommutation rather than commutation relations. This is consistent with the spin-statistics theorem stating that fermions should obey anticommutation relations. Algebras that consist of even (in our case P_n and L_{nm}) and odd (in our case $Q^i_{\hat\alpha}$) elements, which obey commutation relations in the even-even and even-odd sectors and anticommutation relations in the odd-odd sector, are called *Lie superalgebras.*

More formally the definition of a *Lie superalgebra* is as follows. First of all a Lie superalgebra s is a $\mathbb{Z}_2$ graded vector space, which simply means that any element a of s can be (uniquely) represented as a sum $a = a_0 + a_1$ of some *even* element a_0 and an *odd* element a_1. Elements which are either even or odd are called *homogeneous* and are said to have *parity* $\pi(a_i) = i$. The Lie bracket $[a\,, b]_\pm$ on s is such that, for homogeneous elements, *

$$\pi([a\,, b]_\pm) = \pi(a) + \pi(b) \quad mod\ 2\,, \tag{11.4}$$

which means that s is $\mathbb{Z}_2$-graded. The generalization of the Lie algebra axioms is as follows. The Lie bracket has the symmetry properties

$$[a\,, b]_\pm = -(-1)^{\pi(a)\pi(b)}[b\,, a]_\pm\,, \tag{11.5}$$

i.e., it is a antisymmetric when $\pi(a) = 0$ or $\pi(b) = 0$, and symmetric when $\pi(a) = \pi(b) = 1$. The Jacobi identity now also comes with signs:

$$(-1)^{\pi(a)\pi(c)}[a\,, [b\,, c]_\pm]_\pm + (a \to b \to c \to a) + (a \to c \to b \to a) = 0\,. \tag{11.6}$$

Lie superalgebras can be related to *Lie supergroups* which are *supermanifolds.* In practice this means that some of the coordinates of a supermanifold are anticommuting rather than commuting. Usually, supermanifolds are denoted by $M^{p|q}$ where p and q denote, respectively, the number of commuting and anticommuting coordinates. Anticommuting coordinates have properties analogous to the variables $\xi^{\underline{n}}$ in the exterior-algebra formalism, *i.e.*, functions of anticommuting parameters form a Grassmann algebra (see Section 7.2.1). Correspondingly, infinitesimal parameters $\epsilon^{\hat\alpha}$ of transformations associated with the supergenerators are anticommuting. This means that somehow we cannot visualize a real geometric effect of such transformations like we can for a rotation of usual space. Nevertheless, supersymmetric models exhibit many remarkable features that can be tested experimentally.

As a result, an infinitesimal supersymmetry transformation is described by the supergroup element

$$g \cong e - i\epsilon^n P_n - \frac{i}{2}\epsilon^{nm} L_{nm} + \epsilon_i^{\hat{\alpha}} Q_{\hat{\alpha}}^i \,, \tag{11.7}$$

where the *supersymmetry parameters* $\epsilon_i^{\hat{\alpha}}$ are x-independent anticommutative Majorana spinors $\epsilon_i^{\hat{\alpha}}$. We use the convention that

$$\epsilon_i^{\hat{\alpha}} Q_{\hat{\beta}}^j = -Q_{\hat{\beta}}^j \epsilon_i^{\hat{\alpha}} \,. \tag{11.8}$$

As was shown for the Poincaré algebra in Section 3, the form of the super-Poincaré algebra gives a lot of information on the properties of possible supersymmetric models. In particular, the field pattern of well-defined supersymmetric models is determined by unitary modules of the SUSY algebra. The appropriate sets of fields are called *supermultiplets*. Since the Poincaré algebra is a subalgebra of the SUSY algebra, supermultiplets in turn consist of the usual relativistic fields (particles). The pattern of supermultiplets is heavily restricted by the SUSY algebra. In particular, it is not difficult to make sure that any $N \geq 1$ supermultiplet contains an equal number of bosons and fermions.

Let us start with some general properties. First of all we observe that $C_2 = P_n P^n$ is still a Casimir operator of $iso_N(d-1,1)$. This has the immediate consequence that all particles from the same supermultiplet have the same mass. We shall see that, like in the case of the Poincaré algebra, massless and massive supermultiplets have different properties.

As usual for fermions, $\overline{Q} = Q^\dagger \gamma^0$. This has the consequence that

$$\{Q_{\hat{\alpha}}^i \,, (Q_{\hat{\beta}}^j)^\dagger\} = 2\delta^{ij} P_n (\gamma^n \gamma_0)_{\hat{\alpha}}{}^{\hat{\beta}} \,. \tag{11.9}$$

Since

$$tr(\gamma^n \gamma_m) = \delta_m^n tr(I) \tag{11.10}$$

it follows that P_0 is positive definite in consistent supersymmetric models. Indeed, assuming unitarity, *i.e.*, that the supersymmetric quantum field theory in question is formulated in a positive-definite Hilbert space, for any state $|\psi\rangle$ we obtain

$$\langle\psi|P_0|\psi\rangle = 8N \sum_{i,\hat{\alpha}} \left(\left|Q_{\hat{\alpha}}^i|\psi\rangle\right|^2 + \left|(Q_{\hat{\alpha}}^i)^\dagger|\psi\rangle\right|^2 \right) . \tag{11.11}$$

Moreover, any supersymmetric state $|\psi\rangle$ annihilated by $Q_{\hat{\alpha}}^i$ (and $(Q_{\hat{\alpha}}^i)^\dagger$, which is automatic for Majorana spinors) should have zero energy as well as momentum

$$Q_{\hat{\alpha}}^i|\psi\rangle = 0 \qquad \Longrightarrow \qquad P_n|\psi\rangle = 0 \,. \tag{11.12}$$

On the other hand, if the state $|\psi\rangle$ is not supersymmetric its energy is strictly positive

$$Q_{\hat{\alpha}}^i|\psi\rangle \neq 0 \qquad \Longrightarrow \qquad E_\psi > 0 \,. \tag{11.13}$$

This implies that in the case of spontaneously broken supersymmetry, which is the case when the vacuum state is not supersymmetric, the energy of the vacuum should be strictly positive. This simple observation gives a criterion distinguishing between the regimes with broken and unbroken supersymmetry. In addition, it suggests that supersymmetry can only be realized in models with positive-semidefinite energy. In fact, because of this property, supersymmetry distinguishes between De Sitter and anti-De Sitter geometries. The latter can be supersymmetrized, while the former cannot.

Another relevant point is that the cancelation of vacuum-energy contributions in supersymmetric models conform the property that the energy should be zero in the supersymmetric vacuum. In practice this cancelation follows from the property that supersymmetric models contain equal numbers of bosonic and fermionic degrees of freedom. Indeed, boson and fermions contribute to the vacuum energy with opposite signs because

$$\{b_+\,, b_-\} = 2b_+b_- + 1\,, \qquad [f_+\,, f_-] = 2f_+f_- - 1 \tag{11.14}$$

for canonical bosonic and fermionic oscillators which obey

$$[b_-\,, b_+] = 1\,, \qquad \{f_-\,, f_+\} = 1\,, \qquad b_-|0\rangle = f_-|0\rangle = 0\,. \tag{11.15}$$

Since fields in QFT contain infinitely many oscillators due to the dependence of quantum fields on momenta, the total vacuum energy computed this way for any bosonic and fermionic field is infinite. However, in supersymmetric models these contributions cancel even without the substraction of these infinities by normal ordering. This property gives the simplest example of cancelations of divergencies in supersymmetric models of QFT.

In four dimensions the supercharges $Q^i_{\hat\alpha}$ are Majorana, obeying the reality condition (10.33). This allows us to rewrite the anticommutation relation (11.3) in the form

$$\{Q^j_{\hat\alpha}\,, Q^k_{\hat\beta}\} = 2i\delta^{jk} P_n \gamma^n{}_{\hat\alpha\hat\beta}\,. \tag{11.16}$$

Since the left-hand side is symmetric in the spinor indices, for this relation it is important that $\gamma^n{}_{\hat\alpha\hat\beta}$ is symmetric in the spinor indices (and pure imaginary). This is why γ-matrices were demanded to have the property (10.30).

The four-dimensional supersymmetry algebra $iso_N(3,1)$ is conveniently described in terms of two-component spinors. In this language, the mutually conjugate (Weyl) supergenerators Q^i_α and $\bar Q^j_{\dot\alpha}$ obey the relations

$$\{Q^i_\alpha\,, \bar Q^j_{\dot\beta}\} = 2\delta^{ij} P_n \sigma^n_{\alpha\dot\beta}\,, \qquad \{Q^i_\alpha\,, Q^j_\beta\} = 0\,, \qquad \{\bar Q^i_{\dot\alpha}\,, \bar Q^j_{\dot\beta}\} = 0\,. \tag{11.17}$$

Recall that $\sigma^0_{\alpha\dot\beta}$ is positive definite.

The detailed structure of unitary modules of the SUSY algebra can be elaborated using the same Wigner method as for the Poincaré algebra. Unfortunately, due to the lack of space this will not be explained in this paper. Instead, we consider a simplest example of a supermultiplet.

11.2 Wess-Zumino model

The simplest $N = 1$ supersymmetric system in the four-dimensional Minkowski space-time called the Wess-Zumino model contains a scalar $\mathcal{S}(x)$, a pseudoscalar $\mathcal{P}(x)$ and a Majorana spinor $\psi_{\hat{\alpha}}(x)$.[14]

The free Wess-Zumino action is

$$S^{WZ} = \frac{1}{2}\int d^4x\Big(\partial_n\mathcal{S}(x)\partial^n\mathcal{S}(x) - m^2\mathcal{S}^2(x) + \partial_n\mathcal{P}(x)\partial^n\mathcal{P}(x) - m^2\mathcal{P}^2(x)$$
$$+\psi_{\hat{\alpha}}\big(\gamma^{n\hat{\alpha}\hat{\beta}}\partial_n - imC^{\hat{\alpha}\hat{\beta}}\big)\psi_{\hat{\beta}}\Big)\,, \tag{11.18}$$

where, in agreement with the general property of the super-Poincaré algebra, all fields have equal masses. Note that, since we have chosen the mostly-minus signature we use the slightly unusual convention that fermions are real while the γ-matrices are purely imaginary.

As usual, fields of half-integer spins are anticommuting

$$\psi^{\hat{\alpha}}(x)\psi^{\hat{\beta}}(x) = -\psi^{\hat{\beta}}(x)\psi^{\hat{\alpha}}(x)\,. \tag{11.19}$$

This agrees with the fact that the charge conjugation matrix $C_{\hat{\alpha}\hat{\beta}}$ is antisymmetric while $\gamma^n{}_{\hat{\alpha}\hat{\beta}}$ is symmetric in the spinor indices $\hat{\alpha}, \hat{\beta}$. Indeed, otherwise the mass term

$$m\psi_{\hat{\alpha}}\psi_{\hat{\beta}}C^{\hat{\alpha}\hat{\beta}} \tag{11.20}$$

would be zero and the kinetic term

$$\psi_{\hat{\alpha}}\gamma^{n\hat{\alpha}\hat{\beta}}\partial_n\psi_{\hat{\beta}} \tag{11.21}$$

a total derivative.

Let us check if the action (11.18) is invariant under a transformation with the symmetry parameter being an x-independent anticommuting Majorana spinor $\epsilon^{\hat{\alpha}}$. To simplify the analysis let us consider the massless case $m = 0$.

Somewhat surprisingly there exist two such transformations $\delta_{\rm s}$ and $\delta_{\rm p}$ with independent parameters $\epsilon_{\rm s}^{\hat{\alpha}}$ and $\epsilon_{\rm p}^{\hat{\alpha}}$, respectively,

$$\delta_{\rm s}\mathcal{S} = i\epsilon_{\rm s}^{\hat{\alpha}}\psi_{\hat{\alpha}}\,,\qquad \delta_{\rm s}\mathcal{P} = 0\,,\qquad \delta_{\rm s}\psi_{\hat{\alpha}} = i\gamma^n_{\hat{\alpha}\hat{\beta}}\epsilon_{\rm s}^{\hat{\beta}}\partial_n\mathcal{S}\,, \tag{11.22}$$

$$\delta_{\rm p}\mathcal{S} = 0\,,\qquad \delta_{\rm p}\mathcal{P} = i\epsilon_{\rm p}^{\hat{\alpha}}\Gamma_{\hat{\alpha}}{}^{\hat{\beta}}\psi_{\hat{\beta}}\,,\qquad \delta_{\rm p}\psi_{\hat{\alpha}} = -i(\gamma^n\Gamma)_{\hat{\alpha}\hat{\beta}}\epsilon_{\rm p}^{\hat{\beta}}\partial_n\mathcal{P} \tag{11.23}$$

with Γ from (10.7). Note that the appearance of Γ, which is parity odd, is consistent with $\mathcal{P}$ being a pseudoscalar. Also note that both $\delta_{\rm s}$ and $\delta_{\rm p}$ transformations mix bosons and fermions.

Indeed, the variation of the action under the transformation with the parameter $\epsilon_{\rm s}$ is

$$\delta_{\rm s}S = i\int d^4x\Big(\partial_n\mathcal{S}(x)\partial^n\epsilon_{\rm s}^{\hat{\alpha}}\psi_{\hat{\alpha}} - \partial_n\mathcal{S}\,\epsilon_{\rm s}^{\hat{\alpha}}(\gamma^n\gamma^m)_{\hat{\alpha}}{}^{\hat{\beta}}\partial_m\psi_{\hat{\beta}}\Big)\,. \tag{11.24}$$

[14]Recall that a scalar $\mathcal{S}$ and a pseudoscalar $\mathcal{P}$ differ by their transformation law under a spatial reflection P, which is $(P\mathcal{S})(x) = \mathcal{S}(P(x))$ and $(P\mathcal{P})(x) = -\mathcal{P}(P(x))$ respectively.

To prove that this is zero we use that

$$\gamma^n\gamma^m = \gamma^{[n}\gamma^{m]} + \eta^{nm} I\,. \tag{11.25}$$

As a result,

$$\delta_{\rm s} S = -i\int d^4x\,\partial_n\mathcal{S}\,\epsilon_{\rm s}^{\hat\alpha}\gamma^{[nm]}{}_{\hat\alpha}{}^{\hat\beta}\partial_m\psi_{\hat\beta} = -i\int d^4x\partial_n\Big(\mathcal{S}\,\epsilon_{\rm s}^{\hat\alpha}\gamma^{[nm]}{}_{\hat\alpha}{}^{\hat\beta}\partial_m\psi_{\hat\beta}\Big) = 0\,. \tag{11.26}$$

Analogously one can check that the action (11.18) with $m = 0$ is invariant under the $\epsilon_{\rm p}$ transformation.

There are at least two surprising outputs of this analysis. One is that we found two transformations instead of one. Another one is that we actually did not use in this analysis the fact that our space-time is four-dimensional except that there are Majorana spinors in $4d$. This is surprising in many respects. In particular, in higher dimensions, like for example in $d = 4+8n$, the number of components of the spinor field $\psi_{\hat\alpha}$ increases as 2^{4n+2} while the number of components of the (pseudo)scalar remains unchanged. So, the question arises: what are the symmetries (11.22) and (11.23) and where is the supersymmetry $iso_N(d-1,1)$ here?

It is instructive to check the commutator of two transformations $\delta = \delta_{\rm s} + \delta_{\rm p}$ and see if the result is compatible with the supersymmetry algebra (11.16). The computation of the commutator on the bosonic fields gives

$$[\delta_2\,,\delta_1]\mathcal{S}(x) = \delta_2(\delta_1\mathcal{S}(x)) - \delta_1(\delta_2\mathcal{S}(x)) = a^n_{\rm s1,2}\partial_n\mathcal{S}(x) + b^n_{1,2}\partial_n\mathcal{P}(x)\,, \tag{11.27}$$

$$[\delta_2\,,\delta_1]\mathcal{P}(x) = \delta_2(\delta_1\mathcal{P}(x)) - \delta_1(\delta_2\mathcal{P}(x)) = a^n_{\rm p1,2}\partial_n\mathcal{P}(x) - b^n_{1,2}\partial_n\mathcal{S}(x)\,, \tag{11.28}$$

where

$$a^n_{\rm s1,2} := \epsilon_{\rm 2s}^{\hat\alpha}\gamma^n_{\hat\alpha\hat\beta}\epsilon_{\rm 1s}^{\hat\beta} - (1\leftrightarrow 2) = 2\epsilon_{\rm 2s}^{\hat\alpha}\gamma^n_{\hat\alpha\hat\beta}\epsilon_{\rm 1s}^{\hat\beta} \tag{11.29}$$

$$a^n_{\rm p1,2} := \epsilon_{\rm 2p}^{\hat\alpha}\gamma^n_{\hat\alpha\hat\beta}\epsilon_{\rm 1p}^{\hat\beta} - (1\leftrightarrow 2) = 2\epsilon_{\rm 2p}^{\hat\alpha}\gamma^n_{\hat\alpha\hat\beta}\epsilon_{\rm 1p}^{\hat\beta}\,, \tag{11.30}$$

$$b^n_{1,2} := \epsilon_{\rm 1s}^{\hat\alpha}(\gamma^n\Gamma)_{\hat\alpha\hat\beta}\epsilon_{\rm 2p}^{\hat\beta} - (1\leftrightarrow 2)\,. \tag{11.31}$$

The first terms on the right-hand sides of (11.27) and (11.28) have the form of translations anticipated from the supersymmetry algebra (11.16). The second ones do not.

An important property of the four-dimensional γ-matrices is that $(\gamma^n\Gamma)_{\hat\alpha\hat\beta}$ is antisymmetric, *i.e.*, equivalently,

$$\gamma^{nmk}{}_{\hat\alpha\hat\beta} = -\gamma^{nmk}{}_{\hat\beta\hat\alpha}\,. \tag{11.32}$$

This has the consequence that if we consider the transformations with $\epsilon_{\rm s} = \epsilon_{\rm p} = \epsilon$, then $b^n_{1,2} = 0$, and their commutator will give a translation with parameter

$$a^n_{1,2} = 2\epsilon_2^{\hat\alpha}\gamma^n_{\hat\alpha\hat\beta}\epsilon_1^{\hat\beta} \tag{11.33}$$

acting both on $\mathcal{S}$ and on $\mathcal{P}$. This is the result expected from the $iso_1(3,1)$-algebra. Thus we anticipate that Poincaré supersymmetry is a combination of both transformations with $\epsilon^{\hat\beta} = \epsilon_{\rm p}^{\hat\beta} = \epsilon_{\rm s}^{\hat\beta}$. To verify this we have to compute the commutator of the supertransformations acting on the fermion field which yields

$$[\delta_2\,, \delta_1]\psi_{\hat\alpha} = \gamma^n_{\hat\alpha\hat\beta}\epsilon^{\hat\beta}_{2\,\rm s}\epsilon^{\hat\gamma}_{1\,\rm s}\partial_n\psi_{\hat\gamma} - (\gamma^n\Gamma)_{\hat\alpha\hat\beta}\epsilon^{\hat\beta}_{2\,\rm p}\epsilon^{\hat\gamma}_{1\,\rm s}\Gamma_{\hat\gamma}{}^{\hat\delta}\partial_n\psi_{\hat\delta} - (1 \leftrightarrow 2)\,. \tag{11.34}$$

To rewrite this formula in an appropriate form one has to use the so-called *Fierz identities*, which follow from the fact that the matrices with $\gamma_{n_1,\dots,n_p\hat\alpha}{}^{\hat\beta}$ with various p form a basis of the Clifford algebra. (It should be noted that the necessity of using Fierz identities in the spinor formalism is the main source of complications of the analysis in higher dimensions. We will see however how this can be avoided in dimensions $d = 3$ and 4.) This allows us to write

$$\epsilon_1^{\hat\alpha}\epsilon_2^{\hat\beta} = \sum_{p=0}^{4} a(p)\gamma_{n_1,\dots,n_p}{}^{\hat\alpha\hat\beta}\,\epsilon_2^{\hat\gamma}\gamma^{n_1,\dots,n_p}{}_{\hat\gamma\hat\delta}\epsilon_1{}^{\hat\delta} \tag{11.35}$$

for some coefficients $a(p)$ such that, in particular, $a(1) = 1/4$ as can be easily seen by computing the trace with $\gamma^k_{\hat\alpha\hat\beta}$ on the both sides of (11.35).

The antisymmetrization with respect to $(1 \leftrightarrow 2)$, along with the anticommutativity of $\epsilon_1^{\hat\alpha}$ and $\epsilon_2^{\hat\beta}$, implies that only those terms in (11.35) contribute that are symmetric with respect to the spinor indices $\hat\alpha$ and $\hat\beta$. In accordance with (10.30) and (10.35), in the four-dimensional case there are only two such terms in (11.35), namely those with $p = 1$ and $p = 2$. The terms with $p = 2$ cancel between the contributions of $\epsilon_{\rm s}$ and $\epsilon_{\rm p}$ upon setting $\epsilon_{\rm s} = \epsilon_{\rm p} = \epsilon$. The term with $p = 1$ gives

$$[\delta_2\,, \delta_1]\psi_{\hat\alpha} = \frac{1}{2}a^n_{1,2}(\gamma^m\gamma_n)_{\hat\alpha}{}^{\hat\beta}\partial_m\psi_{\hat\beta} = a^n_{1,2}\partial_n\psi_{\hat\alpha} - \frac{1}{2}a^n_{1,2}(\gamma_n\gamma^m)_{\hat\alpha}{}^{\hat\beta}\partial_m\psi_{\hat\beta}\,. \tag{11.36}$$

The first term here indeed describes the anticipated translation. The second term, though being non-geometric, vanishes on the field equations $\gamma^n\partial_n\psi = 0$. In such cases one says that the algebra is *open* or, equivalently, closes on-shell.

To achieve off-shell closure it turns out to be necessary to introduce *auxiliary fields*, which are zero on-shell but have a non-zero transformation law off-shell. For the Wess-Zumino model the appropriate set of auxiliary fields consists of another scalar $\mathcal{F}_{\rm s}$ and pseudoscalar $\mathcal{F}_{\rm p}$ that, in the massless case, enter the action via the additional term

$$\Delta S = \frac{1}{2}\int d^4x\big((\mathcal{F}_{\rm s})^2 + (\mathcal{F}_{\rm p})^2\big). \tag{11.37}$$

Clearly, the equations of motion for the auxiliary fields are just

$$\mathcal{F}_{\rm s} = 0\,, \qquad \mathcal{F}_{\rm p} = 0\,. \tag{11.38}$$

The remaining question is, however, what are the symmetries with parameters $\epsilon_{\rm p}^{\hat\beta}$ and $\epsilon_{\rm s}^{\hat\beta}$ and what are the additional symmetries that appear in the commutator in the case of higher dimensions where some other terms in the expansion (11.35) contribute? The answer is that these are particular higher-spin symmetries. For instance, one can check that the commutator of the s-transformations alone will contain the following transformation of the fermion field

$$\delta\psi = \epsilon_{12}^{kl}\gamma^n\gamma_{kl}\partial_n\psi\,, \qquad \epsilon_{12}^{kl} \sim \epsilon_{1\mathrm{s}}^{\hat\alpha}\gamma^{[kl]}{}_{\hat\alpha\hat\beta}\epsilon_{2\mathrm{s}}^{\hat\beta}\,. \tag{11.39}$$

We leave it to the reader to check that the transformation (11.39) is a symmetry of the massless action for a Majorana fermion. These are also some higher-spin symmetries.

11.2.1 Open algebra

Let us briefly discuss why on-shell trivial terms can appear in the commutator of symmetry transformations. This is in no way because we neglect the terms that are zero on-shell.

Consider any action $S(q)$ that depends on some variables q^i. The transformation

$$\delta q^i = \varepsilon^{ij}\frac{\delta S}{\delta q^j} \tag{11.40}$$

with any antisymmetric matrix $\varepsilon^{ij} = -\varepsilon^{ji}$ is a symmetry of the action S since

$$\delta S = \frac{\delta S}{\delta q^i}\varepsilon^{ij}\frac{\delta S}{\delta q^j} \equiv 0\,. \tag{11.41}$$

Symmetries of the form (11.40) are often called *trivial* since, in particular, they do not generate on-shell nontrivial conserved currents. Trivial symmetries form a generalization of the rotational symmetry reproduced by (11.40) for the special case of $S = q^i q^j \delta_{ij}$.

Being symmetries of the action, trivial symmetries can appear in the commutator of other symmetries. This happens in supersymmetric models in the absence of auxiliary fields. An important property of trivial symmetries is that, as is easy to see, they form an *ideal* of the algebra of symmetries, *i.e.*, the commutator of a trivial symmetry with any other symmetry gives again a trivial symmetry.

Note that at present there are still some supersymmetric models including the famous $N = 4$ super-Yang-Mills and $N = 8$ supergravity where proper auxiliary fields are not known (maybe even do not exist) and the supersymmetry algebra is open.

11.3 Classification of supermultiplets

The Wigner approach to the classification of possible types of particles directly applies to the SUSY superalgebra. As for the Poincaré algebra, the massive and massless cases are different. We start with the massive case.

11.3.1 Massive supermultiplets

Choosing again the rest frame

$$p_n = (m, 0, \ldots, 0) \tag{11.42}$$

in momentum space we have to analyze the Wigner little superalgebra. The latter consists of those Lorentz generators L_{nm} and supergenerators $Q^i_{\hat\alpha}$ that leave invariant the momentum (11.42). Clearly, the Wigner little superalgebra consists of the $o(d-1)$-generators of spatial rotations and all supergenerators $Q^i_{\hat\alpha}$ since the latter commute with the momentum operators.

In the rest frame the relations (11.9) give

$$\{Q^i_{\hat\alpha}\,, (Q^\dagger)^{j\,\hat\beta}\} = 2m\delta^{ij}\delta^{\hat\beta}_{\hat\alpha}\,. \tag{11.43}$$

We see that these relations have the form of a Clifford algebra with the supercharges Q^i_α as generating elements. Let us stress that, from the perspective (11.43), the spinor indices $\hat\alpha$ of the original Lorentz algebra enumerate the Clifford generating elements. Modules of the Clifford algebra can be constructed in the usual manner. From here it follows in particular that any supermultiplet at $N > 0$ contains an equal number of bosonic and fermionic states, simply because the dimensions of the odd and even subspaces of the Fock module coincide.

The analysis of the little superalgebra depends on how many values are taken by the spinor indices $\hat\alpha$. Clearly, the complexity of massive supermultiplets quickly increases with the number N and the dimension of space-time. Indeed, since spinor indices carried by the supercharges in d dimensions take $2^{[d/2]}$ values, where $[n]$ denotes the integer part of n, the dimension of the Fock module associated with the supercharges even for the singlet vacuum state (*cf.* (11.46) below with $s = 0$) is

$$\dim F = 2^{N2^{[d/2-1]}}\,. \tag{11.44}$$

As a result, for higher N or d, a supermultiplet contains more and more states of higher and higher spins.

For definiteness we consider the important particular case of the four-dimensional supersymmetry algebra. In this case the four real supergenerators $Q^i_{\hat\alpha}$ are equivalent to the complex generators Q^i_α and their complex conjugates $\bar Q^i_{\dot\alpha}$ with $\alpha, \beta = 1, 2$, $\dot\alpha, \dot\beta = 1, 2$. In these terms the commutation relations (11.43) take the form (*cf.* (11.17))

$$\{Q^i_\alpha\,, \bar Q^j_{\dot\beta}\} = 2m\delta^{ij}\delta_{\alpha\dot\beta}\,, \qquad \{Q^i_\alpha\,, Q^j_\beta\} = 0\,, \qquad \{\bar Q^i_{\dot\alpha}\,, \bar Q^j_{\dot\beta}\} = 0\,. \tag{11.45}$$

Here Q_α and $\bar Q_{\dot\alpha}$ can be interpreted as conjugated spinors of the algebra $o(3) \simeq su(2)$. As is known from quantum mechanics, spinors in three dimensions are complex. Indeed, the Euclidean Clifford algebra in three dimensions is realized by the Hermitian Pauli matrices, two of which are real (σ_1 and σ_3) and one purely imaginary (σ_2).

In these terms, the little algebra of the supergenerators acquires the form (10.17). Now it is easy to describe all massive supermultiplets of $iso_N(3,1)$ following the method used for the construction of the spinor module of the Clifford algebra. Namely, we start from the Fock vacuum $|s\rangle$ which is annihilated by all operators $\bar{Q}^j_{\dot\beta}$

$$\bar{Q}^j_{\dot\beta}|s\rangle = 0\,. \tag{11.46}$$

The novel point is that, since the Wigner little algebra contains the three-dimensional rotations $o(3)$, the vacuum $|s\rangle$ should form some (not necessarily singlet) spin-s $o(3)$–module. In accordance with the analysis of $o(N)$-modules in Section 3.2.2, this means that $|s\rangle$ is a rank-s symmetric traceless tensor for integer s and a rank-$(s-1/2)$ symmetric γ–transversal tensor-spinor for half-integer s.

Now we have to figure out the content of these supermultiplets in terms of the usual fields of Section 3. A very useful fact, special to $o(3)$-spinors, is that an irreducible spin-s $o(3)$–module can be represented by a totally symmetric *multispinor* of rank $2s$. This property, which extends the relation between a vector $\chi^{\mathbf{n}}$ and a symmetric bispinor $\chi^{\alpha\beta}$ via

$$\chi^{\mathbf{n}} = \sigma^{\mathbf{n}}_{\alpha\beta}\chi^{\alpha\beta}\,, \qquad \chi^{\alpha\beta} = \sigma^{\alpha\beta}_{\mathbf{n}}\chi^{\mathbf{n}}\,, \tag{11.47}$$

can be seen directly. For example, for integer s consider a multispinor of even rank $\chi_{\alpha_1\ldots\alpha_{2s}}$

$$\chi_{\alpha_1\ldots\alpha_{2s}} := \sigma^{\mathbf{n}_1}_{\alpha_1\alpha_2}\ldots\sigma^{\mathbf{n}_s}_{\alpha_{2s-1}\alpha_{2s}}\chi_{\mathbf{n}_1\ldots\mathbf{n}_s} \tag{11.48}$$

where $\mathbf{n} = 1,2,3$ is an $o(3)$-vector index, and

$$\sigma^{\mathbf{n}}_{\alpha\beta} = \sigma^{\mathbf{n}}{}_\alpha{}^\gamma\epsilon_{\gamma\beta}\,, \tag{11.49}$$

where $\sigma^{\mathbf{n}}{}_\alpha{}^\beta$ are the usual Pauli matrices which form the $o(3)$ γ-matrices. By virtue of the simple property of the σ-matrices

$$\sigma^{\mathbf{n}}_{\alpha\beta}\sigma^{\mathbf{m}}_{\gamma\delta}\epsilon^{\alpha\gamma}\epsilon^{\beta\delta} = \delta^{\mathbf{nm}} \tag{11.50}$$

the multispinor $\chi_{\alpha_1\ldots\alpha_{2s}}$ turns out to be totally symmetric for any traceless symmetric $\chi_{\mathbf{n}_1\ldots\mathbf{n}_s}$.

The other way around, the tensor

$$\chi'^{\mathbf{n}_1\ldots\mathbf{n}_s} = \sigma^{\mathbf{n}_1}_{\alpha_1\alpha_2}\ldots\sigma^{\mathbf{n}_s}_{\alpha_{2s-1}\alpha_{2s}}\chi^{\alpha_1\ldots\alpha_{2s}} \tag{11.51}$$

is totally symmetric and traceless for any totally symmetric multispinor $\chi^{\alpha_1\ldots\alpha_{2s}}$. The former property is obvious while the latter follows from the property of σ-matrices

$$\sigma^{\mathbf{n}}_{\alpha\beta}\sigma^{\mathbf{m}}_{\gamma\delta}\delta_{\mathbf{nm}} = \frac{1}{2}(\epsilon_{\alpha\gamma}\epsilon_{\beta\delta} + \epsilon_{\beta\gamma}\epsilon_{\alpha\delta}) \tag{11.52}$$

along with the property that contraction of the antisymmetric matrix $\epsilon_{\alpha\beta}$ with any symmetric multispinor gives zero.

Analogously one can show the equivalence of multispinors $\chi^{\alpha_1\ldots\alpha_{2s}}$ of odd rank (*i.e.*, with half-integer s) to symmetric tensor-spinors $\chi'_{\mathbf{n}_1\ldots\mathbf{n}_{[s]};\alpha}$ obeying the σ-transversality condition

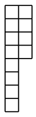
$$\sigma^{\mathbf{n}_1}{}_\alpha{}^\beta\chi'_{\mathbf{n}_1\ldots\mathbf{n}_{[s]};\beta} = 0\,. \tag{11.53}$$

As a result, the full Fock space consists of states of the form

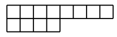
$$|v\rangle = \sum_{n=0}^{2N} v_{i_1\dots i_n}^{\alpha_1\dots\alpha_n\beta_1\dots\beta_{2s}} Q_{\alpha_1}^{i_1}\dots Q_{\alpha_n}^{i_n}|s\rangle_{\beta_1\dots\beta_{2s}}\,. \tag{11.54}$$

The restriction $0 \le n \le 2N$ follows from the anticommutativity of Q_α^i and the fact that α takes two values. The total number of components of $v_{i_1\dots i_n}^{\alpha_1\dots\alpha_n\beta_1\dots\beta_{2s}}$ in $|v\rangle$ is $2^{2N}(2s+1)$, hence growing rapidly with N.

Since $v_{i_1\dots i_n}^{\alpha_1\dots\alpha_n\beta_1\dots\beta_{2s_0}}$ is antisymmetric under the exchange of any two pairs of indices (i_j,α_j) and (i_k,α_k) with $j\neq k$, it follows that the symmetrization of any subset of indices $i_{k_1}, i_{k_2},\dots i_{k_p}$ implies the antisymmetrization of the indices $\alpha_{k_1},\alpha_{k_2},\dots\alpha_{k_p}$ and vice versa. However, since spinor indices α take two values, the antisymmetrization over any three or more spinor indices gives zero. Analogously, the antisymmetrization over any $N+1$ internal indices i gives zero. This implies that with respect to the indices i and α, $v_{i_1\dots i_n}^{\alpha_1\dots\alpha_n\beta_1\dots\beta_{2s_0}}$ can have only the following symmetry types

$i:\quad h_1 = n \le N$ 

. (11.55)

$\alpha\;:$ $l_1 = n \le N$  (11.56)

Since the antisymmetrization of any two spinor indices is equivalent to their contraction with the matrix $\epsilon_{\alpha\beta}$, an $o(3)$-module described by a (spinor) two-row Young diagram with rows of length l_1 and l_2 is equivalent to the $o(3)$-module corresponding to the one-row Young diagram of length $l_1 - l_2$.

Thus, any supermultiplet of the massive $4d$ little Wigner superalgebra at any N (including $N=0$) is characterized by some spin s. Elements of the supermultiplet are

$$v_{i_1\dots i_{l_1},j_1\dots j_{l_2}}^{\alpha_1\dots\alpha_{l_1-l_2};\beta_1\dots\beta_{2s}}\,. \tag{11.57}$$

To work out the full pattern it remains to observe that the decomposition of a multispinor $A_{\alpha_1\dots\alpha_n;\beta_1\dots\beta_m}$ that is symmetric separately in the indices α_i and β_j into irreducible $o(3)$-modules is represented by the quantum-mechanical spin summation (Clebsch-Gordan) rule, which implies that the irreducible $o(3)$-modules contained in $A_{\alpha_1\dots\alpha_n;\beta_1\dots\beta_m}$ consist of symmetric multispinors $B_{\gamma_1\dots\gamma_p}$ with various $|n-m|\le p\le n+m$. Note that this result can most easily be obtained by using the equivalence of multispinors described by two-row Young diagrams and those described by one-row Young diagrams.

The final result therefore is that the pattern of $o(3)$-modules of the massive little superalgebra is represented by the elements

$$w_{[i_1\dots i_{l_1}],[j_1\dots j_{l_2}]}^{\gamma_1\dots\gamma_p} \tag{11.58}$$

which belong to various $gl(N)$-tensors described by the two-column Young diagrams (11.55) with

$$0\le l_2\le l_1\le N\,, \tag{11.59}$$

each component of which is a symmetric multispinor of any rank p obeying

$$p = |2s - l_1 + l_2|, |2s - l_1 + l_2| + 2, \ldots |2s + l_1 - l_2| \,. \tag{11.60}$$

This gives the full list of possible massive supermultiplets in four dimensions.

For example, let $s = 0$ and $N = 1$. In this case we have two spin-zero fields, namely those with $l_1 = l_2 = 0$ and $l_1 = l_2 = 1$, and one spin-1/2 field with $l_1 = 1$, $l_2 = 0$. This is the $N = 1$ scalar supermultiplet.

For $s = 1/2$, $N = 1$ we obtain the massive vector supermultiplet which contains one spin-one field, two spin-1/2 fields and one spin-zero field.

11.3.2 Massless supermultiplets

Massless supermultiplets are most interesting since they contain the various gauge fields. Let us start in d dimensions with Dirac spinors with indices $\hat{\alpha}$.

The standard momentum is convenient to choose in the form

$$\mathbf{p}_n = (\omega, 0, \ldots, 0, \omega)\,. \tag{11.61}$$

The even part of the Wigner little algebra consists of the $o(d-2)$-generators

$$L_{\mathbf{ij}}\,, \qquad \mathbf{i}, \mathbf{j} = 1, \ldots, d-2\,. \tag{11.62}$$

The supergenerators obey

$$\{Q^i_{\hat{\alpha}}\,, (Q^\dagger)^{j\hat{\beta}}\} = 4\omega\delta^{ij}\pi_{+\hat{\alpha}}{}^{\hat{\beta}}\,, \tag{11.63}$$

where

$$\pi_{\pm\hat{\alpha}}{}^{\hat{\beta}} := \frac{1}{2}(I \pm \gamma_{d-1}\gamma_0)_{\hat{\alpha}}{}^{\hat{\beta}}\,. \tag{11.64}$$

Analogously to the projectors (3.20), $\pi_\pm$ form a complete set of projectors that decompose the spinor space into two subspaces of equal dimensions. Introducing

$$Q^i_{\pm\hat{\alpha}} := (\pi_\pm Q^i)_{\hat{\alpha}} \tag{11.65}$$

the relations (11.63) imply

$$\{Q^i_{+\hat{\alpha}}\,, (Q^\dagger)^{j\hat{\beta}}_+\} = 4\omega\delta^{ij}\pi_{+\hat{\alpha}}{}^{\hat{\beta}}\,, \tag{11.66}$$

$$\{Q^i_{-\hat{\alpha}}\,, (Q^\dagger)^j_{\pm\hat{\beta}})\} = 0\,, \qquad \{Q^i_{\pm\hat{\alpha}}\,, (Q^\dagger)^j_{-\hat{\beta}})\} = 0\,, \tag{11.67}$$

i.e., $Q_{+\hat{\alpha}}$ generates a nontrivial Clifford algebra while $Q_{-\hat{\alpha}}$ anticommutes with itself and with $Q_{+\hat{\alpha}}$. Note that the $o(d-2)$-generators (11.62) leave invariant $\gamma_{d-1}\gamma_0$ and hence $\pi_\pm$.

These anticommutation relations allow us to consider modules of the massless little superalgebra where Q_- acts trivially, *i.e.*, by zero. In fact, unitarity requires that Q_- should act trivially, because otherwise it would produce states of zero norm as a consequence of (11.67).

Thus, the Clifford part of the massless little algebra has twice less generating elements than the massive one. As a result massless supermultiplets are usually much smaller than massive.

Let us now consider the four-dimensional case where it is most convenient to use the language of two-component spinors. In these terms, the anticommutation relations (11.66) of the supercharges take the form

$$\{Q^i_\alpha\,, \bar{Q}^j_{\dot\beta}\} = 2\omega\delta^{ij}(\sigma^0_{\alpha\dot\beta} + \sigma^3_{\alpha\dot\beta})\,, \qquad \{Q^i_\alpha\,, Q^j_\beta\} = 0\,, \qquad \{\bar{Q}^i_{\dot\alpha}\,, \bar{Q}^j_{\dot\beta}\} = 0\,. \tag{11.68}$$

This implies that the only nonzero anticommutation relations are those between Q^i_1 and $\bar{Q}^j_{\dot 1}$

$$\{Q^i_1\,, \bar{Q}^j_{\dot 1}\} = 2\omega\delta^{ij}\,. \tag{11.69}$$

All other anticommutation relations are zero, including all those that contain Q^i_2 and/or $\bar{Q}^j_{\dot 2}$ (*cf.* (11.66)). As a result, the states of the module of the massless little algebra are the states of the Fock space

$$Q_1^{[i_1}Q_1^{i_2}\dots Q_1^{i_p]}|s\rangle\,, \qquad \bar{Q}^j_{\dot 1}|s\rangle = 0\,, \tag{11.70}$$

where the vacuum state carries *helicity*, which is an eigenvalue of the single generator L_{12} of the four-dimensional massless little algebra $o(2)$, *i.e.*,

$$L_{12}|s\rangle = s\,|s\rangle\,, \qquad s = 0, \pm 1/2, \pm 1, \pm 3/2, \dots \tag{11.71}$$

Opposite helicities correspond to (anti)self-dual irreducible modules of $o(2)$ discussed in Section 3.4. Since in the four-dimensional field theory massless self-dual fields cannot be defined, every module of the massless little algebra that carries some set of helicities s either has to be self-conjugated, containing helicities $\pm s$ for any s, or be supplemented with the conjugated supermultiplet containing helicities of opposite signs. This phenomenon is a consequence of the CPT symmetry of quantum field theory. An example of the situation with the doubling of the multiplets is provided by any $N = 1$ supermultiplet including the scalar and vector supermultiplets.

The commutation relations of the supercharges with the generator L_{12} give

$$[L_{12}, Q^i_1] = \frac{1}{2}Q^i_1\,, \qquad [L_{12}, \bar{Q}^i_{\dot 1}] = -\frac{1}{2}\bar{Q}^i_{\dot 1}\,. \tag{11.72}$$

This implies that the supergenerator Q^i_1 carries helicity 1/2. As a result, an irreducible supermultiplet contains states with helicities from s to $s + N/2$. The multiplicity of the states of helicity $s + p/2$ is $\frac{N!}{(N-p)!p!}$. The difference between the maximal and minimal helicities is $N/2$. Taking into account that helicity coincides with the projection of spin on the propagation direction it follows in particular that for $N > 8$ every massless supermultiplet should contain a massless state of spin larger than 2. As long as it was believed that interacting theories of massless higher-spin fields do not exist the conclusion was that there is no room for supersymmetries with $N > 8$. The restriction $N \leq 8$ is just the restriction $s \leq 2$.

There is a distinguished class of self-conjugated supermultiplets which contain equal numbers of helicities $\pm h$ for any h. This happens if

$$s_{\max} = \frac{N}{4} \tag{11.73}$$

which is possible for even N. The following two cases are particularly important.

The $N = 4$ self-conjugated supermultiplet underlies the famous four-dimensional $N = 4$ super-Yang-Mills theory which contains one spin-one Yang-Mills field $A_n(x)$, four Majorana spin-1/2 fields $\psi^i_{\hat{\alpha}}(x)$ and six spin-zero fields $\phi^{ij}(x) = -\phi^{ji}(x)$. (Recall that the indices i, j here take four values.) More precisely, in the case of Yang-Mills theory with some non-Abelian algebra h, all fields A_n, $\psi^i_{\hat{\alpha}}(x)$ and $\phi^{ij}(x)$ are valued in the adjoint representation of h. This theory has maximal supersymmetry among theories that do not involve spins $s > 1$. It has remarkable properties providing the only known example of a finite (*i.e.*, free of divergencies) QFT. Also this theory is conformally invariant, playing a fundamental role in the AdS/CFT duality.

The $N = 8$ self-conjugated supermultiplet underlies another prominent theory called $N = 8$ supergravity. This theory contains one spin-two field, eight spin-3/2 fields (*gravitinos*), 28 spin-one fields (*graviphotons*), 56 spin-1/2 fields and 128 spin-zero fields. This maximal supergravity model has remarkable properties as well. It is still debated whether or not it is a finite theory of gravity.

For $N \leq 8$, the supergravitational multiplets have maximal spin two. They contain one massless spin-two field, N massless spin-3/2 fields, $N(N-1)/2$ spin-one fields, etc. The supersymmetric models associated with these supermultiplets are called *N-extended supergravity.*

11.4 Poincaré supergravity

11.4.1 Gauging the supersymmetry algebra

As explained in Section 8 gravity results from the gauging of the Poincaré or (anti-) De Sitter algebra. Analogously, N-extended supergravity results from the gauging of supersymmetry algebra $iso_N(d-1,1)$ or its anti-De Sitter extension defined below.

For definiteness we consider the four-dimensional case. The $iso_N(3,1)$-gauge fields are the one-forms

$$A(x) = -i\Big(e^n(x)P_n + \frac{1}{2}\omega^{nm}(x)L_{nm}\Big) + Q^{\hat{\alpha}}_i\psi^i_{\hat{\alpha}}(x)\,. \tag{11.74}$$

The one-form gauge field $\psi_{\underline{n}\hat{\alpha}}$ associated with the supercharges $\bar{Q}^{\hat{\alpha}}$ is a massless field of spin 3/2 called the *gravitino.* The internal indices $i, j = 1 \ldots N$ are raised and lowered by the Kronecker deltas δ_{ij} and δ^{ij}. The two-form field strengths (curvatures) have the analogous decomposition

$$\mathcal{R}(x) = -i\Big(\mathcal{R}^n(x)P_n + \frac{1}{2}\mathcal{R}^{nm}(x)L_{nm}\Big) + Q^{\hat{\alpha}}_i\mathcal{R}^i_{\hat{\alpha}}(x)\,. \tag{11.75}$$

The definition (7.40) along with the commutation relations of the Poincaré algebra (2.21), (2.22) and (2.23) give $\mathcal{R}^{nm}$ (8.5) and

$$\mathcal{R}^n(x) = D^{\mathrm{L}}e^n(x) - \psi^{\hat{\alpha}}_i(x)\gamma^n{}_{\hat{\alpha}}{}^{\hat{\beta}}\psi^i_{\hat{\beta}}(x)\,, \tag{11.76}$$

$$\mathcal{R}^i_{\hat\alpha} = D^{\mathrm{L}}\psi^i_{\hat\alpha}\,. \tag{11.77}$$

Recall that the Lorentz-covariant derivative is defined as follows

$$D^L A^n(x) = \mathrm{d}A^n(x) + \omega^n{}_m(x)A^m(x)\,, \qquad D^{\mathrm{L}}\psi_{\hat\alpha} = \mathrm{d}\psi_{\hat\alpha} + \frac{1}{4}\omega^{nm}[\gamma_n,\,\gamma_m]_{\hat\alpha}{}^{\hat\beta}\psi_{\hat\beta}\,. \tag{11.78}$$

Showing explicitly the indices of differential forms we have

$$\mathcal{R}_{\underline{nm}}{}^n(x) = D^{\mathrm{L}}_{\underline{n}}e_{\underline{m}}{}^n(x) - \psi^{i\hat\alpha}_{\underline{n}}(x)\gamma^n{}_{\hat\alpha}{}^{\hat\beta}\psi_{\underline{m}\,i\,\hat\beta}(x) - (\underline{n}\leftrightarrow\underline{m})\,, \tag{11.79}$$

$$\mathcal{R}^i_{\underline{nm}\,\hat\alpha} = D^L_{\underline{n}}\psi^i_{\underline{m}\,\hat\alpha} - D^L_{\underline{m}}\psi^i_{\underline{n}\,\hat\alpha}\,. \tag{11.80}$$

The $iso_N(3,1)$ Yang-Mills gauge transformations are (8.9) and

$$\delta e_{\underline{n}}{}^n(x) = D^{\mathrm{L}}_{\underline{n}}\varepsilon^n(x) - \varepsilon^n{}_m(x)e_{\underline{n}}{}^m(x) - 2\psi^{i\,\hat\alpha}_{\underline{n}}(x)\gamma^n{}_{\hat\alpha}{}^{\hat\beta}\varepsilon_{i\,\hat\beta}(x)\,, \tag{11.81}$$

$$\delta\psi^i_{\underline{n}\,\hat\alpha}(x) = D^L_{\underline{n}}\varepsilon^i_{\hat\alpha}(x)\,. \tag{11.82}$$

We will also need the Bianchi identity

$$D^{\mathrm{L}}\mathcal{R}_{\hat\alpha} = \frac{1}{4}\mathcal{R}^{nm}[\gamma_n,\,\gamma_m]_{\hat\alpha}{}^{\hat\beta}\psi_{\hat\beta}\,, \tag{11.83}$$

which follows from the Bianchi identity (7.44) for general Yang-Mills theory. by virtue of (11.2).

The most important novelty is that the frame field has a nontrivial transformation law under the supertransformations with parameter $\varepsilon_{\hat\alpha}$. A related fact is that the anticommutation relation (11.3) implies that the curvature $\mathcal{R}^n$ (11.79) contains the $\bar\psi\psi$-term. This means in particular that setting $\mathcal{R}^n$ to zero implies that the torsion is no longer zero, being expressed via bilinears of the gravitinos. The expression for the Lorentz connection resulting from the constraint $\mathcal{R}^n = 0$ now contains not only the frame field with its derivatives but also terms bilinear in the gravitinos. The explicit expression for the Lorentz connection resulting from the constraint $\mathcal{R}^n = 0$ still has the form (8.37) with

$$\mathcal{F}_{\underline{nm},}{}^n = \partial_{[\underline{n}}e_{\underline{m}]}{}^n(x) - \psi^{i\hat\alpha}_{[\underline{n}}(x)\gamma^n{}_{\hat\alpha}{}^{\hat\beta}\psi_{\underline{m}]\,i\,\hat\beta}(x)\,. \tag{11.84}$$

The Yang-Mills gauge transformations of the curvatures are

$$\delta\mathcal{R}^n(x) = \mathcal{R}^n{}_m(x)\varepsilon^m(x) - \varepsilon^n{}_m(x)\mathcal{R}^m(x) - \mathcal{R}^{i\hat\alpha}(x)\gamma^n{}_{\hat\alpha}{}^{\hat\beta}\varepsilon_{i\,\hat\beta}(x)\,, \tag{11.85}$$

$$\delta\mathcal{R}^{nm}(x) = \mathcal{R}^n{}_k(x)\varepsilon^{km}(x) + \mathcal{R}^m{}_k(x)\varepsilon^{nk}(x)\,, \tag{11.86}$$

$$\delta\mathcal{R}^i_{\hat\alpha}(x) = \frac{1}{4}\Big(\mathcal{R}^{nm}(x)[\gamma_n,\,\gamma_m]_{\hat\alpha}{}^{\hat\beta}\varepsilon^i_{\hat\beta}(x) - \varepsilon^{nm}(x)[\gamma_n,\,\gamma_m]_{\hat\alpha}{}^{\hat\beta}\mathcal{R}^i_{\hat\beta}(x)\Big)\,. \tag{11.87}$$

11.4.2 Free massless field of spin 3/2

In the free-field limit in Cartesian coordinates with $e^n = \xi^n$ and zero Lorentz connection in $D^{\mathrm{L}}_{\underline{n}}$, the field transformation (11.82) takes the form

$$\delta\psi_{n\,\hat\alpha}(x) = \partial_n \varepsilon_{\hat\alpha}(x)\,. \tag{11.88}$$

This gauge transformation corresponds to a massless spin-3/2 field in the Wigner classification. The flat-space action for this field in any dimension can be written in the *Rarita-Schwinger* form

$$S = \int d^d x \bar\psi_n^{\hat\alpha} \gamma_{\hat\alpha}^{[nmk]\,\hat\beta} \partial_m \psi_{k\,\hat\beta}\,, \tag{11.89}$$

which is manifestly invariant under the gauge transformation (11.88).

The equations of motion are

$$\gamma_{\hat\alpha}^{[nmk]\,\hat\beta} \partial_m \psi_{k\,\hat\beta} = 0\,. \tag{11.90}$$

To check that these equations indeed describe a massless spin 3/2-particle we impose the gauge condition

$$\gamma^n \psi_n = 0\,. \tag{11.91}$$

As a result, (11.90) gives

$$\gamma^k \partial_k \psi_n - \gamma_n \partial^k \psi_k = 0\,. \tag{11.92}$$

Multiplying (11.92) by γ^n and using again (11.91) we obtain

$$\partial^k \psi_k = 0\,. \tag{11.93}$$

Finally, (11.92) gives

$$\gamma^k \partial_k \psi_n = 0\,. \tag{11.94}$$

Eqs. (11.91), (11.93) and (11.94) are just the equations for a massless spin-3/2 field in the form of Section 3.3.

In four dimensions $\psi_{n\,\hat\beta}$ is a Majorana spinor. The Rarita-Schwinger action can be written in terms of differential forms as

$$S = \int_{M^4} \psi_{\hat\alpha} e^n (\Gamma\gamma_n)^{\hat\alpha\hat\beta} \mathrm{d}\psi_{\hat\beta} = \frac{1}{4!}\epsilon^{\underline{nmkl}} \int d^4x\, \psi_{\underline{n}\hat\alpha} (\Gamma\gamma_{\underline{m}})^{\hat\alpha\hat\beta} \partial_{\underline{k}} \psi_{\underline{l}\hat\beta}\,, \tag{11.95}$$

where the one-form $\psi_{\hat\beta} = \xi^{\underline{n}} \psi_{\underline{n}\hat\beta}$ has the gauge transformation $\delta\psi_{\hat\alpha}(x) = \mathrm{d}\varepsilon_{\hat\alpha}(x)\,.$

Because $\psi_{\hat\beta}$ are one-forms with anticommuting coefficients $\psi_{\underline{n}\hat\beta}$ they commute with each other

$$\psi_{\hat\alpha}\psi_{\hat\beta} = \psi_{\hat\beta}\psi_{\hat\alpha}\,. \tag{11.96}$$

The fact that $(\Gamma\gamma_n)^{\hat\alpha\hat\beta}$ is antisymmetric in spinor indices then implies that the action (11.95) is not the integral of a total derivative.

11.4.3 Supergravitational multiplets and the admissibility condition

We see that the gauge fields of $iso_N(d-1,1)$ contain one spin-two field and N spin-3/2 fields. This matches the list of of the supergravitational multiplets which indeed contain one spin-two field and N spin-3/2 fields. For definiteness, let us consider the four-dimensional case. We know that for $2 \le N \le 8$ the supergravitational multiplets also contain $\frac{N(N-1)}{2}$ massless fields of spin one along with some lower-spin fields of spin 1/2 and 0. We shall see in the next section that the spin-one massless fields also naturally result from the gauging of the anti-De Sitter supersymmetry algebra which contains the algebra $o(N)$ as a subalgebra. In this setup all gauge fields will be associated with some generators of the supersymmetry algebra. Since the fields with spins $s = 0$ or $1/2$ are not gauge fields, these do not result from the gauging procedure and have to be added by hand.

Thus the strategy is to take a set of fields of the supergravitational supermultiplets and try to construct a non-linear theory with local supersymmetry. Containing a massless spin-two field it should be a theory of gravity called *pure* N-extended supergravity to distinguish it from the models which contain also matter supermultiplets with spins $s \le 1$. Then one can extend pure supergravity by adding interactions with matter supermultiplets. There is a number of subtleties however.

First of all, matter supermultiplets that do not involve fields of $s > 1$ only exist at $N \le 4$. This means that at $N > 4$ only pure supergravities can be considered.

Even more important is that at $N > 8$ any supermultiplet, including the supergravitational one, contains some massless higher-spin fields which are themselves gauge fields. This raises two related issues. Firstly, as long as one faces problems with interacting higher-spin gauge fields, this implies that one should face these difficulties in any attempt to construct an interacting supergravity theory at $N > 8$. Secondly, if not convinced by the no-go higher-spin statements, one should find a larger *higher-spin algebra* that would give rise to the appropriate sets of higher-spin gauge fields upon its gauging. In fact, the next topic after our discussion of supergravity will be just the introduction of the simplest higher-spin algebra.

The example of supergravity illustrates an important criterium called the *admissibility condition* which distinguishes between algebras that have a chance to give rise to consistent non-trivial theories and those that do not. Namely, suppose we have an algebra h that contains some space-time symmetry algebra t as a subalgebra. (Say, t is the Poincaré or anti-De Sitter algebra.) Consider the gauge fields A associated with h and figure out the pattern of spins $\{s_a\}$ associated with A. Then consider unitary h-modules. The admissibility condition is that there should exist a unitary h-module V that, considered as a t-module, decomposes into the direct sum of massless submodules associated with precisely the same set of the gauge fields as in the list $\{s_a\}$. If the admissibility condition is satisfied, there is a chance to construct a consistent nontrivial gauge theory via the gauging of h. Otherwise, there are just no chances for that.

In the case of supergravity, the admissibility condition is satisfied for $N \le 8$ with the module V being the supergravitational supermultiplet. Strictly speaking, in the case of Poincaré supersymmetry one should start with

$$h = iso_N(3,1) \oplus u(1)^{\frac{N(N-1)}{2}}\,, \tag{11.97}$$

where the Abelian algebra $u(1)^{\frac{N(N-1)}{2}}$ is added to give rise to the spin-one fields from the supergravitational supermultiplet. (Alternatively, one can start with the anti-De Sitter supersymmetry algebra considered in the next section.) However, the admissibility condition is not satisfied for $N > 8$ because in this case every supergravitational supermultiplet contains higher-spin states with spins $s > 2$. So, one has to look for a larger algebra h that obeys the admissibility condition with some h-module V that contains higher-spin gauge fields along with the lower-spin fields of spins $s \leq 2$.

It should be stressed that the admissibility condition imposes severe conditions on symmetries h that can serve as gauge symmetries in consistent field-theoretical models. It is this condition that rules out most of the potential candidates for h.

11.4.4 $N = 1$ Poincaré supergravity

Consider the action for the $N = 1$, $d = 4$ supergravity in the form of the sum of the gravitational action (8.56) and the covariantized Rarita-Schwinger action (11.95)

$$S = -\frac{6}{\kappa}\int_{M^4}\Big(\epsilon_{n_1n_2n_3n_4}e^{n_1}e^{n_2}\mathcal{R}^{n_3n_4}(\omega) - 4ie^n\psi^{\hat{\alpha}}(\Gamma\gamma_n)_{\hat{\alpha}\hat{\beta}}\mathcal{R}^{\hat{\beta}}(\omega,\psi)\Big)\,, \tag{11.98}$$

where $\psi_{\hat{\alpha}}(x) = \xi^{\underline{n}}\psi_{\underline{n}\hat{\alpha}}(x)$ is a Majorana gravitino one-form.

Note that containing a dimensionful coupling constant this theory does not lead to a renormalizable quantum field theory. Nevertheless, supersymmetry softens divergencies. For example, any model of gravity with matter fields is generically divergent in the first order of the perturbation theory. However, because of supersymmetry this divergence cancels in $N = 1$, $d = 4$ supergravity. As I already mentioned, it is still an open question whether $N = 8$ supergravity is free of divergencies or not.

We assume that the Lorentz connection is chosen in such a way that

$$\omega^{nm} = \omega^{nm}(e,\psi)\,: \qquad \mathcal{R}^n = 0 \tag{11.99}$$

with $\mathcal{R}^n$ (11.79). The coefficient in front of the spin-3/2 action is adjusted in such a way that the constraint (11.99) solves the field equations for the Lorentz connection

$$\frac{\delta S}{\delta\omega^{nm}} = 0\,: \qquad \mathcal{R}^n = 0\,. \tag{11.100}$$

This convention allows us to use the 1.5-order formalism which simplifies the analysis a lot making it possible to ignore all terms due to complicated $\mathcal{R}$-dependent corrections to the variation of the ω^{nm}.[15]

[15] That such corrections are necessary follows from the transformation law (11.85) of $\mathcal{R}^n$. Indeed, in order to keep invariant the condition (11.99) it is necessary to modify the transformation law of the Lorentz connection by appropriate terms proportional to $\mathcal{R}^{nm}$ and $\mathcal{R}_{\hat{\alpha}}$.

Now we are in a position to check that the action (11.98) is locally supersymmetric. Direct computation gives the following variation of the action under the local supersymmetry transformation

$$\begin{aligned}\delta S = \; & -\frac{6}{\kappa}\int_{M^4}\Big(2\epsilon_{n_1n_2n_3n_4}e^{n_1}\delta e^{n_2}\mathcal{R}^{n_3n_4}(\omega) - 4ie^n(D^{\mathrm{L}}\varepsilon^{\alpha})(\Gamma\gamma_n)_{\hat\alpha}{}^{\hat\beta}\mathcal{R}_{\hat\alpha}(\omega,\psi) \\ & -2i\mathcal{R}^{nm}\psi_{\hat\alpha}e^k(\Gamma\gamma_k)^{\hat\alpha\hat\beta}\gamma_{nm\beta}{}^{\gamma}\varepsilon_\gamma - 4i\delta e^n\psi^{\hat\alpha}(\Gamma\gamma_n)_{\hat\alpha}{}^{\hat\beta}\mathcal{R}_{\hat\beta}(\omega,\psi)\Big)\end{aligned} \tag{11.101}$$

with

$$\delta e^n = -2\psi^{\hat\alpha}(x)\gamma^n{}_{\hat\alpha}{}^{\hat\beta}\varepsilon_{\hat\beta}(x)\,. \tag{11.102}$$ * removed $\underline{n}$

Integrating the second term by parts, using the Bianchi identities (11.83) and that the condition $\mathcal{R}^n = 0$ implies

$$D^{\mathrm{L}}e^n(x) = \psi^{\hat\alpha}(x)\gamma^n{}_{\hat\alpha}{}^{\hat\beta}\,\psi_{\hat\beta}(x)\,. \tag{11.103}$$ * removed .

Due to

$$\gamma_{knm} = i\epsilon_{knml}\Gamma\gamma^l \tag{11.104}$$

along with the symmetry properties of the γ-matrices, it easy to see that all terms linear in the fermions cancel.

The remaining terms are

$$\delta S = \frac{6}{\kappa}\int_{M^4}\Big(8i(\psi^{\hat\alpha}\gamma^n{}_{\hat\alpha}{}^{\hat\beta}\varepsilon_{\hat\beta})(\psi^{\hat\gamma}(\Gamma\gamma_n)_{\hat\gamma}{}^{\hat\delta}\mathcal{R}_{\hat\delta}) - 4i(\psi^{\hat\alpha}\gamma^n{}_{\hat\alpha}{}^{\hat\beta}\psi_{\hat\beta})(\varepsilon^{\hat\gamma}(\Gamma\gamma_n)_{\hat\gamma}{}^{\hat\delta}\mathcal{R}_{\hat\delta})\Big)\,. \tag{11.105}$$

To see that these terms also cancel one should again use the Fierz identity (11.35) along with the simple identity

$$\gamma_n\gamma_{kl}\gamma^n = 0 \tag{11.106}$$

which is true in $d = 4$.

In a certain sense, the most important part of the variation of the action under supersymmetry transformations is that linear in the fermionic field $\psi_{\hat\alpha}$. Namely this part of the gauge variation of the covariantized spin-3/2 action turns out to be proportional to the Einstein tensor, *i.e.*, to the left-hand side of the Einstein equations. This allowed us to cancel it by requiring that the frame field transforms nontrivially under the supertransformations. In fact this is not too surprising because any term constructed from $\psi_{\underline{n}\,\hat\alpha}$, $\varepsilon_{\hat\beta}$ and $R_{\underline{nm}}{}^{nm}$ having the algebraic properties of the Riemann tensor can only involve the Ricci tensor. An equivalent statement is that it is not possible to construct a Lorentz invariant from $\psi_{\underline{n}\,\hat\alpha}$, $\varepsilon_{\hat\beta}$ and the Weyl tensor (which is the traceless part of the Riemann tensor).

In the case of higher-spin gauge fields carrying more tensor indices the terms containing the Weyl tensor do exist and appear in the gauge variation of the covariantized higher-spin action. Their appearance seemingly makes the construction of a gauge invariant higher-spin action in the presence of gravity impossible. However, this problem finds a remarkable solution in the (anti-)De Sitter setup.

The largest dimension where a model of supergravity exists is $d = 11$. Like for $N > 8$ in $d = 4$, supermultiplets in $d > 11$ necessarily involve higher-spin states. Note that this is essentially the same restriction that the total number of components of supergenerators does not exceed 32. Indeed, the minimal Majorana-Weyl spinor in eleven dimensions just contains 32 components. The list of fields of $d = 11$ supergravity contains graviton, gravitino and the rank-three antisymmetric tensor gauge field. This theory is associated with the low-energy sector of mysterious *M-theory* conjectured to underly String theory

11.5 Anti-De Sitter supergravity

11.5.1 Anti-De Sitter supersymmetry

As explained in Section 9,, Poincaré symmetry admits a deformation to De Sitter and anti-De Sitter symmetries. A natural question is whether analogous deformation exists for Poincaré supersymmetry. The answer to this question depends on the conditions imposed on the deformed algebra h. Conventionally it is required that h should contain only space-time generators P_n and L_{nm}, supergenerators $Q^i_{\hat{\alpha}}$ and some internal generators T^a that commute with P_n and L_{nm}. This condition takes its origin in the Coleman-Mandula theorem and rules out higher-spin symmetries, severely restricting the class of possible deformations. In particular, anti-De Sitter deformations of the supersymmetry exist only in $d \leq 7$ while De Sitter deformations of supersymmetry do not exist at all. In this section we focus on the case of $d = 4$.

The N-extended AdS_4 supersymmetry algebras, called $osp(N,4)$, contain the generators

$$\mathcal{T}_{\hat{\alpha}\hat{\beta}} = \mathcal{T}_{\hat{\beta}\hat{\alpha}}\,, \qquad T^{ij} = -T^{ji}\,, \qquad \mathcal{Q}^i_{\hat{\alpha}}\,, \qquad \hat{\alpha},\hat{\beta} = 1,2,3,4\,, \qquad i,j = 1,2,\ldots,N\,, \tag{11.107}$$

which obey the commutation relations

$$\frac{1}{i}[\mathcal{T}_{\hat{\alpha}\hat{\beta}}\,,\mathcal{T}_{\hat{\gamma}\hat{\delta}}] = C_{\hat{\alpha}\hat{\gamma}}\mathcal{T}_{\hat{\beta}\hat{\delta}} + C_{\hat{\beta}\hat{\gamma}}\mathcal{T}_{\hat{\alpha}\hat{\delta}} + C_{\hat{\alpha}\hat{\delta}}\mathcal{T}_{\hat{\beta}\hat{\gamma}} + C_{\hat{\beta}\hat{\delta}}\mathcal{T}_{\hat{\alpha}\hat{\gamma}}\,, \tag{11.108}$$

$$\frac{1}{i}[T^{ij}\,,T^{kl}] = \delta^{jk}T^{il} - \delta^{ik}T^{jl} - \delta^{jl}T^{ik} + \delta^{il}T^{jk}\,, \tag{11.109}$$

$$\{\mathcal{Q}^i_{\hat{\alpha}}\,,\mathcal{Q}^j_{\hat{\beta}}\} = \delta^{ij}\mathcal{T}_{\hat{\alpha}\hat{\beta}} + C_{\hat{\alpha}\hat{\beta}}T^{ij}\,, \tag{11.110}$$

$$\frac{1}{i}[\mathcal{T}_{\hat{\alpha}\hat{\beta}}\,,\mathcal{Q}^i_{\hat{\gamma}}] = C_{\hat{\alpha}\hat{\gamma}}\mathcal{Q}^i_{\hat{\beta}} + C_{\hat{\beta}\hat{\gamma}}\mathcal{Q}^i_{\hat{\alpha}}\,, \tag{11.111}$$

$$\frac{1}{i}[T^{ij}\,,\mathcal{Q}^k_{\hat{\gamma}}] = \delta^{jk}\mathcal{Q}^i_{\hat{\gamma}} + \delta^{ik}\mathcal{Q}^j_{\hat{\gamma}}\,, \tag{11.112}$$

where $C_{\hat{\alpha}\hat{\beta}}$ is antisymmetric and nondegenerate. In the context of supersymmetry, the indices $\hat{\alpha},\hat{\beta}$ are interpreted as spinorial while $C_{\hat{\alpha}\hat{\beta}}$ is the charge-conjugation matrix.

If the indices $\hat{\alpha},\hat{\beta}$ take an even number of values M, the generators $\mathcal{T}_{\hat{\alpha}\hat{\beta}}$ form the *symplectic* Lie algebra $sp(M)$. Symplectic algebras are algebras of infinitesimal linear transformations that leave invariant an antisymmetric bilinear form. Suppose that $\phi_{\hat{\alpha}}$ are anticommuting variables

$$\phi_{\hat{\alpha}}\phi_{\hat{\beta}} = -\phi_{\hat{\beta}}\phi_{\hat{\alpha}}\,. \tag{11.113}$$

Then $sp(M)$ leaves the bilinear form $C^{\hat{\alpha}\hat{\beta}}\phi_{\hat{\alpha}}\phi_{\hat{\beta}}$ invariant. Analogously, $o(N)$ is the algebra of infinitesimal linear transformations that leave invariant a symmetric bilinear form $\delta^{ij}y_i y_j$ with commuting variables y_i, $i,j = 1,2,\ldots N$. These two algebras are unified in $osp(N,M)$, the superalgebra of infinitesimal linear transformations that leave invariant a *graded symmetric* bilinear form

$$A(\phi, y) = \delta^{ij}y_i y_j + C^{\hat{\alpha}\hat{\beta}}\phi_{\hat{\alpha}}\phi_{\hat{\beta}}\,. \tag{11.114}$$

Its even (bosonic) part is the Lie algebra $o(N) \oplus sp(M)$ with generators T^{ij} and $\mathcal{T}_{\hat{\alpha}\hat{\beta}}$. The odd generators $Q^i_{\hat{\alpha}}$ describe those transformations that mix commuting variables y_i with the anticommuting $\phi_{\hat{\alpha}}$. In this case the symmetry parameters are anticommuting being spinors $\varepsilon^{\hat{\alpha}}_i$.

The construction of the $4d$ AdS supersymmetry is based on the isomorphism $sp(4) \simeq o(3,2)$. The latter is most easily understood using the properties of the four-dimensional γ-matrices. Indeed, we know that the matrices $M_{nm} = \frac{i}{4}[\gamma_n\,, \gamma_m]$ obey the commutation relations of the Lorentz algebra $o(3,1)$. Introducing

$$P_n := \frac{i\lambda}{2}\gamma_n \tag{11.115}$$

we obtain that

$$[P_n\,, P_m] = i\lambda^2 M_{nm}\,, \qquad [M_{nm}\,, P_k] = i(\eta_{mk}P_n - \eta_{nk}P_m)\,. \tag{11.116}$$

Together with the commutation relations for M_{nm} these constitute the commutation relations of the anti-De Sitter algebra $o(3,2)$.

Note that the factor of i in (11.115) is chosen in such a way that P_n is Hermitian for the mostly-minus signature. This determines the sign of the right-hand side of the first commutation relation in (11.116), leading to the anti-De Sitter algebra rather than to De Sitter. This is one of the evidences that spinors in any dimension enjoy anti-De Sitter but not De Sitter symmetry.

To prove the isomorphism $sp(4) \simeq o(3,2)$ one next observes that in $d = 4$ the matrices $\gamma_n^{\hat{\alpha}\hat{\beta}}$ and $\gamma_{nm}^{\hat{\alpha}\hat{\beta}}$ form the full set of matrices that are symmetric in the spinor indices $\hat{\alpha}, \hat{\beta}$. (Compare the number of matrices γ_n and γ_{nm} with the number of symmetric 4×4 matrices.)

The explicit relations are

$$P_n = \lambda {\gamma_n}^{\hat{\alpha}\hat{\beta}}\mathcal{T}_{\hat{\alpha}\hat{\beta}}\,, \qquad M_{nm} = -\frac{1}{2}{\gamma_{nm}}^{\hat{\alpha}\hat{\beta}}\mathcal{T}_{\hat{\alpha}\hat{\beta}}\,. \tag{11.117}$$

By this procedure the matrix $C_{\hat{\alpha}\hat{\beta}}$ in the relations (11.108) is indeed identified with the charge conjugation matrix in four dimensions.

Let us now show that $osp(N,4)$ is indeed a deformation of $iso_N(3,1)$ with deformation parameter $\lambda \sim R^{-1}$ where R is the AdS radius. To this end we rewrite the commutation relations (11.108)–(11.110) of $osp(N,4)$ in terms of two-component spinors, setting

$$P_{\alpha\dot{\alpha}} = \lambda\mathcal{T}_{\alpha\dot{\alpha}}\,, \qquad M_{\alpha\beta} = \mathcal{T}_{\alpha\beta}\,, \qquad \bar{M}_{\dot{\alpha}\dot{\beta}} = \mathcal{T}_{\dot{\alpha}\dot{\beta}}\,, \tag{11.118}$$

$$Q^i_\alpha = \sqrt{\lambda}\mathcal{Q}^i_\alpha\,, \qquad \bar{Q}^i_{\dot\alpha} = \sqrt{\lambda}\mathcal{Q}^i_{\dot\alpha}\,, \tag{11.119}$$

where λ is a positive number. In these terms the most interesting relations of the algebra $osp(N,4)$ take the form

$$\frac{1}{i}[P_{\alpha\dot\alpha}\,, P_{\beta\dot\beta}] = \lambda^2\big(\,\epsilon_{\alpha\beta}\bar{M}_{\dot\alpha\dot\beta} + \bar\epsilon_{\,\dot\alpha\dot\beta}L_{\alpha\beta}\big)\,, \tag{11.120}$$ * all ϵ instead of ε

$$\frac{1}{i}[P_{\alpha\dot\beta}\,, Q_\gamma] = \lambda\,\epsilon_{\alpha\gamma}\bar{Q}_{\dot\beta}\,, \qquad \frac{1}{i}[P_{\alpha\dot\beta}\,, \bar{Q}_{\dot\delta}] = \lambda\,\bar\epsilon_{\,\dot\beta\dot\delta}\bar{Q}_\alpha \tag{11.121}$$ *

$$\{Q^i_\alpha\,, Q^j_\beta\} = \lambda\big(\delta^{ij}L_{\alpha\beta} + \epsilon_{\alpha\beta}T^{ij}\big)\,, \qquad \{\bar{Q}^i_{\dot\alpha}\,, \bar{Q}^j_{\dot\beta}\} = \lambda\big(\delta^{ij}\bar{L}_{\dot\alpha\dot\beta} + \bar\epsilon_{\,\dot\alpha\dot\beta}T^{ij}\big)\,, \tag{11.122}$$ *

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$$\{Q^i_\alpha\,, \bar{Q}^j_{\dot\beta}\} = \delta^{ij}P_{\alpha\dot\beta}\,. \tag{11.123}$$

Other nonzero relations just imply covariance under the Lorentz and $o(N)$-transformations. The latter have the same form as in (11.112). The former read

$$\frac{1}{i}[L_{\alpha\beta}\,, L_{\gamma\delta}] = \,\epsilon_{\alpha\gamma}L_{\beta\delta} + \,\epsilon_{\beta\gamma}L_{\alpha\delta} + \,\epsilon_{\alpha\delta}L_{\beta\gamma} + \,\epsilon_{\beta\delta}L_{\alpha\gamma}\,, \tag{11.124}$$ *

$$\frac{1}{i}[\bar{L}_{\dot\alpha\dot\beta}\,, \bar{L}_{\dot\gamma\dot\delta}] = \,\bar\epsilon_{\,\dot\alpha\dot\gamma}\bar{L}_{\dot\beta\dot\delta} + \,\bar\epsilon_{\,\dot\beta\dot\gamma}\bar{L}_{\dot\alpha\bar\delta} + \,\bar\epsilon_{\,\dot\alpha\dot\delta}\bar{L}_{\dot\beta\dot\gamma} + \,\bar\epsilon_{\,\dot\beta\dot\delta}\bar{L}_{\alpha\gamma}\,, \tag{11.125}$$ *

$$\frac{1}{i}[L_{\alpha\beta}\,, P_{\gamma\dot\delta}] = \,\epsilon_{\alpha\gamma}P_{\beta\dot\delta} + \,\epsilon_{\beta\gamma}P_{\alpha\dot\delta}\,, \qquad \frac{1}{i}[\bar{L}_{\dot\alpha\dot\beta}\,, P_{\gamma\dot\delta}] = \,\bar\epsilon_{\,\dot\alpha\dot\delta}\bar{P}_{\gamma\dot\beta} + \,\bar\epsilon_{\,\dot\beta\dot\delta}\bar{P}_{\gamma\dot\alpha}\,, \tag{11.126}$$ *

$$\frac{1}{i}[L_{\alpha\beta}\,, Q^j_\gamma] = \,\epsilon_{\alpha\gamma}Q^j_\beta + \,\epsilon_{\beta\gamma}Q^j_\alpha\,, \qquad \frac{1}{i}[\bar{L}_{\dot\alpha\dot\beta}\,, \bar{Q}^j_{\dot\delta}] = \,\bar\epsilon_{\,\dot\alpha\dot\delta}Q^j_{\dot\beta} + \,\bar\epsilon_{\,\dot\beta\dot\delta}\bar{Q}^j_{\dot\alpha}\,. \tag{11.127}$$ *

We see that at $\lambda = 0$ this algebra indeed reproduces $iso_N(3,1)$ extended by the $o(N)$ generators to the semidirect sum $o(N) \ni\!\!\!+ iso_N(3,1)$ (here $iso_N(3,1)$ is an ideal). Note that this $o(N)$ is usually called *R-symmetry*.

In the De Sitter case the parameter λ^2 in the relation (11.120) is negative. In the absence of supergenerators this is not a problem since the commutation relations contain λ only through λ^2. However the commutation relations of the supergenerators (11.121) contain the parameter λ itself, which has to be purely imaginary in the De Sitter case. This spoils the compatibility of the commutation relations with the Hermiticity properties of the generators, making the De Sitter supersymmetry ill-defined.

Let us note that the algebras $osp(N,M)$ with $M > 4$ (usually $M = 2^n$) are also interesting and appear in different contexts in string theory and higher-spin theory. However, for $M > 4$ they extend usual supersymmetry by additional generators that carry tensor indices which can be interpreted as particular higher-spin generators.

11.5.2 Anti-De Sitter supergravity

The gauging of $osp(N,4)$ gives rise to one-form gauge fields

$$A(x) = i\Big(e^{\alpha\dot\alpha}(x)P_{\alpha\dot\alpha} + \frac{1}{2}\omega^{\alpha\beta}(x)L_{\alpha\beta} + \frac{1}{2}\bar\omega^{\dot\alpha\dot\beta}(x)\bar L_{\dot\alpha\dot\beta} + \frac{1}{2}A^{ij}T_{ij}\Big) + Q^\alpha_i\psi^i_\alpha(x) + \bar\psi^{\dot\alpha}_i(x)\bar Q^i_{\dot\alpha} \tag{11.128}$$

and two-form curvatures

$$R(x) = i\Big(R^{\alpha\dot\alpha}(x)P_{\alpha\dot\alpha} + \frac{1}{2}R^{\alpha\beta}(x)L_{\alpha\beta} + \frac{1}{2}\bar R^{\dot\alpha\dot\beta}(x)\bar L_{\dot\alpha\dot\beta} + \frac{1}{2}F^{ij}T_{ij}\Big) + Q^\alpha_i R^i_\alpha(x) + \bar R^{\dot\alpha}_i(x)\bar Q^i_{\dot\alpha} \tag{11.129}$$

with

$$R_{\alpha\dot\alpha} = D^{\mathrm{L}}e_{\alpha\dot\alpha} + \psi^i_\alpha\bar\psi^j_{\dot\alpha}\delta_{ij}\,, \tag{11.130}$$ * removed (x)

$$R_{\alpha\beta} = \mathrm{d}\omega_{\alpha\beta} + \omega_\alpha{}^\gamma\omega_{\gamma\beta} + \lambda^2 e_\alpha{}^{\dot\gamma}e_{\beta\dot\gamma} + \lambda\psi^i_\alpha\psi^j_\beta\delta_{ij}\,, \tag{11.131}$$

$$\bar R_{\dot\alpha\dot\beta} = \mathrm{d}\bar\omega_{\dot\alpha\dot\beta} + \bar\omega_{\dot\alpha}{}^{\dot\gamma}\bar\omega_{\dot\gamma\dot\beta} + \lambda^2 e^\alpha{}_{\dot\alpha}e_{\alpha\dot\beta} + \lambda\bar\psi^i_{\dot\alpha}\bar\psi^j_{\dot\beta}\delta_{ij}\,, \tag{11.132}$$

$$R^i_\alpha = D^{\mathrm{L}}\psi^i_\alpha + A^i{}_j\psi^j_\alpha + \lambda e_\alpha{}^{\dot\alpha}\bar\psi^i_{\dot\alpha}\,, \qquad \bar R^i_{\dot\alpha} = D^{\mathrm{L}}\bar\psi^i_{\dot\alpha} + A^i{}_j\bar\psi^j_{\dot\alpha} + \lambda e^\alpha{}_{\dot\alpha}\psi^i_\alpha\,, \tag{11.133}$$

$$F_{ij} = \mathrm{d}A_{ij} + A_i{}^k A_{kj}\,, \tag{11.134}$$

where D^{L} is the usual Lorentz-covariant derivative * corrected eqn

$$D^{\mathrm{L}}e_{\alpha\dot\alpha} = \mathrm{d}e_{\alpha\dot\alpha} + \omega_\alpha{}^\beta e_{\beta\dot\alpha} + \bar\omega_{\dot\alpha}{}^{\dot\beta}e_{\alpha\dot\beta}\,, \tag{11.135}$$

$$D^{\mathrm{L}}\psi^i_\alpha = \mathrm{d}\psi^i_\alpha + \omega_\alpha{}^\gamma\psi^i_\gamma\,, \qquad D^{\mathrm{L}}\bar\psi^i_{\dot\alpha} = \mathrm{d}\bar\psi^i_{\dot\alpha} + \bar\omega_{\dot\alpha}{}^{\dot\beta}\bar\psi^i_{\dot\beta}\,. \tag{11.136}$$ * removed D^{L}

The list of gauge fields of $osp(N,4)$ contains one spin-two field, N spin-3/2 fields and $\frac{N(N-1)}{2}$ spin-one fields. Since this precisely matches the $N \le 8$ supergravitational multiplets, $osp(N,4)$ satisfies the admissibility condition from Section 11.4.3.

In terms of the curvatures of $osp(1,4)$, it is easy to write down the action for $N=1$ supergravity in the MacDowell-Mansouri form

$$S^{MM} = \frac{i}{4\kappa\lambda^2}\int_{M^4}\Big(R_{\alpha\beta}R^{\alpha\beta} - \bar R_{\dot\alpha\dot\beta}\bar R^{\dot\alpha\dot\beta} + 2\lambda(R_\alpha R^\alpha - \bar R_{\dot\alpha}\bar R^{\dot\alpha})\Big)\,. \tag{11.137}$$

Here the relative coefficient between the bosonic and fermionic parts of the action is adjusted in such a way that the equations of motion for the Lorentz connection give

$$\frac{\delta S}{\delta\omega^{\alpha\beta}} = 0\,, \qquad \frac{\delta S}{\delta\bar\omega^{\dot\alpha\dot\beta}} = 0 \quad\Longrightarrow\quad R^{\alpha\dot\alpha} = 0\,. \tag{11.138}$$

We suggest to the reader to check this fact.

This allows us to use the 1.5-order formalism assuming that the Lorentz connection solves (11.138) so that we can neglect all terms which contain $R^{\alpha\dot\alpha}$. Using that the supersymmetry transformation law for the curvatures has the form

$$\delta R_{\alpha\beta} = \lambda(\varepsilon_\alpha R_\beta + \varepsilon_\beta R_\alpha)\,, \qquad \delta\bar R_{\dot\alpha\dot\beta} = \bar\varepsilon_{\dot\alpha}\bar R_{\dot\beta}\,, \tag{11.139}$$

$$\delta R_\alpha = \varepsilon_\beta R^\beta{}_\alpha + \lambda\bar\varepsilon_{\dot\beta}R_\alpha{}^{\dot\beta}\,, \qquad \delta\bar R_{\dot\alpha} = \bar\varepsilon_{\dot\beta}\bar R^{\dot\beta}{}_{\dot\alpha} + \lambda\varepsilon_\beta R^\beta{}_{\dot\alpha} \tag{11.140}$$

and neglecting the terms with $R^{\alpha\dot\alpha}$ we see that the variation of the action S^{MM} under the supertransformations vanishes.

It is straightforward to see that the bosonic part of the action (11.137) is the gravitational MacDowell-Mansouri action rewritten in the language of two-component spinors. The factor of i in front of the whole action encodes the Levi-Civita symbol in the tensor notation. The fermionic part of the action contains a term with two derivatives, which is a total derivative (topological invariant), terms with one derivative, forming the covariantized Rarita-Schwinger action, and a mass-like term proportional to $\lambda\psi^{\alpha}_{\underline{n}}\sigma^{\underline{nm}}_{\alpha\beta}\psi^{\beta}_{\underline{m}} + \lambda\bar\psi^{\dot\alpha}_{\underline{n}}\bar\sigma^{\underline{nm}}_{\dot\alpha\dot\beta}\bar\psi^{\dot\beta}_{\underline{m}}$.

To understand the action (11.137) better one can observe the following.

The terms involving four gravitini resulting from the last terms in (11.131)–(11.132) independently vanish.

The terms from (11.131)–(11.132) involving two gravitini and at least one Lorentz connection, along with the terms with two Lorentz-covariant derivatives coming from (11.133), is a total derivative.

The terms containing precisely one Lorentz-covariant derivative of the gravitini yield the covariantized version of the Rarita-Schwinger action (11.89) in the language of two-component spinors. To this end it is useful to take into account that $ie_{\alpha\dot\beta}$ is the two-component realization of $\xi^{\underline{n}}\gamma_{\underline{n}}\Gamma$ since the factor of i shows that $ie_{\alpha\dot\beta}$ is parity odd. The remaining terms of the fermionic part is a mass-like term, proportional to $\lambda\left(e_{\alpha}{}^{\dot\beta}e^{\alpha\dot\gamma}\bar\psi_{\dot\beta}\bar\psi_{\dot\gamma} - e^{\beta}{}_{\dot\alpha}e^{\gamma\dot\alpha}\psi_{\beta}\psi_{\gamma}\right)$.

The bosonic, i.e. ψ-independent, part of (11.137) is the ordinary, i.e. non-supersymmetric, MacDowell-Mansouri action (9.24) in the language of two-component spinors.

Note that mass-like terms for gauge fields of any spin $s > 1$ in anti-De Sitter space are non-zero, being proportional to the inverse radius λ of the anti-De Sitter space for dimensional reasons these mass-like terms tend to zero in the flat limit $\lambda = 0$. The exact coefficient in front of the mass-like term is spin-dependent and is fixed by demanding the gauge invariance under the higher-spin symmetries. Such terms are needed because the Lorentz-covariant derivatives do not commute in the anti-De Sitter background.

The action (11.137) admits a straightforward generalization to higher-spins. The main issue is to find an appropriate higher-spin symmetry algebra that gives rise to the proper set of gauge fields to describe the Fronsdal fields. This is our next topic. Before explaining the structure of the higher-spin algebra, however, let me say a few concluding words on supersymmetry and supergravity.

Unfortunately, at least in its present form, the MacDowell-Mansouri approach is less efficient for the lower spins $s = 1, 1/2$ and 0. This makes the problem of construction of the supergravity actions with $N > 2$ and/or in the presence of matter supermultiplets more involved. In general, the analysis of various models of supergravity is rather involved and differs significantly for the models with different N and in various dimensions.

To finish this discussion of supersymmetric models it should be mentioned that many important ideas underlying these theories were not mentioned. In the first place this concerns the *superspace* approach in which the supersymmetry is realized geometrically in an appropriate superspace with anticommuting coordinates

$\theta^i_{\hat{\alpha}}$. Note that the properties of superspace are not hard to deduce from the form of the supergravity curvatures in full analogy with the description in Section 8.2 of Minkowski space as a solution of the zero-curvature equations for Poincaré curvatures.

12 Frame-like formulation for higher-spin fields

Now we start the analysis of higher-spin connections. The plan is to use some higher-spin algebra h such that its gauge fields describe the higher-spin gauge fields. This means that the higher-spin gauge fields should be described by some set of one-forms analogously to how gravity is described by the frame field e^n and ω^{nm} in the Cartan formalism. This approach to the higher-spin theory is called the *frame-like* formalism. For simplicity we start with a Minkowski background.

12.1 Higher-spin connections

Before introducing the full nonlinear higher-spin algebra let us first analyze how the Fronsdal theory of free higher-spin gauge fields (see Section 4.2) can be reformulated in terms of Yang-Mills-like gauge connections. To this end we replace the spin-s Fronsdal field by a frame-like one-form as follows

$$\varphi_{m_1\cdots m_s} \qquad \Rightarrow \qquad e^{n_1\cdots n_{s-1}} = \xi^{\underline{m}}\, e_{\underline{m}}{}^{\;n_1\cdots n_{s-1}}\,. \tag{12.1}$$

The frame-like field is totally symmetric in the indices n_i $(i = 1, \cdots, s-1)$ and traceless

$$\eta_{n_1 n_2}\, e^{n_1 n_2\cdots n_{s-1}} \;=\; 0\,. \tag{12.2}$$

The Fronsdal field is identified with the symmetric part of the frame-like field

$$\varphi_{m_1\cdots m_s} := e_{(m_1\, m_2 \ldots m_s)}\,, \tag{12.3}$$

where, working in a Cartesian coordinate system, we identify fiber and base indices. As a consequence of (12.2), the so-defined $\varphi_{m_1\cdots m_s}$ is double traceless, *i.e.*, $\varphi^{nk}{}_{nkm_5\cdots m_s} = 0$. Thus the peculiar double-tracelessness property of the Fronsdal field acquires a simple explanation in the frame-like formalism.

However, analogously to the relation between the deviation of the metric field $g_{\underline{nm}}$ from the flat metric

$$g_{\underline{nm}} = \eta_{\underline{nm}} + \varphi_{\underline{nm}} \tag{12.4}$$

and the deviation $e'_{\underline{n}}{}^n$ of the frame field

$$e_{\underline{n}}{}^n = \delta^n_{\underline{n}} + e'_{\underline{n}}{}^n\,, \tag{12.5}$$

the frame-like higher-spin field contains more components than the Fronsdal fields. As for gravity, to make the two descriptions equivalent we should postulate an additional Lorentz-like gauge transformation to gauge away the non–symmetric components of the frame field. As a result, the full gauge-symmetry transformation of the higher-spin vielbein is

$$\delta\, e^{n_1\cdots n_{s-1}} = \mathrm{d}\varepsilon^{n_1\cdots n_{s-1}} + \mathrm{e}^m\, \varepsilon^{n_1\cdots n_{s-1},}{}_m\,, \tag{12.6}$$

where $\mathrm{e}^m = \xi^m$ is the frame-field of the background Minkowski space, $\varepsilon^{n_1\cdots n_{s-1}}(x)$ is the parameter of higher-spin gauge symmetry in the Fronsdal formulation, while the parameter $\varepsilon^{n_1\cdots n_{s-1},m}(x)$ corresponds to the Lorentz-like symmetry. The latter is symmetric in the n_i and has the further property that the total symmetrization in n_i and m vanishes

$$\varepsilon^{(n_1\cdots n_{s-1},m)} = 0\,. \tag{12.7}$$

All gauge parameters are traceless:

$$\eta_{n_1n_2}\,\varepsilon^{n_1n_2\cdots n_{s-1}} \;=\; 0\,; \qquad \eta_{n_1n_2}\,\varepsilon^{n_1n_2\cdots n_{s-1},m} \;=\; 0 \quad \Rightarrow \quad \eta_{n_1m}\,\varepsilon^{n_1n_2\cdots n_{s-1},m} \;=\; 0\,. \tag{12.8}$$

The conditions (12.7), (12.8) implies that $\varepsilon^{n_1n_2\cdots n_{s-1},m}$ has the symmetry properties of the $o(N)$-Young diagram $\begin{smallmatrix}\overbrace{\square\square\square\square\square\square\square}^{s-1}\\ \square\end{smallmatrix}$. As a consequence of (12.7), the gauge transformation for the Fronsdal field takes the anticipated Fronsdal form

$$\delta\,\varphi_{m_1\ldots m_s} = \partial_{(m_s}\,\varepsilon_{m_1\ldots m_{s-1})}\,. \tag{12.9}$$

The next step is to introduce the analog of the spin connection, *i.e.*, the gauge field associated with the new local symmetry:

$$\omega^{n_1\cdots n_{s-1},m} = \xi^{\underline{m}}\,\omega_{\underline{m}}{}^{n_1\cdots n_{s-1},m}\,, \tag{12.10}$$

which transforms as

$$\delta\,\omega^{n_1\cdots n_{s-1},m} = \mathrm{d}\varepsilon^{n_1\cdots n_{s-1},m}\,. \tag{12.11}$$

The gauge field $\omega^{n_1\cdots n_{s-1},m}$ has the same tensor properties as the gauge parameter $\varepsilon^{n_1\cdots n_{s-1},m}$, *i.e.*,

$$\omega^{(n_1\cdots n_{s-1},m)} = 0\,, \tag{12.12}$$

$$\eta_{n_1n_2}\,\omega^{n_1n_2\cdots n_{s-1},m} \;=\; 0 \quad \Rightarrow \quad \eta_{n_1m}\,\omega^{n_1n_2\cdots n_{s-1},m} \;=\; 0\,. \tag{12.13}$$

The gauge-invariant torsion–like tensor is

$$R^{n_1\cdots n_{s-1}} = \mathrm{d}\,e^{n_1\cdots n_{s-1}} + \mathrm{e}_m\omega^{n_1\cdots n_{s-1},m}\,. \tag{12.14}$$

The novelty compared to the spin-two case is that it is also invariant under transformations of ω of the form

$$\delta\omega^{n_1\cdots n_{s-1},l} \;=\; \mathrm{e}_m\,\varepsilon^{n_1\cdots n_{s-1},lm}\,, \tag{12.15}$$

where the parameter $\varepsilon^{n_1\cdots n_{s-1},m_1m_2}$ is symmetric separately in the two sets of n- and m- indices and is subject to the condition $\varepsilon^{(n_1\cdots n_{s-1},m_1)m_2} = 0$, which is tantamount to a $\begin{smallmatrix}\overbrace{\square\square\square\square\square\square\square}^{s-1}\\ \square\square\end{smallmatrix}$ Young projection. This projection eliminates this parameter at $s \leq 2$, simply because the corresponding Young diagrams vanish identically. For $s \geq 3$, however, the additional gauge symmetry plays an important role, eliminating the redundant degrees of freedom of the spin connection $\omega_{\underline{m}}{}^{n_1\ldots n_{s-1},m}$.

The role of the symmetry (12.15) for the Lorentz-like connection is analogous to that of the Lorentz-like symmetry of the frame-like (vielbein) field in the gauge transformation (12.6). Hence we introduce a new gauge field

$$\omega^{n_1\cdots n_{s-1},m_1m_2} = \xi^{\underline{m}}\,\omega_{\underline{m}}{}^{n_1\cdots n_{s-1},m_1m_2}\,. \tag{12.16}$$

It has the symmetry properties of the parameters (12.15), and transforms as

$$\delta\,\omega^{n_1\cdots n_{s-1},m_1m_2} = \mathrm{d}\varepsilon^{n_1\cdots n_{s-1},m_1m_2}\,. \tag{12.17}$$

Using (12.10) and (12.16) one can construct a linearized curvature which is invariant under the transformations (12.11)–(12.17)

$$R^{n_1\cdots n_{s-1},m_1} = \mathrm{d}\,\omega^{n_1\cdots n_{s-1},m_1} + \mathrm{e}_{m_2}\,\omega^{n_1\cdots n_{s-1},m_1m_2}\,. \tag{12.18}$$

The curvature (12.18) is also invariant under the local gauge transformations

$$\delta\,\omega^{n_1\cdots n_{s-1},m_1m_2} = \mathrm{e}_{m_3}\,\varepsilon^{n_1\cdots n_{s-1},m_1m_2m_3} \tag{12.19}$$

where the parameters $\varepsilon^{n_1\cdots n_{s-1},m_1m_2m_3}$ are symmetric within the sets of indices n and m and satisfy the irreducibility condition $\varepsilon^{(n_1\cdots n_{s-1},m_1)m_2m_3} = 0$.

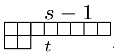
One should now introduce the connection corresponding to this gauge transformation $\omega^{n_1\cdots n_{s-1},m_1m_2m_3}$, and so on, until the number of the indices m and n become equal so that the procedure stops since the corresponding tensors (Young diagrams) with $t > s-1$ simply do not exist.

Summarizing, the higher–spin frame–like construction for spin $s \geq 1$ requires a set of one–form connections labeled by the index $0 \leq t \leq s-1$

$$\omega^{n_1\ldots n_{s-1}\,,m_1\ldots m_t} = \xi^{\underline{m}}\omega_{\underline{m}}{}^{n_1\ldots n_{s-1},\,m_1\ldots m_t}\,, \tag{12.20}$$

which are symmetric in the sets of n_i and m_j fiber indices, satisfy the (anti)symmetry condition (3.52)

$$\omega^{(n_1\ldots n_{s-1}\,,n_s)m_2\ldots m_t} = 0\,, \tag{12.21}$$

and therefore possess the symmetry properties of the Young diagram , where the number of indices in the upper row is fixed for a given spin s while the number of indices t in the lower row ranges from $t=0$ to $t=s-1$. The higher-spin connections are traceless

$$\eta_{n_1n_2}\,\omega^{n_1n_2\cdots n_{s-1},m_1\cdots m_t} = 0\,, \qquad \eta_{n_1m_1}\,\omega^{n_1n_2\cdots n_{s-1},m_1\cdots m_t} = 0\,,$$

$$\eta_{m_1m_2}\,\omega^{n_1n_2\cdots n_{s-1},m_1\cdots m_t} = 0\,. \tag{12.22}$$

The $t=0$ field of this set is the higher–spin frame-like field,

$$e^{n_1\ldots n_{s-1}} := \xi^{\underline{m}}\omega_{\underline{m}}{}^{n_1\ldots n_{s-1}}\,, \tag{12.23}$$

while the next member, with $t=1$, is the Lorentz-like connection (12.10), which is an auxiliary field similar to the usual Lorentz connection for gravity. The remaining fields with $t > 1$, called *extra* fields, appear for $s > 2$. The different names, "auxiliary" and "extra", are meant to stress their different roles in higher–spin dynamics. The auxiliary fields, together with the higher–spin vielbeins, are the only ones which enter two-derivative free higher-spin actions, while the extra fields play an important role at the non-linear level.

The full set of the linearized higher-spin curvatures

$$\begin{aligned} R^{n_1 \dots n_{s-1},\, m_1 \dots m_t} &= \mathrm{d}\omega^{n_1 \dots n_{s-1},\, m_1 \dots m_t} + \mathrm{e}_k\, \omega^{n_1 \dots n_{s-1},\, m_1 \dots m_t\, k} \quad (t < s-1)\,; \\ R^{n_1 \dots n_{s-1},\, m_1 \dots m_{s-1}} &= \mathrm{d}\omega^{n_1 \dots n_{s-1},\, m_1 \dots m_{s-1}} \end{aligned} \tag{12.24}$$

is invariant under the higher–spin gauge transformations

$$\delta\omega^{n_1 \dots n_{s-1},\, m_1 \dots m_t} = \mathrm{d}\,\varepsilon^{n_1 \dots n_{s-1},\, m_1 \dots m_t} + \mathrm{e}_k\, \varepsilon^{n_1 \dots n_{s-1},\, m_1 \dots m_t\, k}\,, \tag{12.25}$$

$$\delta\omega^{n_1 \dots n_{s-1}, m_1 \dots m_{s-1}} = \mathrm{d}\varepsilon^{n_1 \dots n_{s-1}, m_1 \dots m_{s-1}} \tag{12.26}$$

This is easy to check using that $\mathrm{e}_k \mathrm{e}_l = -\mathrm{e}_l \mathrm{e}_k$ since e_k is a one-form. Also one can see that the curvatures (12.24) obey the proper symmetry and tracelessness conditions, analogous to (12.21)–(12.22), with respect to the fiber indices.

12.2 Free action

The frame-like analog of the spin-s Fronsdal action has the form

$$S = \int_{M^d} \mathrm{e}^{k_1} \dots \mathrm{e}^{k_{d-3}}\, \epsilon_{k_1 \dots k_{d-3} pqr}\, (\mathrm{d}\, e^{n_1 \dots n_{s-2} p} + \frac{1}{2}\, \mathrm{e}_m\, \omega^{n_1 \dots n_{s-2} p,\, m})\, \omega_{n_1 \dots n_{s-2}}{}^{q,\, r}. \tag{12.27}$$

The part of this action bilinear in the higher–spin connection is symmetric under the exchange of its components, *i.e.*,

$$\mathrm{e}^{k_1} \dots \mathrm{e}^{k_{d-3}} \mathrm{e}_m\, \epsilon_{k_1 \dots k_{d-3} pqr}\, ({\omega_1}^{n_1 \dots n_{s-2} p,\, m}\, {\omega_2}_{n_1 \dots n_{s-2}}{}^{q,\, r} - {\omega_2}^{n_1 \dots n_{s-2} p,\, m}\, {\omega_1}_{n_1 \dots n_{s-2}}{}^{q,\, r}) = 0 \tag{12.28}$$

as a consequence of the symmetry properties and tracelessness of the connection. Using this it is easy to check that the action is invariant under all gauge symmetries. By virtue of this property, the action (12.27) is invariant under the gauge transformations (12.25) as a consequence of gauge invariance of the curvatures (12.24).

The equations of motion that follow from this action consist of the zero-torsion-like equation

$$\frac{\delta S}{\delta \omega}\ : \qquad R^{n_1 \cdots n_{s-1}} = 0 \tag{12.29}$$

and

$$\frac{\delta S}{\delta e}\ : \qquad \eta^{q(n_{s-1}} R_{mq}{}^{n_1 \cdots n_{s-2})[p,\, m]} = 0\,. \tag{12.30}$$

I believe that this problem has to be worked out in the text.

Since the Fronsdal action (4.13) is uniquely fixed by its field content and the symmetry principles (*cf.* Problem 4.4), it follows that the action (12.27) is equivalent to the Fronsdal action (maybe up to a factor).

12.3 σ_--cohomology analysis and the First on-shell theorem

The curvatures (12.24) and the gauge-transformation law (12.25) can be represented in a coordinate-independent form as follows:

$$R^{n_1\ldots n_{s-1},\, m_1\ldots m_t} = D^{\mathrm{L}}\omega^{n_1\ldots n_{s-1},\, m_1\ldots m_t} + (\sigma_-(\omega))^{n_1\ldots n_{s-1},\, m_1\ldots m_t}\,, \tag{12.31}$$

$$\delta\omega^{n_1\ldots n_{s-1},\, m_1\ldots m_t} = D^{\mathrm{L}}\varepsilon^{n_1\ldots n_{s-1},\, m_1\ldots m_t} + (\sigma_-(\varepsilon))^{n_1\ldots n_{s-1},\, m_1\ldots m_t}\,, \tag{12.32}$$

where

$$\sigma_-(A)^{n_1\ldots n_{s-1},\, m_1\ldots m_t} := \mathrm{e}_p\, A^{n_1\ldots n_{s-1},\, m_1\ldots m_t\, p}\,, \qquad \sigma_-(A)^{n_1\ldots n_{s-1}} := 0 \tag{12.33}$$

and D^{L} and e_n are the Lorentz-covariant derivative and the frame one-form of the background Minkowski space, *i.e.*,

$$D^{\mathrm{L}}\mathrm{e}^n = 0\,, \qquad D^{\mathrm{L}}D^{\mathrm{L}} = 0\,. \tag{12.34}$$

Let V^s_p be the space of p-forms $(A)^{n_1\ldots n_{s-1},\, m_1\ldots m_t}$ valued in two-row traceless tensors. Then σ_- maps V^s_p to V^s_{p+1} and obeys

$$\sigma_-^2 = 0\,. \tag{12.35}$$

Note that σ_- indeed acts on p-forms valued in two-row traceless tensors.

podrobnee In fact, such a formulism can be applied to a very general class of dynamical systems. Independently of a particular system in question, the operator σ_- encodes the information of the dynamical content of the system via its cohomology groups $H^p(\sigma_-) := \mathrm{Ker}\,\sigma_-(\mathrm{V_p})/\mathrm{Im}\,\sigma_-(\mathrm{V_{p-1}})$. (Compare with the de Rham cohomology in the footnote in Section 7.2.1.) Namely, skipping details, it can be shown that, for the one-form connections, there are as many *differential gauge-symmetry parameters* as elements of $H^0(\sigma_-)$, as many *dynamical fields* that can neither be expressed via derivatives of the other fields nor gauged away by the algebraic (Stueckelberg) gauge symmetries as elements of $H^1(\sigma_-)$, and as many *gauge invariant combinations of derivatives of the dynamical fields* as elements of $H^2(\sigma_-)$.

Indeed, consider the curvatures (12.31). All those fields that are not annihilated by σ_- can be expressed via derivatives of the lower fields (in our case, those with smaller t) by setting to zero the corresponding components of the curvatures. It should be stressed that the corresponding expressions respect all gauge symmetries since the curvatures (12.31) are gauge invariant under the gauge transformations (12.32). Thus, independent fields can only be associated with those components of the one-forms $\omega^{n_1\ldots n_{s-1},\, m_1\ldots m_t}$ that are annihilated by σ_-. In the case of interest the only such field is the $t=0$ frame-like higher-spin field $e^{n_1\ldots n_{s-1}}$. However, those components of the σ_--closed fields that are σ_--exact can be gauge-fixed to zero using the shift gauge symmetries associated with the σ_--exact terms in the transformation law (12.32). In the frame-like field sector this is the Lorentz-like gauge symmetry. So, we arrive at the conclusion that $H^1(\sigma_-)$ just describes the Fronsdal field $\varphi_{n_1\ldots n_s}$. This is of course anticipated since our system was designed to describe the Fronsdal field.

The analysis of differential gauge parameters is analogous. All those parameters ε that are not annihilated by σ_- in (12.32) describe Stueckelberg shift symmetries that eliminate the redundant components of the connections ω. The differential symmetries are therefore associated with those zero-form gauge parameters ε that are annihilated by σ_-. Zero-forms cannot be σ_--exact since σ_- is a one-form. Therefore $H^0(\sigma_-)$ consists of all σ_--closed zero-forms. These are just the gauge symmetry parameters of the Fronsdal theory.

The analysis of $H^2(\sigma_-)$ is more complicated. The final result is that $H^2(\sigma_-)$ consists of the two parts. The first one contains the left-hand sides of the Fronsdal equations, *i.e.*, the tensor $G_{n_1,\ldots,n_s}(\varphi)$ (4.20) containing two derivatives of the Fronsdal field $\varphi_{n_1,\ldots,n_s}$. The second part consists of the generalized Weyl tensor containing s derivatives of the Fronsdal field. Note that in the case of spin one the generalized Weyl tensor coincides with the Maxwell tensor, containing one derivative of the vector potential, while in the case of spin two it coincides with the usual Weyl tensor containing two derivatives of the spin-two metric tensor.

The above analysis implies in particular that considering the Fronsdal system on-shell, *i.e.*, on the solutions of the Fronsdal equations $G_{n_1,\ldots,n_s}(\varphi) = 0$, it is possible to express all higher connections $\omega^{n_1\ldots n_{s-1},\, m_1\ldots m_t}$ with $t > 0$ via order-t derivatives of the Fronsdal field in such a way that all components of the higher-spin curvatures will be zero except for those that belong to the generalized Weyl tensors. This is equivalent to the fact that the system of equations

$$\begin{aligned} R^{n_1\ldots n_{s-1},\, m_1\ldots m_t} &= 0 \quad (t < s-1)\,; \\ R^{n_1\ldots n_{s-1},\, m_1\ldots m_{s-1}} &= \mathrm{e}_{n_s}\mathrm{e}_{m_s} C^{n_1\ldots n_s,\, m_1\ldots m_s}\,, \end{aligned} \tag{12.36}$$

where the zero-form $C^{n_1\ldots n_s,\, m_1\ldots m_s}$ is the higher-spin Weyl tensor, contains just the dynamical (Fronsdal) equations together with constraints that express all connections $\omega^{n_1\ldots n_{s-1},\, m_1\ldots m_t}$ with $t > 0$ via order-t derivatives of the Fronsdal field as well as constraints that express the Weyl tensor $C^{n_1\ldots n_s,\, m_1\ldots m_s}$ via order-s derivatives of the Fronsdal field. The equations (12.36) provide a generalization to any spin of the equations (8.49) in the gravity case. They play a very important role in the higher-spin theory, being often referred to as the *first on-shell theorem* and providing the starting point for the study of the *unfolded form* of the nonlinear higher-spin field equations.

13 Higher-spin algebra

The curvatures (12.24) generalize the linearized curvatures of the Poincaré algebra to higher spins. The key problem is to find the higher-spin extension of the Poincaré (in fact anti-De Sitter) algebra. Originally this algebra was found in the four-dimensional case using the language of two-component spinors via a straightforward analysis of the Jacobi identities for the higher-spin algebra, *i.e.*, a possible nonlinear deformation of the linearized higher-spin curvatures of Section 12. Later on it was realized that in the context of the AdS/CFT correspondence it admits a much simpler realization as the conformal higher-spin symmetry of a free massless scalar field in three dimensions. Since this analysis is simple enough and also useful for understanding the AdS/CFT correspondence, I will present it here. I will start however with a discussion of the general case in any dimension.

13.1 Arbitrary dimension

13.1.1 Conformal symmetry

Consider a seemingly simple question: what is the maximal symmetry h of the massless Klein-Gordon equation

$$\Box C(x) = 0\,, \qquad \Box = \eta^{nm}\partial_n\partial_m \tag{13.1}$$

in Minkowski space. By a symmetry we mean a transformation of the field $C(x)$

$$C(x) \to C'(x) = C(x) + \delta C(x)\,, \tag{13.2}$$

such that $C'(x)$ is a solution of (13.1) if the $C(x)$ was.

Of course, h contains the Poincaré transformations. Since (13.1) is free of scale parameters it is also invariant under the *dilation* transformation

$$\delta C(x) = \epsilon D C(x)\,, \qquad D := \Big(x^n \frac{\partial}{\partial x^n} + \frac{d}{2} - 1\Big)\,. \tag{13.3}$$

A less trivial symmetry is generated by the *special conformal transformations*

$$\delta C(x) = \epsilon_n K^n C(x)\,, \qquad K^n = (x^2\eta^{nm} - 2x^n x^m)\frac{\partial}{\partial x^m} + (2-d)x^n\,. \tag{13.4}$$

We leave it to the reader to check that this is a symmetry of the Klein-Gordon equation.

The nonzero commutation relations involving K^n and D are

$$[D\,, P_n] = -P_n\,, \qquad [D\,, K_n] = K_n\,, \qquad [P_n\,, K_m] = \eta_{nm}D + L_{nm} \tag{13.5}$$

Note that these relations determine the form of the constant term in the definition of the dilatation operator (13.3).

P_n, L_{nm}, K_n and D form the *conformal* Lie algebra $o(d,2)$. At the group level, conformal symmetry of a space consist of transformations that preserve the metric tensor up to a factor. (Recall that motions like Poincaré symmetry leave the metric tensor invariant.)

To make sure that P_n, L_{nm}, K_n and D indeed form $o(d,2)$ consider $o(d,2)$-generators M_{AB} with $A,B = 0,1,\dots d, d+1$ where the positive entities of the metric η_{AB} are associated with $A,B = 0, d+1$. Identify the first d components with the Lorentz indices n,m while for the last two introduce $\pm$-combinations such that $\eta_{\pm\pm} = 0$, $\eta_{+-} = \eta_{-+} = 1$. Setting

$$L_{nm} = M_{nm}\,, \qquad P_n = M_{n-}\,, \qquad K_n = M_{n+}\,, \qquad D = M_{-+} \tag{13.6}$$

it is easy to reproduce the commutation relations (13.5) from those of $o(d,2)$.

Comparing this result with that from Section 9.2 we arrive at the important conclusion that the conformal symmetry of d-dimensional flat space-time coincides with the symmetry of the $(d+1)$-dimensional anti-De Sitter space. This fact underlies the prominent AdS/CFT duality conjecturing the equivalence of theories of gravity in $(d+1)$-dimensional anti-De Sitter space and conformal theories in d-dimensional space-time.

This relation goes far beyond the usual conformal symmetry discussed above, working also for the higher-spin symmetries that will be discussed below. Namely it turns out that the higher-spin extension of the conformal symmetry of the Klein-Gordon equation in d dimensions is isomorphic to the higher-spin symmetry of the higher-spin theory in AdS_{d+1}.

13.1.2 Auxiliary problem

The problem of finding symmetries of the Klein-Gordon equation, or any of the other relativistic field equations, is not easy to solve directly. To figure out the structure of the whole conformal higher-spin algebra it is useful to consider an auxiliary problem of the type we have already considered in Section 8.2 for the analysis of global symmetries.

Consider the equations

$$D\mathcal{C}_A(x) = 0\,, \tag{13.7}$$

where $\mathcal{C}_A(x)$ is a set of fields valued in some space V (the index A) and

$$D\mathcal{C}_A(x) := \mathrm{d}\mathcal{C}_A(x) + \omega_A{}^B(x)\mathcal{C}_B(x) \tag{13.8}$$

is a covariant derivative acting in the space V treated as a $gl(V)$-module *i.e.*, $\omega_A{}^B(x)$ is some $gl(V)$-connection, where $gl(V)$ is the Lie algebra of linear transformations of the space V. The covariant derivative D is demanded to be flat, *i.e.*,

$$D^2 = 0\ : \qquad \mathrm{d}\omega_A{}^B(x) + \omega_A{}^C(x)\omega_C{}^B(x) = 0\,. \tag{13.9}$$

Eqs. (13.7) and (13.9) are invariant under the gauge transformation

$$\delta\mathcal{C}_A(x) = -\varepsilon_A{}^B(x)\mathcal{C}_B(x)\,, \tag{13.10}$$

$$\delta\omega(x) = D\varepsilon(x) := \mathrm{d}\varepsilon(x) + \omega(x)\varepsilon(x) - \varepsilon(x)\omega(x)\,, \tag{13.11}$$

where indices are implicit. The condition that the equations remain invariant for some fixed $\omega(x) = \omega_0(x)$ solving (13.9) restricts the gauge parameters $\varepsilon_A{}^B(x)$ to the parameters $\varepsilon_A{}^B(x)$ obeying the conditions

$$\delta\omega_0(x) = 0 \quad \longrightarrow \quad D_0\varepsilon_A{}^B(x) = 0\,, \qquad D_0 := \mathrm{d} + \omega_0\,. \tag{13.12}$$

Since $D_0^2 = 0$, $\varepsilon_A{}^B(x)$ is reconstructed (locally) in terms of $\varepsilon_A{}^B(x_0)$ at any x_0 (*cf.* Exercise 8.3). The $\epsilon_A{}^B := \varepsilon_A{}^B(x_0)$ are the global symmetry parameters of the equation $D_0\mathcal{C}(x) = 0$.

Thus, the system of equations (13.7) is invariant under the global symmetry $gl(V)$. Clearly, this is the maximal symmetry of (13.7). At the first sight the form of the equations (13.7) seems too special to help to find symmetries of the Klein-Gordon equation. In fact, this is not the case. Equations of the form (13.7) have very general applicability, providing an example of *unfolded equations*.

13.1.3 Massless scalar field unfolded

As usual we describe Minkowski space by a flat Poincaré-connection $\omega(x) = -i\big(\mathrm{e}^n(x)P_n + \frac{1}{2}\omega^{nm}(x)L_{nm}\big)$. In Cartesian coordinates $\mathrm{e}^n(x) = \xi^n$ and $\omega^{nm} = 0$.

Introduce an infinite set of zero-forms that are traceless symmetric tensors

$$C_{n_1\dots n_k}(x) = C_{(n_1\dots n_k)}(x)\,, \quad \eta^{ml}C_{mln_3\dots n_k}(x) = 0\,. \tag{13.13}$$

The unfolded system of equations that is equivalent to the Klein-Gordon equation has the form

$$D^{\mathrm{L}}C_{n_1\dots n_k}(x) = \mathrm{e}^m C_{n_1\dots n_k m}(x)\,. \tag{13.14}$$

Since the fields $C_{n_1\dots n_k}(x)$ are symmetric while $\mathrm{e}^n\mathrm{e}^m = -\mathrm{e}^m\mathrm{e}^n$, the system (13.14) is consistent with $(D^{\mathrm{L}})^2 = 0$. In fact, this means that the set of fields $C_{n_1\dots n_k}$ forms a Poincaré-module in which the Lorentz transformations rotate Lorentz indices while translations remove one cell of a tensor.

In Cartesian coordinates the system (13.14) reads

$$\mathrm{d}C_{n_1\dots n_k}(x) = \xi^m C_{n_1\dots n_k m}(x)\,. \tag{13.15}$$

The first two equations of this system imply

$$\partial_n C(x) = C_n(x)\,, \qquad \partial_n C_m(x) = C_{nm}(x) \Longrightarrow C_{nm}(x) = \partial_n\partial_m C(x)\,.$$

The tracelessness of $C_{nm}(x)$ then implies

$$\Box C(x) = 0\,. \tag{13.16}$$

All other equations express higher tensor components via higher derivatives of the scalar field

$$C_{n_1\dots n_k}(x) = \partial_{n_1}\dots\partial_{n_k}C(x)\,. \tag{13.17}$$

This formula explains the meaning of the $C_{n_1\dots n_k}(x)$: these fields form a basis for the space of all on-mass-shell nontrivial derivatives of $C(x)$. The space of $C_{n_1\dots n_k}(x)$ is analogous (in some sense equivalent) to the space of single-particle states for the spin-zero field. Via Eq. (13.15) the set of fields $C_{n_1\dots n_k}(x)$ at any given $x = x_0$ determines $C(x)$ in some neighborhood of x_0 (*cf.* Exercise 8.3), thus providing a locally complete set of "initial data".

We observe that equation (13.14) has the form of the covariant constancy equation (13.7) with a particular connection ω. This allows us to apply the analysis of Section 13.1.2. As a result, we conclude that the conformal higher-spin algebra h in d dimensions is the algebra of linear transformations of the infinite-dimensional space V of various traceless symmetric tensors (13.13), *i.e.*, $h = gl(V)$.

Since the space V is infinite dimensional, such a definition is not fully satisfactory, requiring a more precise specification of the appropriate class of operators. In practice, the idea is that the basis elements of the conformal higher-spin algebra h should reproduce higher-spin symmetry transformations represented by finite-order differential operators. The accurate definition of h was given by Eastwood in 2002 by different methods. The construction for $d = 3$ significantly simplifies in the framework of the spinorial formalism. Since this formulation is most relevant in the context of the AdS_4/CFT_3 higher-spin holography we now explain it in some more detail.

13.2 Conformal higher-spin algebra in $d = 3$

13.2.1 Spinorial form of $3d$ massless equations

The analysis of the three-dimensional equations greatly simplifies in spinorial terms because symmetric multispinors describe all irreducible Lorentz tensors and tensor-spinors. In fact, we have already used this approach for the analysis of $o(3)$-modules in Section 11.3.1. Note that the equivalence of the two languages in $3d$ theories is due to the well-known isomorphisms of the $3d$ Lorentz algebra: $o(2,1) \simeq sp(2,\mathbb{R}) \simeq sl_2(\mathbb{R})$.

Recall that $3d$ spinors in Minkowski signature are real, $\chi^\dagger_\alpha = \chi_\alpha$ ($\alpha = 1,2$), and the spinorial indices are raised and lowered by $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$

$$\chi^\alpha = \epsilon^{\alpha\beta}\chi_\beta\,, \qquad \chi_\alpha = \chi^\beta\epsilon_{\beta\alpha}\,. \tag{13.18}$$

Recall also that, because a two-by-two antisymmetric matrix is unique up to a factor, the antisymmetrization of $3d$ spinor indices is equivalent to their contraction

$$A_{\alpha,\beta} - A_{\beta,\alpha} = \epsilon_{\alpha\beta}A_{\gamma,}{}^{\gamma}\,.$$

As a result, finite-dimensional irreducible modules of the Lorentz algebra are represented by various totally symmetric multispinors $A_{\alpha_1\ldots\alpha_n}$. In particular, a rank-$k$ traceless symmetric tensor in the tensor notations is equivalent to the rank-$2k$ totally symmetric multispinor

$$A_{n_1\ldots n_m} \sim A_{\alpha_1\ldots\alpha_{2m}}\,, \qquad A^m{}_{mn_3\ldots n_m} = 0\,. \tag{13.19}$$

The explicit relation between the two formalisms is established with the help of the three 2×2 real symmetric matrices $\tau^n_{\alpha\beta}$.

Thus, in $d = 3$, the space V of all traceless symmetric tensors is equivalent to the space of symmetric multispinors of all even ranks. The latter can be naturally packed into a single even function of the commuting spinor variables y^α

$$C(y|x) := \sum_{n=0}^{\infty} C_{\alpha_1\ldots\alpha_{2n}}(x)\, y^{\alpha_1}\ldots y^{\alpha_{2n}}\,, \qquad y^\alpha y^\beta = y^\beta y^\alpha\,. \tag{13.20}$$

In these terms, the unfolded equations (13.15) for a massless scalar take the form

$$\xi^{\alpha\beta}\left(\frac{\partial}{\partial x^{\alpha\beta}} + \frac{\partial^2}{\partial y^\alpha \partial y^\beta}\right)C(y|x) = 0 \tag{13.21}$$

with $C(-y|x) = C(y|x)$. The physical scalar field $C(x)$ is identified with $C(0|x)$. One can see that the same equation with odd $C(-y|x) = -C(y|x)$ describes a $3d$ massless spinor field $C_\alpha(x) = \frac{\partial}{\partial y^\alpha}C(y|x)\Big|_{y=0}$.

A remarkable feature of the spinor formulation is that $C(y|x)$ is an unrestricted function of y^α. Eq. (13.21) shows that $C(y|x_0)$ at any point x_0 reconstructs $C(y|x)$ for all x. This situation is typical for the Penrose *twistor theory* where an unrestricted function in a twistor space generates solutions $C(x)$ of relativistic equations in Minkowski space-time. In fact, the unfolded equations (13.21) describes a Penrose-like transform from the twistor space (coordinates y^α) to the space-time (coordinates $x^{\alpha\beta}$).

13.2.2 $3d$ conformal higher-spin algebra

The reformulation of the massless Klein-Gordon equation in the form (13.21) immediately uncovers the structure of the $3d$ higher-spin algebra. Indeed, from the general analysis of Section 13.1.2 it follows that the $3d$ bosonic conformal higher-spin algebra is the algebra of various operators acting in the space V of functions of y^α. To have symmetry transformations in the form of differential operators in space-time we identify $gl(V)$ with the algebra of differential operators $\epsilon(y, \frac{\partial}{\partial y})$ that are even in the sense that

$$\epsilon\Big(-y, -\frac{\partial}{\partial y}\Big) = \epsilon\Big(y, \frac{\partial}{\partial y}\Big)\,, \tag{13.22}$$

which guarantees that the higher-spin algebra acts properly on bosons, mapping even functions in y to even functions. It is not difficult to see that the transformation law is

$$\delta C(y|x) = \varepsilon_{\rm gl}\Big(y, \frac{\partial}{\partial y}\Big|x\Big)C(y|x)\,, \tag{13.23}$$

where

$$\varepsilon_{\rm gl}\Big(y, \frac{\partial}{\partial y}\Big|x\Big) = \exp\left[-x^{\alpha\beta}\frac{\partial^2}{\partial y^\alpha \partial y^\beta}\right]\epsilon\Big(y, \frac{\partial}{\partial y}\Big)\exp\left[x^{\alpha\beta}\frac{\partial^2}{\partial y^\alpha \partial y^\beta}\right]. \tag{13.24}$$

Here $\varepsilon_{\rm gl}\Big(y, \frac{\partial}{\partial y}\Big|x\Big)$ describes the global transformation law in (13.23). It can be x-dependent analogously to the x-dependent form of the global Lorentz transformations (2.16), (2.20). On the other hand $\epsilon\big(y, \frac{\partial}{\partial y}\big)$ is the generating function for the *x-independent* parameters of global higher-spin transformations. In particular, these contain the antisymmetric parameters ϵ^{nm} in the transformation law (2.16), (2.20). Obviously (13.23), (13.24) map any solution of (13.21) to a solution.

When $\epsilon\big(y, \frac{\partial}{\partial y}\big)$ is a polynomial, $\varepsilon_{\rm gl}\big(y, \frac{\partial}{\partial y}\big|x\big)$ is polynomial as well. To derive the transformation law for the scalar field $C(x) = C(0|x)$ one should move the exponentials through $\epsilon(y, \frac{\partial}{\partial y})$, then set $y^\alpha = 0$ and, using (13.21), replace all operators $\frac{\partial^2}{\partial y^\alpha \partial y^\beta}$ by $-\frac{\partial}{\partial x^{\alpha\beta}}$. This gives a final result which is not too complicated, but which may be difficult to guess without the explained techniques.

The $3d$ conformal algebra is $sp(4) \simeq o(3,2)$. As anticipated it is a subalgebra of the conformal higher-spin algebra. Indeed, the generators of $sp(4)$ are given by various bilinears of y^α and $\frac{\partial}{\partial y^\alpha}$:

$$P_{\alpha\beta} = \frac{\partial^2}{\partial y^\alpha \partial y^\beta}\,, \qquad K^{\alpha\beta} = y^\alpha y^\beta\,, \qquad M_{\alpha\beta} = y_\alpha \frac{\partial}{\partial y^\beta} + y_\beta \frac{\partial}{\partial y^\alpha}\,, \qquad D = y^\alpha \frac{\partial}{\partial y^\alpha} + 1\,. \tag{13.25}$$

It is a useful exercise to check how formulae (13.23)–(13.25) reproduce the standard conformal transformations for a massless scalar in three dimensions.

From this analysis it follows that the $3d$ conformal higher-spin algebra is associated with the *Weyl algebra*. Indeed, in general, the Weyl algebra A_M is the associative algebra of polynomials of oscillators $\hat{Y}_{\hat\alpha}$ obeying the commutation relations

$$[\hat{Y}_{\hat\alpha}\,, \hat{Y}_{\hat\beta}] = 2iC_{\hat\alpha\hat\beta}\,, \qquad C_{\hat\alpha\hat\beta} = -C_{\hat\beta\hat\alpha}\,, \qquad \hat\alpha, \hat\beta = 1, \dots 2M \tag{13.26}$$

with a nondegenerate $C_{\hat{\alpha}\hat{\beta}}$. We use notation $\hat{Y}$ to stress that $\hat{Y}_{\hat{\alpha}}$ are operators in the sense that they do not commute. (This hat has nothing to do with that over indices in (13.26) which is introduced to emphasize that indices $\hat{\alpha}, \hat{\beta}, \ldots$ will be interpreted as spinorial.)

Taking into account that

$$\hat{Y}_{\hat{\alpha}} = \begin{pmatrix} y^{\alpha} \\ i\frac{\partial}{\partial y^{\beta}} \end{pmatrix} \tag{13.27}$$

obey the Heisenberg commutation relations (13.26), we conclude that the $3d$ conformal higher-spin algebra (to be identified with the AdS_4 higher-spin algebra) is the Lie algebra associated with the even part A_2^{even} of the Weyl algebra A_2, *i.e.*, elements of the Lie higher-spin algebra are from A_2^{even} while the Lie bracket is defined as the commutator in A_2^{even}, *i.e.*, $[\hat{f}\,,\hat{g}] = \hat{f}\hat{g} - \hat{g}\hat{f}$.

Note that, for the oscillators obeying (13.26), the bilinears

$$\mathcal{T}_{\hat{\alpha}\hat{\beta}} := \frac{1}{4}\{\hat{Y}_{\hat{\alpha}}\,, \hat{Y}_{\hat{\beta}}\} \tag{13.28}$$

form the Lie algebra $sp(2M)$ with the commutation relations (11.108). To check this one can first compute the commutator $[\hat{Y}_{\hat{\alpha}}\,, \mathcal{T}_{\hat{\beta}\hat{\gamma}}]$ and then use the following relation, which holds for any associative algebra:

$$[\{a\,,b\}\,,c] = \{a\,,[b\,,c]\} + \{b\,,[a\,,c]\}\,. \tag{13.29}$$

13.3 Star product

The *star product* provides an efficient tool for manipulations with the oscillators (13.26). Let us again consider a general Weyl algebra A_M whose elements are various polynomials of $\hat{Y}_{\hat{\alpha}}$

$$\hat{f}(\hat{Y}) = \sum_{n\geq 0} \frac{1}{n!} f^{\hat{\alpha}_1\ldots\hat{\alpha}_n} \hat{Y}_{\hat{\alpha}_1} \ldots \hat{Y}_{\hat{\alpha}_n}\,, \tag{13.30}$$

where we take $f^{\hat{\alpha}_1\ldots\hat{\alpha}_n}$ to be symmetric in the indices $\hat{\alpha}_i$. Such polynomials correspond to the totally symmetric (*Weyl*) ordering. In fact, any polynomial in the $\hat{Y}_{\hat{\alpha}}$ can be represented in the form (13.30) because antisymmetrization of any pair of indices carried by $\hat{Y}_{\hat{\alpha}}$ decreases the degree of the polynomial by virtue of the commutation relations (13.26). For example, let $f^{\hat{\alpha},\hat{\beta}}$ be non symmetric. Then

$$f^{\hat{\alpha},\hat{\beta}}\hat{Y}_{\hat{\alpha}}\hat{Y}_{\hat{\beta}} = \frac{1}{2} f^{\hat{\alpha},\hat{\beta}}\big(\{\hat{Y}_{\hat{\alpha}}\,,\hat{Y}_{\hat{\beta}}\} + [\hat{Y}_{\hat{\alpha}}\,,\hat{Y}_{\hat{\beta}}]\big) = f^{(\hat{\alpha},\hat{\beta})}\hat{Y}_{\hat{\alpha}}\hat{Y}_{\hat{\beta}} + f\,, \qquad f = i f^{\hat{\alpha},\hat{\beta}} C_{\hat{\alpha}\hat{\beta}} \tag{13.31}$$

To every Weyl-ordered operator $\hat{f}(\hat{Y})$ (13.30) with symmetric coefficients $f^{\hat{\alpha}_1\ldots\hat{\alpha}_n}$ we assign a function $f(Y)$ of ordinary variables $Y_{\hat{\alpha}}$ ($Y_{\hat{\alpha}}Y_{\hat{\beta}} = Y_{\hat{\beta}}Y_{\hat{\alpha}}$), that has the form

$$f(Y) := \sum_{n\geq 0} \frac{1}{n!} f^{\hat{\alpha}_1\ldots\hat{\alpha}_n} Y_{\hat{\alpha}_1} \ldots Y_{\hat{\alpha}_n}\,. \tag{13.32}$$

This $f(Y)$ is called the *Weyl symbol* of the operator (13.30). By construction, the correspondence between operators and their symbols is one-to-one.

The *Weyl star product* is defined by the rule that $(f * g)(Y)$ is the symbol of $\hat{f}(\hat{Y})\hat{g}(\hat{Y})$. In particular, this implies

$$[Y_{\hat\alpha}, Y_{\hat\beta}]_* = 2iC_{\hat\alpha\hat\beta}\,, \qquad [a\,,b]_* := a*b - b*a\,. \tag{13.33}$$

The Weyl star product is concisely described by the *Weyl-Moyal formula*

$$(f*g)(Y) = f(Y)\,\exp\,[i\overleftarrow{\partial^{\hat\alpha}}C_{\hat\alpha\hat\beta}\overrightarrow{\partial^{\hat\beta}}]\,g(Y)\;, \quad \partial^{\hat\alpha} := \frac{\partial}{\partial Y_{\hat\alpha}}\,, \tag{13.34}$$

where $\overleftarrow{\partial^{\hat\alpha}}$ and $\overrightarrow{\partial^{\hat\beta}}$ act on $f(Y)$ and $g(Y)$, respectively. This formula effectively describes the result of bringing the product $\hat{f}(\hat{Y})\hat{g}(\hat{Y})$ of two Weyl-ordered operators to the Weyl-ordered form again. Doing this by hand quickly becomes a tedious task, so (13.34) is a very useful tool.

For practical computations with the star-product it is often useful to use the Taylor formula in the form

$$f(x+a) = \exp\left[a\frac{\partial}{\partial x}\right]f(x)\,. \tag{13.35}$$

In particular, this gives

$$(f*g)(Y) = f(Y_{\hat\alpha} + iC_{\hat\alpha\hat\beta}\overrightarrow{\partial^{\hat\beta}})g(Y) = f(Y)g(Y_{\hat\beta} + i\overleftarrow{\partial^{\hat\alpha}}C_{\hat\alpha\hat\beta})\,. \tag{13.36}$$

Since the product of the oscillators $\hat{Y}^{\hat\alpha}$ was associative $*$ gives an associative product law for ordinary functions, *i.e.*, $(f*g)*h = f*(g*h)$ as is not too difficult to see directly.

By its definition, the star product of any two polynomials of Y is a polynomial. $f=1$ is the unit element in the star-product algebra, *i.e.*, $1*f = f*1 = f\,,\quad \forall f$.

Using the Weyl-Moyal formula one can check that

$$\{Y_{\hat\alpha}\,, f(Y)\}_* = 2Y_{\hat\alpha}f(Y)\,, \tag{13.37}$$

$$[Y_{\hat\alpha}\,, f(Y)]_* = 2i\frac{\partial}{\partial Y^{\hat\alpha}}f(Y)\,, \tag{13.38}$$

where $Y^{\hat\alpha} = C^{\hat\alpha\hat\beta}Y_{\hat\beta}$.

The components of $(f*g)(Y)$, expanded as in (13.32), are related to the components of $f(Y)$ and $g(Y)$ as

$$(f*g)^{\alpha_1\ldots\alpha_n} = \sum_{p\geq 0}\sum_{r=0}^{n} i^p \frac{n!}{p!\,r!\,(n-r)!}\, f^{(\alpha_1\ldots\alpha_r}{}_{\beta_1\ldots\beta_p}\, g^{\alpha_{r+1}\ldots\alpha_n)\,\beta_1\ldots\beta_p}\,. \tag{13.39}$$

The Moyal formula can be derived as follows. Firstly, one observes that any power $(\hat{Y}(\mu))^n$ of the operator $\hat{Y}(\mu) := \mu^{\hat\alpha}\hat{Y}_{\hat\alpha}$, where $\mu^{\hat\alpha}$ are arbitrary real or complex parameters, is automatically Weyl ordered. Indeed, in this case the coefficients

$f^{\hat\alpha_1\ldots\hat\alpha_n}$, being proportional to $\mu^{\hat\alpha_1}\ldots\mu^{\hat\alpha_n}$, are automatically symmetric in their indices. One can use the exponential $\exp(\mu^{\hat\alpha}\hat Y_{\hat\alpha})$ as a generating function for all Weyl-ordered polynomials using that

$$\hat Y_{(\hat\alpha_1}\ldots\hat Y_{\hat\alpha_n)}=\frac{\partial^n}{\partial\mu^{\hat\alpha_1}\ldots\partial\mu^{\hat\alpha_n}}\exp(\mu^{\hat\alpha}\hat Y_{\hat\alpha})|_{\mu=0}\,. \tag{13.40}$$

Hence, it suffices to derive the Moyal formula for the exponential in order to prove it for any polynomial.

Secondly, one should use the Campbell-Hausdorff formula for the product of exponentials of operators

$$\exp(\hat A)\exp(\hat B)=\exp(\hat A+\hat B)\exp(\frac{1}{2}[\hat A,\hat B]) \tag{13.41}$$

valid at the condition that the commutator $[\hat A,\hat B]$ is central, *i.e.*, $[[\hat A,\hat B],\hat A]=[[\hat A,\hat B],\hat B]=0$.

Applying this formula to the exponentials we obtain that

$$\exp(\mu^{\hat\alpha}\hat Y_{\hat\alpha})\exp(\lambda^{\hat\beta}\hat Y_{\hat\beta})=\exp((\mu^{\hat\alpha}+\lambda^{\hat\alpha})\hat Y_{\hat\alpha})\exp(iC_{\hat\alpha\hat\beta}\mu^{\hat\alpha}\lambda^{\hat\beta})\,. \tag{13.42}$$

The reader can check that this is precisely the result of the application of the Moyal formula (13.34) to the exponentials.

For the proof of (13.41) consider $\hat A=a\hat C$ and differentiate with respect to a showing that the derivative vanishes as a consequence of (13.41). This proves the Cambell-Hausdorff formula at any a provided that it is true at $a=0$, which is obvious. To differentiate it is convenient to use the following general formula

$$\delta\exp(\hat X)=\int_0^1 dt\exp(t\hat X)\delta\hat X\exp((1-t)\hat X), \tag{13.43}$$

which can be checked via a power-series expansion, but is essentially a consequence of the defining property of the exponential

$$\exp(\hat X)=\underbrace{\exp(\hat X/n)\ldots\exp(\hat X/n)}_{n}\,. \tag{13.44}$$

An important property of the Weyl-Moyal star product is that, as is easy to check, it admits a *supertrace operation*

$$str f(y)=f(0) \tag{13.45}$$

that obeys the property

$$str(f(y)*g(y))=str(g(-y)*f(y))=str(g(y)*f(-y)). \tag{13.46}$$

For even functions this implies that

$$str([f\,,g]_*)=0\,,\qquad f(y)=f(-y)\qquad g(y)=g(-y)\,. \tag{13.47}$$

Apart from the Weyl ordering one can use other orderings. From the point of view of the Weyl algebra they all correspond to different choices of a basis in the same Weyl algebra A_N. For example, the P-Q (Wick) ordering is defined as follows. Let $\hat{Y}_{\hat{\alpha}}$ be decomposed into canonical pairs $\hat{p}_i$, $\hat{q}^i$ $(i,j=1,\ldots,N)$ obeying

$$[\hat{p}_i,\hat{q}^j]=\delta_i{}^j\,,\qquad [\hat{p}_i,\hat{p}_j]=0\,,\qquad [\hat{q}^i,\hat{q}^j]=0\,. \tag{13.48}$$

Consider the ordering in which all $\hat{p}$ stand on the left while all $\hat{q}$ stand on the right

$$\hat{f}(\hat{p},\hat{q})=\sum_{k,l=0}^{\infty} f^{i_1\ldots i_k}{}_{,\,j_1\ldots j_l}\hat{p}_{i_1}\ldots\hat{p}_{i_k}\hat{q}^{j_1}\ldots\hat{q}^{j_l}. \tag{13.49}$$

Introduce the P–Q symbol

$$f(p,q)=\sum_{k,l=0}^{\infty} f^{i_1\ldots i_k}{}_{,\,j_1\ldots j_l}p_{i_1}\ldots p_{i_k}q^{j_1}\ldots q^{j_l}, \tag{13.50}$$

where p and q are commuting variables. In this case the P–Q star product, which is of course associative, has the form

$$\big(f\star g\big)(p,q)=f(p,q)\exp\Big(-\frac{\overleftarrow{\partial}}{\partial q^j}\frac{\overrightarrow{\partial}}{\partial p_j}\Big)g(p,q)\,, \tag{13.51}$$

where a different star-product symbol $\star$ is used to stress the difference with the Moyal product.

Note that the relation (13.38) is true in any ordering while the relation (13.37) is specific for the Weyl ordering. To summarize, let me stress again that the star product gives a highly efficient tool for the description of the algebra of oscillators. In particular the star-product formalism is completely self-sufficient requiring no reference to the original operator language of the hatted operators.

14 Higher-spin gauge theory in AdS_4

In Section 13.2 the conformal higher-spin algebra in $d=3$ was shown to be isomorphic to the Lie algebra associated with the Weyl algebra A_2 of the oscillators $\hat{Y}_{\hat{\alpha}}$. Similarly to the lower-spin relation between the conformal algebra in $d=3$ and the AdS_4 algebra we now interpret it as the AdS_4 higher-spin algebra. Higher-spin gauge theory results from gauging this higher-spin algebra. In fact the full story is more involved, and this is just the first step.

14.1 Higher-spin gauge fields and curvatures

The gauge fields obtained by gauging the AdS_4 higher-spin algebra are one-forms $\omega(Y|x)$ where $x^{\underline{n}}$ are four-dimensional space-time coordinates while $Y_{\hat{\alpha}}$ is now interpreted as a four-dimensional Majorana spinor. In practice, in $4d$ it is convenient to use the formalism of two-component spinors with $Y_{\hat\alpha} = (y_\alpha, \bar y_{\dot\alpha})$. Notice that the $4d$ variables y_α and $\bar y_{\dot\alpha}$ can be understood as the symbols (13.32) of the following linear combinations of the $3d$ variables (13.27), which we mark by primes here,

$$y_\alpha = y'_\alpha + i\frac{\partial}{\partial y'^\alpha}\,, \qquad \bar y_{\dot\alpha} = y'_\alpha - i\frac{\partial}{\partial y'^\alpha}\,. \tag{14.1}$$

This means that, being usual commutative variables with respect to the ordinary product, the Moyal star product of y_α and $\bar y_{\dot\alpha}$ reproduces the commutation relations of the expressions on the right-hand sides of (14.1).

In these terms we can expand

$$\omega(y,\bar y \mid x) = \sum_{n,m=0}^{\infty} \frac{i}{2\,n!m!} y_{\alpha_1}\dots y_{\alpha_n} \bar y_{\dot\beta_1}\dots \bar y_{\dot\beta_m}\,\omega^{\alpha_1\dots\alpha_n,\dot\beta_1\dots\dot\beta_m}(x)\,. \tag{14.2}$$

The nonlinear higher-spin curvatures and non-Abelian gauge transformation law acquire the form

$$R(Y|x) = \mathrm{d}\omega(Y|x) + \omega(Y|x) * \omega(Y|x)\,, \tag{14.3}$$

$$\delta\omega(Y|x) = D^{ad}\varepsilon(Y|x) = \mathrm{d}\varepsilon(Y|x) + [\omega(Y|x)\,,\varepsilon(Y|x)]_*\,, \tag{14.4}$$

where $\varepsilon(Y|x)$ is a higher-spin gauge-transformation parameter and the star product (13.34) takes the form

$$(f*g)(y,\bar y) = f(y,\bar y)\,\exp i\big(\overleftarrow{\partial_\alpha}\epsilon^{\alpha\beta}\overrightarrow{\partial_\beta} + \overleftarrow{\partial_{\dot\alpha}}\epsilon^{\dot\alpha\dot\beta}\overrightarrow{\partial_{\dot\beta}}\big)\, g(y,\bar y)\ , \quad \partial_\alpha := \frac{\partial}{\partial y^\alpha}\ , \quad \partial_{\dot\alpha} := \frac{\partial}{\partial \bar y^{\dot\alpha}}\,. \tag{14.5}$$

For the components of the curvature

$$R(y,\bar y \mid x) = \sum_{n,m=0}^{\infty} \frac{i}{2\,n!m!} y_{\alpha_1}\dots y_{\alpha_n} \bar y_{\dot\beta_1}\dots \bar y_{\dot\beta_m} R^{\alpha_1\dots\alpha_n,\dot\beta_1\dots\dot\beta_m}(x) \tag{14.6}$$

this gives

$$R_{\alpha(n),\dot\alpha(m)}(x) = \mathrm{d}\omega_{\alpha(n),\dot\alpha(m)}(x) + \sum_{p,q,r,l,s,t} i^{p+q+1}\frac{\delta^{r+s}_n \delta^{l+t}_m n!m!}{2\,p!q!r!l!s!t!}\,\omega_{\alpha(r)\beta(p),\dot\alpha(l)\dot\beta(q)}(x)\,\omega_{\alpha(s)}{}^{\beta(p)}{}_{,\dot\alpha(t)}{}^{\dot\beta(q)}(x)\,, \tag{14.7}$$

where for simplicity we indicate the number of indices in brackets at the convention that indices denoted by the same letter are automatically symmetrized before contractions. We leave it as an exercise to derive (14.7) with the help of (13.39).

To see that this set of gauge fields precisely matches the set of gauge fields valued in the two-row Young diagrams in the tensor formalism introduced in Section 12.1 we should elaborate a bit on the dictionary between the languages of tensors and two-component spinors.

14.2 $4d$ tensors versus two-component multispinors

Since, for two–component indices, the contraction of any pair of indices is equivalent to their antisymmetrization (see (10.41)), irreducible finite–dimensional representations of the $4d$ Lorentz group can be described by multi–spinors

$$A_{\alpha_1\cdots\alpha_n,\dot\beta_1\cdots\dot\beta_m}\,,\qquad \overline{A_{\alpha_1\cdots\alpha_n,\dot\beta_1\cdots\dot\beta_m}}=A_{\beta_1\cdots\beta_m,\dot\alpha_1\cdots\dot\alpha_n}\,,\tag{14.8}$$

that are symmetric both in the undotted and in the dotted indices.

The one–to–one correspondence between two-component multi–spinors and four–dimensional tensors and spinor–tensors is established making use of the Hermitian matrices $\sigma^n_{\alpha\dot\alpha}$ and of the complex conjugate matrices $\sigma^{mn}_{\alpha\beta}$ and $\bar\sigma^{mn}_{\dot\alpha\dot\beta}$ (10.45) that are antisymmetric in vector indices and symmetric in spinor indices.

In general, a pair of conjugated multi–spinors $A_{\alpha_1\cdots\alpha_k,\dot\beta_1\cdots\dot\beta_l}$ and $A_{\beta_1\cdots\beta_l,\dot\alpha_1\cdots\dot\alpha_k}$, with even $k+l$ and $k\neq l$, in the conventional tensor notation corresponds to a traceless tensor $A_{n_1,\cdots,n_{(k+l)/2},m_1,\cdots,m_{|k-l|/2}}$ that has the symmetry property of the Young diagram $\begin{array}{l}\overbrace{\square\square\square\square\square}^{(k+l)/2}\\ \underbrace{\square\square\square}_{|k-l|/2}\end{array}$. This correspondence can be verified by noting that each pair α, $\dot\alpha$ can be contracted with $\sigma^m_{\alpha\dot\alpha}$, while the remaining set of totally symmetric spinorial indices can be contracted with a number of $\sigma^{mn}_{\alpha\beta}$ and $\bar\sigma^{mn}_{\dot\alpha\dot\beta}$ (10.45).

For instance, a vector A^n is equivalent to a second-rank multispinor of mixed type

$$A^n=\sigma^n_{\alpha\dot\alpha}A^{\alpha\dot\alpha}\,.\tag{14.9}$$

An antisymmetric tensor is equivalent to the mutually conjugate second-rank multispinors $B^{\alpha\beta}$ and $\bar B^{\dot\alpha\dot\beta}$ of non-mixed types

$$B^{nm}=\sigma^{nm}_{\alpha\beta}B^{\alpha\beta}+\bar\sigma^{nm}_{\dot\alpha\dot\beta}\bar B^{\dot\alpha\dot\beta}\,.\tag{14.10}$$

The irreducibility properties in the tensor notation follow from the fact that, being antisymmetric, the contraction of any pair of symmetrized spinor indices gives zero. A self-conjugate multi–spinor $A_{\alpha_1\cdots\alpha_k,\dot\beta_1\cdots\dot\beta_k}$ is equivalent to a rank-k traceless totally symmetric tensor $A_{n_1,\cdots,n_k}$.

From this correspondence between multi–spinors of even ranks and tensors it follows that the $4d$ set of spin-s connections (12.20) is indeed equivalent to the set of one-forms

$$\omega_{\alpha_1\cdots\alpha_k,\dot\alpha_1\cdots\dot\alpha_l}=\xi^{\underline n}\omega_{\underline n\alpha_1\cdots\alpha_k,\dot\alpha_1\cdots\dot\alpha_l}\,,\qquad k+l=2(s-1)\,,\qquad \frac12|k-l|=t\,,\tag{14.11}$$

satisfying the reality conditions

$$\overline{\omega_{\alpha_1\cdots\alpha_k,\dot\beta_1\cdots\dot\beta_l}}=\omega_{\beta_1\cdots\beta_k,\dot\alpha_1\cdots\dot\alpha_l}\,.\tag{14.12}$$

The frame-like field is the self-conjugated field

$$\omega_{\alpha_1\cdots\alpha_{s-1},\dot\beta_1\cdots\dot\beta_{s-1}}\equiv e_{\alpha_1\cdots\alpha_{s-1},\dot\beta_1\cdots\dot\beta_{s-1}}\,.\tag{14.13}$$

Thus the star-product indeed gives an associative algebra that gives rise to the correct set of higher-spin connections in four dimensions. The fact that such an algebra exists is not trivial, opening the way towards the construction of the nonlinear higher-spin gauge theory. This construction also predicts the spectrum of fields in the nontrivial higher-spin theory resulting from the gauging of the higher-spin symmetry algebra in question. Namely, the related higher-spin theory should contain massless fields of all integer spins. Strictly speaking, the appearance of a spin-zero field does not follow from the analysis of the gauge sector since it is not associated with any gauge symmetry. Nevertheless the analysis of higher-spin multiplets (modules) shows that it should also be included. Also this analysis shows that the higher-spin algebra obeys the admissibility condition of Section 11.4.3 admitting a unitary module representing the infinite tower of massless fields of all spins.

14.3 *AdS* vacuum and higher-spin perturbations

To analyze the higher-spin theory perturbatively we set

$$\omega = \omega_0 + \omega_1\,, \tag{14.14}$$

where ω_0 is some solution of the higher-spin equations of motion and ω_1 describes some small fluctuations of the dynamical fields near this solution. Although we do not yet know the full form of the higher-spin field equations, an educated guess is that, whatever they are, they are somehow expressed via the higher-spin curvatures. In that case, if ω_0 is a flat connection, *i.e.*,

$$R(\omega_0) = 0\,, \tag{14.15}$$

it will solve the full system of higher-spin equations. The simplest vacuum solution can be chosen in the form

$$\omega_0(Y|x) = \frac{i}{4}(\omega_0^{\alpha\beta}(x)y_\alpha y_\beta + \bar{\omega}_0^{\dot\alpha\dot\beta}(x)\bar{y}_{\dot\alpha}\bar{y}_{\dot\beta} + 2\lambda \mathrm{e}^{\alpha\dot\beta}(x)y_\alpha\bar{y}_{\dot\beta})\,, \tag{14.16}$$

where ω_0 and e obey the equations

$$R_{0\alpha\dot\alpha}(x) := D_0^{\mathrm{L}}\mathrm{e}_{\alpha\dot\alpha}(x) = 0\,, \tag{14.17}$$

$$R_{0\alpha\beta}(x) := \mathrm{d}\omega_{0\alpha\beta}(x) + \omega_{0\alpha}{}^{\gamma}(x)\omega_{0\gamma\beta}(x) + \lambda^2\mathrm{e}_{\alpha}{}^{\dot\gamma}(x)\mathrm{e}_{\beta\dot\gamma}(x) = 0\,, \tag{14.18}$$

$$\bar{R}_{0\dot\alpha\dot\beta}(x) := \mathrm{d}\bar{\omega}_{0\dot\alpha\dot\beta}(x) + \bar{\omega}_{0\dot\alpha}{}^{\dot\gamma}(x)\bar{\omega}_{0\dot\gamma\dot\beta}(x) + \lambda^2\mathrm{e}^{\gamma}{}_{\dot\alpha}(\mathrm{x})\mathrm{e}_{\gamma\dot\beta}(\mathrm{x}) = 0\,, \tag{14.19}$$

which describe the $4d$ anti-De Sitter space. Note that Eqs. (14.17)–(14.19) follow from (14.7) and (14.16).

Indeed, (14.17)–(14.19) are the curvatures of $sp(4)$ in the two-component spinor notations, *cf.* (11.130)–(11.132). As shown in Section 11.5.1, $sp(4) \simeq o(3,2)$. Hence any Lorentz connection and frame field of AdS_4 rewritten in terms of two-component spinors solve (14.17)–(14.19).

Since $sp(4)$ is a subalgebra of the higher-spin algebra (see (13.25)), any ω_0 of the form (14.16) that solves (14.17)–(14.19), with all other higher-spin connections being zero, automatically solves (14.15). As a result, anti-De Sitter space is a natural solution (vacuum) of the higher-spin theory. Although other solutions can in principle also be considered, this is the simplest meaningful one.

The linearized part of the higher-spin curvature (14.3) is

$$R_1 = D_0^{\mathrm{ad}}\omega_1 := \mathrm{d}\omega_1 + [\omega_0\,,\omega_1]_* \equiv \mathrm{d}\omega_1 + \omega_0 * \omega_1 + \omega_1 * \omega_0\,. \tag{14.20}$$

An elementary computation with the star product gives

$$D_0^{\mathrm{ad}} = \mathrm{d} - \left(\omega_0^{\alpha\beta} y_\alpha \frac{\partial}{\partial y^\beta} + \bar{\omega}_0^{\dot\alpha\dot\beta} \bar{y}_{\dot\alpha} \frac{\partial}{\partial \bar{y}^{\dot\beta}}\right) - \lambda \mathrm{e}^{\alpha\dot\beta}\left(y_\alpha \frac{\partial}{\partial \bar{y}^{\dot\beta}} + \frac{\partial}{\partial y^\alpha}\bar{y}_{\dot\beta}\right)\,. \tag{14.21}$$

To prove (14.21) one can use (13.37), (13.38) repeatedly together with the fact that for any associative algebra (such as the star-product algebra)

$$[\{a,b\},c] = \{a\,,[b\,,c]\} + \{b\,,[a\,,c]\}\,.$$

Because ω_0 obeys (14.15), the linearized curvatures (14.20) are invariant under the higher-spin gauge transformations

$$\delta\omega_1(Y|x) = D_0^{\mathrm{ad}}\varepsilon(Y|x) := \mathrm{d}\varepsilon(Y|x) + [\omega_0, \varepsilon]_*(Y|x) \tag{14.22}$$

due to the simple fact that $D_0^{\mathrm{ad}} D_0^{\mathrm{ad}} = 0$ as a consequence of (14.15).

Since (14.21) is homogeneous in y_α and $\bar{y}_{\dot\alpha}$, the linearized curvatures involving ω_1 of different total degrees in y_α and $\bar{y}_{\dot\alpha}$ are independent. The higher-spin connections $\omega_1(Y|x)$ associated with different homogeneous polynomials in Y describe different spins s. Namely, for any $\nu \in \mathbb{C}$

$$\omega^s(\nu Y|x) = \nu^{2(s-1)}\omega(Y|x)\,. \tag{14.23}$$

Equivalently, in terms of the connections (14.2), spin-s fields are described by the one-forms $\omega^{\alpha_1\ldots\alpha_n,\dot\beta_1\ldots\dot\beta_m}(x)$ with various $n+m = 2(s-1)$. According to the dictionary of Section 14.2, for any s, this set of connections precisely corresponds to the set of two-row Young diagrams with the first row of length $s-1$ in complete correspondence with the set of higher-spin connections of Section 12.1.

The components (14.6) of the linearized curvatures have the form

$$R_{\alpha(n),\dot\alpha(m)}(x) = D_0^{\mathrm{L}}\omega_{\alpha(n),\dot\alpha(m)}(x) + \lambda\big(n\mathrm{e}_\alpha{}^{\dot\beta}\omega_{\alpha(n-1),\dot\alpha(m)\dot\beta}(x) + m\mathrm{e}^\beta{}_{\dot\alpha}\omega_{\alpha(n)\beta,\dot\alpha(m-1)}(x)\big)\,, \tag{14.24}$$

where the Lorentz-covariant derivative is

$$D_0^{\mathrm{L}}\omega_{\alpha(n),\dot\alpha(m)}(x) = \mathrm{d}\omega_{\alpha(n),\dot\alpha(m)}(x) + n\omega_\alpha{}^\beta\omega_{\alpha(n-1)\beta,\dot\alpha(m)}(x) + m\bar\omega_{\dot\alpha}{}^{\dot\beta}\omega_{\alpha(n),\dot\alpha(m-1)\dot\beta}(x)\,. \tag{14.25}$$

The First on-shell theorem in the anti-De Sitter space has the same form as (12.36), with the Minkowski higher-spin curvatures replaced by the linearized curvatures in the anti-De Sitter space, *i.e.*, by the curvatures (14.24). In terms of two-component spinors the result reads

$$R_{\alpha_1\ldots\alpha_n,\dot\alpha_1\ldots\dot\alpha_m} = \delta_n^0 \mathrm{e}_\beta{}^{\dot\alpha_{m+1}}\mathrm{e}^{\beta\dot\alpha_{m+2}}\overline{C}_{\dot\alpha_1\ldots\dot\alpha_{m+2}} + \delta_m^0 \mathrm{e}^{\alpha_{n+1}}{}_{\dot\beta}\mathrm{e}^{\alpha_{n+2}\dot\beta}C_{\alpha_1\ldots\alpha_{n+2}} + U_{\alpha_1\ldots\alpha_n,\dot\alpha_1\ldots\dot\alpha_m}(\mathcal{R})\,, \tag{14.26}$$

where the conjugated multispinors $C_{\alpha_1\dots\alpha_s}$ and $\overline{C}_{\dot\alpha_1\dots\dot\alpha_s}$ describe the higher-spin Weyl tensors and $U_{\alpha_1\dots\alpha_n,\dot\alpha_1\dots\dot\alpha_m}$ is some linear differential operator acting on the left–hand sides of the Fronsdal field equations (4.7) $\mathcal{R} = 0$. In this form the First on-shell theorem expresses all auxiliary and extra connections $\omega_{\alpha_1\cdots\alpha_k,\,\dot\alpha_1\cdots\dot\alpha_l}$ with $k+l = 2(s-1)$ as well as the Weyl tensor $C_{\alpha_1\dots\alpha_s}$, $\overline{C}_{\dot\alpha_1\dots\dot\alpha_s}$ via derivatives of the frame-like field $e_{\alpha_1\cdots\alpha_{s-1},\,\dot\alpha_1\cdots\dot\alpha_{s-1}}$ (14.13), up to terms that vanish on-shell.

Since the expressions for the higher connections result from the resolution of the algebraic (*i.e.*, e-dependent) part of (14.24) on the left-hand side of (14.26) which contains explicitly the parameter of λ, the resulting expressions have the structure

$$\omega_{\alpha_1\cdots\alpha_k,\,\dot\alpha_1\cdots\dot\alpha_l}(e,\partial e,\dots,\partial^{\frac{1}{2}|k-l|}e;\lambda) = \lambda^{-\frac{1}{2}|k-l|}\tilde\omega_{\alpha_1\cdots\alpha_k,\,\dot\alpha_1\cdots\dot\alpha_l}(e,\partial e,\dots,\partial^{\frac{1}{2}|k-l|}e;\lambda)\,, \tag{14.27}$$

where $\tilde\omega_{\alpha_1\cdots\alpha_k,\,\dot\alpha_1\cdots\dot\alpha_l}(e,\partial e,\dots,\partial^{\frac{1}{2}|k-l|}e;\lambda)$ are regular in λ. As we will see shortly, this singularity of the higher-spin connections in λ is the source of the singularity in the nonlinear action that makes the flat limit in the interacting higher-spin theory meaningless.

14.4 Action

14.4.1 Topological action

As a consequence, of the properties(13.46) of the supertrace the action

$$S^{\rm top} = \frac{1}{2}\int_{M^4} str(R*R) \tag{14.28}$$

turns out to be topological, *i.e.*, its variation is identically zero as can be easily seen using Bianchi identities (5.17).

In terms of the components (14.6) $S^{\rm top}$ has the form

$$S = \frac{1}{2}\sum_{n,m=0}^{\infty}\frac{i^{n+m-1}}{n!m!}\int_{M^4} R_{\alpha_1\dots\alpha_n,\,\dot\beta_1\dots\dot\beta_m}(x)R^{\alpha_1\dots\alpha_n,\dot\beta_1\dots\dot\beta_m}(x)\,. \tag{14.29}$$

14.4.2 Free action

The quadratic higher-spin action in AdS_4 has the remarkably simple form

$$-\frac{1}{4\kappa}\sum_{n,m=0}^{\infty}\frac{i^{n+m-1}}{n!m!}\,\mathrm{sign}(n-m)\int_{M^4} R_{1\,\alpha_1\dots\alpha_n,\,\dot\beta_1\dots\dot\beta_m}(x)R_1^{\alpha_1\dots\alpha_n,\dot\beta_1\dots\dot\beta_m}(x)\,, \tag{14.30}$$

where the κ-dependent factor is introduced for the future convenience and

$$\mathrm{sign}(-n) = -\mathrm{sign}(n)\,,\qquad \mathrm{sign}(n) = 1 \quad\text{at}\quad n>0\,. \tag{14.31}$$

Since the linearized curvatures $R_1^{\alpha_1\dots\alpha_n,\dot\beta_1\dots\dot\beta_m}(x)$ are invariant under the higher-spin gauge transformation (14.22), the action (14.30) is manifestly gauge invariant. The coefficients in the action (14.30) are adjusted in such a way that its variation with respect to the "extra fields" $\omega_{\alpha_1\ldots\alpha_n\,,\beta_1\ldots\beta_m}$ with $|n-m|>2$ vanishes identically. This is achieved by requiring the coefficients in the action to be a constant far enough from the line $n=m$ because in this case the variation of the linearized action is the same as for the linearized topological action, *i.e.*, zero, taking into account that the linearized curvatures (14.24) relate the fields with the nearest n,m. Another condition is that the coefficients should be antisymmetric under the exchange of n and m for the action to be parity even because there is a hidden Levi-Civita symbol which contracts indices of differential forms and the parity transformation exchanges dotted and undotted indices. The function $\mathrm{sign}(n-m)$ just obeys both of these conditions.

The variation with respect to the dynamical fields, $\omega_{\alpha_1\ldots\alpha_n\,,\beta_1\ldots\beta_m}$ with $|n-m|\leq 2$, is non-trivial and leads to the correct free massless equations of motion. The action (14.30) is equivalent to a sum of the free Fronsdal actions in AdS_4 for all spin $s>1$. This follows from the fact that the Fronsdal action is the only gauge-invariant action that contains two derivatives of the Fronsdal field.

Note that the action (14.30) also describes all massless gauge fields of half-integer spins $s\geq 3/2$ in AdS_4, generalizing the MacDowell-Mansouri form of the spin-3/2 action (*cf.* the second term in (11.137)) to all massless higher-spin fermions.

14.4.3 Cubic Action

Remarkably, the same action with the full higher-spin curvatures *i.e.*,

$$S=-\frac{1}{4\kappa}\sum_{n,m=0}^{\infty}\frac{i^{n+m-1}}{n!m!}\,\mathrm{sign}(n-m)\int_{M^4}R_{\alpha_1\ldots\alpha_n,\,\dot\beta_1\ldots\dot\beta_m}(x)R^{\alpha_1\ldots\alpha_n,\dot\beta_1\ldots\dot\beta_m}(x)\,,\quad (14.32)$$

provides a solution to the higher-spin interaction problem to the lowest order (including the Aragone-Deser problem of the higher-spin gravitational interactions) at the condition that the extra fields are expressed via derivatives of the dynamical higher-spin gauge fields in accordance with the First on-shell theorem (14.26).

Indeed, the lowest-order part of the gauge variation of the action is quadratic in the fields *i.e.*, $\delta S=\phi^2\epsilon$, where ϕ stands for dynamical fields and their derivatives, since the free part of the action is invariant under the linearized transformations.

The key observation is that any deformation of the transformation laws of the extra fields or of the Lorentz–like auxiliary fields by ϕ–dependent terms does not contribute to the variation of the action at the cubic order. Indeed, by construction, the extra fields do not enter the free (quadratic) part of the action. Therefore, the contribution of the variation coming from the field-dependent deformation of the gauge transformations of the extra fields will be at least of the order ϕ^3. By virtue of the 1.5-order formalism the same argument, which in fact has been used already in the analysis of supergravity, makes it possible to disregard the terms due to the deformation of the transformation law of the Lorentz-like fields.

The question is now whether the transformation law can be modified by adding some extra terms

$$\delta\omega(y,\bar y|x)=d\epsilon(y,\bar y|x)+[\omega(y,\bar y|x),\epsilon(y,\bar y|x)]_*+\Delta(\phi)\qquad(14.33)$$

in such a way that

$$\delta S = O(\phi^3) \tag{14.34}$$

and the variation (14.33) is still nontrivial. The latter property will be guaranteed if $\Delta(\phi)$ is built from the linearized curvatures because terms of this type cannot compensate the original gauge variation.

To show that a deformed gauge transformation (14.33) exists consider the variation of the action (14.32) under the undeformed gauge transformations (14.4)

$$\delta S = -\frac{1}{2\kappa}\sum_{n,m=0}^{\infty}\frac{i^{n+m-1}}{n!m!}\,\mathrm{sign}(n-m)\int_{M^4}[R_1\,,\varepsilon]_{*\alpha_1\ldots\alpha_n,\,\dot\beta_1\ldots\dot\beta_m}R_1^{\alpha_1\ldots\alpha_n,\dot\beta_1\ldots\dot\beta_m}+O(\phi^3)\ , \tag{14.35}$$

where up to the ϕ^3 terms in the variation the two curvature tensors should be linearized. In this order, the field dependent deformation of the transformation law for the frame-like field contributes only through the variation of the quadratic part of the action hence being proportional to the left-hand sides of the Fronsdal field equations $\mathcal{R}$. Thus, to prove the existence of some deformed transformation (14.33) it suffices to show that (14.35) is proportional to $\mathcal{R}$, *i.e.*, is zero on-shell, since all such terms can be compensated by an appropriate deformation $\Delta(\Phi)$. By virtue of the First on-shell theorem (14.26) the only terms that may be non-zero on-shell are those bilinear in the generalized Weyl tensors. The latter terms vanish however.

Indeed, consider the CC-type terms. Since C carries only undotted indices in this case the non-vanishing contributions could only originate from gauge parameters ε that themselves carry only undotted indices. However, in this case $m = 0$ in Eq. (14.35), so that the function $\mathrm{sign}(n-m)$ is effectively a constant. So this part of the gauge variation must be the same as in S^{top} (14.28), which is zero. The $\overline{CC}$-type terms vanish analogously.

The mixed terms with C and $\overline{C}$ all contain the following combination of the AdS_4 frame one-forms

$$H^{(\alpha\beta),(\dot\gamma\dot\delta)} := e^{\alpha\dot\alpha}e^{\beta}{}_{\dot\alpha}e^{\gamma\dot\gamma}e_{\gamma}{}^{\dot\delta}\ . \tag{14.36}$$

However, this expression vanishes. Indeed, any four-form built out of the frame field must be a Lorentz singlet because there is only one (up to a factor) four-form in the four-dimensional space, namely the volume form. On the other hand $H^{(\alpha\beta),(\dot\gamma\dot\delta)}$ (14.36) describes a rank-two traceless symmetric tensor which contains no Lorentz invariant components. Hence it is zero.

Note that the same proof works for the action (14.32) extended to fermions.

Summarizing, the action (14.32) possesses the following properties:

i) it is manifestly general coordinate invariant, since it only involves exterior products of differential forms;

ii) in the spin-2 sector ($n + m = 2$) it reduces to the MacDowell-Mansouri gravitational action;

iii) it is gauge invariant up to the cubic order, provided the constrains (14.26) are imposed. As a result, it provides an explicit solution to the consistency problem for cubic interactions of massless fields of all spins $s > 1$.

(iv) it does not admit a flat limit because of the nonanalyticity in λ related to the property that higher-spin interactions contain higher derivatives. Technically, both the higher-derivative terms and the nonanalyticity in λ result from the dependence of S on the extra fields at the interaction level, which are non-analytic in λ by (14.27).

Let us stress that the action (14.32) has such a simple form due to the use of the higher-spin connections. The substitution of the expressions for the latter in terms of derivatives of the frame-like fields into the action would lead to a mess.

Analogous higher-spin cubic actions are known in any dimension. However, a generalization of the action (14.32) beyond the cubic level is not known yet. There are two essential points. First is that the field spectrum of the action is not complete since the appropriate higher-spin multiplets also contain lower-spin fields with spins $s \leq 1$. To proceed beyond the cubic level these fields have to be included. The second more serious issue is that to go to higher orders it is necessary to know a nonlinear deformation of the constraints for extra fields contained in the First on-shell theorem (14.26). The off-shell version of these constraints, *i.e.*, with the explicit form of the terms proportional to the left-hand sides of field equations is still unknown. What is known however is the on-shell version of the nonlinear higher-spin field equations.

14.5 Central On-Shell Theorem

The starting point for the formulation of nonlinear higher-spin equations is the extension of the First on-shell theorem which is formulated in terms of the one-form $\omega(Y|X)$ and zero-form $C(Y|X)$ as follows:

$$R_1(Y \mid X) = \mathrm{e}_\alpha{}^{\dot\alpha}\mathrm{e}^{\alpha\dot\beta}\frac{\partial^2}{\partial\bar y^{\dot\alpha}\partial\bar y^{\dot\beta}}C(0,\bar y \mid X) + \mathrm{e}^\alpha{}_{\dot\alpha}\mathrm{e}^{\beta\dot\alpha}\frac{\partial^2}{\partial y^\alpha\partial y^\beta}C(y,0 \mid X)\,, \tag{14.37}$$

$$\tilde D_0 C(Y \mid X) = 0\,, \tag{14.38}$$

where R_1 is defined in (14.20),

$$\tilde D_0 = D^{\mathrm{L}} + \lambda \mathrm{e}^{\alpha\dot\beta}\left(y_\alpha \bar y_{\dot\beta} + \frac{\partial^2}{\partial y^\alpha \partial \bar y^{\dot\beta}}\right)\,, \tag{14.39}$$

and the Lorentz covariant derivative is

$$D^{\mathrm{L}}A = d_X - \left(\omega^{\alpha\beta}y_\alpha\frac{\partial}{\partial y^\beta} + \bar\omega^{\dot\alpha\dot\beta}\bar y_{\dot\alpha}\frac{\partial}{\partial \bar y^{\dot\beta}}\right)\,. \tag{14.40}$$

Since the system of equations (14.37) and (14.38) contains the exhaustive information about free massless fields, including all their dual formulations, it is called the *Central on-shell theorem*.

The pattern of the Central on-shell theorem is as follows. Gauge fields of different spins are described by the one-forms ω which in accordance with (14.23) are homogeneous polynomials in Y and the zero-forms C that obey

$$C^s(\nu y, \nu^{-1}\bar y|X) = \nu^{\pm 2s}C(y,\bar y|X)\,. \tag{14.41}$$

This implies that a set of one-forms associated with a massless spin s contains a finite number of components while a set of zero-forms contains an infinite number of components. Altogether, these fields describe an infinite set of spins $s = 0, 1/2, 1, 3/2, 2, 5/2 \ldots$

$$\omega^s_{\alpha_1\ldots\alpha_n\,,\dot\beta_1\ldots\dot\beta_m}: \quad n+m = 2(s-1)\,, \qquad C^s_{\alpha_1\ldots\alpha_n\,,\dot\beta_1\ldots\dot\beta_m}: \quad |n-m| = 2s\,. \tag{14.42}$$

The zero-forms $C(Y|X)$ encode the gauge invariant higher-spin curvatures and spin-zero matter fields along with all their derivatives that remain non-zero on the dynamical field equations. Dynamical fields include the frame-like fields $\omega^s_{\alpha_1\ldots\alpha_{s-1}\,,\dot\beta_1\ldots\dot\beta_{s-1}}$ and the scalar $C(0,0|x)$. In the spin-zero sector (14.38) amounts to the unfolded spin-zero field equations which we have derived in any space-time dimensions. Eq. (14.38) describes the same equations in the language of two-component spinors. Recall that, as explained in Section 14.2, in four space-time dimensions multispinors $A_{\alpha_1\cdots\alpha_k,\,\dot\beta_1\cdots\dot\beta_k}$ are equivalent to rank-k traceless totally symmetric tensors $A_{n_1,\cdots,n_k}$.

All other fields are expressed by Eqs. (14.37) and (14.38) via higher derivatives of the dynamical fields. The derivatives come in the dimensionless combination

$$\lambda^{-1}\frac{\partial}{\partial x}\,, \qquad \lambda^2 = -\Lambda$$

with λ being the inverse radius of the background AdS space. As a result, as we have discussed already, the higher-spin interactions, that contain higher derivatives, turn out to be nonanalytic in the cosmological constant Λ of the background AdS space.

The Central on-shell theorem (14.37) and (14.38) provides a starting point for the search of an interaction system as its nonlinear deformation. The full form of nonlinear higher-spin field equations is known.

14.6 Structure of higher-spin interactions

Let us discuss the structure of higher-spin interactions in some more detail. Spin-s generators are represented by homogeneous polynomials $T_s(y,\bar y)$ of degree $2(s-1)$. The commutation relations have the following structure

$$[T_{s_1}\,, T_{s_2}] = T_{s_1+s_2-2} + T_{s_1+s_2-4} + \ldots + T_{|s_1-s_2|+2}\,.$$

Once a spin $s > 2$ appears, the higher-spin algebra contains an infinite tower of higher spins. Indeed, since $[T_s, T_s]$ gives rise to T_{2s-2}, further commutators then lead to higher and higher spins. Note also that $[T_s, T_s]$ contains the generators T_2 of the AdS_4 algebra $o(3,2) \simeq sp(4)$.

The form of the higher-spin curvatures (14.7) suggests that nonlinear combinations of lower spins contribute to the curvatures of the higher spins and vise versa.

The symmetry algebra of a single boundary scalar field called $hu(1,0|4)$ contains every integer spin in one copy. Conventional symmetries are associated with spins $s \le 2$, forming finite-dimensional subalgebras of the higher-spin algebra. For example, the maximal finite-dimensional subalgebra of $hu(1,0|4)$ is $u(1) \oplus o(3,2)$ where $u(1)$ is associated with the unit element of the star-product algebra.

More generally, there are three series of $4d$ higher-spin superalgebras, namely $hu(n,m|4)$, $ho(n,m|4)$ and $husp(2n,2m|4)$. Spin-one fields of the respective higher-spin theories are the Yang-Mills fields of the Lie groups $G = U(n) \times U(m)$, $O(n) \times O(m)$ and $Usp(2n) \times Usp(2m)$, respectively. Fermions belong to the bifundamental modules of the two components of G. All odd spins are in the adjoint representation of G. Even spins carry the opposite symmetry second rank representation of G. Namely, in the $hu(n,m|4)$ higher-spin theories they are still in the adjoint representation of $U(n) \times U(m)$, while in the $ho(n,m|4)$ and $husp(2n,2m|4)$ higher-spin theories even spins carry rank-two symmetric representation of $O(n) \times O(m)$ and antisymmetric representation of $Usp(2n) \times Usp(2m)$, respectively. The $ho(1,0|4)$ higher-spin theory is the minimal one only containing even spins $s = 0, 2, 4, 6, \ldots$.

The higher-spin theories have the important feature that they always contain a colorless graviton and a colorless scalar which are both invariant under the spin-one Yang-Mills internal symmetries. It is interesting to note that the presence of the colorless scalar field in the spectrum, which is a standard ingredient of the modern cosmological models, is one of the predictions of the higher-spin symmetry.

The important question to address is which of these models are supersymetric in the usual sense. The answer is quite simple. These are the models with $n = m = 2^N$. Indeed, as we know, in these cases the matrix algebras underlying the construction of extended higher-spin algebras are equivalent to the Clifford algebras C_N which can be realized as the algebras of functions of the Clifford elements ϕ_i

$$\{\phi_i\,, \phi_j\} = 2\delta_{ij}\,, \tag{14.43}$$

i.e., gauge connections are now functions $\omega(y,\phi|x)$ that are required to be even in the sense that

$$\omega(Y,\phi|x) = \omega(-Y,-\phi|x). \tag{14.44}$$

This condition implies that fermion (boson) fields which carry odd numbers of spinorial indices contain an odd (even) number of anticommuting Clifford elements ϕ_i. Note that in the case of $N = 1$ it does not matter whether or not we introduce the Clifford element ϕ since $\phi^2 = 1$, *i.e.*, ϕ just indicates whether or not the element is bosonic or fermionic.

The connections associated with bilinears of Y and ϕ are the gauge fields of $osp(N,4)$. Indeed, the connections bilinear in Y like the vacuum connection (14.16) just describe the gravitational Lorentz connection and vierbein which, along with the spin-3/2 connections

$$\psi^i_\alpha(x) y^\alpha \phi_i\,, \qquad \bar{\psi}^i_{\dot\alpha}(x) \bar{y}^{\dot\alpha} \phi_i \tag{14.45}$$

and spin-one connections

$$A^{ij}(x)[\phi_i\,, \phi_j]\,, \qquad A^{ij}(x) = -A^{ji}(x) \tag{14.46}$$

form the $osp(N,4)$ connections of AdS_4 supersymmetry discussed in Section 11.5. We leave it to the reader to check that various bilinears in Y and ϕ form an algebra $osp(N,4)$.

15 Unfolded Dynamics

15.1 Unfolded equations

Both the Central on-shell theorem (14.37) and (14.38) and its nonlinear deformation provide examples of the so-called *unfolded equations*.

15.1.1 General setup

The unfolded form of dynamical equations is a covariant generalization of the first-order form of differential equations

$$\dot{q}^i(t) = \varphi^i(q(t)) ,$$

which is convenient in many respects. In particular, initial values can be given in terms of the values of variables $q^i(t_0)$ at any given point t_0. As a result, in the first-order formulation, the number of degrees of freedom equals to the number of dynamical variables.

Unfolded dynamics is a multidimensional generalization achieved via the replacement of the time derivative by the de Rham derivative

$$\frac{\partial}{\partial t} \to \mathrm{d} = \xi^{\underline{n}} \partial_{\underline{n}}$$

and the dynamical variables q^i by a set of differential forms

$$q^i(t) \to W^\Omega(\xi, x) = \xi^{\underline{n}_1} \dots \xi^{\underline{n}_p} W^\Omega_{\underline{n}_1 \dots \underline{n}_p}(x)$$

to reformulate a system of partial differential equations in the first-order covariant form

$$\mathrm{d}W^\Omega(\xi, x) = G^\Omega(W(\xi, x)) . \tag{15.1}$$

Here $G^\Omega(W)$ are some functions of the "supercoordinates" W^Ω

$$G^\Omega(W) = \sum_n f^\Omega{}_{\Lambda_1 \dots \Lambda_n} W^{\Lambda_1} \dots W^{\Lambda_n} . \tag{15.2}$$

Since $\mathrm{d}^2 = 0$, at $d > 1$ the functions $G^\Lambda(W)$ have to obey the compatibility conditions

$$G^\Lambda(W) \frac{\partial G^\Omega(W)}{\partial W^\Lambda} \equiv 0 . \tag{15.3}$$

(Recall that all products of the differential forms $W(\xi, x)$ are the wedge products due to anticommutativity of $\xi^{\underline{n}}$.) Let us stress that these are conditions on the functions $G^\Lambda(W)$ rather than on W.

As a consequence of the compatibility conditions (15.3) the system (15.1) is manifestly invariant under the gauge transformation

$$\delta W^\Omega = \mathrm{d}\varepsilon^\Omega + \varepsilon^\Lambda \frac{\partial G^\Omega(W)}{\partial W^\Lambda} , \tag{15.4}$$

where the gauge parameter $\varepsilon^{\Omega}(x)$ is a $(p_{\Omega}-1)$-form if W^{Ω} is a p_{Ω}-form. Strictly speaking, this is true for the class of *universal* unfolded systems in which the compatibility conditions (15.3) hold independently of the dimension d of space-time, *i.e.*, (15.3) should be true disregarding the fact that any $(d+1)$-form is zero. Let us stress that all unfolded systems, which appear in higher-spin theories including those considered in these lectures, are universal. We leave it to the reader as a simple exercise to check that the equations (15.1) are invariant under the gauge transformation (15.4).

The idea of the unfolded formulation was put forward when it was realized that the nonlinear equations can be searched in the form (15.1) as a deformation of the Central on-shell theorem.

This method can be applied to the description of invariant functionals of the system in question. Here it is useful to distinguish between the off-shell and on-shell unfolded dynamical systems.

As demonstrated in Section 13.1.3, most of the relations contained in unfolded equations impose constraints expressing some new fields in terms of derivatives of the old ones. In the *off-shell* case the unfolded equations just express all fields in terms of derivatives of some ground fields, imposing no differential restrictions on the latter. In the scalar-field example of Section 13.1.3, to make the system off-shell one should relax the tracelessness condition in (13.13). In this case, the pattern of the unfolded system (13.15) is given by the set of constraints (13.17) which express the higher tensors $C_{n_1\dots n_k}(x)$ via derivatives of the ground scalar field $C(x)$. The *on-shell* unfolded equations not only express all fields in terms of derivatives of the ground fields, but also impose differential restrictions on the latter. In the scalar-field example this is the Klein-Gordon equation (13.1).

Invariant functionals associated with the unfolded equations (15.1) are described by the cohomology of the operator

$$Q = G^{\Omega}(W)\frac{\partial}{\partial W^{\Omega}}\,, \tag{15.5}$$

which obeys

$$Q^2 = 0 \tag{15.6}$$

as a consequence of (15.3).

In these terms, unfolded equations take the form

$$\mathrm{d}F(W) = QF(W)\,, \tag{15.7}$$

where $F(W)$ is an arbitrary function of W. By virtue of (15.1), Q-closed p-form functions $L_p(W)$ are d-closed, giving rise to the gauge invariant functionals

$$S = \int_{\Sigma^p} L_p\,. \tag{15.8}$$

Ob'yasnit' In the off-shell case they can be used to construct invariant action functionals while in the on-shell case they describe conserved charges.

15.1.2 Properties

The unfolded formulation of partial differential equations has a number of remarkable properties.

- First of all, it has general applicability: every system of partial differential equations can be reformulated in the unfolded form.
- Due to the exterior algebra formalism, the system is invariant under diffeomorphisms, being coordinate independent.
- Interactions correspond to nonlinear deformations of $G^\Omega(W)$.
- Degrees of freedom are carried by the subset of zero-forms $\mathcal{C}^I(x_0) \in \{W^\Omega(x_0)\}$ at any $x = x_0$. This is analogous to the fact that $q^i(t_0)$ describe degrees of freedom in the first-order form of ordinary differential equations. The zero-forms $\mathcal{C}^I(x_0)$ realize an infinite-dimensional module dual to the space of single-particle states of the system. In the higher-spin theory it is realized as a space of functions of auxiliary variables like $C(y, \bar{y}|x_0)$.
- Unfolded dynamics provides a tool to control unitarity in presence of higher derivatives via the requirement that the space of zero-forms like $C(y, \bar{y})$ admits a positive-definite norm preserved by the unfolded equations in question.

The above list of remarkable properties of the unfolded formulation is far from being complete. In particular, the unfolded formulation admits a nice interpretation in terms of Lie algebra cohomology. The most striking feature of this formulation is however that it makes it possible to describe one and the same dynamical system in space-times of different dimensions.

15.2 Space-time metamorphoses

Unfolded dynamics exhibits independence of the “world-volume” space-time with coordinates x. Instead, geometry is encoded by the functions $G^\Omega(W)$ in the “target space” of fields W^Ω. Indeed, the universal unfolded equations make sense in any space-time independently of a particular realization of the de Rham derivative d. For instance one can extend space time by adding additional coordinates z

$$\mathrm{d}W^\Omega(x) = G^\Omega(W(x))\,, \quad x \to X = (x, z)\,, \quad \mathrm{d}_x \to \mathrm{d}_X = \mathrm{d}_x + \mathrm{d}_z\,, \quad \mathrm{d}_z = \xi^u \frac{\partial}{\partial z^u}\,.$$

The unfolded equations reconstruct the X-dependence in terms of values of the fields $W^\Omega(X_0) = W^\Omega(x_0, z_0)$ at any X_0. Clearly, to take $W^\Omega(x_0, z_0)$ in space M_X with coordinates X_0 is the same as to take $W^\Omega(x_0)$ in the space $M_x \subset M_X$ with coordinates x.

The problem becomes most interesting provided that there is a nontrivial vacuum connection along the additional coordinates z. This is in particular the case of AdS/CFT correspondence where the conformal flat connection at the boundary is extended to the flat AdS connection in the bulk with z being a radial coordinate of the Poincaré type.

Generally, the unfolding can be interpreted as some sort of a covariant twistor transform

$$\begin{tikzcd} & C(Y|x) \arrow[dl, "\eta"'] \arrow[dr, "\nu"] & \\ M(x) & & T(Y)\,. \end{tikzcd}$$

Here $W^{\Omega}(Y|x)$ are functions on the "correspondence space" C with local coordinates Y, x. The space-time M has local coordinates x. The twistor space T has local coordinates Y.

The unfolded equations reconstruct the dependence of $W^{\Omega}(Y|x)$ on x in terms of the function $W^{\Omega}(Y|x_0)$ on T at some fixed x_0. The restriction of $W^{\Omega}(Y|x)$ or some its Y-derivatives to $Y = 0$ gives dynamical fields $\omega(x)$ in M which, in the on-shell case, solve their dynamical field equations. Hence unfolded equations map functions on T to solutions of the dynamical field equations in M.

In these terms, the holographic duality can be interpreted as the duality between different space-times M that can be associated with the same twistor space. This phenomenon has a number of interesting realizations.

15.3 $sp(8)$-invariant setup

An example of the application of unfolded dynamics is related to the $sp(8)$ extension of conformal symmetry in the theory of massless fields in four dimensions. As was shown by Fronsdal, the tower of all $4d$ massless fields is $sp(8)$ symmetric. The $sp(8)$ symmetry extends conformal symmetry $su(2,2) \subset sp(8)$ that acts on every massless field. The generators in $sp(8)/su(2,2)$ mix fields of different spins in the tower of massless fields of all spins $0 \leq s < \infty$.

In fact, we can easily deduce these facts by means of the unfolded dynamics formulation. Indeed, the unfolded equations (14.38) for $4d$ massless fields of all spins in the flat limit $\lambda \to 0$ take the form

$$\left(\frac{\partial}{\partial x^{\alpha\dot{\alpha}}} + \frac{\partial^2}{\partial y^{\alpha}\partial \bar{y}^{\dot{\beta}}}\right) C(y, \bar{y}|X) = 0\,. \tag{15.9}$$

Note that to obtain these equations it is necessary to rescale the spinor variables

$$y_{\alpha} \to \lambda^{\frac{1}{2}} y_{\alpha}\,, \qquad \bar{y}_{\dot{\alpha}} \to \lambda^{\frac{1}{2}} \bar{y}_{\dot{\alpha}} \tag{15.10}$$

before taking the limit.

We observe that Eqs. (15.9) are the covariant constancy conditions on the space of functions of four spinor variables instead of two in the $3d$ case considered in Section 13.2. Therefore the higher-spin conformal symmetry algebra of the set of all $4d$ massless fields is the Lie algebra associated with the Weyl algebra A_4 generated by $Y^{\hat{\alpha}}$ and $\frac{\partial}{\partial Y^{\hat{\alpha}}}$. This contains $sp(8)$ realized by bilinears (13.25) $Y^{\hat{\alpha}}$ and $\frac{\partial}{\partial Y^{\hat{\alpha}}}$.

15.3.1 From four to ten

Fronsdal has shown that the space-time $\mathcal{M}_4$ appropriate for geometric realization of $Sp(8)$ is ten-dimensional with local coordinates $X^{\hat{\alpha}\hat{\beta}} = X^{\hat{\beta}\hat{\alpha}}$, where $\hat{\alpha} = (\alpha, \dot{\alpha}) = 1, 2, 3, 4$. Applying the construction of Section 15.2 it is easy to derive the equations for massless fields in $\mathcal{M}_4$. Indeed, unfolded $4d$ massless equations can be uplifted to $\mathcal{M}_4$ as follows

$$dX^{\hat{\alpha}\hat{\beta}}(\frac{\partial}{\partial X^{\hat{\alpha}\hat{\beta}}} + \frac{\partial^2}{\partial Y^{\hat{\alpha}} \partial Y^{\hat{\beta}}})C(Y|X) = 0\,, \qquad \hat{\alpha}, \hat{\beta} = 1, \dots 4\,. \tag{15.11}$$

By the general argument in the beginning of Section 15.2 (15.11) describes the same dynamics as the original massless field equations in $4d$ Minkowski space because they consist of the usual $4d$ equations (15.9) for the coordinates $X^{\alpha\dot{\beta}}$ supplemented with the equations describing the evolution along the additional spinning coordinates $X^{\alpha\beta}$ and $X^{\dot{\alpha}\dot{\beta}}$. The key question is what are independent dynamical variables in $\mathcal{M}_4$? From (15.11) it is clear that these are the fields $C(0|X)$ and $Y^{\hat{\alpha}} C_{\hat{\alpha}}(0|X)$. Indeed, all other components of $C(Y|X)$ are expressed by Eq. (15.11) via X-derivatives of $C(0|X)$ and $Y^{\hat{\alpha}} C_{\hat{\alpha}}(0|X)$. It turns out that $C(0|X)$ describes all $4d$ massless fields of integer spins while $C_{\hat{\alpha}}(0|X)$ describes all $4d$ massless fields of half-integer spins.

As is easy to check using the σ_--cohomology analysis, the nontrivial field equations in $\mathcal{M}_4$ are

$$\left(\frac{\partial^2}{\partial X^{\hat{\alpha}\hat{\beta}} \partial X^{\hat{\gamma}\hat{\delta}}} - \frac{\partial^2}{\partial X^{\hat{\gamma}\hat{\beta}} \partial X^{\hat{\alpha}\hat{\delta}}} \right) C(X) = 0 \tag{15.12}$$

for bosons and

$$\left(\frac{\partial}{\partial X^{\hat{\alpha}\hat{\beta}}} C_{\hat{\gamma}}(X) - \frac{\partial}{\partial X^{\hat{\gamma}\hat{\beta}}} C_{\hat{\alpha}}(X) \right) = 0 \tag{15.13}$$

for fermions. In particular, (15.12) and (15.13) are obvious consequences of (15.11).

These equations are interesting in many respects. First of all, they are overdetermined. This is what makes it possible to describe the four-dimensional massless fields by virtue of differential equations in the ten-dimensional space $\mathcal{M}_4$. Another interesting feature is that equations (15.12) and (15.13) contain no index contraction and hence no metric tensor.

15.3.2 From ten to four

It is instructive to see how the usual space-time picture re-appears from the ten-dimensional one. Remarkably, in this setup, the four-dimensional picture results from the identification of a concept of local event simultaneously with the metric tensor. Referring for more detail of the derivation to the original paper [7], we just summarize the final results.

Time in $\mathcal{M}_M$ is a parameter t along a time-like direction in $\mathcal{M}_4$ represented by any positive-definite matrix $T^{\hat{\alpha}\hat{\beta}}$

$$X^{\hat{\alpha}\hat{\beta}} = T^{\hat{\alpha}\hat{\beta}} t\,.$$

Usual space in $\mathcal{M}_M$ is identified with the space of local events at a given time. Coordinates of the space of local events x^n are required to have the property that the differential equations in question admit "initial data" localized at any point of space-time, *i.e.*, represented by the δ-functions $\delta(x^n - x_0^n)$ with various x_0^n. Since the system of equations in question is overdetermined, the analysis of this issue is not quite trivial. The final result is [7] that, for Eqs. (15.12), (15.13), the space of local events in $\mathcal{M}_4$ is represented by a Clifford algebra with

$$X^{\hat{\alpha}\hat{\beta}} = x^n \gamma_n{}^{\hat{\alpha}}{}_{\hat{\gamma}} T^{\hat{\beta}\hat{\gamma}}$$

formed by matrices $\gamma_n{}^{\hat{\alpha}}{}_{\hat{\beta}}$ that obey

$$\{\gamma_n\,, \gamma_m\} = 2g_{nm}\,, \tag{15.14}$$

where g_{nm} is the spatial metric tensor of R^3.

Thus, the three-dimensional space of the $4d$ Minkowski space appears as the space R^3 of local events. In this analysis, the metric tensor appears just after the identification of coordinates that parametrize local events with the generators of the Clifford algebra. In a certain sense, this construction is opposite to the original Dirac's construction where the γ-matrices were introduced as a square root of the metric tensor. Here, the metric tensor appears from the definition of the γ-matrices that represent local events.

Analogous analysis can be performed in some other dimensions. In particular Eqs. (15.11) at $M = 2, 4, 8, 16$ describe free conformal fields of all spins in $d = 3, 4, 6, 10$.

It should be noted that different $sp(2M)$-symmetric field equations in the same space $\mathcal{M}_M$ have spaces of local events of different dimensions. The resulting picture is somewhat analogous to the brane picture in String Theory allowing the co-existence of objects of different dimensions in the same space. The difference is however that the "higher-spin branes" in the $sp(2M)$ setup are not localized as a particular surface embedded into $\mathcal{M}_M$. Instead, different choices of a representative surface is a matter of the gauge choice. This example gives another manifestation of the general property that higher symmetries may affect such fundamental concepts as local event and space-time dimension.

16 Conclusion

In the end of these lectures let me mention that we arrived at the point where really interesting things start happen which are currently under investigation. Although I had no time to discuss many important directions in the field, I believe that those who followed these lectures should be able to study and even to do research in this exciting field filling in the remaining gaps with the help of reviews listed in the references.

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